Problem 1. *Prove that if z_1, $z_2 \in \mathbb{C}$ and $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$.*

Proof. Suppose that $\exists\ z_1,\ z_2 \in \mathbb{C} \wedge z_1 z_2 = 0$. We can express their product as

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_2 y_1 + y_2 x_1) = (0, 0)$$

which implies $x_1 x_2 - y_1 y_2 = x_2 y_1 + y_2 x_1$. From this, rearranging yields

$$\begin{aligned} x_1 x_2 - y_1 y_2 &= x_2 y_1 + y_2 x_1 \\ x_1 x_2 - y_2 x_1 &= x_2 y_1 + y_1 y_2 \\ x_1(x_2 - y_2) &= y_1(x_2 + y_2). \end{aligned}$$

Since there is no choice of $x_2, y_2 \in \mathbb{Z}\backslash\{0\}$ that makes $x_2 - y_2 = x_2 + y_2$ true, $(x_1, y_1) = 0 \vee (x_2, y_2) = 0$. Thus there are no proper zero divisors in $\mathbb{C}$. $\square$

Remark. Is it sufficient to say that $\mathbb{C}$ is a field, all fields do not have proper zero divisors, thus $\mathbb{C}$ has no proper zero divisors?

Problem 2. *Do each of the following:*

(a) Write the complex number $\frac{1+2i}{3+4i}$ in the form $a + bi$.

(b) Find $\operatorname{Re}\left(\frac{2-i}{2+i}\right)$ and $\operatorname{Im}\left(\frac{2-i}{2+i}\right)$.

For the first subproblem, we will re-express the fraction using the denominators conjugate, $3 - 4i$:

$$\begin{aligned} \left(\frac{1+2i}{3+4i}\right)\left(\frac{3-4i}{3-4i}\right) &= \frac{(1+2i)(3-4i)}{9-16i^2} \\ &= \frac{3+2i-8i^2}{25} \\ &= \frac{11}{25} + \frac{2}{25}i. \end{aligned}$$

$\boxed{\text{Therefore } a = \frac{11}{25} \text{ and } b = \frac{2}{25}.}$ For the second subproblem, we will complete the same computation but will present the real and imaginary components separately. Using the denominator's conjugate of $2 - i$:

$$\left(\frac{2-i}{2+i}\right)\left(\frac{2-i}{2-i}\right) = \frac{4-4i+i^2}{4-i^2} = \frac{3-4i}{5}.$$

$\boxed{\text{Therefore } \operatorname{Re}\left(\frac{2-i}{2+i}\right) = \frac{3}{5} \text{ and } \operatorname{Im}\left(\frac{2-i}{2+i}\right) = \frac{-4}{5}.}$

Problem 3. *Prove that if $|z| = 2$, then*

$$\frac{1}{|z^4 - 4z^2 + 3|} \leq \frac{1}{3}.$$

(Hint: factor $z^4 - 4z^2 + 3$ and then use the reverse triangle inequality.)

Proof. Suppose $\exists z \in \mathbb{C}$ where $|z| = 2$. Given a polynomial $z^4 - 4z^2 + 3$, we can determine using the reverse triangle inequality that

$$\begin{aligned} |z^4 - 4z^2 + 3| &\geq |z^4| - |-4z^2| - |3| \\ &\geq |2^4 - 4(2)^2 - 3| \\ &\geq 3 \end{aligned}$$

Therefore $|z^4 - 4z^2 + 3|^{-1} \leq \frac{1}{3}$. □

This result can be confirmed using the modulus-polynomial method—if that accurately refers to the procedure—used in the textbook by finding the moduli of the factors of $|z^4 - 4x^2 + 3|$, $|z^2 - 3|$ and $|z^2 - 1|$, as 1 and 3 respectively. We then take the greatest number of the two as our reciprocal polynomial's upper bound, in this case $\frac{1}{3}$.

Problem 4. *Prove the following:*

(a) z is real if and only if z=$\bar{z}$.

(b) z is either real or purely imaginary if and only if $(\bar{z})^2 = z^2$.

Proof. Suppose $\exists z$ such that $z \in \mathbb{R}$. Such a real number in the complex plane doesn't have an imaginary component and is represented simply as $(x, 0)$; because $\text{Im}(z)$ is the opposite of its conjugate, a real number remains a real number as 0 is neutral. If we are to suppose that $z = \bar{z}$, $\text{Im}(z)$ and $\text{Im}(\bar{z})$ are implied to be the same, which is only true for the neutral zero as $\text{Im}(z) = -\text{Im}(\bar{z})$. Therefore $z \in \mathbb{R} \iff z = \bar{z}$. □

Proof. Suppose $\exists z \in \mathbb{C}$. If we also suppose that such a number represented in its component form $a + bi$ is either completely real or completely imaginary, either $z = a$ or $z = bi$ with corresponding $\bar{z} = a$ or $\bar{z} = -bi$. In the real case, $z^2 = a^2 = (\bar{z})^2$ while in the 'imaginary' case $z^2 = (bi)^2 = (-bi)^2 = (\bar{z})^2$. As clearly demonstrated by the commutativity of the second expression, the reverse relationship is true and thus z is either real or purely imaginary $\iff (\bar{z})^2 = z^2$. □

Problem 5. *With $z = x + yi$, write the equation $|2\bar{z} + i| = 4$ in terms of x and y. Sketch the graph of this equation and identify what kind of equation it is.*

If we are to substitute $z = x + yi$ into $|2\bar{z} + i| = 4$, we can say

$$|2(x - yi) + i| = 4 \implies |2x - 2yi + i| = 4.$$

We can calculate the modulus of this expression as the following:

$$|2x + (1 - 2y)i| = 4 \implies \sqrt{4x^2 + 1 - 4y + 4y^2} = 4 \implies x^2 + y^2 - y - \frac{15}{4} = 0.$$

Further simplification involves completing the square by separating $\frac{1}{4}$ out:

$$x^2 + (y^2 - y + \frac{1}{4}) - \frac{16}{4} = 0 \implies x^2 + (y - \frac{1}{2})^2 = 4.$$

$\boxed{\text{Therefore } |2\bar{z} + 1| = 4 \text{ is a circle of radius 2 centered around } (0, \tfrac{1}{2}).}$