THE JACOBI INVERSION PROBLEM AND THETA DIVISORS

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Abstract. An expository article on solving the Jacobi Inversion problem for compact Riemann surfaces of genus $g \geq 2$ using Theta divisors, following Chapters 2.4-2.7 of Principles of Algebraic Geometry by Philip Griffths and Joe Harris.

Contents

1. Introducing the Jacobi inversion problem

The study of compact Riemann surfaces goes back to the study of integrals of the form

$$\int \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}}.$$

Mathematicians tried to study these but soon ran into an issue: these integrals are not well-defined. One way to see this is to view this integral as a line integral

$$\int \frac{dx}{y}$$

of the everywhere holomorphic differential $\frac{dx}{y}$ on the cubic curve C defined by $y^2 = x^3 + ax^2 + bx + c$. When $x^3 + ax^2 + bx + c$ has distinct roots, C is a smooth curve of genus 1, and topologically a torus. Thus, such an integral is only well defined modulo the periods $\int_\gamma \frac{dx}{y}$ over closed loops $\gamma \in H_1(C, \mathbb{Z})$. While such integrals are only well-defined modulo periods, in 1826 Abel noticed that sums of these integrals follow certain qualitative behaviors. For example, any line $L \subset \mathbb{P}^2$ will intersect C at points $p_1(L), p_2(L), p_3(L)$, not necessarily distinct and of no particular order. Fixing a base point b, and letting $\Lambda \subset \mathbb{C}$ denote the period lattice, Abel found that the sum

$$[\int_b^{p_1(L)} \omega + \int_b^{p_2(L)} \omega + \int_b^{p_3(L)} \omega] \mod \Lambda.$$

is constant modulo the period lattice. This example is a special instance of injectivity of the Abel-Jacobi map, which we now define in generality.

Let us first mention the general setup. A good reference for all of this is the first three sections of Chapter 2 of [GH94]. Let S be a compact Riemann surface of genus g. Let $\gamma_1, \cdots, \gamma_g, \gamma_{g+1}, \cdots, \gamma_{2g} \in H_1(S, \mathbb{Z})$ be a canonical basis of loop generators. Let $\omega_1, \cdots, \omega_g$ be a basis for $H^0(S, \Omega_S)$. Then the period lattice Λ is generated by integral linear combinations of $\lambda_1, \cdots, \lambda_{2n}$, where

$$\lambda_j = \begin{pmatrix} \int_{\gamma_j} \omega_1 & \cdots & \int_{\gamma_j} \omega_g \end{pmatrix}.$$

The Jacobian $\mathcal{J}(S)$ of S is defined to be the g-dimensional torus $\frac{C^g}{\Lambda}$. Choosing $e_1, \cdots, e_g$ to be the standard complex basis of $\mathbb{C}^g$, the period matrix of $\mathcal{J}(S)$ is simply the $g \times 2g$ matrix

$$\begin{pmatrix} \lambda_1^T & \cdots & \lambda_{2n}^T \end{pmatrix}.$$

Choosing a normalized basis for $H^0(S, \Omega_S)$ induces a basis $\lambda_1, \cdots, \lambda_{2n}$ such that the period matrix is of the form $\begin{pmatrix} I & Z \end{pmatrix}$ where Z is symmetric and $\Im(Z)$ is positive definite, which demonstrates that the Jacobian $\mathcal{J}(S)$ of a compact Riemann surface S of genus g is principally polarized.

The Abel-Jacobi map of S is defined to be the map

$$\mathrm{Pic}^0(S) \to \mathcal{J}(S)$$

where for any degree 0 divisor $\sum(p_i - q_i)$,

$$\mu(\sum(p_i - q_i)) = (\sum \int_{q_i}^{p_i} \omega_1, \cdots, \sum \int_{q_i}^{p_i} \omega_g).$$

A celebrated fact is the following.

Theorem 1.1. [GH94, Page 235] *The Abel-Jacobi map is an isomorphism.*

The geometric interpretation of the injectivity of the Abel-Jacobi map is straightforward. Given a collection of paths $\{\tau_1, \cdots, \tau_s | \tau_i : [0,1] \to S\}$ on S and another collection of paths $\{\tau_1', \cdots, \tau_s' | \tau_i' : [0,1] \to S\}$ on S, then integrating along these paths

$$(\sum \int_{\tau_i} \omega_1, \cdots, \sum \int_{\tau_i} \omega_g) = (\sum \int_{\tau_i'} \omega_1, \cdots, \sum \int_{\tau_i'} \omega_g) \mod \Lambda$$

yields the same value in $\mathcal{J}(S)$ if and only if there exists a meromorphic differential η such that $\sum(\tau_i(1) - \tau_i(0)) = (\eta) + \sum(\tau_i'(1) - \tau_i'(0))$.

Surjectivity of the Abel-Jacobi map is typically established by showing that the composite

$$S^{(g)} \to \mathrm{Pic}^0(S) \to \mathcal{J}(S)$$

is surjective, where $S^{(g)}$ is a complex manifold, obtained by quotienting S^g by the natural action of the symmetric group Σ_g on g letters, whose points correspond to effective divisors of S of degree g. Fixing a base point $b \in S$, the composite is then the map

$$\sum_{i=1}^{g} p_i \mapsto (\sum_{i=1}^{g} \int_b^{p_i} \omega_1, \cdots, \sum_{i=1}^{g} \int_b^{p_i} \omega_g).$$

For general $c \in \Lambda$, the points $p_1, \cdots, p_g$ are unique. This is because each fiber of the composite is a projective space, since elements belonging to the same fiber are exactly linearly equivalent effective divisors of degree g, and the general fiber has dimension 0. The Jacobi Inversion problem asks the following question.

Question 1.2. *For general $c \in \mathcal{J}(S)$, can we explicitly find the g points $p_1, \cdots, p_g$, such that*

$$(\sum_{i=1}^{g} \int_b^{p_i} \omega_1, \cdots, \sum_{i=1}^{g} \int_b^{p_i} \omega_g) = c \mod \Lambda?$$

In the case of elliptic curves, or compact Riemann surfaces of genus 1, the Jacobi Inversion problem is solved using the classical theory of Weierstrass $\wp$-functions and the theory of doubly periodic meromorphic functions on the complex plane. Many great mathematicians worked on the Jacobi Inversion problem for compact Riemann surfaces of genus $g \geq 2$, which spurred the development of algebraic geometry.

In this expository note, we introduce the solution to the Jacobi Inversion problem for genus $g \geq 2$. If we let μ denote the precomposition of the Abel-Jacobi map with the map $S \to \mathrm{Pic}^0(S)$, where $p \mapsto p - b$, then the solution will involve intersecting the curve $\mu(S) \subset \mathcal{J}(S)$ with a certain effective divisor Θ_c of $\mathcal{J}(S)$ called a Theta divisor. These Theta divisors Θ_c will naturally arise as effective divisors associated to certain global sections of certain line bundles on the Jacobian $\mathcal{J}(S)$. Thus, to set up our solution, we need to first discuss line bundles on $\mathcal{J}(S)$, and more generally, line bundles on compact complex tori.

2. (Positive) Line bundles on compact complex tori

Let $M = V/\Lambda$ be a complex torus of dimension n. Let $\mathcal{L}$ be a line bundle on M. We will first discuss how every line bundle on M can be realized as a quotient of the trivial line bundle $V \times \mathbb{C}$ on V.

First, note that every line bundle on V is trivial. Taking sheaf cohomology on the exponential exact sequence on V, we obtain

$$\cdots \to H^1(V, \mathcal{O}_V) \to H^1(V, \mathcal{O}_V^\times) \to H^2(V, \mathbb{Z}) \to \cdots,$$

and since $V \cong \mathbb{C}^n$, we have that $H^1(V, \mathcal{O}_V) \cong H^{0,1}(V) = 0$ and $H^2(V, \mathbb{Z}) = 0$. Thus, $H^1(V, \mathcal{O}_V^\times) = 0$, so every line bundle on V is trivial. In particular, the pullback line bundle can be globally trivialized:

$$\psi : \pi^*\mathcal{L} \to V \times \mathbb{C}.$$

If $\lambda_1, \cdots, \lambda_{2n}$ are integral generators for Λ, then note for every z, the fibers $\pi^*\mathcal{L}|_z$ and $\pi^*\mathcal{L}|_{z+\lambda_i}$ are identified with $\mathcal{L}|_{\pi(z)}$. Using the trivialization ψ, we have an identification

$$\mathbb{C} \cong^{\psi} \pi^*\mathcal{L}|_z = \mathcal{L}|_{\pi(z)} = \pi^*\mathcal{L}|_{z+\lambda_i} \cong^{\psi} \mathbb{C}$$

given by multiplication by some scalar $e_{\lambda_i}(z)$. These $e_{\lambda_i}(z)$, which we call multipliers, are everywhere nonzero entire functions. Note that these multipliers satisfy the following relation: for all $\lambda, \lambda' \in \Lambda$, we have

$$e_{\lambda+\lambda'}(z) = e_\lambda(z) \cdot e_{\lambda'}(z+\lambda) = e_{\lambda'}(z) \cdot e_\lambda(z+\lambda').$$

Furthermore, note that these multipliers are not uniquely associated to $\mathcal{L}$. Given some everywhere nonzero entire function $m(z)$, we can simply change the global trivialization ψ by $(Id, m) \circ \psi$. This changes the multipliers to be of the form $\{e_\lambda(z) \cdot \frac{m(z+\lambda)}{m(z)}\}$. Any line bundle on M isomorphic to $\mathcal{L}$ will also produce the same multipliers up to this kind of multiplication.

We can also go in the reverse direction. Given everywhere nonzero entire functions $\{e_\lambda(z)\}_{\lambda\in\Lambda}$ satisfying the aforementioned relation, we can define a line bundle $\mathcal{L}$ on M by imposing an equivalence relation on $V \times \mathbb{C}$, such that

$$(v, c) \sim (v+\lambda, c') \iff c' = e_\lambda(z) \cdot c.$$

A global trivialization of $\pi^*\mathcal{L}$ will produce the same multipliers $\{e_\lambda(z) \cdot \frac{m(z+\lambda)}{m(z)}\}$ up to a everywhere nonzero entire function $m(z)$. Thus we have a one to one correspondence

$$\frac{\{\text{ multipliers } \{e_\lambda(z)\}_{\lambda\in\Lambda}\}}{\sim} \longleftrightarrow \mathrm{Pic}(M)$$

where $\{e_\lambda(z)\} \sim \{e'_\lambda(z)\} \iff \exists m(z) \in \mathcal{O}_V^\times(V)$ such that $m(z+\lambda)e_\lambda(z) = e'_\lambda(z)m(z), \forall \lambda \in \Lambda\}$

Now suppose that M is a projective torus. In other words, by the Kodaira embedding theorem [Huy05, Proposition 5.3.1], there exists a positive line bundle on M. In this next part, we demonstrate two things. Firstly, that positive line bundles are classified by their first Chern class, up to translation. This means that if $\mathcal{L}, \mathcal{L}'$ are positive line bundles such that $c_1(\mathcal{L}) = c_1(\mathcal{L}')$, then there exists $\mu \in M$ such that $\mathcal{L} = \tau^*\mathcal{L}'$. Secondly, that positive line bundles, up to translation, admit a relatively simple description in terms of multipliers.

Fix a positive line bundle $\mathcal{L}$ on M. Then there exists an integral basis $\lambda_1, \cdots, \lambda_{2n}$ for Λ and complex basis $e_1, \cdots, e_n$ such that $c_1(\mathcal{L}) \in H^2(M; \mathbb{Z})$ can be written as

$$c_1(\mathcal{L}) = \sum_{\alpha=1}^{n} \delta_\alpha dx_\alpha \wedge dx_{\alpha+n},$$

where x_j are coordinates of V dual to λ_j ([GH94, Page 305]). Note that the projection $\pi : V \longrightarrow M$ factors as

$$V \to \frac{V}{\mathbb{Z}\langle\lambda_1, \cdots, \lambda_n\rangle} \to^{\pi'} M.$$

Note that $\frac{V}{\mathbb{Z}\langle\lambda_1,\cdots,\lambda_n\rangle} \cong (\mathbb{C}^\times)^n$. Taking the long exact sequence in sheaf cohomology, we find the terms

$$\cdots \to H^1((\mathbb{C}^\times)^n, \mathcal{O}_{(\mathbb{C}^\times)^n}) \to H^1((\mathbb{C}^\times)^n, \mathcal{O}^\times_{(\mathbb{C}^\times)^n}) \to H^2((\mathbb{C}^\times)^n, \mathbb{Z}) \to H^2((\mathbb{C}^\times)^n, \mathcal{O}_{(\mathbb{C}^\times)^n}) \to \cdots$$

where $H^k((\mathbb{C}^\times)^n, \mathcal{O}_{(\mathbb{C}^\times)^n}) \cong H^{0,k}((\mathbb{C}^\times)^n) = 0$ for $k \geq 1$ ([GH94, Page 27]), so that the first Chern class map $c_1 : H^1((\mathbb{C}^\times)^n, \mathcal{O}^\times_{(\mathbb{C}^\times)^n}) \to H^2((\mathbb{C}^\times)^n, \mathbb{Z})$ is an isomorphism. Thus, we see that line bundles on $\frac{V}{\mathbb{Z}\langle\lambda_1,\cdots,\lambda_n\rangle}$ are classified by their first Chern class. Now note that $c_1((\pi')^*\mathcal{L}) = (\pi')^* c_1(\mathcal{L})$, and $x_{\alpha+n}$ is a well-defined function on $\frac{V}{\mathbb{Z}\langle\lambda_1,\cdots,\lambda_n\rangle}$, and since

$$\int_{\lambda_i} dx_j = \delta_{ij},$$

we find $[dx_{\alpha+n}] = 0 \in H^1(\frac{V}{\mathbb{Z}\langle\lambda_1,\cdots,\lambda_n\rangle}; \mathbb{Z})$. Thus,

$$c_1((\pi')^*\mathcal{L}) = 0.$$

This implies $(\pi')^*\mathcal{L}$ is isomorphic to the trivial line bundle on $\frac{V}{\mathbb{Z}\langle\lambda_1,\cdots,\lambda_n\rangle}$. Taking a global trivialization on $\frac{V}{\mathbb{Z}\langle\lambda_1,\cdots,\lambda_n\rangle}$ and extending it to a global trivialization for $\pi^*\mathcal{L}$ on V, the multipliers produced are such that $e_{\lambda_i} \equiv 1$ for $1 \leq i \leq n$. So given a positive line bundle on M, we can choose an integral basis for the lattice Λ so that its multipliers $e_{\lambda_i} \equiv 1$. In fact, we claim that there is a positive line bundle on M with a very simple description of its multipliers which has the same first Chern class.

Fix ω to be some integral invariant positive definite (1,1) form on M (by the Lefschetz (1,1) theorem [Huy05, Proposition 3.3.2] we know ω comes from the first Chern class of some positive line bundle). Again, we can choose an integral basis $\lambda_1, \cdots, \lambda_{2n}$ of Λ and complex basis $e_1, \cdots, e_n$ such that

$$\omega = \sum_{\alpha=1}^{n} \delta_\alpha dx_\alpha \wedge dx_{\alpha+n},$$

where x_j are coordinates on V dual to λ_j, and z_j are complex coordinates on V dual to e_j. We have the following theorem.

Theorem 2.1. [GH94, Page 310] *The line bundle $\mathcal{L}$ on M given by the multipliers*

$$e_{\lambda_\alpha}(z) \equiv 1 \text{ and } e_{\lambda_{\alpha+n}}(z) = e^{-2\pi i z_\alpha}, 1 \leq \alpha \leq n$$

has first Chern class

$$c_1(\mathcal{L}) = [\omega] \in H^2(M, \mathbb{Z}).$$

Proof. It is easy to verify that these functions satisfy the multiplier relation, so that they produce a legitimate line bundle $\mathcal{L}$ on M. To show that $c_1(\mathcal{L}) = [\omega]$, one needs to first define a hermitian metric h on $\mathcal{L}$. Then one calculates the curvature of the Chern connection of $(\mathcal{L}, h)$. If on a local trivialization $\psi : \mathcal{L} \to U \times \mathbb{C}$ we let h denote function on U where $h(z) = h(\psi^{-1}e(z), \psi^{-1}e(z))$ where e is the unit frame for $U \times \mathbb{C}$, then the curvature of the Chern connection is locally $\overline{\partial}\partial \log h$ ([Huy05, Example 4.3.9(iii)]). Thus, the hardest part of this proof is constructing the hermitian metric h on $\mathcal{L}$. We defer the reader to page 310 of Griffths and Harris for the metric construction and Chern class verification. $\square$

Let $\mathcal{L}$ be the line bundle constructed in theorem 2.1. Let $\tau_u : M \to M$ denote the translation automorphism by $u \in M$. What are the multipliers $\{e'_\lambda(z)\}_{\lambda\in\Lambda}$ of $\tau_u^*\mathcal{L}$? We have that

$$e'_{\lambda_\alpha}(z) = e_\lambda(z+u) \equiv 1 \text{ and } e'_{\lambda_{\alpha+n}}(z) = e_{\lambda_{\alpha+n}}(z+u) = e^{-2\pi i z_\alpha(z+u)} = e^{-2\pi i z_\alpha(u)} \cdot e_{\lambda_{\alpha+n}}(z).$$

So the multipliers of $\tau_u^*\mathcal{L}$ have the same simple form up to some constants. Now we show that if $\mathcal{L}'$ is another positive line bundle such that $c_1(\mathcal{L}') = [\omega]$ as well, then there exists $u \in M$ such that $\tau_u^*\mathcal{L}' \cong \mathcal{L}$. Consider the inclusion of exact sequences of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{Z}}_M \arrow[r] & \mathcal{O}_M \arrow[r] & \mathcal{O}_M^\times \arrow[r] & 0 \\
0 \arrow[r] & \underline{\mathbb{Z}}_M \arrow[r] \arrow[u, equal] & \underline{\mathbb{C}}_M \arrow[r] \arrow[u, hook] & \underline{\mathbb{C}^\times}_M \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

This induces a sheaf cohomology diagram

$$\begin{tikzcd}
H^1(M, \mathcal{O}_M) \arrow[r] & H^1(M, \mathcal{O}_M^\times) \arrow[r, "c_1"] & H^2(M, \underline{\mathbb{Z}}_M) \\
H^1(M, \underline{\mathbb{C}}_M) \arrow[r] \arrow[u, "\pi_1"] & H^1(M, \underline{\mathbb{C}^\times}_M) \arrow[r] \arrow[u] & H^2(M, \underline{\mathbb{Z}}_M) \arrow[u, equal]
\end{tikzcd}$$

and we have $H^1(M, \mathcal{O}_M) \cong H^{0,1}(M)$ by Dolbeaut's theorem [Huy05, Corollary 2.6.21], and $H^1(M, \underline{\mathbb{C}}_M) \cong H^1(M, \mathbb{C}) \cong H^{1,0}(M) \oplus H^{0,1}(M)$ by the Hodge decomposition [Huy05, Corollary 3.2.12] since M is compact Kahler. With respect to these identifications, the map π_1 is in fact given by projection, so π_1 is surjective. Then if $c_1(\mathcal{L}") = 0$, it is in the image of $H^1(M, \mathcal{O}_M) \to H^1(M, \mathcal{O}_M^\times)$, but surjectivity of π_1 implies that $\mathcal{L}$ comes from an element of $H^1(M, \underline{\mathbb{C}}_M^\times)$, which means that $\mathcal{L} \in H^1(M, \mathcal{O}_M^\times) \cong \check{H}^1(\underline{U}, \mathcal{O}_M^\times)$ can be identified with constant Cech cocyles, and thus has constant transition functions. This implies that the pullback of $\mathcal{L}"$ onto V can be trivialized to produce constant multipliers.

Now consider $\mathcal{L}^{-1} \otimes \mathcal{L}'$ on M. Then $c_1(\mathcal{L}^{-1} \otimes \mathcal{L}') = 0$. Then $\mathcal{L}^{-1} \otimes \mathcal{L}'$ has constant transition functions, and since $(\pi')^*(\mathcal{L}^{-1} \otimes \mathcal{L}')$ is trivial, this implies that the multipliers for $\mathcal{L}^{-1} \otimes \mathcal{L}'$ are such that $e"_{\lambda_\alpha} \equiv 1$ and $e"_{\lambda_{\alpha+n}} \equiv c_{\lambda_{\alpha+n}}$ are constant for $1 \leq \alpha \leq n$. This implies that $\mathcal{L}'$ has multipliers

$$e'_{\lambda_\alpha}(z) = e_{\lambda_\alpha} \equiv 1, \text{ and } e'_{\lambda_{\alpha+n}}(z) = c_{\lambda_{\alpha+n}} \cdot e_{\lambda_{\alpha+n}}(z)$$

for $1 \leq \alpha \leq n$. Then if we define

$$u = \frac{-1}{2\pi i} \sum_{\alpha=1}^{n} \log c_{\lambda_{\alpha+n}} \cdot e_\alpha,$$

we find $\mathcal{L}' \cong \tau_u^*\mathcal{L}$, as desired.

3. Global sections and Theta divisors

In this section we compute the dimension of the space of global sections of a positive line bundle on M. Afterwards, we discuss positive line bundles on the Jacobian $\mathcal{J}(S)$ of a compact Riemann surface S of genus g, and define Theta divisors.

Fix a positive line bundle $\mathcal{L}$ on $M = V/\Lambda$. Choose an integral basis $\lambda_1, \cdots, \lambda_{2n}$ for Λ such that

$$c_1(\mathcal{L}) = \sum_{\alpha=1}^{n} \delta_\alpha dx_\alpha \wedge dx_{\alpha+n},$$

and let $\{e_\alpha\}_{\alpha=1}^n$ be a complex basis for V where $e_\alpha = \frac{1}{\delta_\alpha}\lambda_\alpha$. The period matrix for M is then $\begin{pmatrix}\Delta_\delta & Z\end{pmatrix}$ where Z is symmetric and $Im(Z)$ is positive definite. Note that since $c_1(\mathcal{L})$ is a Kahler form, $c_1(\mathcal{L})^{\wedge n} \neq 0$, so the δ_α are all nonzero positive integers. We will now prove the following proposition.

Proposition 3.1. *The dimension of the space of global sections of $\mathcal{L}$ is*

$$\dim H^0(M, \mathcal{L}) = \prod_{\alpha=1}^{n} \delta_\alpha.$$

Proof. Note that it suffices to prove this proposition for any line bundle with equal Chern class. This is because any such line bundle is a positive line bundle which, by the work we did in the previous section, differs from $\mathcal{L}$ by only a translation by some $u \in M$. Pulling back $\mathcal{L}$ by translation automorphism will neither affect the elementary divisors, as $\sum_{\alpha=1}^n \delta_\alpha dx_\alpha \wedge dx_{\alpha+n}$ is a translation-invariant 2-form, nor affect the dimension of the space of global sections.

Then consider the line bundle given by multipliers

$$e_{\lambda_\alpha} \equiv 1 \text{ and } e_{\lambda_{\alpha+n}} = e^{-2\pi i z_\alpha}, 1 \leq \alpha \leq n,$$

and pull back this line bundle by τ_u where $u = \frac{1}{2}\sum Z_{\alpha\alpha} \cdot e_\alpha$. Then the multipliers for this new line bundle, which we call $\mathcal{L}$, has multipliers

$$e_{\lambda_\alpha} \equiv 1 \text{ and } e_{\lambda_{\alpha+n}} = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}.$$

Then we see that global sections of $\mathcal{L}$ are given by entire functions θ which satisfy

$$\theta(z + \lambda_\alpha) = e_{\lambda_\alpha}(z) \cdot \theta(z) = \theta(z) \text{ and } \theta(z + \lambda_{\alpha+n}) = e_{\lambda_{\alpha_n}}(z) \cdot \theta(z) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}} \cdot \theta(z).$$

Let $z_\alpha^* = e^{2\pi i \frac{1}{\lambda_\alpha} z_\alpha}$. The condition that $\theta(z + \lambda_\alpha) = \theta(z)$ implies that we can write

$$\theta = \sum_{\ell \in \mathbb{Z}^n} a_\ell (z_1^*)^{\ell_1} \cdots (z_n^*)^{\ell_n} = \sum_{\ell \in \mathbb{Z}^n} a_\ell e^{2\pi i \langle \ell, \Delta_\delta^{-1} z\rangle}.$$

Now we unpack what the condition $\theta(z + \lambda_{\alpha+n}) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}} \cdot \theta(z)$ tells us about this expression of θ.

We have that

$$\theta(z+\lambda_{\alpha+n}) = \sum_{\ell\in\mathbb{Z}^n} a_\ell e^{2\pi i\langle \ell, \Delta_\delta^{-1}(z+\lambda_{\alpha+n})\rangle} = \sum_{\ell\in\mathbb{Z}^n} a_\ell e^{2\pi i\langle \ell,\Delta_\delta^{-1}\lambda_{\alpha+n}\rangle}\cdot e^{2\pi i\langle \ell,\Delta_\delta^{-1}z\rangle},$$

and the condition implies that this is equal to

$$= e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}\cdot \sum_{\ell\in\mathbb{Z}^n} a_\ell e^{2\pi i\langle \ell,\Delta_\delta^{-1}z\rangle} = e^{-\pi i Z_{\alpha\alpha}}\sum_{\ell\in\mathbb{Z}^n} a_{\ell+\delta_\alpha\cdot e_\alpha} e^{2\pi i\langle \ell,\Delta_\delta^{-1}z\rangle},$$

since $e^{-2\pi i z_\alpha} = (z_\alpha^*)^{-\delta_\alpha}$. Altogether, this implies that

$$a_{\ell+\delta_\alpha\cdot e_\alpha} = e^{2\pi i\langle \ell,\Delta_\delta^{-1}\lambda_{\alpha+n}\rangle+\pi i Z_{\alpha\alpha}} a_\ell.$$

This tells us that θ is completely determined by the coefficients

$$\{a_\ell\}_{0\le \ell_\alpha<\delta_\alpha, 1\le\alpha\le n}.$$

Thus, we see that $H^0(M,\mathcal{L}) \le \prod_{\alpha=1}^n \delta_\alpha$. To show $H^0(M,\mathcal{L}) = \prod_{\alpha=1}^n \delta_\alpha$, one must demonstrate that any choice of a_ℓ produces a convergent θ. Let us rewrite our series θ in terms of these $\{a_\ell\}_{0\le\ell_\alpha<\delta_\alpha,1\le\alpha\le n}$. We find

$$\theta(z) = \sum_{\ell_0, 0\le(\ell_0)_\alpha<\delta_\alpha} (\sum_{\ell\in\mathbb{Z}^n} a_{\ell_0+\Delta_\delta \ell} e^{2\pi i\langle \ell_0+\Delta_\delta\ell,\Delta_\delta^{-1}z\rangle}) = \sum_{\ell_0, 0\le(\ell_0)_\alpha<\delta_\alpha} e^{2\pi i\langle \ell_0,\Delta_\delta^{-1}z\rangle}(\sum_{\ell\in\mathbb{Z}^n} a_{\ell_0+\Delta_\delta\ell} e^{2\pi i\langle \ell,z\rangle})$$

If we let $\theta_{\ell_0} = e^{2\pi i\langle \ell_0,\Delta_\delta^{-1}z\rangle}(\sum_{\ell\in\mathbb{Z}^n} a_{\ell_0+\Delta_\delta\ell}e^{2\pi i\langle\ell,z\rangle})$ denote the normalized series where $a_{\ell_0}=1$, then we can write

$$\theta(z) = \sum_{\ell_0,0\le(\ell_0)_\alpha<\delta_\alpha} a_{\ell_0}\cdot\theta_{\ell_0}(z).$$

Each coefficient $a_{\ell_0+\Delta_\delta\ell}$ equals

$$e^{\pi i\langle \ell,Z\ell\rangle+2\pi i\langle\Delta_\delta^{-1}\ell_0,Z\ell\rangle} =: b_\ell.$$

Using the fact that $Z^T = Z$ and $Ze_\alpha = \lambda_{n+a}$, one can verify that that these b_ℓ satisfy the recursive relation imposed on the coefficients [GH94, Page 320]. Using positive-definiteness of $Im(Z)$, one can produce bounds on $|b_\ell|$ for ℓ sufficiently large to show that θ converges uniformly on compact sets in V [GH94, Page 320]. Thus, for any choice of $\{a_\ell\}_{0\le\ell_\alpha<\delta_\alpha,1\le\alpha\le n}$, $\theta(z)$ converges. □

So in general, the global sections of the line bundle given by the multipliers

$$e_{\lambda_\alpha}\equiv 1 \text{ and } e_{\lambda_{\alpha+n}} = e^{-2\pi i z_\alpha-\pi i Z_{\alpha\alpha}}, 1\le\alpha\le n$$

are given by the $\prod_{i=1}^n \delta_\alpha$ entire functions

$$\theta(z) = \sum_{\ell_0,0\le(\ell_0)_\alpha<\delta_\alpha} a_{\ell_0}\cdot\theta_{\ell_0}(z)$$

where

$$\theta_{\ell_0}(z) = e^{2\pi i\langle\ell_0,\Delta_\delta^{-1}z\rangle}\cdot\sum_{\ell\in\mathbb{Z}^n} b_\ell e^{2\pi i\langle\ell,z\rangle}, \text{ and } b_\ell = e^{\pi i\langle\ell,Z\ell\rangle+2\pi i\langle\Delta_\delta^{-1}\ell_0,Z\ell\rangle}.$$

We will now define the Riemann θ-function and Theta divisors. Consider the Jacobian $\mathcal{J}(S) = V/\Lambda$ of a compact Riemann surface of genus g. There exists integral basis $\lambda_1, \cdots, \lambda_{2n}$ for Λ, and complex basis $e_j = \lambda_j$ for $1 \leq j \leq n$, so that the period matrix of $\mathcal{J}(S)$ is of the form $\begin{pmatrix} I & Z \end{pmatrix}$ where Z is symmetric and $Im(Z)$ is positive-definite. In particular, $\mathcal{J}(S)$ is a principally polarized abelian variety. Letting $\{z_\alpha\}$ denote complex coordinates on V dual to $\{e_\alpha\}$, and $\{x_\alpha\}$ denote coordinates on V dual to $\{\lambda_\alpha\}$, there exists an invariant integral Kahler form that may be expressed as

$$\omega = \sum_{\alpha=1}^{n} dx_\alpha \wedge dx_{\alpha+n}.$$

Consider the line bundle $\mathcal{L}$ given by the multipliers

$$e_{\lambda_\alpha} \equiv 1 \text{ and } e_{\lambda_{\alpha+n}} = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}}, \text{ for } 1 \leq \alpha \leq n.$$

Then as previously discussed, $c_1(\mathcal{L}) = \omega$ and in particular, $H^0(M, \mathcal{L}) = 1$. The space of global sections is generated by a particular global section $\tilde{\theta}$ which is given by a particular entire function θ called the *Riemann theta-function*, where

$$\theta = \sum_{\ell \in \mathbb{Z}^n} e^{\pi i \langle \ell, Z\ell \rangle} e^{2\pi i \langle \ell, z \rangle},$$

which satisfies $\theta(z) = \theta(-z)$ and

$$\theta(z + \lambda_\alpha) = \theta(z) \text{ and } \theta(z + \lambda_{\alpha+n}) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}} \cdot \theta(z) \text{ for } 1 \leq \alpha \leq n$$

The effective divisor Θ associated to the global section $\tilde{\theta} \in H^0(M, \mathcal{L})$ given by the Riemann theta function θ is called a *Theta divisor*. If $u \in M$, then we let Θ_u denote the Theta divisor associated to $\tilde{\theta}_u := \tau_u^* \tilde{\theta} \in H^0(M, \tau_u^* \mathcal{L})$ given by the function $\tau_u^* \theta = \theta(z - u)$. Note that $\Theta_u = \Theta + u$.

4. Solving Jacobi Inversion

Using Theta divisors, we will solve our Jacobi Inversion problem. We begin by intersecting Theta divisors with the image of our compact Riemann surface S under the Abel-Jacobi map.

Theorem 4.1. *Let $\mu : S \to \mathcal{J}(S)$. Then for $u \in \mathcal{J}(S)$, either $\mu(S) \subset \Theta_u$ or $\mu(S)$ and Θ_u intersect at exactly g points, counting multiplicity.*

Proof. First, suppose that $\mu(S) \not\subset \Theta$. To understand how $\mu(S)$ intersects Θ, it suffices to pullback the global section $\tilde{\theta} \in H^0(\mathcal{J}(S), \mathcal{L})$ to the section $\mu^* \tilde{\theta} \in H^0(S, \mu^* \mathcal{L})$. Since S is a Riemann surface, the effective divisor associated to $\mu^* \tilde{\Theta}$ will be a finite number of points on S, and these will be the intersection points, counting multiplicity, of $\mu(S)$ and Θ. Now we show that there will always be g points, counting multiplicity.

If we make the usual identification of S with a polygon Δ of $4g$ sides, we have a natural lift $\tilde{\mu} : \Delta \to V$ such that
$$\tilde{\mu}(z) = (\int_b^z \omega_1, \cdots, \int_b^z \omega_g)$$
fits into the diagram
$$\begin{array}{ccc} \Delta & \xrightarrow{\tilde{\mu}} & V \\ \downarrow & & \downarrow \\ S & \xrightarrow[\mu]{} & \mathcal{J}(S) \end{array}$$
Then to calculate the points of intersection of $\mu(S)$ and Θ, it suffices to calculate the zeroes of the holomorphic function $\tilde{\mu}^*\theta$ on Δ. We have that the number of zeroes of $\tilde{\mu}^*\theta$ counting multiplicity is equal to
$$\frac{1}{2\pi i}\int_{\partial\Delta} d\log\theta(\tilde{\mu}(z)).$$
Per the usual technique for integration along the boundary of the polygonal identification of a compact Riemann surface, we first examine integrating only on the sides "γ_α" and "γ_α^{-1}", which are identified under $\Delta \to S$ with the loop γ_α. First, examine the value of
$$\tilde{\mu}(p') - \tilde{\mu}(p)$$
where $p \in \gamma_\alpha$ and $p' \in \gamma_\alpha^{-1}$ are points on the sides of Δ identified under $\Delta \to S$ with the same point on the loop γ_α. We have that the k-th coordinate of this quantity is
$$\tilde{\mu}_k(p') - \tilde{\mu}_k(p) = \int_b^{p'} \omega_k - \int_b^p \omega_k = \int_p^{p'} \omega_k = \int_{\gamma_{\alpha+g}} \omega_k = Z_{k\alpha},$$
so $\tilde{\mu}(p') - \tilde{\mu}(p)$ equals $\lambda_{\alpha+g}$. Then we have
$$\theta(\tilde{\mu}(p')) = e^{-2\pi i z_\alpha(\tilde{\mu}(p')) - \pi i Z_{\alpha\alpha}}\theta(\tilde{\mu}(p)) \implies \log(\theta(\tilde{\mu}(p'))-\log(\theta(\tilde{\mu}(p)) = -2\pi i z_\alpha(\tilde{\mu}(p'))-\pi i Z_{\alpha\alpha}.$$
Thus, we have that
$$\frac{1}{2\pi i}\int_{\gamma_\alpha \cup -\gamma_\alpha^{-1}} d\log\theta(\tilde{\mu}(z)) = \frac{1}{2\pi i}\int_{-\gamma_\alpha^{-1}} d(2\pi i z_\alpha(\tilde{\mu}(z)) + \pi i Z_{\alpha\alpha}) = \int_{\gamma_\alpha} dz_\alpha = 1.$$
Then integrating $d\log(\theta(\tilde{\mu}(z))$ along γ_α and $-\gamma_\alpha^{-1}$ produces the quantity g.

Now we verify that integrating along the other $2g$ sides contributes zero. Consider the sides "$\gamma_{\alpha+g}$" and "$\gamma_{\alpha+g}^{-1}$". Let $p' \in$ "$\gamma_{\alpha+g}^{-1}$" and $p \in$ "$\gamma_{\alpha+g}$" be identified. Then we have
$$\tilde{\mu}_k(p') - \tilde{\mu}_k(p) = \int_b^{p'} \omega_k - \int_b^p \omega_k = \int_p^{p'} \omega_k = \int_{-\gamma_\alpha} \omega_k = -\delta_{*\alpha} = -e_\alpha = -\lambda_\alpha,$$
so
$$\theta(\tilde{\mu}(p')) = \theta(\tilde{\mu}(p)) \implies \log\theta(\tilde{\mu}(p')) - \log\theta(\tilde{\mu}(p)) = 0.$$
Thus,
$$\frac{1}{2\pi i}\int_{\gamma_{\alpha+g} \cup -\gamma_{\alpha+g}^{-1}} d\log\theta(\tilde{\mu}(z)) = 0.$$

Thus, we find that

$$\frac{1}{2\pi i}\int_{\partial\Delta} d\log\theta(\tilde{\mu}(z)) = g,$$

which implies our claim that if $\mu(S)\not\subset\Theta$, which means that if $\mu^*\tilde{\theta}$ is not just the zero section, then $\mu(S)$ and Θ intersect at g points counting multiplicity.

If, however, $\mu(S)\subset\Theta$, which means that $\tilde{\theta}$ pulls back to the zero section of $\mu^*\mathcal{L}$, then we can find some $u\in M$ such that $\mu(S)\not\subset\Theta_u$ and an analagous argument follows. □

For $u\in M$ such that $\mu(S)\not\subset\Theta_u$, let $\mu^*\Theta_u = p_1(u)+\cdots+p_g(u)$ denote the g points. Amazingly, up to a fixed constant, these points $p_i(u)$ are the solution to the Jacobi Inversion problem for $u\in M$.

Theorem 4.2. *There exists a fixed constant $c\in\mathfrak{J}(S)$ such that for all $u\in M$ such that $\mu(S)\not\subset\Theta_u$,*

$$c+\sum_{i=1}^{g}\mu(p_i(u)) = u.$$

Proof. Just as in the proof of Theorem 4.1, let Δ denote the $4g$-polygon whose edges glue to S, and let $\tilde{\mu}:\Delta\to V$ denote the lift of μ. Fix $u\in M$ such that $\mu(S)\not\subset\Theta_u$. Then $p_1(u),\cdots,p_g(u)$ are the zeroes of $\theta_u(\tilde{\mu}(z))$. Fixing the k-th coordinate for $1\le k\le g$, by the residue theorem, we have

$$\sum_{i=1}^{g}\tilde{\mu}_k(p_i)(u) = \frac{1}{2\pi i}\int_{\partial\Delta}\tilde{\mu}_k(z)d\log\theta_u(\tilde{\mu}(z)).$$

Just as before, we unpack the integral by examining what happens on sides which glue. First, we consider the sides "γ_α" and "γ_α^{-1}". Let $p'\in$ "γ_α^{-1}" and $p\in$ "γ_α". Just as in the proof of Theorem 4.1, we have

$$\tilde{\mu}_k(p')-\tilde{\mu}_k(p) = Z_{k\alpha} \implies \tilde{\mu}(p')-\tilde{\mu}(p) = \lambda_{\alpha+g}.$$

Then

$$\theta_u(\tilde{\mu}(p')) = e^{-2\pi i z_\alpha(\tilde{\mu}(p'))-\pi i Z_{\alpha\alpha}+2\pi i z_\alpha(u)}\cdot\theta_u(\tilde{\mu}(p)).$$

Then

$$\log\theta_u(\tilde{\mu}(p'))-\log\theta_u(\tilde{\mu}(p)) = -2\pi i z_\alpha(\tilde{\mu}(p'))-\pi i Z_{\alpha\alpha}+2\pi i z_\alpha(u).$$
$$\implies d\log\theta_u(\tilde{\mu}(p'))-d\log\theta_u(\tilde{\mu}(p)) = -2\pi i\omega_\alpha(p').$$

Then

$$\frac{1}{2\pi i}\int_{\gamma_\alpha\cup-\gamma_\alpha^{-1}}\tilde{\mu}_k(z)d\log\theta_u(\tilde{\mu}(z)) = \frac{1}{2\pi i}\int_{\gamma_\alpha}\tilde{\mu}_k(z)d\log\theta_u(\tilde{\mu}(z))+\frac{1}{2\pi i}\int_{-\gamma_\alpha^{-1}}\tilde{\mu}_k(z)d\log\theta_u(\tilde{\mu}(z))$$
$$= \frac{1}{2\pi i}\int_{\gamma_\alpha}\tilde{\mu}_k(z)d\log\theta_u(\tilde{u}(z))+\frac{1}{2\pi i}\int_{\gamma_\alpha}(-\tilde{\mu}_k(z)-Z_{k\alpha})[d\log\theta_u(\tilde{\mu}(z))-2\pi i\omega_\alpha(z)].$$
$$= \int_{\gamma_\alpha}\tilde{\mu}_k(z)\cdot\omega_\alpha(z)-\frac{Z_{k\alpha}}{2\pi i}\int_{\gamma_\alpha}d\log\theta_u(\tilde{u}(z))+Z_{k\alpha}.$$

The first and third terms are constant and independent of $u \in M$, so they may be absorbed into the k-th coordinate c_k of the constant c. We claim that the second term is also constant and can be absorbed into c_k. Note that if x and x' denote the ordered endpoints of γ_α, then $\tilde{\mu}_k(x')-\tilde{\mu}_k(x) = \int_x^{x'} \omega_k = \int_{\gamma_\alpha} \omega_k = \delta_{k\alpha}$. Then $\tilde{\mu}(x')-\tilde{\mu}(x) = e_\alpha$. Then θ_λ takes the same value at both $\tilde{\mu}(x')$ and $\tilde{\mu}(x)$. Then $\theta_u\tilde{\mu}(\gamma_\alpha)$ is a closed loop in $\mathbb{C}$, so $\frac{1}{2\pi i}\int_{\gamma_\alpha} d\log\theta_u(\tilde{u}(z)) \in \mathbb{Z}$, since the integral calculates a winding number. So the integral

$$\frac{1}{2\pi i}\int_{\gamma_\alpha \cup -\gamma_\alpha^{-1}, 1\leq\alpha\leq n} \tilde{\mu}_k(z) d\log\theta_u(\tilde{\mu}(z))$$

only contributes to the constant c_k.

Now let us integrate around the sides "$\lambda_{\alpha+g}$" and "$\lambda_{\alpha+g}^{-1}$". Let $p \in$ "$\lambda_{\alpha+g}$ and $p' \in \lambda_{\alpha+g}^{-1}$. Just as in the proof of Theorem 4.1, we have

$$\tilde{\mu}_k(p') - \tilde{\mu}_k(p) = -\delta_{\alpha k} \implies \tilde{\mu}(p') - \tilde{\mu}(p) = -\lambda_\alpha = -e_\alpha \implies \theta_u(\tilde{\mu}(p')) = \theta_u(\tilde{\mu}(p)).$$

We have then that

$$\frac{1}{2\pi i}\int_{\gamma_{\alpha+g}\cup-\gamma_{\alpha+g}^{-1}} \tilde{\mu}_k(z) d\log\theta_u(\tilde{\mu}(z)) = \frac{1}{2\pi i}[\int_{\gamma_{\alpha+g}} \tilde{\mu}_k(z) d\log\theta_u(\tilde{\mu}(z)) + \int_{-\gamma_{\alpha+g}^{-1}} \tilde{\mu}_k(z) d\log\theta_u(\tilde{\mu}(z))]$$

$$= \frac{1}{2\pi i}[\int_{\gamma_{\alpha+g}} \tilde{\mu}_k(z) d\log\theta_u(\tilde{\mu}(z)) - \int_{\gamma_{\alpha+g}} (\tilde{\mu}_k(z) - \delta_{\alpha k}) d\log\theta_u(\tilde{\mu}(z))] = \frac{\delta_{\alpha k}}{2\pi i}\int_{\gamma_{\alpha+g}} d\log\theta_u(\tilde{\mu}(z)).$$

If we let x, x' denote the ordered endpoints of $\gamma_{\alpha+g}$, then $\tilde{\mu}(p') - \tilde{\mu}(p) = \lambda_{\alpha+g}$ per the proof of Theorem 4.1. This implies that

$$\theta_u(\tilde{\mu}(p')) = e^{-2\pi i z_\alpha(\tilde{\mu}(p')) - \pi i Z_{\alpha\alpha} + 2\pi i z_\alpha(u)} \cdot \theta_u(\tilde{\mu}(p))$$

$$\implies \frac{1}{2\pi i}\int_{\gamma_{\alpha+g}} d\log\theta_u(\tilde{\mu}(z)) = -z_\alpha(\tilde{\mu}(p')) - \frac{\pi i}{2} Z_{\alpha\alpha} + z_\alpha(u) \mod \mathbb{Z}$$

$$\implies \frac{\delta_{\alpha k}}{2\pi i}\int_{\gamma_{\alpha+g}} d\log\theta_u(\tilde{\mu}(z)) - \delta_{\alpha k} z_\alpha(u) = -\delta_{\alpha k} z_\alpha(\tilde{\mu}(z)) - \frac{\pi i}{2} Z_{\alpha\alpha} \mod \mathbb{Z},$$

and the right hand side are constants that we absorb into c_k. This shows that for the coordinate k, there exist constant c_k such that

$$c_k + \sum_{i=1}^{g} \tilde{\mu}_k(p_i) = u_k \mod \mathbb{Z},$$

which implies that

$$c + \sum_{i=1}^{g} \mu(p_i) = u \in \mathfrak{J}(S).$$

□

Note for general $u \in \mathfrak{J}(S)$, we have $\mu(S) \not\subset \Theta_u$. Thus, the Jacobi Inversion problem can be solved for general $u \in \mathfrak{J}(S)$.

5. Riemann's theorem and conclusion

To conclude our exposition, we state what the constant c in Theorem 4.2 is in terms of the geometry of the Theta divisor Θ. This is otherwise known as Riemann's theorem.

Let $S^{(d)}$ denote the complex manifold which is the quotient manifold of $S^{\times d}$ by the natural action of the symmetric group Σ_d on d letters. In other words, $S^{(d)}$ parameterizes effective divisors of degree d. Consider the map

$$\mu^{(d)} : S^{(d)} \to \mathcal{J}(S)$$

given by

$$p_1 + \cdots + p_d \mapsto (\sum_{i=1}^{d} \int_b^{p_i} \omega_1, \cdots, \sum_{i=1}^{d} \int_b^{p_i} \omega_g).$$

Suppose $D = \sum_{i=1}^d p_i \in S^{(d)}$ is an effective divisor such that all the p_i are distinct. Let z_i denote local coordinates around each p_i. Then locally we can write $\omega_i = \Omega_i(p_j)dz_j$ in terms of the local coordinates z_j around p_j. Then the differential of the map $\mu^{(d)}$ at D is given by

$$\mathcal{J}(\mu^{(d)}) = \begin{pmatrix} \Omega_1(p_1) & \cdots & \Omega_g(p_1) \\ \vdots & & \vdots \\ \Omega_1(p_d) & \cdots & \Omega_g(p_d) \end{pmatrix}.$$

Note that the complete linear system

$$|\Omega_S| : S \to \mathbb{P}^{g-1}$$

given by Ω_S is always base point free for S when $g \geq 2$ by Theorem 6.1. The complete linear system sends

$$p_j \in S \mapsto [\Omega_1(p_j) : \cdots : \Omega_g(p_j)].$$

We see then that $\mathcal{J}(\Phi)$ has full rank at D when the images of the p_i under $|\Omega_S|$ are linearly independent, which we see is generically the case when $d \leq g$. This implies that the image $W_d = \mu^{(d)}(S^{(d)})$ is an analytic subvariety of dimension d. Furthermore, the map $\mu^{(d)}$ is generically one-to-one because the fibers of $\mu^{(d)}$ are linear spaces. Note that since $\Theta \subset \mathcal{J}(S)$ is an effective divisor, it is of dimension g-1. Miraculously, c is determined by the simple relationship between Θ and W_{g-1}.

Theorem 5.1 (Riemann's Theorem)**.** *Let c denote the constant in Theorem 4.2. Then the Theta divisor Θ is exactly the translation of W_{g-1} by c. In other words,*

$$\Theta = W_{g-1} + c.$$

The proof of Riemann's theorem is described in [GH94, Page 338-340]. In particular, one is also able to analyze exactly when $\mu(S) \subset \Theta_u$. Specifically, $\mu(S) \subset \Theta_c + u$ if and only if $u = \mu^{(g)}(D)$ for $D \in S^{(g)}$ such that $h^0(D) > 0$. In other words, there fails to be a solution exactly when the fiber of the Abel-Jacobi map, which is a projective space, has nonzero dimension.

In conclusion, the solution to the Abel-Jacobi Inversion problem is obtained by defining Theta divisors, which are translates of the image W_{g-1}, and intersecting Theta divisors with the curve $\mu(S)$.

6. Appendix

Lemma 6.1. *Let X be a compact Riemann surface of genus $g \geq 2$. The canonical bundle ω_X is basepoint free.*

Proof. Note by Serre duality we have $h^1(X, \omega_X) = h^0(X, \mathcal{O}_X) = 1$, and $h^1(X, \omega_X(-p)) = h^0(X, \mathcal{O}_X(p))$. Since X is genus $g \geq 2$, no two points are linearly equivalent (otherwise $X \cong \mathbb{P}^1$). Thus, we must have $h^0(X, \mathcal{O}_X(p)) = 1$, since there is a global section with associated effective divisor p, and any other indepedent global section would yield an effective divisor of degree 1 i.e. a point q linearly equivalent to p, which would be a contradiction. Then we see that $h^1(X, \omega_X) = h^1(X, \omega_X(-p))$, and thus by Riemann-Roch, we see that $h^0(X, \omega_X(-p)) = h^0(X, \omega_X) - 1$. Thus, ω_X is basepoint free. □

References

[GH94] Philip Griffths and Joe Harris. *Principles of Algebraic Geometry.* 1994.

[Huy05] Daniel Huybrechts. *Complex Geometry.* 2005.