INTERPOLATION FOR DEGREE 2 VERONESES OF ODD DIMENSION

RAY SHANG

Abstract. A classical fact is that through any $d+3$ general points in $\mathbb{P}^d$, there exists a unique rational normal curve of degree d passing through them. We generalize this by proving that, when n is odd, for any $\binom{n+2}{2}+n+1$ general points in $\mathbb{P}^{\binom{n+2}{2}-1}$, there exists at least $2^{n(n-1)}$ degree 2 Veroneses passing through them. This makes substantial progress on a question of Aaron Landesman and Anand Patel.

Contents

1. Introduction

The problem of interpolation is one of the oldest questions in mathematics, dating back to at least Euclid, who postulated in his *Elements* that there exists a unique line passing through any two distinct points in the plane. In the 18th century, interpolation began to be studied with the early tools of algebraic geometry. For example, in 1750, Cramer showed that plane curves of degree n can pass through $\frac{n(n+3)}{2}$ general points [Cra50] and, in 1779, Waring solved the interpolation problem for graphs of polynomial functions [War79]. In its most simple form, the problem of interpolation asks: does a projective variety of a given type always pass through a general collection of points (of a fixed number) in projective space?

The study of interpolation has a wide range of applications – for example, to Gromov-Witten theory, the slope conjecture, constructing degenerations, and understanding Hilbert schemes, as discussed in [LP16, Subsection 1.1], and to Larson's resolution of Severi's 1915 Maximal Rank Conjecture [Lar17]. Outside of mathematics, interpolation is important for Newton-Cotes method for numerical integration, for Shamir's cryptographic secret sharing protocol [Sha79], and for Reed-Solomon error-correcting codes [RS60].

In recent years, the literature on the modern study of interpolation has grown tremendously, especially for the interpolation problem of curves. In roughly chronological order, the development on answering the interpolation problem of curves follows the works of Sacchiero in 1980 [Sac81], Ellingsrud-Hirschowitz in 1984 [EH84], Perrin in 1987 [Per87], Stevens in 1989 and 1996 [Ste89] [Ste96], Ran in 2007 [Ran07], Atanasov in 2015 [Ata15], Atanasov-Larson-Yang in 2019 [ALY19], Vogt in 2018 [Vog18], and Larson-Vogt in 2021 and 2023 [LV21] [LV23]. These developments have led to the first comprehensive answer to the interpolation problem of curves, given by Larson-Vogt [LV23, Theorem 1.2]. See the introduction of Larson-Vogt (2023) for more details on this development [LV23, Introduction].

Much less is known about interpolation problems for higher dimensional varieties. To the author's knowledge, the following represents the current literature. In 1922, Arthur Coble showed that degree 2 Veronese surfaces satisfied interpolation [Cob22], where a modern exposition can be found in [LP16, Remark 5.4]. Weak interpolation was established by David Eisenbud and Sorin Popescu for scrolls of degree d and dimension k with $d \geq 2k - 1$ [EP00] in 2000, and interpolation for 2-dimensional scrolls was established by Izzet Coskun [Cos06] in 2006. These results were extended and unified by Aaron Landesman's work, which proved that smooth varieties of minimal degree satisfy interpolation [Lan16, Theorem 1.1]. Furthermore, Aaron Landesman and Anand Patel showed that all del Pezzo surfaces satisfy weak interpolation [LP16, Theorem 1.1].

1.1. Main Result.

This paper makes substantial progress on a question of Aaron Landesman and Anand Patel [LP16, Question 6, 7] and adds to the literature on interpolation problems for higher dimensional varieties, specifically for Veronese varieties. It is known that there exists a unique rational normal curve of degree d through any $d + 3$ general points in $\mathbb{P}^d$, that there are exactly four degree 2 Veronese surfaces through any 9 general points in $\mathbb{P}^5$, and that there are at least 630 degree 3 Veronese surfaces through any 13 general points in $\mathbb{P}^9$. In this paper, we prove the following.

Theorem 1.1. *Let n be a positive odd integer. There exists at least $2^{n(n-1)}$ degree 2 Veroneses of dimension n through any $\binom{n+2}{2} + n + 1$ general points in $\mathbb{P}^{\binom{n+2}{2}-1}$.*

Our proof strategy involves deformation theory and degeneration, and is inspired by Arthur Coble's work [Cob22]. Our techniques are quite different from those used in [LP16] and [Lan16], but more similar to the normal bundle interpolation approach found in [ALY19] and [LV23].

1.2. **Outline and notation.** In section 2, we state the formal definition of interpolation for varieties and its equivalence to normal bundle interpolation. In section 3, we first interpret Arthur Coble's classical argument from a deformation theory point of view in subsection 3.1 before stating our strategy for proving theorem 1.1 in 3.2. In sections 4 and 5 we carry out our strategy to prove theorem 1.1. We conclude in section 6 with some remarks and further questions.

Unless otherwise specified, always assume that the base field we are working with is an algebraically closed field of characteristic 0. Let $\mathrm{Hilb}^r_{P(t)}$ denote the Hilbert scheme parameterizing closed subschemes with Hilbert polynomial P^r in projective space $\mathbb{P}^r$. If $X \subset \mathbb{P}^r$ is a variety lying on a unique irreducible component of the Hilbert scheme, then let Hilb_X denote that irreducible component. Let $F\,\mathrm{Hilb}^r_{P(t)}$, where $P(t) = (P_1(t), \cdots, P_m(t))$, denote the flag Hilbert scheme parameterizing closed subschemes $X_1 \subset \cdots \subset X_m \subset \mathbb{P}^r$ where X_i has Hilbert polynomial $P_i(t)$. Let $v_{n,2} : \mathbb{P}^n \to \mathbb{P}^{\binom{n+2}{2}-1}$ denote a degree 2 Veronese embedding. Let Ver always denote the image of $v_{n,2}$ in $\mathbb{P}^{\binom{n+2}{2}-1}$.

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2. Interpolation in general

In this section, we recall the formal setup for interpolation problems and assert and provided references for the fact that, under certain hypotheses, interpolation and vector bundle interpolation for normal bundles are the same. A more complete exposition of the formal aspects of interpolation, as well as proofs of various equivalencies between different ways of proving interpolation, can be found in [LP16, Appendix A].

Interpolation problems, in their simplest form, may be thought of in the following way. Let H be the space of projective varieties of a given type. Suppose we wonder whether there always exists a projective variety of this type passing through M general points in $\mathbb{P}^r$. Then we are wondering whether the projection

$$\{(X, p_1, \cdots, p_M) | X \in H, p_i \in X\} \to \{(p_1, \cdots, p_M) | p_i \in \mathbb{P}^r\} \cong (\mathbb{P}^r)^M$$

has dense image. If the image is dense, this would mean for a general collection of M points, we can find a variety of the given type passing through them. The definition of interpolation we are about to present formalizes this idea.

Definition 2.1. *Let $X \subset \mathbb{P}^n$ be an integral projective scheme of dimension k lying on a unique irreducible component of the Hilbert scheme parameterizing closed subschemes with the same Hilbert polynomial. Let $\mathcal{H}_X$ denote the irreducible component, taken to have reduced structure, of the Hilbert scheme on which $[X]$ lies. Let $\mathcal{V}_X$ denote the universal family over $\mathcal{H}_X$.*

Let $\lambda := (\lambda_1, \cdots, \lambda_m)$ be an m-tuple of nonnegative integers such that

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$,
(2) *for all* $1 \leq i \leq m$, *we have* $0 \leq \lambda_i \leq n-k$,
(3) *and* $\sum_{i=1}^m \lambda_i \leq \dim \mathcal{H}_X$.

We call such a λ admissible. Let $\Lambda_i \subset \mathbb{P}^n$ be a plane of dimension $n-k-\lambda_i$, for $1 \leq i \leq m$. Define

$$\Psi := (\mathcal{V}_{\Lambda_1} \times_{\mathbb{P}^n} \mathcal{V}_X) \times_{\mathcal{H}_X} \cdots \times_{\mathcal{H}_X} (\mathcal{V}_{\Lambda_m} \times_{\mathbb{P}^n} \mathcal{V}_X).$$

Noting that $\mathcal{H}_{\Lambda_i} \cong \mathrm{Gr}(n-k-\lambda_i+1, n+1)$, consider the projection

$$\pi : \Psi \to \mathcal{H}_X \times (\prod_{i=1}^m \mathrm{Gr}(n-k-\lambda_i+1, n+1)) \times (\mathbb{P}^n)^m \to (\prod_{i=1}^m \mathrm{Gr}(n-k-\lambda_i+1, n+1)).$$

Defining q and r so that $\dim \mathcal{H}_X = q\dot{(}n-k) + r$ with $0 \leq r < n-k$, we say X, or $\mathcal{H}_X$, satisfies

(1) *λ-interpolation if the projection π is surjective,*
(2) *weak interpolation if $((n-k)^q)$-interpolation is satisfied,*
(3) *interpolation if $((n-k)^q, r)$-interpolation is satisfied,*
(4) *and strong interpolation if λ-interpolation is satisfied for all admissible λ.*

Remark 2.2. *There is a slightly more general version of definition 2.1, which also includes the notion of pointed interpolation (which is equivalent to interpolation, anyway). We have simplified the definition here for the purposes of this paper. The reader should refer to* [LP16, Definition A.3] *for the most general definition*

There is also a notion of vector bundle interpolation.

Definition 2.3. *Let λ be admissible. Let E be a locally free sheaf on a scheme X with $H^1(X, E) = 0$. Choose points $p_1, \cdots, p_m$ on X and vector subspace $V_i \subset E|_{p_i}$, for $1 \leq i \leq m$ with $\operatorname{codim} V_i = \lambda_i$. Then define E' so that*

$$0 \to E' \to E \to \bigoplus_{i=1}^m \frac{E|_{p_i}}{V_i} \to 0$$

is a short exact sequence of coherent sheaves. We say E satisfies λ-interpolation if there exists points $p_1, \cdots, p_n$ and subspaces $V_i \subset E|_{p_i}$ as above so that

$$h^0(E) - h^0(E') = \sum_{i=1}^m \lambda_i.$$

Write $h^0(E) = q \cdot \operatorname{rk} E + r$ *with* $0 \leq r < \operatorname{rk} E$. *We say* E *satisfies*

(1) *weak interpolation if it satisfies* $((\operatorname{rk} E)^q)$ *interpolation,*
(2) *interpolation if it satisfies* $((\operatorname{rk} E)^q, r)$ *interpolation,*
(3) *and strong interpolation if it satisfies* λ*-interpolation for all admissible* λ.

The following tells us that up to certain hypotheses, proving interpolation in the sense of definition 2.1 is equivalent to showing vector bundle interpolation for the normal bundle $\mathcal{N}_{X/\mathbb{P}^n}$ in the sense of definition 2.3.

Theorem 2.4. *Suppose* X *is an integral projective scheme over an algebraically closed field of characteristic 0. Furthermore, suppose* $H^1(X, \mathcal{N}_{X/\mathbb{P}^n}) = 0$ *and* X *is a local complete intersection. Then the following are equivalent:*

(1) X *satisfies interpolation*
(2) *The map* π *is dominant*
(3) X *satisfies strong interpolation*
(4) *The sheaf* $\mathcal{N}_{X/\mathbb{P}^n}$ *satisfies interpolation.*
(5) *The sheaf* $\mathcal{N}_{X/\mathbb{P}^n}$ *satisfies strong interpolation.*

Remark 2.5. *Assuming the hypotheses of theorem 2.4, there are actually 22 different ways of proving (strong) interpolation, as listed in* [LP16, Theorem A.7]. *Our theorem 2.4 is merely a subset of* [LP16, Theorem A.7] *so as to keep our discussion focused. We are mainly interested in item 4 of 2.4.*

Remark 2.6. *The sufficiency of proving interpolation of* $\mathcal{N}_{X/\mathbb{P}^n}$ *to show imply interpolation of* X *crucially depends on the characteristic 0 hypothesis. As shown in* [LP16, Corollary 7.2.9], *the degree 2 Veronese surface over an algebraically closed field of characteristic 2 is an example of a variety which satisfies interpolation, but whose normal bundle does.*

3. Our plan of attack for degree 2 Veroneses

In lieu of the previous section, this section explains our strategy for tackling the interpolation problem of degree 2 Veronese varieties. We begin by rephrasing our interpolation problem in terms of Veronese normal bundle interpolation, then interpret the work of Coble on interpolation for 2-Veronese surfaces from a deformation theoretic point of view, before finally describing our strategy in full. First, we have the following rephrasing of our problem.

Proposition 3.1. *Let Ver denote a smooth Veronese variety of dimension* n *and degree 2 over an algebraically closed field of characteristic 0. If there exists distinct points* p_i, *for* $1 \leq i \leq \binom{n+2}{2} + n + 1$, *on Ver such that*

$$H^0(Ver, \mathcal{N}_{Ver/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_{p_1, \cdots, p_{\binom{n+2}{2}+n+1}}) = 0,$$

then degree 2 Veronese varieties of dimension n satisfy interpolation.

Proof. Since Ver is smooth, it is a local complete intersection. Furthermore, a standard short exact sequence chase using the obvious Euler exact sequences and tangent-normal bundle short exact sequence shows that $H^1(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}) = 0$. Then by theorem 2.4, interpolation for these Veronese varieties is equivalent to interpolation for $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}$.

A standard short exact sequence chase using the obvious Euler exact sequences and tangent-normal bundle short exact sequence shows (assuming $n \geq 2$) that

$$H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}) = \binom{n+2}{2}^2 - (n+1)^2.$$

The rank of the normal bundle is $\binom{n+2}{2} - n - 1$. Thus, we see that the number of points in the vector bundle interpolation setup is $\binom{n+2}{2}+n+1$. Letting $V_i = \{0\}$, for $1 \leq i \leq \binom{n+2}{2}+n+1$, per the setup in definition 2.3, we see that the E' in question, per definition 2.3, is exactly $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_{p_1,\cdots,p_{\binom{n+2}{2}+n+1}}$. This can be seen by tensoring the ideal sheaf short exact sequence

$$0 \to \mathcal{I}_{p_1,\cdots,p_{\binom{n+2}{2}+n+1}} \to \mathcal{O}_{\mathrm{Ver}} \to \bigoplus_{i=1}^{\binom{n+2}{2}+n+1} k(p_i) \to 0$$

by $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}$, where the quotient sheaf is supported at the points p_i. Altogether, the dimension of space of global sections of the normal bundle, the rank of the normal bundle, and the number of points implies that to show interpolation for $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}$, we need to find distinct points $p_i \in \mathrm{Ver}$ such that

$$H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_{p_1,\cdots,p_{\binom{n+2}{2}+n+1}}) = 0.$$

□

Thus, the main objective of this paper is to show that the hypothesis of proposition 3.1 is true. To motivate our strategy for accomplishing this, we discuss Arthur Coble's proof of interpolation for degree 2 Veronese surfaces.

3.1. **Arthur Coble's argument.** In 1922, Arthur Coble proved that there were exactly four degree 2 Veronese surfaces through any 9 general points in $\mathbb{P}^5$. His proof crucially relies on the following lemma.

Lemma 3.2. [Cob22] *There exists a unique elliptic normal sextic (genus 1 and degree 6) curve through any 9 general points in $\mathbb{P}^5$.*

For Coble's argument, an elliptic normal sextic acts as a sort of auxillary curve for the 2-Veronese surfaces. This is made precise by the following.

Proposition 3.3. *Fix 9 general points in $\mathbb{P}^5$. Let C be the unique genus 1 curve in $\mathbb{P}^5$, embedded by a degree 6 line bundle $\mathcal{O}_C(1)$, passing through those 9 general points. There are natural bijections between the following.*

(1) *The set of 2-Veronese surfaces passing through those 9 general points.*
(2) *The set of 2-Veronese surfaces containing C.*
(3) *The set of degree 3 line bundles $\mathcal{L}$ on C such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(1)$.*

Proof. This follows from [LP16, Proposition 5.4, Theorem 5.6]. $\square$

Remark 3.4. *In particular, proposition 3.3 implies that there are exactly four 2-Veronese surfaces through any 9 general points. This is because the cardinality of (3) is the same cardinality as the set of 2-torsion line bundles in the Picard group* $\mathrm{Pic}^0(C)$*. Since C is a compact Riemann surface of genus 1, we have* $\mathrm{Pic}^0(C) \cong \frac{\mathbb{C}}{\Lambda} \cong S^1 \times S^1$.

Let us interpret proposition 3.3 from the viewpoint of deformation theory. In general, let $Y \subset X \subset \mathbb{P}^r$ be smooth subvarieties, such that $[Y]$ and $[X]$ are smooth points of Hilb_Y and Hilb_X. Let $P(t) = (P_Y(t), P_X(t))$ be a tuple of their respective Hilbert polynomials. The flag Hilbert scheme $F\,\mathrm{Hilb}^r_{P(t)}$ then has tangent space

$$T_{(Y,X)}F\,\mathrm{Hilb}^r_{P(t)} = \ker[H^0(X, \mathcal{N}_{X/\mathbb{P}^r}) \oplus H^0(Y, \mathcal{N}_{Y/\mathbb{P}^r}) \to H^0(Y, \mathcal{N}_{X/\mathbb{P}^r}|_Y)]$$

at $([Y],[X])$ by [Ser06, Remark 4.5.4]. Note that both $F\,\mathrm{Hilb}^r_{P(t)}$ and $\mathrm{Hilb}_X \times \mathrm{Hilb}_Y$ are closed subschemes of $\mathrm{Hilb}^r_{P_Y(t)} \times \mathrm{Hilb}^r_{P_X(t)}$. Let

$$F\,\mathrm{Hilb}_{Y,X} := F\,\mathrm{Hilb}^r_{P(t)} \times_{(\mathrm{Hilb}^r_{P_Y(t)} \times \mathrm{Hilb}^r_{P_X(t)})} (\mathrm{Hilb}_X \times \mathrm{Hilb}_Y).$$

Note that $F\,\mathrm{Hilb}^r_{P(t)}$ comes with a natural projection $F\,\mathrm{Hilb}^r_{P(t)} \to \mathrm{Hilb}^r_{P_Y(t)}$, so we can restrict this projection to $F\,\mathrm{Hilb}_{Y,X}$. Then let Hilb^Y_X denote the scheme-theoretic fiber of $F\,\mathrm{Hilb}_{Y,X} \to \mathrm{Hilb}^r_{P_Y(t)}$ over $[Y]$. The tangent space of Hilb^Y_X at $([Y],[X])$ is $H^0(X, \mathcal{N}_{X/\mathbb{P}^r} \otimes \mathcal{I}_Y)$.

Let S be a degree 2 Veronese surface. Let P_9 be the subvariety that is the 9 points $p_1, \cdots, p_9 \in \mathbb{P}^5$ in general position, and C the unique genus 1 curve passing through them, as in proposition 3.3. Then lemma 3.2 implies that $\mathrm{Hilb}^S_{P_9}$ and Hilb^S_C are isomorphic schemes, which can be verified by comparing the functors they represent. Furthermore, set (3) of proposition 3.3 is the underlying closed points of a translate of the kernel subscheme $\ker[\mathrm{sq} : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C)]$ of the squaring endomorphism on the Picard group of C. This kernel subscheme is a reduced scheme of finitely many points. Since proposition 3.3 implies we have a bijection on $\mathbb{C}$-points between Hilb^S_C and $\ker[\mathrm{sq} : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C)]$, and the former is connected and the latter is normal, we have an isomorphism of schemes [hb]. Since a reduced scheme of finitely many points has no tangent vectors, this implies

$$H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^5} \otimes \mathcal{I}_{p_1,\cdots,p_9}) = 0.$$

3.2. **Our strategy.** Recall that, by proposition 3.1, to show interpolation for degree 2 Veronese varieties, it suffices to demonstrate that there exists distinct points p_i, for $1 \leq i \leq \binom{n+2}{2} + n + 1$, such that

$$H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_{p_1,\cdots,p_{\binom{n+2}{2}+n+1}}) = 0.$$

Our strategy is the following.

(1) Find a smooth curve $C \subset \mathrm{Ver} \subset \mathbb{P}^{\binom{n+2}{2}-1}$ of degree $n(n+1)$ and genus $\frac{n(n-1)}{2}$ on the 2-Veronese variety such that $H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_C) = 0$,

(2) and find points $p_1, \cdots, p_{\binom{n+2}{2}+n+1}$ on C such that

$$H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_{p_1,\cdots,p_{\binom{n+2}{2}+n+1}}) \cong H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_C).$$

Let us first explain where the degree and genus specification comes from. It relates to a higher dimensional version of Coble's argument: if smooth C is embedded in $\mathbb{P}^{\binom{n+2}{2}-1}$ by a line bundle $\mathcal{O}_C(1)$, then we would like for there to be a bijection between

$$\{\text{degree 2 Veronese varieties containing } C\} \text{ and } \{\mathcal{L} \in \mathrm{Pic}(C) | \mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(1)\}. \qquad (1)$$

This is because, if Ver is a degree 2 Veronese variety of dimension n containing C, then if this bijection holds, then we have a scheme-theoretic isomorphism between $\mathrm{Hilb}_C^{\mathrm{Ver}}$ and a translate of $\ker[\mathrm{sq} : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C)]$, by the same argument discussed in subsection 3.1. This would imply $H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_C) = 0$. We have this bijection when the following holds.

Proposition 3.5. *Suppose C is a smooth curve in $\mathbb{P}^{\binom{n+2}{2}-1}$ and all of the square roots of $\mathcal{O}_C(1)$ are very ample. We have bijection 1 between*

(1) *$\{$degree 2 Veronese varieties containing $C\}$ and*

(2) *$\{\mathcal{L} \in \mathrm{Pic}(C) | \mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(1)\}$*

when $\dim H^0(C, \mathcal{O}_C(1)) = \binom{n+2}{2}$ and when $\dim H^0(C, \mathcal{L}) = n+1$ for all $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(1)$.

Proof. Suppose we have a degree 2 Veronese variety which contains C. Suppose the Veronese is embedded by $v_{n,2} : \mathbb{P}^n \to \mathbb{P}^{\binom{n+2}{2}-1}$. Then we obtain a line bundle $\mathcal{L} \cong v_{n,2}^* \mathcal{O}_C(1)$ which squares to $\mathcal{O}_C(1)$.

On the other hand, suppose we have a line bundle $\mathcal{L}$ on C which squares to $\mathcal{O}_C(1)$. By assumption, $\dim H^0(C, \mathcal{L}) = n+1$, and $\dim H^0(C, \mathcal{O}_C(1)) = \binom{n+2}{2}$. Then the complete linear system $|\mathcal{L}|$ embeds the curve into $\mathbb{P}^n$, and composing this map with the degree 2 Veronese embedding on $\mathbb{P}^n$, we obtain a degree 2 Veronese variety containing C. We would like to show that this degree 2 Veronese variety is uniquely associated to the isomorphism class of $\mathcal{L}$. So suppose that we have two degree 2 Veronese varieties X_1 and X_2 in $\mathbb{P}^{\binom{n+2}{2}-1}$ which contain C. Then let $v_{n,2}^{(1)}, v_{n,2}^{(2)} : \mathbb{P}^n \to \mathbb{P}^{\binom{n+2}{2}-1}$ denote embeddings whose images are X_1 and X_2, respectively. Furthermore, suppose ϕ_1 and ϕ_2 are embeddings (which may be different, although projectively equivalent) of the curve in $\mathbb{P}^n$ by global sections of $\mathcal{L}$ such that $v_{n,2}^{(1)} \circ \phi_1$ and $v_{n,2}^{(2)} \circ \phi_2$ both yield C. But there is an automorphism σ of $\mathbb{P}^{\binom{n+2}{2}-1}$ which sends X_1 to X_2. If we let $C_2 \subset X_2$ denote the image of C under σ, then if we pull both curves C_2 and C back along $v_{n,2}^{(2)}$, then we see that there is an automorphism of $\mathbb{P}^n$ sending the pull back of C_2 to the pull back of C, and thus there is an automorphism σ' of $\mathbb{P}^{\binom{n+2}{2}-1}$

stabilizing X_2 and sending C_2 to C. Then the images of the embeddings $\sigma' \circ \sigma \circ v_{n,2}^{(1)}$ and $v_{n,2}^{(2)}$ both give X_2, and the compositions $\sigma' \circ \sigma \circ v_{n,2}^{(1)} \circ \phi_1$ and $v_{n,2}^{(2)} \circ \phi_2$ both give C. But altogether, this tells us that we have found an automorphism of $\mathbb{P}^{\binom{n+2}{2}-1}$ which sends X_1 to X_2 and is identity on C. But the span of C is all of $\mathbb{P}^{\binom{n+2}{2}-1}$, which implies that since the automorphism is identity on C, it must be identity everywhere. In particular, this forces $X_1 = X_2$. □

Thus, if we can find a smooth curve C of degree $n(n+1)$ and genus $\frac{n(n-1)}{2}$ embedded by $\mathcal{O}_C(1)$ where $H^1(C, \mathcal{O}_C(1)) = 0$, and for all square roots $\mathcal{L}$ of $\mathcal{O}_C(1)$ we have $H^1(C, \mathcal{L}) = 0$, then by Riemann-Roch, we have

$$\dim H^0(C, \mathcal{O}_C(1)) = \binom{n+2}{2} \text{ and } \dim H^0(C, \mathcal{L}) = n+1.$$

This would then be a smooth curve C such that $H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_C) = 0$, as desired. Explicitly constructing such a curve is hard. Instead, we construct a degenerate curve with at worst nodal singularities and deform it to a smooth curve with the desired properties.

4. Rational normal curve chain

Fix a degree 2 Veronese embedding $v_{n,2} : \mathbb{P}^n \to \mathbb{P}^{\binom{n+2}{2}}$, where n is odd. We now construct our degenerate curve, in $\mathbb{P}^n$, which will have degree $\frac{n(n+1)}{2}$ and arithmetic genus $\frac{n(n-1)}{2}$.

Consider $\frac{n+1}{2}$ rational normal curves R_i, for $1 \leq i \leq \frac{n+1}{2}$. Note $\frac{n+1}{2}$ is a positive integer since n is odd. Glue R_1 and R_2 at $n+1$ nodal intersections, where the $n+1$ points of intersection are in general position. Note we can find such rational normal curves because $n+3$ points in general position determine a unique rational normal curve, and from this we deduce that the space of rational normal curves passing through the $n+1$ points has dimension $2(n-1)$.

Then glue R_2 and R_3 at $n+1$ nodal intersections, where again the points of intersection are in general position with respect to each other, and distinct from the previous $n+1$ nodal intersections between R_1 and R_2. Continue this gluing procedure for R_i and R_{i+1}, where $3 \leq i \leq \frac{n-1}{2}$. We claim that these rational normal curves can be chosen so that they do not intersect anywhere else besides at the specified nodal intersections.

Lemma 4.1. *In general, two rational normal curves in $\mathbb{P}^n$ intersecting nodally at $n+1$ general points do not intersect anywhere else.*

Proof. Fix points $p_1, \cdots, p_{n+1}$ in general position. The space of rational normal curves through these $n+1$ points has dimension $2n-2$. Then the locus Σ of pairs of distinct rational normal curves passing through $p_1, \cdots, p_{n+1}$ has dimension $4n-4$. Let

$$\Phi \subseteq \Sigma \times (\mathbb{P}^n \setminus \{p_1, \cdots, p_{n+1}\})$$

denote the incidence correspondence given by $\{(R, R', p) | p \in R \cap R'\}$. Note $R \cap R'$ is either empty or a point. If we project Φ to $\mathbb{P}^n \setminus \{p_1, \cdots, p_{n+1}\}$, then the fiber of a general point $p_{n+2} \in \mathbb{P}^n \setminus \{p_1, \cdots, p_{n+1}\}$ will be the space of pairs of distinct rational normal curves passing through $p_1, \cdots, p_{n+2}$. But this space has dimension $2n-2$. Thus, $\dim \Phi = n+2n-2 = 3n-2$, so the image of Φ in its projection to Σ has dimension at most $3n-2$, while $\dim \Sigma = 4n-4$. Since we are implicitly in the case of $n \geq 3$, we obtain our conclusion. □

If $n = 3$, then we have finished showing that the rational normal curve chain does not intersect anywhere else besides at the prescribed nodal intersections. Now suppose $n \geq 5$. We prove the following.

Lemma 4.2. *In general, three rational normal curves R_1, R_2, and R_3, where R_1, R_2 are glued nodally at $n+1$ general points and R_2 and R_3 are glued nodally at different $n+1$ general points, do not intersect anywhere else.*

Proof. Suppose we fix R_2 and points $\{p_1, \cdots, p_{n+1}\}$ in general position on R_2, and another set of points $\{q_1, \cdots, q_{n+1}\}$ in general position on R_2. We know by the previous lemma that, in general, the R_1 and R_3 we pick to go through $p_1, \cdots, p_{n+1}$ and $q_1, \cdots, q_{n+1}$, respectively, will both not intersect R_2 anywhere else. Then it remains to show that R_1 and R_3 do not intersect anywhere else.

Let Σ_1 denote the locus of rational normal curves which pass through $p_1, \cdots, p_{n+1}$. Let Σ_3 denote the locus of rational normal curves which pass through $q_1, \cdots, q_{n+1}$. We have $\dim \Sigma_1 = \dim \Sigma_3 = 2n-2$. Let

$$\Phi \subseteq \Sigma_1 \times \Sigma_3 \times (\mathbb{P}^n \setminus \{p_j, q_j\})$$

be the incidence correspondence given by $\{(R_1, R_3, p) | R_i \in \Sigma_i, p \in R_1 \cap R_3\}$. The fiber of the projection Φ to $\mathbb{P}^n \setminus \{p_j, q_j\}$ over a general point will have dimension $2n-2$. Then $\dim \Phi = 2n-2+n = 3n-2$. Then the dimension of the projection of Φ to $\Sigma_1 \times \Sigma_3$ is at most $3n-2$, while $\dim \Sigma_1 \times \Sigma_3 = 4n-4$. Thus, in general, R_1 and R_3 will not intersect anywhere else and will not intersect R_2 anywhere else. □

This case of three rational normal curves shows us how to do the inductive step in general.

Lemma 4.3. *Let n be an odd positive integer such that $n \geq 3$. The rational normal curves $R_1, \cdots, R_{\frac{n+1}{2}}$ in the rational normal curve chain in $\mathbb{P}^n$ can be chosen in such a way that there are no other intersections besides the prescribed nodal intersections.*

Proof. We showed the base case with just two rational normal curves. Suppose we have shown that, in general, a chain of i rational normal curves do not intersect anywhere else for $i < \frac{n+1}{2}$. Now show the same conclusion holds for a chain of $i+1$ rational normal curves. Pick some chain of $i-2$ rational normal curves $R_2, \cdots, R_{i-1}$ which do not intersect anywhere else except at the prescribed nodal intersections. Choose $p_1, \cdots, p_{n+1}$ in general position on R_2 which are distinct from the points of $R_2 \cap R_3$, and choose $q_1, \cdots, q_{n+1}$ in general position on R_{i-1}, which are distinct from the points of $R_{i-2} \cap R_{i-1}$.

In general, we can pick R_1 passing through $p_1, \cdots, p_{n+1}$ which does not intersect $R_2, \cdots, R_{i-1}$ anywhere else, and we can pick R_i passing through $q_1, \cdots, q_{n+1}$ which does not intersect $R_2, \cdots, R_{i-1}$ anywhere else. Then the proof that R_1 and R_i do not intersect anywhere else is analogous to the proof given in lemma 4.2. □

Now that we have constructed the rational normal curve chain $C \subset \mathbb{P}^n$, we verify the following, which will allow us to deform the rational normal curve chain to a smooth curve.

Proposition 4.4. *Let $C \subset \mathbb{P}^n$ be a rational normal curve chain. Then*

$$H^1(C, \mathcal{O}_C(1)) = H^1(C, \mathcal{O}_C(2)) = 0.$$

Proof. Showing $H^1(C, \mathcal{O}_C(2)) = 0$ is the same as showing $H^1(C, \mathcal{O}_C(1)) = 0$, so for brevity we prove the latter. Let v denote the normalization map

$$v : \bigsqcup_{i=1}^{\frac{n+1}{2}} R_i \to C.$$

Note the restriction $\mathcal{O}_{R_i}(1)$ of $\mathcal{O}_C(1) := \mathcal{O}_{\mathbb{P}^n}(1)|_C$ to rational normal curve R_i is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(n)$. The line bundle $\mathcal{O}_C(1)$ is equivalent to the data of the line bundles $\mathcal{O}_{R_i}(1)$ on each R_i along with identifications of these line bundles over the $\frac{n-1}{2} \cdot (n+1)$ nodal intersection points. This identification data is, for each $p_{j,k}$ where $1 \le j \le \frac{n-1}{2}$ and $1 \le k \le n+1$, a nonzero scalar c_{jk} which specifies an isomorphism between the fiber of $\mathcal{O}_{R_j}(1)$ and the fiber of $\mathcal{O}_{R_{j+1}}(1)$ over p_{jk}. Pushing and pulling $\mathcal{O}_C(1)$ along the normalization map, we obtain the short exact sequence

$$0 \to \mathcal{O}_C(1) \to v_* v^* \mathcal{O}_C(1) \to Q \to 0.$$

In particular, $\mathcal{O}_C(1) \to v_* v^* \mathcal{O}_C(1)$ is not just an injective map of sheaves, but an actual injection of vector bundles on fibers. Away from the nodal intersections p_{jk} we have an isomorphism of fibers, but at the points p_{jk} we have an injection of fibers

$$\mathcal{O}_C(1)(p_{jk}) \to \mathcal{O}_{R_j}(1)(p_{jk}) \oplus \mathcal{O}_{R_{j+1}}(1)(p_{jk}).$$

We see that Q is supported only at the nodal intersections p_{jk} and its fiber over p_{jk} is the quotient

$$\frac{\mathcal{O}_{R_j}(1)(p_{jk}) \oplus \mathcal{O}_{R_{j+1}}(1)(p_{jk})}{\mathcal{O}_C(1)(p_{jk})}$$

where the denominator here really means the image of $\mathcal{O}_C(1)(p_{jk})$. Taking the long exact sequence in sheaf cohomology yields

$$0 \to H^0(C, \mathcal{O}_C(1)) \to \bigoplus_{i=1}^{\frac{n+1}{2}} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \to^{\alpha} H^0(C, Q) \to H^1(C, \mathcal{O}_C(1)) \to 0.$$

Then $H^1(C, \mathcal{O}_C(1)) = 0$ if and only if the map α is surjective. Note

$$H^0(C, Q) \cong \bigoplus_{1 \le j \le \frac{n-1}{2}, 1 \le k \le n+1} \frac{\mathcal{O}_{R_j}(1)(p_{jk}) \oplus \mathcal{O}_{R_{j+1}}(1)(p_{jk})}{\mathcal{O}_C(1)(p_{jk})}.$$

Focusing on just the p_{11} term for now, there is certainly a homogeneous degree n polynomial of $H^0(R_1, \mathcal{O}_{R_1}(1)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ which vanishes at p_{11}, and there is certainly a homogeneous degree n polynomial of $H^0(R_2, \mathcal{O}_{R_2}(1)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ which does not vanish at p_{11}. The images of these two sections map to an element of $H^0(C, Q)$ which does not vanish under the quotient. Thus, we have surjectivity over p_{11}. An analagous argument can be made for each p_{jk}, so that α is surjective, and thus $H^1(C, \mathcal{O}_C(1)) = 0$. □

Finally, we show there exists $\binom{n+2}{2} + n + 1$ smooth points (in other words, the points are chosen so that they are not the nodal singularities) on the rational normal curve chain C, such that the following holds.

Proposition 4.5. *There exists smooth points $p_1, \cdots, p_{\binom{n+2}{2}+n+1}$ on the rational normal curve chain C such that*

$$H^0(C, \mathcal{N}_{Ver/\mathbb{P}^{\binom{n+2}{2}-1}}|_C \otimes \mathcal{I}_{p_1, \cdots, p_{\binom{n+2}{2}+n+1}}) = 0.$$

Proof. The vector bundle $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_C$ is equivalent to the data of the vector bundles $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_j}$ and linear isomorphisms ϕ_{jk} identifying the fiber of $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_j}$ over p_{jk} with the fiber of $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_{j+1}}$ over p_{jk}. Note

$$\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_j} \cong \bigoplus_{i=1}^{\frac{n(n+1)}{2}} \mathcal{O}_{\mathbb{P}^1}(2n+2)$$

by [Sha, Theorem 4.3]. Pick any $2n+3$ smooth points on R_1, then pick $n+2$ smooth points on each of the other $\frac{n-1}{2}$ rational normal curves. Twisting down $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_1}$ by $2n+3$ points and, for $j \neq 1$, twisting down $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_j}$ by $n+2$ points yields

$$\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_1} \otimes \mathcal{I}_{p_1, \cdots, p_{2n+3}} \cong \bigoplus_{i=1}^{\frac{n(n+1)}{2}} \mathcal{O}_{\mathbb{P}^1}(-1) \text{ and}$$

$$\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_j} \otimes \mathcal{I}_{p_{2n+3+(n+2)(j-1)+1}, \cdots, p_{2n+3+(n+2)j}} \cong \bigoplus_{i=1}^{\frac{n(n+1)}{2}} \mathcal{O}_{\mathbb{P}^1}(n), \text{ for } 2 \le j \le \frac{n+1}{2}.$$

A global section s of $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_C \otimes \mathcal{I}_{p_1, \cdots, p_{\binom{n+2}{2}+n+1}}$ is equivalent to the data of a global section s_1 of $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_1} \otimes \mathcal{I}_{p_1, \cdots, p_{2n+3}}$ and global sections s_j of $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{R_1} \otimes \mathcal{I}_{p_1, \cdots, p_{2n+3}}$, for $j \neq 1$, where s_i and s_{i+1} are compatible with the identification ϕ_{ik} of fibers over p_{ik}. So given such a global section s, note that s_1 must vanish completely over R_1. Furthermore, since s_2 and s_1 are compatible over the points $\{p_{1k}\}_{1 \le k \le n+1}$, we must have s_2, a tuple of homogeneous polynomial of degree n, vanish at the $n+1$ nodal intersections, which forces

s_2 to be a tuple of zero sections, thus vanishing everywhere on R_2. Again by compatibility, this forces s_3 to vanish everywhere on R_3, and so on. This implies that all the s_j must vanish, thus $s = 0$. $\square$

5. Smoothing the rational normal curve chain

In this section we smooth our rational normal curve chain, and obtain a desired curve in the sense of subsection 3.2. The end of this section will conclude our proof of interpolation for degree 2 Veroneses of odd dimension along with the enumerative lower bound.

Fix a degree 2 Veronese embedding $v_{n,2} : \mathbb{P}^n \to \mathbb{P}^{\binom{n+2}{2}-1}$, where n is odd. In section 4 we constructed a rational normal curve chain $C \subset \mathbb{P}^n$ with degree $\frac{n(n+1)}{2}$ and arithmetic genus $\frac{n(n-1)}{2}$ with $H^1(C, \mathcal{O}_C(1)) = H^1(C, \mathcal{O}_C(2)) = 0$ by proposition 4.4. Since the rational normal curve chain C is reduced with nodal singularities and $H^1(C, \mathcal{O}_C(1)) = 0$, we have by [Har10, Proposition 29.9] that there exists a flat family $\chi \subseteq \mathbb{P}^n \times T$ over an integral scheme of finite type T, such that the fiber of χ over 0 is the rational normal curve chain C, while there are fibers χ_p for $p \neq 0$ such that χ_p is smooth. Furthermore, since T is an integral noetherian scheme, all fibers of the family χ share the same Hilbert polynomial by [Har, Chapter 3 Proposition 9.9]. This implies that we can smooth the rational normal curve chain to a smooth $\tilde{C} \subseteq \mathbb{P}^n$ of degree $n(n+1)$ and genus $\frac{n(n-1)}{2}$. Since the projection $\chi \to T$ is a proper flat morphism of finite presentation with T being irreducible, we have that existence of a smooth fiber implies that the projection is smooth over a dense open subset of T by properness and [Gro67, IV-4-12.2.4]. In particular, this means that there is a dense open subset of the base T, on which the fibers of χ are smooth. By [Har, Chapter 3, Theorem 12.8], the function $\dim H^1(\chi_p, \mathcal{O}_{\chi_p}(1))$ is an upper-semicontinuous function of $p \in T$. We conclude that there is a dense open U of T on which, for $p \in U$, we have χ_p is a smooth curve of degree $\frac{n(n+1)}{2}$ and genus $\frac{n(n-1)}{2}$, embedded in $\mathbb{P}^n$ by a nonspecial line bundle $\mathcal{O}_{\chi_p}(1)$. Furthermore, we can embed this family χ into $\mathbb{P}^{\binom{n+2}{2}-1} \times T$ by the map $v_{n,2} \times \mathrm{id}$. Again, by upper-semicontinuity, we have $H^1(\chi_p, \mathcal{O}_{\chi_p}(2)) = 0$ for p in a dense open subset $U \subseteq T$. This argument shows that we can smooth the rational normal curve chain to a smooth curve $C' \subset \mathbb{P}^n$ with the desired genus and degree, and whose very ample line bundles $\mathcal{O}_{C'}(1)$ and $\mathcal{O}_{C'}(2)$ are nonspecial. However, we would like to find C' which satisfies proposition 1. Thus, we demonstrate the following.

Proposition 5.1. *We can smooth the rational normal curve chain to a smooth curve $\tilde{C} \subset \mathbb{P}^n$ of degree $\frac{n(n+1)}{2}$ and genus $\frac{n(n-1)}{2}$, such that $\mathcal{O}_{\tilde{C}}(2)$ and all of its square roots are very ample nonspecial line bundles.*

Proof. Let $H^n_{g,NS,\frac{n(n+1)}{2}}$ denote the unique irreducible component of the Hilbert scheme parameterizing genus g, nonspecial curves of degree $\frac{n(n+1)}{2}$ in $\mathbb{P}^n$ [Kee22, Page 2]. The family $\chi|_U \to U$ is obtained by pullback along a morphism $U \to H^n_{g,NS,\frac{n(n+1)}{2}}$.

Let $M_{\frac{n(n-1)}{2}}$ denote the moduli space of genus $\frac{n(n-1)}{2}$ curves. There is a universal squaring map between the universal Picard bundles of degrees $\frac{n(n+1)}{2}$ and $n(n+1)$:

$$\begin{tikzcd} \mathrm{Pic}^{\frac{n(n+1)}{2}} \arrow[rr, "sq"] \arrow[dr] & & \mathrm{Pic}^{n(n+1)} \arrow[dl] \\ & M_{\frac{n(n-1)}{2}} & \end{tikzcd}$$

Note that $\mathrm{Pic}^{\frac{n(n+1)}{2}}$ and $\mathrm{Pic}^{n(n+1)}$ are irreducible. Note the locus of nonspecial line bundles is a Zariski open of $\mathrm{Pic}^{n(n+1)}$. The locus which parameterizes pairs (C, L), where C is a genus g curve and L is a line bundle whose square roots are all non-special, is also Zariski open. By irreducibility, their intersection is a dense open parameterizing pairs (C, L) where L is a non-special line bundle whose square roots are all non-special. Denote this intersection as $U_{NS,\sqrt{NS}}$. By irreducibility, $sq^{-1}(U_{NS,\sqrt{NS}})$ is a dense open subset of $\mathrm{Pic}^{\frac{n(n+1)}{2}}$.

Furthermore, consider the natural map $\pi_L : H^n_{g,NS,\frac{n(n+1)}{2}} \to \mathrm{Pic}^{\frac{n(n+1)}{2}}$ which sends closed points $[C]$ to $(C, \mathcal{O}_C(1))$. The image of π_L contains the open set which is the intersection of the locus of very ample bundles and the locus of non-special line bundles in $\mathrm{Pic}^{\frac{n(n+1)}{2}}$. Since this open set is dense by irreducibility, this implies that the image of π_L intersects with $sq^{-1}(U_{NS,\sqrt{NS}})$. Taking preimages, there is an open subset of $U \subset H^n_{g,NS,\frac{n(n+1)}{2}}$ such that for $[C] \in U$, we have $\mathcal{O}_C(2)$ and all of its square roots are non-special very ample line bundles. By irreducibility, U is dense in $H^n_{g,NS,\frac{n(n+1)}{2}}$.

Now note that $T \to H^n_{g,NS,\frac{n(n+1)}{2}}$ is a morphism whose domain is an integral scheme. Then it factors through the reduced locus

$$T \to H^{n,\mathrm{red}}_{g,NS,\frac{n(n+1)}{2}} \to H^n_{g,NS,\frac{n(n+1)}{2}}$$

by [Sta24, lemma 26.12.7]. In particular, the rational normal curve chain is a point in the reduced locus which is open and thus an integral finite type $\mathbb{C}$-scheme. The reduced locus intersects U nontrivially by density. Then this reduced locus provides a family in which we can smooth the rational normal curve chain to smooth curve $\tilde{C} \subset \mathbb{P}^n$ of degree $\frac{n(n+1)}{2}$ and genus $\frac{n(n-1)}{2}$, such that $\mathcal{O}_{\tilde{C}}(2)$ and all of its square roots are very ample nonspecial line bundles. $\square$

Applying Riemann-Roch to $\mathcal{O}_{\tilde{C}}(2)$ and all of its square roots implies that the dimension of their space of global sections are $\binom{n+2}{2}$ and $n+1$, respectively. Then proposition 1 holds for $v_{n,2}(\tilde{C})$. In particular, the discussion thereafter implies

$$H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_{v_{n,2}(\tilde{C})}) = 0.$$

Furthermore, recall that we showed in proposition 4.5 that there exists smooth points $p_1, \cdots, p_{\binom{n+2}{2}+n+1}$ on the rational normal curve chain C such that

$$H^0(C, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_C \otimes \mathcal{I}_{p_1,\cdots,p_{\binom{n+2}{2}+n+1}}) = 0.$$

Since these points are smooth, as we deform the rational normal curve chain to $v_{n,2}(\tilde{C})$, these points will deform to smooth points $\widetilde{p_1}, \cdots, \widetilde{p_{\binom{n+2}{2}+n+1}}$ on $v_{n,2}(\tilde{C})$. Then, again by [Har, Chapter 3, Theorem 12.8], we have

$$H^0(v_{n,2}(\tilde{C}), \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}}|_{v_{n,2}(\tilde{C})} \otimes \mathcal{I}_{\widetilde{p_1},\cdots,\widetilde{p_{\binom{n+2}{2}+n+1}}}) = 0,$$

which implies

$$H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_{\widetilde{p_1},\cdots,\widetilde{p_{\binom{n+2}{2}+n+1}}}) = H^0(\mathrm{Ver}, \mathcal{N}_{\mathrm{Ver}/\mathbb{P}^{\binom{n+2}{2}-1}} \otimes \mathcal{I}_{v_{n,2}(\tilde{C})}) = 0.$$

By proposition 3.1, this implies that degree 2 Veronese varieties of odd dimension satisfy interpolation. If we consider the projection map in the incidence correspondence of definition 2.1,

$$\Psi \to \mathcal{H}_{\mathrm{Ver}} \times (\prod_{i=1}^{\binom{n+2}{2}+n+1} \mathrm{Gr}(1, \binom{n+2}{2})) \times (\mathbb{P}^{\binom{n+2}{2}-1})^{\binom{n+2}{2}+n+1} \to (\prod_{i=1}^{\binom{n+2}{2}+n+1} \mathrm{Gr}(1, \binom{n+2}{2})),$$

then interpolation implies that there is a dense open subset of $(\prod_{i=1}^{\binom{n+2}{2}+n+1} \mathrm{Gr}(1, \binom{n+2}{2}))$, over which this projection has geometric fibers which are nonempty and zero-dimensional. Furthermore, the smooth curve $v_{n,2}(\tilde{C})$ satisfies proposition 1. Since $v_{n,2}(\tilde{C}))$ is smooth, there are $2^{n(n-1)}$ square roots of $\mathcal{O}_{v_{n,2}(\tilde{C})}(1)$ by the general theory of smooth curves and their Jacobians, and thus there are at least $2^{n(n-1)}$ degree 2 Veroneses of odd dimension which contain $v_{n,2}(\tilde{C})$. This demonstrates that over a closed point of $(\prod_{i=1}^{\binom{n+2}{2}+n+1} \mathrm{Gr}(1, \binom{n+2}{2}))$ corresponding to $(\widetilde{p_1}, \cdots, \widetilde{p_{\binom{n+2}{2}+n+1}})$, the zero-dimensional fiber of this projection has at least $2^{n(n-1)}$ points. By [DM69, Theorem 4.17(iii)], the number of connected components of geometric fibers of this projection map is a lower semicontinuous function. This implies that there is an open subset of $(\prod_{i=1}^{\binom{n+2}{2}+n+1} \mathrm{Gr}(1, \binom{n+2}{2}))$, over which the geometric fibers have at least $2^{n(n-1)}$ connected components. Thus, there is a dense open subset of $(\prod_{i=1}^{\binom{n+2}{2}+n+1} \mathrm{Gr}(1, \binom{n+2}{2}))$ over which the interpolation projection has zero-dimensional fibers with at least $2^{n(n-1)}$ points. This completes the proof of theorem 1.1, which makes substantial progress on questions of Aaron Landesman and Anand Patel [LP16, Question 6, 7].

6. Concluding Remarks and Questions

In summary, we proved interpolation for degree 2 Veroneses of odd dimension by proving the vanishing of certain normal vector fields on the Veroneses. Our argument relied on smoothing a degenerate curve (the rational normal curve chain) with the right degree and genus. We constructed our degenerate curve out of rational normal curves, exploiting the fact that we knew precisely the line bundle decomposition of the degree 2 Veronese normal bundles to these rational normal curves by [Sha, Theorem 4.3]. The following still remains open.

Question 6.1. *Aside from surfaces, do degree 2 Veronese varieties of even dimension satisfy interpolation?*

In the case of even dimensions, any rational normal curve chain will not have the desired degree and genus in the sense of subsection 3.2. Knowledge of how Veronese normal bundles restrict to other curves or subvarieties may lead to progress in this direction. We note that, in general, the Veronese normal bundle restricted to a rational normal curve will not be well-balanced, since the degree of the first Chern class of the restriction is $\binom{n+d}{d}nd - n(n+1)$ and the rank is $\binom{n+d}{d} - n - 1$. However, besides the degree 2 case, the numerology of the first Chern class and rank of the restriction does work out in the case of degree 3 Veroneses of dimension 7.

Question 6.2. *Let $\mathbb{P}^7 \to \mathbb{P}^{\binom{10}{3}-1}$ be an embedding for a degree 3 Veronese of dimension 7. Let RNC denote a rational normal curve in $\mathbb{P}^7$. In this case, is the Veronese normal bundle restricted to the rational normal curve well-balanced? In other words, is*

$$\mathcal{N}_{Ver/\mathbb{P}^{\binom{10}{3}-1}}|_{RNC} \cong \bigoplus_{i=1}^{112} \mathcal{O}_{\mathbb{P}^1}(22)?$$

If the answer to 6.2 is yes, then we can deduce interpolation for degree 3 Veroneses of dimension 7 by, essentially word-for-word, the same argument in this paper for degree 2 Veronese varieties of odd dimension. Note, however, that the well-balancedness of the restriction to rational normal curves is not a prerequisite for interpolation. For example, degree 3 Veronese surfaces satisfy interpolation [LP16, Theorem 6.1] but the numerology of their normal bundle restricted to rational normal curves in $\mathbb{P}^2$ immediately implies that they are not well-balanced.

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