Competing Bandits

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We empirically study the interplay between *exploration* and *competition*. Systems that learn from interactions with users often engage in *exploration*: making potentially suboptimal decisions in order to acquire new information for future decisions. However, when multiple systems are competing for the same market of users, exploration may hurt a system's reputation in the near term, with adverse competitive effects. In particular, a system may enter a "death spiral", when the short-term reputation cost decreases the number of users for the system to learn from, which degrades the system's performance relative to competition and further decreases the market share.

We ask whether better exploration algorithms are incentivized under competition. We run extensive numerical experiments in a stylized duopoly model in which two firms deploy multi-armed bandit algorithms and compete for myopic users. We find that duopoly and monopoly tend to favor a primitive "greedy algorithm" that does not explore and leads to low consumer welfare, whereas a temporary monopoly (a duopoly with an early entrant) may incentivize better bandit algorithms and lead to higher consumer welfare. Our findings shed light on the first-mover advantage in the digital economy by exploring the role that data can play as a barrier to entry in online markets.

1 INTRODUCTION

Learning from interactions with users is ubiquitous in modern customer-facing systems, from product recommendations to web search to spam detection to content selection to fine-tuning the interface. Many systems purposefully implement *exploration*: making potentially suboptimal choices for the sake of acquiring new information. Randomized controlled trials, a.k.a. A/B testing, are an industry standard, with a number of companies such as *Optimizely* offering tools and platforms to facilitate them. Many companies use more sophisticated exploration methodologies based on *multi-armed bandits*, a well-known theoretical framework for exploration and making decisions under uncertainty.

Systems that engage in exploration typically need to compete against one another; most importantly, they compete for users. This creates an interesting tension between *exploration* and *competition*. In a nutshell, while exploring may be essential for improving the service tomorrow, it may degrade quality and make users leave *today*, in which case there will be no users to learn from! Thus, users play three distinct roles: they are customers that generate revenue, they generate data for the systems to learn from, and they are self-interested agents which choose among the competing systems.

We initiate a study of the interplay between *exploration* and *competition*. The main high-level question is: **whether and to what extent competition incentivizes adoption of better exploration algorithms**. This translates into a number of more concrete questions. While it is commonly assumed that better learning technology always helps, is this so for our setting? In other words, would a better learning algorithm result in higher utility for a principal? Would it be used in an equilibrium of the "competition game"? Also, does competition lead to better social welfare compared to a monopoly? We investigate these questions for several models, as we vary the capacity of users to make rational decisions (*rationality*) and the severity of competition between

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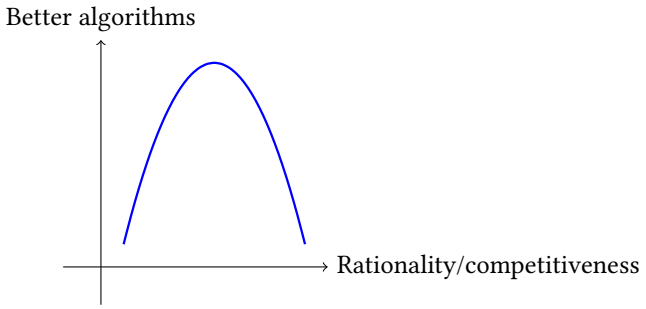


Fig. 1. Inverted-U relationship between rationality/competitiveness and algorithms.

the learning systems (*competitiveness*). The two are controlled by the same "knob" in our models; such coupling is not unusual in the literature, *e.g.*, see [25].

On a high level, our contributions can be framed in terms of the "inverted-U relationship" between rationality/competitiveness and the quality of adopted algorithms (see Figure 5).

Our model. We define a game in which two firms (*principals*) simultaneously engage in exploration and compete for users (*agents*). These two processes are interlinked, as exploration decisions are experienced by users and informed by their feedback. We need to specify several conceptual pieces: how the principals and agents interact, what is the machine learning problem faced by each principal, and what is the information structure. Each piece can get rather complicated in isolation, let alone jointly, so we strive for simplicity. Thus, the basic model is as follows:

- A new agent arrives in each round $t = 1, 2, \ldots$, and chooses among the two principals. The principal chooses an action (*e.g.*, a list of web search results to show to the agent), the user experiences this action, and reports a reward. All agents have the same "decision rule" for choosing among the principals given the available information.
- Each principal faces a very basic and well-studied version of the multi-armed bandit problem: for each arriving agent, it chooses from a fixed set of actions (a.k.a. *arms*) and receives a reward drawn independently from a fixed distribution specific to this action.
- Principals simultaneously announce their learning algorithms before round 1, and cannot change them afterwards. There is a common Bayesian prior on the rewards (but the realized reward distributions are not observed by the principals or the agents). Agents do not receive any other information. Each principal only observes agents that chose him.

Technical results. Our results depend crucially on agents' "decision rule" for choosing among the principals. The simplest and perhaps the most obvious rule is to select the principal which maximizes their expected utility; we refer to it as `HardMax`. We find that `HardMax` is not conducive to adopting better algorithms. In fact, each principal's dominant strategy is to do no purposeful exploration whatsoever, and instead always choose an action that maximizes expected reward given the current information; we call this algorithm `DynamicGreedy`. While this algorithm may potentially try out different actions over time and acquire useful information, it is known to be dramatically bad in many important cases of multi-armed bandits — precisely because it does not explore on purpose, and may therefore fail to discover best/better actions. Further, we show that `HardMax` is very sensitive to tie-breaking when both principals have exactly the same expected utility according to agents' beliefs. If tie-breaking is probabilistically biased — say, principal 1 is

always chosen with probability strictly larger than $\frac{1}{2}$ — then this principal has a simple "winning strategy" no matter what the other principal does.

We relax HardMax to allow each principal to be chosen with some fixed baseline probability. One intuitive interpretation is that there are "random agents" who choose a principal uniformly at random, and each arriving agent is either HardMax or "random" with some fixed probability. We call this model HardMax&Random. We find that better algorithms help in a big way: a sufficiently better algorithm is guaranteed to win all non-random agents after an initial learning phase. While the precise notion of "sufficiently better algorithm" is rather subtle, we note that commonly known "smart" bandit algorithms typically defeat the commonly known "naive" ones, and the latter typically defeat DynamicGreedy. However, there is a substantial caveat: one can defeat any algorithm by interleaving it with DynamicGreedy. This has two undesirable corollaries: a better algorithm may sometimes lose, and a pure Nash equilibrium typically does not exist.

We further relax the decision rule so that the probability of choosing a given principal varies smoothly as a function of the difference between principals' expected rewards; we call it SoftMax. For this model, the "better algorithm wins" result holds under much weaker assumptions on what constitutes a better algorithm. This is the most technical result of the paper. The competition in this setting is necessarily much more relaxed: typically, both principals attract approximately half of the agents as time goes by (but a better algorithm may attract slightly more).

All results extend to a much more general version of the multi-armed bandit problem in which the principal may observe additional feedback before and/or after each decision, as long as the feedback distribution does not change over time. In most results, principal's utility may depend on both the market share and agents' rewards.

Economic interpretation. The inverted-U relationship between the severity of competition among firms and the quality of technologies that they adopt is a familiar theme in the economics literature [*e.g.,* 3, 55].[1] We find it illuminating to frame our contributions in a similar manner, as illustrated in Figure 5.

Our models differ in terms of rationality in agents' decision-making: from fully rational decisions with HardMax to relaxed rationality with HardMax&Random to an even more relaxed rationality with SoftMax. The same distinctions also control the severity of competition between the principals: from cut-throat competition with HardMax to a more relaxed competition with HardMax&Random, to an even more relaxed competition with SoftMax. Indeed, with HardMax you lose all customers as soon as you fall behind in performance, with HardMax&Random you get some small market share no matter what, and with SoftMax you are further guaranteed a market share close to $\frac{1}{2}$ as long as your performance is not much worse than the competition. The uniform choice among principals corresponds to no rationality and no competition.

We identify the inverted-U relationship in the spirit of Figure 5 that is driven by the rationality/competitiveness distinctions outlined above: from HardMax to HardMax&Random to SoftMax to Uniform. We also find another, technically different inverted-U relationship which zeroes in on the HardMax&Random model. We vary rationality/competitiveness inside this model, and track the marginal utility of switching to a better algorithm.

These inverted-U relationships arise for a fundamentally different reason, compared to the existing literature on "competition vs. innovation." In the literature, better technology always

[1]The literature frames this relationship as one between "competition" and "innovation". In this context, "innovation" refers to adoption of a better technology, at a substantial R&D expense to a given firm. It is not salient whether similar ideas and/or technologies already exist outside the firm. It is worth noting that adoption of exploration algorithms tends to require substantial R&D effort in practice, even if the algorithms themselves are well-known in the research literature; see [1] for an example of such R&D effort.

helps in a competitive environment, other things being equal. Thus, the tradeoff is between the costs of improving the technology and the benefits that the improved technology provides in the competition. Meanwhile, we find that a better exploration algorithm may sometimes perform much worse under competition, even in the absence of R&D costs.

[as: Yishay's edits, slightly edited by Alex]

Discussion. We capture some pertinent features of reality while ignoring some others for the sake of tractability. Most notably, we assume that agents do not receive any information about other agents' rewards after the game starts. In the final analysis, this assumption makes agents' behavior independent of a particular realization of the Bayesian prior, and therefore enables us to summarize each learning algorithm via its Bayesian-expected rewards (as opposed to detailed performance on the particular realizations of the prior). Such summarization is essential for formulating lucid and general analytic results, let alone proving them. It is a major open question whether one can incorporate signals about other agents' rewards and obtain a tractable model.

We also make a standard assumption that agents are myopic: they do not worry about how their actions impact their future utility. In particular, they do not attempt to learn over time, to second-guess or game future agents, or to manipulate principal's learning algorithm. We believe this is a typical case in practice, in part because agent's influence tend to be small compared to the overall system. We model this simply by assuming that each agent only arrives once.

Much of the challenge in this paper, both conceptual and technical, was in setting up the right model and the matching theorems, and not only in proving the theorems. Apart from making the modeling choices described above, it was crucial to interpret the results and intuitions from the literature on multi-armed bandits so as to formulate meaningful assumptions on bandit algorithms and Bayesian priors which are productive in our setting.

Open questions. How to incorporate signals about the other agents' rewards? One needs to reason about how exact or coarse these signals are, and how the agents update their beliefs after receiving them. Also, one may need to allow principals' learning algorithms to respond to updates about the other principal's performance. (Or not, since this is not how learning algorithms are usually designed!) A clean, albeit idealized, model would be that (i) each agent learns her exact expected reward from each principal before she needs to choose which principal to go to, but (ii) these updates are invisible to the principals. Even then, one needs to argue about the competition on particular realizations of the Bayesian prior, which appears very challenging.

Another promising extension is to heterogeneous agents. Then the agents' choices are impacted by their idiosyncratic signals/beliefs, instead of being entirely determined by priors and/or signals about the average performance. It would be particularly interesting to investigate the emergence of *specialization*: whether/when an algorithm learns to target specific population segments in order to compete against a more powerful "incumbent".

Map of the paper. We survey related work (Section 2), lay out the model and preliminaries (Section 9), and proceed to analyze the three main models, `HardMax`, `HardMax&Random` and `SoftMax` (in Sections 4, 5, and 6, respectively). We discuss economic implications in Section 7. Appendix **??** provides some pertinent background on multi-armed bandits. Appendix **??** gives a broad example to support an assumption in our model.

2 RELATED WORK

Multi-armed bandits (*MAB*) is a particularly elegant and tractable abstraction for tradeoff between *exploration* and *exploitation*: essentially, between acquisition and usage of information. MAB problems have been studied in Economics, Operations Research and Computer Science for many decades;

see [20, 27, 51] for background on regret-minimizing and Bayesian formulations, respectively. A discussion of industrial applications of MAB can be found in [1].

The literature on MAB is vast and multi-threaded. The most related thread concerns regret-minimizing MAB formulations with IID rewards [7, 37]. This thread includes "smart" MAB algorithms that combine exploration and exploitation, such as UCB1 [7] and Successive Elimination [23], and "naive" MAB algorithms that separate exploration and exploitation, including explore-first and ϵ-Greedy [*e.g.,* see 51].

The three-way tradeoff between exploration, exploitation and incentives has been studied in several other settings: incentivizing exploration in a recommendation system [12, 17, 21, 24, 36, 40, 41], dynamic auctions [*e.g.,* 6, 16, 32], pay-per-click ad auctions with unknown click probabilities [*e.g.,* 10, 11, 22], coordinating search and matching by self-interested agents [35], as well as human computation [*e.g.,* 26, 29, 50].

[18, 28, 33] studied models with self-interested agents jointly performing exploration, with no principal to coordinate them.

There is a superficial similarity (in name only) between this paper and the line of work on "dueling bandits" [*e.g.,* 57, 58]. The latter is not about competing bandit algorithms, but rather about scenarios where in each round two arms are chosen to be presented to a user, and the algorithm only observes which arm has "won the duel".

Our setting is closely related to the "dueling algorithms" framework [31] which studies competition between two principals, each running an algorithm for the same problem. However, this work considers algorithms for offline / full input scenarios, whereas we focus on online machine learning and the explore-exploit-incentives tradeoff therein. Also, this work specifically assumes binary payoffs (*i.e.,* win or lose) for the principals.

Other related work in economics. The competition vs. innovation relationship and the inverted-U shape thereof have been introduced in a classic book [49], and remained an important theme in the literature ever since [*e.g.,* 3, 55]. Production costs aside, this literature treats innovation as a priori beneficial for the firm. Our setting is very different, as innovation in exploration algorithms may potentially hurt the firm.

A line of work on *platform competition*, starting with [48], concerns competition between firms (*platforms*) that improve as they attract more users (*network effect*); see [56] for a recent survey. This literature is not concerned with *innovation*, and typically models network effects exogenously, whereas in our model network effects are endogenous: they are created by MAB algorithms, an essential part of the model.

Relaxed versions of rationality similar to ours are found in several notable lines of work. For example, "random agents" (a.k.a. noise traders) can side-step the "no-trade theorem" [43], a famous impossibility result in financial economics. The `SoftMax` model is closely related to the literature on *product differentiation*, starting from [30], see [45] for a notable later paper.

There is a large literature on non-existence of equilibria due to small deviations (which is related to the corresponding result for `HardMax&Random`), starting with [46] in the context of health insurance markets. Notable recent papers [9, 54] emphasize the distinction between `HardMax` and versions of `SoftMax`.

3 OUR MODEL AND PRELIMINARIES

Principals and agents. There are two principals and T agents. The game proceeds in rounds (we will sometimes refer to them as *global rounds*). In each round $t \in [T]$, the following interaction takes place. A new agent arrives and chooses one of the two principals. The principal chooses a recommendation: an action $a_t \in A$, where A is a fixed set of actions (same for both principals and

all rounds). The agent follows this recommendation, receives a reward $r_t \in [0,1]$, and reports it back to the principal.

The rewards are i.i.d. with a common prior. More formally, for each action $a \in A$ there is a parametric family $\psi_a(\cdot)$ of reward distributions, parameterized by the mean reward μ_a. (The paradigmatic case is 0-1 rewards with a given expectation.) The mean reward vector $\mu = (\mu_a : a \in A)$ is drawn from prior distribution $\mathcal{P}_{\texttt{mean}}$ before round 1. Whenever a given action $a \in A$ is chosen, the reward is drawn independently from distribution $\psi_a(\mu_a)$. The prior $\mathcal{P}_{\texttt{mean}}$ and the distributions $(\psi_a(\cdot): a \in A)$ constitute the (full) Bayesian prior on rewards, denoted $\mathcal{P}$.

Each principal commits to a learning algorithm for making recommendations. This algorithm follows a protocol of *multi-armed bandits* (*MAB*). Namely, the algorithm proceeds in time-steps:[2] each time it is called, it outputs a chosen action $a \in A$ and then inputs the reward for this action. The algorithm is called only in global rounds when the corresponding principal is chosen.

The information structure is as follows. The prior $\mathcal{P}$ is known to everyone. The mean rewards μ_a are not revealed to anybody. Each agent knows both principals' algorithms, and the global round when (s)he arrives, *but not* the rewards of the previous agents. Each principal is completely unaware of the rounds when the other is chosen.

Some terminology. The two principals are called "Principal 1" and "Principal 2". The algorithm of principal $i \in \{1,2\}$ is called "algorithm i" and denoted $\texttt{alg}_i$. The agent in global round t is called "agent t"; the chosen principal is denoted i_t.

Throughout, $\mathbb{E}[\cdot]$ denotes expectation over all applicable randomness.

Bayesian-expected rewards. Consider the performance of a given algorithm $\texttt{alg}_i$, $i \in \{1,2\}$, when it is run in isolation (*i.e.,* without competition, just as a bandit algorithm). Let $\texttt{rew}_i(n)$ denote its Bayesian-expected reward for the n-th step.

Now, going back to our game, fix global round t and let $n_i(t)$ denote the number of global rounds before t in which this principal is chosen. Then:

$$\mathbb{E}[r_t \mid \text{principal } i \text{ is chosen in round } t \text{ and } n_i(t) = n] = \texttt{rew}_i(n+1) \quad (\forall n \in \mathbb{N}).$$

Agents' response. Each agent t chooses principal i_t as as follows: it chooses a distribution over the principals, and then draws independently from this distribution. Let p_t be the probability of choosing principal 1 according to this distribution. Below we specify p_t; we need to be careful so as to avoid a circular definition.

Let $\mathcal{I}_t$ be the information available to agent t before the round. Assume $\mathcal{I}_t$ suffices to form posteriors for quantities $n_i(t)$, $i \in \{1,2\}$, denote them by $\mathcal{N}_{i,t}$. Note that the Bayesian expected reward of each principal i is a function only of the number rounds he was chosen by the agents, so the posterior mean reward for each principal i can be written as

$$\texttt{PMR}_i(t) := \mathbb{E}[r_t \mid \mathcal{I}_t \text{ and } i_t = i] = \mathbb{E}[\texttt{rew}_i(n_i(t)+1) \mid \mathcal{I}_t] = \mathop{\mathbb{E}}_{n \sim \mathcal{N}_{i,t}} [\texttt{rew}_i(n+1)].$$

This quantity represents the posterior mean reward for principal i at round t, according to information $\mathcal{I}_t$; hence the notation PMR. In general, probability p_t is defined by the posterior mean rewards $\texttt{PMR}_i(t)$ for both principals. We assume a somewhat more specific shape:

$$p_t = f_{\texttt{resp}}\left(\texttt{PMR}_1(t) - \texttt{PMR}_2(t)\right). \tag{1}$$

Here $f_{\texttt{resp}} : [-1,1] \to [0,1]$ is the *response function*, which is the same for all agents. We assume that the response function is known to all agents.

[2] These time-steps will sometimes be referred to as *local steps/rounds*, so as to distinguish them from "global rounds" defined before. We will omit the local vs. local distinction when clear from the context.

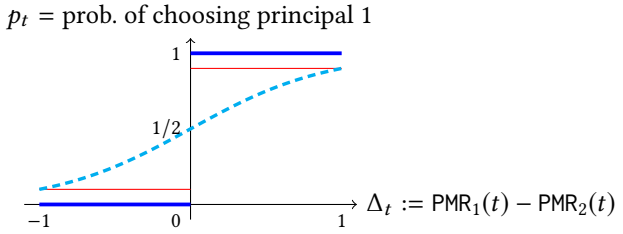


Fig. 2. The three models for agents' response function: HardMax is thick blue, HardMax&Random is slim red, and SoftMax is the dashed curve.

To make the model well-defined, it remains to argue that information $\mathcal{I}_t$ is indeed sufficient to form posteriors on $n_1(t)$ and $n_2(t)$. This can be easily seen using induction on t.

Since all agents arrive with identical information (other than knowing which global round they arrive in), it follows that all agents have identical posteriors for $n_{i,t}$ (for a given principal i and a given global round t). This posterior is denoted $\mathcal{N}_{i,t}$.

Response functions. We use the response function f_{resp} to characterize the amount of rationality and competitiveness in our model. We assume that f_{resp} is monotonically non-decreasing, is larger than $\frac{1}{2}$ on the interval $(0,1]$, and smaller than $\frac{1}{2}$ on the interval $[-1,0)$. Beyond that, we consider three specific models, listed in the order of decreasing rationality and competitiveness (see Figure 2):

- HardMax: f_{resp} equals 0 on the interval $[-1,0)$ and 1 on the interval $(0,1]$. In other words, the agents will deterministically choose the principal with the higher posterior mean reward.
- HardMax&Random: f_{resp} equals ϵ_0 on the interval $[-1,0)$ and $1-\epsilon_0$ on the interval $(0,1]$, where $\epsilon_0 \in (0, \frac{1}{2})$ are some positive constants. In words, each agent is a HardMax agent with probability $1-2\epsilon_0$, and with the remaining probability she makes a random choice.
- SoftMax: $f_{\mathsf{resp}}(\cdot)$ lies in the interval $[\epsilon_0, 1-\epsilon_0]$, $\epsilon_0 > 0$, and is "smooth" around 0 (in the sense defined precisely in Section 6).

We say that f_{resp} is *symmetric* if $f_{\mathsf{resp}}(-x) + f_{\mathsf{resp}}(x) = 1$ for any $x \in [0,1]$. This implies *fair tie-breaking*: $f_{\mathsf{resp}}(0) = \frac{1}{2}$.

MAB algorithms. We characterize the inherent quality of an MAB algorithm in terms of its *Bayesian Instantaneous Regret* (henceforth, BIR), a standard notion from machine learning:

$$\mathsf{BIR}(n) := \mathop{\mathbb{E}}_{\mu\sim\mathcal{P}_{\mathsf{mean}}}\left[\max_{a\in A}\mu_a\right] - \mathsf{rew}(n), \tag{2}$$

where $\mathsf{rew}(n)$ is the Bayesian-expected reward of the algorithm for the n-th step, when the algorithm is run in isolation. We are primarily interested in how BIR scales with n; we treat K, the number of arms, as a constant unless specified otherwise.

We will emphasize several specific algorithms or classes thereof:

- "smart" MAB algorithms that combine exploration and exploitation, such as UCB1 [7] and Successive Elimination [23]. These algorithms achieve $\mathsf{BIR}(n) \leq \tilde{O}(n^{-1/2})$ for all priors and all (or all but a very few) steps n. This bound is known to be tight for any fixed n. [3]
- "naive" MAB algorithms that separate exploration and exploitation, such as Explore-then-Exploit and ϵ-Greedy. These algorithms have dedicated rounds in which they explore by choosing an action uniformly at random. When these rounds are known in advance, the

[3]This follows from the lower-bound analysis in [8].

algorithm suffers constant BIR in such rounds. When the "exploration rounds" are instead randomly chosen by the algorithm, one can usually guarantee an inverse-polynomial upper bound BIR, but not as good as the one above: namely, $\texttt{BIR}(n) \leq \tilde{O}(n^{-1/3})$. This is the best possible upper bound on BIR for the two algorithms mentioned above.

- `DynamicGreedy`: at each step, recommends the best action according to the current posterior: an action a with the highest posterior expected reward $\mathbb{E}[\mu_a \mid \mathcal{I}]$, where $\mathcal{I}$ is the information available to the algorithm so far. `DynamicGreedy` has (at least) a constant BIR for some reasonable priors, *i.e.,* $\texttt{BIR}(n) > \Omega(1)$.
- `StaticGreedy`: always recommends the prior best action,*i.e.,* an action a with the highest prior mean reward $\mathbb{E}_{\mu\sim\mathcal{P}_{\texttt{mean}}}[\mu_a]$. This algorithm typically has constant BIR.

We focus on MAB algorithms such that $\texttt{BIR}(n)$ is non-increasing; we call such algorithms *monotone*. While some reasonable MAB algorithms may occasionally violate monotonicity, they can usually be easily modified so that monotonicity violations either vanish altogether, or only occur at very specific rounds (so that agents are extremely unlikely to exploit them in practice).

More background and examples can be found in Appendix **??**. In particular, we prove that `DynamicGreedy` is monotone.

Competition game between principals. Some of our results explicitly study the game between the two principals. We model it as a simultaneous-move game: before the first agent arrives, each principal commits to an MAB algorithm. Thus, choosing a pure strategy in this game corresponds to choosing an MAB algorithm (and, implicitly, announcing this algorithm to the agents).

Principal's utility is primarily defined as the market share, *i.e.,* the number of agents that chose this principal. Principals are risk-neutral, in the sense that they optimize their expected utility.

Assumptions on the prior. We make some technical assumptions for the sake of simplicity. First, each action a has a positive probability of being the best action according to the prior:

$$\forall a \in A: \quad \Pr_{\mu\sim\mathcal{P}_{\texttt{mean}}}[\mu_a > \mu_{a'}\ \forall a' \in A] > 0. \tag{3}$$

Second, posterior mean rewards of actions are pairwise distinct almost surely. That is, the history h at any step of an MAB algorithm[4] satisfies

$$\mathbb{E}[\mu_a \mid h] \neq \mathbb{E}[\mu_{a'} \mid h] \quad \forall a, a' \in A, \tag{4}$$

except at a set of histories of probability 0. In particular, prior mean rewards of actions are pairwise distinct: $\mathbb{E}[\mu_a] \neq \mathbb{E}[\mu'_a]$ for any $a, a' \in A$.

We provide two examples for which property (4) is 'generic', in the sense that it can be enforced almost surely by a small random perturbation of the prior. Both examples focus on 0-1 rewards and priors $\mathcal{P}_{\texttt{mean}}$ that are independent across arms. The first example assumes Beta priors on the mean rewards, and is very easy.[5] The second example assumes that mean rewards have a finite support, see Appendix **??** for details.

Some more notation. Without loss of generality, we label actions as $A = [K]$ and sort them according to their prior mean rewards, so that $\mathbb{E}[\mu_1] > \mathbb{E}[\mu_2] > \ldots > \mathbb{E}[\mu_K]$.

[4]The *history* of an MAB algorithm at a given step comprises the chosen actions and the observed rewards in all previous steps in the execution of this algorithm.

[5]Suppose the rewards are Bernouli r.v. and the mean reward μ_a for each arm a is drawn from some Beta distribution $\text{Beta}(\alpha_a, \beta_a)$. Given any history that contains h_a number of heads and t_a number of tails from arm a, the posterior mean reward is $\frac{\alpha_a+h_a}{\alpha_a+h_a+\beta_a+t_a}$. Note that h_a and t_a take integer values. Therefore, perturbing the parameters α_a and β_a independently with any continuous noise will induce a prior with property (4) with probability 1.

Fix principal $i \in \{1, 2\}$ and (local) step n. The arm chosen by algorithm $\texttt{alg}_i$ at this step is denoted $a_{i,n}$, and the corresponding BIR is denoted $\texttt{BIR}_i(n)$. History of $\texttt{alg}_i$ up to this step is denoted $H_{i,n}$.

Write $\texttt{PMR}(a \mid E) = \mathbb{E}[\mu_a \mid E]$ for posterior mean reward of action a given event E.

3.1 Generalizations

Our results can be extended compared to the basic model described above.

First, unless specified otherwise, our results allow a more general notion of principal's utility that can depend on both the market share and agents' rewards. Namely, principal i collects $U_i(r_t)$ units of utility in each global round t when she is chosen (and 0 otherwise), where $U_i(\cdot)$ is some fixed non-decreasing function with $U_i(0) > 0$. In a formula,

$$U_i := \sum_{t=1}^{T} \mathbf{1}_{\{i_t = i\}} \cdot U_i(r_t). \tag{5}$$

Second, our results carry over, with little or no modification of the proofs, to much more general versions of MAB, as long as it satisfies the i.i.d. property. In each round, an algorithm can see a *context* before choosing an action (as in *contextual bandits*) and/or additional feedback other than the reward after the reward is chosen (as in, *e.g., semi-bandits*), as long as the contexts are drawn from a fixed distribution, and the (reward, feedback) pair is drawn from a fixed distribution that depends only on the context and the chosen action. The Bayesian prior $\mathcal{P}$ needs to be a more complicated object, to make sure that PMR and BIR are well-defined. Mean rewards may also have a known structure, such as Lipschitzness, convexity, or linearity; such structure can be incorporated via $\mathcal{P}$. All these extensions have been studied extensively in the literature on MAB, and account for a substantial segment thereof; see [20] for background and details.

3.2 Chernoff Bounds

We use an elementary concentration inequality known as *Chernoff Bounds*, in a formulation from [**?**].

Theorem 3.1 (Chernoff Bounds)**.** *Consider n i.i.d. random variables $X_1 \ldots X_n$ with values in $[0, 1]$. Let $X = \frac{1}{n} \sum_{i=1}^{n} X_i$ be their average, and let $v = \mathbb{E}[X]$. Then:*

$$\min\left(\Pr[X - v > \delta v], \quad \Pr[v - X > \delta v] \right) < e^{-vn\delta^2/3} \quad \text{for any } \delta \in (0, 1).$$

4 FULL RATIONALITY (HARDMAX)

In this section, we will consider the version in which the agents are fully rational, in the sense that their response function is HardMax. We show that principals are not incentivized to *explore*—*i.e.,* to deviate from DynamicGreedy. The core technical result is that if one principal adopts DynamicGreedy, then the other principal loses all agents as soon as he deviates.

To make this more precise, let us say that two MAB algorithms *deviate* at (local) step n if there is an action $a \in A$ and a set of step-n histories of positive probability such that any history h in this set is feasible for both algorithms, and under this history the two algorithms choose action a with different probability.

Theorem 4.1. *Assume* HardMax *response function with fair tie-breaking. Assume that* $\texttt{alg}_1$ *is* DynamicGreedy*, and* $\texttt{alg}_2$ *deviates from* DynamicGreedy *starting from some (local) step $n_0 < T$. Then all agents in global rounds $t \geq n_0$ select principal* 1*.*

Corollary 4.2. *The competition game between principals has a unique Nash equilibirium: both principals choose* DynamicGreedy*.*

Remark 4.3. This corollary holds under a more general model which allows time-discounting: namely, the utility of each principal i in each global round t is $U_{i,t}(r_t)$ if this principal is chosen, and 0 otherwise, where $U_{i,t}(\cdot)$ is an arbitrary non-decreasing function with $U_{i,t}(0) > 0$.

4.1 Proof of Theorem 4.1

The proof starts with two auxiliary lemmas: that deviating from DynamicGreedy implies a strictly smaller Bayesian-expected reward, and that HardMax implies a "sudden-death" property: if one agent chooses principal 1 with certainty, then so do all subsequent agents do. We re-use both lemmas in later sections, so we state them in sufficient generality.

Lemma 4.4. *Assume that* $\texttt{alg}_1$ *is* DynamicGreedy, *and* $\texttt{alg}_2$ *deviates from* DynamicGreedy *starting from some (local) step* $n_0 < T$. *Then* $\texttt{rew}_1(n_0) > \texttt{rew}_2(n_0)$. *This holds for any response function* $f_{\texttt{resp}}$.

Lemma 4.4 does not rely on any particular shape of the response function because it only considers the performance of each algorithm without competition.

Proof of Lemma 4.4. Since the two algorithms coincide on the first $n_0 - 1$ steps, it follows by symmetry that histories H_{1,n_0} and H_{2,n_0} have the same distribution. We use a *coupling argument*: w.l.o.g., we assume the two histories coincide, $H_{1,n_0} = H_{2,n_0} = H$.

At local step n_0, DynamicGreedy chooses an action $a_{1,n_0} = a_{1,n_0}(H)$ which maximizes the posterior mean reward given history H: for any realized history $h \in \texttt{support}(H)$ and any action $a \in A$

$$\texttt{PMR}(a_{1,n_0} \mid H = h) \geq \texttt{PMR}(a \mid H = h). \tag{6}$$

[as: Rewrote the rest of the proof to account for positive-prob set of histories.]

By assumption (4), it follows that

$$\texttt{PMR}(a_{1,n_0} \mid H = h) > \texttt{PMR}(a \mid H = h) \quad \text{for any } h \in \texttt{support}(H) \text{ and } a \neq a_{1,n_0}(h). \tag{7}$$

Since the two algorithms deviate at step n_0, there is a set $S \subset \texttt{support}(H)$ of step-n_0 histories such that $\Pr[S] > 0$ and any history $h \in S$ satisfies $\Pr[a_{2,n_0} \neq a_{1,n_0} \mid H = h] > 0$. Combining this with (7), we deduce that

$$\texttt{PMR}(a_{1,n_0} \mid H = h) > \mathbb{E}\left[\mu_{a_{2,n_0}} \mid H = h\right] \quad \text{for each history } h \in S. \tag{8}$$

Using (6) and (8) and integrating over realized histories h, we obtain $\texttt{rew}_1(n_0) > \texttt{rew}_2(n_0)$. □

Lemma 4.5. *Consider* HardMax *response function with* $f_{\texttt{resp}}(0) \geq \frac{1}{2}$. *Suppose* $\texttt{alg}_1$ *is monotone, and* $\texttt{PMR}_1(t_0) > \texttt{PMR}_2(t_0)$ *for some global round* t_0. *Then* $\texttt{PMR}_1(t) > \texttt{PMR}_2(t)$ *for all subsequent rounds* t.

Proof. Let us use induction on round $t \geq t_0$, with the base case $t = t_0$. Let $\mathcal{N} = \mathcal{N}_{1,t_0}$ be the agents' posterior distribution for n_{1,t_0}, the number of global rounds before t_0 in which principal 1 is chosen. By induction, all agents from t_0 to $t-1$ chose principal 1, so $\texttt{PMR}_2(t_0) = \texttt{PMR}_2(t)$. Therefore,

$$\texttt{PMR}_1(t) = \mathop{\mathbb{E}}_{n\sim\mathcal{N}}[\texttt{rew}_1(n+1+t-t_0)] \geq \mathop{\mathbb{E}}_{n\sim\mathcal{N}}[\texttt{rew}_1(n+1)] = \texttt{PMR}_1(t_0) > \texttt{PMR}_2(t_0) = \texttt{PMR}_2(t),$$

where the first inequality holds because $\texttt{alg}_1$ is monotone, and the second one is the base case. □

Proof of Theorem 4.1. Since the two algorithms coincide on the first $n_0 - 1$ steps, it follows by symmetry that $\texttt{rew}_1(n) = \texttt{rew}_2(n)$ for any $n < n_0$. By Lemma 4.4, $\texttt{rew}_1(n_0) > \texttt{rew}_2(n_0)$.

Recall that $n_i(t)$ is the number of global rounds $s < t$ in which principal i is chosen, and $\mathcal{N}_{i,t}$ is the agents' posterior distribution for this quantity. By symmetry, each agent $t < n_0$ chooses a

principal uniformly at random. It follows that $\mathcal{N}_{1,n_0} = \mathcal{N}_{2,n_0}$ (denote both distributions by $\mathcal{N}$ for brevity), and $\mathcal{N}(n_0 - 1) > 0$. Therefore:

$$\begin{aligned}\texttt{PMR}_1(n_0) &= \mathop{\mathbb{E}}_{n\sim\mathcal{N}}[\texttt{rew}_1(n+1)] = \sum_{n=0}^{n_0-1} \mathcal{N}(n)\cdot \texttt{rew}_1(n+1)\\ &> \mathcal{N}(n_0-1)\cdot \texttt{rew}_2(n_0) + \sum_{n=0}^{n_0-2} \mathcal{N}(n)\cdot \texttt{rew}_2(n+1)\\ &= \mathop{\mathbb{E}}_{n\sim\mathcal{N}}[\texttt{rew}_2(n+1)] = \texttt{PMR}_2(n_0) \end{aligned} \tag{9}$$

So, agent n_0 chooses principal 1. By Lemma 4.5 (noting that DynamicGreedy is monotone), all subsequent agents choose principal 1, too. □

4.2 HardMax with biased tie-breaking

The HardMax model is very sensitive to the tie-breaking rule. For starters, if ties are broken deterministically in favor of principal 1, then principal 1 can get all agents no matter what the other principal does, simply by using StaticGreedy.

Theorem 4.6. *Assume* HardMax *response function with* $f_{\texttt{resp}}(0) = 1$ *(ties are always broken in favor of principal* 1*). If* $\texttt{alg}_1$ *is* StaticGreedy*, then all agents choose principal* 1*.*

Proof. Agent 1 chooses principal 1 because of the tie-breaking rule. Since StaticGreedy is trivially monotone, all the subsequent agents choose principal 1 by an induction argument similar to the one in the proof of Lemma 4.5. □

A more challenging scenario is when the tie-breaking is biased in favor of principal 1, but not deterministically so: $f_{\texttt{resp}}(0) > \frac{1}{2}$. Then this principal also has a "winning strategy" no matter what the other principal does. Specifically, principal 1 can get all but the first few agents, under a mild technical assumption that DynamicGreedy deviates from StaticGreedy. Principal 1 can use DynamicGreedy, or any other monotone MAB algorithm that coincides with DynamicGreedy in the first few steps.

Theorem 4.7. *Assume* HardMax *response function with* $f_{\texttt{resp}}(0) > \frac{1}{2}$ *(*i.e.*, tie-breaking is biased in favor of principal* 1*). Assume the prior* $\mathcal{P}$ *is such that* DynamicGreedy *deviates from* StaticGreedy *starting from some step* n_0*. Suppose that principal* 1 *runs a monotone MAB algorithm that coincides with* DynamicGreedy *in the first* n_0 *steps. Then all agents* $t \geq n_0$ *choose principal* 1*.*

Proof. The proof re-uses Lemmas 4.4 and 4.5, which do not rely on fair tie-breaking.

Because of the biased tie-breaking, for each global round t we have:

$$\text{if } \texttt{PMR}_1(t) \geq \texttt{PMR}_2(t) \text{ then } \Pr[i_t = 1] > \tfrac{1}{2}. \tag{10}$$

Recall that i_t is the principal chosen in global round t.

Let m_0 be the first step when $\texttt{alg}_2$ deviates from DynamicGreedy, or DynamicGreedy deviates from StaticGreedy, whichever comes sooner. Then $\texttt{alg}_2$, DynamicGreedy and StaticGreedy coincide on the first $m_0 - 1$ steps. Moreover, $m_0 \leq n_0$ (since DynamicGreedy deviates from StaticGreedy at step n_0), so $\texttt{alg}_1$ coincides with DynamicGreedy on the first m_0 steps.

So, $\texttt{rew}_1(n) = \texttt{rew}_2(n)$ for each step $n < m_0$, because $\texttt{alg}_1$ and $\texttt{alg}_2$ coincide on the first $m_0 - 1$ steps. Moreover, if $\texttt{alg}_2$ deviates from DynamicGreedy at step m_0 then $\texttt{rew}_1(m_0) > \texttt{rew}_2(m_0)$ by Lemma 4.4; else, we trivially have $\texttt{rew}_1(m_0) = \texttt{rew}_2(m_0)$. To summarize:

$$\texttt{rew}_1(n) \geq \texttt{rew}_2(n) \quad \text{for all steps } n \leq m_0. \tag{11}$$

We claim that $\Pr[i_t = 1] > \frac{1}{2}$ for all global rounds $t \le m_0$. We prove this claim using induction on t. The base case $t = 1$ holds by (10) and the fact that in step 1, DynamicGreedy chooses the arm with the highest prior mean reward. For the induction step, we assume that $\Pr[i_t = 1] > \frac{1}{2}$ for all global rounds $t < t_0$, for some $t_0 \le m_0$. It follows that distribution $\mathcal{N}_{1,t_0}$ stochastically dominates distribution $\mathcal{N}_{2,t_0}$.[6] Observe that

$$\mathtt{PMR}_1(t_0) = \mathop{\mathbb{E}}_{n\sim\mathcal{N}_{1,t_0}} [\mathtt{rew}_1(n+1)] \ge \mathop{\mathbb{E}}_{n\sim\mathcal{N}_{2,t_0}} [\mathtt{rew}_2(n+1)] = \mathtt{PMR}_2(t_0). \tag{12}$$

So the induction step follows by (10). Claim proved.

Now let us focus on global round m_0, and denote $\mathcal{N}_i = \mathcal{N}_{i,m_0}$. By the above claim,

$$\mathcal{N}_1 \text{ stochastically dominates } \mathcal{N}_2, \text{ and moreover } \mathcal{N}_i(m_0-1) > \mathcal{N}_i(m_0-1). \tag{13}$$

By definition of m_0, either (i) $\mathtt{alg}_2$ deviates from DynamicGreedy starting from local step m_0, which implies $\mathtt{rew}_1(m_0) > \mathtt{rew}_2(m_0)$ by Lemma 4.4, or (ii) DynamicGreedy deviates from StaticGreedy starting from local step m_0, which implies $\mathtt{rew}_1(m_0) > \mathtt{rew}_1(m_0-1)$ by Lemma **??**. In both cases, using (11) and (13), it follows that the inequality in (12) is strict for $t_0 = m_0$.

Therefore, agent m_0 chooses principal 1, and by Lemma 4.5 so do all subsequent agents. □

5 RELAXED RATIONALITY: HARDMAX & RANDOM

This section is dedicated to the HardMax&Random response model, where each principal is always chosen with some positive baseline probability. The main technical result for this model states that a principal with asymptotically better BIR wins by a large margin: after a "learning phase" of constant duration, all agents choose this principal with maximal possible probability $f_{\mathtt{resp}}(1)$. For example, a principal with $\mathtt{BIR}(n) \le \tilde{O}(n^{-1/2})$ wins over a principal with $\mathtt{BIR}(n) \ge \Omega(n^{-1/3})$. However, this positive result comes with a significant caveat detailed in Section 5.1.

We formulate and prove a cleaner version of the result, followed by a more general formulation developed in a subsequent Remark 5.2. We need to express a property that $\mathtt{alg}_1$ eventually catches up and surpasses $\mathtt{alg}_2$, even if initially it receives only a fraction of traffic. For the cleaner version, we assume that both algorithms are well-defined for an infinite time horizon, so that their BIR does not depend on the time horizon T of the game. Then this property can be formalized as:

$$(\forall \epsilon > 0) \qquad \mathtt{BIR}_1(\epsilon n)/\mathtt{BIR}_2(n) \to 0. \tag{14}$$

In fact, a weaker version of (14) suffices: denoting $\epsilon_0 = f_{\mathtt{resp}}(-1)$, for some constant n_0 we have

$$(\forall n \ge n_0) \qquad \mathtt{BIR}_1(\epsilon_0 n/2)/\mathtt{BIR}_2(n) < \tfrac{1}{2}. \tag{15}$$

We also need a very mild technical assumption on the "bad" algorithm:

$$(\forall n \ge n_0) \qquad \mathtt{BIR}_2(n) > 4\, e^{-\epsilon_0 n/12}. \tag{16}$$

Theorem 5.1. *Assume* HardMax&Random *response function. Suppose both algorithms are monotone and well-defined for an infinite time horizon, and satisfy* (15) *and* (16)*. Then each agent $t \ge n_0$ chooses principal* 1 *with maximal possible probability $f_{\mathtt{resp}}(1) = 1 - \epsilon_0$.*

Proof. Consider global round $t \ge n_0$. Recall that each agent chooses principal 1 with probability at least $f_{\mathtt{resp}}(-1) > 0$.

Then $\mathbb{E}[n_1(t+1)] \ge 2\epsilon_0\, t$. By Chernoff Bounds (Theorem 3.1), we have that $n_1(t+1) \ge \epsilon_0 t$ holds with probability at least $1 - q$, where $q = \exp(-\epsilon_0 t/12)$.

We need to prove that $\mathtt{PMR}_1(t) - \mathtt{PMR}_2(t) > 0$. For any m_1 and m_2, consider the quantity

$$\Delta(m_1, m_2) := \mathtt{BIR}_2(m_2+1) - \mathtt{BIR}_1(m_1+1).$$

[6] For random variables X, Y on $\mathbb{R}$, we say that X *stochastically dominates* Y if $\Pr[X \ge x] \ge \Pr[Y \ge x]$ for any $x \in \mathbb{R}$.

Whenever $m_1 \geq \epsilon_0 t/2 - 1$ and $m_2 < t$, it holds that

$$\Delta(m_1, m_2) \geq \Delta(\epsilon_0 t/2, t) \geq \texttt{BIR}_2(t)/2.$$

The above inequalities follow, resp., from algorithms' monotonicity and (15). Now,

$$\begin{aligned}
\texttt{PMR}_1(t) - \texttt{PMR}_2(t) &= \mathop{\mathbb{E}}_{m_1 \sim \mathcal{N}_{1,t},\ m_2 \sim \mathcal{N}_{2,t}} [\Delta(m_1, m_2)] \\
&\geq -q + \mathop{\mathbb{E}}_{m_1 \sim \mathcal{N}_{1,t},\ m_2 \sim \mathcal{N}_{2,t}} [\Delta(m_1, m_2) \mid m_1 \geq \epsilon_0 t/2 - 1] \\
&\geq \texttt{BIR}_2(t)/2 - q \\
&> \texttt{BIR}_2(t)/4 > 0 \qquad \text{(by (16))}.
\end{aligned}$$

□

Remark 5.2. Many standard MAB algorithms in the literature are parameterized by the time horizon T. Regret bounds for such algorithms usually include a polylogarithmic dependence on T. In particular, a typical upper bound for BIR has the following form:

$$\texttt{BIR}(n \mid T) \leq \text{polylog}(T) \cdot n^{-\gamma} \quad \text{for some } \gamma \in (0, \tfrac{1}{2}]. \tag{17}$$

Here we write $\texttt{BIR}(n \mid T)$ to emphasize the dependence on T.

We generalize (15) to handle the dependence on T: there exists a number T_0 and a function $n_0(T) \in \text{polylog}(T)$ such that

$$(\forall T \geq T_0,\ n \geq n_0(T)) \quad \frac{\texttt{BIR}_1(\epsilon_0 n/2 \mid T)}{\texttt{BIR}_2(n \mid T)} < \frac{1}{2}. \tag{18}$$

If this holds, we say that $\texttt{alg}_1$ *BIR-dominates* $\texttt{alg}_2$.

We provide a version of Theorem 5.1 in which algorithms are parameterized with time horizon T and condition (15) is replaced with (18); its proof is very similar and is omitted.

To state a game-theoretic corollary of Theorem 5.1, we consider a version of the competition game between the two principals in which they can only choose from a finite set $\mathcal{A}$ of monotone MAB algorithms. One of these algorithms is "better" than all others; we call it the *special* algorithm. Unless specified otherwise, it BIR-dominates all other allowed algorithms. The other algorithms satisfy (16). We call this game the *restricted competition game*.

Corollary 5.3. *Assume* HardMax&Random *response function. Consider the restricted competition game with special algorithm* alg*. Then, for any sufficiently large time horizon T, this game has a unique Nash equilibrium: both principals choose* alg*.*

5.1 A little greedy goes a long way

Given any monotone MAB algorithm other than DynamicGreedy, we design a modified algorithm which learns at a slower rate, yet "wins the game" in the sense of Theorem 5.1. As a corollary, the competition game with unrestricted choice of algorithms typically does not have a Nash equilibrium.

Given an algorithm $\texttt{alg}_1$ that deviates from DynamicGreedy starting from step n_0 and a "mixing" parameter p, we will construct a modified algorithm as follows.

(1) The modified algorithm coincides with $\texttt{alg}_1$ (and DynamicGreedy) for the first $n_0 - 1$ steps;
(2) In each step $n \geq n_0$, $\texttt{alg}_1$ is invoked with probability $1-p$, and with the remaining probability p does the "greedy choice": chooses an action with the largest posterior mean reward given the current information collected by $\texttt{alg}_1$.

For a cleaner comparison between the two algorithms, the modified algorithm does not record rewards received in steps with the "greedy choice". Parameter $p > 0$ is the same for all steps.

Theorem 5.4. *Assume symmetric* HardMax&Random *response function. Let* $\epsilon_0 = f_{\mathtt{resp}}(-1)$ *be the baseline probability. Suppose* $\mathtt{alg}_1$ *deviates from* DynamicGreedy *starting from some step* n_0*. Let* $\mathtt{alg}_2$ *be the modified algorithm, as described above, with mixing parameter* p *such that* $(1-\epsilon_0)(1-p) > \epsilon_0$*. Then each agent* $t \geq n_0$ *chooses principal* 2 *with maximal possible probability* $1-\epsilon_0$*.*

Corollary 5.5. *Suppose that both principals can choose any monotone MAB algorithm, and assume the symmetric* HardMax&Random *response function. Then for any time horizon* T*, the only possible* pure *Nash equilibrium is one where both principals choose* DynamicGreedy*. Moreover, no pure Nash equilibrium exists when some algorithm "dominates"* DynamicGreedy *in the sense of* (18) *and the time horizon* T *is sufficiently large.*

Remark 5.6. The modified algorithm performs exploration at a slower rate. Let us argue how this may translate into a larger BIR compared to the original algorithm. Let $\mathtt{BIR}'_1(n)$ be the BIR of the "greedy choice" after after $n-1$ steps of $\mathtt{alg}_1$. Then

$$\mathtt{BIR}_2(n) = \mathop{\mathbb{E}}_{m\sim(n_0-1)+\mathtt{Binomial}(n-n_0+1,1-p)} \left[(1-p)\cdot \mathtt{BIR}_1(m) + p\cdot \mathtt{BIR}'_1(m)\right]. \tag{19}$$

In this expression, m is the number of times $\mathtt{alg}_1$ is invoked in the first n steps of the modified algorithm. Note that $\mathbb{E}[m] = n_0 - 1 + (n-n_0+1)(1-p) \geq (1-p)n$.

Suppose $\mathtt{BIR}_1(n) = \beta n^{-\gamma}$ for some constants $\beta, \gamma > 0$. Further, assume $\mathtt{BIR}'_1(n) \geq c\ \mathtt{BIR}_1(n)$, for some $c > 1-\gamma$. Then for all $n \geq n_0$ and small enough $p > 0$ it holds that:

$$\begin{aligned}
\mathtt{BIR}_2(n) &\geq (1-p+pc)\ \mathbb{E}[\ \mathtt{BIR}_1(m)\ ] \\
\mathbb{E}[\ \mathtt{BIR}_1(m)\ ] &\geq \mathtt{BIR}_1(\ \mathbb{E}[m]\ ) && \text{(By Jensen's inequality)} \\
&\geq \mathtt{BIR}_1(\ (1-p)n\ ) && \text{(since } \mathbb{E}[m] \geq n(1-p)\text{)} \\
&\geq \beta\cdot n^{-\gamma}\cdot(1-p)^{-\gamma} && \text{(plugging in } \mathtt{BIR}_1(n) = \beta n^{-\gamma}\text{)} \\
&> \mathtt{BIR}_1(n)\ \ (1-p\gamma)^{-1} && \text{(since } (1-p)^{\gamma} < 1-p\gamma\text{).} \\
\mathtt{BIR}_2(n) &> \alpha\cdot \mathtt{BIR}_1(n), && \text{where}\quad \alpha = \tfrac{1-p+pc}{1-p\gamma} > 1.
\end{aligned}$$

(In the above equations, all expectations are over m distributed as in (19).)

Proof of Theorem 5.4. Let $\mathtt{rew}'_1(n)$ denote the Bayesian-expected reward of the "greedy choice" after after $n-1$ steps of $\mathtt{alg}_1$. Note that $\mathtt{rew}_1(\cdot)$ and $\mathtt{rew}'_1(\cdot)$ are non-decreasing: the former because $\mathtt{alg}_1$ is monotone and the latter because the "greedy choice" is only improved with an increasing set of observations. Therefore, the modified algorithm $\mathtt{alg}_2$ is monotone by (19).

By definition of the "greedy choice," $\mathtt{rew}_1(n) \leq \mathtt{rew}'_1(n)$ for all steps n. Moreover, by Lemma 4.4, $\mathtt{alg}_1$ has a strictly smaller $\mathtt{rew}(n_0)$ compared to DynamicGreedy; so, $\mathtt{rew}_1(n_0) < \mathtt{rew}_2(n_0)$.

Let alg denote a copy of $\mathtt{alg}_1$ that is running "inside" the modified algorithm $\mathtt{alg}_2$. Let $m_2(t)$ be the number of global rounds before t in which the agent chooses principal 2 *and* alg is invoked; in other words, it is the number of agents seen by alg before global round t. Let $\mathcal{M}_{2,t}$ be the agents' posterior distribution for $m_2(t)$.

We claim that in each global round $t \geq n_0$, distribution $\mathcal{M}_{2,t}$ stochastically dominates distribution $\mathcal{N}_{1,t}$, and $\mathtt{PMR}_1(t) < \mathtt{PMR}_2(t)$. We use induction on t. The base case $t = n_0$ holds because $\mathcal{M}_{2,t} = \mathcal{N}_{1,t}$ (because the two algorithms coincide on the first $n_0 - 1$ steps), and $\mathtt{PMR}_1(n_0) < \mathtt{PMR}_2(n_0)$ is proved as in (9), using the fact that $\mathtt{rew}_1(n_0) < \mathtt{rew}_2(n_0)$.

The induction step is proved as follows. The induction hypothesis for global round $t-1$ implies that agent $t-1$ is seen by alg with probability $(1-\epsilon_0)(1-p)$, which is strictly larger than ϵ_0, the

probability with which this agent is seen by $\texttt{alg}_2$. Therefore, $\mathcal{M}_{2,t}$ stochastically dominates $\mathcal{N}_{1,t}$.

$$\begin{aligned}
\texttt{PMR}_1(t) &= \mathop{\mathbb{E}}_{n\sim\mathcal{N}_{1,t}}\left[\texttt{rew}_1(n+1)\right] \\
&\leq \mathop{\mathbb{E}}_{m\sim\mathcal{M}_{2,t}}\left[\texttt{rew}_1(m+1)\right] && (20)\\
&< \mathop{\mathbb{E}}_{m\sim\mathcal{M}_{2,t}}\left[(1-p)\cdot\texttt{rew}_1(m+1)+p\cdot\texttt{rew}_1'(m+1)\right] && (21)\\
&= \texttt{PMR}_2(t).
\end{aligned}$$

Here inequality (20) holds because $\texttt{rew}_1(\cdot)$ is monotone and $\mathcal{M}_{2,t}$ stochastically dominates $\mathcal{N}_{1,t}$, and inequality (21) holds because $\texttt{rew}_1(n_0) < \texttt{rew}_2(n_0)$ and $\mathcal{M}_{2,t}(n_0) > 0$.[7] □

6 SOFTMAX RESPONSE FUNCTION

This section is devoted to the SoftMax model. We recover a positive result under the assumptions from Theorem 5.1 (albeit with a weaker conclusion), and then proceed to a much more challenging result under weaker assumptions. We start with a formal definition:

Definition 6.1. A response function $f_{\texttt{resp}}$ is SoftMax if the following conditions hold:

- $f_{\texttt{resp}}(\cdot)$ is bounded away from 0 and 1: $f_{\texttt{resp}}(\cdot) \in [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, \frac{1}{2})$,
- the response function $f_{\texttt{resp}}(\cdot)$ is "smooth" around 0:

$$\exists \text{ constants } \delta_0, c_0, c_0' > 0 \qquad \forall x \in [-\delta_0, \delta_0] \qquad c_0 \leq f_{\texttt{resp}}'(x) \leq c_0'. \tag{22}$$

- fair tie-breaking: $f_{\texttt{resp}}(0) = \frac{1}{2}$.

Remark 6.2. This definition is fruitful when parameters c_0 and c_0' are close to $\frac{1}{2}$. Throughout, we assume that $\texttt{alg}_1$ is better than $\texttt{alg}_2$, and obtain results parameterized by c_0. By symmetry, one could assume that $\texttt{alg}_2$ is better than $\texttt{alg}_1$, and obtain similar results parameterized by c_0'.

Our first result is a version of Theorem 5.1, with the same assumptions about the algorithms and essentially the same proof. The conclusion is much weaker: we can only guarantee that each agent $t \geq n_0$ chooses principal 1 with probability slightly larger than $\frac{1}{2}$. This is essentially unavoidable in a typical case when both algorithms satisfy $\texttt{BIR}(n) \to 0$, by Definition 6.1.

Theorem 6.3. *Assume* SoftMax *response function. Suppose* $\texttt{alg}_1$ *has better* BIR *in the sense of* (15)*, and* $\texttt{alg}_2$ *satisfies the condition* (16)*. Then each agent* $t \geq n_0$ *chooses principal* 1 *with probability*

$$\Pr[i_t = 1] \geq \tfrac{1}{2} + \tfrac{c_0}{4}\,\texttt{BIR}_2(t). \tag{23}$$

Proof Sketch. We follow the steps in the proof of Theorem 5.1 to derive

$$\texttt{PMR}_1(t) - \texttt{PMR}_2(t) \geq \texttt{BIR}_2(t)/2 - q, \quad \text{where } q = \exp(-\epsilon_0 t/12).$$

This is at least $\texttt{BIR}_2(t)/4$ by (16). Then (23) follows by the smoothness condition (22). □

We recover a version of Corollary 5.3, if each principal's utility is the number of users (rather than the more general model in (5)). We also need a mild technical assumption that cumulative Bayesian regret (BReg) tends to infinity. BReg is a standard notion from the literature (along with BIR):

$$\texttt{BReg}(n) := n \cdot \mathop{\mathbb{E}}_{\mu\sim\mathcal{P}_{\texttt{mean}}}\left[\max_{a\in A}\mu_a\right] - \sum_{n=1}^{n}\texttt{rew}(n') = \sum_{n'=1}^{n}\texttt{BIR}(n'). \tag{24}$$

[7] If $\texttt{rew}_1(\cdot)$ is strictly increasing, then inequality (20) is strict, too; this is because $\mathcal{M}_{2,t}(t-1) > \mathcal{N}_{1,t}(t-1)$.

Corollary 6.4. *Assume that the response function is* SoftMax*, and each principal's utility is the number of users. Consider the restricted competition game with special algorithm* alg*, and assume that all other allowed algorithms satisfy* $\texttt{BReg}(n) \to \infty$*. Then, for any sufficiently large time horizon T, this game has a unique Nash equilibrium: both principals choose* alg*.*

Further, we prove a much more challenging result in which the condition (15) is replaced with a much weaker "BIR-dominance" condition. For clarity, we will again assume that both algorithms are well-defined for an infinite time horizon. The *weak BIR dominance* condition says there exist constants $\beta_0, \alpha_0 \in (0, 1/2)$ and n_0 such that

$$(\forall n \geq n_0) \quad \frac{\texttt{BIR}_1((1-\beta_0)\, n)}{\texttt{BIR}_2(n)} < 1 - \alpha_0. \tag{25}$$

If this holds, we say that $\texttt{alg}_1$ *weakly BIR-dominates* $\texttt{alg}_2$. Note that the condition (18) involves sufficiently small multiplicative factors (resp., $\epsilon_0/2$ and $\frac{1}{2}$), the new condition replaces them with factors that can be arbitrarily close to 1.

We make a mild assumption on $\texttt{alg}_1$ that its $\texttt{BIR}_1(n)$ tends to 0. Formally, for any $\epsilon > 0$, there exists some $n(\epsilon)$ such that

$$(\forall n \geq n(\epsilon)) \qquad \texttt{BIR}_1(n) \leq \epsilon. \tag{26}$$

We also require a slightly stronger version of the technical assumption (16):for some n_0,

$$(\forall n \geq n_0) \qquad \texttt{BIR}_2(n) \geq \frac{4}{\alpha_0} \exp\left(\frac{-\min\{\epsilon_0, 1/8\} n}{12}\right) \tag{27}$$

Theorem 6.5. *Assume the* SoftMax *response function. Suppose* $\texttt{alg}_1$ *weakly-BIR-dominates* $\texttt{alg}_2$*,* $\texttt{alg}_1$ *satisfies* (26)*, and* $\texttt{alg}_2$ *satisfies* (27)*. Then there exists some* t_0 *such that each agent* $t \geq t_0$ *chooses principal* 1 *with probability*

$$\Pr[i_t = 1] \geq \tfrac{1}{2} + \tfrac{c_0 \alpha_0}{4}\, \texttt{BIR}_2(t). \tag{28}$$

The main idea behind our proof is that even though $\texttt{alg}_1$ may have a slower rate of learning in the beginning, it will gradually catch up and surpass $\texttt{alg}_2$. We will describe this process in two phases. In the first phase, $\texttt{alg}_1$ receives a random agent with probability at least $f_{\texttt{resp}}(-1) = \epsilon_0$ in each round. Since $\texttt{BIR}_1$ tends to 0, the difference in BIRs between the two algorithms is also diminishing. Due to the SoftMax response function, $\texttt{alg}_1$ attracts each agent with probability at least $1/2 - O(\beta_0)$ after a sufficient number of rounds. Then the game enters the second phase: both algorithms receive agents at a rate close to $\frac{1}{2}$, and the fractions of agents received by both algorithms — $n_1(t)/t$ and $n_2(t)/t$ — also converge to $\frac{1}{2}$. At the end of the second phase and in each global round afterwards, the counts $n_1(t)$ and $n_2(t)$ satisfy the weak BIR-dominance condition, in the sense that they both are larger than n_0 and $n_1(t) \geq (1-\beta_0)\, n_2(t)$. At this point, $\texttt{alg}_1$ actually has smaller BIR, which reflected in the PMRs eventually. Accordingly, from then on $\texttt{alg}_1$ attracts agents at a rate slightly larger than $\frac{1}{2}$. We prove that the "bump" over $\frac{1}{2}$ is at least on the order of $\texttt{BIR}_2(t)$.

Proof of Theorem 6.5. Let $\beta_1 = \min\{c_0' \delta_0, \beta_0/20\}$ with δ_0 defined in (22). Recall each agent chooses $\texttt{alg}_1$ with probability at least $f_{\texttt{resp}}(-1) = \epsilon_0$. By condition (26) and (27), there exists some sufficiently large T_1 such that for any $t \geq T_1$, $\texttt{BIR}_1(\epsilon_0 T_1/2) \leq \beta_1/c_0'$ and $\texttt{BIR}_2(t) > e^{-\epsilon_0 t/12}$. Moreover, for any $t \geq T_1$, we know $\mathbb{E}[n_1(t+1)] \geq \epsilon_0\, t$, and by the Chernoff Bounds (Theorem 3.1), we have $n_1(t+1) \geq \epsilon_0 t/2$ holds with probability at least $1 - q_1(t)$ with $q_1(t) = \exp(-\epsilon_0 t/12) < \texttt{BIR}_2(t)$. It

follows that for any $t \geq T_1$,

$$\begin{aligned}\mathtt{PMR}_2(t) - \mathtt{PMR}_1(t) &= \mathop{\mathbb{E}}_{m_1 \sim \mathcal{N}_{1,t},\ m_2 \sim \mathcal{N}_{2,t}} [\mathtt{BIR}_1(m_1+1) - \mathtt{BIR}_2(m_2+1)] \\ &\leq q_1(t) + \mathop{\mathbb{E}}_{m_1 \sim \mathcal{N}_{1,t}} [\mathtt{BIR}_1(m_1+1) \mid m_1 \geq \epsilon_0 t/2 - 1] - \mathtt{BIR}_2(t) \\ &\leq \mathtt{BIR}_1(\epsilon_0 T_1/2) \leq \beta_1/c_0'\end{aligned}$$

Since the response function f_{resp} is c_0'-Lipschitz in the neighborhood of $[-\delta_0, \delta_0]$, each agent after round T_1 will choose $\mathtt{alg}_1$ with probability at least

$$p_t \geq \frac{1}{2} - c_0' \left(\mathtt{PMR}_2(t) - \mathtt{PMR}_1(t)\right) \geq \frac{1}{2} - \beta_1.$$

Next, we will show that there exists a sufficiently large T_2 such that for any $t \geq T_1 + T_2$, with high probability $n_1(t) > \max\{n_0, (1-\beta_0)n_2(t)\}$, where n_0 is defined in (25). Fix any $t \geq T_1 + T_2$. Since each agent chooses $\mathtt{alg}_1$ with probability at least $1/2 - \beta_1$, by Chernoff Bounds (Theorem 3.1) we have with probability at least $1 - q_2(t)$ that the number of agents that choose $\mathtt{alg}_1$ is at least $\beta_0(1/2 - \beta_1)t/5$, where the function

$$q_2(x) = \exp\left(\frac{-(1/2-\beta_1)(1-\beta_0/5)^2 x}{3}\right).$$

Note that the number of agents received by $\mathtt{alg}_2$ is at most $T_1 + (1/2+\beta_1)t + (1/2-\beta_1)(1-\beta_0/5)t$.

Then as long as $T_2 \geq \frac{5T_1}{\beta_0}$, we can guarantee that $n_1(t) > n_2(t)(1-\beta_0)$ and $n_1(t) > n_0$ with probability at least $1 - q_2(t)$ for any $t \geq T_1 + T_2$. Note that the weak BIR-dominance condition in (25) implies that for any $t \geq T_1 + T_2$ with probability at least $1 - q_2(t)$,

$$\mathtt{BIR}_1(n_1(t)) < (1-\alpha_0)\mathtt{BIR}_2(n_2(t)).$$

It follows that for any $t \geq T_1 + T_2$,

$$\begin{aligned}\mathtt{PMR}_1(t) - \mathtt{PMR}_2(t) &= \mathop{\mathbb{E}}_{m_1 \sim \mathcal{N}_{1,t},\ m_2 \sim \mathcal{N}_{2,t}} [\mathtt{BIR}_2(m_2+1) - \mathtt{BIR}_1(m_1+1)] \\ &\geq (1 - q_2(t))\alpha_0 \mathtt{BIR}_2(t) - q_2(t) \\ &\geq \alpha_0 \mathtt{BIR}_2(t)/4\end{aligned}$$

where the last inequality holds as long as $q_2(t) \leq \alpha_0 \mathtt{BIR}_2(t)/4$, and is implied by the condition in (27) as long as T_2 is sufficiently large. Hence, by the definition of our `SoftMax` response function and assumption in (22), we have

$$\Pr[i_t = 1] \geq \frac{1}{2} + \frac{c_0 \alpha_0 \mathtt{BIR}_2(t)}{4}. \qquad \square$$

Similar to the condition (15), we can also generalize the weak BIR-dominance condition (25) to handle the dependence on T: there exist some T_0, a function $n_0(T) \in \mathrm{polylog}(T)$, and constants $\beta_0, \alpha_0 \in (0, 1/2)$, such that

$$(\forall T \geq T_0, n \geq n_0(T)) \quad \frac{\mathtt{BIR}_1((1-\beta_0)\, n \mid T)}{\mathtt{BIR}_2(n \mid T)} < 1 - \alpha_0. \tag{29}$$

We also provide a version of Theorem 6.3 under this more general weak BIR-dominance condition; its proof is very similar and is omitted. The following is just a direct consequence of Theorem 6.3 with this general condition

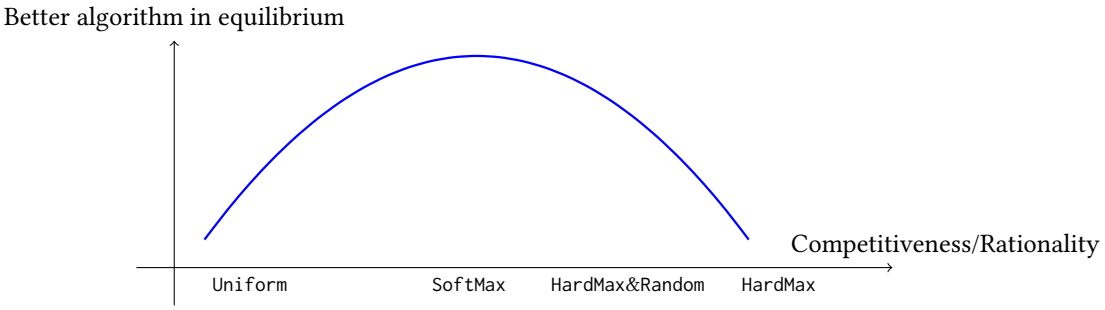


Fig. 3. The stylized inverted-U relationship in the "main story".

Corollary 6.6. *Assume that the response function is* `SoftMax`*, and each principal's utility is the number of users. Consider the restricted competition game in which the special algorithm* `alg` *weakly-BIR-dominates the other allowed algorithms, and the latter satisfy* $\texttt{BReg}(n) \to \infty$*. Then, for any sufficiently large time horizon* T*, there is a unique Nash equilibrium: both principals choose* `alg`*.*

7 ECONOMIC IMPLICATIONS

We frame our contributions in terms of the relationship between *competitiveness* and *rationality* on one side, and adoption of better algorithms on the other. Recall that both *competitiveness* (of the game between the two principals) and *rationality* (of the agents) are controlled by the response function $f_{\texttt{resp}}$.

Main story. Our main story concerns the restricted competition game between the two principals where one allowed algorithm `alg` is "better" than the others. We track whether and when `alg` is chosen in an equilibrium. We vary *competitiveness*/*rationality* by changing the response function from `HardMax` (full rationality, very competitive environment) to `HardMax&Random` to `SoftMax` (less rationality and competition). Our conclusions are as follows:

- Under `HardMax`, no innovation: `DynamicGreedy` is chosen over `alg`.
- Under `HardMax&Random`, some innovation: `alg` is chosen as long as it BIR-dominates.
- Under `SoftMax`, more innovation: `alg` is chosen as long as it weakly-BIR-dominates.[8]

These conclusions follow, respectively, from Corollaries 4.2, 5.3 and 6.4. Further, we consider the uniform choice between the principals. It corresponds to the least amount of rationality and competition, and (when principals' utility is the number of agents) uniform choice provides no incentives to innovate.[9] Thus, we have an inverted-U relationship, see Figure 3.

Secondary story. Let us zoom in on the symmetric `HardMax&Random` model. Competitiveness and rationality within this model are controlled by the baseline probability $\epsilon_0 = f_{\texttt{resp}}(-1)$, which goes smoothly between the two extremes of `HardMax` ($\epsilon_0 = 0$) and the uniform choice ($\epsilon_0 = \frac{1}{2}$). Smaller ϵ_0 corresponds to increased rationality and increased competitiveness. For clarity, we assume that principal's utility is the number of agents.

[8]This is a weaker condition, the better algorithm is chosen in a broader range of scenarios.

[9]On the other hand, if principals' utility is somewhat aligned with agents' welfare, as in (5), then a monopolist principal is incentivized to choose the best possible MAB algorithm (namely, to minimize cumulative Bayesian regret $\texttt{BReg}(T)$). Accordingly, monopoly would result in better social welfare than competition, as the latter is likely to split the market and cause each principal to learn more slowly. This is a very generic and well-known effect regarding economies of scale.

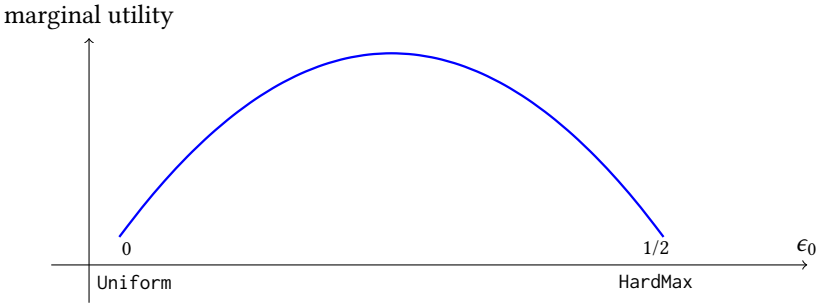


Fig. 4. The stylized inverted-U relationship from the "secondary story"

We consider the marginal utility of switching to a better algorithm. Suppose initially both principals use some algorithm `alg`, and principal 1 ponders switching to another algorithm `alg'` which BIR-dominates `alg`. We are interested in the marginal utility of this switch. Then:

- $\epsilon_0 = 0$ (`HardMax`): the marginal utility can be negative if `alg` is `DynamicGreedy`.
- ϵ_0 near 0: only a small marginal utility can be guaranteed, as it may take a long time for `alg'` to "catch up" with `alg`, and hence less time to reap the benefits.
- "medium-range" ϵ_0: large marginal utility, as `alg'` learns fast and gets most agents.
- ϵ_0 near $\frac{1}{2}$: small marginal utility, as principal 1 gets most agents for free no matter what.

The familiar inverted-U shape is depicted in Figure 4.

8 INTRODUCTION

Many modern online platforms simultaneously compete for users as well as learn from the users they manage to attract. This creates a tension between *exploration* and *competition*: firms experiment with potentially sub-optimal options for the sake of gaining information to make better decisions tomorrow, while they need to incentivize consumers to select them over their competitors today. For instance, Google Search and Bing compete for users in the search engine market yet at the same time need to experiment with their search and ranking algorithms to learn what works best. Similar exploration vs. competition tension arises in other application domains: recommendation systems, news and entertainment websites, online commerce, and so forth.

Online platforms routinely deploy A/B tests, and are increasingly adopting more sophisticated exploration methodologies based on *multi-armed bandits*, a well-known framework for making decisions under uncertainty and trading off exploration and exploitation (making good near-term decisions). While deploying "better" learning algorithms for exploration would improve performance, this is not necessarily beneficial under competition, even putting aside the deployment/maintenance costs. In particular, excessive experimentation may hurt a platform's reputation and decrease its market share in the near term. This would leave the learning algorithm with less users to learn from, which may further degrade the platform's performance relative to competitors who keep learning and improving from *their* users, and so forth. Taken to the extreme, such dynamics may create a "death spiral" effect when the vast majority of customers eventually switch to competitors.

In this paper, we ask how the interplay of exploration and competition affects platforms' incentives. While some bandit algorithms are traditionally considered "better" than others in the literature, ***does competition incentivize the adoption of the better algorithms***? How is this affected by the intensity of competition? We investigate these issues via extensive numerical experiments in a stylized duopoly model.

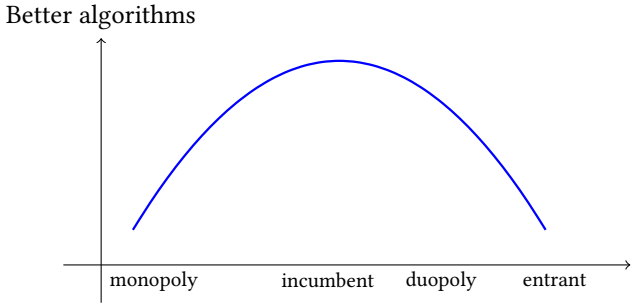


Fig. 5. A stylized "inverted-U relationship" between strength of competition and "level of innovation".

Our model. We consider two firms that compete for users and simultaneously learn from them. Each firm commits to a multi-armed bandit algorithm, and *explores* according to this algorithm. Users select between the two firms based on the current reputation score: rewards from the firm's algorithm, averaged over a recent time window. Each firm's objective is to maximize its market share (the fraction of users choosing this firm).

We consider a *permanent duopoly* in which both firms start at the same time, as well as *temporary monopoly*: a duopoly with a first-mover. Accordingly, the intensity of competition in the model varies from "permanent monopoly" (just one firm) to "incumbent" (the first-mover in temporary monopoly) to permanent duopoly to "entrant" (late-arriver in temporary monopoly).[10]

We focus on three classes of bandit algorithms, ranging from more primitive to more sophisticated: *greedy algorithms* that do not explicitly explore, algorithms that separate exploration and exploitation, and algorithms that combine the two. We know from prior work that in the absence of competition, "better" algorithms are better in the long run, but could be worse initially.

Main findings. We find that in the permanent duopoly, competition incentivizes firms to choose the "greedy algorithm", and even more so if the firm is a late arriver in a market. This algorithm also prevails under monopoly, simply because it tends to be easier to deploy. Whereas the incumbent in the temporary monopoly is incentivized to deploy a more advanced exploration algorithm. As a result, consumer welfare is highest under temporary monopoly. We find strong evidence of the "death spiral" effect mentioned earlier; this effect is strongest under permanent duopoly.

Interpreting the adoption of better algorithms as "innovation", our findings can be framed in terms of the "inverted-U relationship" between competition and innovation (see Figure 5). This is a well-established concept in the economics literature, dating back to [49], whereby too little or too much competition is bad for innovation, but intermediate levels of competition tend to be better. However, the "inverted-U relationship" is driven by different aspects in our model than the ones in the existing literature in economics. In our case, the barriers for innovations arise entirely from the reputational consequences of exploration in competition, as opposed to the R&D costs (which is the more standard cause in prior work).

Additional findings. We investigate the "first-mover advantage" phenomenon in more detail. Being first in the market gives free data to learn from (a "data advantage") as well as a more definite,

[10] We consider the "permanent monopoly" scenario for comparison only, without presenting any findings. We just assume that a monopolist chooses the greedy algorithm, because it is easier to deploy in practice. Implicitly, users have no "outside option": the service provided is an improvement over not having it (and therefore the monopolist is not incentivized to deploy better learning algorithms). This is plausible with free ad-supported platforms such as Yelp or Google.

and possibly better reputation compared to an entrant (a "reputation advantage"). We run additional experiments so as to isolate and compare these two effects. We find that either effect alone leads to a significant advantage under competition. The data advantage is larger than reputation advantage when the incumbent commits to a more advanced bandit algorithm.

Data advantage is significant from the anti-trust perspective, as a possible barrier to entry. We find that even a small amount "data advantage" gets amplified under competition, causing a large difference in eventual market shares. This observation runs contrary to prior work [13, 38, 53], which studied learning without competition, and found that small amounts of additional data do not provide significant improvement in eventual outcomes. We conclude that competition dynamics – that firms compete as they learn over time – are pertinent to these anti-trust considerations.

We also investigate how algorithms' performance "in isolation" (without competition) is predictive of the outcomes under competition. We find that mean reputation – arguably, the most natural performance measure "in isolation" – is sometimes not a good predictor. We suggest a more refined performance measure, and use it to explain some of the competition outcomes.

We also consider an alternative choice rule with explicit noise/randomness: a small fraction of users choose a firm uniformly at random.[11] We confirm the theoretical intuition that better algorithms prevail if the expected number of "random" users is sufficiently large. However, we find that this effect is negligible for some smaller but still "relevant" parameter values.

8.1 Discussion

The present paper is an experimental counterpart to [42], which considered a similar duopoly model and obtained a number of theoretical results with "asymptotic" flavor. For the sake of analytical tractability, that paper makes a somewhat unrealistic simplification that users do not observe any signals about firms' ongoing performance. Instead, users choose between firms according to the firms' Bayesian-expected rewards. The strength of competition is varied in a different way, using assumptions about (ir)rational user behavior. For these reasons, the theorems from [42] have no direct bearing on our simulations. However, their high-level conclusion in is an inverted-U relationship similar to ours.

The present paper provides a more nuanced and "non-asymptotic" perspective. In essence, we look for substantial effects within relevant time scales. Indeed, we start our investigation by determining what time scales are relevant in the context of our model. The reputation-based choice model accounts for competition in a more direct way, allows to separate reputation vs. data advantage, and makes our model amenable to numerical simulations (unlike the model in [42]).

To elucidate the interplay of competition and exploration, our model is stylized in several important respects, some of which we discuss below. Firms compete only on the quality of service, rather than, say, pricing or the range of products. Various performance signals available to the users, from personal experience to word-of-mouth to consumer reports, are abstracted as persistent "reputation scores", which further simplified to average performance over a time window. On the machine learning side, our setup is geared to distinguish between "simple" vs. "better" vs. "smart" bandit algorithms; we are not interested in state-of-art algorithms for very realistic bandit settings.

Even with a stylized model, numerical investigation is quite challenging. An "atomic experiment" is a competition game between a given pair of bandit algorithms, in a given competition model, on a given instance of a multi-armed bandit problem.[12] Accordingly, we have a three-dimensional space of atomic experiments one needs to run and interpret: {pairs of algorithms} x {competition

[11] Reputation scores already introduce some noise into users' choices. However, the amount of noise due to this channel is typically small, both in our simulations and in practice, because reputation signals average over many datapoints.

[12] Each such experiment is run many times to reduce variance.

models} x {bandit instances}, and we are looking for findings that are consistent across this entire space. It is essential to keep each of the three dimensions small yet representative. In particular, we need to capture a huge variety of bandit instances with only a few representative examples. Further, one needs succinct and informative summarization of results within one atomic experiment and across multiple experiments (*e.g.,* see Table 1).

While amenable to simulations, our model appears difficult to analyze. This is for several reasons: intricate feedback loop from performance to reputations to users to performance; mean reputation, most connected to our intuition, is sometimes a bad predictor in competition (see Sections 10 and 13); mathematical tools from regret-minimization would only produce "asymptotic" results, which do not seem to suffice. Further, we believe that resolving first-order theoretical questions about our model would not add much value to this paper. Indeed, we are in the realm of stylized economic models that provide mathematical intuition about the world, and [42] already has an elaborate analysis with similar high-level conclusions.

8.2 Related work

Machine learning. Multi-armed bandits (MAB) is a tractable abstraction for the tradeoff between exploration and exploitation. MAB problems have been studied for many decades, see [20, 39] for background on this immense body of work; we only comment on the most related aspects.

We consider MAB with i.i.d. rewards, a well-studied and well-understood MAB model [7]. We focus on a well-known distinction between "greedy" (exploitation-only) algorithms, "naive" algorithms that separate exploration and exploitation, and "smart" algorithms that combine them. Switching from "greedy" to "naive" to "smart" algorithms involves substantial adoption costs in infrastructure and personnel training [1, 2].

The study of competition vs. exploration has been initiated in [42], as discussed above. Our setting is also closely related to the "dueling algorithms" framework [31], but this framework considers offline / full feedback scenarios whereas we focus on online machine learning problems.

In "dueling bandits" (e.g., [57, 58]), an algorithm sets up a "duel" between a pair of arms in each round, and only learns which arm has "won". While this setting features a form of competition inside an MAB problem, it is very different from ours.

The interplay between exploration, exploitation and incentives has been studied in other scenarios: incentivizing exploration in a recommendation system, *e.g.,* [17, 21, 24, 36, 40], dynamic auctions (see [15] for background), online ad auctions, *e.g.,* [10, 11, 22, 44], human computation [26, 29, 50], and repeated auctions, *e.g.,* [4, 5, 19].

Economics. Our work is related to a longstanding economics literature on competition vs. innovation, *e.g.,* [3, 14, 49]. While this literature focuses on R&D costs of innovation, "reputational costs" seem new and specific to exploration.

Whether data gives competitive advantage has been studied theoretically in [38, 53] and empirically in [13]. While these papers find that small amounts of additional data do not provide significant improvement, they focus on learning in isolation. The first-mover advantage has been well-studied in other settings in economics and marketing, see survey [34].

The most common measures of market "competitiveness" such as the Lerner Index or the Herfindahl-Hirschman Index of a market rely on ex-post observable attributes of a market such as prices or market shares [52]. However, neither is applicable to our setting: in our model, there are no prices, and market shares are endogenous.

9 MODEL AND PRELIMINARIES

We consider a game involving two firms and T customers (henceforth, *agents*). The game lasts for T rounds. In each round, a new agent arrives, chooses among the two firms, interacts with the chosen firm, and leaves forever.

Each interaction between a firm and an agent proceeds as follows. There is a set A of K actions, henceforth *arms*, same for both firms and all rounds. The firm chooses an arm, and the agent experiences a numerical reward observed by the firm. Each arm corresponds to a different version of the experience that a firm can provide for an agent, and the reward corresponds to the agent's satisfaction level. The other firm does not observe anything about this interaction, not even the fact that this interaction has happened.

From each firm's perspective, the interactions with agents follow the protocol of the multi-armed bandit problem (MAB). We focus on i.i.d. Bernoulli rewards: the reward of each arm a is drawn from $\{0, 1\}$ independently with expectation $\mu(a)$. The mean rewards $\mu(a)$ are the same for all rounds and both firms, but initially unknown.

Before the game starts, each firm commits to an MAB algorithm, and uses this algorithm to choose its actions. Each algorithm receives a "warm start": additional T_0 agents that arrive before the game starts, and interact with the firm as described above. The warm start ensures that each firm has a meaningful reputation when competition starts. Each firm's objective is to maximize its market share: the fraction of users who chose this firm.

In some of our experiments, one firm is the "incumbent" who enters the market before the other ("late entrant"), and therefore enjoys a *temporary monopoly*. Formally, the incumbent enjoys additional X rounds of the "warm start". We treat X as an exogenous element of the model, and study the consequences for a fixed X.

Agents. Firms compete on a single dimension, quality of service, as expressed by agents' rewards. Agents are myopic and non-strategic: they would like to choose among the firms so as to maximize their expected reward (i.e. select the firm with the highest quality), without attempting to influence the firms' learning algorithms or rewards of the future users. Agents are not well-informed: they only receive a rough signal about each firm's performance before they choose a firm, and no other information.

Concretely, each of the two firms has a *reputation score*, and each agent's choice is driven by these two numbers. We posit a version of rational behavior: each agent chooses a firm with a maximal reputation score (breaking ties uniformly). The reputation score is simply a sliding window average: an average reward of the last M agents that chose this firm.

MAB algorithms. We consider three classes of algorithms, ranging from more primitive to more sophisticated:

(1) *Greedy algorithms* that strive to take actions with maximal mean reward, based on the current information.
(2) *Exploration-separating algorithms* that separate exploration and exploitation. The "exploitation" choices strives to maximize mean reward in the next round, and the "exploration" choices do not use the rewards observed so far.
(3) *Adaptive exploration*: algorithms that combine exploration and exploitation, and sway the exploration choices towards more promising alternatives.

We are mainly interested in qualitative differences between the three classes. For concreteness, we fix one algorithm from each class. Our pilot experiments indicate that our findings do not change substantially if other algorithms are chosen. For technical reasons, we consider Bayesian versions initialized with a "fake" prior (*i.e.,* not based on actual knowledge). We consider:

(1) a greedy algorithm that chooses an arm with largest posterior mean reward. We call it "Dynamic Greedy" (because the chosen arm may change over time), DG in short.
(2) an exploration-separated algorithm that in each round, *explores* with probability ϵ: chooses an arm independently and uniformly at random, and with the remaining probability *exploits* according to DG. We call it "dynamic epsilon-greedy", DEG in short.[13]
(3) an adaptive-exploration algorithm called "Thompson Sampling" (TS). In each round, this algorithm updates the posterior distribution for the mean reward of each arm a, draws an independent sample s_a from this distribution, and chooses an arm with the largest s_a.

For ease of comparison, all three algorithms are parameterized with the same fake prior: namely, the mean reward of each arm is drawn independently from a $\texttt{Beta}(1,1)$ distribution. Recall that Beta priors with 0-1 rewards form a conjugate family, which allows for simple posterior updates.

Both DEG and TS are classic and well-understood MAB algorithms, see [20, 47] for background. It is well-known that TS is near-optimal in terms of the cumulative rewards, and DEG is very suboptimal, but still much better than DG.[14] In a stylized formula: TS $\gg$ DEG $\gg$ DG as stand-alone MAB algorithms.

MAB instances. We consider instances with $K = 10$ arms. Since we focus on 0-1 rewards, an instance of the MAB problem is specified by the *mean reward vector* $(\mu(a) : a \in A)$. Initially this vector is drawn from some distribution, termed *MAB instance*. We consider three MAB instances:

(1) *Needle-In-Haystack*: one arm (the "needle") is chosen uniformly at random. This arm has mean reward .7, and the remaining ones have mean reward .5.
(2) *Uniform instance*: the mean reward of each arm is drawn independently and uniformly from $[1/4, 3/4]$.
(3) *Heavy-Tail instance*: the mean reward of each arm is drawn independently from $\texttt{Beta}(.6, .6)$ distribution (which is known to have substantial "tail probabilities").

We argue that these MAB instances are (somewhat) representative. Consider the "gap" between the best and the second-best arm, an essential parameter in the literature on MAB. The "gap" is fixed in Needle-in-Haystack, spread over a wide spectrum of values under the Uniform instance, and is spread but focused on the large values under the Heavy-Tail instance. We also ran smaller experiments with versions of these instances, and achieved similar qualitative results.

Terminology. Following a standard game-theoretic terminology, algorithm Alg1 *(weakly) dominates* algorithm Alg2 for a given firm if Alg1 provides a larger (or equal) market share than Alg2 at the end of the game. An algorithm is a (weakly) dominant strategy for the firm if it (weakly) dominates all other algorithms. This is for a particular MAB instance and a particular selection of the game parameters.

Simulation details. For each MAB instance we draw $N = 1000$ mean reward vectors independently from the corresponding distribution. We use this same collection of mean reward vectors for all experiments with this MAB instance. For each mean reward vector we draw a table of realized rewards (*realization table*), and use this same table for all experiments on this mean reward vector. This ensures that differences in algorithm performance are not due to noise in the realizations but due to differences in the algorithms in the different experimental settings.

More specifically, the realization table is a 0-1 matrix W with K columns which correspond to arms, and $T + T_{\max}$ rows, which correspond to rounds. Here $T_{\max}$ is the maximal duration of the

[13] Throughout, we fix $\epsilon = 0.05$. Our pilot experiments showed that different ϵ did not qualitatively change the results.
[14] Formally, TS achieves regret $\tilde{O}(\sqrt{TK})$ and $O(\frac{1}{\Delta}\log T)$, where Δ is the gap in mean rewards between the best and second-best arms. DEG has regret $\tilde{\Theta}(T^{2/3}K^{1/3})$ in the worst case. And DG can have regret as high as $\Omega(T)$. Deeper discussion of these distinctions is not very relevant to this paper.

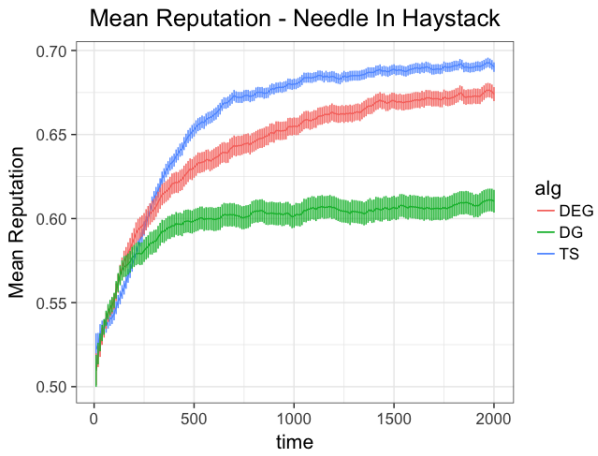


Fig. 6. Mean reputation trajectories for Needle-in-Haystack. The shaded area shows 95% confidence intervals.

"warm start" in our experiments, *i.e.,* the maximal value of $X + T_0$. For each arm a, each value $W(\cdot, a)$ is drawn independently from Bernoulli distribution with expectation $\mu(a)$. Then in each experiment, the reward of this arm in round t of the warm start is taken to be $W(t, a)$, and its reward in round t of the game is $W(T_{\max} + t, a)$.

We fix the sliding window size $M = 100$. We found that lower values induced too much random noise in the results, and increasing M further did not make a qualitative difference. Unless otherwise noted, we used $T = 2000$.

The simulations are computationally intensive. An experiment on a particular MAB instance comprised multiple runs of the competition game: N mean reward vectors times 9 pairs of algorithms times three values for the warm start. We used a parallel implementation over a cluster of 12 2.2 GHz CPU cores, with 8 GB RAM per core. Each experiment took about 10 hours.

Consistency. While we experiment with various MAB instances and parameter settings, we only report on selected, representative experiments in the body of the paper. Additional plots and tables are provided in the appendix. Unless noted otherwise, our findings are based on and consistent with all these experiments.

10 PERFORMANCE IN ISOLATION

We start with a pilot experiment in which we investigate each algorithm's performance "in isolation": in a stand-alone MAB problem without competition. We focus on reputation scores generated by each algorithm. We confirm that algorithms' performance is ordered as we'd expect: `TS` > `DEG` > `DG` for a sufficiently long time horizon. For each algorithm and each MAB instance, we compute the mean reputation score at each round, averaged over all mean reward vectors. We plot the *mean reputation trajectory*: how this score evolves over time. Figure 6 shows such a plot for the Needle-in-Haystack instance; for other MAB instances the plots are similar. We summarize this finding as follows:

Finding 1. *The mean reputation trajectories are arranged as predicted by prior work:* `TS` > `DEG` > `DG` *for a sufficiently long time horizon.*

We also use Figure 6 to choose a reasonable time-horizon for the subsequent experiments, as $T = 2000$. The idea is, we want T to be large enough so that algorithms performance starts to plateau, but small enough such that algorithms are still learning.

The mean reputation trajectory is probably the most natural way to represent an algorithm's performance on a given MAB instance. However, we found that the outcomes of the competition game are better explained with a different "performance-in-isolation" statistic that is more directly connected to the game. Consider the performance of two algorithms, Alg1 and Alg2, "in isolation" on a particular MAB instance. The *relative reputation* of Alg1 (vs. Alg2) at a given time t is the fraction of mean reward vectors/realization tables for which Alg1 has a higher reputation score than Alg2. The intuition is that agent's selection in our model depends only on the comparison between the reputation scores.

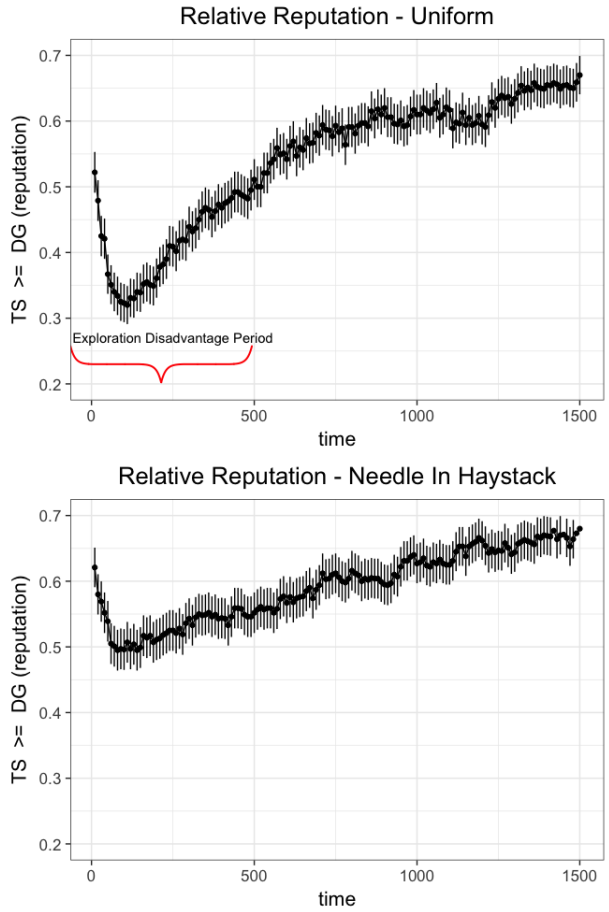


Fig. 7. Relative reputation trajectory for TS vs DG, on Uniform instance (top) and Needle-in-Haystack instance (bottom). Shaded area display 95% confidence intervals.

This angle allows a more nuanced analysis of reputation costs vs. benefits under competition. Figure 7 (top) shows the relative reputation trajectory for TS vs DG for the Uniform instance. The relative reputation is less than $\frac{1}{2}$ in the early rounds, meaning that DG has a higher reputation score

	Heavy-Tail			Needle-in-Haystack		
	$T_0 = 20$	$T_0 = 250$	$T_0 = 500$	$T_0 = 20$	$T_0 = 250$	$T_0 = 500$
TS vs DG	**0.29** ±0.03 EoG 55 (0)	**0.72** ±0.02 EoG 570 (0)	**0.76** ±0.02 EoG 620 (99)	**0.64** ±0.03 EoG 200 (27)	**0.6** ±0.03 EoG 370 (0)	**0.64** ±0.03 EoG 580 (122)
TS vs DEG	**0.3** ±0.03 EoG 37 (0)	**0.88** ±0.01 EoG 480 (0)	**0.9** ±0.01 EoG 570 (114)	**0.57** ±0.03 EoG 150 (14)	**0.52** ±0.03 EoG 460 (79)	**0.56** ±0.02 EoG 740 (628)
DG vs DEG	**0.62** ±0.03 EoG 410 (7)	**0.6** ±0.02 EoG 790 (762)	**0.57** ±0.03 EoG 730 (608)	**0.46** ±0.03 EoG 340 (129)	**0.42** ±0.02 EoG 650 (408)	**0.42** ±0.02 EoG 690 (467)

Table 1. **Permanent duopoly**, for Heavy-Tail and Needle-in-Haystack instances. Each cell describes a game between two algorithms, call them Alg1 vs. Alg2, for a particular value of the warm start T_0. Line 1 in the cell is the market share of Alg 1: the average (in bold) and the 95% confidence band. Line 2 specifies the "effective end of game" (EoG): the average and the median (in brackets). The time horizon is $T = 2000$.

in a majority of the simulations, and more than $\frac{1}{2}$ later on. The reason is the exploration in TS leads to worse decisions initially, but allows for better decisions later. The time period when relative reputation vs. DG dips below $\frac{1}{2}$ can be seen as an explanation for the competitive disadvantage of exploration. Such period also exists for the Heavy-Tail MAB instance. However, it does not exist for the Needle-in-Haystack instance, see Figure 7 (bottom).[15]

Finding 2. *Exploration can lead to relative reputation vs.* DG *going below* $\frac{1}{2}$ *for some initial time period. This happens for some MAB instances but not for some others.*

Definition 10.1. For a particular MAB algorithm, a time period when relative reputation vs. DG goes below $\frac{1}{2}$ is called *exploration disadvantage period.* An MAB instance is called *exploration-disadvantaged* if such period exists.

Uniform and Heavy-tail instance are exploration-disadvantaged, but Needle-in-Haystack is not.

11 COMPETITION VS. BETTER ALGORITHMS

Our main experiments are with the duopoly game defined in Section 9. As the "intensity of competition" varies from permanent monopoly to "incumbent" to permanent duopoly to "late entrant", we find a stylized inverted-U relationship as in Figure 5. More formally, we look for equilibria in the duopoly game, where each firm's choices are limited to DG, DEG and TS. We do this for each "intensity level" and each MAB instance, and look for findings that are consistent across MAB instances. For cleaner results, we break ties towards less advanced algorithms (as they tend to have lower adoption costs [1, 2]). Note that DG is trivially the dominant strategy under permanent monopoly.

Permanent duopoly. The basic scenario is when both firms are competing from round 1. A crucial distinction is whether an MAB instance is exploration-disadvantaged:

Finding 3. *Under permanent duopoly:*

(a) *(*DG,DG*) is the unique pure-strategy Nash equilibrium for exploration-disadvantaged MAB instances with a sufficiently small "warm start".*

(b) *This is not necessarily the case for MAB instances that are not exploration-disadvantaged. In particular,* TS *is a weakly dominant strategy for Needle-in-Haystack.*

[15] We see two explanations for this: TS identifies the best arm faster for the Needle-in-Haystack instance, and there are no "very bad" arms which make exploration very expensive in the short term.

We investigate the firms' market shares when they choose different algorithms (otherwise, by symmetry both firms get half of the agents). We report the market shares for Heavy-Tail and Needle-in-Haystack instances in Table 1 (see the first line in each cell), for a range of values of the warm start T_0. Table 2 reports similarly on the Uniform instance. We find that DG is a weakly dominant strategy for the Heavy-Tail and Uniform instances, as long as T_0 is sufficiently small. However, TS is a weakly dominant strategy for the Needle-in-Haystack instance. We find that for a sufficiently small T_0, DG yields more than half the market against TS, but achieves similar market share vs. DG and DEG. By our tie-breaking rule, (DG,DG) is the only pure-strategy equilibrium.

	$T_0 = 20$	$T_0 = 250$	$T_0 = 500$
TS vs DG	**0.46** ±0.03	**0.52** ±0.02	**0.6** ±0.02
TS vs DEG	**0.41** ±0.03	**0.51** ±0.02	**0.55** ±0.02
DG vs DEG	**0.51** ±0.03	**0.48** ±0.02	**0.45** ±0.02

Table 2. **Permanent duopoly**, for the Uniform MAB instance. Semantics are the same as in Table 1.

We attribute the prevalence of DG on exploration-disadvantaged MAB instances to its prevalence on the initial "exploration disadvantage period", as described in Section 10. Increasing the warm start length T_0 makes this period shorter: indeed, considering relative reputation trajectory in Figure 7 (top), increasing T_0 effectively shifts the starting time point to the right. This is why it helps DG if T_0 is small.

Temporary Monopoly. We turn our attention to the temporary monopoly scenario. Recall that the incumbent firm enters the market and serves as a monopolist until the entrant firm enters at round X. We make X large enough, but still much smaller than the time horizon T. We find that the incumbent is incentivized to choose TS, in a strong sense:

Finding 4. *Under temporary monopoly,* TS *is the dominant strategy for the incumbent. This holds across all MAB instances, if X is large enough.*

The simulation results for the Heavy-Tail MAB instance are reported in Table 3, for a particular $X = 200$. We see that TS is a dominant strategy for the incumbent. Similar tables for the other MAB instances and other values of X are reported in the supplement, with the same conclusion.

	TS	DEG	DG
TS	**0.003**±0.003	**0.083**±0.02	**0.17**±0.02
DEG	**0.045**±0.01	**0.25**±0.02	**0.23**±0.02
DG	**0.12**±0.02	**0.36**±0.03	**0.3**±0.02

User share of row player (entrant), 200 round head-start, Heavy-Tail Instance

DG is a weakly dominant strategy for the entrant, for Heavy-Tail instance in Table 3 and the Uniform instance, but not for the Needle-in-Haystack instance. We attribute this finding to exploration-disadvantaged property of these two MAB instance, for the same reasons as discussed above.

Finding 5. *Under temporary monopoly,* DG *is a weakly dominant strategy for the entrant for exploration-disadvantaged MAB instances.*

Inverted-U relationship. We interpret our findings through the lens of the inverted-U relationship between the "intensity of competition" and the "quality of technology". The lowest level of competition is monopoly, when DG wins out for the trivial reason of tie-breaking. The highest levels are permanent duopoly and "late entrant". We see that DG is incentivized for exploration-disadvantaged MAB instances. In fact, incentives for DG get stronger when the model transitions from permanent duopoly to "late entrant".[16] Finally, the middle level of competition, "incumbent" in the temporary monopoly creates strong incentives for TS. In stylized form, this relationship is captured in Figure 5.

Our intuition for why incumbency creates more incentives for exploration is as follows. During the temporary monopoly period, reputation costs of exploration vanish. Instead, the firm wants to improve its performance as much as possible by the time competition starts. Essentially, the firm only faces a classical explore-exploit tradeoff, and is incentivized to choose algorithms that are best at optimizing this tradeoff.

Death spiral effect. Further, we investigate the "death spiral" effect mentioned in the Introduction. Restated in terms of our model, the effect is that one firm attracts new customers at a lower rate than the other, and falls behind in terms of performance because the other firm has more customers to learn from, and this gets worse over time until (almost) all new customers go to the other firm. With this intuition in mind, we define *effective end of game* (EoG) for a particular mean reward vector and realization table, as the last round t such that the agents at this and previous round choose different firms. Indeed, the game, effectively, ends after this round. We interpret low EoG as a strong evidence of the "death spiral" effect. Focusing on the permanent duopoly scenario, we specify the EoG values in Table 1 (the second line of each cell). We find that the EoG values are indeed small:

Finding 6. *Under permanent duopoly,* EoG *values tend to be much smaller than the time horizon T.*

We also see that the EoG values tend to increase as the warm start T_0 increases. We conjecture this is because larger T_0 tends to be more beneficial for a better algorithm (as it tends to follow a better learning curve). Indeed, we know that the "effective end of game" in this scenario typically occurs when a better algorithm loses, and helping it delays the loss.

Welfare implications. We study the effects of competition on consumer welfare: the total reward collected by the users over time. Rather than welfare directly, we find it more lucid to consider *market regret*:

$$T \max_a \mu(a) - \sum_{t\in[T]} \mu(a_t),$$

where a_t is the arm chosen by agent t. This is a standard performance measure in the literature on multi-armed bandits. Note that smaller regret means higher welfare.

We assume that both firms play their respective equilibrium strategies for the corresponding competition level. As discussed previously, these are:

- DG in the monopoly,
- DG for both firms in duopoly (Finding 3),
- TS for the incumbent (Finding 4) and DG for the entrant in temporary monopoly (Finding 5).

[16] For the Heavy-Tail instance, DG goes from a weakly dominant startegy to a strictly dominant one. For the Uniform instance, DG goes from a Nash equilibrium strategy to a weakly dominant one.

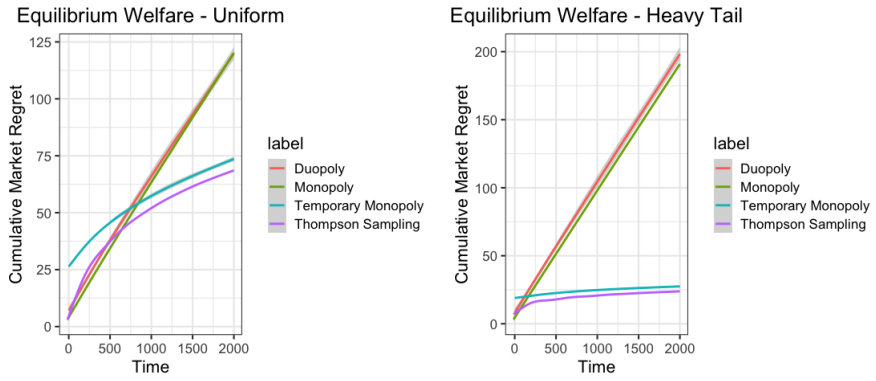


Fig. 8. Smoothed welfare plots resulting from equilibrium strategies in the different market structures. Note that welfare at $t = 0$ incorporates the regret incurred during the incumbent and warm start periods.

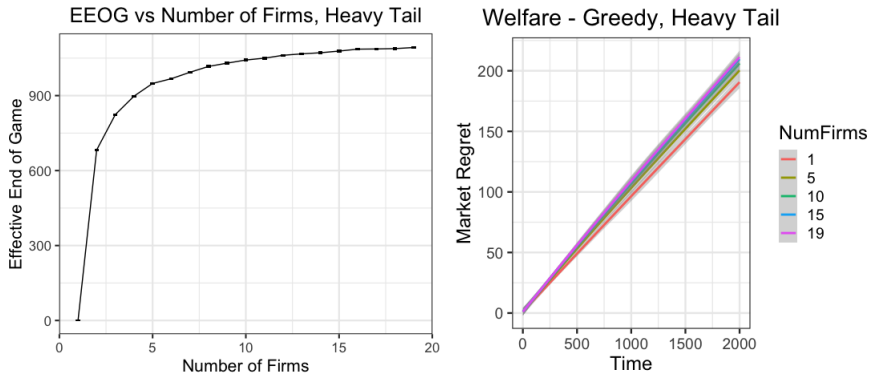


Fig. 9. Average welfare and EoG as we increase #firms playing DG

Figure 8 displays the market regret (averaged over multiple runs) under different levels of competition. Consumers are *better off* in the temporary monopoly case than in the duopoly case. Recall that under temporary monopoly, the incumbent is incentivized to play TS. Moreover, we find that the welfare is close to that of having a single firm for all agents and running TS. We also observe that monopoly and duopoly achieve similar welfare.

Finding 7. *In equilibrium, consumer welfare is (a) highest under temporary monopoly, (b) similar for monopoly and duopoly.*

Finding 7(b) is interesting because, in equilibrium, both firms play DG in both settings, and one might conjecture that the welfare should increase with the number of firms playing DG. Indeed, one run of DG may get stuck on a bad arm. However, two firms independently playing DG are less likely to get stuck simultaneously. If one firm gets stuck and the other does not, then the latter should attract most agents, leading to improved welfare.

To study this phenomenon further, we go beyond the duopoly setting to more than two firms playing DG (and starting at the same time). Figure 9 reports the average welfare across these simulations. Welfare not only does not get better, *but is weakly worse* as we increase the number of firms.

Finding 8. *When all firms deploy* DG*, and start at the same time, welfare is weakly decreasing as the number of firms increases*

We track the average EoG in each of the simulations and notice that it *increases* with the number of firms. This observation also runs counter of the intuition that with more firms running DG, one of them is more likely to "get lucky" and take over the market (which would cause EoG to *decrease* with the number of firms).

12 DATA AS A BARRIER TO ENTRY

Under temporary monopoly, the incumbent can explore without incurring immediate reputational costs, and build up a high reputation before the entrant appears. Thus, the early entry gives the incumbent both a *data* advantage and a *reputational* advantage over the entrant. We explore which of the two factors is more significant. Our findings provide a quantitative insight into the role of the classic "first mover advantage" phenomenon in the digital economy.

For a more succinct terminology, recall that the incumbent enjoys an extended warm start of $X + T_0$ rounds. Call the first X of these rounds the *monopoly period* (and the rest is the proper "warm start"). The rounds when both firms are competing for customers are called *competition period.*

We run two additional experiments to isolate the effects of the two advantages mentioned above. The *data-advantage experiment* focuses on the data advantage by, essentially, erasing the reputation advantage. Namely, the data from the monopoly period is not used in the computation of the incumbent's reputation score. Likewise, the *reputation-advantage experiment* erases the data advantage and focuses on the reputation advantage: namely, the incumbent's algorithm 'forgets' the data gathered during the monopoly period.

We find that either data or reputational advantage alone gives a substantial boost to the incumbent, compared to permanent duopoly. The results for the Heavy-Tail instance are presented in Table 4, in the same structure as Table 3. For the other two instances, the results are qualitatively similar.

	Reputation advantage (only)			Data advantage (only)		
	TS	DEG	DG	TS	DEG	DG
TS	**0.021**±0.009	**0.16**±0.02	**0.21** ±0.02	**0.0096**±0.006	**0.11**±0.02	**0.18**±0.02
DEG	**0.26**±0.03	**0.3**±0.02	**0.26**±0.02	**0.073**±0.01	**0.29**±0.02	**0.25**±0.02
DG	**0.34**±0.03	**0.4**±0.03	**0.33**±0.02	**0.15**±0.02	**0.39**±0.03	**0.33**±0.02

Table 4. Data advantage vs. reputation advantage experiment, on Heavy-Tail MAB instance. Each cell describes the duopoly game between the entrant's algorithm (the **row**) and the incumbent's algorithm (the **column**). The cell specifies the entrant's market share for the rounds in which hit was present: the average (in bold) and the 95% confidence interval. NB: smaller average is better for the incumbent.

We can quantitatively define the data (resp., reputation) advantage as the incumbent's market share in the competition period in the data-advantage (resp., reputation advantage) experiment, minus the said share under permanent duopoly, for the same pair of algorithms and the same problem instance. In this language, our findings are as follows.

Finding 9.
(a) Data advantage and reputation advantage alone are substantially large, across all algorithms and all MAB instances.

(b) The data advantage is larger than the reputation advantage when the incumbent chooses TS.
(c) The two advantages are similar in magnitude when the incumbent chooses DEG *or* DG.

Our intuition for Finding 9(b) is as follows. Suppose the incumbent switches from DG to TS. This switch allows the incumbent to explore actions more efficiently – collect better data in the same number of rounds – and therefore should benefit the data advantage. However, the same switch increases the reputation cost of exploration in the short run, which could weaken the reputation advantage.

13 PERFORMANCE IN ISOLATION, REVISITED

We saw in Section 11 that mean reputation trajectories do not suffice to explain the outcomes under competition. Let us provide more evidence and intuition for this.

Mean reputation trajectories are so natural that one is tempted to conjecture that they determine the outcomes under competition. More specifically:

Conjecture 13.1. If one algorithm's mean reputation trajectory lies above another, perhaps after some initial time interval (*e.g.,* as in Figure 6), then the first algorithm prevails under competition, for a sufficiently large warm start T_0.

However, we find a more nuanced picture. For example, in Figure 1 we see that DG attains a larger market share than DEG even for large warm starts. We find that this also holds for $K = 3$ arms and longer time horizons, see the supplement for more details. We conclude:

Finding 10. *Conjecture 13.1 is false: mean reputation trajectories do not suffice to explain the outcomes under competition.*

To see what could go wrong with Conjecture 13.1, consider how an algorithm's reputation score is distributed at a particular time. That is, consider the empirical distribution of this score over different mean reward vectors.[17] For concreteness, consider the Needle-in-Haystack instance at time $t = 500$, plotted in Figure 10. (The other MAB instances lead to a similar intuition.)

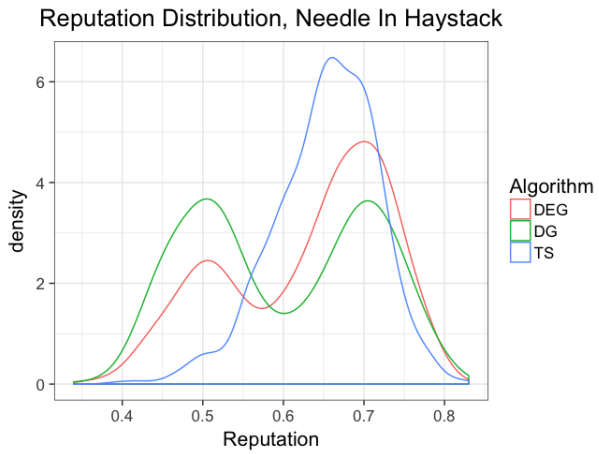


Fig. 10. Reputation scores for Needle-in-Haystack at $t = 500$ (smoothed using a kernel density estimate)

[17] Recall that each mean reward vector in our experimental setup comes with one specific realization table.

We see that the "naive" algorithms DG and DEG have a bi-modal reputation distribution, whereas TS does not. The reason is that for this MAB instance, DG either finds the best arm and sticks to it, or gets stuck on the bad arms. In the former case DG does slightly better than TS, and in the latter case it does substantially worse. However, the mean reputation trajectory may fail to capture this complexity since it simply takes average over different mean reward vectors. This may be inadequate for explaining the outcome of the duopoly game, given that the latter is determined by a simple comparison between the firm's reputation scores.

To further this intuition, consider the difference in reputation scores (*reputation difference*) between TS and DG on a particular mean reward vector. Let's plot the empirical distribution of the reputation difference (over the mean reward vectors) at a particular time point. Figure 11 shows such plots for several time points. We observe that the distribution is skewed to the right, precisely due to the fact that DG either does slightly better than TS or does substantially worse. Therefore, the mean is not a good measure of the central tendency, or typical value, of this distribution.

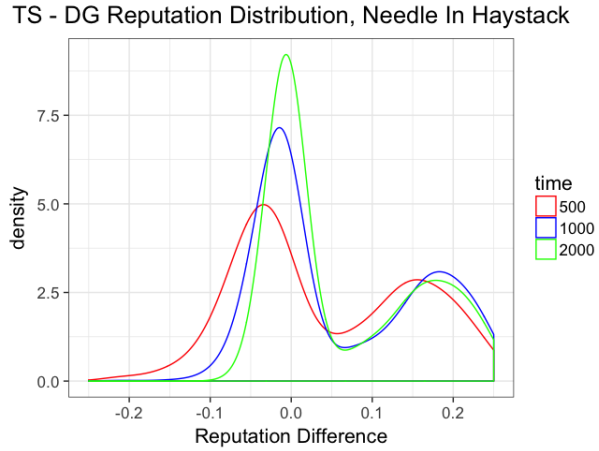


Fig. 11. Reputation difference TS − DG for Needle-in-Haystack (smoothed using a kernel density estimate)

14 NON-DETERMINISTIC CHOICE MODELS

Let us consider an extension in which the agents' choice is no longer deterministic. Recall that in our main model agents deterministically choose the firm with the higher reputation score; call this choice rule *HardMax* (HM). Now, we introduce some randomness: each agent selects between the firms uniformly with probability $\epsilon \in (0, 1)$, and takes the firm with the higher reputation score with the remaining probability; call this choice rule *HardMax with randomness* (HMR).

One can view HMR as a version of "warm start", where a firm receives some customers without competition, but these customers are dispersed throughout the game. The expected duration of this "dispersed warm start" is ϵT. If this quantity is large enough, we expect better algorithms to reach their long-term performance and prevail in competition. We confirm this intuition; we also find that this effect is negligible for smaller (but still relevant) values of ϵ or T.

Finding 11. TS *is weakly dominant under the* HMR *choice rule, if and only if* ϵT *is sufficiently large.* HMR *leads to lower variance in market shares, compared to* HM.

	Heavy-Tail (HMR with $\epsilon = .1$)			Heavy-Tail (HM)		
	TS vs DG	TS vs DEG	DG vs DEG	TS vs DG	TS vs DEG	DG vs DEG
$T = 2000$	**0.43** ± 0.02 Var: 0.15	**0.44** ± 0.02 Var: 0.15	**0.6** ± 0.02 Var: 0.1	**0.29** ± 0.03 Var: 0.2	**0.28** ± 0.03 Var: 0.19	**0.63** ± 0.03 Var: 0.18
$T = 5000$	**0.66** ± 0.01 Var: 0.056	**0.59** ± 0.02 Var: 0.092	**0.56** ± 0.02 Var: 0.098	**0.29** ± 0.03 Var: 0.2	**0.29** ± 0.03 Var: 0.2	**0.62** ± 0.03 Var: 0.19
$T = 10000$	**0.76** ± 0.01 Var: 0.026	**0.67** ± 0.02 Var: 0.067	**0.52** ± 0.02 Var: 0.11	**0.3** ± 0.03 Var: 0.21	**0.3** ± 0.03 Var: 0.2	**0.6** ± 0.03 Var: 0.2

Table 5. HM and HMR choice models on the Heavy-Tail MAB instance. Each cell describes the market shares in a game between two algorithms, call them Alg1 vs. Alg2, at a particular value of t. Line 1 in the cell is the market share of Alg 1: the average (in bold) and the 95% confidence band. Line 2 specifies the variance of the market shares across the simulations. The results reported here are with $T_0 = 20$.

	Uniform (HMR with $\epsilon = .1$)			Needle-In-Haystack (HMR with $\epsilon = .1$)		
	TS vs DG	TS vs DEG	DG vs DEG	TS vs DG	TS vs DEG	DG vs DEG
$T = 2000$	**0.42** ± 0.02 Var: 0.13	**0.45** ± 0.02 Var: 0.13	**0.49** ± 0.02 Var: 0.093	**0.55** ± 0.02 Var: 0.15	**0.61** ± 0.02 Var: 0.13	**0.46** ± 0.02 Var: 0.12
$T = 5000$	**0.48** ± 0.02 Var: 0.089	**0.53** ± 0.02 Var: 0.098	**0.46** ± 0.02 Var: 0.072	**0.56** ± 0.02 Var: 0.13	**0.63** ± 0.02 Var: 0.12	**0.43** ± 0.02 Var: 0.11
$T = 10000$	**0.54** ± 0.01 Var: 0.055	**0.6** ± 0.02 Var: 0.073	**0.44** ± 0.02 Var: 0.064	**0.58** ± 0.02 Var: 0.083	**0.65** ± 0.02 Var: 0.096	**0.4** ± 0.02 Var: 0.1

Table 6. HMR choice model for Uniform and Needle-In-Haystack MAB instances.

Table 5 shows the average market shares under the HM vs HMR choice rule. In contrast to the HM model, TS becomes weakly dominant under the HMR model, as T gets sufficiently large. These findings hold across all problem instances, see Table 6 (with the same semantics as in Table 5).

However, it takes a significant amount of randomness and a relatively large time horizon for this effect to take place. Even with $T = 10000$ and $\epsilon = 0.1$ we see that DEG still outperforms DG on the Heavy-Tail MAB instance as well as that TS only starts to become weakly dominant at $T = 10000$ for the Uniform MAB instance.

15 CONCLUSION

We consider a stylized duopoly setting where firms simultaneously learn from and compete for users. We showed that competition may not always induce firms to commit to better exploration algorithms, resulting in welfare losses for consumers. The primary reason is that exploration may have short-term reputational consequences that lead to more naive algorithms winning in a long-term competition. Allowing one firm to have a head start, a.k.a. the first-mover advantage, incentivizes the first-mover to deploy “better" algorithms, which in turn leads to better welfare for consumers. Finally, we isolate the component of the first-mover advantage that is due to having more initial data, and find that even a small amount of this “data advantage" leads to substantial long-term market power.

REFERENCES

[1] Agarwal, A., Bird, S., Cozowicz, M., Dudik, M., Hoang, L., Langford, J., Li, L., Melamed, D., Oshri, G., Sen, S., and Slivkins, A. Multiworld testing: A system for experimentation, learning, and decision-making, 2016. A white paper, available at https://github.com/Microsoft/mwt-ds/raw/master/images/MWT-WhitePaper.pdf.

[2] Agarwal, A., Bird, S., Cozowicz, M., Hoang, L., Langford, J., Lee, S., Li, J., Melamed, D., Oshri, G., Ribas, O., Sen, S., and Slivkins, A. Making contextual decisions with low technical debt, 2017. Techical report at `arxiv.org/abs/1606.03966`.

[3] Aghion, P., Bloom, N., Blundell, R., Griffith, R., and Howitt, P. Competition and innovation: An inverted u relationship. *Quaterly J. of Economics 120*, 2 (2005), 701–728.

[4] Amin, K., Rostamizadeh, A., and Syed, U. Learning prices for repeated auctions with strategic buyers. In *26th NIPS* (2013), pp. 1169–1177.

[5] Amin, K., Rostamizadeh, A., and Syed, U. Repeated contextual auctions with strategic buyers. In *27th NIPS* (2014), pp. 622–630.

[6] Athey, S., and Segal, I. An efficient dynamic mechanism. *Econometrica 81*, 6 (Nov. 2013), 2463–2485. A preliminary version has been available as a working paper since 2007.

[7] Auer, P., Cesa-Bianchi, N., and Fischer, P. Finite-time analysis of the multiarmed bandit problem. *Machine Learning 47*, 2-3 (2002), 235–256.

[8] Auer, P., Cesa-Bianchi, N., Freund, Y., and Schapire, R. E. The nonstochastic multiarmed bandit problem. *SIAM J. Comput. 32*, 1 (2002), 48–77. Preliminary version in *36th IEEE FOCS*, 1995.

[9] Azevedo, E., and Gottlieb, D. Perfect competition in markets with adverse selection. *Econometrica 85*, 1 (2017), 67–105.

[10] Babaioff, M., Kleinberg, R., and Slivkins, A. Truthful mechanisms with implicit payment computation. *J. ACM 62*, 2 (2015), 10. Subsumes the conference papers in *ACM EC 2010* and *ACM EC 2013*.

[11] Babaioff, M., Sharma, Y., and Slivkins, A. Characterizing truthful multi-armed bandit mechanisms. *SIAM J. on Computing 43*, 1 (2014), 194–230. Preliminary version in *10th ACM EC*, 2009.

[12] Bahar, G., Smorodinsky, R., and Tennenholtz, M. Economic recommendation systems. In *16th ACM EC* (2016).

[13] Bajari, P., Chernozhukov, V., Hortaçsu, A., and Suzuki, J. The impact of big data on firm performance: An empirical investigation. Tech. rep., National Bureau of Economic Research, 2018.

[14] Barro, R. J., and Sala-i Martin, X. Economic growth: Mit press. *Cambridge, Massachusettes* (2004).

[15] Bergemann, D., and Said, M. Dynamic auctions: A survey. In *Wiley Encyclopedia of Operations Research and Management Science, Vol. 2*. Wiley: New York, 2011, pp. 1511–1522.

[16] Bergemann, D., and Välimäki, J. The dynamic pivot mechanism. *Econometrica 78*, 2 (2010), 771–789. Preliminary versions have been available since 2006.

[17] Bimpikis, K., Papanastasiou, Y., and Savva, N. Crowdsourcing exploration. *Management Science 64* (2018), 1477–1973.

[18] Bolton, P., and Harris, C. Strategic Experimentation. *Econometrica 67*, 2 (1999), 349–374.

[19] Braverman, M., Mao, J., Schneider, J., and Weinberg, M. Selling to a no-regret buyer. In *ACM EC* (2018), pp. 523–538.

[20] Bubeck, S., and Cesa-Bianchi, N. Regret Analysis of Stochastic and Nonstochastic Multi-armed Bandit Problems. *Foundations and Trends in Machine Learning 5*, 1 (2012).

[21] Che, Y.-K., and Hörner, J. Optimal design for social learning. *Quarterly Journal of Economics* (2018). Forthcoming. First published draft: 2013.

[22] Devanur, N., and Kakade, S. M. The price of truthfulness for pay-per-click auctions. In *10th ACM EC* (2009), pp. 99–106.

[23] Even-Dar, E., Mannor, S., and Mansour, Y. Action elimination and stopping conditions for the multi-armed bandit and reinforcement learning problems. *J. of Machine Learning Research (JMLR) 7* (2006), 1079–1105.

[24] Frazier, P., Kempe, D., Kleinberg, J. M., and Kleinberg, R. Incentivizing exploration. In *ACM EC* (2014), pp. 5–22.

[25] Gabaix, X., Laibson, D., Li, D., Li, H., Resnick, S., and de Vries, C. G. The impact of competition on prices with numerous firms. *J. of Economic Theory 165* (2016), 1–24.

[26] Ghosh, A., and Hummel, P. Learning and incentives in user-generated content: multi-armed bandits with endogenous arms. In *ITCS* (2013), pp. 233–246.

[27] Gittins, J., Glazebrook, K., and Weber, R. *Multi-Armed Bandit Allocation Indices*. John Wiley & Sons, 2011.

[28] Gummadi, R., Johari, R., and Yu, J. Y. Mean field equilibria of multiarmed bandit games. In *13th ACM EC* (2012).

[29] Ho, C.-J., Slivkins, A., and Vaughan, J. W. Adaptive contract design for crowdsourcing markets: Bandit algorithms for repeated principal-agent problems. *J. of Artificial Intelligence Research 55* (2016), 317–359. Preliminary version appeared in *ACM EC 2014*.

[30] Hotelling, H. Stability in competition. *The Economic Journal 39*, 153 (1929), 41–57.

[31] Immorlica, N., Kalai, A. T., Lucier, B., Moitra, A., Postlewaite, A., and Tennenholtz, M. Dueling algorithms. In *43rd ACM STOC* (2011), pp. 215–224.

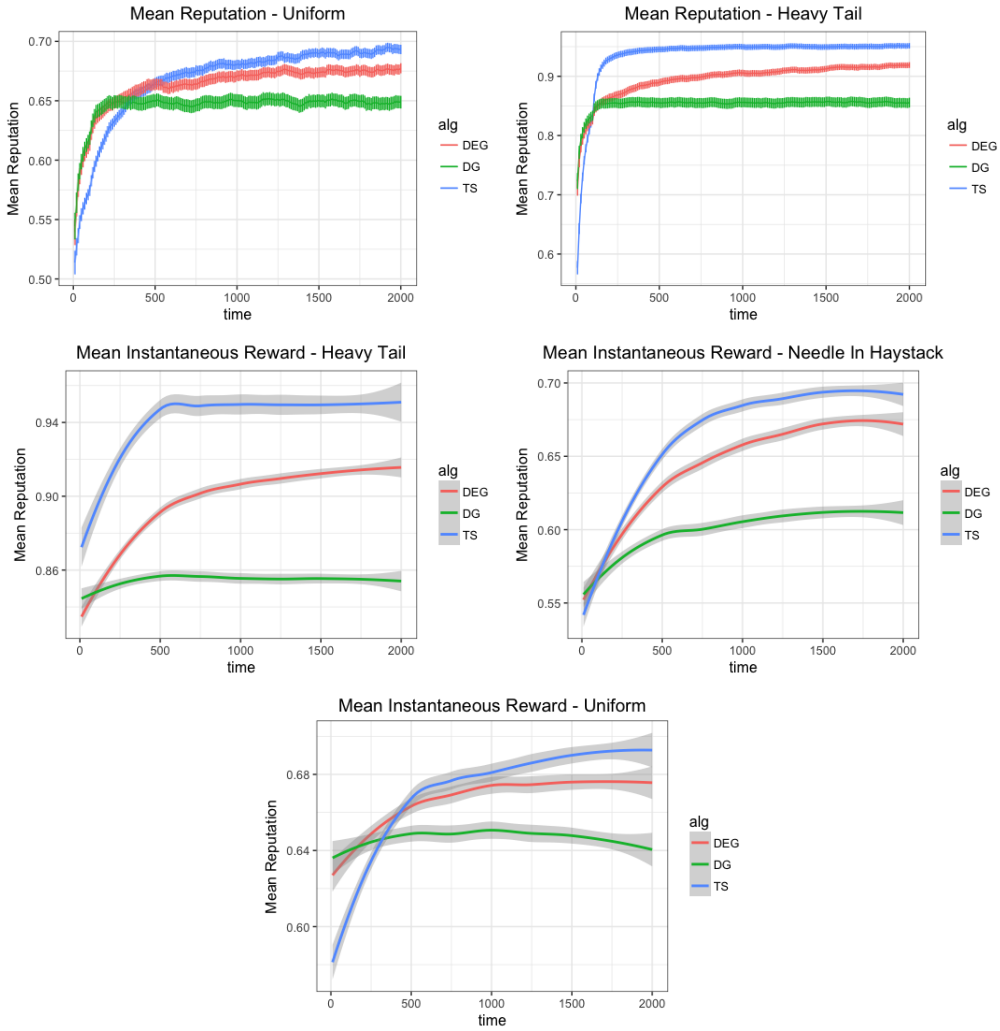
[32] Kakade, S. M., Lobel, I., and Nazerzadeh, H. Optimal dynamic mechanism design and the virtual-pivot mechanism. *Operations Research 61*, 4 (2013), 837–854.
[33] Keller, G., Rady, S., and Cripps, M. Strategic Experimentation with Exponential Bandits. *Econometrica 73*, 1 (2005), 39–68.
[34] Kerin, R. A., Varadarajan, P. R., and Peterson, R. A. First-mover advantage: A synthesis, conceptual framework, and research propositions. *The Journal of Marketing* (1992), 33–52.
[35] Kleinberg, R. D., Waggoner, B., and Weyl, E. G. Descending price optimally coordinates search. Working paper, 2016. Preliminary version in *ACM EC 2016*.
[36] Kremer, I., Mansour, Y., and Perry, M. Implementing the "wisdom of the crowd". *J. of Political Economy 122* (2014), 988–1012. Preliminary version in *ACM EC 2014*.
[37] Lai, T. L., and Robbins, H. Asymptotically efficient Adaptive Allocation Rules. *Advances in Applied Mathematics 6* (1985), 4–22.
[38] Lambrecht, A., and Tucker, C. E. Can big data protect a firm from competition?
[39] Lattimore, T., and Szepesvári, C. *Bandit Algorithms*. Cambridge University Press (preprint), 2019.
[40] Mansour, Y., Slivkins, A., and Syrgkanis, V. Bayesian incentive-compatible bandit exploration. In *15th ACM EC* (2015).
[41] Mansour, Y., Slivkins, A., Syrgkanis, V., and Wu, S. Bayesian exploration: Incentivizing exploration in bayesian games. Working paper, 2018. Available at `https://arxiv.org/abs/1602.07570`. Preliminary version in *ACM EC 2016*.
[42] Mansour, Y., Slivkins, A., and Wu, S. Competing bandits: Learning under competition. In *9th ITCS* (2018).
[43] Milgrom, P., and Stokey, N. Information, trade and common knowledge. *J. of Economic Theory 26*, 1 (1982), 17–27.
[44] Nazerzadeh, H., Saberi, A., and Vohra, R. Dynamic cost-per-action mechanisms and applications to online advertising. In *17th WWW* (2008).
[45] Perloff, J. M., and Salop, S. C. Equilibrium with product differentiation. *Review of Economic Studies LII* (1985), 107–120.
[46] Rothschild, M., and Stiglitz, J. Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *Quaterly J. of Economics 90*, 4 (1976), 629–649.
[47] Russo, D., Roy, B. V., Kazerouni, A., Osband, I., and Wen, Z. A tutorial on thompson sampling. *Foundations and Trends in Machine Learning 11*, 1 (2018), 1–96.
[48] Rysman, M. The economics of two-sided markets. *J. of Economic Perspectives 23*, 3 (2009), 125–144.
[49] Schumpeter, J. *Capitalism, Socialism and Democracy*. Harper & Brothers, 1942.
[50] Singla, A., and Krause, A. Truthful incentives in crowdsourcing tasks using regret minimization mechanisms. In *22nd WWW* (2013), pp. 1167–1178.
[51] Slivkins, A. Introduction to multi-armed bandits, 2018. A book draft, available at `http://research.microsoft.com/en-us/people/slivkins`. To be published with *Foundations and Trends in Machine Learning*.
[52] Tirole, J. *The theory of industrial organization*. MIT press, 1988.
[53] Varian, H. Artificial intelligence, economics, and industrial organization. In *The Economics of Artificial Intelligence: An Agenda*. University of Chicago Press, 2018.
[54] Veiga, A., and Weyl, G. Product design in selection markets. *Quarterly J. of Economics 131*, 2 (2016), 1007–1056.
[55] Vives, X. Innovation and competitive pressure. *J. of Industrial Economics 56*, 3 (2008).
[56] Weyl, G., and White, A. Let the right 'one' win: Policy lessons from the new economics of platforms. *Competition Policy International 12*, 2 (2014), 29–51.
[57] Yue, Y., Broder, J., Kleinberg, R., and Joachims, T. The k-armed dueling bandits problem. *J. Comput. Syst. Sci. 78*, 5 (2012), 1538–1556. Preliminary version in COLT 2009.
[58] Yue, Y., and Joachims, T. Interactively optimizing information retrieval systems as a dueling bandits problem. In *26th ICML* (2009), pp. 1201–1208.

Appendices

We provide plots and tables for our experiments, which were omitted from the main text due to page constraints. In all cases, the plots and tables here are in line with those in the main text, and lead to similar qualitative conclusions.

A PLOTS FOR "PERFORMANCE IN ISOLATION"

We present additional plots for Section 10. First, we provide mean reputation trajectories for Uniform and Heavy-Tail MAB instances. Second, we provide trajectories for instantaneous mean rewards, for all three MAB instances.[18] In all plots, the shaded area represents 95% confidence interval.



B TEMPORARY MONOPOLY

We present additional experiments on temporary monopoly from Section 11, across various MAB instances and various values of the incumbent advantage parameter X.

Each experiment is presented as a table with the same semantics as in the main text. Namely, each cell in the table describes the duopoly game between the entrant's algorithm (the row) and the incumbent's algorithm (the column). The cell specifies the entrant's market share (fraction of

[18] These trajectories are smoothed via a non-parametric regression. More concretely, we use this option in `ggplot`: https://ggplot2.tidyverse.org/reference/geom_smooth.html.

rounds in which it was chosen) for the rounds in which he was present. We give the average (in bold) and the 95% confidence interval. NB: smaller average is better for the incumbent.

Heavy-Tail MAB Instance

	TS	DEG	DG
TS	**0.054** ±0.01	**0.16** ±0.02	**0.18** ±0.02
DEG	**0.33** ±0.03	**0.31** ±0.02	**0.26** ±0.02
DG	**0.39** ±0.03	**0.41** ±0.03	**0.33** ±0.02

Table 7. Temporary Monopoly: Heavy Tail, $X = 50$

	TS	DEG	DG
TS	**0.003** ±0.003	**0.083** ±0.02	**0.17** ±0.02
DEG	**0.045** ±0.01	**0.25** ±0.02	**0.23** ±0.02
DG	**0.12** ±0.02	**0.36** ±0.03	**0.3** ±0.02

Table 8. Temporary Monopoly: Heavy Tail, $X = 200$

	TS	DEG	DG
TS	**0.0017** ±0.002	**0.059** ±0.01	**0.16** ±0.02
DEG	**0.029** ±0.007	**0.23** ±0.02	**0.23** ±0.02
DG	**0.097** ±0.02	**0.34** ±0.03	**0.29** ±0.02

Table 9. Temporary Monopoly: Heavy Tail, $X = 300$

	TS	DEG	DG
TS	**0.002** ±0.003	**0.043** ±0.01	**0.16** ±0.02
DEG	**0.03** ±0.007	**0.21** ±0.02	**0.24** ±0.02
DG	**0.091** ±0.01	**0.32** ±0.03	**0.3** ±0.02

Table 10. Temporary Monopoly: Heavy Tail, $X = 500$

Needle-In-Haystack MAB Instance

	TS	DEG	DG
TS	**0.34** ±0.03	**0.4** ±0.03	**0.48** ±0.03
DEG	**0.22** ±0.02	**0.34** ±0.03	**0.42** ±0.03
DG	**0.18** ±0.02	**0.28** ±0.02	**0.37** ±0.03

Table 11. Temporary Monopoly: Needle-in-Haystack, $X = 50$

	TS	DEG	DG
TS	**0.17** ±0.02	**0.31** ±0.03	**0.41** ±0.03
DEG	**0.13** ±0.02	**0.26** ±0.02	**0.36** ±0.03
DG	**0.093** ±0.02	**0.23** ±0.02	**0.33** ±0.03

Table 12. Temporary Monopoly: Needle-in-Haystack, $X = 200$

	TS	DEG	DG
TS	**0.1** ±0.02	**0.28** ±0.03	**0.39** ±0.03
DEG	**0.089** ±0.02	**0.23** ±0.02	**0.36** ±0.03
DG	**0.05** ±0.01	**0.21** ±0.02	**0.33** ±0.03

Table 13. Temporary Monopoly: Needle-in-Haystack, $X = 300$

	TS	DEG	DG
TS	**0.053** ±0.01	**0.23** ±0.02	**0.37** ±0.03
DEG	**0.051** ±0.01	**0.2** ±0.02	**0.33** ±0.03
DG	**0.031** ±0.009	**0.18** ±0.02	**0.31** ±0.02

Table 14. Temporary Monopoly: Needle-in-Haystack, $X = 500$

Uniform MAB Instance

	TS	DEG	DG
TS	**0.27** ±0.03	**0.21** ±0.02	**0.26** ±0.02
DEG	**0.39** ±0.03	**0.3** ±0.03	**0.34** ±0.03
DG	**0.39** ±0.03	**0.31** ±0.02	**0.33** ±0.02

Table 15. Temporary Monopoly: Uniform, $X = 50$

	TS	DEG	DG
TS	**0.12** ±0.02	**0.16** ±0.02	**0.2** ±0.02
DEG	**0.25** ±0.02	**0.24** ±0.02	**0.29** ±0.02
DG	**0.23** ±0.02	**0.24** ±0.02	**0.29** ±0.02

Table 16. Temporary Monopoly: Uniform, $X = 200$

	TS	DEG	DG
TS	**0.094** ±0.02	**0.15** ±0.02	**0.2** ±0.02
DEG	**0.2** ±0.02	**0.23** ±0.02	**0.29** ±0.02
DG	**0.21** ±0.02	**0.23** ±0.02	**0.29** ±0.02

Table 17. Temporary Monopoly: Uniform, $X = 300$

	TS	DEG	DG
TS	**0.061** ±0.01	**0.12** ±0.02	**0.2** ±0.02
DEG	**0.17** ±0.02	**0.21** ±0.02	**0.29** ±0.02
DG	**0.18** ±0.02	**0.22** ±0.02	**0.29** ±0.02

Table 18. Temporary Monopoly: Uniform, $X = 500$

C REPUTATION VS. DATA ADVANTAGE

This section presents all experiments on data vs. reputation advantage (Section 12).

Each experiment is presented as a table with the same semantics as in the main text. Namely, each cell in the table describes the duopoly game between the entrant's algorithm (the **row**) and the incumbent's algorithm (the **column**). The cell specifies the entrant's market share for the rounds in which hit was present: the average (in bold) and the 95% confidence interval. NB: smaller average is better for the incumbent.

	TS	DEG	DG
TS	**0.0096** ± 0.006	**0.11** ± 0.02	**0.18** ± 0.02
DEG	**0.073** ± 0.01	**0.29** ± 0.02	**0.25** ± 0.02
DG	**0.15** ± 0.02	**0.39** ± 0.03	**0.33** ± 0.02

Table 19. Data Advantage: Heavy Tail, $X = 200$

	TS	DEG	DG
TS	**0.021** ± 0.009	**0.16** ± 0.02	**0.21** ± 0.02
DEG	**0.26** ± 0.03	**0.3** ± 0.02	**0.26** ± 0.02
DG	**0.34** ± 0.03	**0.4** ± 0.03	**0.33** ± 0.02

Table 20. Reputation Advantage: Heavy Tail, $X = 200$

	TS	DEG	DG
TS	**0.25** ± 0.03	**0.36** ± 0.03	**0.45** ± 0.03
DEG	**0.21** ± 0.02	**0.32** ± 0.03	**0.41** ± 0.03
DG	**0.18** ± 0.02	**0.29** ± 0.03	**0.4** ± 0.03

Table 21. Data Advantage: Needle-in-Haystack, $X = 200$

	TS	DEG	DG
TS	**0.35** ± 0.03	**0.43** ± 0.03	**0.52** ± 0.03
DEG	**0.26** ± 0.03	**0.36** ± 0.03	**0.43** ± 0.03
DG	**0.19** ± 0.02	**0.3** ± 0.02	**0.36** ± 0.02

Table 22. Reputation Advantage: Needle-in-Haystack, $X = 200$

	TS	DEG	DG
TS	**0.27** ± 0.03	**0.23** ± 0.02	**0.27** ± 0.02
DEG	**0.4** ± 0.03	**0.3** ± 0.02	**0.32** ± 0.02
DG	**0.36** ± 0.03	**0.29** ± 0.02	**0.3** ± 0.02

Table 23. Reputation Advantage: Uniform, $X = 200$

	TS	DEG	DG
TS	**0.2** ± 0.02	**0.22** ± 0.02	**0.27** ± 0.03
DEG	**0.33** ± 0.03	**0.32** ± 0.03	**0.35** ± 0.03
DG	**0.32** ± 0.03	**0.31** ± 0.03	**0.35** ± 0.03

Table 24. Data Advantage: Uniform, $X = 200$

	TS	DEG	DG
TS	**0.0017** ±0.002	**0.06** ±0.01	**0.18** ±0.02
DEG	**0.04** ±0.009	**0.24** ±0.02	**0.25** ±0.02
DG	**0.12** ±0.02	**0.35** ±0.03	**0.33** ±0.02

Table 25. Data Advantage: Heavy-Tail, $X = 500$

	TS	DEG	DG
TS	**0.022** ±0.009	**0.13** ±0.02	**0.21** ±0.02
DEG	**0.26** ±0.03	**0.29** ±0.02	**0.28** ±0.02
DG	**0.33** ±0.03	**0.39** ±0.03	**0.34** ±0.02

Table 26. Reputation Advantage: Heavy-Tail, $X = 500$

	TS	DEG	DG
TS	**0.098** ±0.02	**0.27** ±0.03	**0.41** ±0.03
DEG	**0.093** ±0.02	**0.24** ±0.02	**0.38** ±0.03
DG	**0.064** ±0.01	**0.22** ±0.02	**0.37** ±0.03

Table 27. Data Advantage: Needle-in-Haystack, $X = 500$

	TS	DEG	DG
TS	**0.29** ±0.03	**0.44** ±0.03	**0.52** ±0.03
DEG	**0.19** ±0.02	**0.35** ±0.03	**0.42** ±0.03
DG	**0.15** ±0.02	**0.27** ±0.02	**0.35** ±0.02

Table 28. Reputation Advantage: Needle-in-Haystack, $X = 500$

	TS	DEG	DG
TS	**0.14** ±0.02	**0.18** ±0.02	**0.26** ±0.03
DEG	**0.26** ±0.02	**0.26** ±0.02	**0.34** ±0.03
DG	**0.25** ±0.02	**0.27** ±0.02	**0.34** ±0.03

Table 29. Data Advantage: Uniform, $X = 500$

	TS	DEG	DG
TS	**0.24** ±0.02	**0.2** ±0.02	**0.26** ±0.02
DEG	**0.37** ±0.03	**0.29** ±0.02	**0.31** ±0.02
DG	**0.35** ±0.03	**0.27** ±0.02	**0.3** ±0.02

Table 30. Reputation Advantage: Uniform, $X = 500$

D MEAN REPUTATION VS. RELATIVE REPUTATION

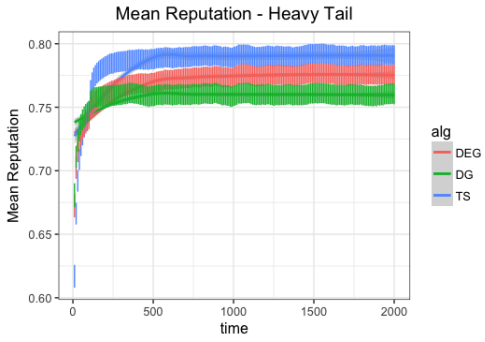
We present the experiments omitted from Section 13. Namely, experiments on the Heavy-Tail MAB instance with $K = 3$ arms, both for "performance in isolation" and the permanent duopoly game. We find that DEG > DG according to the mean reputation trajectory but that DG > DEG according to the relative reputation trajectory *and* in the competition game. As discussed in Section 13, the same results also hold for $K = 10$ for the warm starts that we consider.

The result of the permanent duopoly experiment for this instance is shown in Table 31.

	Heavy-Tail		
	$T_0 = 20$	$T_0 = 250$	$T_0 = 500$
TS vs. DG	**0.4** ±0.02 EoG 770 (0)	**0.59** ±0.01 EoG 2700 (2979.5)	**0.6** ±0.01 EoG 2700 (3018)
TS vs. DEG	**0.46** ±0.02 EoG 830 (0)	**0.73** ±0.01 EoG 2500 (2576.5)	**0.72** ±0.01 EoG 2700 (2862)
DG vs. DEG	**0.61** ±0.01 EoG 1400 (556)	**0.61** ±0.01 EoG 2400 (2538.5)	**0.6** ±0.01 EoG 2400 (2587.5)

Table 31. Duopoly Experiment: Heavy-Tail, $K = 3$, $T = 5000$.
Each cell describes a game between two algorithms, call them Alg1 vs. Alg2, for a particular value of the warm start T_0. Line 1 in the cell is the market share of Alg 1: the average (in bold) and the 95% confidence band. Line 2 specifies the "effective end of game" (EoG): the average and the median (in brackets).

The mean reputation trajectories for algorithms' performance in isolation:



Finally, the relative reputation trajectory of DEG vs. DG: