Tutorial 7

In the solution of our heat equation, we applied a "forward-in-time, centered-in-space" (FTCS) scheme, forward-in-time because of the explicit midpoint rule we used to discretise the time derivative, and centered-in-space because of the central difference method we implemented to discretise the derivative in space. Such schemes work well for a parabolic PDE like the heat equation, but not for hyperbolic PDEs like our conservation laws and advection equations. Today, we will take a look at a simple *finite volume method* that can be used to numerically integrate hyperbolic PDEs.

Recap

We encountered the 1D conservation law in integral form last tutorial,

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(t,x)\ dx = \phi(t,x_1) - \phi(t,x_2). \tag{1}$$

with the flux function $\phi(t,x) = \rho v$ and after some effort, we derived the conservation law in differential form,

$$\partial_t \rho(t,x) + \partial_x \phi(t,x) = 0, \tag{2}$$

Finite volume method

We consider the finite volume method in one dimension. Let us denote the ith cell in the one-dimensional grid by

$$\mathcal{C}_i = (x_{i-1/2}, x_{i+1/2}) \tag{3}$$

with a spatial grid-size of $\Delta x = x_{i+1/2} - x_{i-1/2}$. Then for a quantity $u(t,x)$, the value in the ith cell at time n is

$$U_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(t^n, x)\, dx \equiv \frac{1}{\Delta x}\int_{\mathcal{C}_i} u(t^n,x)\, dx. \tag{4}$$

Here, we assume that we have a uniform grid, i.e. Δx remains the same for each cell. Writing the flux ϕ as a function dependent on u entering from the left edge of the cell at $x - 1/2$ and exiting from the right edge of the cell at $x + 1/2$, (1) becomes

$$\frac{d}{dt}\int_{\mathcal{C}_i} u(t,x)\, dx = f(t, u(x_{i-1/2})) - f(t, u(x_{i+1/2})). \tag{5}$$

To obtain u^{n+1}, we integrate (5) over a single time-step of size $\Delta t = t^{n+1} - t^n$,

$$\int_{\mathcal{C}_i} u(t^{n+1},x)\, dx - \int_{\mathcal{C}_i} u(t^n,x)\, dx = \int_{t^n}^{t^{n+1}} f(u(x_{i-1/2}))\, dt - \int_{t^n}^{t^{n+1}} f(u(x_{i+1/2}))\, dt. \tag{6}$$

Rearranging and dividing by Δx,

$$\frac{1}{\Delta x}\int_{\mathcal{C}_i} u(t^{n+1},x)\, dx = \frac{1}{\Delta x}\int_{\mathcal{C}_i} u(t^n,x)\, dx - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f(u(x_{i+1/2}))\, dt - \int_{t^n}^{t^{n+1}} f(u(x_{i-1/2}))\, dt\right], \tag{7}$$

and inserting (4) into the above,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n), \tag{8}$$

where $F_{i-1/2}^n$ approximates the average flux across the *cell interface* $x_{i-1/2}$,

$$F_{i-1/2}^n \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(t, x_{i-1/2}))\, dt. \tag{9}$$

This process is illustrated in Figure 1. (8) is a general finite volume discretisation in *conservation form* as we derived it from the conservation law in (1).

We assume that, within one time-step, u propagates with finite speed and the flux across $x_{i-1/2}$ can be obtained from the values u_{i-1} and u_i, that is

$$F_{i-1/2}^n = \mathcal{F}(U_{i-1}^n, U_i^n) \tag{10}$$

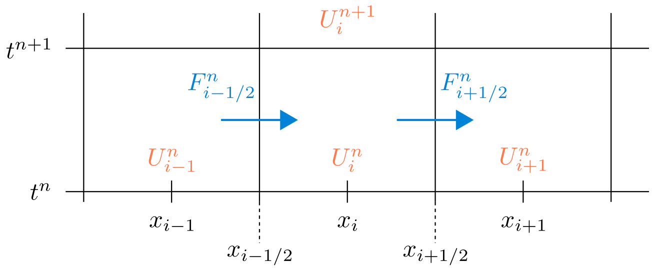

Figure 1: The update of the quantity U_i^n to U_i^{n+1} following the finite volume discretisation in (8) by the fluxes at the cell interfaces.

where $\mathcal{F}$ is some *numerical flux function*. Inserting this into (8) yields

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(U_i^n, U_{i+1}^n) - \mathcal{F}(U_{i-1}^n, U_i^n)\right]. \tag{11}$$

As $\mathcal{F}(U_{i-1}^n, U_i^n)$ is the flux across the cell interface at $x_{i-1/2}$, a simple approximation of this numerical flux function would be the arithmetic middle-point between the two fluxes at the adjacent cell-centres, i.e.,

$$F_{i-1/2}^n = \mathcal{F}(U_{i-1}^n, U_i^n) = \frac{1}{2}[f(U_{i-1}^n) + f(U_i^n))]. \tag{12}$$

Inserting the above into (8), we obtain

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{2\Delta x}\left[f(U_{i+1}^n) - f(U_{i-1}^n)\right] \tag{13}$$

which is unfortunately generally unstable for the hyperbolic problems that we want to tackle.

The Lax-Friedrich method

This brings us to the Lax-Friedrich method,

$$U_i^{n+1} = \frac{1}{2}(U_{i-1}^n + U_{i+1}^n) - \frac{\Delta t}{2\Delta x}\left[f(U_{i+1}^n) - f(U_{i-1}^n)\right], \tag{14}$$

where we now replace U_i^n with the average between U_{i-1}^n and U_{i+1}^n. This method is generally stable for the numerical integration of hyperbolic problems, and we will be using it for our exercise this week. Note that like the FTCS scheme we used to solve the heat equation where we had to discretise the differentials w.r.t. time and space separately, the Lax-Friedrich method in (14) is akin to discretising the conservation law in differential form (2) in both time and space within a single update step.

The CFL condition

The Lax-Friedrich method is stable if the following condition is met,

$$\text{CFL} \equiv \left|\frac{v\Delta t}{\Delta x}\right| \leq 1 \tag{15}$$

where v is the velocity with which u propagates. Intuitively, this means that the signal of u must not exceed the size of one grid-cell, Δx, within one time-step of size Δt.

References

R. J. LeVeque. *Numerical methods for conservation laws*, volume 132. Springer, 1992.

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