# Tutorial 5

Using only our intuitive understanding of basic physics, we derive a simple example for the *conservation law* and the *advection equations*, then we link these two concepts to the heat equation.

## 1 Conservation law

Assume that we have a one-dimensional tube, and we want to model the rate of change of the density of a quantity, e.g. energy, mass, or species, along this tube. We start by denoting the density as $\rho$, and because $\rho$ changes depending on *where* and *when* along the tube, $\rho = \rho(t, x)$. To obtain the amount of the quantity between the points $x_1$ and $x_2$ of our tube, we can integrate our density in space,

$$\int_{x_1}^{x_2} \rho(t, x)\, dx \approx \text{amount of quantity between points } x_1 \text{ and } x_2 \text{ at time } t. \tag{1}$$

Before we proceed, let us give our example a more physical setting. We take our one-dimensional tube to be a pipe with uniform cross section and $\rho(t, x)$ to be the density of gas in the pipe. Equation (1) then gives us the total mass of gas between the points $x_1$ and $x_2$ along the pipe.

Furthermore, we assume that our pipe is closed along $x$, i.e. the amount of gas only changes due to the inflow and outflow along the pipe, and the amount of gas does not change due to external influences. Generally, a *source* models an increase in the amount of a quantity due to the external factors, and a *sink* models a decrease. Here, we assume that we have neither a source nor a sink.

Let us use $\phi(t, x)$ to denote the function that models the flow of gas along the cross section of our pipe at the point $x$ and at time $t$. $\phi(t, x)$ is also known as the flux, and the flux measures the amount of a quantity passing through a surface. Having defined the flux and from our assumptions above, we can now make a statement regarding the conservation of the amount of gas in the pipe:

> The change in the amount of gas over time in the segment of the pipe between points $x_1$ and $x_2$ is due only to the difference between the amount of gas flowing in at point $x_1$ and the amount of gas flowing out at point $x_2$.

Mathematically, the statement we made above can written as

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho(t, x)\ dx = \phi(t, x_1) - \phi(t, x_2). \tag{2}$$

(2) is the *integral form* of our simple conservation law. Now we integrate (2) in time, say from $t_1$ to $t_2$

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \partial_t \rho(t, x)\ dx\, dt = \int_{t_1}^{t_2} \phi(t, x_1) - \phi(t, x_2)\ dt. \tag{3}$$

Supposing that $\rho(t, x)$ and $\phi(t, x)$ are smooth differentiable functions and applying the fundamental theorem of calculus to the right-hand side,

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \partial_t \rho(t, x)\ dx\, dt = -\int_{t_1}^{t_2} \int_{x_1}^{x_2} \partial_x \phi(t, x)\ dx\, dt, \tag{4}$$

and because the above equation holds for any arbitrary $x_1$, $x_2$, $t_1$, and $t_2$,

$$\partial_t \rho(t, x) + \partial_x \phi(t, x) = 0, \tag{5}$$

which gives us the conservation law in *differential form*. Equations (2) and (5) describe the conservation of mass. They say that the rate of change of the density $\rho(t, x)$ in a pipe segment is equal to the difference between the amount of gas entering and leaving the pipe segment, and the mass of the gas is the *conserved quantity*.

We assume that the flow of the gas in the pipe depends on the amount and velocity of the gas at a particular point of the pipe. Then, we can model the flux function $\phi(t, x)$ as

$$\phi(t, x) = \rho(t, x) v(t, x), \tag{6}$$

where $v(t, x)$ is the velocity of the gas at point $x$ of the pipe and at time $t$. Substituting this expression of the flux $\phi(t, x)$ into (5),

$$\partial_t \rho(t, x) + \partial_x [\rho(t, x) v(t, x)] = 0. \tag{7}$$

Let us move momentarily away from our "flow of water in a pipe" problem. Extending (7) to three dimensions,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{8}$$

where now our density is $\rho(t, \mathbf{x})$, with $\mathbf{x} = (x, y, z)$ the three-dimensional Cartesian coordinates, and $\mathbf{v} = (u, v, w)$ the three-dimensional velocity field. (8) is the *continuity equation* that we will come across often in fluid dynamics. In fact, (8) is the equation describing the conservation of mass in the compressible Euler equations!

Equations (7) and (8) are *advection equations*.

### 1.1 Hyperbolic system of equations

Our "flow in a pipe" problem has only one dependent variable $\rho(t, x)$. If we consider a vector of conserved quantities $q$ with $m$ components and a general *flux function* $f(q(t, x))$, then the continuity equation for a first-order system of partial differential equations in $t$ and $x$ is

$$\partial_t q(t, x) + \partial_x f(q(t, x)) = 0, \tag{9}$$

and in *quasilinear form*,

$$\partial_t q(t, x) + f'(q)\, \partial_x q = 0. \tag{10}$$

If the flux Jacobian matrix $f'(q)$ has real eigenvalues and a corresponding set of $m$ linearly independent eigenvectors, then this implies that equation (10) is *hyperbolic*. The conservation laws we are interested in belong to a class of hyperbolic equations.

## 2 The heat equation

For the heat equation, we are not modelling the flow of a substance, e.g. gas, but rather the diffusion of the internal energy in a material from the thermal vibration of the molecules.

Let $u(t, x)$ be the temperature of a one-dimensional rod at point $x$ and at time $t$. Then the density of the internal energy is

$$E(t, x) = \kappa(x) u(t, x), \tag{11}$$

where $\kappa(x)$ is the heat capacity of the material at point $x$. We assume that $\kappa \equiv 1$ and that the internal energy is conserved, i.e. the change in the internal energy in the rod is only due to the flux of energy at the endpoints of the rod. The heat flux in our case is given by Fick's law for diffusion,

$$\phi(t, x) = -D \frac{\partial u(t, x)}{\partial x}, \tag{12}$$

where $D$ is the thermal diffusivity of the material of the rod, which again we set to $D \equiv 1$. Applying a similar argument to the one we used in deriving (2),

$$\frac{d}{dt} \int_{x_1}^{x_2} \kappa\, u(t, x) dx = -D \left[ \frac{\partial u(t, x_1)}{\partial x} - \frac{\partial u(t, x_2)}{\partial x} \right], \tag{13}$$

which leads us to the differential form,

$$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x), \tag{14}$$

where we now explicit set $\kappa = 1$ and $D = 1$. This is akin to inserting (12) directly into (5) and rewriting $\rho(t, x)$ as $u(t, x)$.

## References


R. J. LeVeque. *Numerical methods for conservation laws*, volume 132. Springer, 1992.

R. J. LeVeque. *Finite volume methods for hyperbolic problems*, volume 31. Cambridge university press, 2002.