IB Subject: Mathematics

Extended Essay

Optimizing a Suspension Bridge Design for a Bridge Between Tuas and Jurong Island

Research Question: What is the height and number of towers in a suspension bridge between Tuas and Jurong Island such that it will bear the heaviest possible load and have the lowest cost?

Word Count: 3967

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1 Introduction

When I lived in Somerville, Massachusetts, we used to cross the Leonard P. Zakim Bunker Hill Memorial Bridge whenever we had to travel into the city. I had always wondered why that specific design was chosen. After doing some research on bridge designs, I found out that suspension bridges were one of the most popular and effective designs for relatively long distances. When I moved to Singapore, I noticed that there was no direct road between Tuas and Jurong Island, the country's two major industrial districts. A direct connection between the two major areas would result in more efficient transportation of industrial goods and reduced road pollution. I found this interesting so I decided to investigate what the optimal bridge design would look like for a bridge in this area. A suspension bridge was chosen as they are effective for distances like this. Hence, I came up with the following research question: ***"What is the height and number of towers required in a suspension bridge between Tuas and Jurong Island such that it will bear the heaviest possible load and have the lowest cost?"***

2 Structure of the Suspension Bridge Design

2.1 General Structure of a Suspension Bridge

Suspension bridges are a very commonly used design as it is efficient for long distances and is usually inexpensive. In this case, a suspension bridge will be modelled as being made of two identical sides and a deck. Vertical pillars starting at the seafloor, known as *towers, are* distributed across the lengths of the bridge with equal spacing between them. Each pillar is connected by *main cables*, which are in the shape of a catenary, mathematically known as the hyperbolic cosine function (Calvert), with a sag spanning the height from the deck, h. The lowest point of the main cables will touch the deck, which is a characteristic seen in several suspension bridges. It will be assumed that the main cables perfectly assume the shape of a catenary curve, as external forces will not be considered during the investigation. These main cables are connected to the deck by equally spaced *suspender cables*., so that the sag of the cable is h.

For the optimization process, the height from the deck to the highest point of the towers, *h*, and the number of towers*, n,* will be variable. The minimum value of *h* will be 10m and maximum value of *h* will be 200m, which is slightly higher than the height of the Golden Gate Bridge above its deck. Moreover, the maximum value of *n* will be 10, which is an extreme number of towers, as the average number of towers in a suspension bridge is 2. This is to further investigate the trade-off between cost and strength. The average depth of the Singapore Strait is 22 meters ("Singapore Strait") and the bridge will be built 20 meters above sea level, so the bridge will be 42 meters above the seabed. Assuming that the seafloor is perfectly uniform, the height of one tower will be $h + 42$. Since the main cables exist between two towers, the number of catenary-shaped cables is $n - 1$. As this model requires a central main cable, the smallest possible value for the number of towers, *n*, is 2.

For simplicity, the distance between two towers will be *a*, and the distance between the end of the bridge and the nearest tower will be half that distance, $\frac{a}{2}$. This is justified as most suspension bridges have a similar ratio, such as the Mackinac Bridge which has a ratio of $1 : 0.47 \approx 1 : \frac{1}{2}$ (Mackinac Bridge Authority). Therefore, the total length of the bridge, *l*, is shown by $a(n-1) + \frac{a}{2} + \frac{a}{2} = a \cdot n$. Moreover, the suspender cables are an equal distance away from each other. So, if the number of suspenders is *k*, the spacing between them is $\frac{a}{2} \div (k+1) = \frac{a}{2k+2}$. In the model, there will always be a set number of suspender cables under each half main cable for consistency. A side view of the bridge design for $n = 2$ and $n = 3$ as well as a 3-dimensional model for $n = 2$ is shown below:

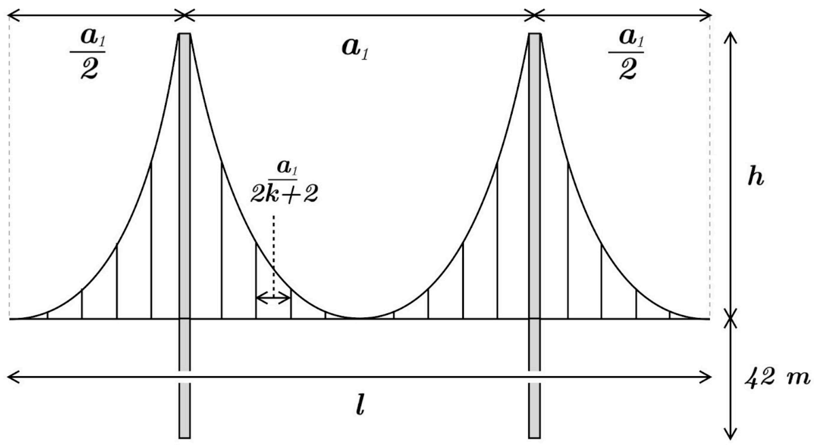

Figure 1. "n=2" Simplest suspension bridge design (not to scale) where k=4.

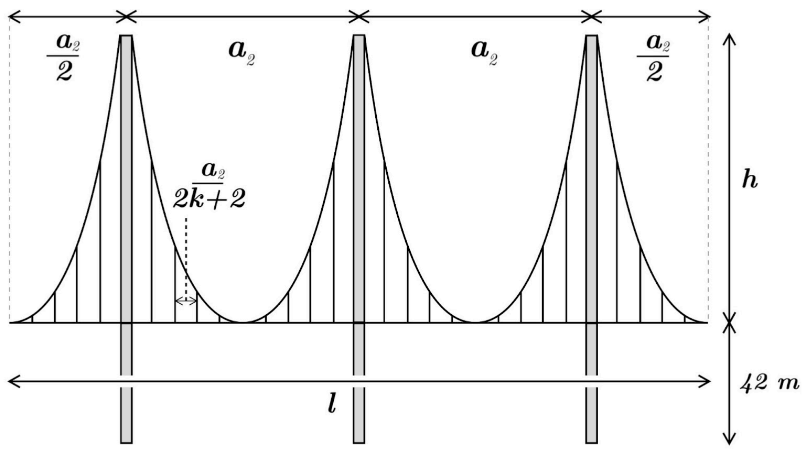

Figure 2. "n=3" Suspension bridge design (not to scale) where k=4.

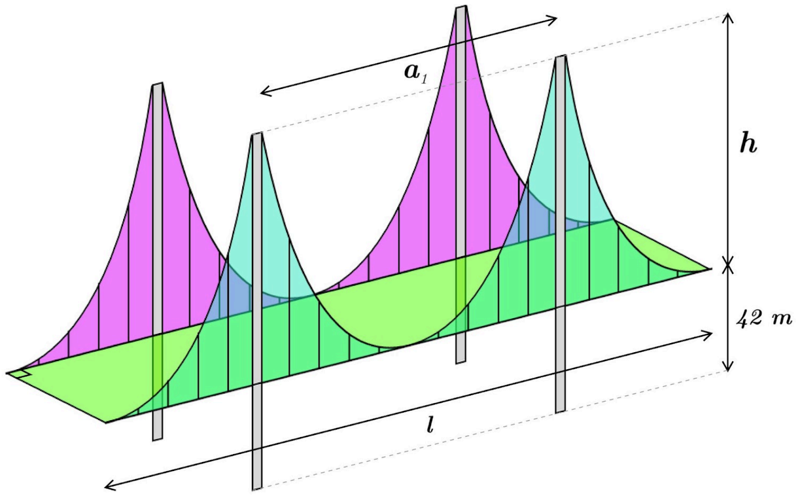

Figure 3. A 3D model of an "n=2" suspension bridge (not to scale) where k=4.

2.2 The Catenary Curve

According to the discovery of Christiaan Huygens, hanging cables are in the shape of a catenary, which is the curve of an ideal cable caused by its own weight when it is supported at its ends ("Catenary"). Mathematically, it is known as the hyperbolic cosine function as it has the general equation $y = cosh\,(x)$ on a cartesian plane. It is also shown by the average of e^x and e^{-x}, with the equation $y = \frac{e^x + e^{-x}}{2}$, where e is Euler's number.

The catenary shape is formed naturally when the cable only has one force, gravity, acting on it. The uniform force of gravity acting on the cable creates a shape that is symmetrical on both sides of the minimum point. As the cable is held up at two stationary points, a tension force also exists. Since the weight, and hence the shape, of the cable depends on its mass, its mass per unit length can affect the shape of the catenary. As gravitational acceleration is constant (9.81 ms^{-2}), the mass per unit length can be multiplied by the gravitational acceleration to obtain the weight per unit length. The equation for a catenary shape in terms of these factors can be written as:

$$y = \left(\frac{T_x}{ug}\right) cosh\left(\frac{x}{\frac{T_x}{ug}}\right) + b$$

where u is the mass per unit length (kg/m), g is gravitational acceleration, T_x is the horizontal tension and b is the constant of integration (Calvert). Hence, ug is the weight per unit length of the cable.

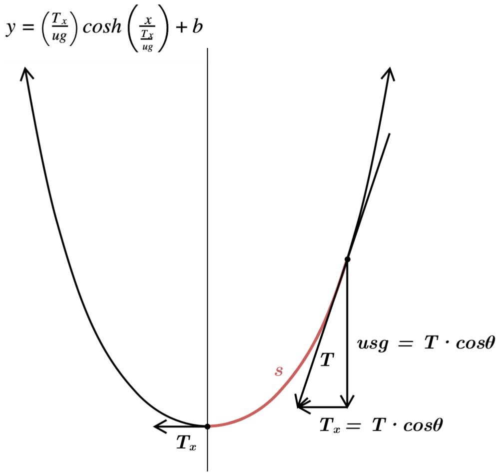

Figure 4. The components of the catenary curve.

By taking $\frac{T_x}{ug}$ as a constant ω, the equation now becomes $y = \omega\, cosh\left(\frac{x}{\omega}\right) + b$. As the minimum point is on the y-axis, it can be found by substituting $x = 0$ into the equation without the constant of integration, b:

$$y = \omega\, cosh\left(\frac{0}{\omega}\right)$$

$$y = \omega \cdot 1 = \omega$$

Hence, the constant will be set to $-\omega$ so that the vertex is always at (0, 0). This is so that the y-value of a point on the curve always refers to the actual height of the point from the lowest y-value of the curve, which is 0. So, the final function for the catenary curve that will be used is:

$$f(x) = \omega\, cosh\left(\frac{x}{\omega}\right) - \omega \quad (1)$$

2.3. Constructing Equations for Price

Prior to finding the price, we must express different components of the bridge in terms of h and n, such as the total length of all main cables in terms of h and n. The length of a main cable in a suspension bridge can be found using the formula for calculating the arc length (of a curve):

$$s(x) = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{2}$$

First, $\frac{dy}{dx}$ will be found in order to be substituted into the formula above. The standard derivative of $cosh(x)$ is $sinh(x)$ ("Derivative and Integral"). As it is a trigonometric function, the same chain rule will apply. Hence, differentiating equation (1):

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\omega \cdot cosh\left(\frac{x}{\omega}\right) - \omega\right) \\ &= \omega \cdot \frac{d}{dx}\left[cosh\left(\frac{x}{\omega}\right)\right] + \frac{d}{dx}[-\omega] \\ &= \omega \cdot sinh\left(\frac{x}{\omega}\right) \cdot \frac{d}{dx}\left[\frac{x}{\omega}\right] + 0 \\ &= \frac{1}{\omega} \cdot \frac{d}{dx}[x] \cdot \omega \cdot sinh\left(\frac{x}{\omega}\right) \\ &= \frac{\omega}{\omega} \cdot 1 \cdot sinh\left(\frac{x}{\omega}\right) \\ &= sinh\left(\frac{x}{\omega}\right) \end{aligned} \tag{3}$$

Hence, by substituting $f'(x)$ into equation (2), the equation for arc length of a catenary, s, between $x = p$ and $x = q$, where $p < q$, is:

$$s(x) = \int_p^q \sqrt{1 + \left(sinh\left(\frac{x}{\omega}\right)\right)^2} dx$$

As the span of the bridge is constant, the width of the catenary shape will decrease as the number of towers increases and the catenary shape will also need to change as h varies, so the formula above as well as the limits, p and q, needs to be expressed in terms of n and h.

As h is the vertical height of the catenary, since the vertex of equation (1) is at the origin, and a is the horizontal distance that each main cables spans, the catenary function will need to intersect $(\frac{a}{2}, h)$ and $(-\frac{a}{2}, h)$ since it is symmetrical across the y-axis.

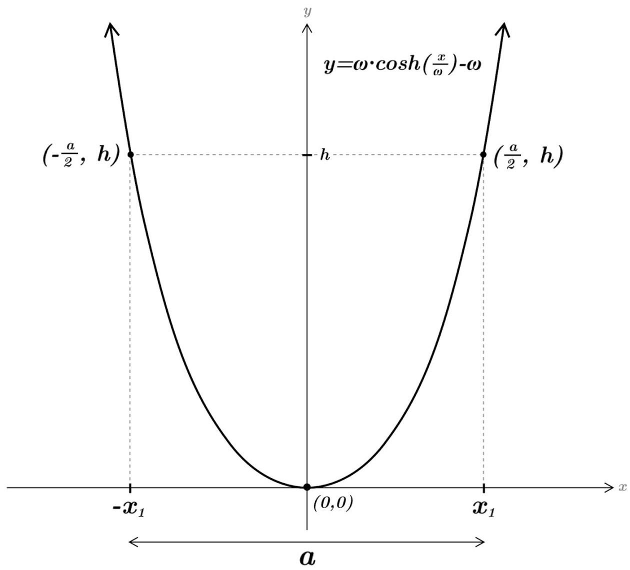

Figure 5. Catenary shape with the equation $y=\omega cosh(\frac{x}{\omega})-\omega$.

As x_1 is a positive value and $f(x_1) = h$, then $a = x_1 + \left|-x_1\right| = 2x_1$, and therefore $x_1 = \frac{a}{2}$. Hence, $h = \omega \cdot cosh\left(\frac{x_1}{\omega}\right) - \omega$. As ω cannot be solved for in terms of x_1 and h algebraically, an alternative method for finding ω will be used. To simplify this task, the ratio of $h : x_1$ will be used, where x_1 is simplified to 1, hence $\frac{h}{x_1} = \frac{h}{1} = h$. The value of ω in $h = \omega\, cosh\left(\frac{1}{\omega}\right) - \omega$ for which the point $(1, h)$ exists on the curve is multiplied by the original value of x_1 to obtain the value of ω for which the point (x_1, h) exists on the curve, as $\frac{a}{a} \cdot a = a$.

The values of ω for which the coordinate $(1, h)$ exists on the catenary curve for different values of h will be found by graphing the function and using the Desmos slider tool. They will be plotted against the $h : a$ ratio, $\frac{h}{a}$. An appropriate function will then be used as a trendline to predict the value of ω given a and h.

$\frac{h}{a}$	ω	$\frac{h}{a}$	ω
0.0001	5000.00	2	0.405433
0.005	100.001	3	0.334416
0.01	50.0017	4	0.297380
0.05	10.0083	5	0.273974
0.1	5.01658	10	0.220864
0.2	2.53265	20	0.185840
0.3	1.71446	30	0.170363
0.4	1.31173	40	0.160963
0.5	1.07432	50	0.154406
1	0.618759	100	0.137257
1.5	0.475681	1000	0.101083

Table 1. Values of ω for ratios of h to a (to six significant figures).

When the values of ω was plotted against $\frac{h}{a}$ from *Table 1*, it closely resembled a rational function as it had both a vertical and horizontal asymptote. By obtaining a trendline of the data using LoggerPro and then adjusting the constants, the equation of the function modelled was $\omega = \frac{8.7(\frac{h}{a})+19.5}{45.6(\frac{h}{a})-0.6}$ where ω would be on the y-axis and $\left(\frac{h}{a}\right)$ would be the y-axis on a cartesian coordinate system. A rational function was used rather than an exponential function because it was visually more diagonally symmetric, which is a characteristic of a rational function. Thus, the formula to find ω for given values of h and a is $\omega = a\left(\frac{8.7(\frac{h}{a})+19.5}{45.6(\frac{h}{a})-0.6}\right)$.

This can be simplified to become $\frac{8.7ah+19.5a^2}{45.6h-0.6a}$.

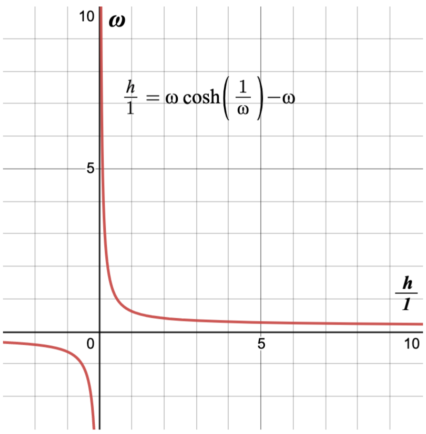

Figure 6.1 The original function for which a model needs to be created.

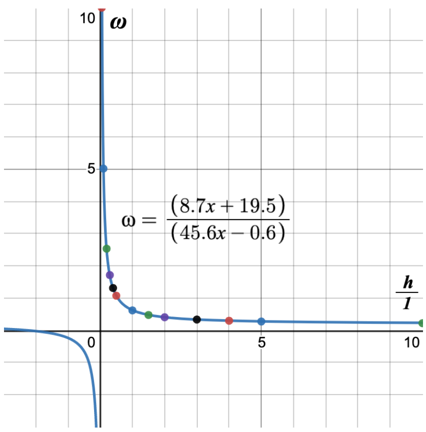

Figure 6.2 The acquired rational function superimposed on points from *Table 1*.

Then, equation (3) can be rewritten as:

$$sinh\left(\frac{x}{\frac{8.7ah+19.5a^2}{45.6h-0.6a}}\right)$$

As previously mentioned, $x_1 = \frac{a}{2}$ and $l = a \cdot n$, so $a = \frac{l}{n}$, which means $x_1 = \frac{a}{2} = \frac{l}{2n}$. This can be substituted into the expression as such:

$$sinh\left(\frac{\frac{l}{2n}}{\frac{8.7ah+19.5a^2}{45.6h-0.6a}}\right)$$

$$= sinh\left(\frac{45.6hl-0.6\frac{l^2}{n}}{8.7(2hl)+19.5\left(\frac{2l^2}{n}\right)}\right)$$

$$= sinh\left(\frac{45.6hl-0.6\frac{l^2}{n}}{17.4hl+39\frac{l^2}{n}}\right)$$

Next, p and q will be written in terms of n, as the restricted domain (vertical width) depends on the number of towers in the design. As *f(x)* is symmetrical across the y-axis and its vertex is at (0, 0), $p = -x_1$ and $q = x_1$. This can be written in terms of n, as it was previously established that $x_1 = \frac{a}{2}$ and $a = \frac{l}{n}$. Hence, $p = -x_1 = -\frac{l}{2n}$ and $q = x_1 = \frac{l}{2n}$.

Thus, the full expression to find the length of one main cable for n towers and a height h is $s(l,n) = \int_{-\frac{l}{2n}}^{\frac{l}{2n}} \sqrt{1+\left(sinh\left(\frac{45.6hl-0.6\frac{l^2}{n}}{17.4hl+39\left(\frac{l^2}{n}\right)}\right)\right)^2}\,dx$. To find the total cost for all main cables in a bridge, this will be multiplied by the number of main cables and then multiplied by the price per unit length of the material used for the main cable.

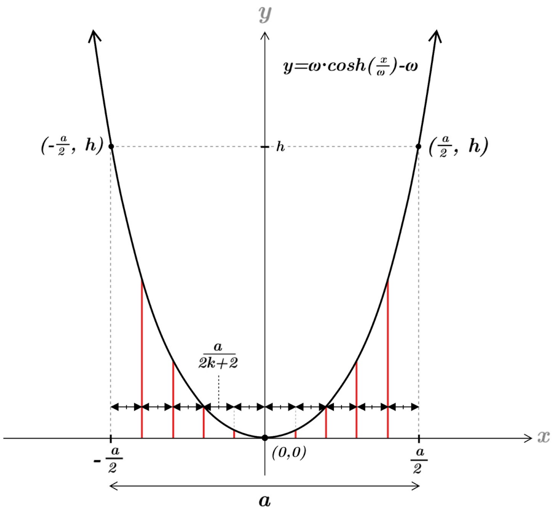

Figure 7. Suspender cables under a main cable where k=4.

As previously mentioned, the number of suspender cables is k and the spacing between them is $\frac{a}{2k+2}$ as shown in Figure 7. The distance between the towers and their nearest suspender cable is $\frac{a}{2k+2}$. This is also the distance between the midpoint of the main cable and the nearest suspender cable.

To find the total length of suspender cables required for a suspension bridge design with height h above the deck and n towers, the total length of suspenders under the function *f(x)* with the domain $0 \leq x \leq \frac{a}{2}$ will first be found. This can be done by summing up the distances from deck to main cable at every relative x-position of a suspender cable.

To find the vertical length of a suspender cable, $h(x)$, the equation for the catenary curve in terms of n and h will be used:

$$h(x) = \omega \cdot cosh\left(\frac{x}{\omega}\right) - \omega = \frac{8.7ah + 19.5a^2}{45.6h - 0.6a} \cdot cosh\left(\frac{x}{\frac{8.7ah + 19.5a^2}{45.6h - 0.6a}}\right) - \frac{8.7ah + 19.5a^2}{45.6h - 0.6a}$$

$$= \frac{8.7ah + 19.5a^2}{45.6h - 0.6a} \cdot cosh\left(\frac{x(45.6h - 0.6a)}{8.7ah + 19.5a^2}\right) - \frac{8.7ah + 19.5a^2}{45.6h - 0.6a}$$

$$= \frac{8.7hln + 19.5l^2}{45.6hn^2 - 0.6ln} \cdot cosh\left(\frac{x(45.6hn^2 - 0.6ln)}{8.7hln + 19.5l^2}\right) - \frac{8.7hln + 19.5l^2}{45.6hn^2 - 0.6ln}$$

As each suspender is always the same distance from the next, the first x-value in the sequence is $\frac{a}{2k+2}$, and each subsequent term will have an x-value that is $\frac{a}{2k+2}$ greater than the previous one for k terms. In summation notation, this is shown as:

$$\sum_{m=1}^{k} h(m) = \sum_{m=1}^{k} \left(\frac{8.7hln + 19.5l^2}{45.6hn^2 - 0.6ln} \cdot cosh\left(\frac{(m \cdot \frac{l}{2kn+2n})(45.6hn^2 - 0.6ln)}{8.7hln + 19.5l^2} \right) - \frac{8.7hln + 19.5l^2}{45.6hn^2 - 0.6ln} \right)$$

In order to keep the optimization simple, it will be given that $k = 4$, meaning that every suspension bridge design regardless of the values of h and n will have 4 suspender cables for every half-catenary. As there are always $n - 1$ full main cables and two half main cables at either end of the bridge, the total length of suspender cables under half a main cable will be multiplied by $2 \cdot (n-1) + 1 + 1 = 2n$. Thus, by substituting k and multiplying the expression by *2n*, the following expression is obtained:

$$2n \cdot \left(\sum_{m=1}^{4} \left(\frac{8.7hln + 19.5l^2}{45.6hn^2 - 0.6ln} \cdot cosh\left(\frac{(m \cdot \frac{l}{8n+2n})(45.6hn^2 - 0.6ln)}{8.7hln + 19.5l^2} \right) - \frac{8.7hln + 19.5l^2}{45.6hn^2 - 0.6ln} \right) \right)$$

$$= 2n \cdot \left(\sum_{m=1}^{4} \left(\frac{8.7hln + 19.5l^2}{45.6hn^2 - 0.6ln} \cdot \left(cosh\left(\frac{(m \cdot \frac{l}{10n})(45.6hn^2 - 0.6ln)}{8.7hln + 19.5l^2} \right) - 1 \right) \right) \right)$$

To find the total cost, p_{total}, of a simple suspension bridge with height h above the deck and n towers, the cost of different parts of the bridge will be summed up. To do this, certain constants will be required:

- p_{deck}: the cost of the deck (US$)
- p_{tower}: the cost per unit length of a tower (US$/meter)
- p_{cable}: the cost per unit length of a main cable (US$/meter)
- $p_{suspender}$: the cost per unit length of a suspender cable (US$/meter)

Thus, the total cost will be the cost of the deck, p_{deck}, added to the total lengths of all towers, main cables, and suspender cables multiplied by their respective costs per unit length. Since the two vertical sides on either side of the deck are identical, the total cost for one side will be multiplied by two. Hence, the following equation is formed:

$$p_{total} = p_{deck} + (2 \cdot (n \cdot (h+42) \cdot p_{tower}) + n \cdot (\int_{-\frac{l}{2n}}^{\frac{l}{2n}} \sqrt{1 + \left(sinh\left(\frac{45.6hl - 0.6\frac{l^2}{n}}{17.4hl + 39(\frac{l^2}{n})}\right)\right)^2} dx) \cdot p_{cable} +$$

$$(2n \cdot \left(\sum_{m=1}^{4} \left(\frac{8.7hln + 19.5l^2}{45.6hn^2 - 0.6ln} \cdot \left(cosh\left(\frac{(m \cdot \frac{l}{10n})(45.6hn^2 - 0.6ln)}{8.7hln + 19.5l^2}\right) - 1\right)\right)\right) \cdot p_{suspender})$$

3 Finding the Maximum Bearable Loads

3.1 Analysis of Relevant Forces

For this model, it will be assumed that the towers are immovable and unbreakable, so the main cables will bear the entire load of the bridge. This is because most bridges have towers which are significantly heavier than the rest of the bridge, hence the maximum bearable loads depend on the main cables. It will also be assumed that the weight of the suspender cables are insignificant, however they transfer the entire force of the load to the main cables. This will make it easier to isolate the main cables as the main determinant of the maximum bearable load.

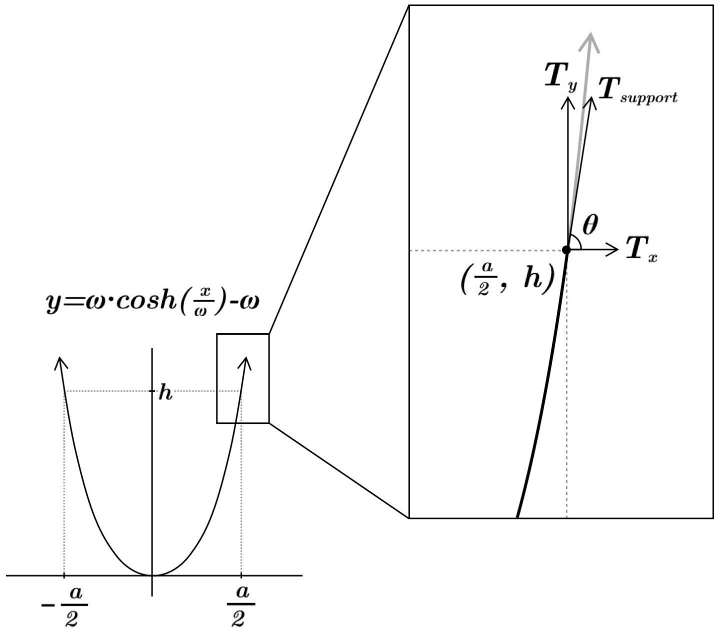

Figure 8. The forces at the ends of the main cables.

$T_{support}$ is the tension at the ends of the main cable from supporting the load underneath it. As previously stated, $\omega = \frac{T_x}{ug} \Longrightarrow T_x = \omega \cdot ug$. As the weight of the load on the bridge, which is assumed to be evenly distributed, is transferred to the main cable through the suspender cables, the weight per unit length of the main cable, ug, increases. The total weight that the main cables would need to support will be $W = W_{Load} + W_{deck}$, which is the sum of the weight of the external load and the weight of the bridge's deck. As the weight is assumed to be evenly distributed, the weight per unit length of the load and the deck is $\frac{W}{l}$.

3.2. Constructing Equations for Maximum Bearable Loads

In order to analyze the bridge structure, a smaller section of the bridge will be analyzed and the findings will be applied to the rest of the structure. To simplify the analysis, we will look at a bridge section with a half-catenary structure:

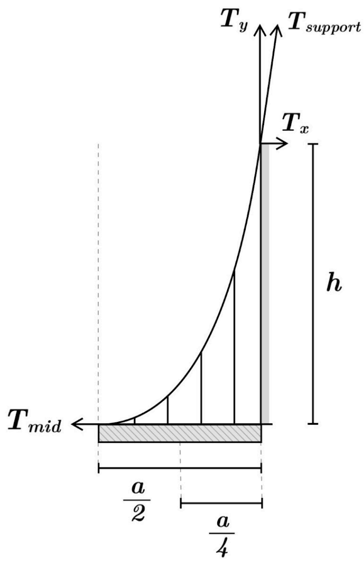

Figure 9. Diagram of the bridge design under half a main cable.

The forces in the main cable must be in equilibrium, otherwise the uneven forces would cause deformation or a collapse of the main cable. Therefore, the sum of moments, *M*, which is the tendency of the force to cause a body to rotate, will need to be equal to 0 so that the structure is rigid and does not deform due to imbalanced forces. A moment is the product of the force acting on a point and the distance from the pivot point. The force used to calculate the moment is always perpendicular to the distance from the pivot point.

Thus, the sum of moments in this case will be the moment of the vertical force, *W*, and the moment of the horizontal force, which is the tension acting at the lowest point of the main cable, T_{mid}. Since T_{mid} is perpendicular to the direction of *W*, it will be a negative value.

The weight of the load per unit length will be multiplied by the length of the section, $\frac{a}{2}$, to find the weight of the load carried by the half main cable. To find the moment, this will be multiplied by $\frac{a}{4}$ as the total force is taken to be acting on the middle of the section since the load is evenly distributed.

$$\Sigma M = \left(\frac{W}{l} \cdot \frac{a}{2}\right) \cdot \left(\frac{a}{4}\right) - T_{mid} \cdot h = 0$$

$$\therefore\ \Sigma M = \frac{a^2 W}{8l} - T_{mid} \cdot h = 0$$

By substituting $a = \frac{l}{n}$ and solving for T_{mid}:

$$\Sigma M = \frac{Wl}{8n^2} - T_{mid} \cdot h$$

$$= 0$$

$$T_{mid} \cdot h = \frac{Wl}{8n^2}$$

$$T_{mid} = \frac{Wl}{8hn^2}$$

Because the system needs to be in equilibrium, $T_x = T_{mid}$ and $T_y = \left(\frac{W}{l} \cdot \frac{a}{2}\right) = \left(\frac{W}{l} \cdot \frac{l}{2n}\right) = \frac{W}{2n}$.

Since T_x is the horizontal component of $T_{support}$, which is the tension of the cables where it is supported by the towers, and T_y is its vertical component, the Pythagorean Theorem can be used to solve for $T_{support}$:

$$T_{support} = \sqrt{T_x^{\,2} + T_y^{\,2}}$$

Substituting $T_x = T_{mid}$ and $T_y = \frac{aW}{2l}$:

$$T_{support} = \sqrt{\left(\frac{Wl}{8hn^2}\right)^2 + \left(\frac{W}{2n}\right)^2}$$

$$= \sqrt{\frac{W^2 l^2}{64h^2 n^4} + \frac{W^2}{4n^2}}$$

Restoring force per unit area is known as “stress”. Every material has a limited allowable stress that it can withstand before major deformation occurs (Mechanics of Materials). From the physics equation $\sigma = \frac{F}{A}$ where σ is the stress of a cable in Pascals, F is the tension force at the end of the cable in Newtons, and A is the cross-sectional area of the cable in square meters, it is evident that a greater force will result in greater stress on the cable.

In this case, the stress of a half-catenary main cable is $\sigma_{cable} = \frac{T_{support}}{\pi(\frac{d}{2})^2} = \frac{\sqrt{\frac{W^2 l^2}{64h^2 n^4} + \frac{a^2 W^2}{4l^2}}}{\pi(\frac{d}{2})^2}$ where d is the diameter of the main cable. Thus, to optimize n and h for the highest bearable load, the combination resulting in the highest value of W must be found with the constraint $\sigma_{cable} < \sigma_{allowable}$.

Allowable stress ($\sigma_{allowable}$) can be calculated by dividing the yield strength of a material by the relevant "factor of safety." In the case of steelwork in bridges, the factor of safety ranges from 5 to 7 ("Factors of Safety"), so a factor of 6 will be used in this case. We will use the yield strength of the Golden Gate Bridge's main cables, which is 182,600 psi ("Design & Construction Stats"). This is approximately equivalent to 1,258,983,000 Pa or N/m^2.

Therefore, $\sigma_{allowable} = \frac{1258983000}{6}$ = 209,830,500 Pa = 209830.5 kN/m^2.

4 Optimization

4.1. Introducing the Situation

The two points chosen for the bridge to be built between are the points on the coastline that are closest to roads as bridges are generally connected to roads. Using the built-in scale on Google Maps, the distance between the two points was found to be roughly 1234 meters, as shown in Figure 10. Hence, $l = 1234$.

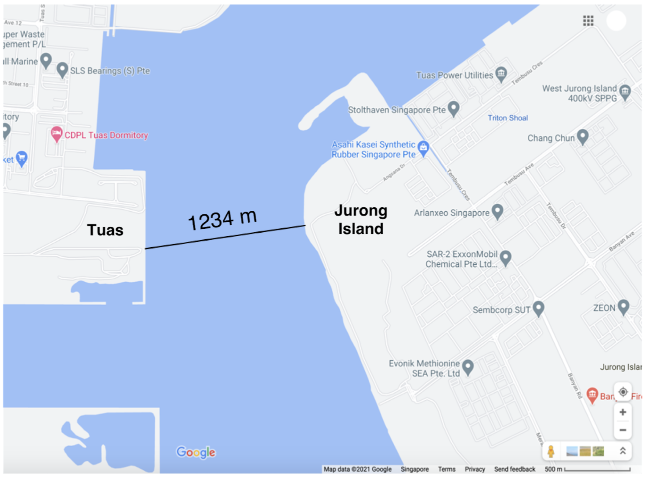

Figure 10. Birds-eye view of the bridge's location (Google Maps).

Prior to optimizing the design, other constants must also be defined. Approximate costs from the construction of the Golden Gate Bridge will be used, as it is regarded as an iconic suspension bridge and also because information on its construction is more accessible than construction information of other bridges.

Bridge Section	***Constants***	***Value and Units***
Deck (base)	Price (*US$*)	*$25,000,000*
	Weight (*kN*)	*1,300,000 kN*
Towers	Price per meter (*US$/m*)	*$23,000/m*
Main cables	Price per meter (*US$/m*)	*$1,200/m*
	Weight per meter (*kN/m*)	*30 kN/m*
	Diameter (*m*)	*0.92 m*
Suspender cables	Price per meter (*US$/m*)	*$260*

Table 2. Prices and other constants ("Design & Construction Stats")

4.2. Graphing the Equations

Using Desmos, a graph will be created where the y-axis represents p_{total} and the x-axis represents h. This results in a function similar to a rational function. However, the vertical asymptote is always a negative x-value, so the function restricted by a positive domain resembles a simple linear function. When the value of n increases, the slope of the diagonal asymptote increases and the y-intercept of the function also increases.

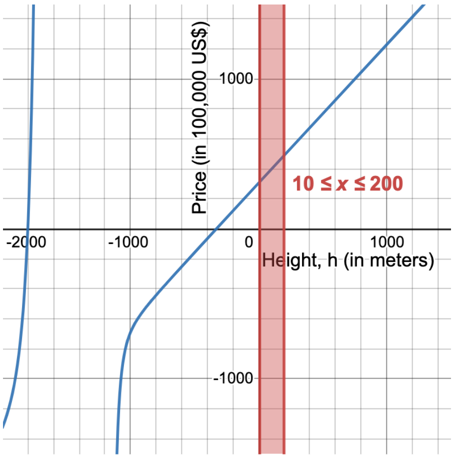

Figure 11.1 p_{total} vs. h (n=2)

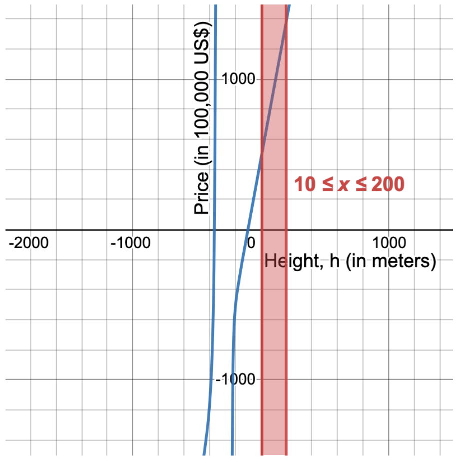

Figure 11.2 p_{total} vs. h (n=10)

Thus, a lower n value would allow for a lower total price for any value of h. Hence, the value of n needs to be as small as possible while also maximizing W_{load}. A smaller value of h would also result in a lower total price, however the change becomes more significant when n has a greater value.

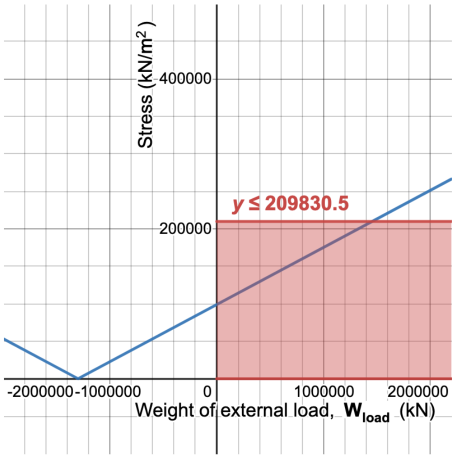

Figure 12.1 Stress vs. W_{Load} (n=10, h=200)

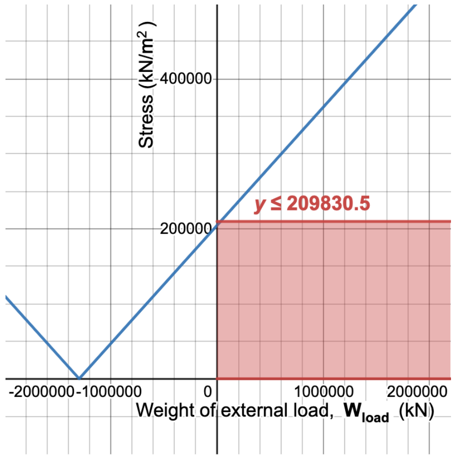

Figure 12.2 Stress vs. W_{Load} (n=5, h=200)

From the graph, it was seen that $n < 5$ where n is an integer will not work as the y-intercept is above $\sigma_{allowable}$ for every value of h where $h \leq 200$ m. Increasing n causes the function to stretch horizontally, with the same x-intercept of being the center of the stretch. Increasing h has the same effect although it causes a less significant change. Therefore, a larger value of n and h would result in a higher maximum bearable load as the x-position of the intercept with the line $y = \sigma_{allowable} = 209830.5$ would increase.

4.3. Optimizing the Equations

In order to optimize the equations, the minimum and maximum values for both p_{total} and W_{load} will be found. From the graphs, it can be deduced that the minimum possible cost would be $31,362,400 at the minimum values where $h = 10$ and $n = 2$ and the maximum possible cost would be $138,117,700 at the maximum values where $h = 200$ and $n = 10$. As for W_{load}, the maximum possible bearable load would be 1,457,135.157 kN when $h = 200$ and $n = 10$ (maximum values) and the minimum would be 0.

The range of p_{total} for all values of h and n is:

$$\$138,117,700 - \$31,362,400 = \$106,755,300$$

The percentage of the range of the total price is shown by:

$$p_{total}\% = \frac{p_{total} - 31,362,400}{106,755,300} \cdot 100\%$$

As for W_{load}, the range is 1,457,135.157 kN. The percentage of the range of the maximum load is shown by:

$$W_{load}\% = \frac{W_{load}}{1,457,135.157} \cdot 100\%$$

The ratio of these two percentages will be used to balance the maximization of W_{load} and the minimization of p_{total}. To do this, a ratio of 1:1 ($= 50\% : 50\%$) will be used to optimize the two equations. 50% of each range will be found:

$$\frac{p_{total} - 31,362,400}{106,755,300} \cdot 100\% = 50\%$$

$$p_{total} - 31362400 = 0.5 \cdot 106755300$$

$$p_{total} = 31362400 + 53377650$$

$$= \$84,740,050$$

$$\frac{W_{load}}{1,457,135.157} \cdot 100\% = 50\%$$

$$W_{load} = 0.5 \cdot 1457135.157$$

$$= 728,567.5785 \text{ kN}$$

By minimizing p_{total} and maximizing W_{load} by the same percent, the optimal values of h and n should result in a total price close to \$84,740,050 and a maximum bearable external load of 728,567.6 kN. To find these values, firstly the value of h and n to achieve the aforementioned W_{load} will be found.

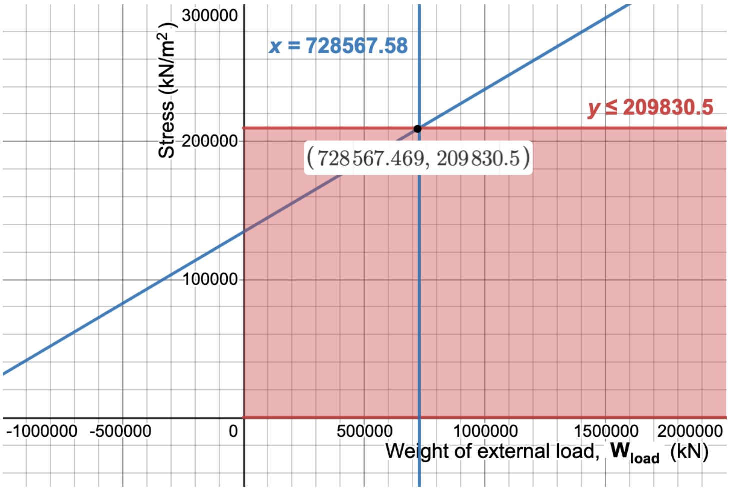

Figure 13. Stress vs. W_{Load} for n=8 and h=84.07

From the graph, it was found that W_{load} for $n < 8$ for all values of h was lower than the desired value of W_{load}. Hence, using the lowest possible value of n to obtain the desired value, the values $n = 8$ and $h = 84.07$ were obtained which resulted in a close value of 728,567.47 kN for the maximum bearable external load. These values can used on the graph showing the total price:

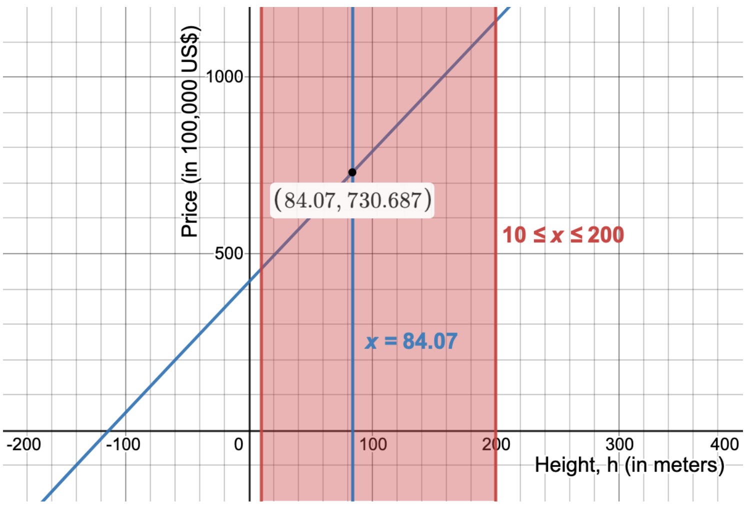

Figure 14. P_{total} vs. height, h for n=8 and h=84.07

5 Conclusion

In conclusion, the total price for an optimal suspension bridge design between Tuas and Jurong Island was found to be $73,068,700, which is lower than the median value of p_{total} which was calculated to be $84,740,050. The design would also have 8 towers and a height of $84.07+42=126.07\,\text{m}$ and a maximum bearable load of 728,567.47 kN. The optimized bridge design is shown below:

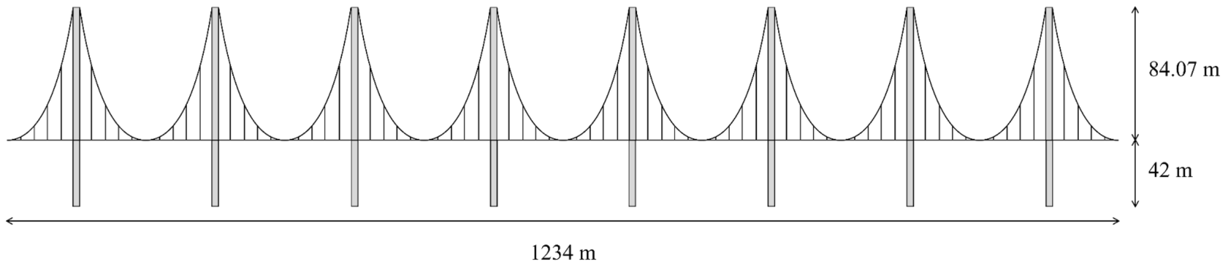

Figure 15. Side view of optimal bridge design.

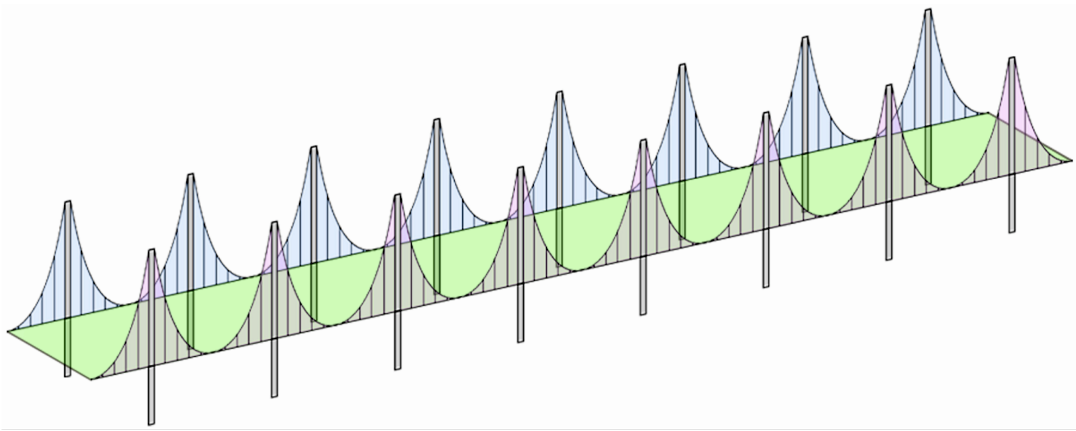

Figure 16. 3D diagram of optimal bridge design.

6 Evaluation

From this investigation, it was found that the cost of the optimized bridge is $73,068,700. This is more than double the cost of the Golden Gate Bridge itself, which was around $35,000,000 (“Design & Construction Stats”). Moreover, the Golden Gate Bridge is more than double the distance of the optimized bridge, however it is less expensive. This may indicate that the method used for optimizing the bridge design was not particularly effective. This may also have been because the maximum number of towers was set to $n = 10$, which is an extreme value.

As two variables, n and h, were altered to optimize both p_{total} and W_{load}, the process was simplified by using a “best balance” method. This meant that the median of the ranges of both p_{total} and W_{load} were found and used to determine the optimal number of towers and the height of the bridge. However, this is not the most effective method as it was shown that a percentage decrease in p_{total} does not cause n and h to change by the same amount as the same percentage decrease in W_{load}. Thus, to improve, more complex multi-objective optimization methods could be used or a computer program could be created to compute all possible combinations and find the result with the optimal ratio of p_{total} to W_{load}.

Moreover, as costs from the Golden Gate Bridge, which was constructed in 1933, were used to calculate the total price, p_{total} does not account for inflation. Using a CPI inflation calculator, $73,068,700 in 1933 is worth $1,200,208,089.64 today (“Inflation Calculator”). Furthermore, the total price, p_{total}, does not include the labour costs or other costs such as for the transportation of materials, which means that the bridge would likely be more expensive.

It was also found that the maximum bearable load for the optimized model was 728,567.47 kN. This can be converted into pounds and compared to the maximum load of the Golden Gate Bridge to determine its strength relative to conventional bridges:

$$\frac{728{,}567.57\times10^{3}}{9.81}\ \text{kg}$$

$$= 74{,}343{,}619.39\ \text{kg}$$

$$= 74343619.39 \cdot 2.205\ \text{pounds}$$

$$\approx 164{,}000{,}000\ \text{pounds}$$

The Golden Gate Bridge can hold 36 million pounds ("How Much Weight"), whereas the optimized bridge, which has a span that is less than half of the Golden Gate Bridge, can hold 164 million pounds. This was unexpected, but understandable, as there are eight towers that support the load on the bridge, while the Golden Gate Bridge only has two. This investigation could help demonstrate an unconventional approach to designing bridges which need to be able to withstand significantly higher loads than conventional bridges, as a greater number of towers could be the solution to the problem.

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