2 Quantum Mechanics

2.1 Photon Polarization

2.1.1 Linear Polarization

Problem 2. *Consider the matrix* $R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

(a) Show that R is an orthogonal matrix.

(b) Apply R to the linear polarization state $|s\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. *Describe in everyday language the effect of this transformation on a state of linear polarization.*

For part (a), we can determine whether R is orthogonal determining whether $RR^T = I$:

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ \cos\phi\sin\phi - \sin\phi\cos\phi & \cos^2\phi + \sin^2\phi \end{pmatrix}$$

and, via the Pythagorean identity, we can simplify the resulting matrix to the identity. Thus R is an orthogonal matrix. Moving to part (b), we know that the resulting polarization state, $|s'\rangle$, will be

$$R|s\rangle = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta \end{pmatrix} = \begin{pmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \end{pmatrix}^{*}$$

which, evidently, is simply an increase in the angle of the linear polarization by ϕ, resulting in a simple rotation.

Problem 4. *Let* $R = \begin{pmatrix} \cos(\pi/(2n)) & -\sin(\pi/(2n)) \\ \sin(\pi/(2n)) & \cos(\pi/(2n)) \end{pmatrix}$.

(a) Compute R^n. That is, compute the product of n factors of R, where the multiplication is matrix multiplication. You may find the following trigonometric identities helpful: $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$; $\cos\alpha\sin\beta + \sin\alpha\cos\beta = \sin(\alpha+\beta)$.

(b) A horizontally polarized photon passes successively through n small containers of sugar water, each of which effects the transformation R. The photon then encounters a polarizing filter whose preferred axis is horizontal. What is the probability of the photon passing the filter?

(c) Another horizontally polarized photon passes through the same n containers of sugar water. But now, just after each container there is a polarizing filter whose preferred axis is horizontal. What is the probability that the photon will pass through all n filters?

(d) Find the limit of your answer to part (c) as n approaches infinity.

*I arrived at this final matrix using trigonometric identities provided in Problem 4 part (a).

Starting with part (a), we can see from the Problem 2 part (b) that the standard rotation operation R_ϕ, when applied to a photon's state, rotates it by ϕ; this operation can be applied n times to rotate the photon's state $n\phi$.[†] In order to apply this operation n times, we simply multiply the left hand side of the vector by R^n. Obfuscating computational steps provided in Problem 2 part (b), we can say, assuming some polarization state $|s\rangle = \left(\begin{smallmatrix}\cos\theta \\ \sin\theta\end{smallmatrix}\right)$:

$$R^n|s\rangle = \begin{pmatrix}\cos(\pi/(2n)) & -\sin(\pi/(2n)) \\ \sin(\pi/(2n)) & \cos(\pi/(2n))\end{pmatrix}^n \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = \begin{pmatrix}\cos(n\pi/(2n)+\theta) \\ \sin(n\pi/(2n)+\theta)\end{pmatrix} = \begin{pmatrix}\cos({}^{\pi}\!/_2+\theta) \\ \sin({}^{\pi}\!/_2+\theta)\end{pmatrix}$$

demonstrating that our newly given R^n will always rotate the polarization state by ${}^{\pi}\!/_2$ radians if n rotations are applied before inspection. Moving to part (b), we know that the photon's polarization state will be vertical after passing through n containers. Because the horizontal polarizing filter is orthogonal to the photon's state, there is zero-chance that the photon will survive after a full rotation by the n containers. For part (c), we first can establish that if a photon survives a container-filter combo according to a probability p, the photon's state polarization will be horizontal. This means that the total probability will be p^n as the photon's chance of survival is equal at each consequtive filter. To determine p^n, we can say

$$p = \langle s|m\rangle = |\cos(\pi/(2n)(1) + \sin(\pi/(2n)(0)|^2 = \cos^2(\pi/(2n))$$

$$p^n = \cos^{2n}(\pi/(2n)).$$

For part (d), we know that $\lim_{n\to\infty} {}^{\pi}\!/_{2n} = 0$ which implies that $\lim_{n\to\infty} \cos^{2n}(\pi/(2n)) = 1$. Intuitively, as the n-th slice gets smaller and smaller as n increases, the photon's state is rotated less and less and a photon will consequently have a better chance to survive a polarized filter, increasing the overall chance of survival.

2.1.3 Circular and Elliptical Polarization

Problem 1. *For each of the following state vectors, find a normalized vector that is orthogonal to the given vector.*

$$\begin{pmatrix}\sqrt{3}/2 \\ 1/2\end{pmatrix} \quad \begin{pmatrix}1/\sqrt{2} \\ i/\sqrt{2}\end{pmatrix} \quad \begin{pmatrix}(1-i)/2 \\ (1+i)/2\end{pmatrix}$$

We proceed with each vector by finding another vector in the same field such that $\langle s|t\rangle = 0$:

$$\begin{pmatrix}\sqrt{3}/2 \\ 1/2\end{pmatrix}^\perp = \begin{pmatrix}-1/2 \\ \sqrt{3}/2\end{pmatrix}; \quad \begin{pmatrix}1/\sqrt{2} \\ i/\sqrt{2}\end{pmatrix}^\perp = \begin{pmatrix}1/\sqrt{2} \\ -i/\sqrt{2}\end{pmatrix}; \quad \begin{pmatrix}(1-i)/2 \\ (1+i)/2\end{pmatrix}^\perp = \begin{pmatrix}(1+i)/2 \\ (1-i)/2\end{pmatrix}$$

Vectors in $\mathbb{R}$ are orthogonal if their dot-product is zero. However, for vectors in $\mathbb{C}$, we must determine if their inner product[‡], $\langle u, v\rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2$, is zero.

[†] This applies to our newly assigned R as the trigonometric structure is the same; i.e., the behavior is the same regardless of the provided arguments to the trigonometric functions as long as their positions and sign are the same.

[‡] This is called the *Hermitian* inner product.

Problem 3. *The rotation operation R_ϕ, defined by*

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

rotates any linear polarization state by an angle ϕ. What does this transformation do to the right-hand circular polarization state? Is the resulting state a state of linear polarization, circular polarization, or elliptical polarization? Does the answer to this question depend on the value of ϕ?

To start, let us define a standard right-hand circular polarization state as $|s\rangle = (1/\sqrt{2})\left(\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right)$.[§] We can then calculate the effect of the rotation operation on the photon's state:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\phi - i\sin\phi \\ \sin\phi + i\cos\phi \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} \\ ie^{-i\phi} \end{pmatrix} = \left(\frac{e^{-i\phi}}{\sqrt{2}}\right)\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Because we are left with the same state vector multiplied by a complex factor, we can confirm that the state remains right-hand circular polarized for all ϕ.

Problem 4. *Consider the elliptical polarization represented by $|s\rangle = \begin{pmatrix} 1/\sqrt{2} \\ (1+i)/2 \end{pmatrix}$. Suppose the measurement M_θ of Example 2.1.6 is applied to a photon in the state $|s\rangle$.*

(a) Find the probabilities of the two outcomes as functions of θ.

(b) For what value of θ do the two probabilities differ the most from each other? The basis defined by M_θ for this value of θ can be thought of as giving the "principal axes" of the elliptical polarization.

Starting with part (a), let's restate the measurement:

$$M_\theta = \left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}\right).$$

For probability p_1, we can say:

$$p_1 = \langle s|m^{(1)}\rangle = \left|\frac{1}{\sqrt{2}}\cos\theta + \frac{1-i}{2}\sin\theta\right|^2.$$

We'll then need to derive its magnitude-squared expansion:[¶]

$$\begin{aligned} p_1 &= \left|\frac{1}{\sqrt{2}}\cos\theta\right|^2 + 2\,\mathrm{Re}\left(\frac{1}{\sqrt{2}}\cos\theta \cdot \frac{1-i}{2}\sin\theta\right) + \left|\frac{1-i}{2}\sin\theta\right|^2 \\ &= \frac{1}{2}\cos^2\theta + \frac{2\cos\theta\sin\theta}{2\sqrt{2}} + \frac{1}{2}\sin^2\theta \\ &= \frac{1}{2} + \frac{\sin 2\theta}{2\sqrt{2}} \;\therefore\; p_2 = \frac{1}{2} - \frac{\sin 2\theta}{2\sqrt{2}}. \end{aligned}$$

[§] We can ensure this is right-handed as $i = i(1)$ according to the quantum physics convention of using the receiver's point-of-view, which is also in-line with the textbook's definition.

[¶] The general formula is $|a^2 + b|^2 = |a|^2 + 2\,\mathrm{Re}(a\bar{b}) + |b|^2$ and is also sometimes referred to as the parallelogram law of complex numbers.

For part (b), we can deduce by inspection that the difference in each probability will be maximized when $\frac{\sin 2\theta}{2\sqrt{2}}$, which is when $\sin 2\theta = \pm 1$. This occurs when $\theta = \pm^{\pi}/_{4}$; thus the principal axes of the elliptical polarization exist at $^{\pi}/_{4}$ and $^{-\pi}/_{4}$.