2 Groups

2.3 Subgroups and Direct Products

2.3.2 Direct Products

Problem 2.63 (Subgroups of $\mathbb{Z}$ and $\mathbb{Z}_n$)**.** *Complete the following:*

(a) *If $H \neq \{0\}$ is a subgroup of $\mathbb{Z}$, let k be the minimal element of $\mathbb{N} \cap H$. Show k exists and that $H = \langle k \rangle$. Hint: If $m \in H$, use the Euclidean Algorithm to write $m = kq + r$ with $0 \leq r < k$ and show $kq \in H$ so that $r \in H$.*

(b) *Conclude that the set of subgroups of $\mathbb{Z}$ is $\{\langle k \rangle = k\mathbb{Z} \mid k \in \mathbb{Z}_{\geq 0}\}$.*

(c) *For $n \in \mathbb{N}$, show that every subset of $\mathbb{Z}_n$ is of the form $\langle [k] \rangle$ for $0 \leq k < n$.*

(d) *Show $\langle [k] \rangle = \langle [(k, n)] \rangle$. Hint: For $\subseteq$, use $(k, n) \mid k$. For $\supseteq$, write $(k, n) = kx + ny$.*

(e) *Conclude that the set of subgroups of $\mathbb{Z}_n$ is $\{\langle [k] \rangle \mid k \in \mathbb{N}$ and $k \mid n\}$.*

(f) *Find all subgroups of $\mathbb{Z}_{15}$.*

For part (a), we'll first show that $\langle k \rangle \subseteq H$ and then show that $H \subseteq \langle k \rangle$. For $\langle k \rangle \subseteq H$, we know that $k \in H$. Because $H \leqslant \mathbb{Z}$, we can say H contains all integer multiples of k: thus $\langle k \rangle \subseteq H$. Next, for $H \subseteq \langle k \rangle$, we write any element $m \in H$ as $kq + r$. We then can say $m, kq \in H \Rightarrow r \in H$. However, because k is defined as the minimal non-zero element, r must be 0; therefore $H \subseteq \langle k \rangle$. Finally, $H \subseteq \langle k \rangle$ and $\langle k \rangle \subseteq H$ imply $\langle k \rangle = H$.

For part (b), we know from part (a) that any subgroup H is equal to the subgroup generated by the minimal positive element k. Because this minimal element can be any $k \in \mathbb{N}$, the comprehensive set of subgroups is simply the set of $\langle k \rangle\ \forall k \in \mathbb{Z}_{>0}$. This definition also includes the trivial case as $\langle 0 \rangle$ is built by $0\mathbb{Z}$, which is simply $\{0\}$. Therefore the comprehensive set of subgroups is the set of $\langle k \rangle\ \forall k \in \mathbb{Z}_{\geq 0}$.

For part (c), we know that any subgroup of $\mathbb{Z}_n$ is a subgroup operating on one of the equivalence classes represented by

2.4 Morphisms

2.4.1 Definitions and Examples

Problem 2.92. *Show the following maps are homomorphisms:*

(b) *For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $\varphi([k]) = m[k]$.*

(d) *For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, $\varphi : U_n \to U_n$ given by $\varphi([k]) = [k]^m$.*

(e) *$\varphi : \mathbb{R} \to \mathbb{R}^+$ given by $\varphi(x) = e^x$.*

(g) *$\theta : GL(n, \mathbb{F}) \to GL(n, \mathbb{F})$ given by $\theta(g) = (g^{-1})^T$.*

Answer here..

Problem 2.93. *Show the following maps are not homomorphisms:*

(a) $\varphi : \mathbb{Z} \to \mathbb{Z}$ *given by* $\varphi(k) = k + 1$.

(b) $\varphi : \mathbb{R} \to \mathbb{R}$ *given by* $\varphi(x) = x^2$.

(c) $\varphi : \mathbb{R} \to \mathbb{R}^\times$ *given by* $\varphi(x) = 2x$.

Answer here...

Problem 2.94. *Show the following groups are isomorphic.*

(a) $\mathbb{Z} \cong 2\mathbb{Z}$.

(c) *For* $n \in \mathbb{N}$, $\{z \in \mathbb{C} \mid z^n = 1\} \cong \mathbb{Z}_n$.

(d) $\mathbb{R} \cong \mathbb{R}^+$. *Hint:* e^x.

(e) $U_7 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

(g) $\mathbb{R} \cong U(2, \mathbb{R}) = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}^\times\}$.

Answer here...

Problem 2.95. *Show the following groups are not isomorphic.*

(a) $\mathbb{Z}_4 \ncong \mathbb{Z}_5$.

(b) $S_3 \ncong \mathbb{Z}_6$.

(c) $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) $R^\times \ncong \mathbb{R}$. *Hint: Count the solutions to* $x^2 = 1$ *in* $\mathbb{R}^\times$ *and to* $2x = 0$ *in* $\mathbb{R}$.

(e) $\mathbb{Z} \ncong \mathbb{Q}$.

Answer here...

2.4.2 Basic Properties

Problem 2.96. *Calculate the kernels and images of the following homomorphisms.*

(a) $\varphi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ *by* $\varphi([k]) = 2[k]$.

(b) $\varphi : U_{10} \to U_{10}$ *by* $\varphi([k]) = [k]^2$.

(c) $\varphi : D_n \to \{\pm 1\}$ *by* $\varphi(R_j) = 1$ *and* $\varphi(W_j) = -1$.

Answer here...

Problem 2.99 (Homomorphisms from $\mathbb{Z}$)**.** *Complete the following:*

(a) *If* G *is a group and* $\varphi : \mathbb{Z} \to G$ *is a homomorphism, show* $\varphi(k) = \varphi(1)^k$ *for* $k \in \mathbb{Z}$.

(b) *If* $\varphi' : \mathbb{Z} \to G$ *is another homomorphism, show* $\varphi = \varphi'$ *if and only if* $\varphi(1) = \varphi'(1)$.

(c) *If* $g \in G$ *show the map* $\varphi : \mathbb{Z} \to G$ *given by* $\varphi(k) = g^k$ *is a homomorphism.*

(d) *Conclude that the set of homomorphisms from $\mathbb{Z}$ to G is in bijection with the set of elements of G.*

Answer here...

Problem 2.104. *Suppose G is a group with $S \subseteq G$ and $G = \langle S \rangle$. If $\varphi, \varphi : G \to H$ are homomorphisms satisfying $\varphi(s) = \varphi'(s)$ for all $s \in S$, show $\varphi = \varphi'$.*

Answer here...