ASP HW1 R11942085 許宸睿

1. (a). let $z_1 = x + jy$ $\sigma_{z_1}^2 = E[z_1 z_1^*]$

$$\mu = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \frac{\sigma_{z_1}^2}{2} & 0 \\ 0 & \frac{\sigma_{z_1}^2}{2} \end{bmatrix}$$

$$f_z(z) = f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}[x, y]\Sigma^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= \frac{1}{2\pi \frac{\sigma_{z_1}^2}{2}} \exp\left(-\frac{1}{2}\frac{x^2+y^2}{\frac{\sigma_{z_1}^2}{2}}\right)$$

$$= \frac{1}{\pi \sigma_{z_1}^2} \exp\left(-\frac{|z|}{\sigma_{z_1}^2}\right)$$

$$\sigma_{z_1}^2 = 5 \Rightarrow f_z(z) = \frac{1}{5\pi} \exp\left(\frac{-|z|}{5}\right)$$

(b) R_z is valid $\longrightarrow$ $R_z = R_z^H$ and $r_{mm} = \text{real}([r_{mm}])$
where m is dim of R_z.

$$\Rightarrow r_{2,1} = -2j \quad r_{3,1} = j \quad r_{3,2} = -1+2j \quad r_{3,3} \in \mathbb{R}$$

(c). $W = [z_1 + z_2 \quad -z_1 + z_2]^T$

$$R_W = E\left([z_1+z_2 \quad -z_1+z_2]^T [(z_1+z_2)^* \quad (-z_1+z_2)^*]\right)$$

$$= E\left(\begin{bmatrix} z_1z_1^* + z_1z_2^* + z_1^*z_2 + z_2z_2^* & -z_1z_1^* + z_1z_2^* - z_1^*z_2 + z_2z_2^* \\ -z_1z_1^* - z_1z_2^* + z_1^*z_2 + z_2z_2^* & z_1z_1^* - z_1z_2^* - z_1^*z_2 + z_2z_2^* \end{bmatrix}\right)$$

$$= \begin{bmatrix} 5+2j-2j+4 & -5+2j+2j+4 \\ -5-2j-2j+4 & 5-2j+2j+4 \end{bmatrix} = \begin{bmatrix} 9 & -1+4j \\ -1-4j & 9 \end{bmatrix}$$

(d) random vector W is a circularly-symmetric vector with zero mean iff $E[W]=0$ & $E[WW^T]=0$.

$$W=\begin{bmatrix} Z_1+Z_2 \\ -Z_1+Z_2 \end{bmatrix}$$

$$E[W]=\begin{bmatrix} E[Z_1+Z_2] \\ E[-Z_1+Z_2] \end{bmatrix}=\begin{bmatrix} E[Z_1]+E[Z_2] \\ -E[Z_1]+E[Z_2] \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$E[WW^T]=E\begin{bmatrix} Z_1Z_1+2Z_1Z_2+Z_2Z_2 & -Z_1Z_1+Z_2Z_2 \\ -Z_1Z_1+Z_2Z_2 & Z_1Z_1-2Z_1Z_2+Z_2Z_2 \end{bmatrix}$$

$$=\begin{bmatrix} 2E[Z_1Z_2] & 0 \\ 0 & -2E[Z_1Z_2] \end{bmatrix}$$

$\because$ R_Z is a correlation matrix.

$$E[ZZ^T]=[0]$$

$$\therefore E\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}\begin{bmatrix} Z_1 & Z_2 & Z_3 \end{bmatrix}=E\begin{bmatrix} Z_1Z_1 & Z_1Z_2 & Z_1Z_3 \\ Z_2Z_1 & Z_2Z_2 & Z_2Z_3 \\ Z_3Z_1 & Z_3Z_2 & Z_3Z_3 \end{bmatrix}=[0]$$

$\because$ $E[Z_1Z_2]$, $E[Z_2Z_1]=0$

$$\Rightarrow E[WW^T]=0 \quad \#.$$

2. $|f(k)| \le f(0) \quad \forall k \in \mathbb{Z}$ —— ①

$$k \begin{cases} 0 & : f(0) = 7 \\ -3, 3 & : |f(k)| = -4 \\ \text{others} & : |f(k)| = 0 \end{cases}$$

satisfies the constraint ①.

$f(-k) = f^*(k)$ —— ②.

$$\begin{cases} f(-k) = f^*(k) = 0 & \text{for } k \neq 0, -3, 3. \\ f(0) = f^*(0) = 7 \\ f(3) = -4e^{j\frac{-\pi}{6}} = f^*(-3) \end{cases}$$

$\Rightarrow$ satisfies the constraint ②.

$\Rightarrow$ $f(k)$ is an autocorrelation function

3. (a). $r_v(k) = \sigma_v^2 \delta(k)$

$v(n)$ is with unit variance

$\Rightarrow r_v(k) = \delta(k)$

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(b) $u(n) + \frac{3}{4} u(n-1) = v(n)$

$a_1 = \frac{3}{4}$

$$H(z) = \frac{1}{1 + \frac{3}{4} z^{-1}} = \frac{1}{1 - (-\frac{3}{4} z^{-1})}$$

#

(c)

$$r_u(k) = h(k) * r_v(k) * h^*(-k)$$

$$S_u(z) = H(z)\, H^*\left(\frac{1}{z^*}\right) S_v(z)$$

$$S_v(z) = \sum_{k=-\infty}^{\infty} \delta(k)\, z^{-k} = 1$$

$$S_v(e^{j2\pi f}) = \frac{1}{1+\frac{3}{4}e^{j2\pi f}} \cdot \frac{1}{1+\frac{3}{4}e^{-j2\pi f}} \cdot 1$$

$$= \frac{1}{\frac{25}{16} + \frac{3}{4}\left(e^{j2\pi f} + e^{-j2\pi f}\right)}$$

$$= \frac{4}{25 + 12\left(e^{j2\pi f} + e^{-j2\pi f}\right)}$$

#

(d)

$$r_u(k) = h(k) * r_v(k) * h^*(-k)$$

$$h(k) = \left(-\frac{3}{4}\right)^k u(k), \quad h^*(-k) = \left(-\frac{3}{4}\right)^{-k} u(-k)$$

$$r_v(k) = \delta(k)$$

h(k)

$$r_u(k) = h(k) * h^*(-k)$$

$$= \left(-\frac{3}{4}\right)^k u(k) * \left(-\frac{3}{4}\right)^{-k} u(-k)$$

$$= \sum_{\tau=0}^{\infty} \left(-\frac{3}{4}\right)^{\tau} \cdot \left(-\frac{3}{4}\right)^{\tau+k}$$

$$= \left(-\frac{3}{4}\right)^k \sum_{\tau=0}^{\infty} \left(-\frac{3}{4}\right)^{2\tau}$$

$$= \left(-\frac{3}{4}\right)^k \sum_{\tau=0}^{\infty} \left(\frac{9}{16}\right)^{\tau} = \left(-\frac{3}{4}\right)^k \frac{1}{1-\frac{9}{16}} = \left(-\frac{3}{4}\right)^k \frac{16}{7}$$

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4. (a)

$$R = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(-1) & r_x(0) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{3} \\ -\frac{1}{3} & 1 \end{bmatrix}$$

$$p = E[\mathbf{X}(n)\, d^*(n)] = \begin{bmatrix} E[x(n)\, d^*(n)] \\ E[x(n-1)\, d^*(n)] \end{bmatrix}$$

$$= \begin{bmatrix} E[x(n)\, x^*(n+2)] \\ E[x(n-1)\, x^*(n+2)] \end{bmatrix}$$

$$= \begin{bmatrix} r_x(-2) \\ r_x(-3) \end{bmatrix} = \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{27} \end{bmatrix}$$

(b)

$$\sigma_d^2 = E[d(n)\, d^*(n)]$$

$$= E[x(n+2)\, x^*(n+2)]$$

$$= r_x(0) = 1$$

(c)

$$\begin{bmatrix} 1 & -\frac{1}{3} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{27} \end{bmatrix}$$

$$\begin{cases} w_1 - \frac{1}{3} w_2 = \frac{1}{9} \\ -\frac{1}{3} w_1 + w_2 = -\frac{1}{27} \end{cases} \Rightarrow \begin{cases} w_1 = \frac{1}{9} \\ w_2 = 0. \end{cases} \Rightarrow w_{opt} = \begin{bmatrix} \frac{1}{9} \\ 0 \end{bmatrix}$$

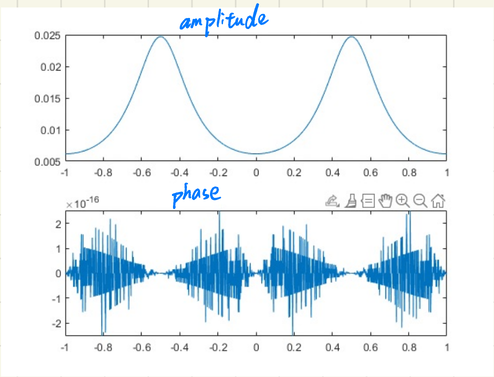
(d)

$$y(n) = W^H X(n)$$

$$= \frac{1}{9} X(n)$$

$$r_y(k) = \frac{1}{81}\left(\frac{-1}{3}\right)^{|k|} = \left(\frac{-1}{3}\right)^{|k|+4}$$

(e)




```
k = [-100:100];
fset = [-1:0.001:1];
rx = (-1/3).^abs(k);
s = zeros(size(fset));
for i = 1:length(s)
    tmp = exp(-j*2*pi*fset(i).*k);
    s(i) = 1/81*sum(tmp.*rx);
end
subplot(2,1,1);
plot(fset,abs(s));
subplot(2,1,2);
plot(fset,angle(s));
```

(f)

$$J_{min} = E[|e(n)|^2]$$

$$= \sigma_d^2 - W_{opt}^H p - p^H W_{opt} + W_{opt}^H R W_{opt}$$

$$= 1 - \frac{1}{81} - \frac{1}{81} + \frac{1}{81} = \frac{80}{81}$$

(g)

$$E[X(n+2)\, e_{opt}^*(n)]$$

$$= E[d(n)\,(d(n) - W^H X(n))^*]$$

$$= E[d(n)\,(d(n) - \tfrac{1}{9} X(n))^*]$$

$$= E[d(n)\,d^*(n) - \tfrac{1}{9} d(n)\, X^*(n)]$$

$$= r_x(0) - \frac{1}{9} r_x(-2) = 1 - \frac{1}{9} \times \frac{1}{9} = \frac{80}{81} \quad \#$$

5. $\min_{W \in \mathbb{C}^N} W^H R W$ subject to $W^H a = g$

$g \in \mathbb{C}$ $R \in \mathbb{C}^N \times \mathbb{C}^N$ $a \in \mathbb{C}^N$

$$\mathcal{L}(w, w^*, \mu_r, \mu_i) = w^H R w + Re\{(\mu_r + j\mu_i)^*(w^H a - g)\}$$

let $\mu = \mu_r + \mu_i$

$$\mathcal{L}(w, w^*, \mu) = w^H R w + \frac{\mu^*}{2}(w^H a - g) + \frac{\mu}{2}(w^T a^* - g^*)$$

$$\frac{\partial \mathcal{L}(w, w^*, \mu)}{\partial w^*} = Rw + \frac{\mu^*}{2} a + 0$$

let $\frac{\partial \mathcal{L}(w, w^*, \mu)}{\partial w^*} = 0 \Rightarrow Rw + \frac{\mu^*}{2} a = 0$

$$Rw = \left(\frac{-\mu^*}{2}\right) a$$

$$Rw = \left(\frac{-\mu_r + j\mu_i}{2}\right) a$$

#

b. $b_m(n) = C_M^H X_{m+1}(n)$

$$\mathbb{E}[b_m(n) x^*(n-k)]$$

$$= \mathbb{E}[C_M^H X_{m+1}(n) x^*(n-k)]$$

$$= \mathbb{E}\left[C_M^H \begin{bmatrix} x(n)\cdot x^*(n-k) \\ x(n-1)\cdot x^*(n-k) \\ \vdots \\ x(n-m)\cdot x^*(n-k) \end{bmatrix}\right]$$

$$= C_M^H \, \mathbb{E}\left[\begin{bmatrix} x(n)\cdot x^*(n-k) \\ x(n-1)\cdot x^*(n-k) \\ \vdots \\ x(n-m)\cdot x^*(n-k) \end{bmatrix}\right]$$

$$= C_M^H \begin{bmatrix} r(k) \\ r(k-1) \\ \vdots \\ r(k-m) \end{bmatrix}$$

from $\begin{bmatrix} R_M & (r_M^B)^* \\ (r_M^B)^T & r_0 \end{bmatrix} C_M = \begin{bmatrix} 0 \\ P_M \end{bmatrix}$

$$\Rightarrow P_M = [r(-M)\ r(-M+1) \cdots r(0)]\, C_M$$

$$P_M = C_M^H \begin{bmatrix} r(M) \\ r(M-1) \\ \vdots \\ r(0) \end{bmatrix}$$

$$0 = C_M^H \begin{bmatrix} r(t) \\ r(t-1) \\ \vdots \\ r(t-M) \end{bmatrix} \quad t \neq M$$

$$\mathbb{E}[b_m(n) x^*(n-k)] = \begin{cases} P_M^* & \text{if } k=m \\ 0 & \text{others} \end{cases}$$

$\because P_M \in \mathbb{R} \Rightarrow P_M^* = P_M$

$$\Rightarrow \mathbb{E}[b_m(n) x^*(n-k)] = \begin{cases} P_M & \text{if } k=m \\ 0 & \text{others} \end{cases} \quad \text{—— } 1$$

題目.

$$\mathbb{E}\left[b_m(n)\, b_i^*(n) \right]$$

$$= \mathbb{E}\left[b_m(n)\, x_{i+1}^H(n)\, C_i \right]$$

$$= \begin{bmatrix} E[b_m(n) x^*(n)] \\ E[b_m(n) x^*(n-1)] \\ \vdots \\ E[b_m(n) x^*(n-i)] \end{bmatrix}^T C_i$$

from ①, $E[b_m(n) x^*(n-i)] = \begin{cases} P_m & \text{if } i=m \\ 0 & \text{if } i= \text{others} \end{cases}$

set $i=m$

$$\begin{bmatrix} E[b_m(n) x^*(n)] \\ E[b_m(n) x^*(n-1)] \\ \vdots \\ E[b_m(n) x^*(n-i)] \end{bmatrix}^T C_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_M \end{bmatrix}^T \begin{bmatrix} -w_{b,1} \\ -w_{b,2} \\ \vdots \\ -w_{b,M} \\ 1 \end{bmatrix}$$

$$= P_M$$

if $i \neq m$ $\begin{bmatrix} E[b_m(n) x^*(n)] \\ \vdots \\ E[b_m(n) x^*(n-i)] \end{bmatrix}^T C_i = \begin{bmatrix} 0 \end{bmatrix}^T \begin{bmatrix} -w_{b,1} \\ -w_{b,2} \\ \vdots \\ -w_{b,M} \\ 1 \end{bmatrix} = 0$

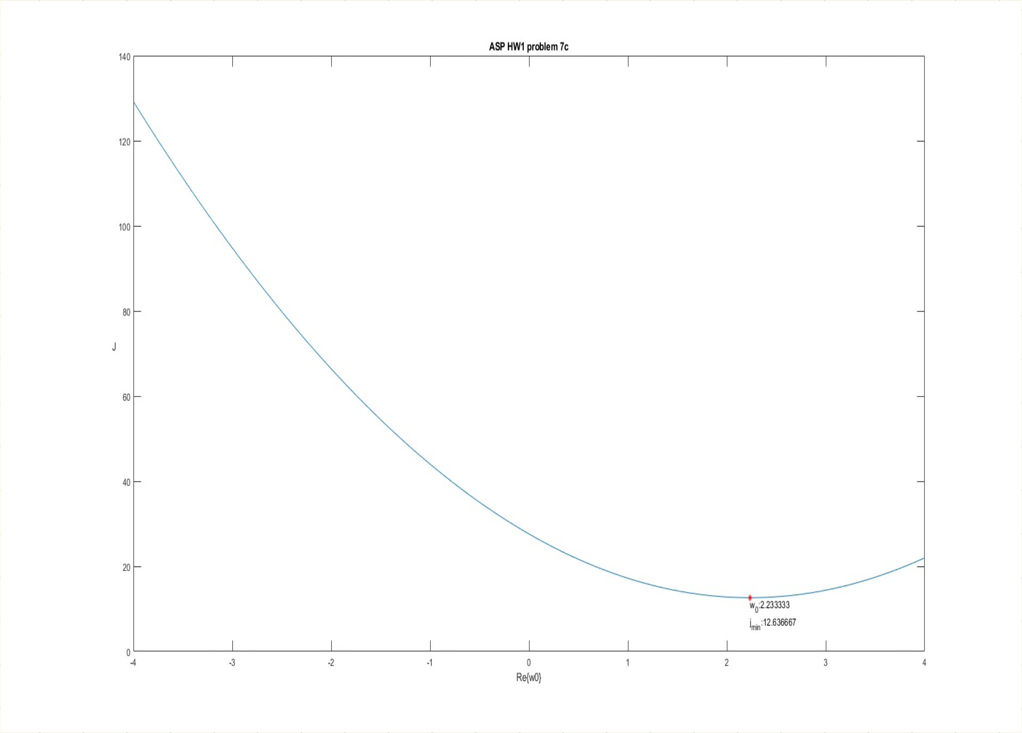
\#

7. (a). 見程式檔.

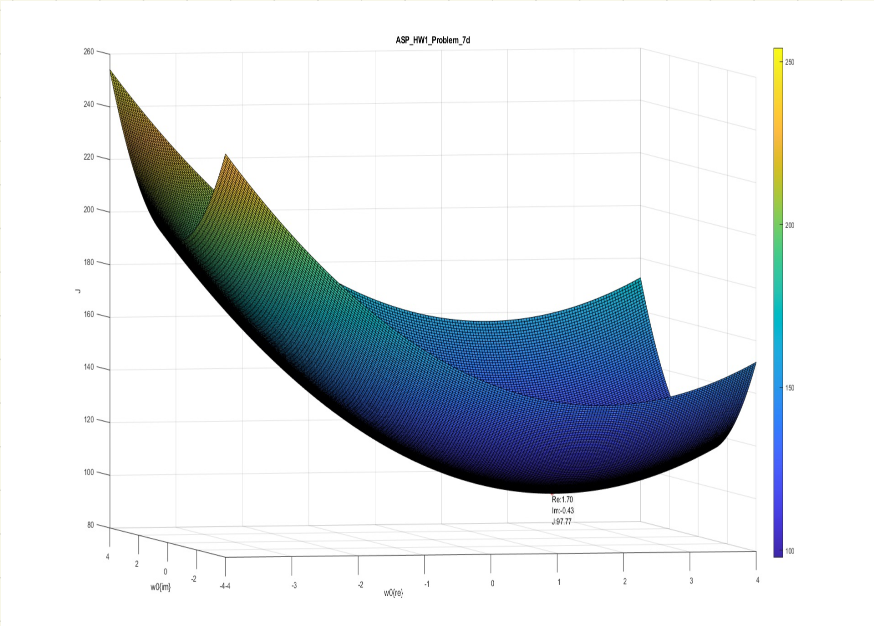
(b). result:

```
>> jmin

jmin =

   1.7116 + 0.0000i
```

(c). result:



(d) result:



(e) result: