Note: unrequired parts of multi-part problems are listed obfuscated as ... to recognize they are multi-parted.

2 Groups

2.2 Basic Properties and Order

Problem 2.38. *For each element g of the listed groups below, find the order of g, $|g|$.*

(a) $[3] \in (\mathbb{Z}_{15}, +)$.

(b) $[3] \in (U_{10}, \cdot)$.

(e) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} \in S_5$.

(f) $R_2 \in D_3$

(g) $\begin{pmatrix} [1] & [1] \\ [0] & [1] \end{pmatrix} \in GL(2, \mathbb{Z}_2)$

For part (a), $|[3]| = 5$ as $3+3+3+3+3 \equiv 0 \pmod{15}$. For part (b), $|[3]| = 4$ as $3^4 = 81 \equiv 1 \pmod{10}$. For part (e), the order of the given $\sigma \in S_5$ is 2 as $\sigma^2 = e$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}^{*} = e.$$

For part (f), $|R_2| = 3$ as a 240° rotation of a triangle must be repeated three times in order for the triangle to reach its original orientation, e. For part (g):

$$\begin{pmatrix} [1] & [1] \\ [0] & [1] \end{pmatrix} \begin{pmatrix} [1] & [1] \\ [0] & [1] \end{pmatrix} = \begin{pmatrix} [1]+[0] & [1]+[1] \\ [0]+[0] & [0]+[1] \end{pmatrix} = \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix} = e;$$

therefore the order of the matrix is 2.

Problem 2.39. *In each infinite group below, find all elements of finite order:*

(a) $(\mathbb{R}, +)$,

(b) $(\mathbb{R}^\times, \cdot)$,

(c) $(\mathbb{C}^\times, \cdot)$,

(d) $D(n, \mathbb{R}) = \{diag(c_1, \ldots, c_n) \mid c_i \in \mathbb{R}^\times, 1 \leq i \leq n\}$.

For part (a), the set of finite-order elements of $(\mathbb{R}, +)$ is $\{0\}$. For part (b), the set of finite-order elements of $(\mathbb{R}^\times, \cdot)$ is $\{1, -1\}$ as $(-1)^2 = 1$. For part (c), the set of finite-order elements contains all n-th roots of unity.[†]

Problem 2.40. *Let G be a group and let $g \in G$.*

(a) *Show* $|g^-1| = g$.

[*] This may be incorrect notation but it gets the idea across.

[†] In research I found that finite-order elements are called *torsion elements* and that groups can be classified as a *torsion group* (where every element is a torsion element).

(b) *For $h \in G$, show $|hgh^{-1}| = |g|$.*

(c) *If $|g| < \infty$, show $g^{-1} = g^{|g|-1}$.*

Answer here...

Problem 2.45. *($g^2 = e \implies$ Abelian). Suppose G is a group so that $g^2 = e$ for every $g \in G$. Show that G is abelian. Hint: Show $g \in G$ implies $g^{-1} = g$ and then apply this fact to the product of two elements.*

2.3 Subgroups and Direct Products

2.3.1 Subgroups

2.3.2 Direct Products