Tutorial 8

Recap

We came across a simple conservation law and the conservation law in differential form is reproduced below,

$$\partial_t \rho(t,x) + \partial_x \phi(t,x) = 0. \tag{1}$$

From the conservation law, we derived a general finite volume discretisation that is unfortunately unstable when applied to hyperbolic problems,

$$\begin{aligned} U_i^{n+1} &= U_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2}^n - F_{i-1/2}^n\right] &\text{(2a)}\\ &= U_i^n - \frac{\Delta t}{2\Delta x}\left[f(U_{i+1}^n) - f(U_{i-1}^n)\right]. &\text{(2b)}\end{aligned}$$

Finally, we introduced the Lax-Friedrich method which involves a slight modification of (2a),

$$\begin{aligned} U_i^{n+1} &= \frac{1}{2}(U_{i-1}^n + U_{i+1}^n) - \frac{\Delta t}{\Delta x}\left[F_{i+1/2}^n - F_{i-1/2}^n\right] &\text{(3a)}\\ &= \frac{1}{2}(U_{i-1}^n + U_{i+1}^n) - \frac{\Delta t}{2\Delta x}\left[f(U_{i+1}^n) - f(U_{i-1}^n)\right]. &\text{(3b)}\end{aligned}$$

Finite difference interpretation

Last tutorial, we made the effort to derive (2) and (3) from the conservation laws. We can also derive (2) from a direct finite difference discretisation of (1). Starting from (1),

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \tag{4}$$

a finite difference discretisation of the cell-averaged values U in time and space yields

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta x} = 0, \tag{5}$$

where we recall that $F_{i\pm1/2}^n$ is the numerical approximation for the flux function at time-level n and cell-interfaces $i \pm 1/2$. Rearranging (5) gives us (2a).

Error in the Lax-Friedrich method

Before we look at the Lax-Friedrich method, let us consider a discretisation of the 1D heat equation that we have encountered before,

$$\partial_t u(t,x) = D\,\partial_{xx} u(t,x). \tag{6}$$

A discretisation of (6) with the explicit Euler method on the left and central difference method on the right leads us to

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left[D\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x}\right]. \tag{7}$$

Now let us move on to the Lax-Friedrich method. We notice that in order to make equation (3a) appear in the form of (2b), we will need to write the numerical fluxes as

$$\begin{aligned} F_{i+1/2}^n &= \frac{1}{2}\left[f(U_i^n) + f(U_{i+1}^n)\right] - \frac{\Delta x}{2\Delta t}(U_{i+1}^n - U_i^n), &\text{(8a)}\\ F_{i-1/2}^n &= \frac{1}{2}\left[f(U_{i-1}^n) + f(U_i^n)\right] - \frac{\Delta x}{2\Delta t}(U_i^n - U_{i-1}^n), &\text{(8b)}\end{aligned}$$

and inserting (8) into (3a),

$$U_i^{n+1} = \frac{1}{2}(U_{i-1}^n + U_{i+1}^n) - \frac{\Delta t}{2\Delta x}\left[f(U_{i+1}^n) - f(U_{i-1}^n)\right] + \frac{\Delta t}{\Delta x}\left[\frac{(\Delta x)^2}{2\Delta t}\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x}\right], \tag{9}$$

and we see that the last term on the right is actually the diffusion term from the heat equation, with $D = \frac{1}{2}(\Delta x)^2/\Delta t$! So even though we derived the Lax-Friedrich method for the advection equation in (4), we are actually modelling an *advection-diffusion equation*,

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = D\,\partial_{xx} u(t,x). \tag{10}$$

This additional diffusion term that arises in the Lax-Friedrich method imposes an artificial *numerical diffusion* that damps away the instabilities present in (2). If we were to fix $\Delta t/\Delta x$ and refine our grid-size, i.e. let $\Delta x \to 0$, then we see that the last term on the right of (9) vanishes. So with a fine-enough grid, we are actually modelling the advection equation only.

References

R. J. LeVeque. *Numerical methods for conservation laws*, volume 132. Springer, 1992.

R. J. LeVeque. *Finite volume methods for hyperbolic problems*, volume 31. Cambridge university press, 2002.