SLOPE SEMISTABILITY FOR VERONESE NORMAL BUNDLES

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Abstract. It is a classical fact that normal bundles of rational normal curves are well-balanced. We generalize this by proving all Veronese normal bundles are slope semistable. The Grauert-Mulich theorem says that the restriction of slope semistable vector bundles on complex projective space to general lines decomposes into a direct sum of line bundles satisfying certain constraints. In the case of degree 2 Veroneses, we determine the line bundle decomposition of the restriction of their normal bundles to lines and rational normal curves.

Contents

1. Introduction

The normal bundle of a smooth projective variety plays a crucial role in many problems of geometry, arithmetic, and commutative algebra. For example, the normal bundle controls deformations of the variety in its ambient projective space. Certain cohomological information about the normal bundle can determine whether the variety satisfies interpolation [LP16, Theorem A.7], [LV23]. The Giesker (semi)stability and slope (semi)stability of normal bundles has received attention since the early 1980s, but has been mostly restricted to normal bundles of curves, especially those living in $\mathbb{P}^3$. This includes the work of Coskun-Larson-Vogt [CLV22], Ein-Lazarsfeld [EL92], Eisenbud-Van de Ven [EVdV81], Atansov-Larson-Yang [ALY19], Ballico-Ellia [BP84], Ellingsrud-Laksov [ED80], Ghione-Sacchiero [GG80], Ran [Ran07], and Newstead [New83]. Notably, Coskun-Larson-Vogt

[CLV22] recently showed that, with few exceptions, Brill-Noether general curves in $\mathbb{P}^3$ have stable normal bundle, and Ein-Lazarsfeld [EL92] showed that an elliptic curve of degree $n+1$ in $\mathbb{P}^n$ has semistable normal bundle. To the author's knowledge, fewer work has been done to investigate Gieseker (semi)stability and slope (semi)stability for normal bundles of higher dimensional varieties (on curves, the two notions coincide). Although, one example is the work of Kleppe-Miro-Roig [KMR16, Theorem 5.3, Theorem 5.7] which shows that when $X \subset \mathbb{P}^n$ is a smooth linear determinantal scheme with certain constraints on its codimension, the normal bundle is slope (semi)stable.

1.1. **Main Results.** In this paper, we expand the literature on varieties with slope semistable normal bundles towards higher dimensional varieties. Specifically, we generalize the classically known fact that normal bundles of rational normal curves are well-balanced, by showing the following.

Theorem 1.1. *All Veronese normal bundles are slope semistable.*

In the theory of coherent sheaves, restrictions theorems for slope semistable sheaves are of considerable interest due to their many useful applications. Thus, for degree 2 Veroneses we also determine the following restrictions.

Theorem 1.2. *For a general line L in $\mathbb{P}^n$, the restriction of the normal bundle of a degree 2 Veronese to a line is isomorphic to*

$$\mathcal{N}_{Ver/\mathbb{P}(\mathrm{Sym}^2 V)}|_L \cong \mathcal{O}_L(2)^{\oplus \frac{(n-1)(n-2)}{2}+(n-1)} \oplus \mathcal{O}_L(3)^{\oplus(n-1)} \oplus \mathcal{O}_L(4).$$

Theorem 1.3. *Let $v : \mathbb{P}^1 \to \mathbb{P}(V)$ denote the embedding of a rational normal curve of degree n in $\mathbb{P}(V)$, where $\dim V = n+1$. Let Ver denote the image of a degree 2 Veronese map $v_{n,2} : \mathbb{P}(V) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 V)$. Then*

$$v^*\mathcal{N}_{Ver/\mathbb{P}(\mathrm{Sym}^2 V)} \cong \bigoplus_{i=1}^{\frac{n(n+1)}{2}} \mathcal{O}_{\mathbb{P}^1}(2n+2).$$

One example of an application of theorem 1.3 can be found in the author's forthcoming work, where it is used to help prove interpolation for degree 2 Veroneses of odd dimension [Sha].

1.2. **Outline and Notation.** This paper is structured as follows. In section 2, we recall the basic theory of Gieseker and slope (semi)stability and present examples of Gieseker semistable sheaves which will be useful for proving theorem 1.1, whose discussion is in section 3. We prove our two restriction theorems in section 4.

Here is some notation. In the literature, μ-(semi)stable and slope (semi)stable are used interchangeably – in this paper, we just say slope (semi)stable. The words (semi)stable will mean Gieseker (semi)stable. Throughout this paper we work over $\mathbb{C}$, unless specified otherwise. For convenience, unless specified otherwise, we fix the embedding of a dimension n and degree d Veronese variety to be given as follows: let V be a complex vector space of dimension $n+1$. Let $v_{n,d} : \mathbb{P}(V) \hookrightarrow \mathbb{P}(\mathrm{Sym}^d V)$ be the d-th power embedding so that $v_{n,d}([L]) = [L^d]$. Let Ver denote the Veronese image under $v_{n,d}$, where the n and d associated to Ver will always be clear from context.

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2. Preliminaries

In this section, we recall the basic theory of Giesker and slope (semi)stability and include an example that we will need to prove theorem 1.1. This entire section follows Chapter 1.1-1.4 of [HL10].

2.1. **Gieseker and Slope Semistability, Harder-Narasimhan filtration.** By the Birkhoff-Grothendieck theorem [Gro57], we understand vector bundles over $\mathbb{P}^1$ quite well. Any vector bundle $\mathcal{V}$ over $\mathbb{P}^1$ decomposes into a direct sum of line bundles, and thus admits a decomposition that looks like

$$\mathcal{V} \cong \bigoplus_{i=1}^{n_1} \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \bigoplus_{i=1}^{n_t} \mathcal{O}_{\mathbb{P}^1}(a_t).$$

where $a_1 > \cdots > a_t$, so that there are n_ℓ copies of the line bundle $\mathcal{O}_{\mathbb{P}^1}(a_\ell)$ in the decomposition. There is a natural filtration here, where the first filtration block is $\bigoplus_{i=1}^{n_1} \mathcal{O}_{\mathbb{P}^1}(a_1)$, the second filtration block is $\bigoplus_{i=1}^{n_1} \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \bigoplus_{i=1}^{n_2} \mathcal{O}_{\mathbb{P}^1}(a_2)$, and so on. Any endomorphism of the vector bundle $\mathcal{V}$ must respect this filtration.

In general, this splitting theorem does not hold for smooth projective k-varieties besides $\mathbb{P}^1$. For example, the tangent bundle bundle $T\mathbb{P}^n$ for projective space does not split for $n > 1$. However, the filtration we have described here generalizes to the Harder-Narasimhan filtration. Before we define the Harder-Narasimhan filtration, let us first define what a Gieseker (semi)stable sheaf is. First, the notion of Gieseker (semi)stability extends to certain coherent sheaves called pure sheaves, and not just sheaves of sections of vector bundles.

Definition 2.1. *Let X be a Noetherian scheme. We say $E \in Coh(X)$ is pure of dimension d if* $\dim E = d$ *and for every nonzero subsheaf $F \subset E$,* $\dim F = d$*. Equivalently, the support of E is equidimensional of dimension d and has no embedded primes.*

When X is a projective k-scheme, where k is an algebraically closed field of characteristic 0, and $E \in Coh(X)$ is a coherent sheaf of dimension d, we can write its Hilbert polynomial as

$$P(E)(m) := \chi(E(m)) = \sum_{i=0}^{d} \alpha_i(E) \frac{m^i}{i!}$$

where $\alpha_i(E)$ are integers and $\alpha_d(E)$ is positive. This follows from lemma 1.2.1 of [HL10, Page 9]. We define the reduced Hilbert polynomial to be $\rho(E) := \frac{P(E)}{\alpha_d(E)}$.

Definition 2.2. *Let X be a projective k-scheme, and $E \in Coh(X)$. Then E is Gieseker semistable if it is pure and for every nonzero subsheaf $F \subset E$,*

$$\rho(F) \leq \rho(E).$$

Replacing the inequality $\leq$ by a strict inequality $<$ is the condition for Gieseker stable.

A closely related notion is μ-(semi)stability or slope (semi)stability.

Definition 2.3. *Let X be a projective k-scheme, and $E \in Coh(X)$ of dimension $d = \dim(X)$. The rank of E is defined to be*

$$\mathrm{rk}(E) := \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)},$$

and the degree of E is defined to be

$$\deg E := \alpha_{d-1}(E) - \mathrm{rk}(E)\alpha_{d-1}(\mathcal{O}_X).$$

The slope of E is defined to be

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Then E is μ-semistable if and only if for all $F \subset E$ such that $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$, we have $\mathrm{rk}(E)\deg(F) \leq \mathrm{rk}(F)\deg(E)$.

Replacing the inequalities $\leq$ by a strict inequalities $<$ is the condition for μ-stable.

These notions are related by the following.

Proposition 2.4. [HL10, lemma 1.2.13, page 14] *When $E \in Coh(X)$ is a pure sheaf of dimension $d = \dim(X)$, then*

$$\mu - stable \implies stable \implies semistable \implies \mu - semistable.$$

On a smooth projective variety, the Hirzebruch-Riemann-Roch formula implies that $\deg E = c_1(E)$, and if X is reduced and irreducible and E is the sheaf of sections of a rank r vector bundle, then $\mathrm{rk}(E) = r$. Thus, for the vector bundles we work with in this paper, their slope is given by the formula $\mu(E) = \frac{c_1(E)}{\mathrm{rk}(E)}$. Furthermore, slope semistability is preserved under the following operations.

Lemma 2.5. [HL10, Chapter 3.2] *Slope semistability is preserved by tensor products and dualizing.*

Lemma 2.6. *Let X be a normal integral projective variety, and let E be a slope semistable vector bundle on X. Then the dual E^* is also slope semistable.*

Proof. Let $F \subseteq E^*$ be a subsheaf with $0 < \mathrm{rk}(F) < \mathrm{rk}\, E^*$. We would like to show that

$$\mathrm{rk}(E^*) \deg(F) \leq \mathrm{rk}(F) \deg(E^*).$$

Note this inequality is equivalent to

$$\alpha_d(E)\alpha_{d-1}(E) \leq \alpha_d(F)\alpha_{d-1}(E),$$

where $d = \dim X$. Let F' denote the saturation of F with respect to E^*, where saturation is defined in [HL10, Definition 1.1.5]. In particular, the cokernel F'/F is supported on a codimension 1 locus of X. Thus, $\alpha_d(F) = \alpha_d(F')$ and $\alpha_{d-1}(F') \geq \alpha_{d-1}(F)$.

Now consider the short exact sequence

$$0 \to F' \to E^* \to E^*/F' \to 0.$$

Note E^*/F' is torsion-free since it is pure of dimension d. Thus, since X is a normal integral variety over an algebraically closed field, by [Ish, Proposition 5.1.7], E^*/F' is locally free on open $U \subset X$ such that $\mathrm{codim}(X \setminus U) \geq 2$. Restricting to U yields the short exact sequence

$$0 \to F'|_U \to E^*|_U \to (E^*/F')|_U \to 0$$

of locally free sheaves. Dualizing we obtain

$$0 \to (E^*/F')|_U^* \to E|_U \to (F'|_U)^* \to 0,$$

and since the Hilbert polynomial coefficients α_d and α_{d-1} are the same over U since $\mathrm{codim}(X \setminus U) \geq 2$, we find that slope semistability of E and additivity of Hilbert polynomials implies

$$\alpha_{d-1}(F')\alpha_d(E^*) \leq \alpha_d(F')\alpha_{d-1}(E^*).$$

Using the fact that $\alpha_d(F) = \alpha_d(F')$ and $\alpha_{d-1}(F') \geq \alpha_{d-1}(F)$, we obtain the desired inequality. □

We are now ready to define the Harder-Narasimhan filtration. First, every pure coherent sheaf admits a maximal destabilizing subsheaf.

Lemma 2.7. [HL10, Lemma 1.3.5, Page 16] *Let E be a pure coherent sheaf of dimension d. Then there exists a subsheaf $F \subset E$ such that for all subsheaves $G \subset E$, one has $\rho(F) \geq \rho(G)$, and in case of equality, $F \subset G$. This F is called the maximal destablizing subsheaf, and is uniquely determined and semistable.*

The existence of the maximal destablizing subsheaf boils down to Zorn's lemma and the additivity of Hilbert polynomials. Now given a pure coherent sheaf E, if $HN_1(E)$ is its maximal destablizing subsheaf, the idea is that $E/HN_1(E)$ will also admit a maximal destablizing subsheaf, and it can be lifted to a subsheaf of E which contains $HN_1(E)$. Doing this inductively, we obtain the Harder-Narasimhan filtration.

Definition 2.8. [HL10, Theorem 1.3.4, Page 16] *Let X be a projective k-scheme and $E \in Coh(X)$ be pure of dimension d. Then E admits a Harder-Narasimhan filtration*

$$0 \subset HN_1(E) \subset \cdots \subset HN_\ell(E) = E,$$

where $HN_i(E)/HN_{i-1}(E)$ is semistable of dimension d for $1 \le i \le \ell$, and defining $\rho_i(E) := \rho_{\frac{HN_i(E)}{HN_{i-1}(E)}}$, we have

$$\rho_1 > \cdots > \rho_\ell.$$

The Harder-Narasimhan filtration exists uniquely.

2.2. **A useful example.** In this subsection, we reproduce the main ideas of a useful example from section 1.4 of [HL10, Page 19-21], namely that $\mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}$ on $\mathbb{P}(V)$, along with certain subbundles, which we denote as K_d^i for $1 \le i \le d+1$, are Gieseker semistable. In section 3, we use the Gieseker semistability of K_d^{d-1} to prove theorem 1.1.

The main idea behind the Gieseker semistability of the K_d^i is the following. Suppose a group G acts on a projective k-scheme X. We say E is G-compatible if the action of $\sigma \in G$ on X lifts to an action $\tilde{\sigma}$ on E such that the diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{\sigma}"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

commutes. By uniqueness of the Harder-Narasimhan filtration on E, each term $HN_i(E)$ of the filtration must be a G-invariant subsheaf of E. This implies that if one can determine all G-invariant subsheaves of E, among these is the maximal destabilizing subsheaf and thus a semistable sheaf.

We now define K_d^i. Consider the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to T_{\mathbb{P}(V)} \to 0$$

on $\mathbb{P}(V)$, where V is a complex vector space of dimension n. Dualizing and twisting up by 1, we obtain the short exact sequence

$$0 \to \Omega_{\mathbb{P}(V)}(1) \to V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(1) \to 0.$$

Symmetrizing this short exact sequence with respect to the injective map yields

$$0 \to \mathrm{Sym}^i \Omega_{\mathbb{P}(V)}(1) \to \mathrm{Sym}^i V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathrm{Sym}^{i-1} V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to 0$$

for every positive integer i. Then define the map $\delta_d^i : \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathrm{Sym}^{d-i} V \otimes \mathcal{O}_{\mathbb{P}(V)}(i)$ to be the composition of quotient maps

$$\mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathrm{Sym}^{d-1} V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \cdots \to \mathrm{Sym}^{d-i} V \otimes \mathcal{O}_{\mathbb{P}(V)}(i),$$

and define K_d^i to be the kernel $\ker \delta_d^i$. There is a natural $\mathrm{PGL}(V)$ action on $\mathbb{P}(V)$ and the sheaves $S^t V \otimes \mathcal{O}_{\mathbb{P}(V)}, \mathcal{O}_{\mathbb{P}(V)}(t)$, and $\Omega_{\mathbb{P}(V)}$, so that the maps δ_d^i are $\mathrm{PGL}(V)$-equivariant and thus the K_d^i are $\mathrm{PGL}(V)$-invariant. In fact, the following is true.

Proposition 2.9. [HL10, Lemma 1.4.4] *The K_d^i are the only* $\mathrm{PGL}(V)$*-invariant subsheaves of $S^dV \otimes \mathcal{O}_{\mathbb{P}(V)}$*

The reason why proposition 2.9 holds is because of lemma 2.10.

Lemma 2.10. [HL10, Lemma 1.4.3] $K_d^1 = \mathrm{Sym}^d\,\Omega_{\mathbb{P}(V)}(1)$ *has no proper* $\mathrm{PGL}(V)$*-invariant subsheaves.*

The idea behind lemma 2.10 is that the isotropy subgroup of $\mathrm{PGL}(V)$ that fixes a point $x \in \mathbb{P}(V)$ acts on the fiber of $\mathrm{Sym}^d\,\Omega_{\mathbb{P}(V)}(1)$ over x. This action is actually an irreducible representation so, combined with the transitivity of the action of $\mathrm{PGL}(V)$ on $\mathbb{P}(V)$, any $\mathrm{PGL}(V)$-invariant subsheaf would have to be 0 or everything.

Then because $\delta_d^{i+1} = \delta_{d-i}^1(i) \circ \delta_d^i$, there exists a short exact sequence

$$0 \to K_d^i \to K_d^{i+1} \to K_{d-i}^1(i) \to 0$$

for $0 < i < i+1 \leq d+1$. Then successive terms K_d^i and K_d^{i+1} are related by $\mathrm{Sym}^{d-i}\,\Omega_{\mathbb{P}(V)}(1) \otimes \mathcal{O}_{\mathbb{P}(V)}(i)$. Thus, proposition 2.9 holds because if there were any other $\mathrm{PGL}(V)$-invariant subsheaves of $S^dV \otimes \mathcal{O}_{\mathbb{P}(V)}$, they would need to be sandwiched between the K_d^i, but the quotients $K_{d-i}^1(i)$ have no $\mathrm{PGL}(V)$-invariant subsheaves by lemma 2.10. A slope analysis then reveals the following.

Lemma 2.11. *The slopes of the K_d^i are such that*

- $\mu(\mathrm{Sym}^d\,\Omega_{\mathbb{P}(V)}(1)) = \frac{-d}{n}$
- *and* $\mu(K_d^1) < \cdots < \mu(K_d^d) < 0$.

Proof. The calculation is done in [HL10, Lemma 1.4.2]. □

Lemmas 2.11 and 2.9 allow us to conclude the following, which we will use when proving that Veronese normal bundles are slope semistable.

Theorem 2.12. *The K_d^i for $1 \leq i \leq d+1$ are Gieseker semistable.*

Proof. Each K_d^j admits a Harder-Narasimhan filtration, and each filtration term must be $\mathrm{PGL}(V)$-invariant. Thus, the Harder-Narasimhan filtration of K_d^j must be comprised of subsheaves among the K_d^i for $i \leq j$. The slope analysis, however, shows that $\mu(K_d^i) < \mu(K_d^j)$ for $i < j$, which implies $\rho(K_d^i) < \rho(K_d^j)$. Thus, each K_d^j is its own maximal destabilizing subsheaf, hence semistable. □

3. Proving Veronese normal bundles are slope semistable

In this section, we prove the following.

Theorem 3.1. *All Veronese normal bundles are μ-semistable.*

Let us fix the dimension and degree of the Veronese to be n and d, respectively. All Veroneses of dimension n and degree d are projectively equivalent. Thus, it suffices to prove Theorem 3.1 for a particular Veronese embedding of dimension n and degree d. Let V denote a complex vector space of dimension $n+1$. Recall that we fix our Veronese embedding to be the d-th power embedding $v_{n,d} : \mathbb{P}(V) \hookrightarrow \mathbb{P}(\mathrm{Sym}^d V)$ such that $v_{n,d}([L]) = [L^d]$. Let Ver denote the Veronese image under $v_{n,d}$. To prove Theorem 3.1, we will express the normal bundle $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)}$ up to a twist as the quotient of certain locally free sheaves. Dualizing the short exact sequence will imply that the dual of the twisted Veronese normal bundle will be isomorphic to one of the semistable sheaves from the previous sections.

Lemma 3.2. *The bundle $\mathcal{N}_{Ver/\mathbb{P}(\mathrm{Sym}^d V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(-d)$ fits into the following short exact sequence of locally free sheaves.*

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1-d) \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{N}_{Ver/\mathbb{P}(\mathrm{Sym}^d V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(-d) \to 0.$$

Proof. First, consider the Euler exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to T_{\mathbb{P}(V)} \to 0, \tag{1}$$

$$0 \to \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^d V)} \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^d V)}(1) \to T_{\mathbb{P}(\mathrm{Sym}^d V)} \to 0. \tag{2}$$

If we pull back short exact sequence 2 along our embedding $v_{n,d}$, then we obtain the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}(d) \to T_{\mathbb{P}(\mathrm{Sym}^d V)}|_{\mathrm{Ver}} \to 0. \tag{3}$$

We can combine short exact sequences 1 and 3 with the short exact sequence

$$0 \to T_{\mathbb{P}(V)} \to T_{\mathbb{P}(\mathrm{Sym}^d V)}|_{\mathrm{Ver}} \to \mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)} \to 0$$

to obtain the short exact sequence

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}(d) \to \mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)} \to 0. \tag{4}$$

More specifically to obtain short exact sequence 4, first note the map

$$\mathcal{O}_{\mathbb{P}(V)} \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$$

in short exact sequence 1 is given by, choosing coordinates $\{Z_i\}_{i=0}^n$ on V dual to basis $\{X_i\}_{i=0}^n$ of V, the global section $\sum_{i=0}^n Z_i \frac{\partial}{\partial X_i} \in H^0(\mathbb{P}(V), V \otimes \mathcal{O}_{\mathbb{P}(V)}(1))$, where we can identify V naturally with $\mathbb{C}\langle \frac{\partial}{\partial X_0}, \cdots, \frac{\partial}{\partial X_n} \rangle$. Letting $\{Y_i\}_{i=1}^{\binom{n+d}{d}}$ be coordinates on $\mathrm{Sym}^d V$ dual to the basis $\{\frac{\partial}{\partial X_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial X_{i_d}}\}_{0 \le i_1 \le \cdots \le i_d \le n}$ of $\mathrm{Sym}^d V$, the map

$$\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^d V)} \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^d V)}(1)$$

in short exact sequence 2 is given by the global section $\sum_{i=1}^{\binom{n+d}{d}} Y_i[\frac{\partial}{\partial X_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial X_{i_d}}] \in H^0(\mathbb{P}(\mathrm{Sym}^d V), \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^d V)}(1))$. Now pull this map back along $v_{n,d} : \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^d V)$. Then there is a commuting diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}(V)} \arrow[r] \arrow[d, "Id"'] & V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \arrow[d, "\Theta"'] \\
\mathcal{O}_{\mathbb{P}(V)} \arrow[r] & \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}(d)
\end{tikzcd}$$

where Θ is given by multiplication by $(\sum_{i=0}^n Z_i \frac{\partial}{\partial X_i})^{d-1}$. The map Θ gives the map in short exact sequence 4.

Twisting down short exact sequence 4 by $\mathcal{O}_{\mathbb{P}(V)}(-d)$, we obtain the desired short exact sequence

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1-d) \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(-d) \to 0.$$

Note the map $V \otimes \mathcal{O}_{\mathbb{P}(V)}(1-d) \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}$ is still given by multiplication by $(\sum_{i=0}^n Z_i \frac{\partial}{\partial X_i})^{\otimes(d-1)}$.

□

In general, twisting short exact sequence 1 down by 1, then symmetrizing to the i-th degree with respect to the quotient map, then twisting down by $i-d$ yields short exact sequences

$$0 \to \mathrm{Sym}^{i-1} V \otimes \mathcal{O}_{\mathbb{P}(V)}(i-d-1) \to \mathrm{Sym}^i V \otimes \mathcal{O}_{\mathbb{P}(V)}(i-d) \to \mathrm{Sym}^i [T_{\mathbb{P}(V)}(-1)] \otimes \mathcal{O}_{\mathbb{P}(V)}(i-d) \to 0,$$

for $1 \le i \le d$. Note that each injection $\mathrm{Sym}^{i-1} V \otimes \mathcal{O}_{\mathbb{P}(V)}(i-d-1) \to \mathrm{Sym}^i V \otimes \mathcal{O}_{\mathbb{P}(V)}(i-d)$ is given by multiplication by $\sum_{i=0}^n Z_i \frac{\partial}{\partial X_i}$. These short exact sequences provide a sequence of injective maps

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-d) \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1-d) \to \cdots \to \mathrm{Sym}^{d-1} V \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}. \quad (5)$$

Note that the composition of injective maps

$$V \otimes \mathcal{O}_{\mathbb{P}(V)}(1-d) \to \cdots \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}$$

is exactly the same as the injective map in lemma 3.2, namely multiplication by $(\sum_{i=0}^n Z_i \frac{\partial}{\partial X_i})^{\otimes(d-1)}$. Dualizing the sequence of injective maps, we obtain a sequence of quotient maps

$$\mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathrm{Sym}^{d-1} V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \cdots \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(d-1) \to \mathcal{O}_{\mathbb{P}(V)}(d) \to 0.$$

We claim that compositions of these quotient maps, starting from $\mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}$ are exactly the maps

$$\delta_d^i : \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathrm{Sym}^{d-1} V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \cdots \to \mathrm{Sym}^{d-i} V \otimes \mathcal{O}_{\mathbb{P}(V)}(i)$$

considered in section 2.2. To see this, we use the following lemma.

Lemma 3.3. *Let*

$$0 \to M \to^{\phi} N \to^{\psi} P \to 0$$

be a short exact sequence of free A-modules. There are two ways to obtain the short exact sequence

$$0 \to \mathrm{Sym}^i P^* \to \mathrm{Sym}^i N^* \to \mathrm{Sym}^{i-1} N^* \otimes M^* \to 0:$$

(1) *by first symmetrizing to obtain*

$$0 \to \mathrm{Sym}^{i-1} N \otimes M \to \mathrm{Sym}^i N \to \mathrm{Sym}^i P \to 0$$

then dualizing to

$$0 \to \mathrm{Sym}^i P^* \to \mathrm{Sym}^i N^* \to \mathrm{Sym}^{i-1} N^* \otimes M^* \to 0$$

(2) *or by first dualizing to obtain*

$$0 \to P^* \to^{\psi^*} N^* \to^{\phi^*} M^* \to 0$$

and then symmetrizing to obtain

$$0 \to \mathrm{Sym}^i P^* \to \mathrm{Sym}^i N^* \to \mathrm{Sym}^{i-1} N^* \otimes M^* \to 0.$$

Proof. Let us first verify that both procedures produce the same map

$$\mathrm{Sym}^i P^* \to \mathrm{Sym}^i N^*.$$

First, let us describe this map through the first procedure. We have $\mathrm{Sym}^i N \to \mathrm{Sym}^i P$ is given

$$e_1 \otimes \cdots \otimes e_i \mapsto \psi(e_1) \otimes \cdots \otimes \psi(e_i).$$

Then $\mathrm{Sym}^i P^* \to \mathrm{Sym}^i N^*$ is given by the composition

$$\mathrm{Sym}^i P^* \cong (\mathrm{Sym}^i P)^* \to (\mathrm{Sym}^i N)^* \cong \mathrm{Sym}^i N^*.$$

Now let us describe the map $Sym^i P^* \to \mathrm{Sym}^i N^*$ through the second procedure. First, ψ^* is the map where given $\ell \in P^*$, we have $\psi^*(\ell) \in N^*$ such that for $n \in N$, we have $\psi^*(\ell)(n) = \ell(\psi(n))$. Then, $\mathrm{Sym}^i P^* \to \mathrm{Sym}^i N^*$ is the map where

$$\ell_1 \otimes \cdots \otimes \ell_i \mapsto \psi^*(\ell_1) \otimes \cdots \otimes \psi^*(\ell_i).$$

Now we check that these are the same maps from $\mathrm{Sym}^i P^*$ to $\mathrm{Sym}^i N^*$. Let $\ell_1 \otimes \cdots \otimes \ell_i \in \mathrm{Sym}^i P^*$, and $v_1 \otimes \cdots \otimes v_i \in \mathrm{Sym}^i N$. By the first procedure, $\ell_1 \otimes \cdots \otimes \ell_i$ is sent to the element of $\mathrm{Sym}^i N^*$ where

$$v_1 \otimes \cdots \otimes v_i \mapsto \psi(v_1) \otimes \cdots \otimes \psi(v_i) \mapsto \frac{1}{i!} \sum_{\sigma \in S_i} \ell_{\sigma(1)}(\psi(v_1)) \cdots \ell_{\sigma(i)}(\psi(v_i)).$$

By the second procedure, $\ell_1 \otimes \cdots \otimes \ell_i$ is sent to the element of $\psi^*(\ell_1) \otimes \cdots \otimes \psi^*(\ell_i)$, which evaluates

$$v_1 \otimes \cdots \otimes v_i \mapsto \frac{1}{i!} \sum_{\sigma \in S_i} \ell_{\sigma(1)}(\psi(v_1)) \cdots \ell_{\sigma(i)}(\psi(v_i)).$$

so we see that these two maps agree.

Now we verify that these two procedures produce the same quotient map $\mathrm{Sym}^i N^* \to \mathrm{Sym}^{i-1} N^* \otimes M^*$. Let $\ell_1 \otimes \cdots \otimes \ell_i \in \mathrm{Sym}^i N^*$. Then the first procedure maps this to the element of $\mathrm{Sym}^{i-1} N^* \otimes M^*$ such that for $[a_1 \otimes \cdots \otimes a_{i-1}] \otimes b \in \mathrm{Sym}^{i-1} N \otimes M$, this element evaluates

$$[a_1 \otimes \cdots \otimes a_{i-1}] \otimes b \mapsto a_1 \otimes \cdots \otimes a_{i-1} \otimes \phi(b) \mapsto \frac{1}{i!} \sum_{\sigma \in S_i} \ell_{\sigma(1)}(a_1) \cdots \ell_{\sigma(i-1)}(a_{i-1}) \ell_{\sigma(i)}(\phi(b)).$$

On the other hand, the second procedure maps

$$\ell_1 \otimes \cdots \otimes \ell_i \mapsto \frac{1}{i}[\ell_2 \otimes \cdots \otimes \ell_i \otimes \phi^* \ell_1 + \cdots + \ell_1 \otimes \cdots \otimes \ell_{i-1} \otimes \phi^* \ell_i],$$

which evaluates

$$\begin{aligned}
[a_1 \otimes \cdots \otimes a_{i-1}] \otimes b \mapsto \frac{1}{i}\Big[\frac{1}{(i-1)!} \sum_{\sigma \in S_{i-1}} \ell_1(\phi(b)) \dot{\sum} \ell_{\sigma(2)}(a_1) \cdots \ell_{\sigma(i)}(a_{i-1}) + \cdots \\
+ \frac{1}{(i-1)!} \sum_{\sigma \in S_{i-1}} \ell_i(\phi(b)) \dot{\sum} \ell_{\sigma(1)}(a_1) \cdots \ell_{\sigma(i-1)}(a_{i-1})\Big] \\
= \frac{1}{i!} \sum_{\sigma \in S_i} \ell_{\sigma(1)}(a_1) \cdots \ell_{\sigma(i-1)}(a_{i-1}) \ell_{\sigma(i)}(\phi(b)),
\end{aligned}$$

so we see that the two procedures produce the same quotient map. □

Lemma 3.3 implies that dualizing the composition

$$\mathrm{Sym}^{d-i} V \otimes \mathcal{O}_{\mathbb{P}(V)}(-i) \to \cdots \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}$$

yields the map

$$\delta_d^i : \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \cdots \to \mathrm{Sym}^{d-i} V \otimes \mathcal{O}_{\mathbb{P}(V)}(d-i).$$

Furthermore, recall the map

$$V \otimes \mathcal{O}_{\mathbb{P}(V)}(1-d) \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}$$

in lemma 3.2 is given exactly by the composition

$$V \otimes \mathcal{O}_{\mathbb{P}(V)}(1-d) \to \cdots \to \mathrm{Sym}^{d-1} V \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)}$$

from short exact sequence 5. This implies that dualizing

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1-d) \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(-d) \to 0$$

yields the short exact sequence

$$0 \to [\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(-d)]^* \to \mathrm{Sym}^d V \otimes \mathcal{O}_{\mathbb{P}(V)} \xrightarrow{\delta_d^{d-1}} V \otimes \mathcal{O}_{\mathbb{P}(V)}(d-1) \to 0.$$

Since the quotient map is exactly δ_d^{d-1}, we can identify

$$[\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(-d)]^* \cong K_d^{d-1}.$$

We know that K_d^{d-1} is semistable from theorem 2.12 and thus μ-semistable by lemma 2.4. Then $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(-d)$ is μ-semistable by lemma 2.6, and thus $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)}$ is μ-semistable by lemma 2.5.

4. Restrictions of 2-Veronese normal bundles to lines and rational normal curves

We know, by the Grothendieck-Birkhoff theorem, that Veronese normal bundles restricted to a line will be isomorphic to a direct sum decomposition of line bundles. The μ-semistability of Veronese normal bundles imposes certain restrictions on this direct sum decomposition for general lines.

Theorem 4.1. [HL10, Theorem 3.0.1, Page 57]

Let E be a μ-semistable locally free sheaf of rank r on complex projective space. If L is a general line in $\mathbb{P}^n$ and $E|_L \cong \mathcal{O}_L(b_1) \oplus \cdots \oplus \mathcal{O}_L(b_r)$ with integers $b_1 \geq b_2 \geq \cdots \geq b_r$, then

$$0 \leq b_i - b_{i+1} \leq 1$$

for all $i = 1, \cdots, r-1$.

This is known as the Grauert-Mulich theorem. A straightforward calculation shows that, if ξ is the hyperplane class in the Chow ring $CH(\mathbb{P}^n)$, then the Chern class of the normal bundle for a degree d Veronese of dimension n is

$$c(\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)}) = \frac{(1+d\xi)^{\binom{n+d}{d}}}{(1+\xi)^{n+1}}.$$

In particular, the degree of the first Chern class of the Veronese normal bundle restricted to a line is $\binom{n+d}{d}d-(n+1)$. Combining this Chern class information with the Grauert-Mulich theorem tells us that if

$$\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)}|_L \cong \mathcal{O}_L(b_1) \oplus \cdots \oplus \mathcal{O}_L(b_{\binom{n+d}{d}-n-1})$$

with integers $b_1 \geq b_2 \geq \cdots \geq b_{\binom{n+d}{d}-n-1}$, then $0 \leq b_i - b_{i+1} \leq 1$ for $1 \leq i \leq \binom{n+d}{d} - n - 2$, and $\sum_{i=1}^{\binom{n+d}{d}-n-1} b_i = \binom{n+d}{d}d - n - 1$. However, we can actually precisely pin down the decomposition for $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\mathrm{Sym}^d V)}|_L$ in the case of degree 2 Veroneses.

Theorem 4.2. *For a general line L in $\mathbb{P}^n$, the restriction of the normal bundle of a degree 2 Veronese to a line is isomorphic to*

$$\mathcal{N}_{Ver/\mathbb{P}(\mathrm{Sym}^2 V)}|_L \cong \mathcal{O}_L(2)^{\oplus \frac{(n-1)(n-2)}{2}+(n-1)} \oplus \mathcal{O}_L(3)^{\oplus(n-1)} \oplus \mathcal{O}_L(4).$$

Proof. Symmetrizing the Euler exact sequence

$$0 \to \mathcal{O} \to V \otimes \mathcal{O}(1) \to T_{\mathbb{P}(V)} \to 0$$

with respect to the quotient map, we obtain the short exact sequence

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \operatorname{Sym}^2 V \otimes \mathcal{O}_{\mathbb{P}(V)}(2) \to \operatorname{Sym}^2 T_{\mathbb{P}(V)} \to 0.$$

We also have short exact sequence 4 in the case of degree 2 Veroneses:

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \operatorname{Sym}^2 V \otimes \mathcal{O}_{\mathbb{P}(V)}(2) \to \mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\operatorname{Sym}^2 V)} \to 0.$$

Just as discussed in the proof of lemma 3.2, if we fix the Veronese embedding $\mathbb{P}(V) \to \mathbb{P}(\operatorname{Sym}^2 V)$ to be given by $[L] \mapsto [L^2]$, then both of the injective maps in these short exact sequences are given by multiplication by $\sum_{i=0}^n Z_i \frac{\partial}{\partial X_i}$. This implies that $\operatorname{Sym}^2 T_{\mathbb{P}(V)}$ and $\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\operatorname{Sym}^2 V)}$ are isomorphic. Then

$$\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\operatorname{Sym}^2 V)}|_L \cong \operatorname{Sym}^2 T_{\mathbb{P}(V)}|_L \cong \operatorname{Sym}^2(T_{\mathbb{P}(V)})|_L.$$

Thus, it suffices to identify the decomposition of $T_{\mathbb{P}(V)}|_L$ into line bundles. Restricting the Euler exact sequence on $\mathbb{P}(V)$ to the line L, we have

$$0 \to \mathcal{O}_L \to V \otimes \mathcal{O}_L(1) \to T_{\mathbb{P}(V)}|_L \to 0.$$

Twisting down by one then dualizing yields

$$0 \to K \to V \otimes \mathcal{O}_L \to \mathcal{O}_L(1) \to 0,$$

where $K \cong (T_{\mathbb{P}(V)}|_L \otimes \mathcal{O}_L(-1))^*$. We know K is a rank n vector bundle isomorphic to $\bigoplus_{i=1}^n \mathcal{O}_L(a_i)$ where $\sum a_i = -1$. But since $K \to V \otimes \mathcal{O}_L$ is injective, we must have $K \cong \mathcal{O}_L(-1) \oplus \bigoplus_{i=1}^{n-1} \mathcal{O}_L$. This implies that $T_{\mathbb{P}(V)}|_L \cong \mathcal{O}_L(1)^{\oplus(n-1)} \oplus \mathcal{O}_L(2)$. Then

$$\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\operatorname{Sym}^2 V)}|_L \cong \mathcal{O}_L(2)^{\oplus \frac{(n-1)(n-2)}{2}+(n-1)} \oplus \mathcal{O}_L(3)^{\oplus(n-1)} \oplus \mathcal{O}_L(4).$$

□

Analogously, we can determine the restriction of degree 2 Veronese normal bundles to rational normal curves.

Theorem 4.3. *Let $v : \mathbb{P}^1 \to \mathbb{P}(V)$ denote the embedding of a rational normal curve of degree n in $\mathbb{P}(V)$, where $\dim V = n+1$. Let Ver denote the image of a degree 2 Veronese map $v_{n,2} : \mathbb{P}(V) \hookrightarrow \mathbb{P}(\operatorname{Sym}^2 V)$. Then*

$$v^*\mathcal{N}_{Ver/\mathbb{P}(\operatorname{Sym}^2 V)} \cong \bigoplus_{i=1}^{\frac{n(n+1)}{2}} \mathcal{O}_{\mathbb{P}^1}(2n+2).$$

Proof. Just as in the proof of Theorem 4.2, it suffices to identify the decomposition of $v^*T_{\mathbb{P}(V)}$ into line bundles. Pulling back the Euler exact sequence on $\mathbb{P}(V)$ along v, we obtain the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to V \otimes \mathcal{O}_{\mathbb{P}^1}(n) \to v^*T_{\mathbb{P}(V)} \to 0.$$

We claim that $v^*T_{\mathbb{P}(V)} \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(n+1)$. To see this, note that we have an evaluation map of global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

Symmetrizing with respect to the quotient map yields

$$0 \to \operatorname{Sym}^{n-1} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \operatorname{Sym}^n H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(n) \to 0.$$

Then twisting down by n and dualizing yields

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \operatorname{Sym}^n H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(n) \to \operatorname{Sym}^{n-1} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(n+1) \to 0.$$

We claim this short exact sequence is isomorphic to the Euler exact sequence on $\mathbb{P}(V)$ pulled back along the rational normal curve map. It suffices to demonstrate that the maps

$$\mathcal{O}_{\mathbb{P}^1} \to V \otimes \mathcal{O}_{\mathbb{P}^1}(n) \text{ and } \mathcal{O}_{\mathbb{P}^1} \to \operatorname{Sym}^n H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(n)$$

are the same. First, note that if we identify V with basis $\mathbb{C}\langle X_0, \cdots, X_n\rangle$ and take the dual basis $\mathbb{C}\langle Z_0, \cdots, Z_n\rangle$ as coordinates, and if we define the rational normal curve map to be

$$[Z_0 : Z_1] \mapsto [Z_0^n : Z_0^{n-1}Z_1 : \cdots : Z_1^n]$$

on coordinates, then the map $\mathcal{O}_{\mathbb{P}^1} \to V \otimes \mathcal{O}_{\mathbb{P}^1}(n)$ is given by the global section

$$Z_0^n \frac{\partial}{\partial X_0} + Z_0^{n-1} Z_1 \frac{\partial}{\partial X_1} + \cdots + Z_1^n \frac{\partial}{\partial X_n}.$$

On the other hand, if we identify $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ with the basis Z_0, Z_1, then the map $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)$ is given by $(1,0) \mapsto Z_0$ and $(0,1) \mapsto Z_1$. Then the map $\operatorname{Sym}^n H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(n)$ is the expected map where the induced basis of "1"s are sent to $Z_0^n, Z_0^{n-1}Z_1, \cdots,$ and Z_1^n. Tensoring this map by $\mathcal{O}_{\mathbb{P}^1}(-n)$ and dualizing, we see that the two desired maps agree. This implies the short exact sequences are isomorphic, and thus

$$v^*T_{\mathbb{P}(V)} \cong \operatorname{Sym}^{n-1} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(n+1) \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(n+1).$$

Then

$$v^*\mathcal{N}_{\mathrm{Ver}/\mathbb{P}(\operatorname{Sym}^2 V)} \cong \operatorname{Sym}^2 v^*T_{\mathbb{P}(V)} \cong \bigoplus_{i=1}^{\frac{n(n+1)}{2}} \mathcal{O}_{\mathbb{P}^1}(2n+2).$$

$\square$

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