Tutorial 10

Shallow water equations

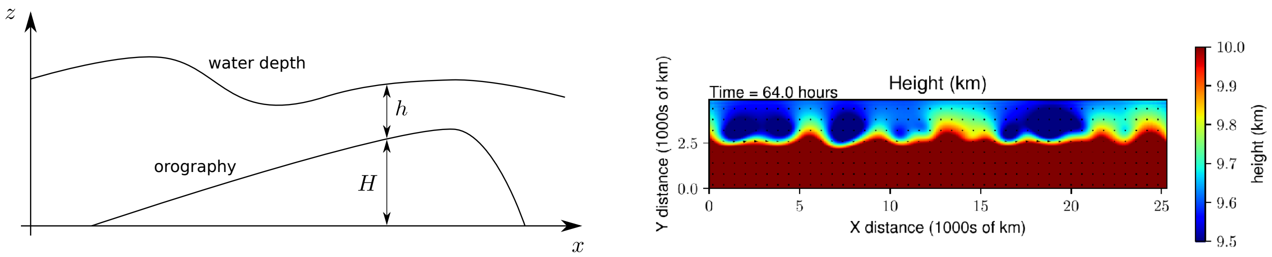
We start with the 2D shallow water equations in conservative form,

$$h_t + \nabla \cdot (h\mathbf{u}) = 0, \tag{1a}$$

$$(h\mathbf{u})_t + \nabla \cdot (h\mathbf{u} \circ \mathbf{u}) + \frac{g}{2}\nabla h^2 = h(-g\nabla H + f\hat{\mathbf{k}} \times \mathbf{u}). \tag{1b}$$

Here, $h(t, x, y)$ is the water depth, $\mathbf{u} = (u(t, x, y), v(t, x, y))$ are the horizontal components of the velocity, g is the acceleration of gravity, f is the Coriolis strength, and $H(x, y)$ is the orography. See Figure 1 for more details. $\hat{\mathbf{k}}$ is the unit vector in the vertical direction, $\cdot$ represents the dot product, and $\circ$ the outer product. The subscript $_t$ denotes the partial derivative with respect to time, i.e. $h_t = \partial_t h = \partial h/\partial t$, and ∇ subsumes the horizontal derivatives, i.e. $\nabla = (\partial_x, \partial_y)$. The three primary variables of our shallow water equation are h, u, and v, and we note that, in a way, the 2D shallow water equations model a *quasi-3D* system.

Recall that (1a) describes the conservation of mass that we are familiar with from tutorial 6, and (1b) describes the conservation of momentum. Elaboration on (1b) should be covered in the lectures, and Connolly (2017) gives a concise derivation of these equations from the Navier-Stokes equations.



(a) Illustration depicting the side view of a 2D shallow water model. h is the water depth, and H is the orography height.

(b) Top view of a 2D shallow water model on a domain with Cartesian coordinates in the (x, y) direction and a heatmap for the water depth h.

Figure 1: Side view (left) and top view (right) depictions of a 2D shallow water problem.

Richtmyer two-step Lax-Wendroff method

Let us now turn our attention to the numerical method that we will be using to solve (1). For a 1D problem, the two-step Richtmyer Lax-Wendroff method (RLW) is as follows,

$$q_{i+1/2}^{n+1/2} = \frac{1}{2}\left(q_{i+1}^n + q_i^n\right) - \frac{\Delta t}{2\Delta x}\left[f(q_{i+1}^n) - f(q_i^n)\right], \tag{2a}$$

$$q_{i-1/2}^{n+1/2} = \frac{1}{2}\left(q_i^n + q_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left[f(q_i^n) - f(q_{i-1}^n)\right]. \tag{2b}$$

The update in (2) gives us the value of q at time $(n + 1/2)$ and at the cell interfaces $(i - 1/2)$ and $(i + 1/2)$. We then use the solutions obtained from (2) to update q from time n to $n + 1$,

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}\left[f(q_{i+1/2}^{n+1/2}) - f(q_{i-1/2}^{n+1/2})\right]. \tag{3}$$

Notice that the two-step RLW method resembles the explicit midpoint update we saw in the heat equation exercise (week 4), with substep (2) being similar to the Lax-Friedrich method we used in the exercise on the Burger's equation (week 7) and substep (3) being similar to the general finite volume update we derived in the notes of week 7, cf. equation (13) in `w7.pdf`.

For a 2D problem, equation (3) becomes

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}\left[f(q_{i+1/2}^{n+1/2}) - f(q_{i-1/2}^{n+1/2})\right] - \frac{\Delta t}{\Delta y}\left[f(q_{j+1/2}^{n+1/2}) - f(q_{j-1/2}^{n+1/2})\right], \tag{4}$$

where now we will have to compute $q_{j\pm1/2}^{n+1/2}$ as well. (Our experience with solving the 2D Burger's equation will come in handy here.)

Discretisation

If we were to ignore the source terms on the RHS of (1), we can rewrite (1) as a set of three subequations, each of which is a conservation equation in differential form,

$$h_t + \partial_x(hu) + \partial_y(hv) = 0, \tag{5a}$$

$$(hu)_t + \partial_x(hu^2 + \frac{g}{2}h^2) + \partial_y(hvu) = 0, \tag{5b}$$

$$(hv)_t + \partial_x(huv) + \partial_y(hv^2 + \frac{g}{2}h^2) = 0, \tag{5c}$$

where we will be applying a time update for h in (5a), hu in (5b), and hv in (5c). Let us now try to rewrite (5) in the form of the RLW update.

To integrate h in (5a) from time n to $n+1$ with the formula (4), we will need the fluxes $(hu)^{n+1/2}_{i\pm1/2}$ and $(hv)^{n+1/2}_{j\pm1/2}$. For the update of hu in (5b), we will need $(hu^2)^{n+1/2}_{i\pm1/2}$, $(hvu)^{n+1/2}_{j\pm1/2}$, and $(h^2)^{n+1/2}_{i\pm1/2}$, and for the update of hv in (5c), we will need $(hv^2)^{n+1/2}_{j\pm1/2}$, $(huv)^{n+1/2}_{i\pm1/2}$, and $(h^2)^{n+1/2}_{j\pm1/2}$. Since $hu^2 = (hu)^2/h$ and so on, we will actually only need

$$(h)^{n+1/2}_{i\pm1/2}, \quad (h)^{n+1/2}_{j\pm1/2}, \quad (hu)^{n+1/2}_{i\pm1/2}, \quad (hu)^{n+1/2}_{j\pm1/2}, \quad (hv)^{n+1/2}_{i\pm1/2}, \quad (hv)^{n+1/2}_{j\pm1/2}, \tag{6}$$

and we will be able to reconstruct all the fluxes that we need to update (5) using the RLW update in (4).

Let us take a closer look at two examples. To obtain $h^{n+1/2}_{i+1/2}$ using the RLW substep (2a), we write

$$h^{n+1/2}_{i+1/2} = \frac{1}{2}\left(h^n_{i+1} + h^n_i\right) - \frac{\Delta t}{2}\left[\frac{(hu)^n_{i+1} - (hu)^n_i}{\Delta x}\right], \tag{7}$$

and to obtain $(hv)^{n+1/2}_{j-1/2}$ using the RLW substep (2b), we write

$$(hv)^{n+1/2}_{j-1/2} = \frac{1}{2}\left((hv)^n_j + (hv)^n_{j-1}\right) - \frac{\Delta t}{2}\left[\frac{(hv^2 + gh^2/2)^n_j - (hv^2 + gh^2/2)^n_{j-1}}{\Delta y}\right]. \tag{8}$$

Once we have obtained all the quantities in (6), we will have the fluxes we need in the RHS of the RLW substep (4). Following which, the update of h, hu, and hv from time n to $n+1$ is straightforward, and we can easily compute the primary variables h, u and v at the new time level $n+1$.

References

P. Connolly. *Shallow water equations*, 2017. https://personalpages.manchester.ac.uk/staff/paul.connolly/teaching/practicals/shallow_water_equations.html [Accessed: 2 January 2021].

R. J. LeVeque. *Finite volume methods for hyperbolic problems.* Cambridge university press, 2002.