# Tutorial 7

## Recap

We encountered the 1D conservation law in the differential form last tutorial,

$$\partial_t \rho(t,x) + \partial_x \phi(t,x) = 0, \tag{1}$$

with the flux function $\phi(t,x) = \rho v$. Today, we choose a function of $v$ such that the we can model a traffic flow problem. The Burgers' equation is similar to the traffic flow problem, so we are going to use it to help us develop an intuition of the model.

## Burgers' equation

For the traffic flow problem, we model the velocity as

$$v(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right), \tag{2}$$

where if there are no other vehicles along a particular stretch of the road, i.e. $\rho = 0$, then vehicles tend to drive at the speed-limit $v = v_{\max}$. On the other hand, if there is a traffic jam and the road is congested, i.e. $\rho = \rho_{\max}$, then the vehicles will come to a stop and $v = 0$. In between these two extreme situations, the velocity of a vehicle decreases linearly with increasing number of vehicles in a given road segment.

Substituting (2) into the differential form of our conservation law (1) and using the flux function we defined above,

$$\partial_t \rho(t,x) + \partial_x \left[\rho(t,x)\, v_{\max}\left(1 - \frac{\rho(t,x)}{\rho_{\max}}\right)\right] = 0. \tag{3}$$

### Non-dimensionalisation

In (3), we have a couple of dimensions of interests, and as we have seen briefly above, we have three fundamental units in our problem, [length], [time], and [number of cars]. All quantities depend in one way or another on these fundamental units. Now we can write our reference length as $L$ and reference time as $T$, chosen such that $v_{\max} = L/T$. We also non-dimensionalise our density by dividing it with a reference $\rho_{\max}$. Then the non-dimensionalised quantities, denoted by asterisks, can be written as

$$x^* = \frac{x}{L}, \qquad t^* = \frac{t}{T}, \qquad u^* = \left(1 - 2\frac{\rho}{\rho_{\max}}\right). \tag{4}$$

Substituting our non-dimensionalised quantities in (4) into (3), the first term in a non-dimensionalised form is

$$\partial_t \rho(t,x) = \frac{1}{T}\partial_{t^*}\frac{\rho_{\max}}{2}\left(1 - u^*\right) = -\frac{\rho_{\max}}{2T}\partial_{t^*} u^*, \tag{5}$$

and the second term is

$$\begin{aligned}\partial_x \left[\rho(t,x)\, v_{\max}\left(1 - \frac{\rho(t,x)}{\rho_{\max}}\right)\right] &= \frac{1}{L}\partial_{x^*}\frac{\rho_{\max}}{2}\left(1 - u^*\right)\frac{L}{T}\left(1 - \frac{1}{2}\left(1 - u^*\right)\right) \\ &= \frac{\rho_{\max}}{2T}\partial_{x^*}(1 - u^*)\frac{1}{2}(1 + u^*) = -\frac{\rho_{\max}}{2T}\partial_{x^*}\frac{(u^*)^2}{2}. \end{aligned} \tag{6}$$

### Governing equation

Dropping the asterisks, writing (5) and (6) together, and cancelling off the common factor,

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0, \tag{7a}$$

$$u(0,x) = u^0(x), \tag{7b}$$

which, along with the initial conditions in (7b), gives us the *Burgers' equation*.

## References


A. Jüngel. Modeling and numerical approximation of traffic flow problems. *Lecture Notes (preliminary version)*, 2002.

D. I. Ketcheson, R. J. LeVeque, and M. J. del Razo, 2020. URL https://www.clawpack.org/riemann_book/html/Burgers.html.