2 Quantum Mechanics

2.1 Photon Polarization

2.1.1 Linear Polarization

Problem 2. *Consider the matrix* $R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

(a) Show that R is an orthogonal matrix.

(b) Apply R to the linear polarization state $|s\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. *Describe in everyday language the effect of this transformation on a state of linear polarization.*

For part (a), we can determine whether R is orthogonal determining whether $RR^T = I$:

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ \cos\phi\sin\phi - \sin\phi\cos\phi & \cos^2\phi + \sin^2\phi \end{pmatrix}$$

and, via the Pythagorean identity, we can simplify the resulting matrix to the identity. Thus R is an orthogonal matrix. Moving to part (b), we know that the resulting polarization state, $|s'\rangle$, will be

$$R|s\rangle = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta \end{pmatrix} = \begin{pmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \end{pmatrix} *$$

which, evidently, is simply an increase in the angle of the linear polarization by ϕ, resulting in a simple rotation.

Problem 4. *Let* $R = \begin{pmatrix} \cos(\pi/(2n)) & -\sin(\pi/(2n)) \\ \sin(\pi/(2n)) & \cos(\pi/(2n)) \end{pmatrix}$.

(a) Compute R^n. That is, compute the product of n factors of R, where the multiplication is matrix multiplication. You may find the following trigonometric identities helpful: $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$; $\cos\alpha\sin\beta + \sin\alpha\cos\beta = \sin(\alpha+\beta)$.

(b) A horizontally polarized photon passes successively through n small containers of sugar water, each of which effects the transformation R. The photon then encounters a polarizing filter whose preferred axis is horizontal. What is the probability of the photon passing the filter?

(c) Another horizontally polarized photon passes through the same n containers of sugar water. But now, just after each container there is a polarizing filter whose preferred axis is horizontal. What is the probability that the photon will pass through all n filters?

(d) Find the limit of your answer to part (c) as n approaches infinity.

*I arrived at this final matrix using trigonometric identities provided in Problem 4 part (a).

Starting with part (a), we can see from the Problem 2 part (b) that the standard rotation operation R_ϕ, when applied to a photon's state, rotates it by ϕ; this operation can be applied n times to rotate the photon's state $n\phi$.[†] In order to apply this operation n times, we simply multiply the left hand side of the vector by R^n. Obfuscating computational steps provided in Problem 2 part (b), we can say, assuming some polarization state $|s\rangle = \left(\begin{smallmatrix}\cos\theta \\ \sin\theta\end{smallmatrix}\right)$:

$$R^n|s\rangle = \begin{pmatrix}\cos(\pi/(2n)) & -\sin(\pi/(2n)) \\ \sin(\pi/(2n)) & \cos(\pi/(2n))\end{pmatrix}^n \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = \begin{pmatrix}\cos(n\pi/(2n)+\theta) \\ \sin(n\pi/(2n)+\theta)\end{pmatrix} = \begin{pmatrix}\cos(\pi/2+\theta) \\ \sin(\pi/2+\theta)\end{pmatrix}$$

demonstrating that our newly given R^n will always rotate the polarization state by $\pi/2$ radians.

2.1.3 Circular and Elliptical Polarization

Problem 1. *For each of the following state vectors, find a normalized vector that is orthogonal to the given vector.*

$$\begin{pmatrix}\sqrt{3}/2 \\ 1/2\end{pmatrix} \quad \begin{pmatrix}1/\sqrt{2} \\ i/\sqrt{2}\end{pmatrix} \quad \begin{pmatrix}1-i/2 \\ 1+i/2\end{pmatrix}$$

Answer here...

Problem 3. *The rotation operation R_ϕ, defined by*

$$R_\phi = \begin{pmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{pmatrix}$$

rotates any linear polarization state by an angle ϕ. What does this transformation do to the right-hand circular polarization state? Is the resulting state a state of linear polarization, circular polarization, or elliptical polarization? Does the answer to this question depend on the value of ϕ?

Answer here...

Problem 4. *Consider the elliptical polarization represented by $|s\rangle = \begin{pmatrix}1/\sqrt{2} \\ 1+i/2\end{pmatrix}$. Suppose the measurement M_θ of Example 2.1.6 is applied to a photon in the state $|s\rangle$.*

(a) Find the probabilities of the two outcomes as functions of θ.

(b) For what value of θ do the two probabilities differ the most from each other? The basis defined by M_θ for this value of θ can be thought of as giving the "principal axes" of the elliptical polarization.

Answer here...

[†]This applies to our newly assigned R as the trigonometric structure is the same; i.e., the behavior is the same regardless of the provided arguments to the trigonometric functions as long as their positions and sign are the same.