Tutorial 11

Recap

Last week, we looked at the 2D shallow water equations,

$$h_t + \nabla \cdot (h\mathbf{u}) = 0, \tag{1a}$$

$$(h\mathbf{u})_t + \nabla \cdot (h\mathbf{u} \circ \mathbf{u}) + \frac{g}{2}\nabla h^2 = h(-g\nabla H + f\hat{\mathbf{k}} \times \mathbf{u}). \tag{1b}$$

and rewriting (1) explicitly,

$$h_t + \partial_x(hu) + \partial_y(hv) = 0, \tag{2a}$$

$$(hu)_t + \partial_x(hu^2 + \frac{g}{2}h^2) + \partial_y(hvu) = 0, \tag{2b}$$

$$(hv)_t + \partial_x(huv) + \partial_y(hv^2 + \frac{g}{2}h^2) = 0, \tag{2c}$$

where we have conveniently ignored the source terms on the right-hand side of the equations involving the orography $H(t, x, y)$ and Coriolis forces $f(z)$, i.e. we have set $H \equiv 0$ and $f = 0$. We discretised (2) with the two-steps Richtmyer Lax-Wendroff method, with the first substep as follows,

$$q^{n+1/2}_{i+1/2,j} = \frac{1}{2}\left(q^n_{i+1,j} + q^n_{i,j}\right) - \frac{\Delta t}{2\Delta x}\left[f(q^n_{i+1,j}) - f(q^n_{i,j})\right], \tag{3a}$$

$$q^{n+1/2}_{i-1/2,j} = \frac{1}{2}\left(q^n_{i,j} + q^n_{i-1,j}\right) - \frac{\Delta t}{2\Delta x}\left[f(q^n_{i,j}) - f(q^n_{i-1,j})\right], \tag{3b}$$

and the second substep is as follows,

$$q^{n+1}_{i,j} = q^n_{i,j} - \frac{\Delta t}{\Delta x}\left[f(q^{n+1/2}_{i+1/2,j}) - f(q^{n+1/2}_{i-1/2,j})\right] - \frac{\Delta t}{\Delta y}\left[f(q^{n+1/2}_{i,j+1/2}) - f(q^{n+1/2}_{i,j-1/2})\right], \tag{4}$$

where we can modify (3) to compute $q^{n+1/2}_{i,j\pm1/2}$.

The boundary conditions

We assume periodic boundary condition in the x-direction of the domain, and no-flux boundary condition at the top and bottom of the domain, i.e.

$$\frac{\partial h}{\partial \mathbf{n}} = 0, \tag{5}$$

where $\mathbf{n}$ is the normal to the top and/or bottom boundaries.

Dealing with the source terms

Now let us rewrite (1) explicitly with the source terms on the right-hand side of equations. This gives us

$$h_t + \partial_x(hu) + \partial_y(hv) = 0, \tag{6a}$$

$$(hu)_t + \partial_x(hu^2 + \frac{g}{2}h^2) + \partial_y(hvu) = h(fv - g\,\partial_x H), \tag{6b}$$

$$(hv)_t + \partial_x(huv) + \partial_y(hv^2 + \frac{g}{2}h^2) = h(-fu - g\,\partial_y H). \tag{6c}$$

Observe from (6) that our time update step for a primary variable q involves the advection terms on the left-hand side (coloured in green) and the source terms on the right (coloured in blue). So we need to modify (4) to include the source terms as follows,

$$q^{n+1}_{i,j} = q^n_{i,j} - \frac{\Delta t}{\Delta x}\left[f(q^{n+1/2}_{i+1/2,j}) - f(q^{n+1/2}_{i-1/2,j})\right] - \frac{\Delta t}{\Delta y}\left[f(q^{n+1/2}_{i,j+1/2}) - f(q^{n+1/2}_{i,j-1/2})\right] + \Delta t\, Q^n_{i,j} h^{n+1/2}_{i,j}, \tag{7}$$

where $Q^n_{i,j}$ is the discrete value of the terms in the brackets on the right-hand side of (6b) or (6c).

Notice that we do not have to modify substep (3) at all, since this substep only prepares the values we need to solve the advection part of the update (in green).

References

P. Connolly. *Shallow water equations*, 2017. https://personalpages.manchester.ac.uk/staff/paul.connolly/teaching/practicals/shallow_water_equations.html [Accessed: 2 January 2021].

R. J. LeVeque. *Finite volume methods for hyperbolic problems.* Cambridge university press, 2002.