EE5027 Adaptive Signal Processing
Homework Assignment #1

Notice

- $\boxed{\textbf{Due at 9:00pm, October 17, 2022 (Monday)}}$ $= T_d$ for the electronic copy of your solution.
- Please submit your solution to NTU COOL (`https://cool.ntu.edu.tw/courses/15994`)
- ***All answers have to be fully justified***.
- No extensions, unless granted by the instructor one day before T_d.

Problems

1. (Complex Gaussian random vectors) Assume that the random vector $\mathbf{z} = [Z_1, Z_2, Z_3]^T$ follows the complex multivariate circularly-symmetric Gaussian distribution with mean $\mathbf{0}$ and correlation matrix $\mathbf{R_z}$. The correlation matrix $\mathbf{R_z}$ has the following form

$$\mathbf{R_z} = \begin{bmatrix} 5 & 2j & -j \\ r_{2,1} & 4 & -1-2j \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix}. \tag{1}$$

We define the random vector $\mathbf{w} \triangleq \mathbf{Az}$, where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}. \tag{2}$$

   (a) (2 points) What is the probability density function of Z_1?

   (b) (8 points) What are the values of $r_{2,1}$, $r_{3,1}$, $r_{3,2}$, and $r_{3,3}$, such that $\mathbf{R_z}$ is a valid correlation matrix? If there are multiple values, give the range of these values.

   (c) (5 points) Find the correlation matrix of $\mathbf{w}$.

   (d) (5 points) Determine whether the random vector $\mathbf{w}$ is a circularly-symmetric random vector with zero mean.

2. (Autocorrelation functions, 5 points) Determine whether $f(k)$ is a valid autocorrelation function. State the reasons to support your answer.

$$f(k) = -4e^{j\pi/6}\delta(k+3) + 7\delta(k) - 4e^{-j\pi/6}\delta(k-3). \tag{3}$$

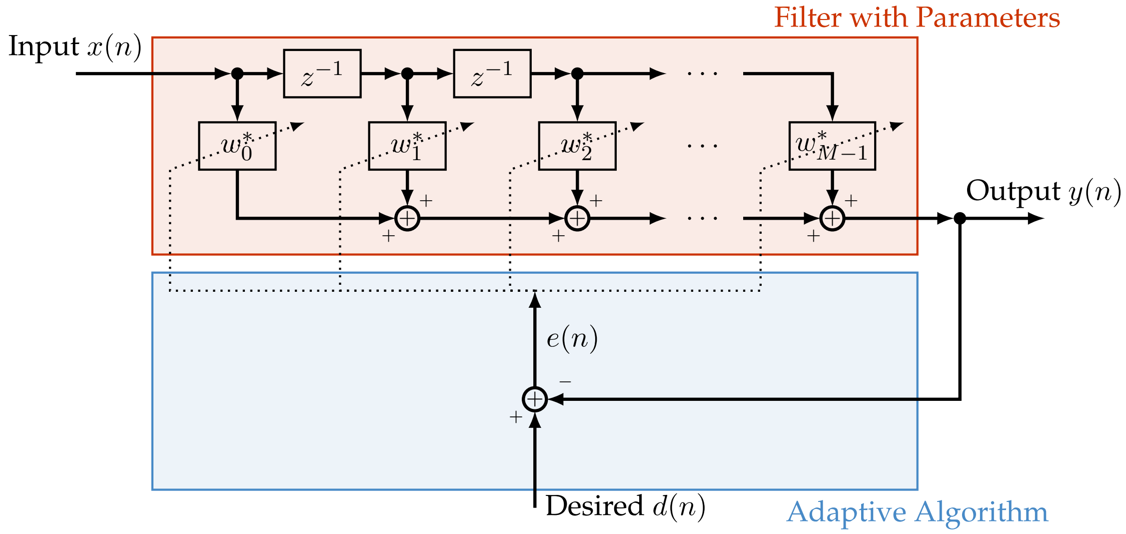

Figure 1: A block diagram for Problem 4.

3. (Stochastic models) Consider an AR process with the following difference equation

$$u(n) + 0.75u(n-1) = v(n), \tag{4}$$

where $v(n)$ is a zero-mean, circularly-symmetric complex Gaussian, white, wide-sense stationary random process with unit variance.

   (a) (5 points) Find the autocorrelation function of $v(n)$.

   (b) (5 points) Find the transfer function $H(z)$ which relates $v(n)$ and $u(n)$.

   (c) (5 points) Find the power spectral density of $u(n)$.

   (d) (5 points) Find the autocorrelation function of $u(n)$.

4. (Wiener filters) We consider a block diagram associated with the Wiener filter in Figure 1, where the number of taps M is 2. The input signal $x(n)$ is a WSS random process with zero mean and the autocorrelation function

$$r_x(k) = \left(-\frac{1}{3}\right)^{|k|}. \tag{5}$$

The desired signal is given by

$$d(n) = x(n+2). \tag{6}$$

In the lecture, it was shown that the optimal weight vector of the Wiener filter is a solution to the Wiener-Hopf equation

$$\mathbf{R}\mathbf{w}_{\text{opt}} = \mathbf{p}, \tag{7}$$

where $\mathbf{R} = \mathbb{E}[\mathbf{x}(n)\mathbf{x}^H(n)]$ and $\mathbf{p} = \mathbb{E}[\mathbf{x}(n)d^*(n)]$.

(a) (4 points) Find $\mathbf{R}$ and $\mathbf{p}$.

(b) (3 points) Find $\sigma_d^2 = \mathbb{E}[d(n)d^*(n)]$.

(c) (3 points) Find the optimal weight vector $\mathbf{w}_{\text{opt}}$.

(d) (5 points) Find the autocorrelation of $y(n)$, when $\mathbf{w} = \mathbf{w}_{\text{opt}}$.

(e) (5 points) Plot the power spectral density $S_y(e^{j2\pi f})$ using MATLAB over $-1 \leq f \leq 1$, when $\mathbf{w} = \mathbf{w}_{\text{opt}}$.

(f) (2 points) Find $J_{\min}$.

(g) (3 points) Let $e_{\text{opt}}(n)$ be the error signal when $\mathbf{w} = \mathbf{w}_{\text{opt}}$. Find $\mathbb{E}\left[x(n+2)e^*_{\text{opt}}(n)\right]$.

5. (Optimization problems, 5 points) Let $\mathbf{R} \in \mathbb{C}^{N\times N}$ be a positive definite matrix. The vector $\mathbf{a} \in \mathbb{C}^N$ and the scalar $g \in \mathbb{C}$. Find the solution to the optimization problem

$$\min_{\mathbf{w}\in\mathbb{C}^N} \mathbf{w}^H\mathbf{R}\mathbf{w} \qquad \text{subject to} \qquad \mathbf{w}^H\mathbf{a} = g. \tag{8}$$

6. (Orthogonality of backward prediction error filters, 5 points) Show that

$$\mathbb{E}[b_m(n)b_i^*(n)] = \begin{cases} P_m, & \text{if } m = i, \\ 0, & \text{if } m \neq i. \end{cases} \tag{9}$$

Hints: Consider $\mathbb{E}[b_m(n)x^*(n-k)]$ for all k.

7. (The error-performance surface) Consider a Wiener filter with a complex input process $x(n)$, a complex desired process $d(n)$, and the weight vector $\mathbf{w} = [w_0, w_1, w_2]^T$. Suppose the covariance matrix $\mathbf{R}$, the cross-correlation vector $\mathbf{p}$, and the variance of $d(n)$ are given by

$$\mathbf{R} = \begin{bmatrix} 3 & 0.7 & 0.3j \\ 0.7 & 3 & 0.7 \\ -0.3j & 0.7 & 3 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} 6 \\ -3j \\ 1+3j \end{bmatrix}, \qquad \sigma_d^2 = 25. \tag{10}$$

In this problem, we will generate the error-performance surface using MATLAB. The real part and the imaginary part of a complex number are denoted by $\text{Re}\{z\}$ and $\text{Im}\{z\}$, respectively.

(a) (3 points) Write a MATLAB function that computes the mean square error (MSE) J given the covariance matrix $\mathbf{R}$, the weight vector $\mathbf{w}$, the cross-correlation vector $\mathbf{p}$, and σ_d^2 (`sd2`). The syntax is as follows:

```
J = ASP_Wiener_MSE(R, w, p, sd2);
```
(11)

This function throws error messages if any of the following occurs: 1) $\mathbf{R}$ is not positive semidefinite, 2) The dimensions of these input arguments are not suitable, or 3) `sd2` is negative or complex-valued.

Note: **We will have additional test data for these exceptions**.

(b) (2 points) Compute the MSE $J_{\min}$ using the MATLAB function in (11).

(c) (5 points) Generate a 1-D plot of the MSE using the MATLAB function `plot`:

- The horizontal axis is $\mathrm{Re}\{w_0\}$, consisting of 201 uniform samples from -4 to 4.
- The vertical axis is the MSE J.
- We assume that $\mathrm{Im}\{w_0\} = 0$, $w_1 = -1$, and $w_2 = 1$.
- Include appropriate labels for the x-axis and the y-axis.
- Mark the extreme point and the associated J on this plot.

(d) (5 points) Generate a surface plot of the MSE using the MATLAB function `surf`:

- The parameter $w_1 = 3 + j$ and $w_2 = 2 + 4j$.
- The x-axis is the real part of w_0, taking 201 uniform samples from -4 to 4.
- The y-axis is the imaginary part of w_0, taking 201 uniform samples from -4 to 4.
- The z-axis is the MSE J.
- Include the axis labels and the color bar in your plot.
- Mark the extreme point and the associated J on this plot.

(e) (5 points) Generate a contour plot of the MSE using the MATLAB function `contour`:

- We set

$$\mathrm{Im}\{w_0\} = 0.3, \qquad \mathrm{Re}\{w_1\} = -0.6, \qquad w_2 = 0.5 + 1.6j. \tag{12}$$

- The x-axis is $\mathrm{Re}\{w_0\}$. We take 201 uniform samples from -3 to 3
- The y-axis is $\mathrm{Im}\{w_1\}$. We take 201 uniform samples from -3 to 3.
- The contour levels are $2, 3, 4, 5, 10, 20, 50, 100$.
- Also include labels on the contour lines and labels for the axes.
- Mark the extreme point and the associated J on this plot.

Note: In localizing the extreme points in Problems 7c, 7d, and 7e, there are two approaches in general.

- The first approach is *grid-based*. For instance, in the plot of Problem 7c, we consider all these 201 samples of the MSE, find the minimum value among these samples, and finally identify the associated $\mathrm{Re}\{w_0\}$.
- The second approach is *grid-less*. Instead of taking discrete grid points, we consider the explicit form of the function and find the exact extreme point(s) of this function. *Hints:* Taking (matrix) derivatives of the MSE to find the extreme point(s) in closed forms. Do not use optimization solvers, such as `fminsearch`, `fmincon`, `cvx`, or Optimization Toolbox in MATLAB.

In Problems 7c, 7d, and 7e, **please implement the second approach (grid-less) for the extreme points**. You can use the MATLAB command `text` to mark these points.

Note: Please include the following MATLAB scripts and figure files in your submission

(a) **`ASP_Wiener_MSE.m`**

(b) **`ASP_HW1_Problem_7b.m`** for the MATLAB codes.

(c) **`ASP_HW1_Problem_7c.m`** for the MATLAB codes and **`ASP_HW1_Problem_7c.fig`** for the plot.

(d) **`ASP_HW1_Problem_7d.m`** for the MATLAB codes and **`ASP_HW1_Problem_7d.fig`** for the plot.

(e) **`ASP_HW1_Problem_7e.m`** for the MATLAB codes and **`ASP_HW1_Problem_7e.fig`** for the plot.

Last updated September 19, 2022.