1 Division Algorithm

Problem 5. *Use old-fashioned long division to implement the Division Algorithm and write $a = bq + r$, $0 \leq r < b$, for each a and b listed below:*

(a) $a = 20$, $b = 3$.

(b) $a = 54$, $b = 7$.

Starting with the first subproblem, we can start by finding the maximal q in the set of possible divisors S. By examination we can quickly deduce

$$\boxed{20 = 3(6) + 2, 0 \leq 2 < 3.}$$

We can do the same for the second subproblem:

$$\boxed{54 = 7(7) + 5, 0 \leq 5 < 7.}$$

Problem 8. *Show that when the square of an odd integer is divided by 8, the remainder is 1. (Hint: remember $2|n(n+1)$.)*

We will restate the problem as a proposition and prove it using the Division Algorithm.

Proposition. *Given the square of an odd integer, it's remainder is always 1 when divided by 8.*

Proof. We start with a simple expression of any odd integer, $2n + 1$. We can restate our proposition as $\exists q \in \mathbb{Z}_{\geq 0}$ such that $(2n + 1)^2 = 8q + 1$ $\forall n \in \mathbb{Z}_{\geq 0}$. We can rearrange the statement to determine whether q is an integer:

$$\begin{aligned}
(2n+1)^2 &= 8q + 1 \\
4n^2 + 4n + 1 &= 8q + 1 \\
4n(n+1) &= 8q \\
\frac{1}{2}n(n+1) &= q
\end{aligned}$$

Because $n(n+1)$ is even (or 0) we now know that $\forall n \in \mathbb{Z}_{\geq 0} \implies \exists q \in \mathbb{Z}_{\geq 0}$ given a remainder of 1. This is important as it ensures our divisor is an integer. Thus $(2n + 1)^2/8$ will yield a remainder of 1 $\forall n \in \mathbb{Z}_{\geq 0}$. $\square$

Remark. Is this too discursive? There is probably a more elegant solution.

Problem 10. *Let $n, m \in \mathbb{N}$ with $m \neq 1$. Show n can be uniquely written in the form $n = \sum_{k=0}^{N} a_k m^k$ for some $N \in \mathbb{Z}_{\geq 0}$ and $a_k \in \{0, 1, \ldots, (m-1)\}$ with $a_N \neq 0$. Hint: Use induction on n and begin by choosing the largest $N \in \mathbb{Z}_{\geq 0}$ so that $m^N \leq n$. Use the Division Algorithm to write $n = a_N m^N + r$ and then apply the inductive hypothesis to r.*

2 Divisors

Problem 12. *List all of the divisors of the following:*

(a) *52,*

(b) ...

Problem 14. *Evaluate the following:*

(a) *(42,56),*

(b) ...

Problem 17. *Let* $b, q, r \in \mathbb{Z}$ *and let* $a = bq + r$ *with* a *and* b *not both* 0*.*

(a) *Show a common divisor of* a *and* b *is a divisor of* r *and that a common divisor of* b *and* r *is a divisor of* a*.*

(b) *Conclude that* $(a, b) = (r, b)$

Problem 19. *Let* $a, b \in \mathbb{Z}$*, not both 0.*

(a) *If* $(a, b) = d$*, then* $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$*. Hint: Write* $ax + by = d$ *so that* $\frac{a}{d}x + \frac{b}{d}y = 1$*.*

(b) ...