2 Groups

2.2 Basic Properties and Order

Problem 2.38. *For each element g of the listed groups below, find the order of g, $|g|$.*

(a) $[3] \in (\mathbb{Z}_{15}, +)$.

(b) $[3] \in (U_{10}, \cdot)$.

(e) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} \in S_5$.

(f) $R_2 \in D_3$

(g) $\begin{pmatrix} [1] & [1] \\ [0] & [1] \end{pmatrix} \in GL(2, \mathbb{Z}_2)$

For part (a), $|[3]| = 5$ as $3+3+3+3+3 \equiv 0 \pmod{15}$. For part (b), $|[3]| = 4$ as $3^4 = 81 \equiv 1 \pmod{10}$. For part (e), the order of the given $\sigma \in S_5$ is 2 as $\sigma^2 = e$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}^* = \text{Id}.$$

For part (f), $|R_2| = 3$ as a 240° rotation of a triangle must be repeated three times in order for the triangle to reach its original orientation, e. For part (g):

$$\begin{pmatrix} [1] & [1] \\ [0] & [1] \end{pmatrix} \begin{pmatrix} [1] & [1] \\ [0] & [1] \end{pmatrix} = \begin{pmatrix} [1]+[0] & [1]+[1] \\ [0]+[0] & [0]+[1] \end{pmatrix} = \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix};$$

therefore the order of the matrix is 2.

Problem 2.39. *In each infinite group below, find all elements of finite order:*

(a) $(\mathbb{R}, +)$,

(b) $(\mathbb{R}^\times, \cdot)$,

(c) $(\mathbb{C}^\times, \cdot)$,

(d) $D(n, \mathbb{R}) = \{diag(c_1, \ldots, c_n) \,|\, c_i \in \mathbb{R}^\times, 1 \le i \le n\}$.

For part (a), the set of finite-order elements of $(\mathbb{R}, +)$ is $\{0\}$. For part (b), the set of finite-order elements of $(\mathbb{R}^\times, \cdot)$ is $\{1, -1\}$ as $(-1)^2 = 1$. For part (c), the set of finite-order elements contains all n-th roots of unity. For part (d), the set of finite-order elements of the group of diagonals composed of $\mathbb{R}$ is the trivial set composed of 1s along the diagonal of the $n \times n$ matrix.[†]

Problem 2.40. *Let G be a group and let $g \in G$.*

(a) *Show* $|g^{-1}| = |g|$.

[*]This may be incorrect notation but it gets the idea across.

[†]In research, I found that finite-order elements are called *torsion elements* and that groups can be classified as a *torsion group* if it only contains torsion elements. Reportedly this term comes from algebraic topology, but the connection is so complex that I can't understand how they're related—something about twisting a space?

(b) For $h \in G$, show $|hgh^{-1}| = |g|$.

(c) If $|g| < \infty$, show $g^{-1} = g^{|g|-1}$.

The following proofs will address each part respectively:

Proof. We know that by Theorem 2.11 part (2) that $g^{n_1} = g^{n_2}$ if and only if $n_1 \equiv n_2 \pmod{|g|}$. By the definition of an element's order, $g^n = e$ for some minimal positive exponent n. We can then say $g^n = e \Rightarrow (g^n)^{-1} = e^{-1} \Rightarrow g^{-n} = e$ which can only be true if n and $-n$ are equivalent $\pmod{n}$. Therefore, because $-n \equiv n \equiv 0 \pmod{n}$, $|g^{-1}| = |g|$. $\square$

Proof. We can start by rewriting $|hgh^{-1}| = |g|$ as $(hgh^{-1})^n$ which can be simplified:

$$hg(h^{-1}h)g(h^{-1}h)gh^{-1} \cdots = hg^n h^{-1}.$$

We can also say that $hg^n h^{-1} = e$ is true only when $g^n = e$ as it allows

$$hg^n h^{-1} = heh^{-1} = hh^{-1} = e.$$

To demonstrate they are both the minimal exponent, consider $(hgh^{-1})^m = e$. We can simplify it to e as $h^{-1}(hgh^{-1})^m h = g^m = e$, thus $m = n$ and $|hgh^{-1}| = |g|$. $\square$

Proof. By definition, $g^{|g|} = e$. Multiplying both sides by g^{-1}, we get $g^{|g|}g^{-1} = eg^{-1} \Rightarrow g^{|g|-1} = g^{-1}$ by Theorem 2.9 part (1). $\square$

Problem 2.45. *($g^2 = e \implies$ Abelian). Suppose G is a group so that $g^2 = e$ for every $g \in G$. Show that G is abelian. Hint: Show $g \in G$ implies $g^{-1} = g$ and then apply this fact to the product of two elements.*

Proof. Suppose G is a group such that $g^2 = e \ \forall g \in G$. Thus $g^2 g^{-1} = eg^{-1} \Rightarrow g = g^{-1}$ by Theorem 2.9 part (1). Next, using another element h and Theorem 2.9 part (2), we can say:

$$(gh)^{-1} = h^{-1}g^{-1} = hg \text{ and } (gh)^{-1} = (gh)$$

as every element is its own inverse in this given G. Therefore G must be abelian. $\square$

2.3 Subgroups and Direct Products

2.3.1 Subgroups

Problem 2.53. *Show the following subsets are groups:*

(b) $\{5^a \mid a \in \mathbb{Q}\} \subseteq \mathbb{R}^\times$.

(c) $k\mathbb{Z}_n = \{k[m] \mid [m] \in \mathbb{Z}_n\} \subseteq \mathbb{Z}_n$ where $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

(g) $SL(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$.

(h) $\{T \in GL(n, \mathbb{R}) \mid Tv_0 = \lambda v_0, \lambda \in \mathbb{R}^+\} \subseteq GL(n, \mathbb{R})$ for some fixed $v_0 \in \mathbb{R}^n$.

Answer here...

Problem 2.54. *Show the following are not subgroups:*

(a) $\{5^a \mid a \in \mathbb{Q}^+\} \subseteq \mathbb{R}^\times$.

(e) For $v_0 \in \mathbb{R}^n$, $\{T \in GL(n, \mathbb{R}) \mid Tv_0 = 2v_0\} \subseteq GL(n, \mathbb{R})$.

Answer here...

Problem 2.55. *Calculate the centralizers of the following subsets:*

(b) $\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\} \subseteq GL(2, \mathbb{R})$.

Answer here...

Problem 2.56. *Show the following:*

(a) $\langle [2] \rangle = U_5$.

(b) $\langle 3, 11 \rangle = \mathbb{Z}$.

Answer here...

Problem 2.63. *(Subgroups of $\mathbb{Z}$ and $\mathbb{Z}_n$) Complete the following:*

(a) If $H \neq \{0\}$ is a subgroup of $\mathbb{Z}$, let k be the minimal element of $\mathbb{N} \cap H$. Show k exists and that $H = \langle k \rangle$. Hint: If $m \in H$, use the ~~Euclidean~~ Division Algorithm to write $m = kq + r$ with $0 \leq r < k$ and show $kq \in H$ so that $r \in H$.

(b) Conclude that the set of subgroups of $\mathbb{Z}$ is $\{\langle k \rangle = k\mathbb{Z} \mid k \in \mathbb{Z}_{\geq 0}\}$.

(c) For $n \in \mathbb{N}$, show that every ~~subset~~ subgroup of $\mathbb{Z}_n$ is of the form $\langle k \rangle$ for $0 \leq k < n$.

(d) Show $\langle [k] \rangle = \langle [(k, n)] \rangle$. Hint: For $\subseteq$, use $(k, n) \mid k$. For $\supseteq$, write $(k, n) = kx + ny$.

(e) Conclude that the set of subgroups of $\mathbb{Z}_n$ is $\{\langle [k] \rangle \mid k \in \mathbb{N} \text{ and } k \mid n\}$.

(f) Find all subgroups of $\mathbb{Z}_{15}$.

Answer here..

2.3.2 Direct Products

Problem 2.57. *Evaluate the following:*

(a) $([3]_7, [5]_6) \cdot ([2]_7, [5]_6) \in U_7 \times U_6$.

(b) $([2]_4, [3]_5, [6]_7) + ([3]_4, [2]_5, [1]_7) \in \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_7$.

Answer here...