The Covering Radius of the Reed–Muller Code $RM(m - 4, m)$ in $RM(m - 3, m)$ [*]

Randall Dougherty, R. Daniel Mauldin, and Mark Tiefenbruck

Draft: June 29, 2021 @ 1:25 am

Abstract

We present methods for computing the distance from a Boolean polynomial on m variables of degree $m - 3$ (i.e., a member of the Reed–Muller code $RM(m - 3, m)$) to the space of lower-degree polynomials ($RM(m - 4, m)$). The methods give verifiable certificates for both the lower and upper bounds on this distance. By applying these methods to representative lists of polynomials, we show that the covering radius of $RM(4, 8)$ in $RM(5, 8)$ is 26 and the covering radius of $RM(5, 9)$ in $RM(6, 9)$ is between 28 and 32 inclusive, and we get improved lower bounds for higher m. We also apply our methods to various polynomials in the literature, thereby improving the known bounds on the distance from 2-resilient polynomials to $RM(m - 4, m)$.

Index Terms—Reed–Muller code, covering radius, nonlinearity, minimum weight, certificate, resilient.

1   Introduction

Let $RM(r, m)$ be the rth order Reed-Muller code of length 2^m, which is the set of truth tables of all Boolean polynomials of degree at most r on m variables. The Reed-Muller codes are one of the best-understood families of codes and have favorable properties such as being relatively easy to decode [15], but many questions regarding their covering radii remain open.

We will be studying distances from members of $RM(r, m)$ to $RM(r - 1, m)$, where the distance between two truth tables is defined as the number of inputs on which they differ. The maximum distance from any element of $RM(r, m)$ to the closest element of $RM(r - 1, m)$ is the covering radius of $RM(r - 1, m)$ in $RM(r, m)$. We will concentrate on the case $r = m - 3$ in this paper. This is the largest unsolved case; the easy cases $r = m$ and $r = m - 1$ are in [8, Sec. 9.1], and $r = m - 2$ is settled in [17].

Schatz [22] showed that the covering radius of $RM(2, 6)$ (and, in particular, the covering radius of $RM(2, 6)$ in $RM(3, 6)$) is 18. Hou [12] later gave a noncomputer proof of this result. Wang, Tan, and Prabowo [25] showed that the covering radius of $RM(3, 7)$ in $RM(4, 7)$ is 20. Hou [12] showed that the covering radius of $RM(4, 8)$ in $RM(5, 8)$ is at least 22. McLoughlin [17] showed that the covering radius of $RM(m - 4, m)$ in $RM(m - 3, m)$ is at least $2m + 2$ for odd m and $2m$ for even m; asymptotically the sphere-covering bound [15] gives a quadratic lower bound on this covering radius.

In this paper, we present (in Sections 3–6) a method for proving a lower bound on the distance from f to $RM(m - 4, m)$, where f is a given member of $RM(m - 3, m)$. The method can require substantial computation to generate a proof but produces a "certificate" that can be used to verify the lower bound with far less computation. We also describe (in Section 7) simple methods for producing an upper bound on the distance (by finding a member of $RM(m - 4, m)$ close to f, so the upper bound is also easily verifiable). By applying these methods to suitable representative lists of Boolean polynomials, we show (in Sections 8–9) that the covering radius of $RM(4, 8)$ in $RM(5, 8)$ is 26 and the covering radius of $RM(5, 9)$ in $RM(6, 9)$ is between 28 and 32 inclusive. (The certificates and other data will be made available online.) We also show (in Section 10) that certificates for polynomials on m variables can be modified to give certificates for certain polynomials on more than m variables, thus giving improved lower bounds on the covering radius of $RM(m' - 4, m')$ in $RM(m' - 3, m')$ for higher m'.

Once we have the distance to $RM(m - 4, m)$ (or lower and upper bounds on it) for each of the representative polynomials, we can get this information for any polynomial in $RM(m - 3, m)$ by looking up which representative polynomial it is equivalent to. In Section 11, we use this method to compute this distance for a number of polynomials from the literature that have desirable cryptographic properties. This lets us give improved bounds on several values of the function $\hat{\rho}(t, r, n)$, the maximum distance between a t-resilient function and $RM(r, n)$.

2   Preliminaries

An m-variable Boolean function is a function from $\mathbf{F}_2^m$ to $\mathbf{F}_2$, where $\mathbf{F}_2$ is the two-element field; we also refer to such functions as truth tables. Any m-variable Boolean function has a unique representation as an m-variable Boolean polynomial, a polynomial over $\mathbf{F}_2$ where no variable occurs to a power higher than 1 (because $\mathbf{F}_2$ satisfies the identity $X^2 = X$). The Reed-Muller code $RM(r, m)$ is the set of truth tables of all Boolean polynomials of degree at most r on m variables.

Note: For the rest of this paper we will not distinguish between a polynomial and its truth table.

The book of MacWilliams and Sloane [15] is a good reference for basic results on Reed–Muller codes. In addition to the book [8], papers giving general upper and/or lower bounds on the covering radii of Reed–Muller codes include [4], [6], [9], and [17].

For a given Boolean polynomial f on m variables, the *weight* of f is the number of input vectors $v \in \mathbf{F}_2^m$ such that $f(v) = 1$ (i.e., the number of 1s in the truth table for f). The *distance* from f to a set S is the minimum of the weights of polynomials $f + p$ (which is the same as $f - p$) for p in S.

The distance from f to $RM(r - 1, m)$ is also known as the nonlinearity of order $r - 1$ of f or the minimum weight of the coset $f + RM(r - 1, m)$. We will use the term "minimum weight" here; in fact, we will sometimes abuse terminology slightly by saying "minimum weight of f" as an abbreviation for "minimum weight of the coset $f + RM(r - 1, m)$".

We refer to polynomials of degree $m - 3$ as cocubic polynomials; similarly, we will use the words "coquadratic" for degree $m - 2$ and "coquartic" for degree $m - 4$. (We will not use the word "colinear.")

The most straightforward (but usually not the most efficient) way to compute the covering radius of $RM(r - 1, m)$ in $RM(r, m)$ is to compute the minimum weight of every coset $f + RM(r - 1, m)$ where f is homogeneous of degree r.

The work here can be cut down considerably by considering the action of the general linear group $GL(m, 2)$ on Boolean polynomials: an element h of $GL(m, 2)$ sends polynomial f to $f \circ h$. This action is linear on f and preserves the degree of f and also the weight of f (since h just permutes the inputs to f). It follows that h maps the coset $f + RM(r - 1, m)$ to the coset $(f \circ h) + RM(r - 1, m)$ (although $f \circ h$ may not be homogeneous even if f is), and the action preserves minimum weights of cosets. Therefore, this action partitions the cosets $f + RM(r - 1, m)$ into equivalence classes, and it suffices to compute the minimum weight of one representative coset from each equivalence class.

Another useful operation is complementation: given a polynomial f on m variables that is homogeneous of degree r, one can get the complementary polynomial f^c of degree $m - r$ by replacing each r-variable monomial in f with the product of the other $m - r$ variables. It turns out, as shown in Hou [11], that homogeneous degree-r polynomials f and g are in $GL(m, 2)$-equivalent cosets of $RM(r - 1, m)$ if and only if f^c and g^c are in $GL(m, 2)$-equivalent cosets of $RM(m - r - 1, m)$. Hence, given a list of homogeneous cubic polynomials that (together with 0) are representatives for the $GL(m, 2)$-equivalence classes of $RM(3, m)/RM(2, m)$, we can complement these polynomials to get a list of homogeneous cocubic polynomials

that (together with 0) are representatives for the $GL(m, 2)$-equivalence classes of $RM(m - 3, m)/RM(m - 4, m)$.

Hou [11] gives a list of representative cubics in up to 8 variables (six in up to 6 variables, six more in 7 variables, and twenty more in 8 variables). Brier and Langevin [2] produced a list of 349 representative 9-variable cubics, which is available from Langevin's web site [3]. We can complement the polynomials in these lists to get lists of representative cocubic polynomials. (In the case of 6 variables, where cubic is the same as cocubic, we will not bother to complement.)

Two more facts will be useful to us later. One is that an m-variable Boolean polynomial has degree m if and only if its weight is odd [15, Chapter 13, Problem 5]. The other is the following proposition.

Proposition 2.1. *If the m-variable Boolean polynomial p is such that the coset $p + RM(r, m)$ has minimum weight w, and Y is a new variable, then the coset $Yp + RM(r+1, m+1)$ has minimum weight w.*

This is included in the proof of [8, Lemma 9.3.1], but we provide a proof here for convenience.

Proof. Let $\mathrm{wt}_m(q)$ denote the weight of a Boolean polynomial q on m variables. If p' is in $p + RM(r, m)$ and has weight w, then Yp' is in $Yp + RM(r+1, m+1)$ and has weight w. It remains to show that any polynomial in $Yp + RM(r+1, m+1)$ has weight at least w. Any such polynomial has the form $Yp + Yq + q'$ where q and q' are polynomials in the original m variables, q has degree at most r, and q' has degree at most $r + 1$. There are $\mathrm{wt}_m(q')$ solutions to $Yp + Yq + q' = 1$ with $Y = 0$, and $\mathrm{wt}_m(p + q + q')$ solutions with $Y = 1$. So $\mathrm{wt}_{m+1}(Yp + Yq + q') = \mathrm{wt}_m(p + q + q') + \mathrm{wt}_m(q') \geq \mathrm{wt}_m(p + q + q' + q') = \mathrm{wt}_m(p + q) \geq w$, as desired. ∎

3   Initial lower bound results

We start by showing that most cocubic Boolean polynomials on m variables have weight at least $2m + 2$.

Theorem 3.1. *Any Boolean polynomial f of degree at most $m - 3$ in m variables that has no affine (degree-1) factors has weight at least $2m + 2$.*

Proof. For any such f, construct an $(m + 1) \times 2^m$ matrix T over $\mathbf{F}_2$ as follows. We label the rows with the polynomials $f, X_1f, X_2f, \ldots, X_mf$, and we label the columns with the 2^m possible settings for the input variables $X_1, \ldots, X_m$; the entry for row g and column v is the value $g(v)$. (So this is a joint truth table for the listed polynomials.)

We first show that the rows of T are linearly independent; this is equivalent to saying that the label polynomials are linearly independent. Suppose the label polynomials are not linearly independent; then some nonempty subset of them has sum 0, which means that $af = 0$ where a is a nonzero polynomial of degree at most 1. We cannot have $a = 1$, since this would give $f = 0$, and the polynomial 0 has affine factors. So a has degree 1, and so

does $a + 1$, and we have $(a + 1)f = af + f = f$, so f has an affine factor, contradicting the assumptions of the theorem.

Next, we show that the rows of T are orthogonal to themselves and to each other. The dot product of truth table rows labeled g_1 and g_2 is the number of input vectors v such that $g_1(v) = g_2(v) = 1$, reduced modulo 2; this is the same as the weight of $g_1 g_2$ modulo 2. But the product of any two of our label polynomials (distinct or not) is a polynomial of the form f, $X_i f$, or $X_i X_j f$, since $f^2 = f$; all of these polynomials have degree less than m and hence have even weight. So all dot products of rows of T are 0.

This shows that the row space of T has dimension $m + 1$ and codimension at least $m + 1$, so the dimension of the ambient space (i.e., the number of columns of T) is at least $2m + 2$.

Now let T' be T with its all-0 columns (those labeled v where $f(v) = 0$) deleted. The all-0 columns do not affect the independence or orthogonality of rows, so the arguments above show that the number of columns of T' is at least $2m + 2$. The columns of T' are labeled with v such that $f(v) = 1$, so the number of columns of T' is the weight of f, and we are done. ∎

This result gives a lower bound on the weight of a cocubic polynomial f, but we are interested in the minimum weight of f. Thus, we will have to consider polynomials $f + p$ where p has degree at most $m - 4$. In this case, instead of trying to show that all such polynomials $f + p$ have no affine factor, it will be more convenient to impose a linear independence assumption directly (especially since verifying that assumption is probably the most convenient way to prove that f has no affine factor anyway).

Theorem 3.2. *If a Boolean polynomial f of degree $m - 3$ in m variables has the property that the degree-$(m-2)$ parts of the products $X_i f$ $(i = 1, 2, \ldots, m)$ are linearly independent, then $f + RM(m - 4, m)$ has minimum weight at least $2m + 2$.*

Proof. We want to show that, for any polynomial p of degree at most $m - 4$, the polynomial $f + p$ has weight at least $2m + 2$. So construct the matrices T and T' as in the proof of Theorem 3.1, except we now label the rows $f + p$ and $X_i(f + p)$ for $i = 1, 2, \ldots, m$.

For any sum of one or more of the label polynomials that includes at least one polynomial $X_i(f + p)$, the coquadratic part of this sum (which just involves f, since $X_i p$ has degree less than $m - 2$) is nonzero by our assumption on f, so the sum is nonzero. If a sum of one or more label polynomials does not include any of the polynomials $X_i(f + p)$, then the sum is just $f + p$, which has degree $m - 3$ and is therefore nonzero. Therefore, the label polynomials, and hence the rows of T, are linearly independent.

Any product of two of the label polynomials (distinct or not) has the form $f + p$, $X_i(f + p)$, or $X_i X_j(f + p)$; all of these have degree less than m, so the rows of T are orthogonal to themselves and each other.

Deleting the all-zero columns of T to produce T' does not affect either of these conclusions, so as before, the number of columns of T' is at least twice the number of rows of T'. Therefore, the weight of $f + p$ is at least $2m + 2$, as desired. ∎

To get lower bounds better than $2m + 2$ requires an extension of this method, which we will demonstrate in the next section on 6-variable polynomials.

4   The 6-variable case

As we saw earlier, in order to compute the covering radius of $RM(2,6)$ in $RM(3,6)$, we only need to compute the minimum weights of a list of representative polynomials from which every polynomial in $RM(3,6)$ can be obtained by linear transformations of the variables and/or adding terms of degree less than 3.

Hou [11] lists a complete set of six representative cubic polynomials on 6 variables, but three of them essentially reduce (by use of Proposition 2.1) to fewer variables:

- $f_1 = 0$: The zero polynomial on any number of variables obviously has weight 0 and minimum weight 0.

- $f_2 = X_1X_2X_3$: This reduces to the case of the constant polynomial 1 on three variables, which has weight 8 and minimum weight 8.

- $f_3 = X_1X_2X_3 + X_2X_4X_5$: This reduces to the case of the polynomial $X_1X_3 + X_4X_5$ on five variables, which has weight 12 and (by Theorem 3.2) minimum weight 12.

We now consider the remaining three representative cubics. First, let

$$f = f_4 = X_1X_2X_3 + X_4X_5X_6.$$

One can verify directly that f has weight 14; by Theorem 3.2, f has minimum weight 14.

We next look at

$$f = f_5 = X_1X_2X_3 + X_2X_4X_5 + X_3X_4X_6.$$

This has weight 16 by direct computation. Theorem 3.2 gives a lower bound of 14 for its minimum weight; to improve the lower bound to 16, we will add one more row to the truth table in the proof of Theorem 3.1.

Let us list the degree-4 terms from $X_i f$ for this case:

$$
\begin{array}{ll}
X_1f: & X_1X_2X_4X_5 + X_1X_3X_4X_6 \\
X_2f: & X_2X_3X_4X_6 \\
X_3f: & X_2X_3X_4X_5 \\
X_4f: & X_1X_2X_3X_4 \\
X_5f: & X_1X_2X_3X_5 + X_3X_4X_5X_6 \\
X_6f: & X_1X_2X_3X_6 + X_2X_4X_5X_6
\end{array}
$$

All of the monomials here are distinct, so these coquadratic polynomials are linearly independent.

We now add to the truth table a new row given by $(X_1X_2 + X_4X_6)f$.

First note that, when one multiplies $X_1X_2 + X_4X_6$ by f, the only degree-5 term that occurs is $X_1X_2X_3X_4X_6$, and it occurs twice and hence cancels out, so $(X_1X_2 + X_4X_6)f$ has degree 4. This is important because it implies that the truth table row for $(X_1X_2 + X_4X_6)f$ is orthogonal to itself and to the rows for f and $X_i f$.

5

The degree-4 part of $(X_1X_2 + X_4X_6)f$ is $X_1X_2X_4X_5 + X_2X_4X_5X_6$, which is linearly independent of the degree-4 parts of X_if listed above. A quick way to verify this is to note that, if coef($rstu$) is the coefficient of the monomial $X_rX_sX_tX_u$ for a polynomial we are considering, then coef(1245) + coef(1346) is 0 (in $\mathbf{F}_2$) for all of the X_if but is 1 for $(X_1X_2 + X_4X_6)f$.

Putting these facts together, we can follow the argument of Theorem 3.1 using the matrix T augmented with the new row for $(X_1X_2 + X_4X_6)f$: the augmented matrix has a row space of dimension 8 and codimension at least 8, so it must have at least 16 nonempty columns. This shows (again) that the weight of f is at least 16.

However, we cannot yet conclude that the minimum weight of f is at least 16. For Theorem 3.2, the addition of lower-degree terms (degree at most $m - 4$) to f did not affect the arguments; this is no longer the case now that we are multiplying f by a quadratic.

So suppose we instead look at

$$g = f + \sum_{r<s} c_{rs}X_rX_s + \text{lower-degree terms,} \qquad (1)$$

where the lower-degree terms (degree at most 1) will have no effect on the following arguments.

How much of the preceding proof still goes through? The degree-4 parts of X_ig are the same as those of X_if, so these are still linearly independent. The product $(X_1X_2 + X_4X_6)g$ still has degree at most 4, which takes care of the orthogonality conditions.

This leaves the proof of linear independence, and again we use the coefficient combination coef(1245) + coef(1346) to prove this. It turns out that, for $(X_1X_2 + X_4X_6)g$, we have

$$\text{coef(1245)} + \text{coef(1346)} = 1 + c_{13} + c_{45}.$$

This is not general enough to handle all possibilities for g, but we at least have a partial result: if g is as in (1) with $c_{13} + c_{45} = 0$, then the weight of g is at least 16.

It turns out we can handle the case $c_{13} + c_{45} = 1$ by repeating the entire argument with T augmented by the row for $(X_1X_3 + X_4X_5)g$ instead of $(X_1X_2 + X_4X_6)g$. We again find that $(X_1X_3 + X_4X_5)f$ has degree 4, so $(X_1X_3 + X_4X_5)g$ has degree at most 4 and the orthogonality conditions are met. Of course, the X_ig are still linearly independent, so that leaves the question of $(X_1X_3 + X_4X_5)g$'s independence. It turns out that we can show independence for $(X_1X_3 + X_4X_5)g$ by looking at the monomial coefficient coef(1345). This comes out to 0 for X_ig and $c_{13}+c_{45}$ for $(X_1X_3+X_4X_5)g$; since we are assuming $c_{13}+c_{45} = 1$, we have the desired independence, so the weight of g is at least 16.

Putting these two cases together to handle all possible g, we see that the minimum weight of f is at least (and hence exactly) 16.

Finally, consider

$$f = f_6 = X_1X_2X_3 + X_1X_4X_5 + X_2X_4X_6 + X_3X_5X_6 + X_4X_5X_6.$$

Up to renaming variables, this is the polynomial shown by Schatz [22] to have minimum weight 18.

In order to prove a lower bound of 18 on the minimum weight of f, we need to add *two* rows to the truth table in Theorem 3.2. The argument is similar in flavor to the f_5 case, only more complicated, so we will omit some of the easy steps. We again must handle all g of the form (1).

We begin by considering T augmented by the pair of rows for the polynomials $(X_1X_5 + X_2X_6)g$ and $(X_1X_2 + X_1X_5 + X_5X_6)g$. We first check that the products $(X_1X_5 + X_2X_6)g$ and $(X_1X_2 + X_1X_5 + X_5X_6)g$ have degree at most 4; this shows that the two corresponding rows of the truth table are self-orthogonal and orthogonal to all of the X_ig rows. We still have to verify that these two rows are orthogonal to each other.

For independence, we first check that the X_ig rows have linearly independent coquadratic parts. Next, we consider the coefficient combinations $\mathrm{coef}(1245) + \mathrm{coef}(2356)$ and $\mathrm{coef}(1235) + \mathrm{coef}(1245) + \mathrm{coef}(2456)$. Both of these are zero for the X_ig. We also find:

- the degree-6 coefficient of $(X_1X_5 + X_2X_6)(X_1X_2 + X_1X_5 + X_5X_6)g$ is c_{34};

- the combination $\mathrm{coef}(1245) + \mathrm{coef}(2356)$ for $(X_1X_5 + X_2X_6)g$ is $1 + c_{24} + c_{35}$;

- the combination $\mathrm{coef}(1235) + \mathrm{coef}(1245) + \mathrm{coef}(2456)$ for $(X_1X_5 + X_2X_6)g$ is $c_{23} + c_{24} + c_{45}$; and

- the combination $\mathrm{coef}(1235) + \mathrm{coef}(1245) + \mathrm{coef}(2456)$ for $(X_1X_2 + X_1X_5 + X_5X_6)g$ is $1 + c_{23} + c_{35} + c_{45}$.

Thus, the two new rows will be orthogonal to each other if $c_{34} = 0$; $(X_1X_5 + X_2X_6)g$ will be independent of the X_ig if $c_{24} + c_{35} = 0$; and $(X_1X_2 + X_1X_5 + X_5X_6)g$ will be independent of the X_ig and $(X_1X_5 + X_2X_6)g$ if $c_{23} + c_{35} + c_{45} = 0$ and $c_{23} + c_{24} + c_{45} = 0$.

Putting these pieces together and noting that

$$c_{23} + c_{35} + c_{45} = (c_{24} + c_{35}) + (c_{23} + c_{24} + c_{45}),$$

we see that g must have weight at least 18 so long as

$$\begin{aligned} c_{34} &= 0, \\ c_{24} + c_{35} &= 0, \text{ and} \\ c_{23} + c_{24} + c_{45} &= 0. \end{aligned} \tag{2}$$

To handle the remaining possibilities for g, we need to use different quadratic multipliers and/or coefficient combinations.

For the next case, we may repeat the same arguments using the quadratics X_1X_6 and $X_1X_5 + X_2X_6$ and the coefficient combinations $\mathrm{coef}(1346)$ and $\mathrm{coef}(1234) + \mathrm{coef}(1345) + \mathrm{coef}(3456)$. Then, we find:

- the degree-6 coefficient of $X_1X_6(X_1X_5 + X_2X_6)g$ is 0;

- the coefficient $\mathrm{coef}(1346)$ for X_1X_6g is c_{34};

- the combination $\mathrm{coef}(1234) + \mathrm{coef}(1345) + \mathrm{coef}(3456)$ for X_1X_6g is 0; and

- the combination $\text{coef}(1234) + \text{coef}(1345) + \text{coef}(3456)$ for $(X_1X_5 + X_2X_6)g$ is c_{34}.

Hence, g must have weight at least 18 if

$$c_{34} = 1. \tag{3}$$

A similar computation using quadratics X_1X_6 and $X_1X_5 + X_2X_6$ (the same as the preceding case) and coefficient combinations $\text{coef}(1236) + \text{coef}(1246) + \text{coef}(1456)$ and $\text{coef}(1235) + \text{coef}(1245) + \text{coef}(2456)$ shows that g must have weight at least 18 if it satisfies

$$c_{23} + c_{24} + c_{45} = 1. \tag{4}$$

Yet another such computation using quadratics X_1X_6 and $X_1X_2 + X_1X_5 + X_5X_6$ and coefficient combinations $\text{coef}(1246) + \text{coef}(1356)$ and $\text{coef}(1245) + \text{coef}(2356)$ shows that g must have weight at least 18 if

$$\begin{aligned} c_{24} + c_{35} &= 1 \text{ and} \\ c_{23} + c_{24} + c_{45} &= 0. \end{aligned} \tag{5}$$

It is easy to see that, no matter what g is, at least one of the conditions (2), (3), (4), and (5) must hold, so g has weight at least 18. Therefore, the minimum weight of f is 18.

The results in this section give a new noncomputer proof that the covering radius of $RM(2,6)$ in $RM(3,6)$ is 18.

5   Certificates

For a proof of the form given for f_5 and f_6 in the previous section, it should be clear that *verifying* such a proof may be much easier than *finding* it. While verifying those two proofs *may* be within grasp of a human, some of our later results are certainly not. However, the skeptical reader does not simply have to trust us, as with previous computational results mentioned herein. By conveying all the necessary pieces for these proofs in a *certificate*, the reader may verify our claims with a relatively modest computation.

Here is a brief summary of the proofs. We start with a cocubic polynomial f in m variables, and we want to show that it has minimum weight $2m + 2 + 2k$ for some k. We thus must show that every g with the same cocubic terms as f must have weight at least $2m + 2 + 2k$. Then, we divide the proof into a number of cases. For each case, we choose k quadratic multiples of g and append their truth tables to T, as defined in Theorem 3.1. Provided the coefficients of g satisfy certain constraints, we can prove that all the rows of T are linearly independent and orthogonal, which establishes the bound on all such g. When all of the cases, taken together, cover all possible g, our proof is complete.

We now formalize the method by defining a *level-k certificate* that contains all the necessary data to verify such a proof.

Definition 5.1. Let f be a Boolean polynomial of degree $m - 3$ on m variables $X_1, \ldots, X_m$. A *level-k certificate* for f is a sequence of triples $\langle C, q, r \rangle$ (which we will often refer to as "subproofs") where:

- C is a list of affine equations on $\mathbf{F}_2$ variables c_M where M is a monomial of degree $m-4$ (C will be viewed as a condition to be assumed on the values of these variables);

- $q = [q_1, \ldots, q_k]$ is a sequence of k quadratics;

- $r = [r_1, \ldots, r_k]$ is a sequence of k linear combinations of coefficient specifiers $\mathrm{coef}(M)$ where M is a monomial of degree $m-2$;

meeting the following requirements:

1. The degree-$(m-2)$ parts of the polynomials $X_1 f, \ldots, X_m f$ are linearly independent. (This says nothing about the certificate; it is a restriction on the polynomial f.)

2. For each triple $\langle C, q, r \rangle$ and each $j \le k$, $q_j f$ has degree at most $m-2$.

3. For each triple $\langle C, q, r \rangle$ and each pair $j' < j \le k$, if condition C holds, then the coefficient of $X_1 X_2 \cdots X_m$ in $q_{j'} q_j (f + \sum_M c_M M)$ is 0.

4. For each triple $\langle C, q, r \rangle$ and each $i \le m$ and $j \le k$, the coefficient combination r_j for $X_i f$ is 0.

5. For each triple $\langle C, q, r \rangle$ and each pair $j' \le j \le k$, if condition C holds, then the coefficient combination r_j for $q_{j'}(f + \sum_M c_M M)$ is 0 if $j' < j$ but is 1 if $j' = j$.

6. The affine subspaces (flats) defined by the conditions C for all triples $\langle C, q, r \rangle$ in the sequence cover the entire space of possible values of the variables c_M (i.e., any assignment to these variables satisfies at least one condition C).

In particular, a simple level-0 certificate is just a single triple of three null sequences; to verify that it is in fact a certificate for f, one just has to check requirement 1 above. This is a certificate for f_4.

Based on the arguments in the preceding section, a level-1 certificate for f_5 is

$$\langle [c_{13} + c_{45} = 0], [X_1 X_2 + X_4 X_6], [\mathrm{coef}(1245) + \mathrm{coef}(1346)] \rangle,$$
$$\langle [c_{13} + c_{45} = 1], [X_1 X_3 + X_4 X_5], [\mathrm{coef}(1345)] \rangle,$$

and a level-2 certificate for f_6 is

$$\langle [c_{34} = 0, c_{24} + c_{35} = 0, c_{23} + c_{24} + c_{45} = 0], [X_1 X_5 + X_2 X_6, X_1 X_2 + X_1 X_5 + X_5 X_6],$$
$$[\mathrm{coef}(1245) + \mathrm{coef}(2356), \mathrm{coef}(1235) + \mathrm{coef}(1245) + \mathrm{coef}(2456)] \rangle,$$
$$\langle [c_{34} = 1], [X_1 X_6, X_1 X_5 + X_2 X_6],$$
$$[\mathrm{coef}(1346), \mathrm{coef}(1234) + \mathrm{coef}(1345) + \mathrm{coef}(3456)] \rangle,$$
$$\langle [c_{23} + c_{24} + c_{45} = 1], [X_1 X_6, X_1 X_5 + X_2 X_6],$$
$$[\mathrm{coef}(1236) + \mathrm{coef}(1246) + \mathrm{coef}(1456), \mathrm{coef}(1235) + \mathrm{coef}(1245) + \mathrm{coef}(2456)] \rangle,$$
$$\langle [c_{24} + c_{35} = 1, c_{23} + c_{24} + c_{45} = 0], [X_1 X_6, X_1 X_2 + X_1 X_5 + X_5 X_6],$$
$$[\mathrm{coef}(1246) + \mathrm{coef}(1356), \mathrm{coef}(1245) + \mathrm{coef}(2356)] \rangle.$$

Of course, we have abbreviated the notation by writing, say, c_{34} instead of $c_{X_3 X_4}$ and coef(1245) instead of coef($X_1 X_2 X_4 X_5$).

The general result is:

Theorem 5.2. *If f is a Boolean polynomial on m variables of degree $m-3$ that has a level-k certificate, then the minimum weight of $f + RM(m-4, m)$ is at least $2m + 2k + 2$.*

Proof. Fix a level-k certificate for f. Let g be an arbitrary member of the coset $f + RM(m-4, m)$; we must show that g has weight at least $2m + 2k + 2$. Write g in the form

$$g = f + \sum_M c_M M + \text{lower-degree terms}, \tag{6}$$

where M varies over all coquartic monomials. By certificate requirement 6, there is a triple $\langle C, q, r \rangle$ in the certificate such that the condition C is satisfied by this g. Now form a truth table whose rows are given by g, $X_i g$ for $1 \le i \le m$, and $q_j g$ for $1 \le j \le k$. As in Theorem 3.1 and Theorem 3.2, the first $m+1$ rows of this table are linearly independent (using requirement 1), self-orthogonal, and orthogonal to each other. The last k rows are self-orthogonal since $q_j g$ has degree less than m; these rows are orthogonal to each other by requirement 3 and orthogonal to the first $m+1$ rows by requirement 2. Requirements 4 and 5 ensure that each row $q_j g$ is independent of the rows g and $X_i g$ and the preceding rows $q_{j'} g$. Therefore, the row space of the table has dimension $m + k + 1$ and codimension at least $m + k + 1$, so the dimension of the ambient space is at least $2m + 2k + 2$. But all of this goes through if we delete the entirely-0 columns of the table (where g is 0), leaving an ambient space whose dimension is the weight of g, so the weight of g is at least $2m + 2k + 2$, as desired. ∎

Remark. If a cocubic polynomial f does not satisfy requirement 1, then there is a nonzero linear polynomial $a = X_{i_1} + \cdots + X_{i_j}$ such that af has degree less than $m-2$. We can apply a linear transformation of the variables that sends a to X_m; this transformation maps f to a cocubic polynomial f' such that $X_m f'$ has degree less than $m-2$. It follows that the degree-$(m-3)$ part of f' is of the form $X_m p$ where $p \in RM(m-4, m-1)$. Polynomials f and $X_m p$ have the same minimum weight, and one may apply Proposition 2.1 to see that the minimum weight of $X_m p$ is equal to the minimum weight of p. If p still does not satisfy requirement 1, we can repeat this process, eventually obtaining a polynomial q satisfying requirement 1 whose minimum weight is equal to the minimum weight of f. A certificate for q may be used in lieu of a certificate for f. (Note that, if f and $X_m p$ above are homogeneous cocubics, then f^c and $(X_m p)^c$ are homogeneous cubics that are linearly equivalent modulo $RM(2, m)$, and $(X_m p)^c$ does not use variable X_m. Therefore, if f is a homogeneous cocubic on m variables such that f^c is known not to be linearly equivalent modulo $RM(2, m)$ to a polynomial on fewer variables, then f must satisfy requirement 1.)

6   Algorithms for verifying and producing certificates

Checking the validity of a given certificate entails verifying six requirements. All of these verifications are straightforward except for requirement 6: the affine subspaces or flats of the

subproofs cover the space of coquartic coefficients. (This space is usually far too large to simply exhaust over.) This is like a validity problem (the dual of the satisfiability or SAT problem) for a proposition in conjunctive normal form, so we can borrow a technique from SAT solvers to solve it.

Namely, we start by making an assumption (an affine equation) and using it to eliminate a variable from all of the flats; this yields a reduced problem on one fewer variable, where each new flat either has one fewer equation than the corresponding old flat (if the assumption actually followed from the equations defining the flat), has the same number of equations (if the assumption is independent of the flat equations), or is discarded altogether (if the assumption contradicts the flat equations). We apply the algorithm recursively to this reduced problem. Then, if that was successful, we apply the algorithm to the other reduced problem obtained by assuming the negation of the original assumption. If that also succeeds, then the verification is complete.

The efficiency of this recursive procedure depends heavily on how well each assumption is chosen. We currently look at the flats with the smallest number of equations and try to choose an assumption that occurs in as many of them as possible (preferably both positively and negatively).

What if such a verification fails? Failure occurs when, within some number of recursive calls, a state is reached where we have no flats remaining because all of the flats have been contradicted by current assumptions. This means that the set of current assumptions defines a flat (call it a "failure flat") that is disjoint from all of the flats in the purported certificate.

While the verification procedure above is somewhat fuzzy, our procedure for *producing* a level-k certificate is really more art than science. As with SAT solvers, it is difficult to settle on one best technique, and it is probably the case that different strategies work better for different classes of polynomials f. However, we will try to describe our general strategy below.

First, generate a list of quadratics q such that qf has degree at most $m - 2$. There will be many such quadratics; the map sending a quadratic q to the degree-$(m-1)$ part of qf is a linear map from a space of dimension $m(m - 1)/2$ to a space of dimension m, so its kernel, the set of suitable quadratics q, has dimension at least $m(m - 3)/2$. In fact, this will be too many to work with conveniently; it will help to initially restrict to a smaller set of quadratics, such as a basis of the kernel. (If we only had independence requirements, then we could always limit ourselves to such a basis. But orthogonality considerations may require the use of non-basis quadratics; it may be that $q_1 g$ and $q_2 g$ are not orthogonal, and $q_1 g$ and $q_3 g$ are not orthogonal, but $q_1 g$ and $(q_2 + q_3)g$ are orthogonal.)

Next, generate new subproofs. Given a current list of subproofs (which is initially empty), check whether their flats cover the coquartic space. If they do, then we are done; if they don't, then we get a failure flat, from which we can select a coquartic p. Consider the particular polynomial $f + p$, and try to find quadratics $q_1, \ldots, q_k$ from our list such that the products $q_j(f + p)$ are orthogonal to each other and (along with the products $X_i f$) have linearly independent coquadratic parts. If we have such quadratics, then we can select coefficient specifier combinations $r_1, \ldots, r_k$ that demonstrate the linear independence. Now we can generalize from $f + p$ to g, as in (6), and determine which equations C on the

coefficients c_M will ensure that certificate requirements 3 and 5 are satisfied.

There are usually numerous ways to choose $q_1, \ldots, q_k$ and $r_1, \ldots, r_k$ as above; we can search through these to find one for which the success condition C requires as few equations as possible, and therefore the corresponding flat covers as much of the coquartic space as possible. (This search is one reason why we prefer to work with a relatively short list of quadratics.) The result is a triple $\langle C, q, r \rangle$ that we can add to our list of subproofs; then we start a new iteration, and continue until the proof is complete or we find a coquartic p that cannot be handled as above. (There are various refinements that can speed this up, such as concentrating on one failure flat and producing several new subproofs covering parts of it before starting a new full iteration of the algorithm.)

If we find a coquartic p for which there is no suitable list $q_1, \ldots, q_k$, then one of three things has occurred:

- Our current list of quadratics is insufficient; add one or more quadratics to handle this coquartic and resume the algorithm.

- The lower bound we are trying to prove isn't true; see Section 7.

- The lower bound we are trying to prove is true, but there is no certificate proving it; see Section 12.

If the algorithm succeeds in producing a certificate, then this certificate is likely to have a large number of redundant subproofs, because the flat for a particular subproof may end up being covered by flats from earlier and/or later subproofs. A final pass to remove such redundant subproofs can result in a considerably shorter certificate. But this final pass may be quite time-consuming; also, surprisingly, the shorter certificate may require more time to verify than the longer certificate, possibly because the heuristic for choosing assumptions does not perform as well when presented with less information. (Certificates using fewer quadratics tend to reduce better.)

7   Algorithms for upper bounds

Certifying an upper bound w on the minimum weight of a cocubic polynomial f is easy: just produce a coquartic p and verify that the weight of $f + p$ is at most w. This leaves the problem of finding a good p.

One simple approach is a basic hill climb: start with f (or any other polynomial in its coset), and try adding monomials of degree at most $m - 4$ to it in the hopes of finding a new polynomial of lower weight than the given one; if successful, start at this new polynomial and repeat the process. This method tends to succeed if the original f has very few terms, but is less likely to work in other cases.

This leads to another approach: try to modify f by using linear transformations of the variables, in the hope of getting another polynomial f' such that it is easier to find a low-weight member of $f' + RM(m - 4, m)$. One can view this as a second hill climb, this time using linear transformations as the steps (so a basic step might be swapping two variables or

adding one variable to another) in order to optimize some property of the resulting f' (which might be the number of distinct variables, the number of terms, some other property, or a combination of these); of course, one will probably want to throw away any terms of degree less than $m-3$ that are produced during this process. Each polynomial f' encountered during this second hill climb can be tried as a starting point for the first hill climb; if a low-weight polynomial $f' + p' \in f' + RM(m-4, m)$ is found in this way, and if f' is $f \circ h$ (or the degree-$(m-3)$ part of $f \circ h$) for some $h \in GL(m, 2)$, then $(f' + p') \circ h^{-1}$ is a low-weight member of $f + RM(m-4, m)$.

A third approach is to try to prove that the minimum weight of f is greater than w and see where the attempt fails. Let $k = w/2 - m$, and apply the algorithm from Section 6 to try to produce a level-k certificate for f. It could be that f "almost" has minimum weight greater than w, meaning that the flats from the subproofs one can produce cover almost all of the coquartic space. The coquartics that are not covered by these flats are the only possible degree-$(m-4)$ parts for coquartic polynomials p such that $f + p$ has weight at most w. If the number of such coquartics is small enough, then one can try adding f to each of them and running the first hill climb (modified so as to alter only terms of degree less than $m-4$) starting from each such sum. We used this method to improve the upper bounds on the minimum weights of two of the polynomials in Section 8. Note that, even if we fix the coquartic terms, there will probably be too many polynomials of degree less than $m-4$ to run a full exhaust on the remaining part of the coset $f + RM(m-4, m)$.

8   Polynomials in 7 or 8 variables

Hou [11] provides lists of cubic polynomials on up to 8 variables that include a representative from each equivalence class of $RM(3, m)/RM(2, m)$ under the action of $GL(m, 2)$ for $m \le 8$; in addition to the polynomials $f_1, \ldots, f_6$ given earlier, there are polynomials $f_7, \ldots, f_{12}$ on 7 variables and $f_{13}, \ldots, f_{32}$ on 8 variables. We take the complementary forms to get a representative list $f_1^c, \ldots, f_{12}^c$ of 7-variable cocubic polynomials and a representative list $f_1^c, \ldots, f_{32}^c$ of 8-variable cocubic polynomials. (Note that $f_1, \ldots, f_6$ are all equivalent to their 6-variable complements under a permutation of variables, so we do not have to redo the work of Section 4 in complementary form.)

By Proposition 2.1, we only need to handle the polynomials $f_7^c, \ldots, f_{12}^c$ for 7 variables, and then the polynomials $f_{13}^c, \ldots, f_{32}^c$ for 8 variables.

Table 1 shows the results obtained for polynomials $f_7^c, \ldots, f_{12}^c$. It shows the weight and minimum weight for each polynomial. One can verify that the specified value is an upper bound for the minimum weight by checking that the weight of $f_i^c + p_i$ is the specified value, where p_i is the polynomial in the "Closest in $RM(3, 7)$" column. To get lower bounds, we produced certificates of the levels given in the "Cert. level" column; the last column gives the number of subproofs in each certificate (after full simplification).

These computations give a new (and quickly verifiable) proof of the result of Wang, Tan, and Prabowo [25] that the covering radius of $RM(3, 7)$ in $RM(4, 7)$ is 20.

Table 2 shows the results obtained for polynomials $f_{13}^c, \ldots, f_{32}^c$. The columns are the same as for Table 1, except that the "Closest in $RM(4, 8)$" column has been omitted because

Polynomial	Weight	Min. weight	Closest in $RM(3,7)$	Cert. level	Subproofs
f_7^c	20	16	$X_1X_2X_3$	0	1
f_8^c	18	16	$X_1X_2X_3$	0	1
f_9^c	20	20	0	2	12
f_{10}^c	18	18	0	1	2
f_{11}^c	22	20	$X_1X_2X_7 + X_1X_5X_7$ $+ X_1X_6X_7 + X_4X_6X_7$	2	29
f_{12}^c	24	20	$X_3X_4X_5$	2	13

Table 1: Minimum weights of cocubics on 7 variables

several of these polynomials are too large to fit. (It may be of interest to note that, for f_{20}^c and f_{25}^c, the closest polynomial in $RM(4,8)$ cannot be a homogeneous coquartic; lower-degree terms are required. We verified this by using failed attempts to get a level-3 certificate for f_{20}^c and a level-2 certificate for f_{25}^c; the failures produced explicit lists of what homogeneous coquartics could possibly be added to these polynomials to get the desired weight, and we checked that none of the coquartics on these lists actually worked.)

The initial level-4 certificate for f_{27}^c took several weeks on 90 processors to produce (although program improvements were being made simultaneously, so repeating the computation would probably take less time). This initial certificate had 64534 subproofs; 27 different quadratics were used. Later processing simplified the certificate to 10022 subproofs. Verifying this certificate in Magma took about 4.7 hours (almost all of which is spent verifying certificate requirement 6; a separate C program verified this one requirement in 33 minutes, while a CUDA version running on a GPU took 4 minutes).

These computations show that the covering radius of $RM(4,8)$ in $RM(5,8)$ is 26.

9   Polynomials in 9 variables

We used the Brier–Langevin [3] list of 349 representative 9-variable cubic polynomials (actually, we produced simplified versions of them via linear transformations, which we will call f_i^{BL*} for $1 \leq i \leq 349$), and took their complementary forms to give representative cocubic (degree-6) polynomials on 9 variables.

All have minimum weight at most 32, and all but six (numbers 107, 148, 165, 274, 301, and 329) have minimum weight at most 30. (We have coquartics that can be added to these polynomials to verify this.)

We have produced a level-4 certificate for the complementary form f_{311}^{BL*c} of the polynomial

$$f_{311}^{BL*} = X_1X_2X_3 + X_1X_4X_6 + X_1X_7X_9 + X_2X_4X_9 + X_2X_5X_6 + X_3X_5X_9 + X_6X_8X_9.$$

The original certificate had 21697 subproofs using 33 quadratics; further computation simplified this to 6134 subproofs. So the minimum weight of f_{311}^{BL*c} is at least 28; in fact, we have a coquartic that demonstrates that this minimum weight is exactly 28.

14

Polynomial	Weight	Min. weight	Cert. level	Subproofs
f^c_{13}	20	18	0	1
f^c_{14}	18	18	0	1
f^c_{15}	20	20	1	2
f^c_{16}	22	20	1	2
f^c_{17}	24	22	2	4
f^c_{18}	28	22	2	35
f^c_{19}	26	22	2	33
f^c_{20}	30	22	2	20
f^c_{21}	28	22	2	41
f^c_{22}	28	22	2	81
f^c_{23}	26	22	2	29
f^c_{24}	28	22	2	52
f^c_{25}	28	20	1	4
f^c_{26}	28	24	3	478
f^c_{27}	32	26	4	10022
f^c_{28}	24	22	2	29
f^c_{29}	20	20	1	4
f^c_{30}	22	20	1	2
f^c_{31}	22	20	1	2
f^c_{32}	28	24	3	179

Table 2: Minimum weights of cocubics on 8 variables

So the covering radius of $RM(5,9)$ in $RM(6,9)$ is at least 28 and at most 32. (We conjecture that the actual value is 32; we are just unable to produce a level-6 certificate at present.)

10   Polynomials in more variables

We have also produced a level-4 certificate for the complementary form of the ten-variable polynomial

$$X_1X_3X_5 + X_1X_4X_{10} + X_2X_3X_6 + X_2X_4X_8 + X_2X_5X_{10} + X_3X_4X_7 + X_8X_9X_{10}. \quad (7)$$

The certificate has 65059 subproofs using 37 quadratics (before simplification, which is in progress). So the minimum weight of the complement of (7) is at least 30; again, we have a coquartic that demonstrates that this minimum weight is exactly 30.

We now show how to use the examples already produced to get similar examples on higher numbers of variables.

Theorem 10.1. *If the cocubic Boolean polynomial f on variables $X_1, \ldots, X_m$ has a level-k certificate and the cocubic polynomial f' on new variables $Y_1, \ldots, Y_{m'}$ has a level-0 certificate, then the cocubic polynomial $f^* = Y_1 \cdots Y_{m'} f + X_1 \cdots X_m f'$ has a level-k certificate.*

15

The proof of Theorem 10.1 proceeds by constructing a new certificate for f^* and verifying that requirements 1–6 hold for it; the details are given in the Appendix.

Given an m-variable polynomial $f(X_1, \ldots, X_m)$ of degree $m-3$ with a level-k certificate, we can apply Theorem 10.1 with $f' = 1$ on variables Y_1, Y_2, Y_3 to get an $(m+3)$-variable polynomial f^* with a level-k certificate. Since we already have 8-, 9-, and 10-variable polynomials with level-4 certificates, we can now generate m-variable polynomials with level-4 certificates for all $m \geq 8$. Therefore, Theorem 5.2 gives:

Theorem 10.2. *The covering radius of $RM(m-4, m)$ in $RM(m-3, m)$ is at least $2m+10$ for all $m \geq 8$.*

This improves a lower bound result of McLoughlin [17], stating that the covering radius of $RM(m-4, m)$ in $RM(m-3, m)$ is at least $2m+2$ for odd m and $2m$ for even m. It is not as good asymptotically as the sphere-covering bound [15], which gives a quadratic lower bound on this covering radius. The most basic way to apply the sphere-covering bound gives that the covering radius of $RM(m-4, m)$ in $RM(m-3, m)$ is bounded below by the least k such that

$$\sum_{i=0}^{k} \binom{2^m}{i} \geq 2^{\binom{m}{3}}. \tag{8}$$

One can improve this using the fact that the code $RM(m-3, m)$ has minimum distance 8, and hence the balls of radius 3 centered at the codewords are disjoint; this yields as a lower bound the least k such that

$$\sum_{i=0}^{k+3} \binom{2^m}{i} \geq \left(1 + 2^m + \binom{2^m}{2} + \binom{2^m}{3}\right) 2^{\binom{m}{3}}. \tag{9}$$

These bounds exceed $2m+10$ for $m \geq 15$. (However, the sphere-covering bound is nonconstructive and does not yield explicit examples of cosets of high minimum weight.)

Remark. Inequality (9) implies inequality (8) because

$$\sum_{i=0}^{k+3} \binom{2^m}{i} \leq \left(\sum_{i=0}^{3} \binom{2^m}{i}\right) \left(\sum_{i=0}^{k} \binom{2^m}{i}\right),$$

which is true because any length-2^m bit word of Hamming weight at most $k+3$ can be expressed in at least one way (usually many) as the sum of a word of Hamming weight at most 3 and a word of Hamming weight at most k.

It would be nice to get a better result combining a level-k certificate for f and a level-k' certificate for f' to get a level-$(k+k')$ certificate for f^*, but the straightforward way of doing that (by combining each triple of the first certificate with each triple of the second certificate) does not yield independence or orthogonality between quadratics from the first certificate and quadratics from the second certificate (certificate requirements 3 and 5).

However, one can get another partial result:

16

Theorem 10.3. *If f, f', and f^* are as in Theorem 10.1 and f and f' both have level-1 certificates, then f^* has a level-2 certificate.*

Again the proof proceeds by using the two given certificates to construct a new certificate for f^* and verifying that requirements 1–6 hold for it; the details are given in the Appendix.

11   Cryptographic properties and testing polynomials

A number of properties of Boolean polynomials have been identified as relevant for measuring the strength of these polynomials in cryptographic applications; these properties include balance, resiliency, high nonlinearity, and algebraic immunity. This topic is far too broad to even summarize here; overviews of it are given in [5], [10], and [19]. But here is a brief review of a few of the definitions. An m-variable Boolean function (polynomial) f is *balanced* if it has weight 2^{m-1}. Polynomial f is tth-*order correlation immune* if, for any k with $1 \leq k \leq t$ and any distinct variables $X_{i_1}, \ldots, X_{i_k}$, $f + X_{i_1} + \cdots + X_{i_k}$ is balanced. A polynomial that is both balanced and tth-order correlation immune is called t-*resilient*. The *nonlinearity* of f is the distance from f to $RM(1, m)$.

One may wonder how the minimum weight of a cocubic polynomial interacts with these cryptographic properties. There are two main questions one might ask:

1. Do the representative cocubic polynomials or their minimum-weight coset members from the previous sections exhibit any of these cryptographic properties?

2. What are the minimum weights of cocubic polynomials that do exhibit those cryptographic properties?

We will address these questions in turn, starting with Question 1.

The representative cubic polynomials on the lists from [11] and [3] were chosen at least partially on the basis of simplicity (lower numbers of terms), which means that the corresponding cocubic polynomials have very low weight, even before we modify them to get minimum-weight members of their cosets. It then follows from known results (see the references above) that the polynomials we work with here cannot have most of the cryptographic properties listed above. The exception is tth-order correlation immunity, which is possible even for highly unbalanced polynomials. We checked and found that none of the representative cocubics or their minimum-weight versions is first-order correlation immune, except for the zero polynomial.

These representative polynomials seem unlikely to be of cryptographic interest directly, so we now turn our attention to Question 2. Given a cryptographically-interesting polynomial, we can test its minimum weight by using invariants from [11] and [2] to determine which representative polynomial it is equivalent to. Here and for the rest of this section, "equivalent" means "linearly equivalent modulo coquartic polynomials" or "linearly equivalent over $RM(m-4, m)$."

Remark. In [2], the direct product of two new invariants is used to discriminate the 349 equivalence classes of 9-variable cubic polynomials, but we found that the second new invariant alone suffices.

Parameters	Source	Equivalent to	Min. weight
(7,0,4,44)	Symmetric	f_7^c	16
[8,0,5,64]	Symmetric	f_7^c	16
[9,0,6,120]	Symmetric	$f_{346}^{BL*^c}$	20
(8,2,5,112)	[20, Th. 10(d)]	f_2^c	8
[8,1,5,116]	[16, p. 1828]	f_2^c	8
(9,2,6,232)	[21, Pr. 4(1)]	$f_{348}^{BL*^c} = f_2^c$	8
(9,2,6,240)	[7, App. B]	$f_{321}^{BL*^c} = f_{14}^c$	18
(6,2,3,24)	[1, Lemma 3.3]	f_5	16
(8,2,5,16)	[1, Th. 5.8]	f_7^c	16
(7,1,4,40)	[1, Pr. 5.13]	f_9^c	20
(7,2,4,56)	[24, Table 4]	f_9^c	20
(10,2,7,488)	[14, Table 1]		28–38

Table 3: Minimum weights of various known cocubic polynomials

Table 3 shows the results we got by testing various cocubic polynomials from the literature. In each case (except the last), we found out which representative cocubic was equivalent to the given one, and looked up the corresponding minimum weight.

The "Parameters" column uses a notation appearing in [16] and other sources. An (n, m, d, x) polynomial (or function) is a Boolean polynomial on n variables of degree d which is m-resilient and has nonlinearity x. The unbalanced version of this is: An $[n, m, d, x]$ polynomial (or function) is a Boolean polynomial on n variables of degree d which is mth-order correlation immune and has nonlinearity x.

Other notes on Table 3:

- "Symmetric" means the symmetric homogeneous cocubic polynomial on m variables. These polynomials were used in [17] to get the lower bound mentioned in Section 10.

- The equivalences and minimum weights for the polynomials from [1] were given in [1].

- All 72 of the rotationally symmetric (7,2,4,56) polynomials in [24, Table 4] are in the same equivalence class.

- For the two 10-variable polynomials from [14], since we do not have a precomputed catalog for 10 variables, we applied our methods directly to get lower and upper bounds on the minimum weight; the same bounds were obtained for both. (We verified using the invariants in [11] that these two polynomials are not equivalent.)

For convenience, we show the relevant representative cubic polynomials (other than those

in Section 4) here:

$$f_7 = X_1X_2X_7 + X_3X_4X_7 + X_5X_6X_7,$$
$$f_9 = X_1X_2X_3 + X_1X_4X_7 + X_2X_4X_5 + X_3X_4X_6,$$
$$f_{14} = X_1X_2X_3 + X_1X_7X_8 + X_4X_5X_6 + X_4X_7X_8,$$
$$f_{346}^{BL*} = X_1X_2X_9 + X_3X_4X_9 + X_5X_6X_9 + X_7X_8X_9.$$

A new type of covering radius was introduced in [13]: $\hat{\rho}(t, r, n)$ is defined to be the maximum distance between a t-resilient function and $RM(r, n)$. Since we are working with $RM(m-4, m)$ in this paper, and since the Siegenthaler bound [23, Th. 1] states that a t-resilient function on m variables has degree at most $m - t - 1$, the instances of $\hat{\rho}$ that are most relevant here are of the form $\hat{\rho}(2, m-4, m)$.

It is known that $\hat{\rho}(2, 1, 5) = 8$ [13] and $\hat{\rho}(2, 2, 6) = 16$ [1]. Reference [1] also gives the bounds $16 \le \hat{\rho}(2, 3, 7) \le 22$ and $16 \le \hat{\rho}(2, 4, 8)$, while [25] gives $\hat{\rho}(2, 3, 7) \le 20$.

The Siegenthaler bound implies that $\hat{\rho}(2, m-4, m)$ is at most the covering radius of $RM(m-4, m)$ in $RM(m-3, m)$; therefore, our upper bound results imply $\hat{\rho}(2, 4, 8) \le 26$ and $\hat{\rho}(2, 5, 9) \le 32$. For $\hat{\rho}(2, 6, 10)$ we do not know of an upper bound better than 50, obtained from the facts that the covering radius of $RM(6, 10)$ is at most 51 [8, Table 9.1 and (9.3.4)] and that $\hat{\rho}(2, 6, 10)$ must be even because all polynomials involved have degree less than 10.

Our computations in Table 3 give three improvements to the known lower bounds on $\hat{\rho}(2, m-4, m)$: the (7,2,4,56) polynomials from [24] give $\hat{\rho}(2, 3, 7) \ge 20$, the (9,2,6,240) polynomial from [7] gives $\hat{\rho}(2, 5, 9) \ge 18$, and the (10,2,7,488) polynomials from [14] give $\hat{\rho}(2, 6, 10) \ge 28$.

So the updated bounds on $\hat{\rho}(2, m-4, m)$ for $7 \le m \le 10$ are:

$$\hat{\rho}(2, 3, 7) = 20,$$
$$16 \le \hat{\rho}(2, 4, 8) \le 26,$$
$$18 \le \hat{\rho}(2, 5, 9) \le 32,$$
$$28 \le \hat{\rho}(2, 6, 10) \le 50.$$

12   Conclusion and open questions

We have given a method for producing verifiable certificates of lower bounds for the minimum weights of cosets of $RM(m-4, m)$ in $RM(m-3, m)$, and described a simple method for searching for upper bounds on these minimum weights. Using these methods, we have improved the known bounds on the covering radius of $RM(m-4, m)$ in $RM(m-3, m)$ for $8 \le m \le 14$, with an exact value of 26 for $m = 8$. We have also improved the known bounds on the modified covering radius $\hat{\rho}(2, m-4, m)$ for $7 \le m \le 10$, with an exact value of 20 for $m = 7$.

It would be a massive project to apply these methods to compute the covering radius of $RM(6, 10)$ in $RM(7, 10)$; the list of representative polynomials would have length

3691561 [11], and we would need a good upper bound for each, although we only need a lower bound for one of them. For $m > 10$, the situation is even worse: the number of cosets of $RM(m-4, m)$ in $RM(m-3, m)$ is $2^{\binom{m}{3}}$, while the size of $GL(m, 2)$ is less than 2^{m^2}, so the number of representatives needed is greater than $2^{\binom{m}{3} - m^2} = 2^{(m^2(m-9)+2m)/6}$.

Do there exist certificates of arbitrarily high level? In particular, can one combine a level-k certificate with a level-k' certificate to form a level-$(k+k')$ certificate as in Section 10, even for a restricted family of polynomials f and f'?

Do the methods extend to handle $(m-4)$th-order nonlinearity for m-variable polynomials of degree greater than $m-3$? In other words, can they be used to compute the covering radius of $RM(m-4, m)$ within the entire space $RM(m, m)$? This is not clear at present; the basic lower bound results would need substantial revision.

Are these methods sufficient in general? Given a cocubic polynomial f on m variables at distance d from $RM(m-4, m)$, will there always exist a certificate for f giving a lower bound of d? Might one need to look for other truth-table rows besides those of the form qf with q quadratic?

Are the problems studied here provably difficult? It is known that the problem of verifying an upper bound on the covering radius of a general binary linear code is complete for the second level of the polynomial-time hierarchy [18]. Can one prove a lower bound on complexity that is specific to Reed–Muller codes? For instance, is the problem of verifying a given upper bound on the distance from a given polynomial to $RM(r, m)$ NP-complete?

A   Appendix: Proofs of Theorems 10.1 and 10.3

Proof of Theorem 10.1. We get a level-k certificate for f^* by changing each triple $\langle C, q, r \rangle$ of the level-k certificate for f into a new triple $\langle C^*, q^*, r^* \rangle$ as follows:

- Each variable c_M occurring in C (where M is a coquartic monomial in the variables $X_1, \ldots, X_m$) is replaced in C^* with c_{M^*}, where $M^* = MY_1 \cdots Y_{m'}$.

- $q^* = q$.

- Each coefficient specifier $\operatorname{coef}(M)$ occurring in r (where M is a coquadratic monomial in the variables $X_1, \ldots, X_m$) is replaced in C^* with $\operatorname{coef}(M^*)$, where $M^* = MY_1 \cdots Y_{m'}$.

We now verify that all of the certificate requirements are met by the new certificate.

Requirement 1: The degree-$(m+m'-2)$ parts of the polynomials $X_i f^*$ are just $Y_1 \cdots Y_{m'}$ times the degree-$(m-2)$ parts of the polynomials $X_i f$, since X_i times $X_1 \cdots X_m f'$ is just $X_1 \cdots X_m f'$, which has degree $m + m' - 3$. Similarly, the degree-$(m+m'-2)$ parts of the polynomials $Y_i f^*$ are just $X_1 \cdots X_m$ times the degree-$(m'-2)$ parts of the polynomials $Y_i f'$. These two collections of polynomials are separately linearly independent, and they are supported on disjoint sets of monomials, so they are jointly linearly independent.

Requirement 2: We have $q_j f^* = Y_1 \cdots Y_{m'} q_j f + X_1 \cdots X_m q_j f'$, where $q_j f$ has degree at most $m-2$ by assumption and $X_1 \cdots X_m q_j f'$ is either $X_1 \cdots X_m f'$ or 0, depending on the number of terms in q_j, so $q_j f^*$ has degree at most $m + m' - 2$.

Requirement 3: Any sum $\sum_{M^*} c_{M^*} M^*$ of coquartic terms can be split into two parts: the part S_1 which is the sum of those terms for which M^* has the form $MY_1 \cdots Y_{m'}$ for some coquartic monomial M on $X_1, \ldots, X_m$, and the part S_2 which is the sum of those terms for which M^* is not a multiple of $Y_1 \cdots Y_{m'}$. The product $q_{j'} q_j (Y_1 \cdots Y_{m'} f + S_1)$ has degree less than $m + m'$ by requirement 3 for the given certificate, while $q_{j'} q_j S_2$ has no terms that are multiples of $Y_1 \cdots Y_{m'}$, and $q_{j'} q_j X_1 \cdots X_m f'$ is again either $X_1 \cdots X_m f'$ or 0. Therefore, $q_{j'} q_j (f^* + \sum_{M^*} c_{M^*} M^*)$ has degree less than $m + m'$.

Requirement 4: Since every monomial referred to by the coefficient combination r_j^* has $Y_1 \cdots Y_{m'}$ as a factor, r_j^* for $X_i X_1 \cdots X_m f'$ or $Y_i X_1 \cdots X_m f'$ is 0. The combination r_j^* for $Y_i Y_1 \cdots Y_{m'} f$ is 0 since $Y_i Y_1 \cdots Y_{m'} f = Y_1 \cdots Y_{m'} f$ has degree less than $m + m' - 2$, while r_j^* for $X_i Y_1 \cdots Y_{m'} f$ is 0 by requirement 4 for the given certificate. Therefore, r_j^* for $X_i f^*$ or for $Y_i f^*$ is 0.

Requirement 5: Let us again split the sum $\sum_{M^*} c_{M^*} M^*$ of coquartic terms into two parts S_1 and S_2 as above. Since every monomial referred to by r_j^* has $Y_1 \cdots Y_{m'}$ as a factor, r_j^* for $q_{j'} X_1 \cdots X_m f'$ is 0, and so is r_j^* for $q_{j'} S_2$. And r_j^* for $q_{j'} (Y_1 \cdots Y_{m'} f + S_1)$ is 0 if $j' < j$ or 1 if $j' = j$ by requirement 5 for the given certificate. Therefore, r_j^* for $q_{j'} (f^* + \sum_{M^*} c_{M^*} M^*)$ is 0 if $j' < j$, 1 if $j' = j$, as required.

Requirement 6: The conditions C^* are the same as the conditions C except for a global renaming of variables; since every assignment to the old variables satisfies at least one condition C, every assignment to the new variables satisfies at least one condition C^*. (The fact that there are additional new variables not mentioned by any C^* at all does not affect this.)

So the new certificate is valid, and we are done. ∎

Proof of Theorem 10.3. We construct the new certificate as follows. First, for each subproof $\langle C, q, r \rangle$ in the certificate for f and each subproof $\langle C', q', r' \rangle$ in the certificate for f', create a subproof $\langle C^*, q^*, r^* \rangle$ where:

- C^* consists of the equations in C with all subscripts multiplied by $Y_1 \cdots Y_{m'}$, the equations in C' with all subscripts multiplied by $X_1 \cdots X_m$, and two new equations described below.

- $q^* = [q_1, q_1']$.

- $r^* = [r_1^*, r_2^*]$ where r_1^* is r_1 with all monomials multiplied by $Y_1 \cdots Y_{m'}$ and r_2^* is r_1' with all monomials multiplied by $X_1 \cdots X_m$.

The two new equations are of the form

$$\sum_{i,j,i',j'} c_{(X_i X_j Y_{i'} Y_{j'})^c} = 0; \tag{10}$$

in the first equation the sum runs over all $i < j$ and $i' < j'$ where $X_i X_j$ is a monomial of q_1 and $Y_{i'} Y_{j'}$ is a monomial of q_1', while in the second equation the sum runs over all $i < j$ and $i' < j'$ where $X_i X_j$ is a monomial of q_1 and $(Y_{i'} Y_{j'})^c$ is a monomial mentioned in r_1'. These two equations are needed to ensure that certificate requirements 3 and 5 hold for the new subproof. But now, in order to get certificate requirement 6 to hold for the new certificate,

we need some additional subproofs: for all (i, j, i', j') such that $c_{(X_i X_j Y_{i'} Y_{j'})^c}$ is mentioned in one of the new equations (10), add the new triple

$$\langle [c_{(X_i X_j Y_{i'} Y_{j'})^c} = 1], [X_i Y_{i'}, X_i Y_{j'}], [\mathrm{coef}((X_j Y_{j'})^c), \mathrm{coef}((X_j Y_{i'})^c)] \rangle \tag{11}$$

to the new certificate. We will now see that combining the subproofs $\langle C^*, q^*, r^* \rangle$ above with the new triples (11) gives a complete level-2 certificate for f^*.

Requirement 1: The same proof as for Theorem 10.1 works here.

Requirement 2: For $\langle C^*, q^*, r^* \rangle$, the proof for Theorem 10.1 works for q_1, and the same with the roles of X and Y reversed works for q_1'. For (11), just note that $X_i Y_{i'} f^* = X_i Y_1 \cdots Y_{m'} f + Y_{i'} X_1 \cdots X_m f'$.

Requirement 3: For $\langle C^*, q^*, r^* \rangle$, note that $q_1 q_1' (f^* + \sum_{M^*} c_{M^*} M^*)$ is the sum of parts $q_1 q_1' Y_1 \cdots Y_{m'} f$, $q_1 q_1' X_1 \cdots X_m f'$, and $q_1 q_1' \sum_{M^*} c_{M^*} M^*$; the first part has no terms that are multiples of $X_1 \cdots X_m$, the second part has no terms that are multiples of $Y_1 \cdots Y_{m'}$, and, in the third part, the coefficient of $X_1 \cdots X_m Y_1 \cdots Y_{m'}$ is the left-hand side of the first equation (10), which is assumed to be 0. For (11) the argument is the same, except that the third part $X_i Y_{i'} X_i Y_{j'} \sum_{M^*} c_{M^*} M^*$ has no terms of degree $m + m'$ because $X_i Y_{i'} X_i Y_{j'} = X_i Y_{i'} Y_{j'}$ has degree less than 4.

Requirement 4: For $\langle C^*, q^*, r^* \rangle$, the proof for Theorem 10.1 works for r_1^*, and the same with the roles of X and Y reversed works for r_2^*. For (11), note that every monomial in f^* is missing either at least three X's or at least three Y's, so every monomial in $X_\ell f^*$ or $Y_\ell f^*$ is missing either at least two X's or at least two Y's.

Requirement 5: For $\langle C^*, q^*, r^* \rangle$, the proof that r_1^* for $q_1(f^* + \sum_{M^*} c_{M^*} M^*)$ is 1 is the same as for Theorem 10.1, and the proof that r_2^* for $q_1'(f^* + \sum_{M^*} c_{M^*} M^*)$ is 1 is the same with the roles of X and Y reversed. To see that r_2^* for $q_1(f^* + \sum_{M^*} c_{M^*} M^*)$ is 0, separate this product into parts $q_1 Y_1 \cdots Y_{m'} f$, $q_1 X_1 \cdots X_m f'$, and $q_1 \sum_{M^*} c_{M^*} M^*$; the first part has no terms that are multiples of $X_1 \cdots X_m$, the second part has degree at most $m + m' - 3$, and r_2^* for the third part is the left-hand side of the second equation (10), which is assumed to be 0. For (11), note that all terms of $X_i Y_{i'} f^*$ or $X_i Y_{j'} f^*$ are multiples of either $X_1 \cdots X_m$ or $Y_1 \cdots Y_{m'}$, so we compute that $\mathrm{coef}((X_j Y_{j'})^c)$ for $X_i Y_{i'}(f^* + \sum_{M^*} c_{M^*} M^*)$ is $c_{(X_i X_j Y_{i'} Y_{j'})^c}$, $\mathrm{coef}((X_j Y_{i'})^c)$ for $X_i Y_{i'}(f^* + \sum_{M^*} c_{M^*} M^*)$ is 0, and $\mathrm{coef}((X_j Y_{i'})^c)$ for $X_i Y_{j'}(f^* + \sum_{M^*} c_{M^*} M^*)$ is $c_{(X_i X_j Y_{i'} Y_{j'})^c}$, which is the desired result since we assumed $c_{(X_i X_j Y_{i'} Y_{j'})^c} = 1$.

Requirement 6: Suppose we have assigned a value from $\mathbf{F}_2$ to each variable c_{M^*} where M^* is a coquartic monomial in $X_1, \ldots, X_m, Y_1, \ldots, Y_{m'}$; we must show that this assignment satisfies the precondition of at least one of the triples in the new certificate. First, assign to each coquartic monomial M in $X_1, \ldots, X_m$ the value $c_M = c_{M Y_1 \cdots Y_{m'}}$, and find a subproof $\langle C, q, r \rangle$ from the certificate for f such that this assignment satisfies C. Then assign to each coquartic monomial M' in $Y_1, \ldots, Y_{m'}$ the value $c_{M'} = c_{M' X_1 \cdots X_m}$, and find a subproof $\langle C', q', r' \rangle$ from the certificate for f' such that this assignment satisfies C'. Let $\langle C^*, q^*, r^* \rangle$ be created from $\langle C, q, r \rangle$ and $\langle C', q', r' \rangle$ as above. If all of the coefficients mentioned in the two new equations in C^* are assigned the value 0, then the assignment satisfies C^*; if one of these coefficients $c_{(X_i X_j Y_{i'} Y_{j'})^c}$ is assigned the value 1, then the assignment satisfies the precondition of the corresponding triple (11).

So the new certificate is valid, and we are done. ∎

References

[1] Y. Borissov, A. Braeken, S. Nikova, and B. Preneel, "On the covering radii of binary Reed–Muller codes in the set of resilient Boolean functions," *IEEE Trans. Inform. Theory* 51 (2005), 1182–1189.

[2] E. Brier and P. Langevin, "Classification of Boolean cubic forms of nine variables," *ITW 2003*, Paris, France, 2003, 179–182.

[3] E. Brier and P. Langevin, "Boolean cubic forms in 9 variables,"

`http://langevin.univ-tln.fr/project/cubics/`

(accessed on March 2, 2021).

[4] C. Carlet, "Recursive lower bounds on the nonlinearity profile of Boolean functions and their applications," *IEEE Trans. Inform. Theory* 54 (2008), 1262–1272.

[5] C. Carlet, "Boolean functions for cryptography and error correcting codes," in *Boolean Models and Methods in Mathematics, Computer Science, and Engineering* (Y. Crama and P. L. Hammer, eds.), Cambridge Univ. Press, New York, 2010, 257–397.

[6] C. Carlet and S. Mesnager, "Improving the upper bounds on the covering radii of binary Reed–Muller codes," *IEEE Trans. Inform. Theory* 53 (2007), 162–173.

[7] J. A. Clark, J. L. Jacob, S. Maitra, and P. Stănică, "Almost Boolean functions: The design of Boolean functions by spectral inversion," *Computational Intelligence* 20 (2004), 450–462.

[8] G. Cohen, I. Honkala, S. Litsyn, and A. Lobstein, *Covering Codes*, North-Holland, Amsterdam, 1997.

[9] G. Cohen and S. Litsyn, "On the covering radius of Reed–Muller codes," *Discrete Mathematics* 106/107 (1992), 147–155.

[10] T. W. Cusick and P. Stănică, *Cryptographic Boolean functions and applications*, Elsevier, London, 2017.

[11] X. D. Hou, "$GL(m, 2)$ acting on $R(r, m)/R(r-1, m)$," *Discrete Mathematics* 149 (1996), 99–122.

[12] X. D. Hou, "Some results on the covering radii of Reed–Muller codes," *IEEE Trans. Inform. Theory* 39 (1993), 366–378.

[13] K. Kurosawa, T. Iwata, and T. Yoshiwara, "New covering radius of Reed–Muller codes for t-resilient functions," *IEEE Trans. Inform. Theory* 50 (2004), 468–475.

[14] W. M. Liu and A. Youssef, "On the existence of (10,2,7,488) resilient functions," *IEEE Trans. Inform. Theory* 55 (2009), 411–412.

[15] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, 1977.

[16] S. Maitra and E. Pasalic, "Further constructions of resilient Boolean functions with very high nonlinearity," *IEEE Trans. Inform. Theory* 48 (2002), 1825–1834.

[17] A. McLoughlin, "The covering radius of the $(m-3)$rd order Reed Muller codes and a lower bound on the $(m-4)$th order Reed Muller codes," *SIAM J. Appl. Math.*, 37 (1979), 419–422.

[18] A. McLoughlin, "The complexity of computing the covering radius of a code," *IEEE Trans. Inform. Theory* IT-30 (1984), 800–804.

[19] S. Mesnager, *Bent Functions: Fundamentals and Results*, Springer, Switzerland, 2016.

[20] P. Sarkar and S. Maitra, "Construction of nonlinear Boolean functions with important cryptographic properties," in *Advances in Cryptology — EUROCRYPT 2000* (B. Preneel, ed.), Springer-Verlag, Berlin, 2000, 485–506.

[21] P. Sarkar and S. Maitra, "Construction of nonlinear resilient Boolean functions using 'small' affine functions," *IEEE Trans. Inform. Theory* 50 (2004), 2185–2193.

[22] J. Schatz, "The second order Reed–Muller code of length 64 has covering radius 18," *IEEE Trans. Inform. Theory* IT-27 (1981), 529–530.

[23] T. Siegenthaler, "Correlation-immunity of nonlinear combining functions for cryptographic applications," *IEEE Trans. Inform. Theory* IT-30 (1984), 776–780.

[24] P. Stănică and S. Maitra, "Rotation symmetric Boolean functions—Count and cryptographic properties," *Discrete Applied Mathematics* 156 (2008) 1567–1580.

[25] Q. Wang, C. H. Tan, and T. F. Prabowo, "On the covering radius of the third order Reed–Muller code $RM(3,7)$," *Des. Codes Cryptogr.* 86 (2018), 151–159.