COMPLEX GEOMETRY NOTES

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Abstract. A year long course (Math 232) in complex geometry taught by Professor Mihnea Popa at Harvard. All virtues of these notes should be attributed to the instructor, and all vices to the notetaker.

Contents

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1. 9/6/23: basic several complex variables

Roughly weekly homewokr starting next week. Analytic introduction to algebraic geometry. Start by understanding complex manifolds. Eventually Kahler manifolds, hodge theory. Even before that, there's a local aspect we need to discuss. Very briefly, the local theory is several complex variables. Weierstrass preparation theorem. The main interest is the global theory (compact complex manifolds, (p,q)-forms, harmonic forms, kahler manifolds, hodge theory).

The main theorems we are aiming for are Hodge decomposition theorem, which happens on compact Kahler manifolds (in particular, smooth projective varieties). Furthermore, we are aiming for Kodaira embedding, which comes with the famous Kodaira vanishing theorem. We'll learn this from a differential geometric point of view, a topological view, etc. Also Chow's theorem, every compact complex submanifold of projective space is actually algebraic, i.e apriori defined locally by analytic functions, actually globally described by algebraic equations.

The standard complex geometry book is Griffiths-Harris. Some that are heavier is Voisin; another good one is by Huybrechts. About the local theory: Gunning-Rossi analytic functions of several complex variables is good. Huybrechts also has an introduction.

Definition 1.1. Let $U \subseteq \mathbb{C}^n$ open. A function $f : U \to \mathbb{C}$ is holomorphic if for every point $p = (p_1, \cdots, p_n)$ has a neighborhood $V \subseteq U$ such that on V, we can write f as a convergent power series:

$$f(z) = \sum_{i_1,\cdots,i_n=0}^{\infty} a_{i_1\cdots i_n}(z_1 - p_1)^{i_1} \cdots (z_n - p_n)^{i_n}.$$

To simplify notation, sometimes we write $I = (i_1, \cdots, i_n)$ so that

$$f(z) = \sum_I a_I (z-p)^I.$$

We'll say that $\mathcal{O}(U) = \{$holomorphic functions on U $\}$ which forms a $\mathbb{C}$-algebra.

We can write $f : U \to V$ where $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ as $f = (f_1, \cdots, f_m)$ as an m-tuple of $\mathbb{C}$-valued functions, where f is holomorphic if all f_i are so.

Now $f : U \to \mathbb{C}$ being holomorphic in several complex variables is equivalent to being holomorphic in each variable. We'll introduce an analog of Cauchy integral's formula in $\mathbb{C}^n$.

Proposition 1.2 (Osgood's Lemma)**.** *Let $U \subseteq \mathbb{C}^n$. Then $f : U \to \mathbb{C}$ is holomorphic $\iff$ f is continuous and holomorphic in each variable separately.*

Proof. In the forward direction, this is trivial. Let's do the reverse direction. Pick $p = (p_1, \cdots, p_n) \in U \subseteq \mathbb{C}^n$. Fix a "multi-radius" $r = (r_1, \cdots, r_n)$. Fix a polydisk (which form a basis for the topology)

$$\overline{\Delta(p,r)} = \{z \in \mathbb{C}^n | |z_i - p_i| \leq r, \forall i\}.$$

Pick $z \in \Delta(p, r)$. Apply Cauchy's formula successively. So we have that

$$f(z) = \frac{1}{2\pi i} \int_{|w_n - p_n| = r_n} \frac{f(z_1, \cdots, z_{n-1}, w_n)}{w_n - z_n} dw_n = \cdots$$

$$= \frac{1}{(2\pi i)^n} \int_{|w_1 - p_1| = r_1} \cdots \int_{|w_n - p_n| = r_n} \frac{f(w_1, \cdots, w_n)}{(w_1 - z_1) \cdots (w_n - z_n)} dw_1 \cdots dw_n.$$

Since f is continuous, by Fubini's, to get:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{T(p,r)} \frac{f(w_1, \cdots, w_n)}{(w_1 - z_1) \cdots (w_n - z_n)} dw_1 \cdots dw_n$$

where $T(p, r) = \{z | |z_i - p_i| = r_i\}$. We've called it T for torus, because $T(p, r)$ looks more like a product of a bunch of S^1. Note $T(p, r)$ is not $\partial\Delta(p, r)$.

We want to get a power series out of this integral, just as we do in the one variable case. We have the formula

$$\frac{1}{(w_1 - z_1) \cdots (w_n - z_n)} = \sum_{i_1 \cdots i_n)} = \sum_I \frac{(z_1 - p_1)^{i_1} \cdots (z_n - p_n)^{i_n}}{(w_1 - p_1)^{i_1+1} \cdots (w_n - p_n)^{i_n+1}}$$

Substitute to get

$$f(z) = \sum_I a_{i_1 \cdots i_n} (z_1 - p_1^{i_1}) \cdots (z_n - p_n)^{i_n}$$

where

$$a_{i_1 \cdots i_n} = \frac{1}{(2\pi i)^n} \int_{T(p,r)} \frac{f(w_1, \cdots, w_n)}{(w_1 - p_1)^{i_1+1} \cdots (w_n - p_n)^{i_n+1}} dw_1 \cdots dw_n.$$

□

A remark: you don't need to assume continuity, but it is a hard theorem of Hartog.

Let's also discuss Cauchy Riemann in several complex variables. In the one variable situation, we have $f : U \to \mathbb{C}$ where $f = f_1 + if_2$, and we say that f is holomorphic $\iff$ they satisfy $\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}$ and $\frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}$. If $z = x + iy$, then $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. So in other words, f is holomorphic $\iff$ $\frac{\partial}{\partial \bar{z}} = 0$.

Now if we go to n variables, $f : U \to \mathbb{C}$ where $U \subseteq \mathbb{C}^n$. By Osgood's lemma 1.2, we have f holomorphic $\iff$

$$\frac{\partial f}{\partial \overline{z_1}} = \cdots = \frac{\partial f}{\overline{z_n}} = 0.$$

Viewing f as its power series around p, it is easy to see (believe) that just by taking partials and evaluating at p, we have

$$a_I = \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1 + \cdots + i_n} f}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}(p).$$

Theorem 1.3 (Identity principle)**.** *Let $U \subseteq \mathbb{C}^n$ be a connected open set, and $f, g : U \to \mathbb{C}$ holomorphic functions such that there exists open $\emptyset \neq V \subseteq U$ with $f(z) = g(z)$, $\forall z \in V$, then $f = g$.*

Proof. Enough to assume $g = 0$. Let

$$B = \{z \in U | f(z) = 0\}.$$

This is nonempty since f is zero on V. Now we want to show that $int(B)$ is closed. If so, this would violate connectedness of U. Let $z_0 \in \overline{int(B)}$. We want to show that $z_0 \in int(B)$. Note that we can find some element $w_0 \in int(B)$ such that $\Delta(w_0, r)$ is a small open which contains z_0. On this open $\Delta(w_0, r)$, g has a power series expansion which is 0 everywhere. But f also has a power series expansion at w_0, and shrinking this neighborhood so it is contained in U, we see that all the coefficients of this power series of f at w_0 must also be zero. Then this implies that $f(z_0) = 0$, so $z_0 \in int(B)$. □

In the Zariski topology, this is easy to accomplish, because the open sets are so huge. But in the complex analytic topology, this is harder to achieve. Many times there will be things, such as coherence of direct image sheaves, that is easier in algebraic geometry but much much harder in complex geometry.

We also have an analog of the maximum modulus principle for several complex variables.

Theorem 1.4 (maximum principle)**.** *Let $U \subseteq \mathbb{C}^n$ and $f : U \to \mathbb{C}$ holomorphic function such that there $\exists z_0 \in U$ satisfying*

$$|f(z)| \leq |f(z_0)|$$

for every $z \in U$. Then f is constant.

Proof. Pick $z \in \Delta(z_0, r) \subset U$. Then define

$$g : V \subset \mathbb{C} \to \mathbb{C}$$

for some sufficient V where $g(w) = f(z_0 + w(z - z_0))$. Note g is holomorphic. Furthermore, we see that $|g(0)| \geq |g(w)|$ for every $w \in V$. Thus, by MMP in single variables, we have that $g(0) = g(1)$, i.e $f(z_0) = f(z)$. Thus, we see that f is constant. □

Theorem 1.5 (Hartog's Theorem)**.** *Let $n \geq 2$. Let $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ and $\epsilon' = (\epsilon_1', \cdots, \epsilon_n')$ such that $\epsilon_i' < \epsilon_i$. Then a holomorphic*

$$f : \Delta(0, \epsilon) \setminus \overline{\Delta(0, \epsilon')} \to \mathbb{C}$$

can be uniquely extended to a $\tilde{f}$ holomorphic on all of $\Delta(0, \epsilon)$.

Proof. We can assume that $\epsilon = (1, \cdots, 1)$. Then there exist δ such that

$$V = \{z | 1 - \delta < |z_1| < 1, |z_{i \neq 1}| < 1\} \cup \{z | 1 - \delta < |z_2| < 1, |z_{i \neq 2}| < 1\}$$

is an open set in the complement. Note that f is holomorphic on V, so fixing $w = (w_2, \cdots, w_n)$ where $|w_i| < 1$, we have for $1 - \delta < |z_1| < 1$ there is a laurent series expansion

$$f_w(z_1) = \sum a_n(w) z_1^n.$$

Note that

$$a_n(w) = \frac{1}{2\pi i} \int_{|\xi| = 1 - \frac{\delta}{2}} \frac{f_w(\xi)}{\xi^{n+1}} d\xi$$

which implies that $a_n(w)$ is holomorphic as a function of w.

Note, however, that f is also holomorphic on the region $(z_1, w_2, \cdots, w_n)$ where $1 - \delta < |z_1|, |w_2| < 1$. Then on this region, we have that $a_n(w) = 0$ for $n < 0$. Then by the identity principle, $a_n(w) = 0$ for $|w_i| < 1$ and $n < 0$. Thus, we can extend f holomorphic on $\{1 - \delta < z_1 < 1\} \times \{|w_i| < 1\}$ to $\tilde{f}$ which is holomorphic on all of $\Delta(0, (1, \cdots, 1))$. □

Theorem 1.6 (Riemann Extension Theorem)**.** *Let $f \in \mathcal{O}(U)$ be a holomorphic function, and let $g : U \setminus Z(f) \to \mathbb{C}$ also be a holomorphic function locally bounded on $Z(f)$. Then g can be uniquely extended to a holomorphic function $\tilde{g} : U \to \mathbb{C}$.*

Proof. We can assume U is some ϵ polydisk around the origin $B_\epsilon(0)$. We can also assume that $Z(f)$ does not contain $U \cap \{(z_1, 0, \cdots, 0)\}$ (surely f does not vanish on every line, otherwise $U \setminus Z(f) = \emptyset$). Furthermore, if we restrict f to $U \cap \{(z_1, 0, \cdots, 0)\}$, then f becomes a one variable analytic function and thus has isolated zeroes. We can thus reduce to the case that this restriction of f vanishes only at the origin. Then there exists ϵ_1' s.t. for $|z_1| = \epsilon_1$, we have $f(z_1, 0, \cdots, 0) \neq 0$. Then there exists a polydisc given by $\epsilon_2', \cdots, \epsilon_n'$, such that $f(z_1, w_2, \cdots, w_n) \neq 0$ for $|z_1| = \epsilon_1'$, $|w_i| = \epsilon_i'$ for $i = 2 \cdots n$. We can replace $B_\epsilon(0)$ with this $B_{\epsilon'}(0)$, whichever is smaller.

Then fixing w s.t. $|w_i| < \epsilon_i'$, we have $f_w(z_1)$ does not vanish at all for $|z_1| = r$, so g_w is defined on the boundary of this one-dimensional disc. Furthermore, it has only isolated zeroes on this one-dimensional disc, and it is locally bounded, thus it can be analytically continued to some

$$\tilde{g}_w(p) = \frac{1}{2\pi} \int_{|\xi|=\epsilon_1'} \frac{g_w(\xi)}{\xi - p} d\xi.$$

This is also holomorphic in terms of w. Thus, we see that g can be uniquely extended to a $\tilde{g}$. □

2. 9/11/23: Germ of holomorphic functions, Weierstrass preparation theorem

The topic today is germs of holomorphic functions. We've defined holomorphic function on an open set of $\mathbb{C}^n$. Now we want to study an invariant that arises in a neighborhood of a point.

Definition 2.1. Let $z_0 \in U, V$ open sets in $\mathbb{C}^n$. Taking $f \in \mathcal{O}(U), g \in \mathcal{O}(V)$, we say f, g are equivalent (at z_0) if there exists a nonempty open set $W \subset U \cap V$ such that $f|_U = g|_U$. The equivalence class is called the **germ of f** at z_0.

We write $\mathcal{O}_{z_0}$ for the ring of germs of holomorphic functions at z_0. If $V \subseteq U$, then there exist restriction maps $\mathcal{O}(U) \to \mathcal{O}(V)$. These form an inductive system, and the stalk is the direct limit

$$\mathcal{O}_{z_0} = \varinjlim \mathcal{O}(U).$$

What we are really talking about here is the sheaf of holomorphic functions over $\mathbb{C}$, and the stalk of the sheaf at z_0.

By a change of coordinates, we can always assume that z_0 is the origin, so that $\mathcal{O}_{z_0} \cong \mathcal{O}_{origin} =: \mathcal{O}_n$, where n denotes the dimension of $\mathbb{C}^n$. This ring

$$\mathcal{O}_n = \text{ ring of convergent power series in n variables} = \mathbb{C}[[z_1, \cdots, z_n]].$$

This is because we defined a function is holomorphic at a point if there's a neighborhood where there's a power series expansion. By identity principle, this unique in maybe some smaller open neighborhood.

Now we want to understand this ring of convergent power series, from the point of view of commutative algebra. That is our main goal for a while.

Proposition 2.2. *$\mathcal{O}_n$ is a local ring with maximal ideal $\underline{m}_n = (z_1, \cdots, z_n)$. The formal way of defining the maximal ideal is*

$$\underline{m}_n = \{f | f(z_0) = 0\}.$$

Proof. One way of showing $\underline{m}_n$ is a maximal ideal is showing that everything outside of it, is invertible. Clearly, $f \notin \underline{m}_n \iff f(0) \neq 0 \iff \frac{1}{f}$holo in some nbhd of 0. □

Remark: you can show that $\mathcal{O}_n$ is an integral domain also through the identity principle, as opposed to showing $\mathcal{O}_n \cong \mathbb{C}[[z_1, \cdots, z_n]]$.

Lemma 2.3. *$\underline{m}_n / \underline{m}_n{}^2$ is a $\mathbb{C}$-vector space of dimension n. In other words, it is a regular local ring of dimension n.*

Proof. $\underline{m}_n / \underline{m}_n^2 \cong m_n \otimes_{\mathcal{O}_n} \mathcal{O}_n / m_n \cong \mathbb{C}^{\text{num of generators of } m_n} \cong \mathbb{C}^n$. □

The goal of the next lecture and a half is to show that $\mathcal{O}_n$ is a Noetherian UFD. The work we need to do here is to essentially pretend here that we are working with polynomials. These are two famous Weierstrass theorems.

Here's some motivation. We can do something like

$$\mathcal{O}_{n-1} \subseteq \mathcal{O}_{n-1}[z_n] \subseteq \mathcal{O}_n.$$

To show Noetherian, note if we know $\mathcal{O}_{n-1}$ is Noetherian, then by the Hilbert basis theorem the middle term is Noetherian, and then compare the last two to conclude $\mathcal{O}_n$ is Noetherian.

Definition 2.4. An element $h \in \mathcal{O}_{n-1}[z_n]$ of the form

$$z_n^d + c_{d_1} z_n^{d_1} + \cdots + c_1 z + c_0 | c_i \in \mathcal{O}_{n-1}$$

is a Weierstrass polynomial if $\forall c_i \in m_{n-1}$.

Definition 2.5. Let $o \in U \subseteq \mathbb{C}^n$, $f \in \mathcal{O}(U)$. Say f is regular in z_n if $f(0, z_n) \not\cong 0$. It is regular of degree d if $f(0, z_n) = z_n^d * u(z_n)$, where $u(0) \neq 0$.

Example 2.6. Let h be a Weierstrass polynomial

$$z_n^d + c_{n-1} z_n^{d_1} + \cdots + c_0$$

where $c_i \in m_{n-1}$. Then $h(0, z_n) = z_n^d$. So the Weierstrass polynomials are the "obvious" examples of regular of degree d.

Can we construct other regular of degree d functions that are not the Weierstrass polynomials? The answer is no, and this is the Weierstrass preparation theorem.

Example 2.7. Note $f(z) = z_i$ $i \neq n$ is not regular in z_n. But can always change of coordinates.

Lemma 2.8. *If $f_1, \cdots, f_k \in \mathcal{O}_n \setminus \{0\}$, then there exists a change of coordinates such that f_i are all regular w.r.t to z_n.*

Proof. If $f = f_1 \cdots f_k$, this reduces to the case $k = 1$. There exists $w_0 \in \mathbb{C}^n$ such that for $t \in V \subseteq \mathbb{C}$ neighborhood of 0, have $f(t * w_0) \neq 0$. Change coordinates to make $w_0 = (0, \cdots, 0, 1)$. This implies $f(0, z_n) = f(z_n * w_0) \not\cong 0$. $\square$

Theorem 2.9 (Weierstrass Preparation Theorem)**.** *Let $f \in \mathcal{O}_n$ which is regular of degree d in z_n. Then there $\exists!$ Weierstrass polynomial h of degree d such that $f = uh$, where u is invertible around 0 (i.e unit in $\mathcal{O}_n$).*

Proof. We're looking for a polynomial. If you know the roots, you can get back that polynomial. That's what we're going to try to do.

Suppose f is regular of degree d in z_n. We may assume that $d > 0$, otherwise f is a unit. Take $f \in \mathcal{O}(U)$ in this equivalence class. So there $\exists r > 0$ and $\delta > 0$ such that $|f(0, z_n)| \geq \delta$ for $|z_n| = r$. By continuity, there exists $\epsilon > 0$ such that $|f(w, z_n)| \geq \delta'$ if $|z_n| = r$ and $|w| \leq \epsilon$, where $w = (w_1, \cdots, w_{n-1})$.

Let $w \in \mathbb{C}^{n-1}$, $|w| \leq \epsilon$, fixed and define

$$N(w) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{\partial f / \partial z_n(w, \xi)}{f(w, \xi)} d\xi.$$

By the residue theorem, N counts the number of zeroes of $f(w, \xi)$ seen as a single variable function in ξ (with multiplicities) inside $|\xi| < r$. But $N(0) = d$ by hypothesis ($f(0, z_n) = z_n^d u(z_n)$). in fact, we have $N(w) = d, \forall |w| \leq \epsilon$. Call the zeroes $\xi_1(w), \cdots, \xi_d(w)$.

Define $h(w, z_n) = \prod_{j=1}^{d}(z - \xi_j(w)) = z_n^d - \sigma_1(w)z_n^{d-1} + \cdots + (-1)^d\sigma_d(w)$, where $\sigma_i(w)$ are the elementary symmetric functions in $\xi_j(w)$. The symmetric functions $\sigma_i(w)$ actually vary holomorphically. This is an application of the Residue theorem, applied to a slightly different function:

- Let $g(z)$ holomorphic one variable, $g(a) = 0$ and $m =$ multiplicity of a as a zero. Then
$$Res_{z=a}\frac{z^k f'(z)}{f(z)} = ma^k.$$
Then the Residue theorem implies that
$$\xi_1(w)^k + \cdots + \xi_d(w)^k = \frac{1}{2\pi i}\int_{|\xi|=r}\frac{\xi^k \frac{\partial f}{\partial z_n}(w, \xi)}{f(w, \xi)}d\xi$$
holomorphic in w, by some dominated convergence?
You can always write things like
$$2\sum_{i\neq j} x_i x_j = (\sum x_i)^2 - (\sum x_i^2)$$
so this implies that $\sigma_i(w)$ are holomorphic in w. So $h \in \mathcal{O}_{n-1}[z_n]$.

Now consider (for $|w| < \epsilon$, $|z_n| < r$) we have
$$\mu(w, z_n) = \frac{f(w, z_n)}{h(w, z_n)}$$
. We want to show that μ is holomorphic. Away from the zeroes of h, there's no problem. At the zeroes of h, we have the same zeroes with same multiplicities in f, so they cancel and so they're all removable singularities. So add the removable singularities, so μ still holomorphic in z_n. Then
$$\mu(w, z_n)\frac{1}{2\pi i}\int\frac{u(w, \xi}{\xi - z_n}d\xi$$
this shows that u is holomorphic in w variables as well. Details in lemma 1.1.3

So u is a unit, because
$$u(0, z_n) = \frac{f(0, z_n)}{h(0, z_n)} = \frac{z_n^d u_n}{z_n^d}$$
where $u_n(0) \neq 0$. So $f = uh$.

Uniqueness follows immediately from how we constructed the polynomial. The elementary symmetric functions $\sigma_i(w)$ in terms of $\xi_j(w)$, so these were determined from the zeroes of f, so h is uniqluey determined from f by construction, so u is determined by f. □

The main corollary of the Weierstrass preparation theorem is:

Proposition 2.10. *$\mathcal{O}_n$ is a UFD.*

Proof. For $n = 0$, this is $\mathbb{C}$, done. Proceed by induction, and suppose our claim holds for $\mathcal{O}_{n-1}$. Then by Gauss' lemma, $\mathcal{O}_{n-1}[z_n]$ is also a UFD. Take $f \in \mathcal{O}_n$. By the lemma, we may assume that f is regular in z_n. Then the Weierstrass preparation theorem implies that $f = u * h$ where u is a unit, and h is a Weierstrass polynomial in $\mathcal{O}_{n-1}[z_n]$. If h is irreducible, we're done. If not, say $h = f_1 f_2$. Then apply the Weierstrass preparation theorem to f_1 and f_2. So $f_i = u_i h_i$. Then
$$uh = f = (u_1 u_2)(h_1 h_2)$$

and h_1h_2 is another Weierstrass polynomial, but by the uniqueness part of Weierstrass preparation theorem, we have $h = h_1h_2$. So we have unique factorization because h lives in $\mathcal{O}_{n-1}[z_n]$. □

So to continue with properties of polynomials: Weierstrass divison theorem. It's a sort of version of the Euclidean divison algorithm. This is how we're going to get Noetherian.

Afterwards, we'll talk about analytic sets and complex manifolds.

3. 9/13/23: Weierstrass division theorem, analytic germs and sets

Last time we did the Weierstrass preparation theorem, and we deduced that $\mathcal{O}_n$ is a UFD. This has a companion theorem, called the Weierstrass divison theorem.

Theorem 3.1 (Weierstrass Division Theorem)**.** *Let $h \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial of degree d. Then for every $f \in \mathcal{O}_n$, there $\exists!$ formula $f = gh + r$ with $g \in \mathcal{O}_n$ and $r \in \mathcal{O}_{n-1}[z_n]$ with $\deg r < d$. Moreover, if $f \in \mathcal{O}_{n-1}[z_n]$, then $g \in \mathcal{O}_{n-1}[z_n]$.*

Proof. Recall from the proof of the Weierstrass preparation theorem, we can choose $r, \epsilon > 0$ sufficiently small such that $\forall w \in \mathbb{C}^{n-1}$, with $|w| < \epsilon$, implies $h(w, z_n)$ has exactly d zeroes as a function of z_n for any fixed w in the range $|w| < \epsilon$ and $|z_n| < r$.

In this range, we can define

$$g(w, z_n) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(w,\xi)}{h(w,\xi)} \frac{1}{\xi - z_n} d\xi$$

which is holomorphic since the inside is continuous (?). Define $r = f - gh$. By cauchy's integral formula:

$$r(w, z_n) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(w,\xi) - h(w,z_n)\frac{f(w,\xi)}{h(w,\xi)}}{\xi - z_n} d\xi = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(w,\xi)}{h(w,\xi)} g(w,\xi,z_n) d\xi$$

where $g(w, \xi, z_n) = \frac{h(w,\xi)-h(w,z_n)}{\xi - z_n}$. If you let $h(w, z_n) = z_n^d + c_{n-1}z_n^{d-1} + \cdots$, you'll see that $g(w, \xi, z_n)$ is actually a polyomial of degree $\leq d-1$. So $g \in \mathcal{O}_{n-1}[z_n]$. So we can write

$$g = a_0(w,\xi)z_n^{d-1} + a_1(w,\xi)z_n^{d-2} + \cdots$$

so that

$$r(w, z_n) = b_0(w)z_n^{d-1} + b_1(w)z_n^{d-2} + \cdots$$

where $b_j(w) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(w,\xi)}{h(w,\xi)} a_j(w,\xi) d\xi$.

For uniqueness, you can assume that $f = 0$. If $f = qh + r$, then $r(w, z_n) = -q(w, z_n)h(w, z_n)$. But h has exactly d roots in the domain that we're considering as a function of z_n. But $r(w, z_n)$ is a degree $d-1$ polynomial in terms of z_n. Contradiction, so we must have $r = 0$, so $g = 0$.

If $f \in \mathcal{O}_{n-1}[z_n] \implies g \in \mathcal{O}_{n-1}[z_n]$. Then perform division again, so $f = g'h + r'$, but by uniqueness, $g = g'$. □

A corollary of this is:

Proposition 3.2. *$\mathcal{O}_n$ is Noetherian.*

Proof. Induct on n. $n = 0$ is done, since $\mathbb{C}$ is Noetherian. Say $\mathcal{O}_{n-1}$ is Noetherian. This immediately implies $\mathcal{O}_{n-1}[z_n]$ is also Noetherian. Take any ideal $I \subseteq \mathcal{O}_n$. Let $f \in I$. Fix $h \in I$. Up to an invertible function, h is a Weierstrass polynomial, so assume it is a Weierstrass polynomial. By the Weierstrass divsion theorem, we have $f = gh + r$, where $r \in \mathcal{O}_{n-1}[z_n]$. We can do this for every element in I. So $I = J + (h)$ where $J = I \cap \mathcal{O}_{n-1}[z_n]$. But J is finitely generated because $\mathcal{O}_{n-1}[z_n]$ is Noetherian. Thus, I is also finitely generated. □

We'll move on to analytic sets now. In this section, we'll primarily take the local perspective on analytic germs. There's the global definition of an analytic subset, but we'll need to talk about complex manifolds to properly say something about the global perspective.

There is the notion of a germ of a set. We say two sets $X, Y \subset \mathbb{C}^n$ are equivalent at z_0 if there is an open neighborhood U of z_0 such that $U \cap X = U \cap Y$. From now on, for convenience, always assume the point we're taking germs with respect to is the origin, which we can do up to change of coordinates.

An analytic germ $[X]$ is a subset X such that there exists an open U around the origin such that $U \cap X$ is cut out by holomorphic functions $f_1, \cdots, f_k \in \mathcal{O}(U)$.

Given $f \in \mathcal{O}_n$, we can consider the germ $Z(f)$. The germ $Z(f)$ is well-defined; equivalence is maintained by different representatives of f. We can define $Z(f_1, \cdots, f_k) := \cap Z(f_i)$. If f is a unit, then $Z(f)$ is the empty set. We also have the notion of an inclusion of germs, $X \hookrightarrow Y$. This means that there exists an open U such that $X \cap U \subseteq Y \cap U$.

Here's a global definition:

Definition 3.3. Let $U \subseteq \mathbb{C}^n$ be a open set. A closed subset $V \subseteq U$ is an analytic subset if for every $x \in U$, there exists a nbhd U' of x such that $U' \cap V$ is cut out by holomorphic functions $f_1, \cdots, f_k$ on U'.

Analytic germs are locally cut out by elements of $\mathcal{O}_n$. It will be useful to have a notion to work with these functions.

Definition 3.4. Let $X \subset \mathbb{C}^n$ be a germ around the origin. Then define

$$I(X) = \{f \in \mathcal{O}_n | X \subseteq Z(f)\}.$$

Proposition 3.5. *Let $X \subset \mathbb{C}^n$ and $I(X) \subset \mathcal{O}_n$.*

(1) I is an ideal
(2) Let $A \subset \mathcal{O}_n$ finite set, and (A) the ideal it generates. Then $Z(A) = Z((A))$. In particular, $Z(A)$ is an analytic germ.
(3) If $Z_1 \subset Z_2$, then $I(Z_2) \subset I(Z_1)$.
(4) If $I_1 \subset I_2$, then $Z(I_2) \subset Z(I_1)$.
(5) If X is an analytic germ, then $X = Z(I(X))$.
(6) For any ideal $J \subset \mathcal{O}_n$, $J \subset I(Z(J))$.

Definition 3.6. An analytic germ $X \subset \mathbb{C}^n$ containing the origin is irreducible if $X = X_1 \cup X_2$ implies $X = X_1$ or $X = X_2$, where X_i are analytic germs. Note that $X = X_1 \cup X_2$, is a local statement, i.e. there exist an open neighborhood around the oirigin such that $X = X_1 \cup X_2$.

Proposition 3.7. *Suppose $X \subset \mathbb{C}^n$ is an analytic germ containing the origin. Then X is irreducible $\iff$ $I(X) \subset \mathcal{O}_n$ is prime.*

Proof. Suppose X is irreducible. Suppose we have $fg \in I(X)$. Then in some open neighborhood, $X \subseteq V(fg) = V(f) \cup V(g)$. But $V(f)$ and $V(g)$ are analytic germs. So we must have $X = V(f)$ or $X = V(g)$. This implies one of f, g is in $I(X)$.

Now suppose $I(X)$ is prime. Let $X = X_1 \cup X_2$, where X_i are analytic germs. Then we can let $X_1 = V(f_i)$ and $X_2 = V(g_i)$. WLOG if all of $f_i \in I(X)$, then $X \subseteq X_1$ and then we're done. So assume WLOG $f_1, g_1 \notin I(X)$. But $f_1 g_1 \in I(X)$, contradicting $I(X)$ being prime. So we must have X is irreducible. □

This is another one of the differences between complex geometry and algebraic geometry. In complex geometry, the open sets are so fine that sometimes shapes that are irreducible in the Zariski topology are actually reducible in the analytic topology, because we can find an open around the reducible portion(s).

Example 3.8. Consider the vanishing loci of $y^2 - x^2 - x^3 \subset \mathbb{C}^2$. This is a prime ideal in $\mathbb{C}[x, y]$. But it is not prime in $\mathbb{C}[[x, y]]$. It is an irreducible algebraic hypersurface. But it is not an irreducible analytic set. In the neighborhood of 0 in the analytic topology, there's a neighborhood where it is reducible. This neighborhood doesn't exist in the Zariski topology.

To show that it is reducible in the analytic set, we want to write $y^2 - x^2 - x^3 = fg$ in some neighborhood, where $f, g \in \mathbb{C}[[x, y]]$. Pictorially, this seems obviously possible. Algebraically, eyeballing the lowest degree terms $y^2 - x^2$ gives us hope.

Look for $f = y - x + f_2 + f_3 + \cdots$ and $g = y + x + g_2 + g_3 + \cdots$. Where f_i, g_i are homogenous degree i in X, Y. Want $(y - x)g_2 + (y + x)f_2 = -x^3$. We can always find f_2, g_2 because $y - x$ and $y + x$ generate x, y. Then you need $(y - x)g_3 + (y + x)f_3 = -f_2 g_2$. And this gives you some g_3, f_3. And so on.

You can do a coordinate change, where you swap f with x, and g with y. So in a need coordinate system, the set looks like $Z(XY)$.

A similar argument shows that $h = h_r + h_{r+1} + h_{r+2} + \cdots$ such that $h_r = f_s g_t$ where f_s, g_t degrees s, t and $s + t = r$. Then we can do the same thing. There exists $f = f_s + f_{s+1} + \cdots$ and $g = g_t + g_{t+1} + \cdots$ s.t. $h = fg$ locally.

Remark: $x \in X$ smooth point of a variety. Then $\mathcal{O}_{X,x}$ is regular local ring. If m is the maximal ideal. Then you can complete $\mathcal{O}_{X,x}$ with respect to the m-adic topology, $\widetilde{\mathcal{O}_{X,x}}$. The Cohen structure theorem says that $\mathbb{C}[[x_1, \cdots, x_n]]$.

4. 9/18/23: IMPLICIT FUNCTION THEOREM, COMPLEX MANIFOLD

The following is true globally for algebraic varieties, and the key tool for proving this is Noetherianity of the local ring at hand. The same is true for analytic germs.

Proposition 4.1. *Let Z (containing origin) be an analytic germ at 0. Then we claim that in some neighborhood of zero, there exists a unique decomposition*

$$Z = Z_1 \cup \cdots \cup Z_r$$

such that Z_i are irreducible analytic germs, and $Z_i \nsubseteq Z_j$ for any $i \neq j$.

Proof. Suppose we do not have existence. Then Z is reducible in a neighborhood of 0. So $Z = Z_1 \cup Z_2$ nontrivially, where Z_1, Z_2 are analytic germs. So again, one of the Z_i is not irreducible, say Z_1. Then continue. Then there exists a chain

$$Z \supset Z_1 \supset Z_2 \supset \cdots .$$

which corresponds to a chain

$$I(Z) \subset I(Z_1) \subset I(Z_2) \subset \cdots$$

in $\mathcal{O}_n$, which we proved is Noetherian. Contradiction. So we must have existence.

To prove uniqueness, enforce non-redundancy. Then suppose we had two non-redundant decompositions

$$Z = Z_1 \cup \cdots \cup Z_r = Z'_1 \cup \cdots \cup Z'_s.$$

Then we could write, say,

$$Z_1 = (Z_1 \cap Z'_1) \cup \cdots \cup (Z_1 \cap Z'_s).$$

But since Z_1 is an irreducible analytic germ, one of these Z'_i must contain Z_1. And by symmetry, $Z'_i \subset Z_j$. Stitched together, this would violate non-redudancy $Z_i \not\subset Z_j$ for $i \neq j$. $\square$

We'll return to ideals of analytic sets once we develop some manifold language. Before we dive into manifolds, we'll mention the implicit function theorem; the proof is usually given as a corollary of the real version, and the reader need only verify holomorphicity for peace of mind. However, we'll give a proof here using the Weierstrass preparation theorem.

Theorem 4.2 (Implicit function theorem). *Suppose $Z = Z(f_1, \cdots, f_m) \subseteq U \subseteq \mathbb{C}^n$ for $f_1, \cdots, f_m \in \mathcal{O}(U)$. Define $f = (f_1, \cdots, f_m) : U \to \mathbb{C}^{m-1}$. The Jacobain of f is*

$$J(f) = (\frac{\partial f_i}{\partial z_j})_{1 \leq i \leq m, 1 \leq j \leq n} = (J'(f)\ J"(f)),$$

where $J'(f) = (\frac{\partial f_i}{\partial z_j})_{1 \leq i,j \leq m}$.

If $J'(f)$ is invertible, then there exists polydisks

$$\Delta(0, r) = \Delta(0, r') \times \Delta(0, r") \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$$

and map $\phi : \Delta(0, r") \to \Delta(0, r')$ such that $\phi(0) = 0$ and

$$f(z', z") = 0 \iff \phi(z") = z'.$$

Proof. We induct on m. Let $m = 1$, so $f : U \to \mathbb{C}$. We can assume $f(0) = 0$. Then $\frac{\partial f}{\partial z_1} \neq 0$. Thus, f is regular of degree 1 with respect to z_1. Then by the Weierstrass preparation theorem, $f(z) = u(z)(z_1 - a_0(z_2, \cdots, z_n))$. In a sufficiently small polydisk $\Delta(0, r)$, $u(z)$ is nowhere zero. Then we allow $\phi = a_0 : \Delta(0, r_2, \cdots, r_n) \to \Delta(0, r_1)$, and we see that $a_0(0) = 0$ and $f(z) = 0 \iff z_1 = a_0(z_2, \cdots, z_n)$.

Suppose our claim holds for $m-1$. We now show it holds for m. We have $J'(f)$ is invertible. Then perform a change of coordinates so that $J'(f) = I_m$. Then $\frac{\partial f_1}{\partial z_1} \neq 0$. Then apply the implicit function theorem to f_1. Then we obtain a polydisk around the origin $\Delta(0, r_1, \cdots, r_n)$ and a map $\phi_1 : \Delta(r_2, \cdots, r_n) \to \Delta(0, r_1)$ such that $\phi_1(0, \cdots, 0) = 0$ and $f_1(z_1, z_2, \cdots, z_n) = 0 \iff \phi_1(z_2, \cdots, z_n) = z_1$.

Now define $g = (g_2, \cdots, g_m) : \Delta(0, r_2, \cdots, r_n) \to \mathbb{C}^{m-1}$ where

$$g_i(z_2, \cdots, z_n) = f_i(\phi_1(z_2, \cdots, z_n), z_2, \cdots, z_n).$$

Note that $J'(g) = \frac{\partial g_2, \cdots, g_m}{\partial z_2, \cdots, z_m} = I_{m-1}$ is invertible since $J'(f) = I_m$. Then applying the implicit function theorem we find that there is a polydisk $\Delta(0, r_2, \cdots, r_n)$ (shrinking r_i's if necessary) and a map

$$\psi : \Delta(0, r_{m+1}, \cdots, r_n) \to \Delta(0, r_2, \cdots, r_m)$$

such that $\psi(0) = 0$ and $g(z_2, \cdots, z_n) = 0 \iff \psi(z_{m+1}, \cdots, z_n) = (z_2, \cdots, z_m)$. But

$$g(z_2, \cdots, z_n) = 0 \iff f_i(\phi_1(z_2, \cdots, z_n), z_2, \cdots, z - n) = 0 \forall i.$$

Then define $\phi : \Delta(0, r_{m+1}, \cdots, r_n) \to \Delta(0, r_1, \cdots, r_m)$ such that

$$\phi(z_{m+1}, \cdots, z_n) = (\phi_1(\psi(z_{m+1}, \cdots, z_n), z_{m+1}, \cdots, z_n), \psi(z_{m+1}, \cdots, z_n), z_{m+1}, \cdots, z_n).$$

We see then that

$$f(z_1, \cdots, z_n) = 0 \iff \phi(z_{m+1}, \cdots, z_n) = (z_1, \cdots, z_m).$$

□

The intuition here is that, if locally (since the jacobian is linear approximation) the map is surjective, then we expect the fiber to be of dimension $n - m$. Indeed, the implicit function theorem shows that we can parameterize $f^{-1}(0)$ by a $n - m$ dimensional polydisc. The preimage of regular values of holomorphic maps are often the first examples of complex manifolds.

Let X be a topological space, Hausdorff, countable basis. Let $U \subseteq X$ be an open. Let $\mathcal{C}(U)$ be the set of continuous functions $f : U \to \mathbb{C}$.

Definition 4.3. A geometric structure $\mathcal{O}_X$ on X is an assignment of subrings $\mathcal{O}_X(U) \subseteq C(U)$ on each open $U \subseteq X$ such that:

(1) the constant functions $\mathbb{C} \subseteq \mathcal{O}_X(U)$
(2) for every $f \in \mathcal{O}_X(U)$ and $V \subseteq U$ open, then $f|_V \in \mathcal{O}_X(V)$
(3) if $U = \bigcup_{i \in I} U_i$, and $f_i \in O_X(U_i)$ for every $i \in I$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every i, j, then there exists unique $f \in \mathcal{O}_X(U)$ such that $f|_{U_i} = f_i, \forall i$.

In other words, let $C(X, \mathbb{C})$ denote the sheaf of complex-valued functinuous functions on X. Then a "geometric structure" $\mathcal{F}$ on X is a subsheaf of rings $\mathcal{F} \subseteq C(X, \mathbb{C})$, with the constant complex functions included in the subsheaf.

Example 4.4. Here are our main examples.

(1) $X \subseteq \mathbb{R}^n$, can consider the sheaf of C^∞ functions.
(2) $X \subseteq \mathbb{C}^n$ open, can consider sheaf of holomorphic functions.

Definition 4.5. A morphism of geometric spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ such that it preserves the function structures of these spaces, i.e we have a map of sheaves $\mathcal{O}_Y \to f_* \mathcal{O}_X$.

f is an isomorphism if it has a inverse which is also a morphism of geometric structures.

Example 4.6. If $f : U \to V$ where $U \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m$, then f is a morphism of geometric structures $\iff$ f is holomorphic.

Now if $(X, \mathcal{O}_X)$ is a geometric structure, then there is an induced geometric structure $(U, \mathcal{O}_X|_U)$, and we have a canonical geometric morphism $(U, \mathcal{O}_X|_U) \hookrightarrow (X, \mathcal{O}_X)$.

Definition 4.7. A complex manifold $(X, \mathcal{O}_X)$ is a geometric space such that for every $x \in X$, there is a neighborhood $U \subseteq X$ such that there exists an isomorphism of geometric structures

$$(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_V)$$

where $V \subseteq \mathbb{C}^n$ open.

So the basic building blocks of complex manifolds is open sets of $\mathbb{C}^n$ equipped with their sheaf of holomorphic functions. So schemes are to complex manifolds as affine schemes are to open sets of complex space with sheaf of holomorphic functions.

An alternative (more classical) definition of manifolds is:

Definition 4.8. An atlas of X is an open cover $(U_i)_{i\in I}$ of X with homeomorphisms $\phi_i : U_i \to V_i \subset \mathbb{C}^n$ open (coordinate charts), such that for every i, j the transition functions

$$g_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

are holomorphic isomorphisms (note holomorphic maps are biholomorphic $\iff$ they are bijective; this is proposition 1.1.13 in Huybrechts).

To go from the first definition to second, just take an open cover and this gives you an atlas. To go from second definition to first definition, take the atlas (U_i, ϕ), then send this to $\mathcal{O}_X(U_i) = \{f \in C(U) | f|_{U\cap U_i} \circ \phi_i^{-1} \text{ holomorphic on } \phi_i(U \cap U_i), \forall i\}$.

Example 4.9. Any $U \subseteq \mathbb{C}^n$ is a complex manifold.

Any connected 1-dimensional manifold is something called a Riemann surface.

Projective space $\mathbb{P}^n_{\mathbb{C}}$. This is the set of lines in $\mathbb{C}^{n+1}$ through the origin. So you look at the quotient space

$$\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times.$$

So we have $g : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, and this map is via quotient a group action by $\mathbb{C}^\times$. So $U \subseteq \mathbb{P}^n \iff g^{-1}(U)$ is open by definition of quotient topology. Can see $\mathbb{P}^n$ is Hausdorff and compact. Convince yourself $S^{2n+1} \to \mathbb{P}^n_{\mathbb{C}}$ is surjective for every n.

5. 9/20/23: GROUP ACTION QUOTIENT

Recall the definition of a complex manifold $(X, \mathcal{O}_X)$. For every point $p \in X$, there is a neighborhood U_p such that $(U_p, \mathcal{O}_X|_{U_p}) \cong (V, \mathcal{O}_V)$ where $V \subseteq \mathbb{C}^n$.

Note that the stalk

$$\mathcal{O}_{X,x} := \varinjlim_{x\in U} \mathcal{O}_X(U)$$

is isomorphic to $\mathcal{O}_n$, through the map to $(V, \mathcal{O}_V)$. Recall that this is a regular local ring, with maximal ideal

$$m_x = \{f \in \mathcal{O}_X(U) | f(x) = 0\}$$

and m_x/m_x^2 a $\mathbb{C}$-vector space of dimension n. Note that the map

$$p \mapsto \dim_p X$$

is locally connected. If X is connected, then this map is constant. Here's our first example of a complex manifold.

Example 5.1 (projective space)**.** $\mathbb{P}^n_{\mathbb{C}}$. The projective space is topologized as the quotient space $g : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

The manifold charts on these are quite nice, in that they're isomorphic to $\mathbb{C}^n$. If we denote the points of $\mathbb{P}^n$ as

$$[x_0 : \cdots : x_n]$$

then the charts are

$$U_i = \{[x_0 : \cdots : x_n] | x_i \neq 0\}.$$

We see that $U_i \cong \mathbb{C}^n$ via the maps

$$[x_0 : \cdots : x_n] \mapsto (\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}).$$

We have the sheaf $\mathcal{O}_{\mathbb{P}^n}$ is given by

$$\mathcal{O}_{\mathbb{P}^n}(U) := \{f \in C(U) | f \circ g \text{ holomorphic on } g^{-1}(U)\}.$$

One should check that this gives the same thing as the sheaf given by the coordinate charts.

We have surjections $S^{2n+1} \to \mathbb{P}^n$. We have $\mathbb{P}^1$ diffeomorphic to S^2 and $\mathbb{P}^n$ is simply connected for all n.

Note $\mathbb{P}^n \setminus U_i \cong \mathbb{P}^{n-1}$. So $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$. Get a sort of cellular decomposition, can get topological information.

Note that any map $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ plus precomposition with some chart's map on X and postcomposing with some chart's map on Y, will produce a holomorphic map between open subsets of $\mathbb{C}$-space. This is becuase maps of open subsets of $\mathbb{C}$-space which preserve their structure sheaves must be holomorphic.

First let's discuss quotients. Then we'll talk about hypersurfaces and complete intersections. Let X be a complex manifold. We call a holomorphic map $\phi : X \to X$ an endomorphism, and if the map is biholomorphic an automorphism. We have a group

$$Aut(X) = \{\phi : X \to X | \phi \text{ automorphism }\}.$$

Let $G \subseteq Aut(X)$. You can consider the action

$$G \times X \to X, (\phi, x) \mapsto \phi(x).$$

Then consider the quotient space / orbit space

$$X/G = \{[x] | x \sim y \text{ if } y = \phi(x) \text{ for some } \phi\}.$$

Even for very simple examples, the quotient space is not nice. Consider the action of $\mathbb{C}^\times$ on $\mathbb{C}$ via $(\lambda, z) \mapsto \lambda z$. Then considering the projection $q : \mathbb{C} \to \mathbb{C}/\mathbb{C}^*$, you see that $q^{-1}([0]) = 0$ and $q^{-1}([1]) = \mathbb{C}^\times$. So $\mathbb{C}/\mathbb{C}^*$ is not Hausdorff. Other bad examples that are slightly more acceptable, sometimes the quotient space is not a manifold.

Definition 5.2. G acts *properly discontinuously* on X if for every compact subsets $K_1, K_2 \subseteq X$, $\phi(K_1) \cap K_2 \neq \emptyset$ only for finitely many $\phi \in G$.

G acts *freely* on X (or without fixed points) if $\forall x \in X$, if $\phi(x) = x$ then $\phi = Id_X$.

An example of a properly discontinuous action is actions by finite groups. An example of a group not acting freely is $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}$ via $(x, y) \mapsto (-x, -y)$.

Example 5.3. Say $\Lambda \subseteq \mathbb{C}$ is a lttice, so a Λ is a free abelian subgroup that generates $\mathbb{R}^2$ underlying $\mathbb{C}$. By some coordinate change, we can assume the generators are $1, \tau \in \mathbb{C}$. You can repeat this fundamental parallelogram and tessellate $\mathbb{C}$.

We have

$$\Lambda \times \mathbb{C} \to \mathbb{C}, (a, z) \mapsto z + a$$

a group action by translation. This is clearly free, since this is simply addition. This is also properly discontinuous because compact subsets of $\mathbb{C}$ are bounded, and thus contain finitely many lattice points.

More generaly, lattices $\mathbb{Z}^{2n} \cong \Lambda \subseteq \mathbb{C}^n$ acts properly discontinuous by translations.

Let $G \subseteq Aut(X)$ and $q : X \to X/G$. When can we put a geometric structure on X/G? When is X/G a complex manifold?

Lemma 5.4. *The map $q : X \to X/G$ is an open map (note it's continuous by definition).*

Proof. If $V \subseteq X$ is open, we'd like to show that $q(V) \subseteq X/G$ is open. Then we have $q(V)$ is open $\iff$ $q^{-1}q(V)$ is open. But

$$q^{-1}q(V) = \bigcup_{\phi \in G} \phi(V).$$

□

Proposition 5.5. *Let X be a complex manifold, $G \subseteq Aut(X)$ acts freely and properly discontinuously on X. Then X/G is a complex manifold and*

$$\pi : X \to X/G$$

is holomorphic and locally biholomorphic.

Proof. First, let's define the sheaf $\mathcal{O}_{X/G}$ of holomorphic functions on X/G. Naturally, it is:

$$\mathcal{O}_{X/G}(U) := \mathcal{O}_X(\pi^{-1}(U))^G.$$

The fact that it is an actual sheaf follows from $\mathcal{O}_X$ being a sheaf.

Note X/G has a countable basis. This is because $\pi : X \to X/G$ is an open map. So taking a countable basis for X, the image of each will form a countable basis for X/G.

Furthermore, X/G is Hausdorff. Note this means that we want to show that diagonal $\Delta_{X/G} \subset X/G \times X/G$ is closed. But from the quotient topology, this is equivalent to proving that the set

$$\{(x, y) | x \sim y\} \subseteq X \times X$$

is closed. One sees this from the fact that $X/G \times X/G$ is the quotient space of the action $(G \times G) \times (X \times X) \to X \times X$, where $(\phi_1, \phi_2) \times (g_1, g_2) = (\phi_1(g_1), \phi_2(g_2))$. To show that $\{(x, y)|x \sim y\} \subseteq X \times X$ is closed, suppose we have any $(a, b) \in X \times X$ such that $a \not\sim b$. Then we'd like to show that there exists neighborhoods U, V such that $\phi(U) \cap \psi(V) = \emptyset, \forall \phi, \psi \in G$. This is equivalent to showing that $\phi(U) \cap V = \emptyset, \forall \phi \in V$. But indeed, since X is Hausdorff we can intially begin with closed balls of radius ϵ around a, b such that they're disjoint. Since G acts properly discontinuously, $\phi\overline{B(a, \epsilon)} \cap \overline{B(b, \epsilon)} \neq \emptyset$ for only finitely many $\phi \in G$. Then we can find a smaller neighborhood U of a contained in $B(a, \epsilon)$ such that $\phi(U) \cap B(b, \epsilon)$ for all $\phi \in G$. Thus, X/G is Hausdorff. Altogether, this shows X/G is a complex manifold. Clearly $\pi : X \to X/G$ is holomorphic, with $\mathcal{O}_{X/G}(U) \to \mathcal{O}_X(\pi^{-1}(U))$ just being inclusion.

To show that $\pi : X \to X/G$ is locally biholomorphic, pick any $p \in X$. Pick some initial ϵ-ball around p. Since G acts properly discontinuously, there exists only finitely many ϕ such that the image of this ball under ϕ intersects the original ϵ-ball. Since G acts freely, for every $\phi \in G$, $\phi(p) \neq p$. Then for those ϕ where intersection occurs, p does not map to p, and there are finitely many of them, so we can choose a small enough neighborhood U of p that avoids these intersections, and thus $\phi(U)$ is disjoint from U for all $\phi \in G$.

We have that $\pi|_U : U \to \pi(U)$ is biholomorphic. Why? Because it is an isomorphism of geometric spaces. The map is a homeomorphism on topological spaces, and $\mathcal{O}_{X/G}|_{\pi(U)} \cong (\pi|_U)_* \mathcal{O}_X|_U$. $\square$

Back to our example.

Example 5.6. We have $\mathbb{Z}^2 \cong \Lambda \subseteq \mathbb{C}$. Then $\mathbb{C}/\Lambda$ is an elliptic curve, a 1-dimensional compact complex torus. This is diffeomorphic to $S^1 \times S^1$. As complex manifolds, these elliptic curves are very different as you vary τ, the second generator of the lattice. Topologically, these are all the same.

Here's a fact: τ_1, τ_2 give isomorphic complex manifold structures

$$\iff \exists M \in SL(2, \mathbb{Z}) \text{ such that } Az_1 = z_2.$$

In fact, there's a one-parameter family of isomorphic elliptic curves. (j-invariant).

Elliptic curves are always algebraic varieties.

As soon as we goto higher dimensions,

$$\Lambda \cong \mathbb{Z}^{2n} \subseteq \mathbb{C}^n,$$

have $T := \mathbb{C}^n/\Lambda$ is the complex torus, diffeomorphic as $(S^1)^{2n}$. You can classify when there is isomorphic complex structures, and this leads to –something hodge structures–.

Also observe $[0,1]^{2n} \to \mathbb{C}^n/\Lambda$

$$(a_1, \cdots, a_{2n}) \mapsto [a_1\lambda_1 + \cdots + a_{2n}\lambda_{2n}]$$

These complex tori have abelian group structures.

What prevents group quotients from being complex manifolds is fixed points. Consider again the example of $\mathbb{Z}/2\mathbb{Z} = \{1, i\}$ acting on $\mathbb{C}^2$. Where $i : \mathbb{C}^2 \to \mathbb{C}^2$ where $(x, y) \mapsto (-x, -y)$. We see that 0 is a fixed point.

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "{(x,y)\mapsto(x^2,xy,y^2)}"] \arrow[d] & \mathbb{C}^3 \\
\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\cong"'] & Z = Z(z_1^2 = z_0 z_2) \arrow[u]
\end{tikzcd}$$

Something about a cone over a conic.

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "{(x,y)\mapsto(x^2,xy,y^2)}"] \arrow[d] & \mathbb{C}^3 \\
\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

You have this factorization, and the image is $V(z_1^2 = z_0 z_2)$. At 0 this map is special, it goes there with multiplicity 2. So this is going to be the vertex of this cone. Its an isolated singularity on a hypersurface. What's special about Gorenstein. On a normal variety, the points where its singular is codimension at least 2, then push forward the canonical line bundle onto the normal variety, then the condition that it is locally free is called Gorenstein. This example is not phenomenally nice though.

Because this example has Weil divisors line through vertex, but its not cartier divisors.

Consider again the example:

Example 5.7. $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{C}^3$. Have $(x, y, z) \mapsto (-x, -y, -z)$. Fact: $\mathbb{C}^3/(\mathbb{Z}/2)$ is not Gorenstein. In algebraic geometry, we call something nice if its at least Gorenstein because it has the canonical line bundle.

So: even the simplest examples of group actions lead to quotient spaces that are very much not nice.

Example 5.8. We have $\mathbb{C}^\times$ acting on $\mathbb{C}^{n+1} \setminus \{0\}$ which gives us as the quotient $\mathbb{P}^n$. This is clearly not properly discontinuous, but the quotient is still a complex manifold.

If G has a topology (ok for discrete groups, $\mathbb{C}^\times$, in general for Lie groups) acting on your space, then you can say the action is *proper* if $G \times X \to X \times X$ where $(\phi, x) \mapsto (\phi(x), \phi(x))$ is proper (preimage of compact is compact). The earlier result about quotients being complex manifolds, works for proper actions. This action $\mathbb{C}^\times$ acting on $\mathbb{C}^{n+1} \setminus \{0\}$ is a proper action.

Example 5.9 (Hopf manifolds)**.** Fix $0 < \lambda < 1$, $\lambda \in \mathbb{R}$. Consider the action

$$\mathbb{Z} \times (\mathbb{C}^n \times \{0\}) \to \mathbb{C}^n \setminus \{0\}$$

where $(k, (z_1, \cdots, z_n)) \mapsto (\lambda^k z_1, \cdots, \lambda^k z_n)$. This action is free and properly discontinuous. Thus, $X = \frac{\mathbb{C}^n \setminus \{0\}}{\mathbb{Z}}$ is a manifold.

Facts: X is diffeomorphic to $S^{2n-1} \times S^1$.

For $n = 1$, X is an elliptic curve.

X is a manifold that does not have a Kahler structure. This is why these are famous examples. Compact complex manifold that is not Kahler. All smooth projective varieties are Kahler. So these Hopf manifolds are also not algebraic varieties. Once we establish Hodge theory, we'll see certain numerical invariants that must hold for Kahler structures, will not hold for Hopf manifolds.

Example 5.10 (Hypersurfaces in $\mathbb{C}^n$ and $\mathbb{P}^n$)**.** Lemma: if you have a function $f : \mathbb{C}^n \to \mathbb{C}$, such that $0 \in \mathbb{C}$ is a regular value, then $f^{-1}(0) = Z(f)$ is a complex (sub)manifold of dimension $n - 1$. This is what we usually call a hypersurface. So locally around 0, we can write $Z(f)$ as the graph of a function.

6. 9/25/23: SUBMANIFOLDS

Let $f : \mathbb{C}^n \to \mathbb{C}$ holomorphic, and let $Z = f^{-1}(0)$, where 0 is a regular value. This means that for every p s.t. $f(p) = 0$, have $\frac{\partial f}{\partial z_i}(p) \neq 0$ for some i. By the implicit function theorem, locally Z is isomorphic to some open neighborhood $\mathbb{C}^{n-1}$. Thus, Z is a complex manifold. Let us also mention the notion of submanifolds.

Example 6.1. Let $k \leq n$. Then we have

$$\mathbb{C}^k \hookrightarrow \mathbb{C}^n$$

where $(z_1, \cdots, z_k) \mapsto (z_1, \cdots, z_k, 0, \cdots, 0)$. If $V \subseteq \mathbb{C}^k$, then $f : V \to \mathbb{C}$ extends to a holomorphic map $V \times \mathbb{C}^{n-k} \to \mathbb{C}$.

Linear subspaces will be our local model for complex submanifolds.

Definition 6.2. Let X be a complex manifold, and $Y \subseteq X$ (Y has the induced topology). Define $\mathcal{O}_X|_Y$ to be the sheaf such that for $V \subseteq Y$, $\mathcal{O}_X|_Y(V) =$

$$\{f : V \to \mathbb{C} | \forall y \in V, \exists \text{ open neighborhood } y \in U_y \subseteq X, \exists f_y \in \mathcal{O}_X(U_y) \text{ such that } f|_{V \cap U_y} = f_y|_{V \cap U_y}\}.$$

Definition 6.3. A subset $(Y, \mathcal{O}_X|_Y) \subseteq (X, \mathcal{O}_X)$ as above is a complex submanifold if $\forall y \in Y$, there exists chart for X around y, $\phi : U \to V$ s.t. $\phi(U \cap Y) = V \cap (\text{linear subspace} \mathbb{C}^k \hookrightarrow \mathbb{C}^n)$.

Example 6.4. Suppose $f : \mathbb{C}^n \to \mathbb{C}$. By the implicit function theorem, this is a complex manifold of dimension $n-1$. But this is also a submanifold of $\mathbb{C}^n$ of dimension $n-1$.

We have the chart $(z_1, \cdots, z_n) \mapsto (f(z_1, \cdots, z_n), z_2, \cdots, z_n)$. Why a chart? Because locally, this is an isomorphism for $Z(f)$. Should be able to say this remains local isomorphism??

Now consider hypersurfaces in $\mathbb{P}^n$. The immediate issue is that the polynomial $f \in \mathbb{C}[x_0, \cdots, x_n]$ may not be homogenous. The function f may taken on different values for different scalings. Thus to define hypersurfaces of $\mathbb{P}^n$, we require that $F \in \mathbb{C}[X_0, \cdots, X_n]$ is homogenous. While a "value" of homogenous F on $\mathbb{P}^n$ doesn't make sense, the notion of a zero does.

Let F be homogenous. Then $F : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ is a holomorphic function. If 0 is a regular value for F, then we get a hypersurface, namely $F^{-1}(0)$ is a submanifold of $\mathbb{C}^{n+1} \setminus \{0\}$. But also, $Z = F^{-1}(0)/\mathbb{C}^\times$ is a complex submanifold of $\mathbb{P}^n$ of dimension $n-1$. If f is degree d, then this is a hypersurface of degree d in $\mathbb{P}^n$. To show Z is a submanifold of $\mathbb{P}^n$, consider the chart $D_+(x_i) = U_i$,

$$\begin{tikzcd} Z \cap U_i \arrow[r, "\cong"'] \arrow[d, hook] & (\frac{F_i}{x_i^d})^{-1}(0) \arrow[d, hook] \\ U_i \arrow[r, "\cong"] & \mathbb{C}^n \end{tikzcd}$$

and note that 0 is still a regular value of $\frac{F_i}{x_i^d}$. Then use the fact that: if you have an open cover of a subset, such that on each open set the subset is a submanifold, then the subset is a submanifold.

One of the main theorems of this class is that a compact complex submanifold of $\mathbb{P}^n$ is algebraic: it has to be defined by finitely many polynomials. If it is a hypersurface, it's defined by one polynomial. This is the first instance of this fact. This will be a serious theorem towards the end of the semester.

Let us also briefly mention complex lie groups. We've already seen one, namely compact complex tori. They're manifolds, but also have a group structure. Precisely:

Definition 6.5. A complex lie group G is a complex manifold that is a group object in the category of complex manifolds. Concretely, there are holomorphic maps

$$m : G \times G \to G, \iota : G \to G$$

satisfying the appropriate diagrams.

Example 6.6. Let $M_{n,n}(\mathbb{C})$ denote matrices. Isomorphic to $\mathbb{C}^{2n}$. Then

$$GL_n(\mathbb{C}) = \{ \text{ invertible matrices}\}$$

is a Lie group. (Cramer's rule $\implies$ holomorphic? Ask later about this).

Example 6.7. Subgroups of $GL_n(\mathbb{C})$ that are also closed submanifolds:

(1) The special linear groups $SL_n(\mathbb{C})$ is those that are determinant 1. This is a submanifold because it is a hypersurface. For the determinant map, 1 is a regular value, so can use implicit function theorem. So dimension is $n^2 - 1$.
(2) Symplectic groups $Sp(2n, \mathbb{C}) \subset GL_{2n}(\mathbb{C})$, comprised of the symplectic matrices $\{A | A^T \Omega A = \Omega\}$ where $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
(3) $O(n)$
(4) $SO(n)$

In general, showing that something is a lie group is quite hard. There are some tools. There's a very hard theorem that says a subgroup of $GL_n(\mathbb{C})$ that is closed, is automatically a Lie group. Can also prove showing these are complex submanifolds using exponential maps. https://en.wikipedia.org/wiki/Closed-subgroup_theorem.

All affine lie groups are non-abelian. Fact: if a complex lie group is compact, then it is abelian, and in fact a torus. Note: there are many real lie groups that are not complex lie groups. $U(1)$, $SU(2)$ cannot be complex lie groups for dimension reasons, but are real lie groups.

Blowing up manifolds. Let X be a complex manifold of dimension n. Let $x \in X$. We want to blow up X at x. What we want to do is replace x by a copy of $\mathbb{P}^{n-1}$ leave the set undamaged. We'll obtain a projection map $\tilde{X} : Bl(X) \to^{\pi} X$, where $E = \pi^{-1}(x) \cong \mathbb{P}^{n-1}$.

There is a natural $\mathbb{P}^{n-1}$ through $x \in X$ a complex manifold of dimension n. Locally around x, the manifold looks like $\mathbb{C}^n$. So we can take all the lines through x. So to define blow ups at a point, let's worry about the standard local example.

Definition 6.8. Let $x = 0 \in \mathbb{C}^n = X$. Define

$$Bl_0\mathbb{C}^n = \{(z, L) | z \in L \text{ where L is a line through 0 }\} \subseteq \mathbb{C}^n \times \mathbb{P}^{n-1}.$$

For $z \neq 0$, $\pi^{-1}(z)$ is (z, L) where L passes through the origin and z, thus there is a unique such line. So $\pi^{-1}(z) = \{(z, L)\}$. For $z = 0$, $\pi^{-1}(0) = \mathbb{P}^{n-1}$. So at least set-theoretically, we see that this incidence correspondence is exactly what we want.

Proposition 6.9. *$Bl_0(\mathbb{C}^n)$ is a complex manifold of dimension n, and $\pi : Bl_0\mathbb{C}^n \to \mathbb{C}^n$ is holomorphic. The exceptional set E is a submanifold of dimension $n - 1$.*

Proof. Use coordinates $(z_1, \cdots, z_n)$ on $\mathbb{C}^n$ and coordinates $[w_1 : \cdots : w_n]$ on $\mathbb{P}^{n-1}$. So the condition to cut out $Bl_0(\mathbb{C}^n)$ from $\mathbb{C}^n \times \mathbb{P}^{n-1}$ is defined by the equations $z_i w_j = z_j w_i$, for all $i, j \leq n$.

$$\begin{tikzcd} & Bl_0\mathbb{C}^n \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C}^n & & \mathbb{P}^{n-1} \end{tikzcd}$$

Consider the projection π_2. One sees at least set theoretically that this is "tautalogical line bundle." So to show $Bl_0\mathbb{C}^n$ has manifold structure, we'll exploit its nice map to $\mathbb{P}^{n-1}$ which is also a manifold. Consider the charts of $\mathbb{P}^{n-1}$: $U_i = \{w_i \neq 0\}$. Cover $Bl_0\mathbb{C}^n$ by $V_1 \cup \cdots \cup V_n$ where $V_i = \pi_2^{-1}(U_i)$, so

$$V_i = \{(z,w) \in \mathbb{C}^n \times \mathbb{P}^{n-1} | w_i \neq 0, z_j = z_i \frac{w_j}{w_i}, i \neq j\}.$$

The claim is that these V_i are charts. Have $f_i : V_i \to \mathbb{C}^n$ via

$$(\underline{z}, \underline{w}) \mapsto (\frac{w_1}{w_i}, \cdots, \frac{w_{i-1}}{w_i}, z_i, \frac{w_{i+1}}{w_i}, \cdots, \frac{w_n}{w_i})$$

and the map the other way is

$$(v_1, \cdots, v_n) \mapsto \underline{z} = v_i \underline{w}$$

where $\underline{w} = (v_1 : \cdots : v_{i-1} : 1 : v_{i+1} : \cdots : v_n)$. Check that these are inverses. Now we also need to specify the transition functions. Say $i \neq j$. Have $g_{ij} = f_i f_j^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ and

$$g_{ij}(v_1, \ldots, v_n) = (\mu_1, \cdots, \mu_n)$$

where μ_k is given by v_k/v_i when $k \neq i, j$, $v_i v_j$ when $k = i$, and $\frac{1}{v_i}$ for $k = j$ (follow the maps and you should get this).

Look at $\pi : Bl_0\mathbb{C}^n \to \mathbb{C}^n$, have

$$\pi f_i^{-1}(v_1, \cdots, v_n) = (v_i v_1, \cdots, v_i v_{i-1}, v_i, v_i v_{i+1}, \cdots, v_i v_n).$$

This shows that the projection $Bl_0\mathbb{C}^n \to \mathbb{C}^n$ is holomorphic.

Finally, $\pi^{-1}(0) \cap V_i \cong^{f_i} (\{v_i = 0\} = \mathbb{C}^{n-1} \subset \mathbb{C}^n$. □

In fact, $Bl_0(\mathbb{C}^n)$, since it is cut out by algebraic equations, is actually an algebraic variety. The fact that it is a manifold implies it is a smooth algebraic variety (local ring is UFD).

For exercise: play around with blowing up a point $\mathbb{P}^2$ to really understand blow ups, working with all these indices is probably not the most efficient way to reach enlightenment.

7. 9/27/23: BLOWING UP PLANE AT POINT, BUNDLES

Let's finish the story about blow ups at manifolds. You can blow up pretty much anything, but we'll stick to points. Last time we discussed $Bl_0\mathbb{C}^n$, equipped with two projections

$$\begin{tikzcd} & Bl_0\mathbb{C}^n \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C}^n & & \mathbb{P}^{n-1} \end{tikzcd}$$

and we showed that $Bl_0\mathbb{C}^n$ is a complex manifold by pulling back the charts on $\mathbb{P}^{n-1}$ and the exceptional divisor is a submanifold. Note if $0 \in V \subseteq \mathbb{C}^n$ is an open, then we'll define $Bl_0(V) = \pi^{-1}(V)$.

Now let X be a complex manifold, $x \in X$. We'll define the blow up $\pi : Bl_x X \to X$, where $E = \pi^{-1}(x)$ is the exceptional divisor. Take coordinate chart $f : U \to V \subseteq \mathbb{C}^n$ where $x \in U \mapsto 0 \in V$. Let $\tilde{V} := Bl_0(V)$. Note

$$U \setminus \{x\} \cong V \setminus \{0\} \cong \tilde{V} \setminus E.$$

Then glue $X \setminus \{x\}$ with $\tilde{V}$ along the identification $U \setminus \{x\}$ with $\tilde{V} \setminus E$, so that

$$Bl_x X := \frac{X \setminus \{x\} \bigsqcup \tilde{V}}{\sim}.$$

One needs to show that this is independent of the chart. This will be true because of the universal property of blow-ups, which we explain now.

Lemma 7.1 (universal property of blow ups)**.** *Let $f = (f_1, \cdots, f_n) : X \to \mathbb{C}^n$ holomorphic map, with X a complex manifold. Assume $f(X) \neq \{0\}$ and $\forall x \in X$ such that $f(x) = 0$, the ideal $(f_1, \cdots, f_n) \in \mathcal{O}_{X,x}$ is principal. Then $\exists!$ holomorphic map*

$$\tilde{f} : X \to Bl_0\mathbb{C}^n$$

making the following diagram commute:

$$\begin{tikzcd} & Bl_0\mathbb{C}^n \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "f"'] & \mathbb{C}^n \end{tikzcd}$$

Proof. Uniqueness: say we have lifts $\tilde{f}', \tilde{f}$. Then they agree on $X \setminus f^{-1}(0)$, which is dense in X. But by the identity principle, this implies $\tilde{f} \cong \tilde{f}'$.

Existence: since we have uniqueness, existence is local (because we can glue and use uniqueness). We may assume $X = U \subseteq \mathbb{C}^m$, an open neighborhood of 0, and $f(0) = 0$. By hypotheses, $\exists g \in \mathcal{O}_m$ such that $(f_1, \cdots, f_n) = (g)$ in $\mathcal{O}_m$. This means there exists an open neighborhood W around the origin such that

$$g = \sum_{i=1}^{n} h_i f_i \text{ and } f_i = t_i g.$$

$$\implies g = h_1 t_1 g + \cdots + h_n t_n g \implies 1 = h_1 t_1 + \cdots + h_n t_n$$

so for every $x \in W$, there exists j such that $t_j(x) \neq 0$. This implies that

$$[t_1(x) : \cdots : t_n(x)] \in \mathbb{P}^{n-1}$$

and as a point, this equals $(f_1(x) : \cdots : f_n(x))$. So simply define $\tilde{f} : W \to Bl_0\mathbb{C}^n$ where

$$x \mapsto ((f_1(x), \cdots, f_n(x)), [t_1(x) : \cdots : t_n(x)]).$$

□

If you think about it, this condition on the map from $X \to \mathbb{C}^n$ is forced upon you. You know what the lift will look like for $x \in X$ such that $f(x) \neq 0$. But those values x such that $f(x) = 0$, the lift needs to map into $\mathbb{P}^{n-1}$, which is also the space parameterizing hypersurfaces in $\mathbb{C}^n$.

Now let's finish showing that the definition of $Bl_x X$ does not depend on the choice of chart around x. We started with a coordinate chart $f : U \to V$ where $f(x) = 0$. Take another $f' : U \to V'$ where $f'(x) = 0$. Let $\phi = (f') \circ f^{-1}$. Note

$\phi : V \to V'$ biholomorphic and $\phi(0) = 0$. Claim: ϕ induces biholomorphic map $Bl_0(V) \to Bl_0V'$.

$$\begin{tikzcd} & & Bl_0V' \arrow[d, "\pi'"] \\ Bl_0V \arrow[r, "\pi"'] & V \arrow[r, "\phi"] & V' \end{tikzcd}$$

Now use the universal property. Use fact $\phi \circ \pi$ generate a principal ideal in each $\mathcal{O}_{Bl_0V,y}$. Cover Bl_0V by coordinate charts, $f_i : V_i \to \mathbb{C}^n$ where $V_i \subseteq Bl_0V \subseteq Bl_0\mathbb{C}^n$. Let $\psi := \phi \circ \pi f_i^{-1} : \mathbb{C}^n \to V'$, so

$$\psi(w) = \phi(w_iw_1, \cdots, w_iw_{i-1}, w_i, \cdots, w_iw_n).$$

Note $\psi = (\psi_1, \cdots, \psi_n)$. Remember, we are trying to show local principality. Fix $y = (w_1, \cdots, w_n) \in \mathbb{C}^n$, $I = (\psi_1, \cdots, \psi_n) \in \mathcal{O}_n$ Claim that $I = (w_i)$. Then $\phi(0) = 0$ can write

$$(\phi_1(z), \cdots, \phi_n(z)) = A(z) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

where $A(z)$ is matrix of holomorphic functions. Subsitute in $\psi(w) = \phi(w_iw_1, \cdots, w_iw_{i-1}, w_i, \cdots, w_iw_n)$, so we get that $\psi_j(w)$ is divisible by w_i. This implies $I \subseteq (w_i)$. But $A(0)$ is invertible, $\phi : V \to V'$ is biholomorphism. So around 0,

$$(z_1, \cdots, z_n) = (\phi_1(z), \cdots, \phi_n(z))A^{-1}(z).$$

By plugging in w_i being a linear combination of $\psi_1(w), \cdots, \psi_n(w)$, we have $I = (w_i)$.

**At some point, learn blow ups along higher dimensional spaces.

Moving on to **Vector bundles.** Let X be a topological space, k a field ($\mathbb{R}$ or $\mathbb{C}$).

Definition 7.2. A k-vector bundle on X of rank r is a topological space V with a continuous map $\pi : V \to X$ such that

- $\forall x \in X$, the fiber $\pi^{-1}(x) = V_x$ is a k-vector space of dimension r
- $\forall x \in X$, there $\exists U$ and homeomorphism

$$\phi : \pi^{-1}(U) \cong U \times k^{\oplus r}$$

with $\phi(V_x) = \{x\} \times k^{\oplus r}$ being a linear isomorphism. The pair (U, ϕ) is called a local trivialization.

Definition 7.3. Let (U_i, ϕ_i) and (U_j, ϕ_j) be local trivializations. The diagram

$$\begin{tikzcd} U_i \cap U_j \times k^{\oplus r} \arrow[r, "\phi_j^{-1}"] \arrow[dr] & \pi^{-1}(U_i \cap U_j) \arrow[r, "\phi_i"] \arrow[d] & U_i \cap U_j \times k^{\oplus r} \arrow[dl] \\ & U_i \cap U_j & \end{tikzcd}$$

shows we have a continuous map

$$g_{ij} = U_i \cap U_j \to GL_r(k)$$

called transitions functions of V. These satisfy the identities

- $g_{ii} = Id$
- (cocycle) $g_{ij}g_{jk}g_{ki} = Id$ on $U_i \cap U_j \cap U_k$

Definition 7.4. If X is a C^∞-manifold, and the local trivializations are diffeomorphisms, then call V a C^∞ vector bundle. Equivalently, the transition functions g_{ij} are C^∞ functions, where $GL_r(\mathbb{R})$ are considered as real lie groups.

A holomorphic vector bundle $V \to X$ is such that V, X are complex manifolds and the local trivailizations are holomorphic (or equivalently, the transitions fuctions are holomorphic).

Less important: a holomorphic vector bundle is a complex vector bundle, but a complex vector bundle (C^∞ with fibers complex vector spaces). is not necessarily a holomorphic vector bundle. The local trivilization maps of a complex vector bundle can be C^∞, but not holomorphic.

Lemma 7.5. *Let X be a topological space. The data of transition functions (U_i, g_{ij}) is equivalent to the data of a vector bundle V up to isomorphism. If X is C^∞/holomorphic, and g_{ij} are C^∞/holomorphic, then V is C^∞/holomorphic.*

Proof. Look at $\tilde{V} = \bigsqcup (U_i \times k^{\oplus r})$, then define an equivalene relation $(x, v) \sim (y, w)$ via $x = y, v = g_{ij}(w)$, and this is an actual equivalence relation because of the cocycle condition. Gluing together will give you the vector bundle. □

The blow up $\pi : Bl_0\mathbb{C}^{n+1} \to \mathbb{P}^{n+1}$ is the tautological line bundle. This will be the famous $\mathcal{O}_{\mathbb{P}^n}(-1)$.

8. 10/2/23: LINE BUNDLES OF PROJECTIVE SPACE

Let's continue talking about vector bundles today. Note rank 1 vector bundles are called line bundles. Any linear algebra construction has an analog for vector bundles. For example, one can take direct sums, tensor products, exterior powers, symmetric powers, and dualize vector bundles.

What are the transition functions for each of these constructions?

The determinant of a bundle $\det V_i = \bigwedge^r V$, where r is the rank of vector bundle V. So the determinant bundle is a line bundle.

Remember that a holomorphic bundle is a very strong condition on a C^∞-bundle on a real manifold. Can we give examples of bundles that are complex but not holomorphic? One would need a bundle that doesn't have holomorphic structure. Fact: complex vector bundles are classified by their first Chern class $\in H^2(X; \mathbb{Z})$. If X is compact Kahler, holomorphic vector bundles have $c_1(X) \in H^2(X; \mathbb{Z}) \cap H^{1,1}(X)$. Both sit in $H^2(X; \mathbb{C})$. The obstruction to being holomorphic is that you need to be a $(1, 1)$-class. In $\mathbb{P}^4$ and higher, it is not known whether complex and holomorphic are different or the same. For $n \geq 3$, in $\mathbb{P}^n$ complex vector bundles $\iff$ holomorphic. In $n \geq 4$, this is an open problem. This is called the algebrization. Very deep question.

Definition 8.1. A holomorphic vector bundle homomorphism is a holomorphic map $\phi : V_1 \to V_2$ making the diagram

$$\begin{tikzcd} V_1 \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & V_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

commute., such that the rank of $\phi_x : V_{1,x} \to V_{2,x}$ is constant.

We want the rank of the map on fibers to be constant, so that we can form a suitable category. Be very careful about distinguishing between maps of holomorphic vector bundles, and distinguishing between maps of sheaves. For example, we'll see soon that the tautological line bundle admits an inclusion $\mathcal{O}_{\mathbb{P}^n}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^n} \cong \mathbb{P}^n \times \mathbb{C}$. But this is not a holomorphic bundle homomorphism! This is simply a map of sheaves. The rank is zero on some hypersurface of $\mathbb{P}^n$.

A fact for later is that every short exact sequence of C^∞-bundles splits. But this is false for holomorphic vector bundles. Sometimes, complex geometry is about the splitting of vector bundles, and they are very difficult problems.

Definition 8.2. Let $\pi : V \to X$ be a (topological, C^∞, holomorphic) vector bundle. A (continuous, C^∞, holomorphic) section of the vector bundle is a (continuous, C^∞, holomorphic) map $s : X \to V$ where $\pi \circ s = Id_X$.

Example 8.3. There is always a zero section, $s(x) = 0$ for every x.

The trivial bundle $X \times \mathbb{C}^r \to X$ has many sections. One has r linearly independent sections.

Global sections will soon become our main objects of interest. We'll see there is a correspondence between line bundles and divisors.

This is one of the first examples of a nontrivial vector bundle and also has no global sections.

Example 8.4 (Tautological line bundle)**.** Recall that we computed the blow up $Bl_0\mathbb{C}^{n+1}$, which came with a projection π to $\mathbb{P}^n$. We showed that $Bl_0\mathbb{C}^{n+1}$ was a complex manifold of dimension n. In fact,

$$\pi : Bl_0\mathbb{C}^{n+1} \to \mathbb{P}^n$$

is a holomorphic line bundle. It is called the tautalogical line bundle, and commonly denoted as $\mathcal{O}_{\mathbb{P}^n}(-1)$. Let's prove that it is a holomorphic line bundle.

Let $U_i \subset \mathbb{P}^n$ be the i-th standard chart. This lifts to the chart $V_i \subset Bl_0\mathbb{C}^{n+1}$, which will provide us with the local trivalization: we have a map

$$V_i \to U_i \times \mathbb{C}$$

where $(v, \ell) \mapsto (\ell, v_i)$. We see that this is holomorphic and fiberwise is a linear isomorphism. Note that the transition function $g_{ij} : U_i \cap U_j \to \mathbb{C}^\times$ is given by $g_{ij}(z) = \frac{z_i}{z_j}$. This is because if you follow $\phi_i\phi_j^{-1}$ over z, we see that

$$(\ell, v_j) \mapsto (v, \ell) \mapsto (\ell, v_i),$$

and $\frac{v_i}{v_j} = \frac{\ell_i}{\ell_j}$.

This leads us to the famous line bundles of $\mathcal{O}_{\mathbb{P}^n}$.

Definition 8.5 (line bundles of projective space)**.** Define $\mathcal{O}_{\mathbb{P}^n}(1) := \mathcal{O}_{\mathbb{P}^n}(-1)^*$. Fix $m > 0$. Define

$$\mathcal{O}_{\mathbb{P}^n}(-m) = \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes m},$$

and define

$$\mathcal{O}_{\mathbb{P}^n}(m) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m}.$$

Here, the transition function comes in handy to explicity describe these line bundles. Fix the transition functions of the tautalogical line bundle to be

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^\times \text{ where } z \mapsto \frac{z_i}{z_j}.$$

Thinking of what the transition functions look like for tensor products, the transition functions of $\mathcal{O}_{\mathbb{P}^n}(-m)$ are

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^\times \text{ where } z \mapsto \frac{z_i^m}{z_j^m}.$$

How about $\mathcal{O}_{\mathbb{P}^n}(1)$? Let's think about what the dual tensor product does. If g_{ij} is the transition function for the bundle before dualizing, then the transition function of the dual should map a dual vector space to a dual vector space. This map between duals should use the data of g_{ij}. We are forced to take a linear functional ϕ and precompose it with g_{ij}^{-1} evaluated at the point corresponding to the dual vector space that ϕ lives in. Then the transition functions of $\mathcal{O}_{\mathbb{P}^n}(1)$ are

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^\times \text{ where } z \mapsto \frac{z_j}{z_i}$$

and for $\mathcal{O}_{\mathbb{P}^n}(m)$ in general, they are

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^\times \text{ where } z \mapsto \frac{z_j^m}{z_i^m}.$$

Let's discuss global sections of these line bundles. In general, global sections of vector bundles are obtained by gluing together compatible local sections. More precisely, suppose $V \to X$ is a C^∞ bundle and s is a global section. Suppose we have local trivializations $\{U_i, \phi_i\}$. Then let $s_i := \phi_i \circ s|_{U_i}$. Then $s_i : U_i \to U_i \times k^r$ is a local section of the trivial bundle over U_i. But note that we have local compatability: if g_{ij} are the transition functions of V, then we must have

$$g_{ij} s_j = s_i$$

over $U_i \cap U_j$. A global section is equivalent to this data. Suppose we are given the bundle's transition function data $(U_i \times k^r, g_{ij})$, and we have sections $s_i : U_i \to U_i \times k^r$ such that $g_{ij} s_j = s_i$. Then these s_i glue together to some global section of the bundle.

Proposition 8.6. *Global sections of line bundles over projective space:*

1. $\mathcal{O}_{\mathbb{P}^n}(-1)$ *has no nonzero global sections via its blow-up property.*
2. *For all* $m > 0$, $\mathcal{O}_{\mathbb{P}^n}(-m)$ *has no nonzero global sections.*
3. *For all* $m \geq 1$, $\mathcal{O}_{\mathbb{P}^n}(m)$ *has global sections corresponding to the homogenous degree* m *polynomials in* $n+1$ *variables.*

Proof. (1) We prove this first using the universal property. Suppose there was a global section $s : \mathbb{P}^n \to Bl_0\mathbb{C}^{n+1}$. Then this descends to a map $\pi \circ s : \mathbb{P}^n \to \mathbb{C}^{n+1}$ that fits into the diagram

$$\begin{tikzcd} & Bl_0\mathbb{C}^{n+1} \arrow[d] \\ \mathbb{P}^n \arrow[ur] \arrow[r] & \mathbb{C}^{n+1} \end{tikzcd}$$

But note that the map $\mathbb{P}^n \to \mathbb{C}^{n+1}$ is constant, since any global function on $\mathbb{P}^n$ must be constant, and we can project onto the factors. The only way we could have a global section that is constant to $\mathbb{C}^{n+1}$ is the zero section.

(2) Suppose $s : \mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(-m)$ is a global section (note it's holomorphic). Then we have holomorphic local sections $s_i : U_i \to U_i \times \mathbb{C}$, where $g_{ij}s_j = s_i$ over $U_i \cap U_j$. Then fixing any point on $U_i \cap U_j$ and a sufficiently small neighborhood, we have that s_i is the power series

$$s_i = \sum_I c_I z_0^{e_0} \cdots z_{i-1}^{e_{i-1}} z_{i+1}^{e_{i+1}} \cdots z_n^{e_n} \text{ and } s_j = \sum_I c_I z_0^{e_0} \cdots z_{j-1}^{e_{j-1}} z_{j+1}^{e_{j+1}} \cdots z_n^{e_n}$$

and $g_{ij}s_j = \frac{x_i^m}{x_j^m} s_j = s_i$, but there are no powers of x_i in s_i. Thus, we must have $s_i, s_j = 0$. This is true locally everywhere on $U_i \cap U_j$, thus its true on all of $U_i \cap U_j$, and by the identity principle, true for each s_i on all of U_i, and thus s is the zero section.

(3) First we show that a homogenous degree m polynomial F defines a global section of $\mathcal{O}_{\mathbb{P}^n}(m)$. This is quite easy to see, as locally, F on $U_i \to U_i \times \mathbb{C}$ is the map $z \mapsto (z, \frac{F}{x_i^m}(z))$, and we see immediately that $g_{ij}\frac{F}{x_j^m} = \frac{x_j^m}{x_i^m}\frac{F}{x_j^m} = \frac{F}{x_i^m}$. Now let's show that these define all global sections.

Let s be a global section of $\mathcal{O}_{\mathbb{P}^n}(m)$. Let $s_i : U_i \to U_i \times \mathbb{C}$. So we know that $g_{ij}s_j = \frac{z_j^m}{z_i^m} s_j = s_i$. Fix a point in the intersection of all the standard opens, and consider a sufficiently small neighborhood. Then since s_i, s_j are holomorphic, we can write

$$s_i = \sum_I c_I z_0^{e_0} \cdots z_{i-1}^{e_{i-1}} z_{i+1}^{e_{i+1}} \cdots z_n^{e_n} \text{ and } s_j = \sum_I c_I z_0^{e_0} \cdots z_{j-1}^{e_{j-1}} z_{j+1}^{e_{j+1}} \cdots z_n^{e_n}.$$

So we have $z_j^m s_j = z_i^m s_i$. We see that the exponent of z_j in s_i cannot exceed m. Thus, all the s_i are multivariate polynomials. The fact that $z_j^m s_j = z_i^m s_i$ implies that these s_i come from a homogenous polynomial. By the identity principle, this must be true over all the opens. So indeed, our global section comes from a homogenous polynomial of degree m.

□

Let's now discuss an extremely important example of a vector bundle: the tangent bundle of a complex manifold. First, let's discuss the tangent bundle of a C^∞-manifold.

Recall that when we have have an open $U \subset \mathbb{R}^n$, then the tangent vectors of U at p is all of $\mathbb{R}^n$. Each of these tangent vectors provides us with a directional derivative: namely, given a germ of C^∞ functions at p, once can take the directional derivative along the vector. This gives a derivation in $Der_{\mathbb{R}}(C^\infty(U)_p, \mathbb{R})$. In general, we have a vector space isomorphism between tangent vectors at p and derivations of the stalk of smooth functions at p to $\mathbb{R}$.

This allows us to describe tangent vectors on an abstract C^∞-manifold. Let X be a C^∞-manifold of dimension n. Then then tangent vectors of X at p are the derivations $Der_{\mathbb{R}}(\mathcal{O}_{X,p}, \mathbb{R})$. However, we can describe them even more explicitly. Suppose $f : U \to V \subseteq \mathbb{R}^n$ is a chart around $p \in X$.

We can choose coordinates $x_1, \cdots x_n$ for V, so that $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ are a basis for the tangent vectors at V in $f(p)$. Then we can pull back these derivations to become derivations around p on U. For example, if $\phi \in \mathcal{O}_{X,p}$, then the pullback of $\frac{\partial}{\partial x_1}$ maps

$$\phi \mapsto \frac{\partial(\phi \circ f^{-1})}{\partial x_1}(f(p))$$

Since these tangent vectors are derivations of stalks, change of coordinates do not matter. Suppose $g : U \to V'$ was another chart, with coordinates $y_1, \cdots, y_n$. Let $h = f \circ g^{-1} : V' \to V$. What is the pullback of $\frac{\partial}{\partial y_j}$ along h^{-1}?

Let ϕ be some germ of a C^∞ function around $f(p) \in V$. Then abusing notation, we have

$$\frac{\partial \phi}{\partial y_j}(g(p)) = \frac{\partial(\phi \circ h)}{\partial y_j}(g(p)) = \sum \frac{\partial \phi}{\partial x_i}(f(p)) \frac{\partial h_i}{\partial y_j}(g(p)).$$

Here's one more example before we get to tangent bundles. Note that a vector field of V is a global section of $V \times \mathbb{R}^n$, where the tangent vectors $\mathbb{R}^n$ are identified with the vector space of derivations at that point. Then we can write a vector field of V as $\sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i}|_p$. Similarly, a vector field of V' can be written as $\sum_{i=1}^n b_i(q) \frac{\partial}{\partial y_i}|_q$. Pulling the vector field on V' along h^{-1} gives us a vector field on V.

This is the vector field such that for $\phi \in C^\infty(V)$, and $x \in V$,

$$x \mapsto \sum_{i=1}^n (b_i \circ h^{-1})(x) \frac{\partial(\phi \circ h)}{\partial y_i} = \sum_{i=1}^n (b_i \circ h^{-1})(x) \sum_{j=1}^n \frac{\partial \phi}{\partial x_j}(x) \frac{\partial h_j}{\partial y_i}(h^{-1}(x)).$$

Thus we see that if we write the pullback of $\sum_{i=1}^n b_i(q) \frac{\partial}{\partial y_i}|_q$ along h^{-1} as $\sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}|_x$, we'd obtain that

$$a_j(x) = \sum_{i=1}^n (b_i \circ h^{-1})(x) \frac{\partial h_j}{\partial y_i}(h^{-1}(x))$$

Finally, let's define the tangent bundle on T_X. We have (U_α, f_α) charts on X, where $f_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$ with coordinates $x_1^\alpha, \cdots, x_n^\alpha$. Then define $h_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$; T_X has transition functions $g_{\alpha\beta} = \mathcal{J}(h_{\alpha\beta}) \circ f_\beta : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ where

$$\mathcal{J}(h_{\alpha\beta}) = \{\frac{\partial h_{\alpha\beta}}{\partial x_i^\beta}\}$$

is the real jacobian matrix. With the right intuition, this should become "obvious." The transition function should identify tangent vectors on V' with tangent vectors on V. What should the association should be? It should be given by the jacobian of $f_\alpha \circ f_\beta^{-1}$.

9. 10/4/23: DIFFERENTIAL FORMS, DERHAM, (P,Q)-FORMS

Let X be a complex manifold. Let $x \in X$. And let $f : U \to V$ be a chart around x, where $f(x) = 0$. Let $z_1, \cdots, z_n$ be coordinates of V. We can write $z_n = x_n + iy_n$ in terms of real coordinates. The tangent space we discussed last time was

$$T_xX = \mathbb{R}\langle \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n} \rangle.$$

Complexifying $T_{\mathbb{C},x}X = \mathbb{C} \otimes_{\mathbb{R}} T_xX$, we can perform a change of basis and write

$$T_{\mathbb{C},x}X = \mathbb{C}\langle \frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z_1}}, \cdots, \frac{\partial}{\overline{z_n}} \rangle,$$

where $\frac{\partial z_k}{=} \frac{1}{2}(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k})$ and $\frac{\partial \overline{z_k}}{=} \frac{1}{2}(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k})$.

This complexified tangent space admits a decomposition

$$T_{\mathbb{C},x}X = T'_xX \oplus T"_xX$$

where $T'_xX = \mathbb{C}\langle \frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n} \rangle$ is the *holomorphic tangent space* and $T"_xX = \langle \frac{\partial}{\partial \overline{z_1}}, \cdots, \frac{\partial}{\overline{z_n}} \rangle$ is the *anti-holomorphic tangent space*. These are conjugate to each other.

Remark: there is an isomorphism of $\mathbb{R}$-vector spaces via

$$T_xX \hookrightarrow T_{\mathbb{C},x}X \to T'_xX$$

where $\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial y_j} \mapsto i\frac{\partial}{\partial z_j}$.

Recall last time the we had the tangent bundle T_X with transition functions $g_{\alpha\beta} = \mathcal{J}(h_{\alpha\beta}) \circ f_\beta$.

Definition 9.1. T'_X with fibers T'_xX at each $x \in X$ is the holomorphic tangent bundle of X. Let $h_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$. Then the transition functions are

$$g_{\alpha\beta} = \mathcal{J}(h_{\alpha\beta}) \circ f_\beta$$

where $\mathcal{J}(h_{\alpha\beta})$ is the holomorphic jacobian $(\frac{\partial h_{\alpha\beta}}{\partial z_i^\beta})$.

Sections of $T'X$ are called holomorphic vector fields.

Let's discuss differentials. Let $f : X \to Y$, $f(x) = y$, with charts $x \in U$ and $y \in V$, with coordinates $z_1, \cdots, z_n$ on U and coordinates $w_1, \cdots, w_m$ on V. So on charts, we have

$$f(x) = (f_1(z), \cdots, f_m(z))$$

where $z = (z_1, \cdots, z_n) \in U$. Then we have $df_x : T_{\mathbb{C},x}X \to T_{\mathbb{C},y}Y$ where

$$df_x(\frac{\partial}{\partial z_j}) = \sum_{k=1}^{n} \frac{\partial f_k}{\partial z_j}\frac{\partial}{\partial w_k} + \sum_{k=1} \frac{\partial \overline{f_k}}{\partial z_j}\frac{\partial}{\partial \overline{z_j}} = \sum_{k=1}^{n} \frac{\partial f_k}{\partial z_j}\frac{\partial}{\partial w_k}.$$

So the differential on the complexified tangent spaces induces the differential map on the holomorphic tangent spaces

$$df_x : T'_xX \to T'_yY$$

given by the holomorphic Jacobian $\mathcal{J}(f) = (\frac{\partial f_i}{\partial z_j})_{i,j}$.

An application of the implicit function theorem is the following:

Theorem 9.2. *Let $f : X \to Y$ be a holomorphic map of complex manifolds. Assume that $df_x : T'_x X \to T'_y Y$ has constant full rank $= \dim Y$, for every $x \in X$. Then for every $y \in Y$, either $f^{-1}(y)$ is either empty, or $f^{-1}(y)$ is a complex submanifold of X. Moreover, $\dim_x f^{-1}(y) = \dim_x X - r$.*

By the way, it is exceedingly rare to be in this situation. Given some parameter space with compact base, it's rare to find fibers that are complex submanifolds.

Aside: analytic subvarieties in complex manifolds. Let $Z \subseteq X$ be locally defined on U by vanishing of finitely many $f_k \in \mathcal{O}_X(U)$. Recall that locally, Z admits a decomposition into irreducible components. The aforementioned theorem about constant rank differentials applied to this situation... if $f_1, \cdots, f_k$ defined Z locally and $\frac{\partial f_1, \cdots, f_k}{\partial z_1, \cdots, z_n}$ has constant rank k on U, then Z is a submanifold of dimension $n-k$. This doesn't really add anything new, as we already know all this from the implicit function theorem. But we've just rephrased what we already knew from this theorem about differentials.

Definition 9.3. We say $x \in Z \subseteq X$ isa smooth point if Z is a submanifold of X at x. Otherwise, x is called singular.

Example 9.4. Let $Z = \{y^2 = x^2 + x^3\} \subseteq \mathbb{C}^2$. Here

$$\frac{\partial f}{\partial x} = 2x + 3x^2, \frac{\partial f}{\partial y} = 2y.$$

The point $\{(0,0)\}$ is a singular point.

Homework: show that $Z_{sing} = \{x \in Z |$ x singular $\}$ is contained in an analytic subvariety of X that does not contain Z. More strongly: Z_{sing} is an analytic subvariety, and its proper in Z. This finishes our aside. Let's go back to calculus on manifolds.

Differential forms. Let's briefly review the C^∞-case. Assume X is a smooth manifold of dimension n.

Definition 9.5. A differential k-form on X is a global section of the vector bundle $\bigwedge^k (TX)^*$.

In other words, given a smooth vector field $(v_1, \cdots, v_k)$ where v_i are smooth vector fields, this form ω maps $(v_1, \cdots, v_k) \mapsto \omega(v_1, \cdots, v_k)$ that is multi-linear and alternating in $v_1, \cdots, v_k$. And on $x \in X$, our k-form ω is the evaluation map

$$\omega(x) : \bigwedge^k T^*_{\mathbb{R},x} X \to \mathbb{R}.$$

Let $A^k(X)$ denote the differential k-forms on X. Let's provide a local expression of the k-forms. Say $x_1, \cdots, x_n$ are coordinates around $x \in X$. So $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ is a basis for $T_{\mathbb{R},x} X$. This induces a dual basis of 1-forms $dx_1, \cdots, dx_n$ given by $dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$.

If $\omega \in A^1(U)$, then locally around a neighborhood of x, a 1-form ω looks like

$$\omega = \phi_1 dx_1 + \cdots + \phi_n dx_n, \phi_i \in C^\infty(U).$$

Then a form $\omega \in A^k(U)$ is locally is

$$\omega = \sum_{i_1 < \cdots < i_k} \phi_{i_1 \cdots i_k}(\underline{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Note that applying ω to a tangent vector immediately tells you what happens. Have $\phi_{i_1,\cdots,i_k}(\underline{x}) = \omega(\frac{\partial}{\partial x_{i_1}}, \cdots, \frac{\partial}{\partial x_{i_k}})$.

Definition 9.6. Let $\omega \in A^k(X)$. The exterior derivative of $\omega = \sum_I \phi_I(\underline{x}) dx_I$ is $d\omega \in A^{k+1}(X)$ is given locally by:

$$d\omega = \sum_{j=1}^{n} \sum_{I} \frac{\partial \phi_I}{\partial x_j} dx_j \wedge dx_I.$$

The exterior derivative is well-defined.

Example 9.7. Suppose $f \in C^\infty(X)$. We get a 1-form $df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j$.

Proposition 9.8. *Properties of the exterior derivative:*

(1) $d(f * \omega) = df \wedge \omega + f d\omega$, $f \in A^0(X), \omega \in A^k(X)$.
(2) $d \circ d = 0$.
(3) Suppose $f : X \to Y$ *is a* C^∞ *map of smooth manifolds. We can define the pullback map of forms* $f^* : A^k(Y) \to A^k(X)$ *in the following way: let* $x_1, \cdots, x_n$ *be local coordinates coordinates on* X *around* x *and* $y_1, \cdots, y_m$ *local coordinates on* Y *around* $f(x) = y$*. Then*

$$f^* dy_i = df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j,$$

is the map $A^1(Y) \to A^1(X)$*, and everything else follows from linearity.*

What we obtain from these properties is the DeRham complex:

$$0 \longrightarrow A^0(X) \longrightarrow A^1(X) \longrightarrow A^2(X) \longrightarrow \cdots \longrightarrow A^n(X) \longrightarrow 0$$

Definition 9.9. The deRham cohomology of X is the cohomology of $A^*(X)$, i.e

$$H^i_{dR}(X) := \frac{\ker(A^i(X) \to A^{i+1}(X))}{Im(A^{i-1}(X) \to A^i(X))}.$$

The closed forms ω are those such that $d\omega = 0$. The exact forms are those in the image. DeRham cohomology is given by closed forms modulo exact.

The d-bar Poincare lemma will tell us that on very small neighborhoods of a point, the deRham cohomology is zero except in degree 0 (?).

Theorem 9.10 (deRham)**.** *Let* X *be a real smooth manifold. Then singular cohomology and deRham cohomology agree:*

$$H^i(X, \mathbb{R}) \cong H^i_{dR}(X), \forall i \geq 0.$$

We won't prove this theorem for real smooth manifolds. But we'll do the analog of all this for complex manifolds.

Let X be a complex manifold, local coordinates $z_1, \cdots, z_n$. Let $z_j = x_j + iy_j$. Let $A^k(U)$ denote the k-forms on U. Then we can complexify to obtain $A^k_{\mathbb{C}}(U)$. Then any $\omega \in A^k(U)$ can be written as

$$\omega = \sum_{|I|+|J|=k} \phi_{IJ} dz_I \wedge d\overline{z}_J$$

where the $\phi_{IJ} \in C^\infty(U)$.

Definition 9.11. A (p,q)-form (or a form of type (p,q)) in $A^k(X)$ is an $\omega \in A^k_{\mathbb{C}}(X)$ that can be written entirely as

$$\omega = \sum_{|I|=p} \sum_{|J|=q} \phi_{IJ} dz_I \wedge d\bar{z}_J.$$

So we have $A^{p,q}(X) \subseteq A^k_{\mathbb{C}}(X)$, where $p+q=k$.

The definition of type p,q doesn't depend on the choice of charts. Suppose $\omega \in A^{p,q}(X)$ with respect to some covering by atlases $\{U_i, \phi_i\}$. Suppose we have some (U', ϕ'). Suppose we have $U_i \cap U'$. On $U_i \cap U' \to \phi_i(U_i \cap U')$, we have

$$\omega = \sum_{IJ} f_{IJ} dz^i_I \wedge d\overline{z^i_J}.$$

But note that that the transition map $\phi' \circ \phi_i^{-1}$ is holomorphic, thus it will map linear combinations of dz^i_I to linear combinations of dz'_I, and linear combinations of $d\overline{z^i_J}$ to linear combinations of $d\overline{z'_J}$. This is true for every $U_i \cap U'$, showing that ω remains a (p,q)-form on U'.

Note that exterior derivative $d : A^k(X) \to A^{k+1}(X)$, applied to (p,q)-forms $A^{p,q}(X) \subseteq A^k(X)$ maps to $A^{p+1,q}(X) \oplus A^{p,q+1}(X)$. We see that if $\omega = \sum \phi_{IJ} dz_I \wedge d\overline{z_J}$, then

$$d\omega = \sum_{i=1}^n \frac{\partial \phi_{IJ}}{\partial z_i} dz_i \wedge dz_I \wedge d\overline{z_J} + \sum_{i=1}^n \frac{\partial \phi_{IJ}}{\partial \overline{z_i}} d\overline{z_i} \wedge dz_I \wedge d\overline{z_J}.$$

Then we can write the exterior derivative restricted to (p,q) forms in terms of two extremely important operators. We write $d = \partial + \overline{\partial}$, where $\partial : A^{p,q} \to A^{p+1,q}$ and $\overline{\partial} : A^{p,q} \to A^{p,q+1}$. Remember that on functions, this $\overline{\partial}$ was very important in that it recognized holomorphicity. We'll want to continue to concentrate on this operator.

10. 10/11/23: dBar Poincare lemma, integration, orientability

Let X be a complex manifold of complex dimension n. Using the C^∞ structure of X, we have the deRham complex.

$$d : A^*(X) \to A^{*+1}(X).$$

Complexifing each vector space $A^k(X)$, we have the (p,q) forms

$$A^{p,q}(X) \subset A^k_{\mathbb{C}}(X) = A^k(X) \otimes_{\mathbb{R}} \mathbb{C},$$

such that the exterior derivative restricts to

$$A^{p,q}(X) \to A^{p+1,q}(X) \oplus A^{p,q+1}(X).$$

We can write $d = \partial + \overline{\partial}$. Note that $\overline{\partial}\overline{\partial} = 0$.

Definition 10.1. The Dolbeault complexes $A^{p,*}(X)$ of X for every $p \geq 0$ are

$$0 \to A^{p,0}(X) \to^{\overline{\partial}} \to A^{p,1}(X) \to^{\overline{\partial}} \cdots \to^{\overline{\partial}} A^{p,n}(X) \to 0$$

where $0 \leq p, q \leq n$.

The (p,q)-Dolbeault cohomology of X is

$$H^{p,q} := H^q(A^{p,*}(X)).$$

The formers in the kernel are called $\bar{\partial}$-closed forms, and those in the image are called the $\bar{\partial}$-exact forms.

Note that we could have formed a complex with ∂ as the differential. But the result is something that is conjugate to the Dolbeaut complex as we've defined them. Since they're the same up to conjugation, we prefer $\overline{\partial}$ because it detects holomorphicity. Note that these dolbeaut complexes will come from sheaf cohomology of coherent sheaves of forms on X.

Example 10.2. $H^{p,0}(X) = \ker \bar{\partial} : A^{p,0}(X) \to A^{p,1}(X)$.

If $\omega = \sum_I \phi_I dz_I$, then

$$\bar{\partial}\omega = \sum_{I,k} \frac{\partial \phi_I}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I.$$

So we see that we need the ϕ_I to be holomorphic. So we see that

$$H^{p,0}(X)$$

are exactly the holomorphic p-forms on X. In other words, they are global sections of the holomorphic tangent bundle on X.

The $\bar{\partial}$-Poincare lemma tells us that on very nice subsets, the higher Dolbeaut cohomologies vanish.

First we do the one-dimensional case.

Theorem 10.3 (One-variable dee bar Poincare)**.** *Let $B_\epsilon \subset \overline{B_\epsilon} \subset U$. Suppose $\alpha = f d\overline{z} \in A^{0,1}(U)$. Then*

$$g = \frac{1}{2\pi i} \int_{B_\epsilon} \frac{f(w)}{w - z} dw \wedge d\overline{w}$$

is well-defined over B_ϵ and $\overline{\partial} g = \alpha$.

Proof. First let's show that g is well-defined. We'll show it is defined on a neighborhood of every point in B_ϵ. Pick $z_0 \in B_\epsilon$, and let V be an open neighborhood of z_0 such that there exist $\psi : B_\epsilon \to \mathbb{R}$ with compact support, such that $\psi|_V = 1$. Then define $f_1 = \psi f$, and $f_2 = (1-\psi) f$. Then we can split g into

$$g_1 = \frac{1}{2\pi i} \int_{B_\epsilon} \frac{f_1(w)}{w - z} dw \wedge d\overline{w}, g_2 = \frac{1}{2\pi i} \int_{B_\epsilon} \frac{f_2(w)}{w - z} dw \wedge d\overline{w}.$$

Note that g_2 is zero everywhere on V. Consider g_1 on V. We have

$$g_1 = \frac{1}{2\pi i} \int_{B_\epsilon} \frac{f_1(w)}{w - z} dw \wedge d\overline{w} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(z+u)}{u} du \wedge d\overline{u} =$$

by compact support of f_1, and letting $u = re^{i\theta}$, we find that

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(z+u)}{u} du \wedge d\overline{u} = \frac{1}{\pi} \int_{\mathbb{C}} f(z + re^{i\theta}) e^{-i\theta} d\theta \wedge dr.$$

Thus, g_1 is well-defined everywhere. Thus, g is well-defined on V, and hence is well-defined on all of B_ϵ.

Now we show that $\overline{\partial} g = \alpha$ on B_ϵ. It suffices to show it holds over V. Over V, note that $\overline{\partial g} = \overline{\partial g_1}$. Then $\overline{\partial} g_1 =$

$$\frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{\partial}}{\partial \overline{z}} f(z + re^{i\theta}) e^{-i\theta} d\theta \wedge dr = \frac{1}{\pi} \int_{\mathbb{C}} [\frac{\partial f}{\partial \overline{w}} \frac{\overline{z} + re^{-i\theta}}{\partial \overline{z}} + \frac{\partial f}{\partial w} \frac{z + re^{i\theta}}{\partial \overline{z}}] e^{-i\theta} d\theta \wedge dr$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial f}{\partial \overline{w}} (z + re^{i\theta}) e^{-i\theta} d\theta \wedge dr = \frac{1}{2\pi i} \int_{B_\epsilon} \frac{\partial f_1}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z}.$$

We claim that for $z \in V$, $\frac{1}{2\pi i}\int_{B_\epsilon} \frac{\partial f_1}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w-z} = f_1(z) = f(z)$. Observe that

$$\frac{1}{2\pi i}\int_{B_\epsilon} \frac{\partial f_1}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w-z} = \frac{1}{2\pi i}\lim_{\delta \to 0}\int_{B_\epsilon \setminus B_\delta} \frac{\partial f_1}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w-z} = \frac{1}{2\pi i}\lim_{\delta \to 0}\int_{\partial B_\delta} \frac{f_1}{w-z} dw$$

since $supp(f_1) \subset B_\epsilon$ and using Stokes' theorem. But this is $f_1(z)$. And on V this is $f(z)$. $\square$

There is also mention of this in Griffiths-Harris page 5.

Utilizing the one-variable $\overline{\partial}$-Poincare, we can prove the analog for several variables.

Lemma 10.4 ($\overline{\partial}$-Poincare lemma)**.** *Let $\Delta_\epsilon \subset \overline{\Delta_\epsilon} \subset U$. Let $\omega \in A^{p,q+1}(U)$ such that $\overline{\partial}\omega = 0$. Then there exist $\alpha \in A^{p,q}(\Delta_\epsilon)$ such that*

$$\overline{\partial}\alpha = \omega$$

on Δ_ϵ.

Proof. First, we reduce to the case $A^{0,q+1}(U)$. Note that if $\omega \in A^{p,q+1}(U)$, then we can write $\omega = \sum \phi_I dz_I \wedge d\overline{z_J} = \sum \phi_I \wedge \omega_I$ where $\omega_I = \sum_J \phi_I d\overline{z_J}$. Then we see that $\overline{\partial}\omega = 0 \iff \overline{\partial}\omega_I = 0$ for every I. So if we prove the statement for $A^{0,q+1}(U)$, then we see it will be proven for $A^{p,q+1}(U)$ as well.

So suppose $\omega \in A^{0,q+1}(U)$ such that $\overline{\partial}\omega = 0$. We can write $\omega = \phi_I d\overline{z_I}$. Induct on the largest entry k occuring in any index set occuring in ω.

The base case is $k = q+1$. Then $\omega = \phi d\overline{z_1} \wedge \cdots \wedge d\overline{z_{q+1}}$. The fact that $\overline{\partial}\omega = 0 \implies \phi$ is holomorphic in $z_{q+2}, \cdots, z_n$. If we apply the one-variable $\overline{\partial}$-Poincare lemma to coordinate z_{k+1}, we find that

$$g(z_1, \cdots, z_q, z, z_{q+1}, \cdots, z_n) = \frac{1}{2\pi i}\int_{\Delta_\epsilon} \frac{\phi(z_1, \cdots, z_q, w, z_{q+2}, \cdots, z_n)}{w-z} dz_{q+1} \wedge d\overline{z_{q+1}}.$$

This $g(z)$ is holomorphic in $z_{q+2}, \cdots, z_n$ and $\frac{\partial}{\partial \overline{z_{q+1}}} g(z) = \phi(z_1, \cdots, z_q, z, z_{q+2}, \cdots, z_n) d\overline{z_{q+1}}$. Then we can consider

$$(-1)^q g(z) d\overline{z_1} \wedge \cdots \wedge d\overline{z_q}.$$

We see that $\overline{\partial}$ applied to this yields ω as desired.

Now suppose our claim holds for k. We now show it holds for $k+1$. Note that we can write $\omega = \omega_1 + \omega_2 \wedge d\overline{z_{k+1}}$, where $\omega_1 \in A^{0,q+1}(U)$ has largest index k and $\omega_2 \in A^{0,q}(U)$ has largest index k.

Write $\omega_2 = \sum_J f_{2,J} d\overline{z_J}$. Since $\overline{\partial}\omega = 0$, note this imples that each $f_{2,J}$ is holomorphic in variables $z_{k+2}, \cdots, z_n$. If we consider each $f_{2,J} d\overline{z_{k+1}}$, by the one-variable $\overline{\partial}$-Poincare lemma, we have there exist

$$g_{2,J} = \frac{1}{2\pi i}\int_{\Delta_\epsilon} \frac{f_{2,J}}{w - z_{k+1}} dw \wedge d\overline{w}$$

such that $\frac{\partial}{\partial \overline{z_{k+1}}} g_{2,J} = f_{2,J} d\overline{z_{k+1}}$. Note that $g_{2,J}$ are holomorphic in $z_{k+2}, \cdots, z_n$. Then we see that $\overline{\partial}$ applied to

$$(-1)^q \sum g_{2,J} d\overline{z_J}$$

yields $\omega_2 \wedge d\overline{z_{k+1}} + \omega_3$ where $\omega_3 \in A^{0,q+1}(B_\epsilon)$ (check at some point this detail about ω_3's domain).

By the inductive hypothesis, we could find $\alpha_1, \alpha_3 \in A^{0,q}(\Delta_\epsilon)$ such that $\overline{\partial}\alpha_i = \omega_i$. Then let

$$\alpha = (-1)^q \sum g_{2,J} d\overline{z_J} + \alpha_1 - \alpha_3.$$

□

There is a more analytic argument, involving some limiting process. Page 25-27 in Griffiths Harris. One can actually show that the $\overline{\partial}$-Poincare lemma holds on any $\Delta(0,r)$ inclusing when some radius are ∞.

Similar arguments also show that an analog of $\overline{\partial}$-Poincare lemma holds for $(\mathbb{C}^*)^\ell \times \mathbb{C}^k$ for any k, ℓ.

Famous vanishing theorems: Kodaira vanishing for example. What do they mean? From the point of view in analysis, it means solving the $\overline{\partial}$ problem. The Dolbeaut theorem will relate these different cohomologies to Dolbeaut cohomology.

We'll see a different kind of proof of the Kodaira vanishing using topology plus analytic geometry, that aren't just analysis and solving some PDE. One can prove this also by reducing mod p, restricting to $Spec(Z)$ and look at primes, uppersemi-continuity... interesting that there are arithmetic proofs.

Let's talk more about calculus on manifolds. We'll at some point talk about harmonic forms, which are very important, and they'll work well with Kahler manifolds.

Let X be a smooth manifold of dimension n. Then X is called orientable if there $\exists$ a consistent choice of orientation for each tangent space T_xX.

Let's remind ourselves what this means. Let V be a finite dimensional $\mathbb{R}$-vector space. An orientation on V is an equivalence class of bases such that

$$b_1 \sim b_2 \iff Ab_1 = b_2, \text{ where } \det A > 0.$$

So we have two orientations on any finite dimensional real vector space, $+1$ positive bases and -1 negative bases.

Let X be a C^∞-manifold. Then $x_1, \cdots, x_n$ local coordinates is a positive coordinate system if the basis

$$\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$$

is a positive basis. Determinant being continuous guarantees you can find such an open around x.

Let (U_i, ϕ_i) be an atlas for X, transition functions. Let $g_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$. Then our C^∞-manifold is orientable if the g_{ij} are orientation-preserving, i.e the determinant of the real jacobian of g_{ij} is positive (might want to check well-defined up to charts?).

Now let's discussion integration on C^∞-manifolds. Let $\omega \in A^k(X)$, and assume this has compact support. What does this mean? The support is

$$Supp(\omega) = \{x | \omega(x) \neq 0\}.$$

and compactly supported means that $\overline{Supp(\omega)}$ is compact.

For example, if X is compact, then the support of any form is immediately compactly supported.

Have chart (U_i, ϕ_i). Pick partition of unity subordinate to the cover X, so that $\rho_i : X \to \mathbb{R}$, $supp\rho_i \subseteq U_i$, and $\sum_i \rho_i(x) = 1$ where finitely many of the ρ_i are nonzero at any x.

Assume that X is orientable. Have $(\rho_i\omega)|_{U_i} = f_i dx_1^i \wedge \cdots dx_n^i$. Then we define

Definition 10.5. $\int_X \omega := \sum_i \int_{V_i} f_i d\mu$ where μ is the Lebesgue measure, and $\phi_i : U_i \to V_i$

11. 10/16/23: Kahler manifolds

Let X be a smooth orientable manifold of dimension n. Let $\omega \in A^n(X)$ be an n-form with compact support. Take a cover $\{\phi_i : U_i \to V_i\}$ of X and a partition of unity subordinate to the cover. We have $\rho_i\omega|_{U_i} = f_i dx_1^i \wedge \cdots \wedge dx_n^i$.

$$\int_X \omega := \sum_i \int_{V_i} f_i d\mu.$$

Bott-Tu is a good reference for integration of differential forms. For example via change of coordiantes through $U_i \cap U_j$, one finds that

$$dx_1^i \wedge \cdots \wedge dx_n^i = (det J_{\mathbb{R}}(g_{ij})g_{ij}^{-1})dx_1^j \wedge \cdots \wedge dx_n^j.$$

The sign is constant because X is orientable. And the integral is the same because of change of variables formula.

Theorem 11.1 (Stokes'). *If $\eta \in A^{n-1}(X)$, then $\int_X d\eta = \int_{\partial X = \emptyset} = 0$.*

Let X be a complex manifold of dimension n. Then X is always orientable. Why? Because looking at the transition functions g_{ij}, note that the real jacobian will be of the form

$$\begin{pmatrix} J_{\mathbb{C}}(g_{ij}) & 0 \\ 0 & \overline{J_{\mathbb{C}}(g_{ij})} \end{pmatrix}$$

but the determinant of this is $|\det J_{\mathbb{C}}(g_{ij})|^2 > 0$. This proves that every complex manifold is orientable.

Let's restrict ourselves to local coordinates $z_j = x_j + iy_j$ where $j = 1, \cdots, n$. Take $x_1, y_1, \cdots, x_n, y_n$ be a positive basis for orientation. Then

$$dz \wedge d\bar{z} = -2idx \wedge dy.$$

So locally, the "lebesgue measure" we are integrating with respect to is

$$\frac{i^n}{2^n}(dz_1 \wedge d\overline{z_1}) \wedge \cdots \wedge (dz_n \wedge d\overline{z_n}) = (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n).$$

Let's talk about Hermitian forms. We're moving towards Kahler manifolds. Let V be $\mathbb{R}$-vector space.

Definition 11.2. Let V be a $\mathbb{R}$-vector space. Then $g : V \times V \to \mathbb{R}$ is

- symmetric if $g(v_1, v_2) = g(v_2, v_1), \forall v_1, v_2 \in V$.
- positive definite if $g(v, v) > 0$ for all $v \neq 0$.

The symmetric bilinear form g can be represented as a matrix A, so that $\langle v, w\rangle = v^T Aw$. If we choose a basis $\langle e_1, \cdots, e_n\rangle$ for A, then $A_{ij} = \langle e_i, e_j\rangle$. If g is symmetric and positive-definite, then A is symmetric and $v^T Av > 0$ for all $v \neq 0$.

Definition 11.3. Let X be a smooth manifold of dimension n. A Riemannian metric g on X is a collection of symmetric positive definite bilinear forms

$$g_x : T_{\mathbb{R},x}X \times T_{\mathbb{R},x}X \to \mathbb{R}, \forall x \in X$$

which vary smoothly, i.e. for all $U \subseteq X$ open and for all u, v C^∞-vector fields on U, we have that $g(u, v)$ is a C^∞-function on U.

Example 11.4. Let $x_1, \cdots, x_n$ local coordinates. Have $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ basis for $T_{\mathbb{R},x}X$, where $g_{ij}(x) = g_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and we can let $G = (g_{ij})$.

For every $X \subseteq \mathbb{R}^n$, there is an induced metric obtained by restriction (have to be careful, coordinate changes happen on the manifold, not one hundred percent just restriction?)

Now we want to extend this Riemannian metric to an analog for complex manifolds. We want a hermitian metric. Let V be a $\mathbb{C}$-vector space, and $\dim_{\mathbb{C}} V = n$. Let $\mathcal{J} : V \to V$ denote $\mathcal{J}(v) = iv$ for all v.

Definition 11.5 (Hermitian form)**.** A hermitian form h on V is a mapping $h : V \times V \to \mathbb{C}$ that is

- $\mathbb{C}$-linear in the first factor $h(\lambda v, w) = \lambda h(v, w)$.
- $h(v, w) = \overline{h(w, v)}$, $\forall v, w \in V$.

These two conditions imply that h is conjugate-linear in the second factor, i.e $h(v, \lambda w) = \overline{\lambda} h(v, w)$.

- (positive-definite) Furthermore, note that $h(v, v) = \overline{h(v, v)}$, so $h(v, v) \in \mathbb{R}, \forall v$. We require $h(v, v) > 0$.

We can write $g(v, w) = Reh(v, w)$ and $\omega(v, w) = -Imh(v, w)$.

Lemma 11.6. *If h is a positive definite Hermitian form on V, then*

(1) $g(v, w) = Reh(v, w)$ is a symmetric positive definite bilinear form on V
(2) ω is a real bilinear alternative form on V

Proof. Have $g(v, w) = \frac{1}{2}(h(v, w) + \overline{h(v, w)}) = \frac{1}{2}(h(v, w) + h(w, v))$. So from this, it is symmetric bilinear. And it is positive definite.

Similarly, $\omega(v, w) = \frac{i}{2}(h(v, w) - h(w, v))$. □

Remark: $h(v, w) = g(v, w) + ig(v, \mathcal{J}w)$, and where $\omega(v, w) = -g(v, \mathcal{J}w)$. In other words, g determines h uniquely. The same goes for ω, since we can write ω in terms of g. Furthermore, h is hermitian $\iff$ g is compatible with $\mathcal{J}$, i.e $g(\mathcal{J}v, \mathcal{J}w) = g(v, w)$ for all $v, w \in V$.

Now let X be a complex manifold, $\dim_{\mathbb{C}} X = n$. Recall that we had a map

$$T_{\mathbb{R},x}X \hookrightarrow T_{\mathbb{C},x}X \to T'_x X.$$

This composition is an isomorphism of $\mathbb{R}$-vector spaces, where $\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial y_j} \mapsto i\frac{\partial}{\partial z_j}$, and $T_{\mathbb{C},x}X = T_{\mathbb{R},x}X \otimes_{\mathbb{R}} \mathbb{C}$ comes with a canonical $\mathcal{J}$.

Definition 11.7. A hermitian metric on X is a collection of hermitian forms

$$h_x : T'_x X \times T'_x X \to \mathbb{C}$$

such that the real part g_x induces a Riemannian metric on X.

Locally, if $z_j = x_j + iy_j$, then $h_{k\ell} := h(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_\ell})$. So $H = (h_{k\ell})$ where $h(v, w) = v^t H \overline{w}$. So H is a symmetric-hermitian, positive definite $n \times n$ matrix. So $H = \overline{H}^T$.

Note

$$g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}) = Reh(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_\ell}) = Reh_{k\ell}$$

and

$$g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_\ell}) = Reh(\frac{\partial}{\partial z_k}, i\frac{\partial}{\partial z_\ell}) = Imh_{k\ell}.$$

So $G = \begin{pmatrix} ReH & ImH \\ -ImH & ReH \end{pmatrix}$.

We have

$$\omega(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}) = -Imh_{k\ell}$$

$$\omega(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_\ell}) = Reh_{k\ell}.$$

We want to express ω in terms of dz_k and $d\overline{z_k}$. We have $4\omega(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \overline{z_\ell}}) = \omega(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}, \frac{\partial}{\partial x_\ell} + i\frac{\partial}{\partial y_\ell})$

$$= -Imh_{k,\ell} + iReh_{k,\ell} + iReh_{k,\ell} - Imh_{k\ell} = 2ih_{k,\ell}.$$

Do the same calculation with

$$\omega(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_\ell}) = \omega(\frac{\partial}{\partial \overline{z_k}}, \frac{\partial}{\partial \overline{z_\ell}}) = 0.$$

The conclusion is that $\omega = \frac{i}{2}\sum_{k,\ell=1}^n h_{k\ell} dz_k \wedge d\overline{z_\ell}$ is the associated (1,1)-form of hermitian metric h. It is a (1,1)-form that keeps track of the Hermitian structure.

Definition 11.8. A hermitian metric h on X is Kahler if $d\omega = 0$. X is called Kahler if it has a Kahler metric.

Next time we will see that projective space is metric. It's not easy to show this – need to find a metric. Note submanifolds of Kahler manifolds are Kahler as well. Thus, once we show projective varieties are Kahler, we'll be on our way to showing smooth projective varieties are thus Kahler.

Abelian varieties are not Kahler.

The Kahler form on a Kahler manifold is like the hyperplane in $\mathbb{P}^n$.

Note H will correspond to $\mathcal{O}_{\mathbb{P}^n}(1)$, there will be a Chern class living in $H^2(\mathbb{P}^n; \mathbb{Z})$ and by the deRham theorem this will live in $H^{1,1}$. So we could forget about the hyperplane, and we could just consider this form ω. That's what Kahler geometry is. But usually a Kahler form is not with integral coefficients. But one of the main theorems is that: when the Kahler form lives in integral coefficients, the manifold is actually projective. Then you're in the realm of algebraic geometry. That will be one version of the Kodaira embedding theorem.

Note that ω $(1,1)$-form in general is called positive definite $\iff$ the matrix of coefficients is positive definite.

Note that X is Kahler $\iff$ there exists a closed positive $\omega \in A^{1,1}(X)$. This is because ω determines h, and also h is hermitian at each point if ω on each vector space is compatible with the complex structure. But the fact that $\omega(Jv, Jw) = \omega(v, w)$ follows precisely from ω being a $(1,1)$-form.

Example 11.9. On $\mathbb{C}^n$, with coordinates $z_1, \cdots, z_n$. Then $h_{k\ell} = h(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_\ell}) = \delta_{k,\ell}$. Have $H = Id$.

Have g is standard euclidean metric on $\mathbb{R}^{2n}$, and $\omega = \frac{i}{2}\sum_{k=1}^n dz_k \wedge d\overline{z_k} = \sum_{k=1}^n (dx_k \wedge dy_k)$. This ω is a Kahler form on $\mathbb{C}^n$.

Example 11.10. Let (X, h_x) be Kahler. Let $Y \subseteq X$ be a complex submanifold. Then there exists an induced Hermitian metric h_Y on Y, via $\iota : Y \hookrightarrow X$. Have $\iota^*\omega_X = \omega_Y$. So Y is Kahler. Note $d(\iota^*\omega_X) = \iota^* d\omega_X = 0$.

So every submanifold of $\mathbb{C}^n$ is Kahler.

Example 11.11. $\mathbb{C}^n/\Lambda$ is kahler. Every complex manifold has Hermitian structure. If the manifold is one-dimensional, Riemann surface, then it is Kahler by default.

12. 10/18/23: FUBINI-STUDY METRIC, VOLUME FORMS

Every complex manifold admits a Hermitian metric. Exercise. Condition to be Kahler is there exists a closed (1,1)-form. Immediate for one-dimensional complex manifolds.

Example 12.1. Let $X = \mathbb{C}^n/\Lambda$, compact complex torus. The standard Kahler metric/form on $\mathbb{C}^n$ is $\frac{i}{2}\sum dx_i \wedge dy_j$. This X is symmetric enough so that the standard Kahler form descends to X. Note that elements of the lattice Λ give translations

$$t_a : \mathbb{C}^n/\Lambda \to \mathbb{C}^n/\Lambda, x \mapsto x + a.$$

So $t_a^*\omega = \omega$.

Here is our main example: the Fubini-Study metric on $\mathbb{P}^n$. This will show that $\mathbb{P}^n$ is a compact Kahler manifold. Consider homogenous coordinates on $\mathbb{P}^n$, $(z_0 : \cdots : z_n)$. Recall that $\mathbb{P}^n = \bigcup_{i=0}^n U_i$, each $U_i \cong \mathbb{C}^n$. We're going to define $(1,1)$-forms on each of these $U_i \cong \mathbb{C}^n$, and show they all glue together.

Define $w_i \in A^{1,1}(U_i)$ to be $\frac{i}{2\pi}\partial\overline{\partial}\log(\sum_{j=0}^n |\frac{z_j}{z_i}|^2))$. (Note the inside function is called a *Kahler potential*.) The claim is that these glue to an $\omega \in A^{1,1}(\mathbb{P}^n)$. In other words, we want to show that

$$\omega_i|_{U_i\cap U_j} = \omega_j|_{U_i\cap U_j}.$$

Note that on $U_i \cap U_j$, we have

$$\log(\sum_{k=0}^n |\frac{z_k}{z_i}|^2) = \log(|\frac{z_j}{z_i}|^2 \sum_{k=0}^n |\frac{z_k}{z_j}|^2 = \log(|\frac{z_j}{z_i}|^2) + \log(\sum_{k=0}^n |\frac{z_k}{z_j}|^2).$$

It suffices to show that $\partial\overline{\partial}\log(|\frac{z_j}{z_i}|^2) = 0$. Note that pullback commutes with both ∂ and $\overline{\partial}$ because it commutes with the exterior derivative, and this map is holomorphic, so it has the expected structure which gives commutation with ∂ and $\overline{\partial}$. In general,

$$\partial\overline{\partial}\log|x|^2 = \partial\overline{\partial}\log(x\overline{x}) = \partial\frac{1}{x\overline{x}}\overline{\partial}(x\overline{x})$$

$$\partial(\frac{1}{x\overline{x}}xd\overline{x}) = \partial(\frac{d\overline{x}}{\overline{x}}) = \partial(\overline{\frac{dx}{x}}) = 0.$$

Therefore, the forms glue to a ω_{FS}.

Proposition 12.2. *Properties of $\omega_{FS} \in A^{1,1}(\mathbb{P}^n)$.*

1. *(Real) ω_{FS} takes values in $\mathbb{R}$.*
2. *(Closed) $d\omega_{FS} = 0$.*
3. *(Positive) ω_{FS} is positive*

Altogether, these properties tell you that ω_{FS} is the $(1,1)$-form associated to a Kahler metric on $\mathbb{P}^n$, called the Fubini-Study metric.

Proof. (1) This follows from $\overline{\partial\overline{\partial}} = \overline{\partial}\partial = -\partial\overline{\partial}$. The latter follows from $d^2 = (\partial + \overline{\partial})^2 = 0$.

(2) See that $\partial\partial\overline{\partial} = 0$, and $\overline{\partial}\partial\overline{\partial} = -\partial\overline{\partial}\overline{\partial} = 0$.

(3) It suffices to work locally. Restrict to U_i with coordinates in $\mathbb{C}^n$ given by $(w_1, \cdots, w_n)$. So changing coordinates, our form is

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\log(\sum_{k=1}^n |w_k|^2 + 1).$$

This is equal to

$$\frac{\sum_{i,j=1}^n h_{ij} dw_i \wedge d\overline{w_j}}{(\sum_{k=1}^n |w_k|^2 + 1)^2}$$

where

$$h_{ij} = (\sum_{k=1}^n |w_k|^2 + 1)\delta_{ij} - \overline{w_i}w_j.$$

One needs to check that $H = (h_{ij})$ is positive definite. Let $\langle , \rangle$ denote the ordinary Euclidean inner product on $\mathbb{C}^n$. Then

$$v^T H\overline{v} = \langle v, v\rangle + \langle w, w\rangle\langle v, v\rangle - v^T\overline{w}w^T\overline{v}$$

where $v^T\overline{w}w^T\overline{v} = \langle w, v\rangle\langle v, w\rangle = \langle w, v\rangle\overline{\langle w, v\rangle} = |\langle w, v\rangle|^2$. So Cauchy Schwarz inequality tells you that $\langle w, w\rangle\langle v, v\rangle \geq |\langle w, v\rangle|^2$. So $v^T H\overline{v} \geq \langle v, v\rangle > 0$ for $v \neq 0$. This $H = (h_{ij})$ is a part of the theory of plurisubharmonic forms. Analysts would immediately recognize that this is nice and is positive semi-definite. Another reason to study more analysis.

□

Every submanifold of $\mathbb{P}^n$ is Kahler. We'll later show that compact complex submanifolds $\iff$ smooth projecive varieties. This will show that all smooth projective varieties are Kahler, and this will reveal information that is hard to deduce purely through the algebraic study of smooth projective varieties.

Note that how we constructed this $(1, 1)$-form associated to a Kahler metric was by defining it locally first and showing compatability. One might wonder whether there is a global definition for $\mathbb{P}^n$. Indeed, one can look at the projection

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n,$$

and look at the metric on $\mathbb{C}^{n+1} \setminus \{0\}$ via the form

$$\frac{i}{2\pi}\partial\overline{\partial}\log(|z_0|^2 + \cdots + |z_n|^2).$$

This global form on $\mathbb{C}^{n+1} \setminus \{0\}$ descends to $\mathbb{P}^n$. Thus, ω_{FS} on $\mathbb{P}^n$ is the unique $(1, 1)$-form such that $\pi^*\omega_{FS} = \frac{i}{2\pi}\partial\overline{\partial}\log(|z_0|^2 + \cdots + |z_n|^2)$. Why is it clear that this form is invariant under projection to lines? It's because

$$\log(|\lambda z_0|^2 + \cdots + |\lambda z_n|^2) = \log(|\lambda|^4) + \log(\sum |z_i|^2),$$

and we showed before that $\partial\overline{\partial}\log(|\lambda|^4) = 0$. Remark: note that ω_{FS} is also invariant under the action of $U(n + 1)$, the unitary transformations. These are the matrices $\{A \in GL_{n+1}(\mathbb{C}) | AA^* = A^*A = I_n\}$. The action is $U_{n+1} \times \mathbb{P}^n \to \mathbb{P}^n$ where $(A, [z]) \mapsto [Az]$.

Story time about where we're headed: note that $d\omega_{FS} = \overline{\partial}\omega_{FS} = 0$. Note that

$$[\omega_{FS}] \in H^2(\mathbb{P}^n, \mathbb{R})$$

via the fact that it is a 2-form, and forgetting the $(1,1)$-structure. But also with the $(1,1)$-structure, we have

$$[\omega_{FS}] \in H^2(\mathbb{P}^n, \mathbb{R}) \cap H^{1,1}(\mathbb{P}^n) \cong \mathbb{R} \cap \mathbb{C} \subseteq H^2(\mathbb{P}^n, \mathbb{C})$$

Fact: $H^2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$. Iterate $\mathbb{P}^n = \mathbb{C} \cup \mathbb{P}^{n-1}$ to get cellular decomposition.

So this ω_{FS} is going to give us a generator of cohomology. Every kahler form on a kahler manifold is going to give a nonzero cohomology class. This $0 \neq [\omega_{FS}] \in H^2(\mathbb{P}^n, \mathbb{R})$ is a distinguished generator. One has that $\int_{\mathbb{P}^n} \omega_{FS} = 1$ (note we're abusing notation here. Really you from ω_{FS} to the Hermitian form it gives, to the real part which is a Riemannian metric, then you take the volume form w.r.t to this Riemannian metric). The volume with respect to this form is 1. When we have the language, this will be saying $c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = 1$.

To move forward, and establish the next parts of the aboves story, we need to discuss volume forms and the Wirtinger theorem. Let X be a real oriented manifold of dimension n, with a Riemannian metric g. The volume form on X is locally given by

$$vol(g)|_U = \sqrt{\det G(x)} dx_1 \wedge \cdots \wedge dx_n$$

where $G = (g_{ij})$ where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. Why does the volume form have this form?

First let's discuss some linear algebra. Suppose we have a linear map $\phi : V \to V$. What is $\det \phi$? Let $e_1, \cdots, e_n$ is a basis of V. Then it is a basis of $\bigwedge^n V$, and assume that $e_1 \wedge \cdots \wedge e_n =$ "1". Then $\phi : \bigwedge^n V \to \bigwedge^n V$ sends $1 \mapsto \det \phi$. Intuitively, the determinant measures the change in signed volume.

Now recall that $\bigwedge^k V^* \cong (\bigwedge^k V)^*$. What is this isomorphism? This map sends

$$\lambda_1 \wedge \cdots \wedge \lambda_k \mapsto \phi(\lambda_1 \wedge \cdots \wedge \lambda_k)$$

which evaluates $v_1 \wedge \cdots \wedge v_k \mapsto \det(\lambda_i(v_j))$. Where does this determinant come up? Roughly speaking, if we fix a basis $e_1, \cdots, e_n$ of V, then we have a dual basis $e_1^*, \cdots, e_n^*$. Given a vector $v \in V$, e_i^* measures the shadow of v onto e_i. Then $e_1^* \wedge \cdots \wedge e_k^*$ maps to the linear functional on $\bigwedge^k V$, where given $v_1 \wedge \cdots \wedge v_k$, consider the linear map $e_i \mapsto v_i$, and take the matrix with respect to the basis e_i. This matrix is $(e_i^*(v_j))_{1 \le i,j \le k}$. So the determinant denotes the change in signed volume when mapping $e_i \mapsto v_i$.

Now suppose we have an positive-definite symmetric bilinear form i.e an inner product $g : V \times V \to \mathbb{R}$. This induces an inner product $\langle , \rangle : \bigwedge^k V \times \bigwedge^k V \to \mathbb{R}$, such that

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(g(v_i, w_j)).$$

Where does this come from? Well note that the inner product g induces a map $V \to V^*$ where $v \mapsto g(v, -)$. This is injective since positive-definite implies nondegenerate, and thus isomorphism by same dimension. This induces an isomorphism $\bigwedge^k V \to \bigwedge^k V^*$ where $v_1 \wedge \cdots \wedge v_k \mapsto g(v_1, -) \wedge \cdots \wedge g(v_k, -)$. Then note we can identify $\bigwedge^k V^* \cong (\bigwedge^k V)^*$. Then $\bigwedge^k V \to (\bigwedge^k V)^*$ maps $v_1 \wedge \cdots \wedge v_k$ to the linear functional where

$$w_1 \wedge \cdots \wedge w_k \mapsto \det(g(v_i, w_j)).$$

We're building up to something called the fundamental element, which will be crucial to understanding the volume form. Let V a real vector space, dimension n.

Fix orientation so that $\{v_1, \cdots, v_n\}$ is a positive basis if $v_1 \wedge \cdots \wedge v_n \in \bigwedge^n V \cong \mathbb{R}$ is positive.

Given inner product g on V, we get induced inner product on $\bigwedge^n V$. Since this is a map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, there exists a positive basis $u_1, \cdots, u_n$ such that

$$\langle u_1 \wedge \cdots \wedge u_n, u_1 \wedge \cdots \wedge u_n \rangle = \det(g(u_i, u_j)) = 1.$$

But note that we can choose u_i to be an orthonormal basis w.r.t g, so that the matrix $(g(u_i, u_j)) = Id$. The element $g(u_1, -) \wedge \cdots \wedge g(u_n, -) \in \bigwedge^n V^*$ is called the **fundamental element**.

Let (X, g) be an oriented Riemannian manifold of dimesnion n. Then the volume form of (X, g) is the unique form $vol(g) \in A^n(X)$ such that in the fiber $\bigwedge^n T_x^* X$, $vol(g)$ is the fundamental element.

What does the fundamental element in $\bigwedge^n (T_x X)^*$ look like? We can write it as $\phi dx_1 \wedge \cdots \wedge dx_n$. Note that if $u_1, \cdots, u_n$ is an orthonormal basis of $T_x X$ w.r.t g, then the fundamental element in $\bigwedge^n (T_x X)^*$ looks like $g(u_1, -) \wedge \cdots \wedge g(u_n, -)$.

Let's write $u_k = \sum c_{jk} \frac{\partial}{\partial x_j}$. Then $A = (c_{jk})$ is the matrix associated to the linear map $\frac{\partial}{\partial x_j} \mapsto u_j$ with respect to the $\frac{\partial}{\partial x_i}$ as a basis. Note that the fundamental element, considered as an element of $(\bigwedge^n T_x X)^*$, evaluates $u_1 \wedge \cdots \wedge u_n$ to 1. Then note that $\phi dx_1 \wedge \cdots \wedge dx_n$, considered as an element of $(\bigwedge^n T_x X)^*$ will map

$$u_1 \wedge \cdots \wedge u_n \mapsto \phi \det(dx_j(u_k)) = \phi \det A = 1.$$

Furthermore, note that $g(u_k, u_\ell) = \delta_{kl}$. This implies that $A^T G A = I_n \implies (\det A)^2 \det G = 1$, where $G = (g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}))$. This implies that $\phi = \sqrt{\det G}$. Thus, we see that the volume form as we have defined it, is the unique form such that on the fiber, it is the fundamental element.

Definition 12.3. (X, g) riemannian manifold. Then $vol(X) = \int_X vol(g)$.

Something about the volume form is the fundamental element in each fiber. For fun: if X is the sphere of radius r in $\mathbb{R}^3$ and g is induced by the euclidean metric. Then $vol(X) = 4\pi r^2$.

13. 10/23/23: wirtingers theorem, basic obstruction to Kahler, diffgeo POV on kahler metric

Last time we talked about a volume form of a Riemannian manifold. Now we'll talk abotu volume forms when we have a complex manifold equipped with a hermitian metric. This will lead us to Wirtinger's theorem.

Let (X, h) be a complex manifold with a hermitian metric, $\dim_{\mathbb{C}} X = n$. Let g be the associated Riemannian metric. Last time we defined the volume form $vol(g)$ and saw that locally, it looked like

$$vol(g)|_U = \sqrt{\det G} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

where G is the matrix of g applied to pairs among $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$. Recall that how we obtained this was by examining fundamental elements in each fiber. The fundamental element is $g(u_1, -) \wedge \cdots \wedge g(u_{2n}, -)$ where the u_k forms an orthonormal basis in $T_{\mathbb{R},x} X$.

Thinking of another way to express both the Kahler (1,1)-form and the volume leads us to Wirtinger's theorem. Let's go back to working with orthonormal bases.

Locally around each $x \in X$, pick an orthonormal basis $\partial_1, \dots, \partial_n$ of $T'_x X$ with respect to the hermitian metric h. So $h(\partial_k, \partial_\ell) = \delta_{k\ell}$. Now let

$$s_k = \frac{1}{2}(\partial_k + \overline{\partial_k}) \in T_{\mathbb{R},x}X,$$

$$it_k = \frac{1}{2}(\partial_k - \overline{\partial_k}) \in iT_{\mathbb{R},x}X.$$

Note that $s_k = Re(\partial_k)$ and $t_k = Im(\partial_k)$. But note that $s_1, t_1, \cdots, s_n, t_n$ forms an orthonormal basis for $T_{\mathbb{R},x}X$ with respect to the Riemannian metric $g = Re(h)$, where

$$T_{\mathbb{R},x} \cong T'_X \text{ via } s_k \mapsto \partial_k, t_k \mapsto i\partial_k.$$

Note this is compatible with the identifications $\frac{\partial}{\partial x_k} \mapsto \frac{\partial}{\partial z_k}, \frac{\partial}{\partial y_k} \mapsto i\frac{\partial}{\partial z_k}$.

To see that $\{s_k, t_k\}$ is an orthonormal basis, note that

$$g(s_k, s_\ell) = Re(h(\partial_k, \partial_\ell)) = \delta_{k\ell}, g(s_k, t_\ell) = Reh(\partial_k, i\partial_\ell) = 0, g(t_k, t_\ell) = Reh(i\partial_k, i\partial_\ell) = \delta_{k\ell}.$$

Then given our Kahler manifold (X, h), recall the process in which we obtained the local expression

$$\omega = \frac{i}{2}\sum h_{k\ell} dz_k \wedge d\overline{z_\ell}.$$

With the above discussion, we find a second local expression for ω :

$$\frac{i}{2}\sum d\alpha_k \wedge d\overline{\alpha_k}.$$

Furthermore, since the s_k, t_k form an orthonormal basis note that on each fiber, the volume form looks like

$$\begin{aligned} g(s_1, -) \wedge g(t_1, -) \wedge \cdots \wedge g(s_n, -) \wedge g(t_n, -) &= ds_1 \wedge dt_1 \wedge \cdots \wedge ds_n \wedge dt_n \\ &= \frac{i^n}{2^n}\alpha_1 \wedge \overline{\alpha_1} \wedge \cdots \wedge \alpha_n \wedge \overline{\alpha_n}. \end{aligned}$$

This leads us to:

Theorem 13.1 (Wirtinger's Theorem)**.** *Let (X, h) be a Kahler manifold, let g be the Riemannian metric associated to h, and $\omega_X \in A^{1,1}(X)$ the Kahler form. Then*

$$vol(g) = \frac{\omega_X^{\wedge n}}{n!}.$$

Proof. We have locally

$$\omega_X^{\wedge n} = (\frac{i}{2}\sum \alpha_k \wedge \overline{\alpha_k})^{\wedge n} = n!\frac{i^n}{2^n}\alpha_1 \wedge \overline{\alpha_1} \wedge \cdots \wedge \alpha_n \wedge \overline{\alpha_n}.$$

Furthermore, the volume form locally is

$$= \frac{i^n}{2^n}\alpha_1 \wedge \overline{\alpha_1} \wedge \cdots \wedge \alpha_n \wedge \overline{\alpha_n}.$$

Thus,

$$vol(g) = \frac{\omega_X^{\wedge n}}{n!},$$

as desired. ∎

Proposition 13.2. *Wirtinger's implies the following corollaries:*

- *The volume of X is*

$$vol(X) = \int_X vol(g) = \frac{1}{n!}\int_X \omega^{\wedge n}.$$

- *If $Y \subseteq X$ is a submanifold of dimension m, then*
$$vol(Y) = \frac{1}{m!}\int_Y \iota^*\omega^{\wedge m}.$$

This is important to mention because this behavior is extremely special to complex manifolds, that there exists a unique globally defined form on the ambient space X, which allows you to compute the volume of every submanifold.

Example 13.3. Let $X = \mathbb{C}^n/\Lambda$ be a compact complex torus with fundamental domain D for the lattice Λ. This D gives a dense open subset of the torus, i.e $int(D) \cong int(X \setminus \partial D)$ (biholomorphism). Note complement ∂D has measure zero. Let g_X be the induced metric from $\mathbb{C}^n$ on X (descends to torus by symmetry). Then
$$vol(X) = \int_X vol(g_X) = \int_D vol(g) = vol(D) = \int_D d\mu.$$
where $d\mu$ is the usual lebesgue measure.

Let's continue discussing some properties of Kahler manifolds now that we have Wirtinger's theorem.

Recall that

Proposition 13.4. *If X is compact Kahler manifold, then all even betti numbers of X are nonzero, i.e $b_{2k}(X) \neq 0$ for $0 \leq k \leq n$. Recall that $b_{2k}(X) = \dim H^{2k}(X;\mathbb{R})$.*

Proof. Let ω be the Kahler (1,1)-form. Note $d\omega = 0 \implies d(\omega^{\wedge k}) = 0$ by the Leibniz rule. By the deRham theorem, we know that $H^{2k}(X,\mathbb{R}) \cong H^{2k}_{dR}(X)$. Note we have
$$[\omega^k] \in H^{2k}_{dR}(X).$$
It suffices to show that ω^k is not exact, $1 \leq k \leq n$ (the 0th deRham cohomology is always nonzero). Suppose it was. Then $d\eta = \omega^k$ for some η. We have
$$\omega^{\wedge n} = \omega^{\wedge(n-k)} \wedge d\eta = d(\omega^{\wedge(n-k)} \wedge \eta)$$
by Leibniz rule and $d\omega^{\wedge(n-k)} = 0$. But integrating the left hand side, for X compact, by Wirtinger
$$0 < vol(X) = \frac{1}{n!}\int_X \omega^{\wedge n} = \frac{1}{n!}\int_X d(\omega^{\wedge(n-k)} \wedge \eta) = 0$$
where the last equality is by Stokes theorem. □

Remark: the argument really gives us an injective ring homomorphism from
$$\mathbb{C}[x]/(x^{n+1}) \hookrightarrow H^*(X;\mathbb{R}).$$
Note the ring structure of $H^*(X;\mathbb{R})$ is given by cup product. One can also think of this ring as the deRham cohomology ring, i.e
$$H^k_{dR}(X) \times H^\ell_{dR}(X) \to H^{k+\ell}_{dR}(X)$$
where $[\eta_1] \times [\eta_2] \mapsto [\eta_1 \wedge \eta_2]$. The ring homomorphism sends $[X] \to [\omega]$.

Proposition 13.5. *If $Y \subseteq X$ is a compact submanifold of a Kahler manifold. Then Y is not a boundary in X.*

Proof. Hint: use Stokes' theorem. □

Even betti numbers serve as obstructions to when a complex manifold is compact kahler. Here's an example of a non-Kahler compact complex manifold.

Example 13.6. Consider the Hopf surface

$$\mathbb{Z} \times (\mathbb{C}^2 \setminus \{0\}) \to \mathbb{C}^2 \setminus \{0\}$$

where for $\lambda \in \mathbb{R}, 0 < \lambda < 1$,

$$(k, (z_1, z_2)) \mapsto (\lambda^k z_1, \lambda^k z_2).$$

Because this action is properly discontinuous and fixed point free, the quotient X of this action on $\mathbb{C}^2 \setminus \{0\}$ is a compact complex surface.

As a C^∞ manifold, X is diffeomorphic to $S^3 \times S^1$. To tell whether this is Kahler, one can first test whether it has nonzero betti numbers.

Once can uilize the Kunneth formula, so that

$$H^k(S^3 \times S^1, \mathbb{R}) \cong \bigoplus_{p+q=k} H^p(S^3, \mathbb{R}) \otimes H^q(S^1, \mathbb{R}).$$

This implies that $H^k(S^3 \times S^1; \mathbb{R}) \cong \mathbb{R}$ for $k = 0, 1, 3, 4$, and is 0 everywhere else. We see that for $b_2 = 0$. Thus, the Hopf surface is not Kahler.

This not the only way to show that the Hopf surface is not Kahler. The fact it is not Kahler is a consequence of a deeper fact, which is that if X is compact Kahler, we must have $b_i(X)$ is even for i odd. This will come from the Hodge decomposition theorem. Once we have more tools like the Hodge decomposition theorem and the Lefschetz theorems and cohomology of compact Kahler manifolds, we will be able to do more things.

To obtain these tools, we need to establish the Kahler identites. We're going to study some operators defined via the Kahler form, which act on cohomology. In order to do this, we need to understand the Kahler metric better from a differential geometric point of view.

Recall for the Euclidean metric ($h_{ij} = \delta_{ij}$) on $\mathbb{C}^n$, we have the associated (1,1)-form

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z_j}.$$

In fact, every Kahler metric looks *almost* like this. More precisely, it looks like this if you ignore the terms of degree 2 or higher in some taylor series-esque way. Let's try to make this precise.

Definition 13.7. A hermitian metric on X osculates to order k to the euclidean metric on $\mathbb{C}^n$ if $\forall x \in X$, there exists $x \in U \subseteq X$ with local coordinates $z_1, \cdots, z_n$ such that

$$h_{ij} = \delta_{ij} + \psi_{ij}$$

where $\psi_{ij} \in \mathcal{O}_X(U)$ such that all partial derivatives up to order k vanish at x.

Proposition 13.8. *(X, h) is Kahler $\iff$ h osculates to order 2 towards the euclidean metric.*

Proof. Let's take care of the reverse direction. Suppose h osculates to order 2 towards the euclidean metric. Then at each $x \in X$, we can find local coordinates so $h_{ij} = \delta_{ij} + \psi_{ij}$ where the partials of ψ_{ij} vanish up to order 2. We have the $(1,1)$-form associated to h is

$$\frac{i}{2}\sum h_{k\ell} dz_k \wedge d\overline{z_\ell}.$$

Then

$$d\omega = \frac{i}{2}\sum\sum \frac{\partial \psi_{k\ell}}{\partial z_j} dz_j dz_k \wedge d\overline{z_\ell} = 0.$$

Forward direction: fix $x \in X$, and choose local coordinates $z_1, \cdots, z_n$. By translation we can let x be the origin in local coordinates. Furthermore, since the matrix of $(h_{k\ell})$ is invertible at 0, we can perform a change of coordinates so that $h_{k\ell} = \delta_{k\ell}$. Then locally around x, we can write

$$h_{k\ell} = \delta_{k\ell} + L_{k\ell} + O(z^2)$$

where $L_{k\ell}$ is some linear combination of $z_k, \overline{z_k}$. Since $h_{k\ell} = \overline{h_{\ell k}}$, we have $L_{k\ell} = a_{k\ell} + \overline{a_\ell k}$. Then since $d\omega = 0$, we have by lemma 13.9

$$\frac{\partial h_{jk}}{\partial z_\ell} = \frac{\partial h_{\ell k}}{\partial z_j} \implies \frac{\partial a_{jk}}{\partial z_\ell} = \frac{\partial a_{\ell k}}{\partial z_j}.$$

Staring at this, this implies that for $k = 1, \cdots, n$, there exists quadratic $q_k(z)$ such that

$$a_{jk} = \frac{\partial q_k}{\partial z_j} \forall j, k$$

and $q_k(0) = 0$ for every k. We'll use these to perform the correct change of coordinates to obtain what we need to show.

Perform the coordinate change $z_k + q_k(z) \mapsto w_k$ around x. Is this a valid change of coordinates? Yes, because the Jacobian of this map is identity at 0, since q_k is quadratic, so the map is a local isomorphism around 0. Note we have

$$dw_k = dz_k + \sum \frac{\partial q_k}{\partial z_j} dz_j + \sum \frac{\partial q_k}{\partial \overline{z_j}} d\overline{z_j} = dz_k + \sum a_{jk} dz_j.$$

Then note that

$$dw_k \wedge d\overline{w_k} = (dz_k + \sum a_{jk} dz_j) \wedge (d\overline{z_k} + \sum \overline{a_{jk}} d\overline{z_j}) = dz_k \wedge d\overline{z_k} + \sum \overline{a_{jk}} dz_k \wedge d\overline{z_j} + \sum a_{jk} dz_j \wedge d\overline{z_k} + \sum a_{jk}\overline{a_{jk}} dz_j \wedge d\overline{z}$$

Then

$$\frac{i}{2}\sum dw_k \wedge d\overline{w_k} = \frac{i}{2}\sum dz_k \wedge d\overline{z_k} + \sum_{j,k} a_{jk} dz_j \wedge d\overline{z_k} + \overline{a_{jk}} dz_k \wedge d\overline{z_j} + \sum a_{jk}\overline{a_{jk}} dz_j \wedge d\overline{z_j}.$$

Now recall that the hermitian (1,1)-form was

$$\frac{i}{2}\sum dz_k \wedge d\overline{z_k} + \sum_{j,k} a_{jk} dz_j \wedge d\overline{z_k} + \sum_{j,k} \overline{a_{kj}} dz_j \wedge d\overline{z_k} + \sum O(z^2) dz_j \wedge d\overline{z_k}.$$

So up to terms that don't vanish when you take second order partials, these are the same. $\square$

Lemma 13.9. *(X, h) is Kahler $\iff$ $\frac{\partial h_{jk}}{\partial z_\ell} = \frac{\partial h_{\ell k}}{\partial z_j}$ for every $j, k, \ell \in \{1, \cdots, n\}$.*

Proof. If $\omega = \frac{i}{2}\sum h_{jk} dz_j \wedge d\overline{z_k}$, then

$$d\omega = \frac{i}{2}\sum_{j,k,\ell} \frac{\partial h_{jk}}{\partial z_\ell} dz_\ell \wedge dz_j \wedge d\overline{z_k} + \frac{i}{2}\sum_{j,k,\ell} \frac{\partial h_{jk}}{\partial \overline{z_\ell}} dz_j \wedge d\overline{z_k} \wedge d\overline{z_\ell}.$$

So $d\omega = 0 \iff \frac{\partial h_{jk}}{\partial z_\ell} = \frac{\partial h_{\ell k}}{\partial z_j}$ from the first type of terms. And from the second type of terms, $\frac{\partial h_{jk}}{\partial \overline{z_\ell}} = \frac{\partial h_{\ell k}}{\partial \overline{z_j}}$. But note we just need the first, since the second is true by conjugation, since $h_{kj} = \overline{h_{jk}}$. □

The condition 13.8 will be extremely useful when we discuss the Kahler identities. We'll mention another perspective, which is how perhaps Riemannina geometers think of the Kahler condition.

Theorem 13.10. *Let (X, h) be a hermitian complex manifold. TFAE:*

(1) h is Kahler

(2) The complex structure $J : T_{\mathbb{R}}X \to T_{\mathbb{R}}X$ is flat for the Levi-Civita connection associated to the Riemannian metric g associated to h.

Flat here means that for every $v, \eta \in T_{\mathbb{R}}X$, have the complex structure commutes with the covariant derivative. We'll return to these ideas later. Next time, we'll look at sheaf cohomology and relate it to Dolbeaut cohomology. After that, we'll have all the background necessary to attack the Hodge decomposition theorem. Combining the Hodge decomposition theorem and the Dolbeaut theorem, we will attain one of the most powerful tools in the study of projective manifolds.

14. 10/25/23: MITTAG-LEFFLER PROBLEM, EXPONENTIAL EXACT SEQUENCE

Today we'll talk about sheaves and cohomology, motivated by the Mittag-Leffler problem followed Griffiths-Harris.

The Mittag-Leffler problem is a problem in one-dimensional complex analsyis. Let X be a Riemann surface, e.g. $\mathbb{C}$. Let $f : X \to \mathbb{C} \cup \{\infty\}$ be meromorphic. Thus, locally $f = \frac{g}{h}$ where g, h are holomorphic and h is not identically zero.

For every $x \in X$, fix local coordinates z around x. Then locally f has a laurent series expansion

$$f(z) = \sum_{k \geq -m} a_k z^k.$$

Let $P_x(f) := \sum_{-m \leq k \leq 0} a_k z^k$ is called the polar part of f at x. The Mittag-Leffler problem is the following: fix a discrete set of points $x_1, x_2, x_3, \cdots$ on X, and polar parts $P_1, P_2, P_3, \cdots$. Can one find a meromorphic function f on X with specified polar parts

$$P_{x_i}(f) = P_i, \forall i$$

and holomorphic everywhere else outside of the x_i? There are two ways to approach this problem, both cohomological.

First approach. For each x_i, let U_i denote a neighborhood that doesn't contain any other x_j. Let $U_0 = X \setminus \{x_1, x_2, \cdots\}$ and $P_0 = 0$. Note that $g_{ij} := P_i - P_j \in \mathcal{O}_X(U_i \cap U_j)$, and on $U_i \cap U_j \cap U_k$, we have $g_{ij} + g_{jk} + g_{ki} = 0$.

If the Mittag-Leffler problem is solved by some meromorphic f, note that $f - P_i \in \mathcal{O}_X(U_i)$, and $(f - P_j) - (f - P_i) = g_{ij} \in \mathcal{O}_X(U_i \cap U_j)$.

On the other hand, if we have $f_i \in \mathcal{O}_X(U_i)$ such that $f_j - f_i = g_{ij} \in \mathcal{O}_X(U_i \cap U_j)$, then $\{f_i + P_i, U_i\}$ glue together to a meromorphic function, which the specified principal parts and holomorphic everywhere else.

Thus, consider $g = \prod g_{ij} \in \prod \mathcal{O}_X(U_i \cap U_j)$. Since $g_{ij} + g_{jk} + g_{ki} = g_{jk} - g_{ik} + g_{jk} = 0$, we have $[g] \in \check{H}^2(\underline{U}, \mathcal{O}_X)$. Then the Mittag-Leffler problem is solved $\iff$ $[g] = 0$.

Second approach. This approach is a bit more DG. Use the same notation as above. Let ρ_i be a bump function on U_i, i.e has compact support on U_i and ρ_i is identically 1 on a neighborhood of x_i. We can consider the following form:

$$\omega := \sum_{i=0}^{\infty} \overline{\partial}(\rho_i P_i) = \sum_{i=0}^{\infty} \overline{\partial}(p_i) P_i$$

where the last equality is by the Leibniz rule. Note although this looks like an infinite sum, in a neighborhood of each point, this is actually finite. Note because ρ_i is constant in a neighborhood of x_i, then this form is 0 on a neighborhood of each x_i. This implies that

$$\omega \in A^{(0,1)}(X)$$

and is identically zero on each neighborhood of x_i. Note $\overline{\partial}\omega = 0$, so ω gives a class in Dolbeaut cohomology. We wonder whether this class is zero, i.e $\omega = \overline{\partial}\phi, \phi \in C^\infty(X)$. But this happens $\iff f = \sum_{i=0}^{\infty} \rho_i P_i - \phi$ is holomorphic, so ϕ holomorphic in a neighborhood of each x_i. So ϕ holomorphic in a neighborhood of each x, with polar part P_i at x_i. Conclusion: Mittag Leffler problem has a solution $\iff [\omega] = 0$ in the Dolbeaut cohomology group $H^{0,1}(X)$. SEEMS LIKE SOMETHING IS WRONG HERE; How can ϕ be $C^\infty(X)$ and also have polar parts?

Remarks:

- we've seen from the $\overline{\partial}$-Poincare lemma, $H^{0,1}(\mathbb{C}) = 0$. So the Mittag-Leffler problem can be solved on $\mathbb{C}$. Often, people try to prove vanishing theorems to solve concrete problems like Mittag-Leffler problems.
- The problem can always be solvely locally. The obstruction is gluing, and lies in an H^1-cohomology group. We'll see later that

  $$H^1(\underline{U}, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X) \cong H^{0,1}(X).$$

  The first isomorphism is sometimes called the Leray theorem, the second isomorphism is a special case of the Dolbeaut theorem.

Thus, classical problems such as the Mittag-Leffler problem motivate sheaf cohomology. Now we'll dive into some abstract nonsense to build the machinery of sheaf cohomology.

Definition 14.1. Let X be a topological space. A sheaf of abelian groups $\mathcal{F}$ on X is a contravariant functor from the category of open sets of X to the category of abelian groups, satisfying the following conditions:

- If $s_i \in \mathcal{F}(U_i)$ satisfying

  $$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

  then there $\exists! s \in \mathcal{F}(U)$ where $U = \bigcup U_i$ such that $s|_{U_i} = s_i$ for every i.

Example 14.2. In a geometric structure $(X, \mathcal{O}_X)$, $\mathcal{O}_X$ is a sheaf of rings, where $\mathcal{O}_X(U)$ is the ring of holomorphic functions on U.

Example 14.3. Let X be a topological space. A locally constant sheaf $\mathcal{F}$ is defined to be

$$\mathcal{F}(U) = \{f : U \to G\}$$

that are locally constant (so constant in each connected component), where G may be taken to be $\mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$

For example, we have locally constant sheaves

$$\underline{\mathbb{Z}}_X, \underline{\mathbb{Q}}_X, \underline{\mathbb{R}}_X, \underline{\mathbb{C}}_X.$$

For example, $\Gamma(X, \underline{\mathbb{C}}_X)$ is a $\mathbb{C}$-vector space with dimension equal to the number of connected components of X.

Locally constant sheaves are very useful and important in the analytic topology. They are not so useful in the Zariski topology, where the open sets are extremely large (they're all dense).

Example 14.4. Let $\pi : V \to X$ be a (continuous, C^∞, holomorphic) vector bundle on X. Define the sheaf of sections $\mathcal{V}$ on X to be:

$$\mathcal{V}(U) := \{s : U \to V | s \text{ (continuous, smooth, holomorphi) section over } U\}.$$

Note $\mathcal{V}(U)$ is a module over the functions over U. We have that $\mathcal{V}$ is an $\mathcal{O}_X$-module.

For example, if $V = T_{\mathbb{R}}(X)$, then the sheaf of sections $\mathcal{V}$ is the sheaf of C^∞-vector fields.

If $V = \bigwedge^p T^*_{\mathbb{R}}(X)$, then the sheaf of sections $\mathcal{V}$ is the sheaf of smooth p-forms, i.e $\mathcal{V}(U) = A^p(U)$.

If $V = T^{p,q}X$, then the sheaf of sections $\mathcal{V}$ is the sheaf of smooth (p, q)-forms on X. So $\mathcal{V}(U) = A^{p,q}(U)$.

Example 14.5. Sheaves coming from holomorphic structure:

(1) $\mathcal{O}_X$ is the sheaf of holomorphic functions
(2) $\mathcal{O}^*_X$ is the sheaf of nowhere vanishing holomorphic functions, so $\mathcal{O}^*_X(U)$ is the collection of nowhere vanishing hol. functions on U
(3) Ω^p_X is the sheaf of holomorphic p-forms, $U \mapsto \Omega^p_X(U)$, forms of type $(p, 0)$ over U. When $\dim X = n$, we'll see that $\Omega^n_X = \omega_X$ is called the canonical line bundle. This will be very important later on.

Definition 14.6. Let $\mathcal{F}$ be a sheaf on X, $x \in X$. The stalk of $\mathcal{F}$ at x is:

$$\mathcal{F}_x := \varinjlim F(U) = \text{ germs of sections at x}$$

Example 14.7. Let X be a complex manifold. Then $\mathcal{O}_{X,x}$ is the ring of germs of functions at x, which we saw was isomorphic to the power series $\mathbb{C}(z_1, \cdots, z_n)$.

Definition 14.8. A morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is a natural transformation between $\mathcal{F}, \mathcal{G}$ as contravariant functors.

If each map $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective, then we say $\mathcal{F}$ is a subsheaf of $\mathcal{G}$.

The kernel of a morphism of sheaves $\ker f$ is always a sheaf. The image is more complicated. One must take the sheafification of the presheaf $U \to f_U(U)$ (see Vakil). Concretely, $\Gamma(U, Im(f)) = \{s \in \mathcal{G}(U) | s|_{U_i} \in Im f_{U_i}$ for some open cover$U = \bigcup U_i\}$.

Definition 14.9. We say that $f : \mathcal{F} \to \mathcal{J}$ is injective if $\ker f = 0$, and surjective if $Im(f) = \mathcal{G}$.

Definition 14.10. A sequence of morphisms of sheaves

$$\mathcal{F}^0 \to^{d^0} \mathcal{F}^1 \to^{d^1} \mathcal{F}^2 \to \cdots \to^{d^{k-1}} \mathcal{F}^k$$

is a complex if $d^i \circ d^{i-1} = 0$ and is exact, if

$$\ker d^i = Im(d^{i-1}).$$

Note that it is not the case that exactness on opens immediately implies exactness as a complex (of course, on quasicoherent sheaves this is true). We'll see a counterexample soon. It does suffice to check on stalks though. In other words, the above sequence is exact $\iff \mathcal{F}_x^0 \to \cdots \to F_x^k$ is exact $\forall x \in X$.

Here's one of our main examples of a complex of sheaves. This is called the exponential sequence. It is very analytic in nature. It works well in the Euclidean topology. Very analytic tool. Later, we'll understand how beneficial it is to algebraic geometry. This is yet another example of analytic ideas that lead to important consequences for algebraic geometric objects, which we cannot do in the Zariski topology.

Example 14.11 (Exponential sequence)**.** Let X be a complex manifold; we have the SES:

$$0 \to \underline{Z}_X \to \mathcal{O}_X \to^{exp} \mathcal{O}_X^* \to 0.$$

Note the 0 on the right hand side is really a "1", because the unit in $\mathcal{O}_X^*$ is 1, while the unit in $\mathcal{O}_X$ is 0. The exponential map

$$exp_U : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U), f \mapsto e^{2\pi i f}.$$

Now it is clearly a complex up to $\mathcal{O}_X$. Now we check exactness at $\mathcal{O}_X$. So we want to show that $\ker(\exp) = Im(\underline{Z}_X)$. If $e^{2\pi i f} = 1$, then f is integer valued, but it is holomorphic and hence continuous, thus it must be constant on connected components.

Check exactness at $\mathcal{O}_X^*$, i.e exp is surjective. It suffices to check exactness on the level of stalks. Suppose we have $[g] \in \mathcal{O}_{Xp}^*$. Then we can consider a representative $g \in \mathcal{O}_X^*(U)$, so that $g : U \to \mathbb{C}^\times$. Now we want to take the logarithm, so $\log g(z) = \log |g(z)| + i \arg g(z)$. Since argument is multi-valued, we can pick some branch. Then we also want to ensure this argument is holomorphic (and at least continuous), so we can pick a sufficiently small neighborhood of p, so that the arguments of the outputs of $g(z)$ remain in some sufficiently small neighborhood of the branch. Then we set $2\pi i f = \log g(z)$. Then we have that there is indeed some element in $[f] \in O_{X,p}$ which maps to $[g]$.

Remark: if $X = \mathbb{C}$, and $U = \mathbb{C} \setminus 0$, then we could consider $g(z) = z \in \mathcal{O}_{\mathbb{C}}^*$. But note that there is no element $f \in \mathcal{O}_X(\mathbb{C} \setminus 0)$ such that $e^{2\pi i f} = z$. Thus, the exponential sequence is an example of a SES of sheaves where we have exactness, but not exactness on every open.

This exponential sequence is very useful. It will help us in establishing Chow's theorem, which says that every compact complex submanifold of $\mathbb{P}^n$ is a projective variety. This sequence will help us to deduce something about cohomology groups, picard groups, and so on. We'll work a lot with cohomology of analytic sheaves. A priori, it seems like a lot of what we're doing only applies to complex geometry; after all, this exponential sequence is heavily referencing the analytic topology. But there is Serre's amazing GAGA theorem, which says that much of this translates to algebraic sheaf cohomology. This will let us exploit the exponential sequence in the algebraic setting.

15. 10/30/23: basic sheaf cohomology, exponential exact sequence

We continue our discussion of sheaf cohomology. (Note: Appendix B in Huybrechts has some nice references to very classical references on sheaf cohomology, done in the very general setting.)

Lemma 15.1. *If $0 \to F_1 \to F_2 \to F_3 \to 0$ is a SES of sheaves, then $\forall U \subseteq X$ open, we have an exact sequence*

$$0 \to F_1(U) \to F_2(U) \to F_3(U).$$

In other words, the global sections functor is left-exact. In particular, we'll repair its failure to be right exact using sheaf cohomology. The sheaf cohomology groups are the right derived functors with respect to $\Gamma(X, -)$.

Definition 15.2. A sheaf $\mathcal{F}$ is flabby/flasque if $\forall U \subseteq X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. Note this implies every restriction map is surjective.

Lemma 15.3. *Suppose*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a SES of sheaves, such that $\mathcal{F}_1$ is flasque. Then we have a SES

$$0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3) \to 0.$$

Proof. Such a primitive consequence, such as in the beginning of commutative algebra, will require Zorn's lemma. Since the global sections functor is left-exact, it suffices to show surjectivity of

$$\Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_)3 \to 0.$$

Let $t \in \Gamma(X, \mathcal{F}_3)$. Let

$$A = \{(U, s) | s \in \mathcal{F}_2(U) \mapsto t|_U\}.$$

Exactness on stalks immediately implies that A is nonempty. Endow A with a partial order, where $(U_1, s_1) \leq (U_2, s_2) \iff U_1 \subseteq U_2$ and $s_2|_{U_1} = s_1$. Now suppose we have some chain

$$(U_1, s_1) \leq (U_2, s_2) \leq \cdots.$$

Is there a maximal element of this chain? Yes. Take $U = \bigcup U_i$. The s_i are compatible on overlaps, thus by sheaf axioms they glue to $s \in \mathcal{F}_2(U)$ such that (U, s) is maximal with respect to this chain. With the hypotheses satisfied, by Zorn's lemma we have that there exist some maximal element (U', s'). We claim that $U' = X$. To do this, let $p \in X$. We show that $p \in U'$ as well.

Since we have surjectivity on stalks (in particular at p), there exist some $(U, s) \in A$ such that $p \in U$. Consider $U' \cap U$. If $U' \cap U$ is empty, then we can glue (U', s') and (U, s) to $(U' \sqcup U, s")$ which will map to $t|_{U' \sqcup U}$. Thus, $(U' \sqcup U, s") \in A$. But $(U', s') \leq (U' \sqcup U, s")$, contradicting the maximality of (U', s'). Thus, we cannot have $U' \cap U = \emptyset$.

Then let $V = U' \cap U$. Note that $s'|_V - s|_V$ maps to $0 \in \mathcal{F}_3(V)$. Then there exist $w \in \mathcal{F}_1(V)$ such that $w \mapsto s'|_V - s|_V$. Since $\mathcal{F}_1$ is flabby/flasque, there exist $w' \in \mathcal{F}_1(U)$ such that $w'|_V = w$. Let $\phi(w') \in \mathcal{F}_2(U)$ denote the image of w'. Then note that $\phi(w') + s \in \mathcal{F}_2(U)$ and $s' \in \mathcal{F}_3(U')$ agree on $U \cap U' = V$. Then they glue to some element $s"$ in $U' \cup U$. But $s"$ mapsto $t|_{U \cup U'}$. But then we have $(U', s') \leq (U' \cup U, s")$. Then by maximality, we must have that $U \subset U' \implies p \in U'$. Thus, $U' = X$. Since $(X, s') \in A$, this implies that $s \mapsto t \in \Gamma(X, \mathcal{F}_3)$. □

In particular, if $\mathcal{F}_1$ is flasque, then we have $\mathcal{F}_1|_U$ is flasque for every U. This implies that we have exactness over every open.

Definition 15.4. A (cohomologically indexed) resolution of a sheaf $\mathcal{F}$ is a complex

$$0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$$

which is exact except at $\mathcal{F}^0$ where the cohomology is $\mathcal{F}$, i.e

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$$

is exact.

Proposition 15.5 (Godement resolution)**.** *Every sheaf has a resolution with flasque sheaves.*

Proof. It is enough to show that every sheaf embeds into a flasque sheaf. Knowing this we can construct a resolution as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}^0 \arrow[rr] \arrow[dr] & & \mathcal{F}^1 \arrow[rr] \arrow[dr] & & \cdots \\
& & & \mathcal{F}^0/\mathcal{F} \arrow[ur] \arrow[dr] & & \mathcal{F}^1/(\mathcal{F}^0/\mathcal{F}) & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & &
\end{tikzcd}$$

Let $ds(\mathcal{F})$ denote the sheaf of discontinuous functions. This is the sheaf where

$$ds(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x.$$

Clearly this is a flasque sheaf since the restriction maps are just projections. Indeed, we have that $\mathcal{F} \hookrightarrow ds(\mathcal{F})$, where

$$s \in \mathcal{F}(U) \mapsto \prod_{x \in U} s_x \in \prod_{x \in U} \mathcal{F}_x.$$

This is injective because if $s_1, s_2 \in \mathcal{F}(U)$ map to the same element in $ds(\mathcal{F})(U)$, this implies that s_1, s_2 are the same element in the stalk at every point of U. But this implies that for each point of U, there is an open neighborhood where s_1, s_2 agree. Covering U by these open neighborhoods, the sheaf axioms imply $s_1 = s_2$.

Remark: $ds(F)$ is called the sheaf of discontinuous sections of $\mathcal{F}$, because elements of $\mathcal{F}(U)$ are exactly those tuples of elements in $\prod_{x \in U} \mathcal{F}(U)$ which are compatible. Allowing for arbitrary tuples allow for tuples of germs which are not compatible with one another, i.e allowing for "discontinuities." ☐

Definition 15.6. Let $\mathcal{F}$ be a sheaf on X. Take a Godement resolution by flasque sheaves

$$\mathcal{F} \to \mathcal{F}^*$$

We obtain a complex

$$0 \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \Gamma(X, \mathcal{F}^2) \to \cdots$$

because $\Gamma(X, -)$ is left-exact. The n-th sheaf cohomology of $\mathcal{F}$ is

$$H^n(X, \mathcal{F}) := H^i \Gamma(X, \mathcal{F}^*).$$

Note $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. If $\mathcal{F}$ is a sheaf of k-vector spaces, then $H^i(X, \mathcal{F})$ is a k-vector space.

Note that flasque resolutions are also acyclic resolutions. So by the good ol' theory of acylic resolutions, which is a more general way of defining sheaf cohomology, this is well-defined.

Proposition 15.7. *If $\mathcal{F}$ is flasque, then $H^i(X, \mathcal{F}) = 0$ for every $i > 0$, i.e $\mathcal{F}$ is acyclic w.r.t to $\Gamma(X, -)$ (for every Godement resolution of $\mathcal{F}$).*

Proof.

$$
\begin{tikzcd}
 & & & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & & & g^1 \arrow[ur] \arrow[dr] & & & & g^4 \arrow[ur] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}^0 \arrow[rr] \arrow[dr] & & \mathcal{F}^1 \arrow[rr] \arrow[ur] & & \mathcal{F}^2 \arrow[rr] \arrow[dr] & & F^3 \arrow[r] \arrow[ur] & \cdots & \\
 & & & g^0 \arrow[ur] \arrow[dr] & & & & g^3 \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

In a SES, if the first two sheaves are flasque, then the third sheaf is flasque. This follows from the fact that the fist sheaf being flasque implies exactness on sections, combined with the second sheaf being flasque.

We have $\mathcal{F}$ is flasque and $\mathcal{F}^0$ is flasque. Then g^0 is flasque. Then g^0 and $\mathcal{F}^1$ is flasque, so g^1 is flasque. Iterating, we see that all sheaves the appear in the diagram are flasque. This implies that taking the global sections functor is exact, i.e

$$0 \to \Gamma(X, g^i) \to \Gamma(X, \mathcal{F}^i) \to \Gamma(X, g^{i+1}) \to 0$$

is exact by the earlier lemma. This implies

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \cdots$$

is exact. $\square$

Proposition 15.8. *A SES of sheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ gives rise to a long exact sequence on cohomology*

$$
\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathcal{F}_1) \arrow[r] & H^0(X, \mathcal{F}_2) \arrow[r] & H^0(X, \mathcal{F}_3) \arrow[dll] \\
 & H^1(X, \mathcal{F}_1) \arrow[r] & H^1(X, \mathcal{F}_2) \arrow[r] & H^1(X, \mathcal{F}_3) \arrow[dll] \\
 & \ldots & &
\end{tikzcd}
$$

In practice, sheaf cohomology is useless for actually computing something, and one actually computes something using Čech cohomology, which we'll talk about next time.

To wrap up this section, let's apply sheaf cohomology to the exponential exact sequence. This will preview some important notions that we'll discuss later on.

Example 15.9. Let X be a complex manifold, and consider the exponential exact sequence

$$0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \to^{\exp(2\pi i)} \mathcal{O}_X^* \to 0.$$

Taking the LES in sheaf cohomology, one obtains

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \underline{\mathbb{Z}}_X) \arrow[r] & H^0(X, \mathcal{O}_X) \arrow[r] & H^0(X, \mathcal{O}_X^*) \arrow[dll] \\
& H^1(X, \underline{\mathbb{Z}}_X) \arrow[r] & H^1(X, \mathcal{O}_X) \arrow[r] & H^1(X, \mathcal{O}_X^*) \arrow[dll] \\
& H^2(X, \underline{\mathbb{Z}}_X) \arrow[r] & \cdots &
\end{tikzcd}$$

If X is compact and connected, then the first three terms fit into the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to^{\exp(2\pi i)} \mathbb{C}^* \to 0.$$

Let's look at the remaining terms

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, \underline{\mathbb{Z}}_X) \arrow[r] & H^1(X, \mathcal{O}_X) \arrow[r] & H^1(X, \mathcal{O}_X^*) \arrow[dll, "c_1"] \\
& H^2(X, \underline{\mathbb{Z}}_X) \arrow[r] & \cdots &
\end{tikzcd}$$

It turns out that $H^1(X, \mathcal{O}_X^*)$ is isomorphic to the **Picard group** $Pic(X)$ of X, the group of line bundles, with group operation being tensor products. The map

$$c_1 : H^1(X, \mathcal{O}_X^*) \cong Pic(X) \to H^2(X, \underline{\mathbb{Z}}_X)$$

is called the first Chern class map, and it sends an isomorphism class of line bundles $[\mathcal{L}]$ to its **first Chern class** $c_1(\mathcal{L})$.

The kernel of the first Chern class map gives $Pic^0(X)$, which is a subgroup of $Pic(X)$. By exactness, we see that

$$Pic^0(X) \cong \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \underline{\mathbb{Z}}_X)},$$

and this turns out to always be a lattice! This extra structure on line bundles leads to something called the **Picard variety**.

An amusing aside: if X is simply connected then $H^1(X, \mathbb{Z}) = 0$, so you get a short exact sequence in global sections. This means you don't need to take a branch cut to define logarithm on simply connected space. A fun reminder that there is truly analytic data going on here!

A sidetangent fact: if X is a topological space, locally contractible, maybe paracompact (?), then $H^i(X, \underline{Z}_X) \cong H^i(X, \mathbb{Z})$. (? See Artin comparison theorem.)

16. 11/1/23: Cech cohomology, Dolbeaut's theorem

How to actually compute sheaf cohomology? Answer: Čech cohomology. Let X be a topological space, $\mathcal{F}$ sheaf of abelian groups, and $\underline{U} = \{U_i\}_{i \in I}$ an open cover of X. First, we define the Čech complex.

Definition 16.1. The p-th degree of the Čech complex is

$$\check{C}^p(\underline{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}).$$

Define the Čech diferential to be

$$d^p : \check{C}^p \to \check{C}^{p+1}$$

where

$$g \mapsto h = (h_{i_0 \cdots i_{p+1}})$$

where

$$h_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k g_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}}|_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}}.$$

Definition 16.2. The p-th Čech cohomology of $\mathcal{F}$ with respect to $\underline{U}$ is

$$\check{H}^p(\underline{U}, \mathcal{F}) = H^p(\check{C}^*(\underline{U}, \mathcal{F})).$$

Note that $\check{H}^0(\underline{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$. The kernel of d^0 is exactly those tuples of sections over each U_i which are compatible on overlaps.

Proposition 16.3 (Picard group)**.** *We have*

$$\check{H}^1(\underline{U}, \mathcal{O}_X^*) \cong Pic(X)$$

where the Picard group $Pic(X)$ of X is the group of isomorphism classes of line bundles.

Proof. Note that a line bundle is equivalent to the data of its transition functions. More precisely, a line bundle $\mathcal{L}$ is equivalent to specifying $\underline{U} = \{U_i\}$ and $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*$. Note that the transition functions over $U_i \cap U_j \cap U_k$ satisfy the cocycle condition: $g_{ij} g_{jk} g_{ki} = 1$. Thus, we see that $\{U_i \cap U_j, g_{ij}\} \in \check{C}^2(\underline{U}, \mathcal{O}_X^*)$ is in the kernel of the d^2 differential. So we obtain a class

$$[g] \in \check{H}^2(\underline{U}, \mathcal{O}_X^*).$$

Finally, we check that $[g] = 0 \iff$ the line bundle is trivial. If $[g] = 0$, this implies that there exist (U_i, s_i) such that over $U_i \cap U_j$, we have $g_{ij} = \frac{s_j}{s_i}$, inverting, we have $g_{ij} t_j = t_i$ where $t_j = s_j^{-1} \in \mathcal{O}_X(U_i \cap U_j)^*$. These give nowhere vanishing sections which glue together to a nowhere vanishing global section. Hence, the line bundle is trivial! Similarly, if the line bundle is trivial, we can pick a nowhere vanishing global section, restrict it to the opens U_i, pass along local trivialization to obtain t_i, invert to get s_i, s_j, and we find that g is the image of these (U_i, s_i).

Altogether, this implies the isomorphism. $\square$

This is slightly unsatisfactory; we are relating the Picard group to something that a priori depends on the choice of cover. We'll see that this choice of cover won't matter. The following was discovered by Leray in prison (and maybe independently discovered by Cartan?)

Theorem 16.4 (Leray's theorem / Cartan's lemma)**.** *Let $\underline{U} = \{U_i\}_{i \in I}$ be an open cover of X which is acylic for a sheaf $\mathcal{F}$, i.e*

$$H^i(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{F}|_{U_{i_0} \cap \cdots \cap U_{i_p}}) = 0, \forall i > 0, \forall p \geq 0.$$

Then there exist natural isomorphism

$$\check{H}^i(\underline{U}, \mathcal{F}) \cong H^i(X, \mathcal{F}), \forall i \geq 0.$$

Example 16.5. Let $U = (U_0, U_1)$ be the standard open cover of $\mathbb{P}^1$. Then

$$\check{H}^0(U, \mathbb{P}^1) \cong \mathbb{C}, \check{H}^i(U, \mathbb{P}^1) = 0 \forall i > 0.$$

Furthermore, in a moment we'll show via Dolbeaut's theorem that

$$H^i(\mathbb{C}, \mathcal{O}_{\mathbb{C}}) = 0 \text{ and } H^i(\mathbb{C}^*, \mathcal{O}_{\mathbb{C}^*}) = 0, \forall i \geq 1$$

which implies that the cover is acylic, which implies that

$$\check{H}^i(\underline{U}, \mathcal{O}_{\mathbb{P}^1}) \cong H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$$

so

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}, H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0, \forall i > 0.$$

In general, $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$ for all $i > 0$ and for every n. Serre wrote FAC where he computed sheaf cohomology for $\mathbb{P}^n$, where he showed that it amounts to proving something about Koszul complexes. There are many ways to prove these facts, though. We'll see that this vanishing is a special case of Kodaira vanishing.

One can also calculate $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) = 0, H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \cong \mathbb{C}$.

So to make these calculations more tractable and relate Cech cohomology to sheaf cohomology, let's talk about Dolbeaut's theorem. Recall that the (p, q)-Dolbeaut cohomology was

$$H^{p,q}(X) := H^q(A^{p,*}(X)).$$

Theorem 16.6 (Dolbeaut theorem)**.** *For every $p, q \in \mathbb{N}$, there exist natural isomorphism*

$$H^{p,q}(X) \cong H^q(X, \Omega^p_X),$$

where Ω^p_X is the sheaf of holomorphic p-forms on X.

This is the first step towards Hodge theory. Note $H^q(X, \Omega^p_X)$ is purely analytic data. And $H^{p,q}(X)$ will sit inside singular cohomology. So we'll have some connection between purely analytic data, and the topology of the object.

Example 16.7. For $i \geq 1$, $H^i(\mathbb{C}, \mathcal{O}_{\mathbb{C}}) \cong H^{0,i}(\mathbb{C}) = 0$ by the $\overline{\partial}$-Poincare lemma. Same for $\mathbb{C}^*$. Same for $\mathbb{C}^k \times (\mathbb{C}^*)^\ell$, any polydisk...

Some preparation before we prove the Dolbeaut theorem. For every $p \geq 0$, there exist a complex of sheaves

$$0 \to \Omega^p_X \to^{\overline{\partial}} A^{p,0}_X \to^{\overline{\partial}} A^{p,q}_X \to^{\overline{\partial}} A^{p,2}_X \to \cdots$$

where on each $U \subseteq X$, the map is $\overline{\partial} : A^{p,q}(U) \to A^{p,q+1}(U)$. The kernel of $A^{p,0}_X \to A^{p,1}_X$ is Ω^p_X. Furthermore, the complex is exact at $A^{p,0}$. It suffices to check this on stalks. Fix a point and neighborhood $x \in U$, and you take a form in this neighborhood $\omega \in A^{p,q}(U)$, so that $\overline{\partial}\omega = 0$. If $q = 0$, then clearly it lives in $\Omega^p_X(U)$. If $q \geq 1$, the $\overline{\partial}$-Poincare lemma tells us that there is open $x \in V \subset U$ such that $\omega|_V = \overline{\partial}\eta$. So on stalks, this is fine. This gives exactness. Thus,

$$0 \to \Omega^p_X \to^{\overline{\partial}} A^{p,0}_X \to^{\overline{\partial}} A^{p,q}_X \to^{\overline{\partial}} A^{p,2}_X \to \cdots$$

is a resolution.

Proposition 16.8. *$A^{p,q}_X$ is an acyclic sheaf for all p, q, i.e*

$$H^i(X, A^{p,q}_X) = 0, \forall i > 0.$$

To show this, it will require more work; we'll need to show that these $A_X^{p,q}$ belong to a class of sheaves called fine sheaves, and these sheaves have a partition of unity. We'll do this next time. But for now, we'll assume this and prove Dolbeaut's theorem.

Consider

$$\begin{tikzcd}
 & & & & 0 \arrow[dr] & & 0 & & \\
 & & & & & K^2 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[r] & \Omega_X^p \arrow[r] & A_X^{p,0} \arrow[rr, "\overline{\partial}"] \arrow[dr, two heads] & & A_X^{p,q} \arrow[rr, "\overline{\partial}"'] \arrow[ur] & & A_X^{p,2} \arrow[rr] \arrow[dr, two heads] & & \cdots \\
 & & & K^1 \arrow[ur] & & & & K^3 & \\
 & & 0 \arrow[ur] & & & & & &
\end{tikzcd}$$

Let $K^q = \ker(\overline{\partial} : A^{p,q} \to A^{p,q+1}) = Im(\overline{\partial} : A^{p,q-1} \to A^{p,q})$. For every $q \geq 1$, we have the SES

$$0 \to K^{q-1} \to A^{p,q-1} \to K^q \to 0.$$

Now pass to the long exact sequence in cohomology:

$$\begin{tikzcd}
 & \cdots \arrow[r] & H^0(X, A^{p,q-1}) \arrow[r] & H^0(X, K^q) \arrow[dll] \\
H^1(X, K^{q-1}) \arrow[rr] & & H^1(X, A^{p,q-1}) \arrow[r] & H^1(X, K^q) \arrow[dll] \\
 & \cdots & &
\end{tikzcd}$$

Note that $H^i(X, A_X^{p,q}) = 0$ for $i > 0$. This immediately implies that

$$H^i(X, K^{q+1}) \cong H^{i+1}(X, K^q), \forall q, \forall i \geq 1.$$

Note that since $H^1(X, A_X^{p,q}) = 0$, this implies we have the exact sequence

$$0 \to H^0(X, K^q) \to H^0(X, A_X^{p,q}) \to H^0(X, K^{q+1}) \to H^1(X, K^q) \to 0.$$

Then we have that

$$H^1(X, K^q) \cong \frac{H^0(X, K^{q+1})}{H^0(X, A_X^{p,q})} \cong H^{p,q+1}(X).$$

Then we have that

$$H^q(X, \Omega_X^p) = H^q(X, K^0) \cong H^{q-1}(X, K^1) \cong \cdots \cong H^1(X, K^{q-1}) \cong H^{p,q}(X),$$

which proves the Dolbeaut theorem. All we really used here, besides formal nonsense, was acyclicity. This established that $H^1(X, K^{q-1})$ was dolbeaut cohomology, and acyclicity let us slide this down to sheaf cohomology of Ω_X^p.

So put our result on firmer ground and discuss fine and soft sheaves. Let X be a paracompact space (this means if you have any open cover, you can refine it to a locally finite open cover).

- (Aside) Every locally compact Hausdorff space with countable basis is paracompact.
- E.g. manifolds

Definition 16.9. A sheaf $\mathcal{F}$ on paracompact X is *fine* if, for every $\underline{U} = \{U_i\}_{i\in I}$ locally finite open cover of X, there exists sheaf endomorphisms

$$\phi_i : \mathcal{F} \to \mathcal{F}$$

such that

(1) $\forall i$, $X \setminus U_i \subseteq V_i$ open such that $\phi_{i,x} : \mathcal{F}_x \to \mathcal{F}_x$ is identically zero for $x \in V_i$.
(2) as morphisms of sheaves $\sum_{i\in I} \phi_i = Id_{\mathcal{F}}$. Note this sum is finite around any point so it makes sense.

This is a partition of unity in the sheaf theory sense. Some remarks: Note condition 1 is equivalent to: $\forall s \in \mathcal{F}(U)$, $supp(\phi_i(s)) \subseteq U_i$, and condition 2 is equivalent to $\forall s, s = \sum_{i\in I} \phi_i(s)$.

Small observation: can start with $s_i \in \mathcal{F}(U_i)$, and applying $\phi_i(s_i) \in \mathcal{F}(U_i)$, but this time, it is close to zero around boundary of U_i. So glue it with zero section. So you can think of $\phi_i(s_i)$ as a global section.

Next time, we'll show that $A^{p,q}$ is fine, and we'll show that fine implies acyclic (to do this we'll need to talk about soft sheaves).

17. 11/6/23: wrapping up Dolbeault, fine soft sheaves, cousin problem

Our goal is to show that $H^i(X, A_X^{p,q}) = 0$, $\forall i > 0$.

Definition 17.1. Let X paracompact. A sheaf $\mathcal{F}$ on X is fine if for every locally finite open cover $\underline{U} = \{U_i\}$, there exists $\phi_i : \mathcal{F} \to \mathcal{F}$ such that ϕ_i is identically zero on a neighborhood $V_i \subset X \setminus U_i$ and $\sum_{i\in I} \phi_i = Id_{\mathcal{F}}$.

Remark: For $s_i \in \mathcal{F}(U_i)$, we can think of $\phi_i(s_i) \in \Gamma(X, \mathcal{F})$.

Example 17.2. If X is a complex manifold, then $A_X^{p,q}$ is fine $\forall p, q$. First, note X is a complex manifold, so its Hausdorff plus second countable. Futhermore, it being locally euclidean, it is locally compact. Thus, it is paracompact.

Fix some $\underline{U}$ locally finite cover. Since X is a complex manifold, we can find a partition of unity $\{\rho_i\}$ subordinate to this cover. Then we define $\phi_i :=$ to be multiplication by ρ_i. We see that this satisfies the desired conditions, so that $A_X^{p,q}$ is fine.

Example 17.3. For every $\mathcal{F}$ on paracompact X, the sheaf of discontinuous sections $ds(\mathcal{F})$ of $\mathcal{F}$ is fine.

Fix a locally finite cover $\underline{U}$. We can find an open cover $\{U_i'\}$ such that $U_i' \subseteq U_i$ for every i, and $\overline{U_i'} \subseteq U_i$. We want to construct endomorphism $\phi_i : ds(\mathcal{F}) \to ds(\mathcal{F})$ with respect to U_i' with the desired properties. Note that to specify ϕ_i, it's enough to specify an endomorphism for each $\mathcal{F}_x$.

For each $x \in X$, there exist finitely many U_i' covering it. Suppose x is covered by $U_{i_1}, \cdots, U_{i_n}$. Then let ϕ_{i_k} Pick one of these indices i_k so that $\phi_{i_k,x} : ds(\mathcal{F})_x \to ds(\mathcal{F})_x$ is identity, and the rest are zero. Clearly we can pick this in such a way that for any ϕ_i, we have that ϕ_i is identically zero on some open $V_i \supseteq X \setminus U_i$.

Next we need to discuss soft sheaves. Recall that when we had a complex manifold $(X, \mathcal{O}_X)$, we defined a submanifold $(Z, \mathcal{O}_X|_Z)$ in such a way that $\mathcal{O}_X|_Z$

for open $U \subset Z$ is all the functions on U that locally on X come from restrictions of local functions on X. We can formalize this in a sheaf-theoretic way.

Definition 17.4. Let $Z \subseteq X$ be closed, and $\mathcal{F}$ a sheaf on X. The $\mathcal{F}|_Z$ is the sheaf such that, for open $V \subseteq Z$, we have

$$\mathcal{F}|_Z(V) = \{s \in \prod_{p \in V} \mathcal{F}_p | \forall p \in V, \exists t \in \mathcal{F}(U) \text{ where } t|_{U \cap V} = s|_{U \cap V}\}.$$

The pushforward $\iota_* \mathcal{F}|_Z$ is a sheaf on X, whose stalk at $x \notin Z$ is 0, and at $x \in Z$ is $\mathcal{F}_x$.

When we have a notion of an ideal sheaf, this fits into a SES

$$0 \to \mathcal{F} \otimes \mathcal{I}_Z \to \mathcal{F} \to \iota_* \mathcal{F}|_Z \to 0.$$

Definition 17.5. A sheaf $\mathcal{F}$ is soft if $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|_Z)$ surjective for every Z closed.

Intuitively, this is saying that a sheaf is soft if, for every closed set, if you look at the functions on the closed set, they all come from just global functions which restrict to that closed set.

Example 17.6. $ds(F)$ is soft, $\forall \mathcal{F}$.

We're introducing soft sheaves because they behave well with respect to exact sequences. We have the following key lemma.

Lemma 17.7. *Let X be paracompact. If $\mathcal{F}$ is fine, then $\mathcal{F}$ is soft.*

Proof. Idea of the proof: we start with a function on closed Z that locally is restrictions of functions from $\mathcal{F}$. Then we'll use partition of unity to glue these to a global function.

Let $Z \subseteq X$ be a closed set. Let $t \in \Gamma(Z, \mathcal{F}|_Z)$. Then there exist (U_i, s_i) such that $Z \subseteq \bigcup U_i$ and $s_i \in \mathcal{O}_X(U_i)$ such that

$$t|_p = s_i|_p, \forall p \in Z \cap U_i.$$

Refine this $\underline{U} = \{U_i\}$ to be locally finite. Furthermore, throw $U_0 = X \setminus Z$ and let ϕ_0 be identically 0. Now since $\mathcal{F}$ is fine, there exist endomorphisms $\phi_i : \mathcal{F} \to \mathcal{F}$ with the partition of unity properties. Note that each $\phi_i s_i$ can be considered as a global section $\in \Gamma(X, \mathcal{F})$. Then consider $\sum \phi_i s_i$. Note that this sum is finite at each point, and for every $p \in Z$, we have

$$(\sum \phi_i s_i)|_p = \sum \phi_{i,p} s_i|_p = \sum \phi_{i,p} t|_p = t|_p.$$

Thus, $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|_Z)$ is surjective. □

Lemma 17.8. *If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is SES with $\mathcal{F}_1$ soft, then*

$$0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3) \to 0$$

is exact.

Proof. Since Γ is left-exact, we just need to show surjectivity on

$$\Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3).$$

Let $t \in \Gamma(X, \mathcal{F}_3)$. Since we have surjectivity on stalks, at every point in X, there exist neighborhood U_i and $s_i \in \mathcal{F}_2(U_i)$ such that $s_i \mapsto t|_{U_i}$. Let $\underline{U} = \{U_i\}$ be this open cover, and refine it so that it is locally finite (which we can do since X is paracompact). Then find open cover $\{U_i'\}$ such that $U_i' \subseteq \overline{U_i'} \subseteq U_i$ for every i.

Now for $J \subseteq I$, let $V_J = \bigcup \overline{U_i'}$. Let

$$A = \{(J,s) | s \mapsto t|_{V_J}, s \in \mathcal{F}(V_J)\}.$$

Note that $s \in \mathcal{F}(V_J)$ means that locally on V_J, s is the restriction of some section of $\mathcal{F}$. Note that A is nonempty, since $s_i|_{V_i} \mapsto t|_{V_i}$, so $(\{i\}, s_i|_{V_i}) \in A$ for every i. Now endow A with a partial order, so that

$$(J_1, s_1) \leq (J_2, s_2) \iff J_1 \subseteq J_2, s_2|_{V_{J_1}} = s_1.$$

Now given any chain $(J_1, s_1) \leq (J_2, s_2) \leq \cdots$, there exist a bound. Let $J = \bigcup J_i$, and the s_i glue to some s, where $s \mapsto t|_{V_J}$. By Zorn's lemma, there exist a maximal element (J, s). If $J = I$, then we're done, since $V_J = X$. So suppose $J \neq I$. Let $i \in I \setminus J$. We know that $(\{i\}, s_i) \in A$. Then note that

$$s|_{V_J \cap V_i} - s_i|_{V_J \cap V_i} \mapsto 0.$$

Then there exists

$$w \in \mathcal{F}_1(V_J \cap V_i) \mapsto s|_{V_J \cap V_i} - s_i|_{V_J \cap V_i} \in \mathcal{F}_2(V_J \cap V_i).$$

By softness of $\mathcal{F}_1$, we can find global section w' such that $w'|_{V_J \cap V_i} = w$. Then w' maps to $s' \in \mathcal{F}_2(X)$. Consider $s_i - s'|_{V_i}$. This agrees with $s|_{V_J \cap V_i}$ on $V_J \cap V_i$. Then they glue together to a global section which maps to t. □

Lemma 17.9. *If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ SES with $\mathcal{F}_1, \mathcal{F}_2$ soft. Then $\mathcal{F}_3$ is soft.*

Proof. Consider the diagram

$$\begin{tikzcd} \Gamma(X, \mathcal{F}_2) \arrow[r, two heads] \arrow[d, two heads] & \Gamma(X, \mathcal{F}_3) \arrow[d] \\ \Gamma(Z, \mathcal{F}_2|_Z) \arrow[r] & \Gamma(Z, \mathcal{F}_3|_Z) \end{tikzcd}$$

. If you have $W \subseteq Z$, then you have the composition

$$\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|_Z) \to \Gamma(W, \mathcal{F}|_W)$$

is surjective, so that the right hand map is surjective thus $\mathcal{F}$ soft implies $\mathcal{F}|_Z$ is soft. Then $\Gamma(Z, \mathcal{F}_2|_Z) \to \Gamma(Z, \mathcal{F}_3|_Z)$ is surjective. Altogether, this diagram implies $\Gamma(X, \mathcal{F}_3) \to \Gamma(Z, \mathcal{F}_3|_Z)$ is surjective. □

Proving this finally finishes the proof of the Dolbeaut theorem.

Proposition 17.10. *If $\mathcal{F}$ fine $\implies H^i(X, \mathcal{F}) = 0, \forall i > 0$ (i.e $\mathcal{F}$ is acyclic).*

Proof. Consider a Godement resolution

$$\begin{tikzcd}
 & & & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & & & g^1 \arrow[ur] \arrow[dr] & & & & g^4 \arrow[ur] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}^0 \arrow[rr] \arrow[dr] & & \mathcal{F}^1 \arrow[rr] \arrow[ur] & & \mathcal{F}^2 \arrow[rr] \arrow[dr] & & F^3 \arrow[r] \arrow[ur] & \cdots & \\
 & & & g^0 \arrow[ur] \arrow[dr] & & & & g^3 \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Since $\mathcal{F}$ is fine, it is soft. The $\mathcal{F}^i$ are of type $ds(-)$ which implies they are soft. Then we have by the last lemma g^i are also soft. So by applying above lemma, you have

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \cdots$$

is exact $\implies H^i(X, \mathcal{F}) = 0, \forall i > 0$. □

Thus, we know that by exploiting the analytic topology of the complex manifold, we know $A^{p,q}(X)$ is fine. By the above, this shows that $A^{p,q}(X)$ is acyclic, and this completes the proof of Dolbeaut's theorem. Note that what we've done here is very unique to complex geometry – in algebraic geometry, we wouldn't have been able to use the techniques we've done here to show that $A^{p,q}(X)$ is acyclic via it being fine i.e admitting a partition of unity.

We can use everything we've talked about all of our sheaf cohomology technology to prove a nontrivial result in complex geometry, that is usually taken as simply a definition in algebraic geometry: the Cousin problem.

Before that, we prove an important lemma that links sheaf cohomology to singular cohomology.

Lemma 17.11. *Let X be a locally contractible topological space. Then*

$$H^i(X, \underline{\mathbb{Z}}_X) \cong H^i_{sing}(X, \mathbb{Z}).$$

Proof. **Part(1): the resolution.** Let C^q denote the sheafification of the presheaf C^q_{sing}, where $C^q_{sing}(U) = C^q(U, \mathbb{Z})$. We claim that

$$0 \to \underline{\mathbb{Z}}_X \to C^0 \to C^1 \to \cdots$$

is a flasque resolution. First, let's verify exactness. Exactness in degree ¿ 0 follows from thinking about how, for contractible U, we have that

$$C^{q-1}(U, \mathbb{Z}) \to C^q(U, \mathbb{Z}) \to C^{q+1}(U, \mathbb{Z})$$

is exact. Since X is locally contractible, we can find a cover of contractible opens. Then for every $p \in X$, following a direct limit along contractible opens containing p, we have

$$C^{q-1}_p \to C^q_p \to C^{q+1}_p$$

is exact. Now we show exactness at C^0. Why is it that

$$\ker(C^0 \to C^1) = \underline{\mathbb{Z}}_X?$$

Suppose U is contractible, and $\phi \in \ker(C^0(U,\mathbb{Z}) \to C^1(U,\mathbb{Z}))$. Then for every $x, y \in U$, there exists a path $\alpha : x \to y$. Then we have that

$$0 = (\partial\phi)(\alpha) = \phi(\partial\alpha) = \phi(y) - \phi(x).$$

So ϕ must be constant on U. Then if we look at $\phi \in \ker(C^0(U) \to C^1(U))$, ϕ is a tuple of compatible germs $\prod_{p\in U} \phi_p$ of $C^0_{sing}(U)$. To be compatible means for every p a neighborhood U_p and $s_p \in C^0_{sing}(U_p)$ so that $s_p|_x = \phi_x, \forall x \in U_p$. But U_p can be taken to be contractible and sufficiently small so that s_p is constant. This gives an element of $\mathbb{Z}_X(U)$.

Finally, why is C^q flasque? We want to show for every U that

$$C^q(X) \to C^q(U)$$

is surjective. Take an element of $C^q(U)$. Again, this is a tuple of compatible germs $\prod_{p\in U} \phi_p$, and again this means we have local cochains $s_i \in C^q_{sing}(U_i)$ where U_i cover U. But note we can extend each of these cochains s_i to a cochain in $C^q_{sing}(X)$. Just let it map all other chains, whose image is not entirely contained in U_i, to 0. This information gives an element of $C^q(X)$ which restricts to $\prod_{p\in U} \phi_p \in C^q(U)$.

Part 2: Intermediate step to singular cochains. Next we show there is a natural isomorphism

$$\frac{C^q_{sing}(V)}{C^q_{sing}(V)_0} \cong C^q(V).$$

Note that $C^q_{sing}(V,\mathbb{Z})_0$ denotes singular cochains that locally vanish at every point. In other words, for every $\phi \in C^q_{sing}(V,\mathbb{Z})_0$, there exist a covering $\{V_i\}$ of V such that $\phi_{C^q_{sing}(V_i)} = 0$. So if a q-chain σ maps entirely into some V_i, then $\phi : \sigma \mapsto 0$.

First, there is a natural map

$$\phi \in C^q_{sing}(V) \to \phi^+ \in C^q(V)$$

where $\phi^+ = \prod_{p\in V} \phi_p$. Note that $\phi^+ = 0 \iff \phi|_{C^q_{sing}(V_i)} = 0$ where V_i cover V. So we see that the kernel of this map is indeed $C^q_{sing}(V)_0$.

Furthermore, this map is surjective. Take some $\beta \in C^q(V)$, and $\tilde{\beta}_i \in C^q_{sing}(V_i)$ so that $\beta|_p = \tilde{\beta}_i|_p, \forall p \in V_i$, where the V_i cover V. Then define $\tilde{\beta} \in C^q_{sing}(V)$ where for $\sigma \in C_{sing,q}(V)$, $\tilde{\beta}(\sigma) = \tilde{\beta}_i(\sigma)$ if $\sigma(\Delta^q) \subseteq V_i$, and 0 otherwise. This $\tilde{\beta}$ is well-defined since $\tilde{\beta}_i$ and $\tilde{\beta}_j$ agree on $V_i \cap V_j$. This $\tilde{\beta}$ maps to β under the map.

Part 3: Getting singular cohomology. What we have so far is that there is a flasque resolution

$$0 \to \underline{\mathbb{Z}}_X \to \frac{C^0_{sing}}{(C^0_{sing})_0} \to \frac{C^1_{sing}}{(C^1_{sing})_0} \to \cdots .$$

Taking global sections, we have that

$$H^i(X, \underline{\mathbb{Z}}_X) \cong H^i(\frac{C^*_{sing}(X)}{(C^*_{sing}(X))_0}).$$

Then we'd like to show there is a natural isomorphism between

$$H^i(\frac{C^*_{sing}(X)}{(C^*_{sing}(X))_0}) \cong H^i C^*_{sing}(X).$$

Looking at the short exact sequence

$$0 \to (C^*_{sing}(X))_0 \to C^*_{sing}(X) \to \frac{C^*_{sing}(X)}{(C^*_{sing}(X))_0},$$

this amounts to showing that the complex $(C^*_{sing}(X))_0$ is exact. Let ϕ be a locally vanishing i-cocycle, and take $\{U_i\}$ to be an open cover of contractible opens that ϕ vanishes with respect to. Let σ be a $i-1$ simplex. Using barycentric coordinates, construct i simplex c_σ, where the boundary of c_σ is σ along with a collection of $i-1$ simplices that lie entirely in some U_i. Then define an $i-1$ cochain that is generated by sending $\sigma \mapsto \phi(c_\sigma)$. Note that this map is well-defined. Because if c'_σ also was construted, then $c_\sigma - c'_\sigma$ would be homologous to sums of i-simplices entirely contained in some U_i where ϕ vanishes. So $\phi(c_\sigma - c'_\sigma) = 0$. This gives an $i-1$ cochain whose image is ϕ. $\square$

We now solve the Cousin problem. The main idea is that, by definition, a hypersurface is cut out locally by a holomorphic functions. A priori we don't know whether they glue. Using cohomological techniques, we can find local holomorphic functions with the same zero sets which do actually glue together, thus giving a global function which cuts out our hypersurface.

Theorem 17.12 (Cousin problem)**.** *Let $X \subseteq \mathbb{C}^n$ be a hypersurface, i.e. an analytic subset locally defined by the vanishing of a holomorphic function. Then there exists a global function $f \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ such that $X = Z(f)$.*

Proof. By definition, there exists an open cover $\underline{U} = (U_i)$ and $f_i \in \mathcal{O}_{\mathbb{C}^n}(U_i)$ such that $X \cap U_i = Z(f_i)$. By shrinking the U_i, we may assume that

$$U_i = \{z \in \mathbb{C}^n | |x_j - a_j| < r_j, |y_j - a_j| < s_j\}.$$

For all finite intersections of such U_i is again a polydisk, so in particular it is contractible and we can appeal to the $\overline{\partial}$-Poincare lemma. So $\underline{U}$ is acyclic for Ω^p_X for all $p \geq 0$ (also using Dolbeaut theorem). By Leray's theorem, we have

$$\check{H}^q(\underline{U}, \Omega^p_X) \cong H^q(\mathbb{C}^n, \Omega^p_{\mathbb{C}^n}) \cong H^{p,q}(\mathbb{C}^n)$$

which equals 0 for $q \geq 1$. Note we're cleverly going the other way – usually we go from Cech to sheaf cohomology, but not we're going from sheaf to Cech cohomology.

In particular, we have $\check{H}^q(\underline{U}, \mathcal{O}_{\mathbb{C}^n}) = 0, \forall q \geq 1$.

On $U_j \cap U_k : f_j = g_{jk} f_k$, with $g_{jk} \in \mathcal{O}^*(U_j \cap U_k)$ since $Z(f_j) = Z(f_k)$ on $U_j \cap U_k$. Consider the exponential sequence:

$$0 \to \underline{Z}_{\mathbb{C}^n} \to \mathcal{O}_{\mathbb{C}^n} \to^{\exp(2\pi i)} \mathcal{O}^*_{\mathbb{C}^n} \to 0.$$

Restrict to open sets in U (or intersection) and pass to cohomology. Use $H^1(U_j \cap U_k, \mathbb{Z}) = 0 \implies$

$$\mathcal{O}_{\mathbb{C}^n}(U_j \cap U_k) \to \mathcal{O}^*_{\mathbb{C}^n}(U_j \cap U_k)$$

is surjective for every j, k. So there exist h_{jk} such that $e^{2\pi i h_{jk}} = g_{jk}$, and we see that the g_{jk} satisfy the cocycle condition, and $h_{jk} + h_{k\ell} + h_{\ell j} \in \mathbb{Z}$.

Define $a_{jk\ell} := h_{j\ell} - h_{jk} - h_{k\ell}$. These $[a_{jk}]$ define a class in $\check{H}^2(\underline{U}, \underline{\mathbb{Z}}_{\mathbb{C}^n}) \cong H^2(\mathbb{C}^n, \mathbb{Z}_{\mathbb{C}^n}) = 0$ since $\underline{U}$ acyclic. So we know that $[a_{ijk}] = 0$. So there exist integers b_{ij} such that

$$a_{jk\ell} = b_{k\ell} - b_{j\ell} + b_{jk}.$$

replace h_{jk} by $h_{jk}+b_{jk}$ which implies we may assume $h_{jk}+h_{k\ell}=h_{j\ell}$ on $U_j\cap U_k\cap U_\ell$. (Note adding integers to h_{jk} doesn't change $e^{2\pi i h_{jk}} = g_{jk}$) This implies we're looking at another cocycle

$$\underline{h} = (h_{jk}), [h] \in \check{H}^1(\underline{U}, \mathcal{O}_{\mathbb{C}^n})$$

which is zero because $\check{H}^q(\underline{U}, \mathcal{O}_{\mathbb{C}^n}) = 0, \forall q \geq 1$. This is because by $\overline{\partial}$-Poincare and Dolbeaut theorem $\mathcal{O}_{\mathbb{C}^n}$, $\mathcal{O}_{\mathbb{C}^n}$ is acyclic over this cover (each intersection is contractible), so

$$\widehat{H}^q(\underline{U}, \mathcal{O}_{\mathbb{C}^n}) \cong H^q(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \cong H^{0,q}(\mathbb{C}^n).$$

So there exist j,k such that $h_{jk} = h_k - h_h$, for $h_\ell \in \mathcal{O}_{\mathbb{C}^n}(U_\ell)$. Recall $f_j = g_{jk}f_k \implies f_j e^{2\pi i h_j} = f_k e^{2\pi i h_k}$ on $U_j \cap U_k$. We're done since the exponential function has no zeroes. Same zeroes implies they glue to a function $f \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$, and $Z(f) \cap U_j = X \cap U_j = Z(f_j), \forall j \implies Z(f) = X$. $\square$

Griffiths and Harris do this a bit quicker using sheaf cohomology of $\mathcal{O}^*_{\mathbb{C}^n}$. By $\overline{\partial}$-Poincare lemma, we have $H^{p,q}((\mathbb{C}^*)^n) = 0, \forall q \geq 1$ and these are isomorphic to $H^q(\Omega^p_{(\mathbb{C}^*)^n})$.. insert picture here.

Remember that we are working our way towards the Hodge decomposition theorem. It will be a consequence of important results about elliptic differential operators, which we'll only state.

Eventually the main point of Hodge theory will be clear, as well as the point of studying cohomology with differential geometric methods. Among all the representatives of a cohomology class, we'll want to specify a distinguished class. On manifolds, either Riemannian or Hermitian, there will always be a distinguished form, and those will be the harmonic forms.

Let us also establish the connection with deRham cohomology.

Proposition 17.13. *Let X be a smooth manifold. Then*

$$H^i(X, \underline{\mathbb{Z}}) \cong H^i_{dR}(X; \mathbb{Z}).$$

Proof. We claim the following is an acylic resolution:

$$0 \to \underline{\mathbb{Z}}_X \to A^0_X \to A^1_X \to \cdots$$

Note since X is a smooth manifold, it is Hausdorff + second conutable and thus paracompact. Thus, we can always obtain a locally finite open cover and define a partition of unity. Because of this, $A^i(X)$ are sheaves which admit a partition of unity, and thus are fine/soft sheaves (Popa calls them soft sheaves, but some other mathematicians prefer fine), and thus are acyclic.

Furthermore, this is actually a resolution. Clearly the kernle of $A^0_X \to A^1_X$ is $\underline{\mathbb{Z}}_X$. Furthermore, we have exactness on stalks $A^{p-1}_X \to A^p_X \to A^{p+1}_X$. Given a germ $A^p_X|_z$ at z, we can find a representative which is a local p-form on a contractible open. By Poincare lemma 17.14, the deRham cohomology of this open is zero, and thus if the local p-form disappears under the differential, it is the image of some $p-1$ form. Thus we have exactness.

Since we have an acylic resolution of $\underline{\mathbb{Z}}_X$, we have the desired statement. $\square$

The following is not exactly the Poincare lemma, but is more general and immediately implies the Poincare lemma.

Lemma 17.14 (poincareLemma)**.** *Let D and M be smooth manifolds, and $\phi : D \times [0,1] \to M$ be a homotopy between smooth maps ϕ_1, ϕ_0. Then there exists a chain homotopy between ϕ_1 and ϕ_0, i.e.*

$$(\phi_1)^* = (\phi_0)^* : H^*_{dR}(M) \to H^*_{dR}(D).$$

Proof. Let $e_t : D \to D \times [0,1]$ where $x \mapsto (x,t)$. Then note that $\phi_1 = \phi \circ e_1$, and $\phi_2 = \phi \circ e_0$. Then it suffices to exhibit a chain homotopy between

$$(e_1)^*, (e_0)^* : H^*_{dR}(D \times [0,1]) \to H^*_{dR}(D).$$

In particular, how should we define $h : A^p(D \times [0,1]) \to A^{p-1}(D)$? Ideally, from a p form on $D \times [0,1]$, we would obtain a $p-1$ form on $D \times [0,1]$, then pull back by some e_t. T

The natural choice for the first step is to use interior product with respect to the global vector field $X = (0, \frac{\partial}{\partial t})$. In other words, given $\omega \in A^p(D \times [0,1])$, which is a function of p smooth vector fields, we obtain a $p-1$ form $\iota_X \omega$ where we substitute in X, and thus $\iota_X \omega$ is a function of $p-1$ smooth vector fields.

Furthermore, the choice of pulling back by e_t should not depend on t. So we take the average of all the $p-1$ forms obtained via pullback along e_t. Thus, we define

$$h : \omega \mapsto \int_0^1 e_t^*(\iota_X \omega) dt.$$

One can check that this defines a chain homotopy between e_1^* and e_0^*. More details in

https://www.math.ucla.edu/~sdqunell/poincare_lem.pdf

The intuition really comes from the Lie derivative. A good source for intuition on the Lie derivative is

https://www.math.brown.edu/cdaly2/Notes/Lie_Derivative.pdf ∎

18. 11/8/23: HODGE THEOREM REAL

Let's discuss some real harmonic theory. Let V be a finite dimensional $\mathbb{R}$-vector space, $g : V \times V \to \mathbb{R}$ an inner product. The inner product induces an isomorphism $\phi : V \to V^*$.

Let $e_1, \cdots, e_n$ be an orthonormal basis for V w.r.t this inner product, so $g(e_i, e_j) = \delta_{ij}$. Then $\phi(e_1), \cdots, \phi(e_n)$ is an orthonormal basis for V^* with respect to $g^*(\phi(v_1), \phi(v_2)) = g(v_1, v_2)$.

We have induced inner product

$$\bigwedge^k V \times \bigwedge^k V \to \mathbb{R}$$

where $(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) \mapsto \det(g(v_i, w_j))$, and extend by linearity. We have that $e_1, \cdots, e_n$ extends to an orthonormal basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ for $\bigwedge^k V$ w.r.t to the induced inner product.

Fixing an orientation of V such that $\{e_1, \cdots, e_n\}$ positive basis. Then $\Phi := e_1 \wedge \cdots \wedge e_n$ is the fundamental element (or $\phi(e_1) \wedge \cdots \wedge \phi(e_n)$).

Next time we'll define the star operator $\bigwedge^k V \to \bigwedge^{n-k} V$.

Next day: let V be a finite dimensional $\mathbb{R}$-vector space. Let $g : V \times V \to \mathbb{R}$ be an inner product, $e_1, \cdots, e_n$ an orthonormal basis with respect to g.

We have induced inner product

$$\bigwedge^k V \times \bigwedge^k V \to \mathbb{R}$$

where $(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) \mapsto \det(g(v_i, w_j))$, and extend by linearity. We have that $e_1, \cdots, e_n$ extends to an orthonormal basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ for $\bigwedge^k V$ w.r.t to the induced inner product.

Fixing an orientation of V such that $\{e_1, \cdots, e_n\}$ positive basis. Then $\Phi := e_1 \wedge \cdots \wedge e_n$ is the fundamental element (or $\phi(e_1) \wedge \cdots \wedge \phi(e_n)$ in wedge of dual).

Definition 18.1 (The Hodge Star operator)**.** The Hodge $*$-operator on $(\bigwedge^k V, g)$ $(k \geq 1)$ is the unique linear transformation

$$* : \bigwedge^k V \to \bigwedge^{n-k} V$$

such that $\alpha \wedge *\beta = g(\alpha, \beta)\Phi$, for every $\alpha, \beta \in \bigwedge^k V$.

In fact, it is defined by: $\forall \sigma \in S_n$, have

$$*(e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}) = sgn(\sigma) e_{\sigma(k+1)} \wedge \cdots \wedge e_{\sigma(n)}.$$

In essence, the Hodge $*$-operator sends $e_{i_1} \wedge \cdots \wedge e_{i_k}$ to an $(n-k)$-wedge of the remaining basis vectors, with $sgn(\sigma)$ guaranteeing well-definedness.

Remark: $e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)} = sgn(\sigma) e_1 \wedge \cdots \wedge e_n = sgn(\sigma)\Phi$. Furthermore, we see that $*$ takes orthonormal basis to orthonormal basis, and $g(*\alpha, *\beta) = g(\alpha, \beta)$.

Lemma 18.2. *For* $\alpha \in \bigwedge^k V$*,* $**\alpha = (-1)^{k(n-k)}\alpha$*.*

Proof. Take any $\beta \in \bigwedge^k V$. Then

$$\begin{aligned} **\alpha \wedge *\beta &= (-1)^{k(n-k)} *\beta \wedge *(*\alpha) = (-1)^{k(n-k)} g(*\beta, *\alpha)\Phi = g(\beta, \alpha)\Phi = g(\alpha, \beta)\Phi. \\ &= (-1)^{k(n-k)} \alpha \wedge *\beta, \forall \beta \implies **\alpha = (-1)^{k(n-k)}\alpha \end{aligned}$$

□

Corollary: The Hodge star operator $* : \bigwedge^k V \to \bigwedge^{n-k} V$ is an isomorphism. Note that these vector spaces are already abstractly isomorphic – they have the same dimension. And we already have a perfect pairing $\bigwedge^k V \times \bigwedge^{n-k} V \to \mathbb{R}$. But this is not the point of what we're doing. The point is that $*$ induces an *isometry* bewteen these two spaces. This is the abstract version of Poincare duality:

$$H^k(X; \mathbb{R}) \cong H^{n-k}(X; \mathbb{R}).$$

Let's globalize the Hodge $*$-operator. Let (X, g) be an oriented Riemannian manifold. Each g_x inner product on $T_x X$ induces inner product for $\bigwedge^k T_x^* X$.

Definition 18.3. We have a bilinear map $(,)_X : A^k(X) \times A^k(X) \to \mathbb{R}$, so that

$$(\alpha, \beta)_X := \int_X g(\alpha, \beta) vol(g).$$

The global Hodge $*$-operator is a collection of

$$* : \bigwedge^k T_x^* X \to \bigwedge^{n-k} T_x^* X$$

varying with x, to give:

$$* : A^k(X) \to A^{n-k}(X)$$

such that $\forall \alpha, \beta \in A^k(X)$, we have

$$\alpha \wedge *\beta = g(\alpha, \beta) vol(g).$$

How does an analyst think of the exterior derivative? If you applied it to a function, you'd get

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

This is a linear differential operator. There's a general theory in functional analysis where given a linear differential operator and some measure of size, you get an adjoint operator. Since $A^k(X)$ has measure of size, we'll get an adjoint for the exterior derivative.

Proposition 18.4. *The exterior derivative*

$$d : A^k(X) \to A^{k+1}(X)$$

has an adjoint operator

$$d^* : A^{k+1}(X) \to A^k(X)$$

with respect to $(,)_X$*, given by*

$$d^* = (-1)^{k(n-k)+k+1} * d * .$$

This means that for every $\alpha \in A^k(X), \beta \in A^{k+1}(X)$*, we have*

$$(d\alpha, \beta)_X = (\alpha, d^*\beta)_X.$$

Proof. Note that by Stokes' theorem and Leibniz rule, we have

$$0 = \int_X d(\alpha \wedge *\beta) = \int_X d\alpha \wedge *\beta + (-1)^k \int_X \alpha \wedge d(*\beta).$$

Note that since $** = (-1)^{k(n-k)}*$,

$$(-1)^k \int_X \alpha \wedge d(*\beta) = (-1)^k \int_X \alpha \wedge (-1)^{k(n-k)} * *d(*\beta).$$

Then

$$(d\alpha, \beta)_X = \int_X d\alpha \wedge *\beta = \int_X \alpha \wedge *d^*\beta = (\alpha, d^*\beta)_X$$

where $d^* = (-1)^{k(n-k)+k+1} * d*$. $\square$

Definition 18.5. For every $0 \leq k \leq n$, the Laplace operator

$$\Delta : A^k(X) \to A^k(X)$$

is $\Delta = dd^* + d^*d$.

Example 18.6. If $X = \mathbb{R}^n$ with the usual euclidean metric, then the Laplace operator $\Delta : A^0(X) \to A^1(X)$ is the usual laplace operator:

$$\Delta f = -\sum \frac{\partial^2 f}{\partial x_i^2}.$$

To see this, note that $d^* = - * d*$. Then

$$\Delta f = (dd^* + d^*d)f = -d * d * f - *d * df = - * d * df.$$

$$= - * d * \sum_i \frac{\partial f}{\partial x_i} dx_i = - * d \sum_i \frac{\partial f}{\partial x_i} * dx_i$$

$$= - * \sum_{i,j} \frac{\partial^2 f}{\partial x_i x_j} dx_j \wedge * dx_i = - * \sum_{i,j} \frac{\partial^2 f}{\partial x_i x_j} g(dx_j, dx_i)\Phi$$

$$- * 2 \sum_i \frac{\partial^2 f}{\partial x_i^2}\Phi = -2 \sum \frac{\partial^2 f}{\partial x_i^2}.$$

Definition 18.7. A form $\omega \in A^k(X)$ is harmonic if $\Delta\omega = 0$. Let

$$\mathcal{H}^k(X) = \ker \Delta = \{ \text{ space of harmonic k-forms } \}$$

Note that

$$(\Delta\alpha, \beta)_x = ((dd^* + d^*d)\alpha, \beta)$$

and by adjointness of d, d^*, this is equal to

$$= (d^*\alpha, d^*\beta) + (d\alpha, d\beta) = (\alpha, dd^*\beta) + (\alpha, d^*d\beta) = (\alpha, \Delta\beta).$$

So the Laplace operator is self-adjoint. Some further analysis facts which we'll blackbox: the laplace operator is

- 2nd order linear differential operator
- elliptic operator. This is nondegenarcy of the "symbol" of the operator, where for example, the symbol of $\sum \frac{\partial^2 f}{\partial x_i^2}$ is $\sum x_i^2$ is nonzero for all nonzero inputs.

Here is a big theorem in PDE's:

Theorem 18.8. $\mathcal{H}^k(X) = \ker \Delta$ *is finite dimensional. Furthermore,* $A^k(X) \cong \ker \Delta \oplus Im(\Delta)$.

Assuming this, we prove the main theorem of Hodge theory for smooth manifolds.

Theorem 18.9. *Let* (X, g) *be a compact oriented Riemannian manifold. Then for every* k *there exists a natural isomorphism*

$$\mathcal{H}^k(X) \to^{\phi} H^k(X; \mathbb{R}).$$

(in both the technical and non-technical sense)

Proof. We have ϕ is defined by

$$\omega \in \mathcal{H}^k(X) \mapsto [\omega] \in H^k_{dR}(X).$$

and extending by linearity. First, for this to make sense, we need $d\omega = 0$. But we have a measure of size, which is not something once has in algebraic geometry. We have that

$$0 = (\Delta\omega, \omega)_X = ((dd^* + d^*d)\omega, \omega) = (d\omega, d\omega)_X + (d^*\omega, d^*\omega)_X = ||d\omega||^2_X + ||d^*\omega||^2_X,$$

which forces $d\omega = 0 = d^*\omega$.

Next we show ϕ is injective. Say $[\omega] = 0$, i.e $\omega = d\eta$, where $\eta \in A^{k-1}(X)$. Have

$$||\omega||^2_X = (\omega, d\eta)_X = (d^*\omega, \eta)_X = (0, \eta)_X = 0 \implies \omega = 0.$$

So the map is injective. Now we show surjectivity. This is where the main content of Hodge theory is. Every cohomology class has a unique harmonic representative. Let $[v] \in H^k(X; \mathbb{R})$, so we know $dv = 0$. By the theorem of PDEs that $A^k(X) = \ker \Delta \oplus Im(\Delta)$, we have

$$v = \omega + \Delta\mu, \omega \in \mathcal{H}^k(X), \mu \in A^k(X).$$

Then it suffices to show that $d^*d\mu = 0$. Then we're done, because then $[v] = [\omega]$, so we've proven surjectivity. We have

$$||d^*d\mu||_X^2 = (d^*d\mu, d^*d\mu)_X = (d\mu, dd^*d\mu)_X = (d\mu, 0)_X = 0$$

because $dv = d\omega + dd^*d\mu$, but both $dv = d\omega = 0$. □

Note that this immediately implies that the singular cohomology of a oriented compact Riemannian manifold is finite dimensional, since the space of Harmonic forms is.

How did people arrive to this? There's a piece of fundamental calculus that leads us to harmonic forms. There's a reason for considering harmonic forms and the harmonic condition. If you measure something, if you have a notion of size, you might wonder whether this has minima or maxima. Harmonic forms have minima with respect to this measure of size.

More precisely: say ω is representative of the class $[\omega] \in H^k(X;\mathbb{R})$ minimizing $||-||_X^2$. Look at any other representative $\omega + d\mu \in A^k(X)$. Look radially, pick some point and scale the point, do some directional derivative, so look at $\omega + td\mu$:

$$||\omega + td\mu||_X^2 = ||\omega||_X^2 + 2t(\omega, d\mu)_X + t^2||d\mu||_X^2.$$

If we want to minimize this, then we take the derivative at $t = 0$ is $(\omega, d\mu)_X = 0$, and this is for every μ. So by adjointness, we obtain

$$(d^*\omega, \mu)_X = 0 \iff d^*\omega = 0, d\omega = 0 \iff \Delta\omega = 0.$$

So the next thing to do is complex harmonic theory. There's a lot to digest here, look at Huybrechts section 1.2 for the linear algebra.

If V is a $\mathbb{C}$-vector space of dimension n, it has underlying real vector space structure $V_\mathbb{R}$ of dimension $2n$. Have $\mathcal{J} : V_\mathbb{R} \to V_\mathbb{R}$ linear map $\mathcal{J}^2 = -Id$ where $v \mapsto iv$. This is called the almost structure. But you can also take

$$V_\mathbb{C} = V_\mathbb{R} \otimes \mathbb{R}\mathbb{C}$$

so this is now 4n dimensional, and it has an induced linear map

$$\mathcal{J}_\mathbb{C} = \mathcal{J} \otimes id : V_\mathbb{C} \to V_\mathbb{C}, \mathcal{J}_\mathbb{C}(v \otimes z) = \mathcal{J}(v) \otimes z.$$

The i here and $\mathcal{J}_\mathbb{C}$ are not the same. Play around with

$$\mathcal{J}_\mathbb{C}(v \otimes 1 - \mathcal{J}v \otimes i) = i(v \otimes 1 - \mathcal{J}v \otimes i) = -i(v \otimes 1 + \mathcal{J}v \otimes i)$$

so $\pm i$ are the only eigenvalues of $\mathcal{J}_\mathbb{C}$ on $V_\mathbb{C}$. So try to understand the decomposition $V_\mathbb{C} = V^{1,0} + V^{0,1}$. This will be the starting point of next time.

19. 11/13/23: HODGE THEOREM COMPLEX

We're coming up on the Hodge decomposition theorem. Afterwards, a whole new world will open up for us, and we'll be able to do concrete examples.

First, let's formalize some manuevers that we've been doing all along. These manuevers were done when we complexifying tangent spaces, decomposing them into holomorphic and antiholomorphic parts, and the process of obtaining a $(1,1)$-form from a hermitian metric.

Definition 19.1. A finite dimensional real vector space has an almost complex structure $\mathcal{J}$ if $\mathcal{J}$ is an endomorphism such that $\mathcal{J}^2 = -Id$.

If V is a complex vector space, then $V_{\mathbb{R}}$ has almost complex structure given by $\mathcal{J}v = -iv$. Similarly, given a real vector space with almost complex structure, we can give it the structure of a $\mathbb{C}$-vector space with the almost complex structure.

Now let V be a complex vector space, and $V_{\mathbb{R}}$ the underlying real vector space and $\mathcal{J}$ its almost complex structure. Let $V_{\mathbb{C}} := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Note $\mathcal{J}$ extends $\mathbb{C}$-linearly to an endomorphism $\mathcal{J} : V_{\mathbb{C}} \to V_{\mathbb{C}}$. What are the eigenspaces of $\mathcal{J}$ on $V_{\mathbb{C}}$? Note $\mathcal{J}v = \lambda v \implies -v = \lambda^2 v \implies \lambda^2 = -1$. So $\lambda = \pm i$.

Let

$$V^{1,0} = \{v \in V_{\mathbb{C}} | \mathcal{J}v = iv\}, V^{0,1} = \{v \in V_{\mathbb{C}} | \mathcal{J}v = -v\}.$$

Lemma 19.2. *The interaction between the almost complex structure $\mathcal{I}$ of $V_{\mathbb{R}}$ and the complex structure i of $V_{\mathbb{C}}$ yields the decomposition*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

Furthermore, $V^{1,0} \cong V^{0,1}$ via conjugation in $V_{\mathbb{C}}$, and

$$(V_{\mathbb{R}}, \mathcal{J}) \cong V^{1,0}$$

are isomorphic as $\mathbb{R}$-vector spaces.

Proof. Since $V^{1,0} \cap V^{0,1} = 0$, we have the obvious injection

$$V^{1,0} \oplus V^{0,1} \hookrightarrow V_{\mathbb{C}}.$$

Surjection follows from the fact that this map has an inverse $V_{\mathbb{C}} \to V^{1,0} \oplus V^{0,1}$ via

$$v \mapsto \frac{1}{2}(v - i\mathcal{J}v) \oplus \frac{1}{2}(v + i\mathcal{J}v).$$

Now given $\frac{1}{2}(v - i\mathcal{J}v) \in V^{1,0}$, if we write $v = x + iy$, where $x, y \in V_{\mathbb{R}}$, we have the conjugate is

$$x - iy + i(\mathcal{J}x - i\mathcal{J}y) = \frac{1}{2}(\overline{v} + i\mathcal{J}\overline{v}).$$

Finally, we have an isomorphism as $\mathbb{R}$-vector spaces, where $v \mapsto \frac{1}{2}(v - i\mathcal{J}v)$ and $\mathcal{J}v \mapsto \frac{1}{2}(\mathcal{J}v + iv) = i\frac{1}{2}(v - i\mathcal{J}v)$. □

The example that one should keep in mind when thinking about this decomposition is how we separated the $\frac{\partial}{\partial z}$ from the $\frac{\partial}{\partial \overline{z}}$, or dz from $d\overline{z}$. From this decomposition, for every $k \geq 1$, we have

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}.$$

Note that if g is an inner product on $V_{\mathbb{R}}$, then we can extend g to a hermitian form h on $V_{\mathbb{C}}$ via

$$h(v_1 \otimes \lambda_1, v_2 \otimes \lambda_2) = \lambda_1 \overline{\lambda_2} g(v_1, v_2).$$

In particular, if h is a hermitian form on $V^{1,0}$, then we know that $g = Re(h)$ gives an inner product on $V_{\mathbb{R}}$, where for $v, w \in V_{\mathbb{R}}$, we have

$$g(v, w) = Re[h(\frac{1}{2}(v - i\mathcal{J}v), \frac{1}{2}(w - i\mathcal{J}w))].$$

This inner product on $V_{\mathbb{R}}$ then induces a hermitian form on $V_{\mathbb{C}}$. We can extend this form to all $\bigwedge^k V_{\mathbb{C}}$ via the usual determinant formulas.

Lemma 19.3. *Suppose h is a hermitian form on $V^{1,0}$. The decomposition*

$$\bigwedge^k V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$$

is orthogonal with respect to the hermitian form $h_{\mathbb{C}}$ extended to $\bigwedge^k V_{\mathbb{C}}$.

Proof. Note that h on $V^{1,0}$ induces g on $V_{\mathbb{R}}$ which induces hermitian metric $h_{\mathbb{C}}$ on $V_{\mathbb{C}}$. It suffices to show that the decomposition

$$V_{\mathbb{C}} \cong V^{1,0} \oplus V^{0,1}$$

is orthogonal with respect to $h_{\mathbb{C}}$. We have that

$$h_{\mathbb{C}}(\frac{1}{2}(v-i\mathcal{J}v), \frac{1}{2}(w+i\mathcal{J}w)) = \frac{1}{4}[h_{\mathbb{C}}(v,w)-ih_{\mathbb{C}}(v,\mathcal{J}w)-ih_{\mathbb{C}}(\mathcal{J}v,w)-h_{\mathbb{C}}(\mathcal{J}v,\mathcal{J}w)]$$

$$\frac{-i}{4}[h_{\mathbb{C}}(v,\mathcal{J}w) + h_{\mathbb{C}}(\mathcal{J}v,w)] = \frac{-i}{4}[g(v,\mathcal{J}w) + g(\mathcal{J}v,w)] = 0.$$

□

Note that if $v_1, \cdots, v_n \in V_{\mathbb{R}}$ map to elements in $V^{1,0}$ which give an orthonormal basis with respect to h on $V^{1,0}$, then

$$v_1, \mathcal{J}v_1, \cdots, v_n, \mathcal{J}v_n$$

gives a positively-oriented orthonormal basis for $V_{\mathbb{R}}$ with respect to g induced from h. This extends to an orthonormal basis for $V_{\mathbb{C}}$ with respect to $h_{\mathbb{C}}$ induced by g.

Recall that we had the hodge $*$-operator

$$* : \bigwedge^k V_{\mathbb{R}} \to \bigwedge^{2n-k} V_{\mathbb{R}}.$$

This extends $\mathbb{C}$-linearly to

$$* : \bigwedge^k V_{\mathbb{C}} \to \bigwedge^{2n-k} V_{\mathbb{C}},$$

so that $\alpha \wedge *\overline{\beta} = h(\alpha, \beta)\Phi$, where $h(\alpha,\beta)$ is $g(\alpha,\beta)$ but extended so that it is hermitian. Note that we need the conjugation over $\overline{\beta}$, so that, for example,

$$\alpha \wedge *i\beta = h(\alpha, -i\overline{\beta})\Phi = ih(\alpha, \overline{\beta})\Phi = i\alpha \wedge *\beta,$$

since $*$ is $\mathbb{C}$-linear.

Lemma 19.4. *Suppose we have hermitian form on $V^{1,0}$ which induces h on $V_{\mathbb{C}}$. Extend the Hodge $*$-operator $\mathbb{C}$-linearly. Then*

(1) $* : V^{p,q} \to V^{n-q,n-p}$
(2) $**\alpha = (-1)^{p+q}$, for every $\alpha \in V^{p,q}$

Proof. (1) Let $\beta \in V^{p,q}$. Let $\alpha \in V^{p',q'}$ such that $p'+q' = p+q$. Then

$$\alpha \wedge *\beta = h(\alpha, \overline{\beta})\Phi.$$

Note $\overline{\beta} \in V^{q,p}$. Recall the decomposition $V_{\mathbb{C}} \cong \bigoplus_{p+q=k} V^{p,q}$ is orthogonal w.r.t h. Then $h(\alpha, \overline{\beta}) = 0$ unless $q = p', p = q'$. So $\alpha \in V^{q,p}$. Then $*\beta \in V^{n-q,n-p}$.

(2) We already have that $** = (-1)^{k(2n-k)}$. Then $** = (-1)^{-k^2} = (-1)^k = (-1)^{p+q}$ when applied to $\alpha \in V^{p,q}$.

□

Now let's discuss the global version of all this. Let X be a compact complex manifold with hermitian metric. This hermitian metric is a collection of hermitian forms on the holomorphic tangent spaces T'_pX. We get an associated Riemannian metric g, a collection of inner products on $T_{p,\mathbb{R}}X$, where $T_{p,\mathbb{R}}X$ is isomorphic as $\mathbb{R}$-vector spaces to T'_pX via

$$\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial z}, \frac{\partial}{\partial y} \mapsto i\frac{\partial}{\partial z},$$

noting that $\frac{\partial}{\partial y} = \mathcal{J}\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\mathcal{J}\frac{\partial}{\partial x})$. Note that this an example of the isomorphism between $V_{\mathbb{R}} \cong V^{1,0}$.

Now we can extend these inner products to get hermitian forms h_p on $T_{p,\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$ via $h_p(v \otimes \lambda_1, w \otimes \lambda_2) = \lambda_1\overline{\lambda_2}g(v_1, v_2)$. Dualizing and wedging induces hermitian forms for $\bigwedge^k T^*_{p,\mathbb{C}}X$.

Thus, on compact complex manifold X with hermitian metric, we attain a hermitian form on $A^k(X) \otimes \mathbb{C}$, where

$$\langle \alpha, \beta \rangle_X := \int_X \alpha \wedge *\overline{\beta}\Phi = \int_X h(\alpha, \beta)\Phi,$$

and the hodge star operator restricts to $* : A^{p,q}(X) \to A^{n-q,n-p}(X)$.

Note that the exterior derivative and its adjoint in the real case extend $\mathbb{C}$-linearly to

$$d : A^k(X) \otimes \mathbb{C} \to A^{k+1} \otimes \mathbb{C}, d^* : A^{k+1} \otimes \mathbb{C} \to A^k \otimes \mathbb{C}.$$

We have

$$d^* = (-1)^{k(2n-k)+k+1} * d* = - * \partial * - * \overline{\partial} * = \overline{\partial}^* + \partial^*.$$

We have

$$\begin{aligned} \langle d\alpha, \beta \rangle_X &= \langle \partial\alpha, \beta \rangle_X + \langle \overline{\partial}\alpha, \beta \rangle_X \\ &= \langle \alpha, d^*\beta \rangle_X = \langle \alpha, \overline{\partial}^*\beta \rangle + \langle \alpha, \partial^*\beta \rangle. \end{aligned}$$

Restricting our attention to p, q forms, and using orthogonality of the decomposition $A^k(X) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} A^{p,q}$ with respect to h, we see that

Proposition 19.5. *The adjoint of*

$$\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$$

with respect to $\langle , \rangle_X$ *is*

$$\overline{\partial}^* = - * \partial * : A^{p,q+1}(X) \to A^{p,q}(X).$$

The adjoint of $\partial : A^{p,q}(X) \to A^{p+1,q}(X)$ *with respect to* $\langle , \rangle_X$ *is*

$$\partial^* = - * \overline{\partial} * : A^{p+1,q}(X) \to A^{p,q}(X).$$

Definition 19.6. The anti-holomorphic Laplacian is

$$\overline{\square} : A^{p,q}(X) \to A^{p,q}(X)$$

where $\overline{\square} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. We say that ω is $\overline{\square}$-harmonic if $\overline{\square}\omega = 0$. We define

$$\mathcal{H}^{p,q}_{\overline{\square}} := \ker \overline{\square}.$$

The holomorphic Laplacian is

$$\square : A^{p,q}(X) \to A^{p,q}(X)$$

where $\square := \partial\partial^* + \partial^*\partial$. We say ω is $\square$-harmonic if $\square\omega = 0$. We define

$$\mathcal{H}^{p,q}_{\square} := \ker \square.$$

Proposition 19.7. *The operator $\overline{\square}$ is*

- *2nd order linear differential operator*
- *self-adjoint, elliptic*
- *has a map $\mathcal{H}^{p,q}_{\overline{\square}} \to H^{p,q}_{\overline{\partial}}(X)$*
- *$\mathcal{H}^{p,q}_{\overline{\square}}$ is finite dimensional $\mathbb{C}$-vector space (requires X is compact), and*

$$A^{p,q} = \mathcal{H}^{p,q}_{\overline{\square}} \oplus Im\overline{\square}.$$

Similarly, the operator $\square$ is

- *2nd order linear differential operator*
- *self-adjoint, elliptic*
- *has a map $\mathcal{H}^{p,q}_{\square} \to H^{p,q}_{\partial}(X)$*
- *$\mathcal{H}^{p,q}_{\square}$ is finite dimensional $\mathbb{C}$-vector space (requires X compact) and*

$$A^{p,q} = \mathcal{H}^{p,q}_{\square} \oplus Im\square$$

Theorem 19.8 (Hodge theorem for (p,q) forms)**.** *Let X be a compact complex manifold with hermitian metric. Then the natural map*

$$\mathcal{H}^{p,q}_{\overline{\square}} \to H^{p,q}_{\overline{\partial}}(X)$$

is an isomorphism; each dolbeaut cohomology class has a unique harmonic representative.

Proof. The map $H^{p,q}_{\overline{\square}} \to H^{p,q}_{\overline{\partial}}(X)$ is given by

$$\omega \mapsto [\omega].$$

First we check that this map is well-defined, i.e $\overline{\partial}\omega = 0$. We have $\overline{\square}\omega = 0$, so

$$0 = \langle \overline{\square}\omega, \omega\rangle_X = \langle \overline{\partial}\overline{\partial}^*\omega, \omega\rangle_X + \langle \overline{\partial}^*\overline{\partial}\omega, \omega\rangle_X$$

$$\implies ||\overline{\partial}\omega||^2_X + ||\overline{\partial}^*\omega||^2_X = 0 \implies \overline{\partial}\omega = \overline{\partial}^*\omega = 0.$$

Now we check that the map is injective. Suppose $\omega = \overline{\partial}\eta$. Then

$$\langle \omega, \omega\rangle_X = \langle \omega, \overline{\partial}\eta\rangle_X = \langle \overline{\partial}^*\omega, \eta\rangle_X = \langle 0, \eta, \rangle_X = 0.$$

Now we check surjectivity. We can write $\alpha = \omega + \overline{\square}\eta = \omega + \overline{\partial}\overline{\partial}^*\eta + \overline{\partial}^*\overline{\partial}\eta$, where ω is $\overline{\square}$-harmonic. Then it suffices to show that $\overline{\partial}^*\overline{\partial}\eta = 0$. We have

$$\langle \overline{\partial}^*\overline{\partial}\eta, \overline{\partial}^*\overline{\partial}\eta\rangle_X = \langle \overline{\partial}\eta, \overline{\partial}\overline{\partial}^*\overline{\partial}\eta\rangle_X = 0,$$

since $\overline{\partial}\alpha = \overline{\partial}\omega + \overline{\partial}\overline{\partial}^*\overline{\partial}\eta \implies \overline{\partial}\overline{\partial}^*\overline{\partial}\eta = 0$. □

Theorem 19.9 (Holomorphic Hodge theorem for (p,q) forms)**.** *For compact complex manifold X with hermitian metric, we have an isomorphism*

$$\mathcal{H}^{p,q}_{\square} \cong H^{p,q}_{\partial}(X).$$

Proof. Proof is same idea. □

There is no clear relation between $\Delta, \overline{\Box}$, or $\Box$. For example, a $\overline{\partial}$-harmonic form might not even be d-closed. But when X is Kahler, these operators will be related and in fact same up to constants, and this will result in a lot of useful identities.

Even when X is not Kahler, there are some very nice duality statements that hold, which we discuss. As one sees, these duality statements really underline the importance of the Hodge $*$-operator. The proofs of all these duality statements are more or less the same. Duality holds because of how the adjoints are constructed with the $*$-operator, and the fact that $**$ is an isomorphism up to a sign.

We said before the the linear algebra Hodge $*$-operator should be thought of as abstract Poincare duality. We make this precise.

Proposition 19.10 (Poincare duality)**.** *Let X be a compact connected oriented Riemannian manifold. Then the $*$-operator induces an isomorphism*

$$H^k(X;\mathbb{R}) \cong H^{n-k}(X;\mathbb{R})$$

and we have a pairing

$$H^k(X;\mathbb{R}) \times H^{n-k}(X;\mathbb{R}) \to \mathbb{R}, (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$$

nondegenerate in each argument.

Proof. Note that by the real Hodge theorem, we have $\mathcal{H}_d^k \cong H^k(X;\mathbb{R})$. So it suffices to prove that the $*$-operator induces an isomorphism

$$\mathcal{H}_d^k \to \mathcal{H}_d^{n-k}$$

between d-harmonic forms. Suppose $\omega \in A^k(X)$ is Harmonic. Note

$$0 = \Delta\omega \implies d\omega, d^*\omega = 0.$$

Then

$$\begin{aligned}\Delta{*}\omega = \Delta(*\omega) = (dd^*{+}d^*d)(*\omega) &= (-1)^{(n-k-1)(k+1)+n-k-1+1} d{*}d{*}(*\omega){+}(-1)^{(n-k)(k)+n-k+1} {*}d{*}d{*}\omega \\ = (-1)^{(n-k-1)(k+1)+n-k-1+1+k(n-k)} d * d\omega &+ (-1)^{(n-k)(k)+n-k+1} * d * d * \omega \\ &= (-1)^{(n-k)(k)+n-k+1} * d * d * \omega = 0.\end{aligned}$$

We clearly have an inverse. Thus, we have the isomorphism, and via Hodge theorem we obtain $H^k(X;\mathbb{R}) \cong H^{n-k}(X;\mathbb{R})$.

We'll implicitly be appealing to the Hodge isomorphism between harmonic forms and singular cohomology. Checking nondegeneracy is very simple. In the first argument, for nonzero α, we always have

$$(\alpha, *\alpha) \mapsto \int_X \alpha \wedge *\alpha = \int_X g(\alpha,\alpha) vol(g) > 0$$

where g is the Riemannian metric. In the second argument, for nonzero β, there's some $\alpha \in H^k$ where $*\alpha = \beta$. Then

$$(\alpha, \beta) = (\alpha, *\alpha) \mapsto \int_X \alpha \wedge *\alpha = \int_X g(\alpha,\alpha) vol(g) > 0.$$

□

Restricting ourselves to p, q forms, we'll work our way towards a special case of Kodaiara-Serre duality. The idea here is that $*$ induces a duality on the level of p, q forms, and since we know dolbeaut cohomology is isomorphic to sheaf cohomology of Ω_X^p, we obtain a duality statement in the setting of sheaf cohomology.

Proposition 19.11. *Let X be a compact complex hermitian manifold. Show that for any p, q the $*$-operator induces isomorphisms*

$$*: \mathcal{H}_d^{p,q} \to \mathcal{H}_d^{n-q,n-p}$$

and

$$*: \mathcal{H}_{\overline{\partial}}^{p,q} \to \mathcal{H}_{\partial}^{n-q,n-p}.$$

Show also that complex conjugation interchanges $\mathcal{H}_{\overline{\partial}}^{p,q}$ and $\mathcal{H}_{\partial}^{q,p}$.

Proof. Note that $[\Delta, *] = 0$, so $*$ preserves d-harmonic forms. Thus, we indeed have a map $\mathcal{H}_d^{p,q} \to \mathcal{H}_d^{n-q,n-p}$. But $*$ is invertible. Thus, we obtain an isomorphism

$$*: \mathcal{H}_d^{p,q} \to \mathcal{H}_d^{n-q,n-p}.$$

Now we show the $*$-operator induces isomorphism

$$*: \mathcal{H}_{\overline{\partial}}^{p,q} \to \mathcal{H}_{\partial}^{n-q,n-p}.$$

Suppose $\alpha \in A^{p,q}(X)$ is a $\overline{\partial}$-harmonic form, $\overline{\Box}\alpha = 0$. This means

$$(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\alpha = -(\overline{\partial} * \partial * + * \partial * \overline{\partial})\alpha = 0.$$

Certainly $*\alpha \in A^{n-q,n-p}(X)$. Then

$$\Box * \alpha = -(\partial * \overline{\partial} * + * \overline{\partial} * \partial) * \alpha = -(\partial * \overline{\partial} * * + *\overline{\partial} * \partial *)\alpha$$
$$= -((-1)^{(p+q)(2n-p-q)}\partial * \overline{\partial} + *\overline{\partial} * \partial *)\alpha = -(sgn) * (*\partial * \overline{\partial} + \overline{\partial} * \partial *)\alpha = 0.$$

A similar proof shows that $*: \mathcal{H}_{\partial}^{n-q,n-p} \to \mathcal{H}_{\overline{\partial}}^{p,q}$ and this gives an inverse, thus establishing the isomorphism.

Suppose $\alpha \in \mathcal{H}_{\overline{\partial}}^{p,q}$. Then note that

$$\overline{\Box}\alpha = -(\overline{\partial} * \partial * + * \partial * \overline{\partial})\alpha = 0.$$

But conjugating yields

$$-(\partial * \overline{\partial} * + * \overline{\partial} * \partial)\overline{\alpha} = 0 \implies \Box\overline{\alpha} = 0.$$

Conjugating back gives an isomorphism. So conjugation swaps $\mathcal{H}_{\overline{\partial}}^{p,q}$ and $\mathcal{H}_{\partial}^{q,p}$. $\Box$

This is something that works for all compact complex manifolds. This is a special case of what's more generally called Serre duality.

Theorem 19.12 (Special Kodaira-Serre Duality)**.** *Let X be a compact hermitian manifold. Then*

(1) $H^n(X, \Omega_X^n) \cong \mathbb{C}$, noting that the canonica line bundle $\omega_X := \Omega_X^n$.
(2) For every p, q we have a pairing

$$H^q(X, \Omega_X^p) \times H^{n-q}(X, \Omega_X^{n-p}) \to H^n(X, \Omega_X^n) \cong \mathbb{C}.$$

In particular,

$$H^q(X, \Omega_X^p) \cong H^{n-q}(X, \Omega_X^{n-p})^\vee.$$

Proof. (1) We have

$$H^n(X, \Omega_X^n) \cong H_{\overline{\partial}}^{n,n}(X) \cong \mathcal{H}_{\overline{\partial}}^{n,n} \cong \mathcal{H}_{\partial}^{0,0} \cong \mathcal{H}_{\overline{\partial}}^{0,0} \cong H^{0,0}(X) \cong H^0(X; \mathbb{C}) \cong \mathbb{C}$$

where the isomorphisms, in order, are because of Dolbeault, Hodge, duality from $*$-operator, conjugation, Hodge, and deRham. Note that $\omega_X^{\wedge n} = vol(g) \in H_{\overline{\partial}}^{n,n}(X)$. Thus, $H^n(X, \Omega_X^n)$ is generated by the image of the volume form.

(2) By Dolbeaut and Hodge's theorem, we have $H^q(X, \Omega_X^p) \cong H_{\overline{\partial}}^{p,q}(X) \cong \mathcal{H}_{\overline{\partial}}^{p,q}$. Then the pairing is simply given by

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

Note that this is really coming from the inner product on $A^{p,q}(X)$. For elements in $\mathcal{H}_{\overline{\partial}}^{p,q}$, we have

$$\langle \alpha, \beta \rangle_X := \int_X \alpha \wedge *\overline{\beta},$$

and note that $*\overline{\beta} \in \mathcal{H}_{\overline{\partial}}^{n-p,n-q}$. Since this inner product is non-degenerate, this shows that our pairing

$$H^q(X, \Omega_X^p) \times H^{n-q}(X, \Omega_X^{n-p}) \to \mathbb{C}$$

is non-degenerate, since for $\omega \in H^q(X, \Omega_X^p)$, we have $*\overline{\omega} \in H^{n-q}(X, \Omega_X^{n-p})$ and $\int \omega \wedge *\overline{\omega} = \langle \omega, \omega \rangle_X$. This nondegeneracy implies

$$H^q(X, \Omega_X^p) \cong H^{n-q}(X, \Omega_X^{n-p})^\vee.$$

□

Definition 19.13. Let X be a compact complex manifold of dimension n. The Hodge numbers of X are:

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X) = \dim_{\mathbb{C}} \mathcal{H}_{\overline{\partial}}^{p,q}(X) = \dim_{\mathbb{C}} H^q(X, \Omega_X^p).$$

These Hodge numbers are extremely important invariants. Although not well-behaved in general, they are for Kahler manifolds and thus complex projective varieties. Hodge numbers may be viewed as refinements of the Betti numbers coming from topology. This is precisely true when X is Kahler. Then apply the Hodge decomposition theorem.

What do we already know about Hodge numbers?

- From the theory of elliptic PDEs, we know that $h^{p,q}(X) < \infty$
- $h^{p,q}(X) = 0$ for $p \notin [0, n]$ or $q \notin [0, n]$
- $h^{n,n}(X) = h^{0,0}(X) = 1$
- $h^{p,q}(X) = h^{n-p,n-q}(X)$ (Kodaira-Serre duality)
- $h^{p,q} = h^{q,p}$ (because $H^{p,q} = \overline{H^{q,p}}$. (only true if X is Kahler).

We also have an analog of the Kunneth formula from algebraic topology:

$$h^{p,q}(X \times Y) = \sum_{a+b=p, c+d=q} h^{a,c}(X) h^{b,d}(Y).$$

To prove this, note that $\Omega_{X \times Y}^p$ has some expression in terms of pullbacks, and there's some Kunneth thing for sheaf cohomology.

The following is misleading, because these are Kahler and says nothing about non-Kahler.

Example 19.14 (Compact Riemann Surfaces)**.** We have $0 \leq p, q \leq 1$. We have

$$h^{0,0} = 1$$

$$\text{genus} := \dim_{\mathbb{C}} H^0(X, \Omega_X^1) = h^{1,0} \qquad\qquad h^{0,1} = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$$

$$h^{1,1} = 1$$

So these hodge numbers for compact riemann surfaces contain their only sort of invariant, their genus. When $h^{1,0} = 0$, then this is a $\mathbb{P}^1$. When $h^{1,0} = 1$, it is elliptic curve. If $g = h^{1,0}(X) \geq 2$, then the next are "general type" and "hyperbolic."

20. 11/15/23: KAHLER IDENTITIES, HODGE DECOMPOSITION

Check out Bruns-Herzog "Cohen-Macaulay rings and modules." Very good book on homological methods.

Last time, we saw that even without the Kahler condition, we got the hodge numbers, some duality, etc. Now we'll insert the Kahler condition and see what happens.

Let (X, h) be a Kahler manifold. Let $\omega \in A^{1,1}(X)$ be the associated $(1,1)$-form.

Definition 20.1 (Lefschetz operator)**.** The Lefschetz operator $L : A^k(X) \otimes \mathbb{C} \to A^{k+2}(X) \otimes \mathbb{C}$ where

$$\alpha \mapsto \omega \wedge \alpha.$$

Note that L restricts to

$$A^{p,q}(X) \to A^{p+1,q+1}(X).$$

With respect to hermitian metric on $A^k(X) \otimes \mathbb{C}$ induced by the Riemannian metric g, we have an adjoint Λ for L, so that

$$h(L\alpha, \beta) = h(\alpha, \Lambda\beta), \forall \alpha \in A^k(X), \beta \in A^{k+2}(X).$$

The Lefschetz operator is a more general analog of intersecting varieties with a hyperplane. It's an analog for intersecting Kahler manifolds with hypersurfaces. *Come back here.*

Lemma 20.2. *The adjoint* $\Lambda : A^{k+2}(X) \otimes \mathbb{C} \to A^k(X) \otimes \mathbb{C}$ *to the Lefschetz operator* L *is*

$$\Lambda = (-1)^k * L * .$$

Proof. We would like for every $\alpha \in A^k(X) \otimes \mathbb{C}, \beta \in A^{k+2}(X) \otimes \mathbb{C}$,

$$\langle L\alpha, \beta \rangle_X = \int_X \omega \wedge \alpha * \overline{\beta} = \int_X \alpha \wedge * \overline{\Lambda\beta} = \langle \alpha, \Lambda\beta \rangle_X.$$

Note that $\omega \wedge \alpha * \overline{\beta} = \alpha \wedge \omega \wedge * \overline{\beta}$. Note $\omega * \overline{\beta} \in A^{2n-k}(X) \otimes \mathbb{C}$. So $\alpha \wedge \omega \wedge * \overline{\beta} = \alpha \wedge (-1)^k * * \omega \wedge * \overline{\beta}$. We want

$$\alpha \wedge (-1)^k * * \omega \wedge * \overline{\beta} = \alpha \wedge * \overline{\Lambda\beta}.$$

So $\overline{\Lambda\beta} = \Lambda\overline{\beta} = (-1)^k * \omega \wedge \overline{\beta}$. Then $\Lambda = (-1)^k * L *$. $\square$

We want to find more relations between all the operators we've defined. The consequences that follow the Kahler identities are somewhat magical.

Theorem 20.3 (Kahler Identities)**.** *Let (X, h) be Kahler. Then*

$$[\Lambda, \overline{\partial}] = -i\partial^* \text{ and } [\Lambda, \partial] = i\overline{\partial}^*.$$

Proving one implies the other by conjugation. Furthermore, by adjoints, we have that

$$[\partial^*, L] = -i\overline{\partial} \text{ and } [\overline{\partial}^*, L] = i\partial.$$

Proof. Explained very nicely in every book. See Griffiths and Harris. Proof sketch. Main idea: how do these operators depend on the choice of metric h? The formulas we are trying to prove only involve the metric h to order ≤ 1 ($h_{k\ell}$'s and first partials). But remember that h is kahler $\iff$ osculates to order 2 to euclidean. So you can make a change of basis, and get this euclidean form. $h_{k\ell} = \delta_{k\ell} + \psi_{k\ell}$. Enough to prove the identities for $\mathbb{C}^n$ with euclidean metric.

So the $(1, 1)$ form for euclidean metric is

$$\omega = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z_j}.$$

So $L\alpha = \omega \wedge \alpha = \frac{i}{2}\sum_j dz_j \wedge d\overline{z_j} \wedge \alpha$. Define

$$e_j := dz_j \wedge - : A^{p,q}(X) \to A^{p+1,q}$$
$$\overline{e_j} := d\overline{z_j} \wedge - : A^{p,q}(X) \to A^{p,q+1}(X).$$

Let $L = \frac{i}{2}\sum_{j=1}^{n} e_j\overline{e_j}$. So

$$L = \frac{i}{2}\sum_{j=1}^{n} e_j\overline{e_j}, \Lambda = \frac{-i}{2}\sum_{j=1}^{n} \overline{e_j}^* e_j^*.$$

Note

$$\partial(\sum \phi_{jk} dz_j \wedge d\overline{z_k}) = \sum_{\ell,j,k} \frac{\partial \phi_{jk}}{\partial z_\ell} dz_k \wedge dz_j \wedge d\overline{z_k}.$$

So also define

$$\partial_j := A^{p,q}(X) \to A^{p,q}, \sum \phi_{JK} dz_J \wedge d\overline{z}_K \mapsto \sum \frac{\partial_{jk}}{\partial z_j} dz_j \wedge d\overline{z_k}.$$

do something similar for $\overline{\partial_j} : A^{p,q}(X) \to A^{p,q}(X)$, so we can write $\partial = \sum_{j=1}^{n} \partial_j e_j, \overline{\partial} \sum_{j=1}^{n} \overline{\partial_j}\overline{e_j}$. So we'll just prove for $[\Lambda, \partial] = i\overline{\partial}^*$, and we'll obtain the other expression just by conjugation. Use the lemma that $e_k e_j^* + e_j^* e_k = 2\delta_{jk} Id$.

Furthermore, $\partial_j, \overline{\partial_j}$ commutes with e_j, e_j^* also with each other. And $\partial_j^* = -\overline{\partial_j}$ and $\overline{\partial_j}^* = -\partial_j$. (See Griffiths and Harris, integration by parts). $\Lambda = \frac{-i}{2}\sum \overline{e_j}^* e_j^*$. Then $\partial^* = -\sum \overline{\partial_j} e_j^*$ since operators commute. And $\overline{\partial}^* = -\sum \partial_j \overline{e_j}^*$. So putting this all together, we obtain

$$[\Lambda, \partial] = \Lambda\partial - \partial\Lambda = \frac{-i}{2}\sum_{jk}(\overline{e_j}^* e_j^* \partial_k e_k - \partial_k e_k \overline{e_j}^* e_j^*) = \frac{-i}{2}\sum_{jk} \partial_k(\overline{e_j}^* e_j^* e_k - e_k \overline{e_j}^* e_j^*).$$

Apply the formula that $e_k e_j^* + e_j^* e_k = 2\delta_{jk} Id$, and we obtain

$$= -\frac{i}{2}\sum 2\partial_k \delta_{jk} \overline{e_j}^* = -i\sum_{j=1}^{n} \partial_j \overline{e_j}^* = i\overline{\partial}^*.$$

□

Now this is where the magic starts.

Proposition 20.4. *If X is Kahler, then the laplace operators coincide:*

$$\frac{1}{2}\Delta = \square = \overline{\square}.$$

Note this means Δ restricts to $A^{p,q}(X) \to A^{p,q}(X)$.

Proof. First, note that we have the Kahler identites:

$$[\Lambda, \overline{\partial}] = -i\partial^*, [\Lambda, \partial] = i\overline{\partial}^* \implies$$
$$i\Lambda\overline{\partial} - i\overline{\partial}\Lambda = \partial^*$$
$$i\partial\Lambda - i\Lambda\partial = \overline{\partial}^*.$$

Now note that since $d^2 = (\partial + \overline{\partial})^2 = 0$, we have $\partial\overline{\partial} = -\overline{\partial}\partial$. Then

$$\square = \partial\partial^* + \partial^*\partial = \partial[i\Lambda\overline{\partial} - i\overline{\partial}\Lambda] + [i\Lambda\overline{\partial} - i\overline{\partial}\Lambda]\partial = i\partial\Lambda\overline{\partial} + i\overline{\partial}\partial\Lambda - i\Lambda\partial\overline{\partial} - i\overline{\partial}\Lambda\partial$$
$$= \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^* = \overline{\square}.$$

Furthermore, we have

$$\Delta = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial})$$
$$= (\partial + \overline{\partial})(\partial^* + i\partial\Lambda - i\Lambda\partial) + (\partial^* + i\partial\Lambda - i\Lambda\partial)(\partial + \overline{\partial})$$
$$= \partial\partial^* - i\partial\Lambda\partial + \overline{\partial}\partial^* + i\overline{\partial}\partial\Lambda - i\overline{\partial}\Lambda\partial + \partial^*\partial + \partial^*\overline{\partial} + i\partial\Lambda\partial + i\partial\Lambda\overline{\partial} - i\Lambda\partial\overline{\partial}.$$
$$= \partial\partial^* + i\overline{\partial}\partial\Lambda - i\overline{\partial}\Lambda\partial + \partial^*\partial + i\partial\Lambda\overline{\partial} - i\Lambda\partial\overline{\partial},$$

noting from the Kahler identities that $\overline{\partial}\partial^* = -\partial^*\overline{\partial}$. Then

$$\Delta = \square + \overline{\square} \implies \frac{1}{2}\Delta = \square = \overline{\square}.$$

$\square$

Challenge: find some intuition for all this. Is there a converse? Not sure. Suspcion is no there is not a converse at least in low dimensions, because there's Hodge decomposition for some surfaces despite not being Kahler. Unclear though. It would be really nice to have some intuition for this whole calculation, maybe from some other field. It could be very valuable. Another corollary:

Proposition 20.5. *Let X be compact oriented smooth manifold. Then*

$$[\Delta, *] = 0.$$

If X is Kahler, then furthermore we have

$$[\Delta, L] = [\Delta, \Lambda] = 0.$$

This implies that the operators $L, \Lambda, *$ all preserve harmonic forms.

Proof. First we show $[\Delta, *] = 0$. Note that $\Delta = dd^* + d^*d$, and $d^* = (-1)^{k(2n-k)+k+1} * d*$. So $\Delta = (-1)^{k(2n-k)+k+1}[d * d * + * d * d]$. It suffices to show then that

$$d * d * * + *d * d* = *d * d * + * * d * d.$$

But indeed, $d * d * * = d * d(-1)^{k(2n-k)} = d * d(-1)^k$, and $* * d * d = (-1)^{(2n-k)k} = (-1)^k d * d$, so they agree. (Note, this doesn't use X is Kahler).

Note that

$$\langle(\Delta L - L\Delta)\alpha, \beta\rangle_X = \angle\alpha, (\Lambda\Delta - \Delta\Lambda)\beta\rangle_X$$

so $[\Delta, L] = 0 \iff [\Delta, \Lambda] = 0$. Thus, it suffices to prove that $[\Delta, L] = 0$. Since X is Kahler, it suffices to prove that $[\Box, L] = 0$. From the Kahler identities, we know that

$$[\Lambda, \partial] = i\overline{\partial}^* \implies [\Lambda, \partial]^* = \partial^* L - L\partial^* = -i\overline{\partial}.$$

Thus, $L\partial^* = \partial^* L + i\overline{\partial}$. Then

$$L\Box = L\partial\partial^* + L\partial^*\partial = L\partial\partial^* + \partial^* L\partial + i\overline{\partial}\partial.$$

Note $d\omega = 0 \iff \partial\omega, \overline{\partial}\omega = 0$. So $\partial L\alpha = \partial(\omega \wedge \alpha) = \omega \wedge \partial\alpha = L\partial\alpha$. Thus, $L\partial = \partial L$. Then

$$L\Box = L\partial\partial^* + \partial^* L\partial + i\overline{\partial}\partial = \partial\partial^* L + i\partial\overline{\partial} + \partial^*\partial L + i\overline{\partial}\partial = (\partial\partial^* L + \partial^*\partial L) = \Box L,$$

where we note $\partial\overline{\partial} + \overline{\partial}\partial = 0$ since $d^2 = 0$. $\square$

Proposition 20.6. *The associated $(1,1)$-form ω is harmonic. Because $\omega = L(1)$, because 1 is clearly harmonic, and L preserves harmonic forms.*

Note that holomorphic forms are $\overline{\partial}$-harmonic. Then on Kahler manifolds, holomorphic forms are also d-harmonic.

Theorem 20.7. *Let X be compact Kahler. Then there exist a decomposition on the level of harmonic forms*

$$\mathcal{H}^k \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

where the left hand side is d-harmonic forms, and the right hand side is comprised of $\overline{\partial}$-harmonic forms.

Proof. Let $\alpha \in \mathcal{H}^k \otimes_{\mathbb{R}} \mathbb{C}$. We can write $\alpha = \sum \alpha^{p,q}$, where $\alpha^{p,q} \in A^{p,q}(X)$. But note that $\Delta\alpha = 0$, and since X is Kahler, Δ restricts to $A^{p,q}(X) \to A^{p,q}(X)$, so $\Delta\alpha = 0 \implies \Delta\alpha^{p,q} = 0$. Then since each $\alpha^{p,q}$ are d-harmonic, they are also $\overline{\partial}$-harmonic.

Conversely, given a tuple of $\overline{\partial}$-harmonic forms $\sum \alpha^{p,q}$, we have $\Delta\alpha = \sum \Delta\alpha^{p,q} = 0$. $\square$

A priori, a drawback of this decomposition is that it implicitly depends on the metric. This dependence of the metric comes from determining which forms are harmonic, which comes from construction an adjoint to our specified differential operator, which comes from the Hodge $*$-operator and its reference to the metric. We've shown before that $\mathcal{H}^k \cong H^k(X; \mathbb{C})$ and $\mathcal{H}^{p,q} \cong H^{p,q}(X)$. If we fix a Kahler metric, given a topological cohomology class $[\alpha] \in H^k(X; \mathbb{C})$, there is a unique harmonic representative α. This α decomposes into a direct sum of (p,q) forms: $\alpha = \bigoplus \alpha^{p,q}$, and each $\alpha^{p,q}$ is harmonic as well since X is Kahler so Δ restricts to (p,q) forms. Then each $\alpha^{p,q}$ will give some cohomology class in $H^{p,q}(X)$. We want this to be well-defined. If we started with the same cohomology class $[\alpha] \in H^k(X; \mathbb{C})$, if we have a different metric we'll have a different harmonic representative α, and we want to know that in the decomposition $\bigoplus \alpha^{p,q}$, each bigraded element will remain in the same Dolbeaut cohomology class.

To make things simpler, we'll prove it in the other direction.

$$\begin{tikzcd}
\mathcal{H}^k_{\Delta_1,g_1} \arrow[d] & \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\overline{\Box}_1,g_1} \arrow[l] \\
H^k(X;\mathbb{C}) & \bigoplus_{p+q=k} H^{p,q}(X) \arrow[l] \arrow[u] \arrow[d] \\
\mathcal{H}^k_{\Delta_2,g_2} \arrow[u] & \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\overline{\Box}_2,g_2} \arrow[l]
\end{tikzcd}$$

Note that apriori, all arrows except the middle are well-defined isomorphisms. Our task is to show that following along the top or the bottom arrows give the same middle arrow.

Theorem 20.8. *Let X be compact Kahler. Then the decomposition*

$$H^k(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

does not depend on the choice of metric.

Proof. Suppose we have two Kahler metrics on X. Begin with some dolbeaut cohomology class in $H^{p,q}(X)$. Then with respect to the two Kahler metrics, we obtain two $\overline{\partial}$-harmonic representatives α_1, α_2. Our task is to show that α_1, α_2 will give the same cohomology class in $H^k(X;\mathbb{C})$. Let us fix our metric to be g_1, the metric α_1 is harmonic with respect to.

We have $\alpha_2 = \alpha_1 + \overline{\partial}\gamma$. But we also have that $\alpha_2 = \beta + \overline{\Box}\eta'$ where β is harmonic. Note that we can write

$$\alpha_2 = \beta + \overline{\partial}\overline{\partial}^*\eta + \overline{\partial}^*\overline{\partial}\eta,$$

and note $\overline{\partial}\alpha_2 = \overline{\partial}\beta = 0$. Thus, $\overline{\partial}\overline{\partial} * \overline{\partial}\eta = 0$. But $\ker \overline{\partial} \cap Im\overline{\partial}^* = 0$. So we must have $\alpha_2 = \beta + \overline{\partial}\overline{\partial}^*\eta$. But this implies that

$$\alpha_1 - \beta = \overline{\partial}[\overline{\partial}^*\eta - \gamma],$$

so by injectivity we must have $\alpha_1 = \beta$. This implies that $\overline{\partial}\gamma$ is in the image of $\overline{\Box}$. But then this implies that $\overline{\partial}\gamma$ is in the image of Δ.

Note that $d\alpha_2 = d\alpha_1 = 0$ since they are harmonic (and using Kahler property that all the laplacians coincide). Then $d\overline{\partial}\gamma = 0$. Furthermore,

$$\overline{\partial}\gamma = dd^*\xi + d^*d\xi$$

and this implies $dd^*d\xi = 0$. But $\ker d \cap Im(d) = 0$. So $d^*d\xi = 0$. Thus, $\overline{\partial}\gamma = dd^*\xi$. This implies that $\overline{\partial}\gamma \in d(A^{k-1})$. Thus, α_1, α_2 are the same cohomology class in $H^k(X;\mathbb{C})$. $\Box$

Keyword for splitting of $A^{p,q}$ and A^k into laplacian and image: Green operator.

21. 11/20/23: Hodge diamonds and examples

Recall the Hodge decomposition of a compact Kahler X:

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

The hodge numbers are defined to be $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$. We have

$$b_k(x) = \sum_{p+q=k} h^{p,q}(X).$$

This is one of the most fundamental equalities in algebraic geometry.

Proposition 21.1. *If X is compact Kahler, then $b_{2k+1}(X) = 0 \mod 2$.*

Proof. By the hodge decomposition,

$$b_{2k+1}(X) = \sum_{p+q=2k+1} h^{p,q}(X).$$

But note that $h^{p,q} = h^{q,p}$ because conjugation induces isomorphism $\mathcal{H}^{p,q}_{\overline{\partial}} \cong \mathcal{H}^{q,p}_{\partial}$, and X being Kahler implies $\mathcal{H}^{q,p}_{\partial} \cong \mathcal{H}^{q,p}_{\overline{\partial}}$. But note that if $p+q = 2k+1$, we always have pairs. So indeed $b_{2k+1}(X) = \sum_{p \le j} 2h^{p,q} = 0 \mod 2$. ☐

Proposition 21.2. *$h^{k,k}(X) \neq 0$ for every k. So $b_{2k}(X) \neq 0$ for every $k \le n$.*

Proof. ω kahler form is harmonic $\implies$ $\omega^{\wedge k}$ harmonic for every k. Furthermore, Wirtinger's implies $0 \neq [\omega^{\wedge k}] \in H^{2k}$. ☐

Example 21.3 (Compact complex surfaces)**.** Suppose X is a compact complex surface. We know that $h^{0,0} = h^{2,2} = 1$. Note by Poincare duality, we know that $b_1 = b_3$. If X was Kahler, then $b_1 = b_3$ must be even. Surprisingly, the converse is also true. That is, a compact complex surface X is Kahler $\iff$ b_1 is even!

- This was originally a conjecture of Kodaira, and was first shown by using the Enriques-Kodaira classification, with the final case of K3 surfaces done by Siu in 1983. In 1999, Buchdal and Lamari independently gave direct proofs which did not rely on the classification.

As $b_1 = h^{1,0} + h^{0,1}$ for compact complex surfaces (Chapter IV Theorem 2.7(i) in Compact Complex Surfaces by Barth, Hulek, Peters, Van de Ven) we can restate the result as: let X be a compact complex surface. then X is Kahler if and only if $h^{1,0} = h^{0,1}$.

Further discussion/reference: https://math.stackexchange.com/questions/145920/compact-complex-surfaces-with-h1-0-h0-1/1668216#1668216.

Example 21.4 (Hopf surface)**.** Consider the Hopf surface

$$\mathbb{Z} \times (\mathbb{C}^2 \setminus \{0\}) \to \mathbb{C}^2 \setminus \{0\}$$

where for $\lambda \in \mathbb{R}, 0 < \lambda < 1$,

$$(k, (z_1, z_2)) \mapsto (\lambda^k z_1, \lambda^k z_2).$$

Because this action is properly discontinuous and fixed point free, the quotient X of this action on $\mathbb{C}^2 \setminus \{0\}$ is a compact complex surface.

As a smooth manifold, the Hopf surface is diffeomorphic to $S^3 \times S^1$. Then by the Kunneth formula,

$$H^k(S^3 \times S^1; \mathbb{R}) \cong \bigoplus_{p+q=k} H^p(S^3; \mathbb{R}) \otimes H^q(S^1; \mathbb{R}).$$

Then $b_k = 1$ for $k = 0, 1, 3, 4$ and 0 for all other betti numbers. We see that the Hopf surface is far from being Kahler. The betti numbers b_1, b_3 are odd instead of even. And b_2 is 0 when it would need to be nonzero.

Let us mention the $\partial\overline{\partial}$-lemma. This can be thought of as an analog to the $\overline{\partial}$-Poincare lemma and its role in Dolbeaut cohomology calculations. Similarly, the $\partial\overline{\partial}$-lemma is important for something called *Bott-Chern cohomology*. On compact Kahler manifolds, Bott-Chern agrees with Dolbeaut. But on non-Kahler manifolds, Bott-Chern can provide additional information.

Proposition 21.5. *Let X be a compact Kahler manifold, $\alpha \in A^K(X)$ such that $\partial\alpha = \overline{\partial}\alpha$. If α is also ∂-exact or $\overline{\partial}$-exact, then there exists $\beta \in A^{k-2}(X)$ such that*

$$\alpha = \partial\overline{\partial}\beta.$$

Proof. WLOG suppose α is $\overline{\partial}$-exact, so we can write $\alpha = \overline{\partial}\gamma$. Note that $\gamma = \eta + \Delta\vee$, with $\Delta\eta = 0$. Note X Kahler implies $\overline{\Box}\eta = 0 \implies \overline{\partial}\eta = 0$. So

$$\alpha = \overline{\partial}\gamma = \overline{\partial}\eta + 2\overline{\partial}(\partial\partial^* + \partial^*\partial)\vee = 0 + 2\overline{\partial}(\partial\partial^* + \partial^*\partial)\vee$$

where we use $\Delta = 2\Box$. Note also that $\overline{\partial}\partial = -\partial\overline{\partial}, \overline{\partial}\partial^* = -\partial^*\overline{\partial}$. So this equals

$$-2\partial\overline{\partial}\partial^* \vee -2\partial^*\overline{\partial}\partial \vee .$$

We want this to be zero. We know this because

$$\partial\alpha = 0 \implies \partial\partial^*\overline{\partial}\partial\vee = 0 \implies \partial^*\overline{\partial}\partial\vee \in Im(\partial^*) \cap \ker\partial = \{0\}$$

where the last part is because $(\partial\partial^*\beta, \beta) = ||\partial^*\beta||_X^2$. Use also $\overline{\partial}\alpha = 0$. □

Proposition 21.6. *Let X be compact Kahler. Let $\alpha \in A^k(X)$ be $\overline{\partial}$ or ∂ exact, and suppose $\partial\alpha = \overline{\partial}\alpha = 0$. Then there exist $\beta \in A^{k-2}(X)$ such that*

$$\alpha = \partial\overline{\partial}\beta.$$

Proof. WLOG suppose α is $\overline{\partial}$-exact. Then we can write $\alpha = \overline{\partial}\eta$. We can write $\eta = \beta + \Delta\gamma$ where β is harmonic. Then using $2\Delta = \overline{\Box}$, note that since β is harmonic this also means $\overline{\partial}\beta = 0$. Then

$$\alpha = \overline{\partial}\eta = 0 = 2\overline{\partial}\Box\gamma = 2\overline{\partial}(\partial\partial^* + \partial^*\partial)\gamma.$$

Note that $\overline{\partial}\partial = -\partial\overline{\partial}$, and using the Kahler identity $[\Delta, \overline{\partial}] = -i\partial^*$, we get $\overline{\partial}\partial^* = -\partial^*\overline{\partial}$. Then we get

$$\alpha = -2\partial\overline{\partial}\partial^*\gamma - 2\partial^*\overline{\partial}\partial\gamma$$
$$\implies \partial\alpha = 0 = -2\partial\partial^*\overline{\partial}\partial\gamma.$$

But $\ker\partial \cap Im\partial^* = 0$, by observing $\langle\partial\partial^* v, v\rangle = ||\partial^* v||_X^2$. So $\partial^*\overline{\partial}\partial\gamma = 0$, so

$$\alpha = -2\partial\overline{\partial}\partial^*\gamma.$$

□

Recommended reading, appendix in Huybrechts: formality of Kahler manifolds.

Example 21.7 (Compact Riemann Surfaces)**.** A compact riemann surface of genus g has $\pi_1 = \mathbb{Z}^{2g}$. This is because for each hole, we have a loop around and a loop through. In general, H_1 is the abelianization of the fundamnetal group. So by this fact and Poincare-Alexander duality we have

$$H_1(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

We have $H^1 \cong H^{1,0} \oplus H^{0,1}$, and these are isomorphic. So this is why we call $H^{1,0} \cong H^0(X, \Omega_X) = g$ the genus.

For any smooth algebraic variety, one defines its geomtric genus to be $p_g(X) = \dim_k H^0(X, \omega_X)$. For analytic spaces, singular things, one often looks at $H^{0,1}$ defined to be the arithmetic genus.

For compact riemann surfaces, the Hodge diamond is $1, g, g, 1$.

Example 21.8 (compact Kahler surface)**.** Look at hodge diamond:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

Have $h^{0,0} = h^{2,2} = 1$ (remember we're always assuming our manifolds are connected). Have $h^{p,q} = h^{q,p}$. But also by Serre duality, have $h^{p,q} = h^{n-p,n-q} = h^{n-q,n-p} = h^{q,p}$. So $h^{1,0} = h^{0,1} = h^{1,2} = h^{2,1}$. A ton of symmetry in Hodge diamond. But we don't necessarily know what

$$h^{1,0} = h^{0,1} = h^{1,2} = h^{2,1} =: q(x)$$

is. This quantity $q(x)$ is called the **"irregularity of X"**. Note $h^{1,0} = h^0(X, \Omega^1_X)$, number of linearly independent global holomorphic forms. Also equal to $h^1(X, \mathcal{O}_X)$.

We haven't touched the middle row. Here comes the geometric genus. Have $h^{2,0} = h^{0,2} = p_g(X)$, geometric genus $= h^0(X, \omega_X)$. Have $h^{1,1}(X) > 0$.

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ p_g & & h^{1,1} > 0 & & p_g \\ & g & & g & \\ & & 1 & & \end{array}$$

The story doesn't end here. For specific surfaces, you can find inequalities between these quantities. For example, if you have a surface that doesn't have any fibrations over curves of genus at least 2, then you get something like $p_g \geq 2g - 3$. here.

Example 21.9 ($\mathbb{P}^2$)**.** Have $g(\mathbb{P}^2) = 0, p_g(\mathbb{P}^2) = 0$. Have $h^{1,1}(\mathbb{P}^2) = 1$.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Example 21.10. If X is a torus of dimension 2. Then the tangent bundle T_X is trivial. So Ω^1_X is trivial. So Ω^p_X trivial for every p. So

$$q(x) = h^0(X, \Omega^1_X) = h^0(X, \mathcal{O}_X^{\oplus 2}) = 2.$$

And $p_g(X) = h^0(X, \omega_X) = h^0(X, \bigwedge^2 \Omega_X) = h^0(X, \mathcal{O}_X) = 1$.

Have $h^{1,1}(X) = h^1(X, \Omega^1_X) = 2h^1(X, \mathcal{O}_X) = 2q = 4$. So hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

Tangent bundle of torus is trivial.

Example 21.11 (projective space)**.**

Lemma 21.12. *$H^i(\mathbb{P}^n, \mathbb{Z})$ is $\mathbb{Z}$ where $0 \leq i \leq 2n$ is even, and 0 when i is odd.*

Proof. Induct on n:

$$\mathbb{P}^n = \mathbb{P}^{n-1} \cup \mathbb{C}^n.$$

Note $\mathbb{C}^n$ is contractible, so $H_i(\mathbb{C}^n, \mathbb{Z}) = 0, \forall i > 0$. We have a Poincare-Alexander duality, which says that for pairs $(\mathbb{P}^n, H)$, get

$$H^i(\mathbb{P}^n, H; \mathbb{Z}) \cong H_{2n-i}(\mathbb{P}^n \setminus H, \mathbb{Z}).$$

The standard Poincare duality takes H to be empty set. Using the standard exact sequence in singular cohomology, get

$$\cdots \to H^i(\mathbb{P}^n, H; \mathbb{Z}) \to H^i(\mathbb{P}^n, \mathbb{Z}) \to H^i(H, \mathbb{Z}) \to H^{i+1}(\mathbb{P}^n, H; \mathbb{Z}) \to \cdots$$

So $i \leq 2n-2 \implies H^i(\mathbb{P}^n, \mathbb{Z}) \cong H^i(\mathbb{P}^{n-1}, \mathbb{Z})$ using induction. Also $H^{2n-1}(\mathbb{P}^n, \mathbb{Z}) = 0$ and $H^{2n}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$. $\square$

We know $h^{k,k}(\mathbb{P}^n)0$. Get $h^{k,k}(\mathbb{P}^n) = 1$, for all k. And $h^{p,q}(\mathbb{P}^n) = 0, p \neq q$. In hodge diamond, 1 down the middle, 0 everywhere else.

If you have $f : X \to Y$ surjective holomorphic map and X Kahler. If you look at $f^* : H^k(X;\mathbb{R}) \hookrightarrow H^k(Y;\mathbb{R})$. Betti numbers of X have to be at least as small as betti numbers of Y. This shows it is very hard to have a surjection $\mathbb{P}^n \to Y$!! The betti numbers of $\mathbb{P}^n$ are so small, that they're as small as they can get. "Smallest hodge diamond" entry by entry that you can get. It is a very hard theorem that if $\mathbb{P}^n$ surjects to another manifold, then it must be another $\mathbb{P}^n$.

By Bezout's theorem, can't get maps $\mathbb{P}^n \to C$ where C curve. But when Y is arbitrary dimension, it's much harder. Varieties of general type, that have exactly the same Betti numbers as $\mathbb{P}^n$, they're called "fake projective spaces." Same hodge diamond as $\mathbb{P}^n$. This theorem that $\mathbb{P}^n \to X$ surjection means X must be $\mathbb{P}^n$ uses Mori's technique.

Note just from the topology, we learned a lot. We learned $H^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) = 0$, for all $p > 0$. Have $H^p(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0, \forall p > 0$.

Example 21.13 (Complex tori)**.** Compact complex tori $X \cong \mathbb{C}^n/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2n}$ is lattice. Have projection $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda$. Have

$$\pi^* : A^{p,q}(X) \hookrightarrow A^{p,q}(\mathbb{C}^n)$$

forms that are invariant under translation by elements in Λ. Standard Kahler form is

$$\omega = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z_j}.$$

In $\mathbb{P}^n$, it was convenient to study the topology. In complex tori case, it's easier to study forms. This ω descends to a Kahler form ω on X.

Lemma 21.14. *With respect to the induced metric h on X, we have*

$$\Delta(\sum_{J,K} \phi_{JK} dz_J \wedge d\overline{z_k}) = \sum_{J,K} (\Delta\phi_{JK}) dz_J \wedge d\overline{z}_K$$

where Δ on the right hand side is $-\sum_{j=1}^n(\frac{\partial^2 f}{\partial x_j^2} + \frac{\partial 62 f}{\partial y_j^2}$, so to check harmonicity just need to check functions are harmonic.

Corollary: the space of harmonic forms on the torus

$$\mathcal{H}^{p,q}(X) = \{\sum_{|J|=p,|K|=q} a_{JK} dz_J \wedge d\overline{z_K} | a_{JK} \in \mathbb{C}\}.$$

Harmonic $\iff$ coefficients harmonic $\iff$ function on torus is constant. Know that $\mathcal{H}^0(X) \cong H^0(X;\mathbb{R}) \cong \mathbb{R}$. Maybe a more humble proof is that bounded harmonic functions on all of $\mathbb{C}^n$ are constant.

So $H^1(X,\mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X)$ where $H^{1,0}(X) = \mathbb{C}\langle dz_1, \cdots, dz_n\rangle$, $H^{0,1}(X) = \mathbb{C}\langle d\overline{z_1}, \cdots, d\overline{z_n}\rangle$.

Singular cohomology of torus determined by first cohomology via wedging:

$$H^k(X,\mathbb{C}) \cong \bigwedge^k H^1(X,\mathbb{C}) \cong \bigwedge^k (H^{1,0}(X) \oplus H^{0,1})$$

. So $b_i(X) = \binom{2n}{i}$ for every i. And the hodge decomposition becomes

$$H^k(X,\mathbb{C}) \cong \bigwedge^p H^{1,0} \otimes \bigwedge^q H^{0,1}.$$

So $h^{p,q}(X) = \binom{n}{p}\binom{n}{q}$. So the special thin gabout torus was that the harmonic functions were constant. And cohomology determined by H^1.

So for $n = 2$.

True calabi yau, need $H^{1,0} = 0$. Simply connected. The thing that people consider in physics. Quartic surface in $\mathbb{P}^3$ will be Calabi Yau, K3 surfaces.

22. 11/27/23: HODGE NUMBERS OF PROJECTIVE HYPERSURFACE

Today we'll compute the hodge numbers of a hypersurface in $\mathbb{P}^n$. This is a practical setting for a circle of ideas due to Lefschetz. We will talk about the Lefschetz hyperplane theorem, otherwise known as the weak Lefschetz theorem, which makes precise the notion that the geometry of a hyperplane section should reflect the geometry of the ambient variety. We will also see the utility of the Lefschetz decomposition and primitive cohomology as a way of getting a handle on the cohomology of varieties.

The Lefschetz hyperplane theorem that we'll see is a part of a more general principle: the geometry (cohomology, homology, homotopy groups) of a general hyperplane section of a variety refelcts the geometry (cohomology, homology, homotopy groups) of that variety.

The proof of hodge numbers of projective hypersurface is a bit too long for the remaining days we have left, but even the statement requires a lot of effort. Some hypersurfaces will be K3 surfaces, some Calabi Yau, some general type – lots of examples here.

Let $X = Z(F) \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$, where F is homogenous of degree d.

Remark 22.1. Note that we are not losing any information here by discussing hypersurfaces which are cut out by a homogenous polynomial F. By the Cousin problem discussion, an analytic hypersurface is cut out by a single global holomorphic function. We will discuss next semester Chow's theorem, which will show that, in fact, hypersurfaces are algebraic – so the global analytic function is actually a polynomial.

Note that X is Kahler since it is a submanifold of $\mathbb{P}^{n+1}$. Its Kahler 1,1 form can be taken to be the pullback of the Fubini-Study 1,1 form.

Lemma 22.2. *The restriction $H^k(\mathbb{P}^{n+1}, \mathbb{C}) \to H^k(X, \mathbb{C})$ is injective for every k.*

Proof. We have $H^k(\mathbb{P}^n; \mathbb{C}) \cong \mathbb{C}$ for $0 \le k \le 2n+2$ even and 0 for k odd. For $k = 2j$ even, note $H^{2j}(\mathbb{P}^n, \mathbb{C}) \cong \mathbb{C}[\omega_{FS}^j]$, and the map on cohomologies is given by

$$\omega_{FS} \mapsto \iota^*\omega_{FS},$$

which is nonzero by Wirtinger's. □

This leads to something called the **Lefschetz hyperplane theorem** or **weak Lefschetz** theorem.

Theorem 22.3. *The restrictions*

$$H^k(\mathbb{P}^{n+1}, \mathbb{C}) \to H^k(X, \mathbb{C})$$

are isomorphisms for $0 \le k < n$.

We know what the cohomology groups of $\mathbb{P}^{n+1}$ are. Then by Poincare duality, we know all the cohomology groups of X except the middle cohomology $H^n(X;\mathbb{C})$.

Remark 22.4. We will prove the Weak Lefschetz theorem next semester, but we will do the Hard Lefschetz theorem this semester. We will also prove the Kodaira vanishing theorem, an analytic statement, and show that this topological statement is a special case of Kodaira vanishing. You can also do Morse theory to prove this topological statement (Weak Lefschetz) and then prove Kodaira (in Lazarsfield Positivity). But we will prove Kodaira first.

Assuming weak Lefschetz, we know that all the cohomologies $H^k(X,\mathbb{C})$ are 0 or $\mathbb{C}$, and we are left with the middle cohomology $H^n(X,\mathbb{C})$. By Lefschetz decomposition theorem, we have

$$H^n(X;\mathbb{C}) = H_0^n(X;\mathbb{C}) \oplus LH_0^{n-2}(X;\mathbb{C}) \oplus \cdots \oplus L^{\lfloor n/2 \rfloor} H_0^{n-2\lfloor n/2 \rfloor}(X;\mathbb{C}).$$

Note that $H_0^{n-k}(X;\mathbb{C}) \cong H^{n-k}(\mathbb{P}^{n+1};\mathbb{C})$ by the Lefschetz hyperplane theorem. But note that since the Kahler form on X is $\iota^*\omega_{FS}$, the restriction of the Lefschetz operator on $\mathbb{P}^{n+1}$ to X is the Lefschetz operator on X. By the theory of operators and their adjoints, since $H^{n-k}(X;\mathbb{C})$ is entirely generated by $\iota^*\omega_{FS}$, we have $H_0^{n-k}(X;\mathbb{C}) = 0$. This implies that

$$H^n(X;\mathbb{C}) \cong H_0^n(X;\mathbb{C}) \oplus H^n(\mathbb{P}^{n+1};\mathbb{C}).$$

Since the Lefschetz decomposition is compatible with the Hodge decomposition, we obtain

$$H_0^n(X,\mathbb{C}) = H_0^{n,0} \oplus H_0^{n-1,1} \oplus \cdots$$

Then it suffices to understand primitive dolbeaut cohomology.

Theorem 22.5 (Griffiths). *In this setting, for every p we have*

$$H_0^{p,n-p} \cong \frac{A^{n+1}(X, n+1-p)}{A^{n+1}(X, n-p) + dA^n(X, n-p)}$$

where d is the exterior derivative and $A^k(X,\ell)$ = rational k-forms on $\mathbb{P}^{n+1}$ with a pole of order at most ℓ along X and smooth everywhere else.

Note on $\mathbb{P}^n$, rational functions are ratios of polynomials. Terminology: let $z_1, \cdots, z_{n+1}$ be local coordinates for $\mathbb{C}^{n+1}$ and $\omega = \sum_I \frac{P_I(z_1,\cdots,z_n)}{Q_I(z_1,\cdots,z_n)} dz_1 \wedge \cdots \wedge dz_{n+1}$, where ω has poles along the zeroes of Q_I (after simplification). Now we want to do this in $\mathbb{P}^{n+1}$. For example if we identify $\mathbb{C}^{n+1}$ with $\{[z_0 : \cdots : z_n] | z_0 \neq 0\}$, we homogenize ω so that $z_i \mapsto \frac{z_i}{z_0}$.

Example 22.6. Consider a top form $(k = n+1)$, $\omega = \frac{P(z_0,\cdots,z_{n+1})}{Q(z_0,\cdots,z_{n+1})}\omega'$, where

$$d\left(\frac{z_i}{z_0}\right) = \frac{z_0 dz_i - z_i dz_0}{z_0^2}.$$

Homogenzing the fraction of polynomials gives

$$\frac{z_0^{\deg Q} P(z_0, \cdots, z_{n+1})}{z_0^{\deg P} Q(z_0, \cdots, z_{n+1})}.$$

Then under the identification $z_i \mapsto \frac{z_i}{z_0}$,

$$dz_1 \wedge \cdots \wedge dz_{n+1} \mapsto \frac{1}{z_0^{n+2}}\omega'$$

where

$$\omega' = \sum_{j=0}^{n+1} (-1)^j z_j dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1}.$$

So homogenizing gives

$$\frac{z_0^{\deg Q} P}{z_0^{\deg P} Q} \frac{1}{z_0^{n+2}} \omega'.$$

Have to have that the $\deg P + n + 2 = \deg Q$ to get a well-defined rational function on $\mathbb{P}^n$. Furthermore, we want poles of order at most ℓ along the hypersurface X. So $Q = F^\ell$, where $X = Z(f)$.

So a rational form

$$\omega = \frac{P(z_0, \cdots, z_n)}{Q(z_0, \cdots, z_{n+1}} \omega'$$

of degree $n+1$ with poles of order at most ℓ along Z must have the condition that $\deg P + n + 2 = \deg Q$, $Q = F^{\leq \ell}$.

If $\deg F = d, \deg Q = \ell d$, this means $\deg P = \ell d - (n+2)$. So $A^{n+1}(X, n+1-p) \cong \{\frac{P}{F^{n+1-p}}\omega' | P$ has homogenous polynomial of degree $\ell d - (n+2)\} \cong \mathbb{C}[X_0, \cdots, X_{n+1}]_{(n_1-p)d-(n+2)}$.

Now we'll do $k = n$. The calculations are similar:

$$v = \sum_{0 \leq j < k \leq n+1} (-1)^{j+k} \frac{z_k P_j - z_j P_k}{F^\ell} dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1}$$

where P_j homogenous polynomial of degree $\ell d - (n+1)$. Looking at $dA^n(X, \ell)$, you get

$$d\vee = \frac{F \sum_j \frac{\partial P_j}{\partial z_j} - \ell \sum_j P_j \frac{\partial F}{\partial z_j}}{F^{\ell+1}} \omega'.$$

Recall $A^{n+1}(X, n+1-p) \cong S_{(n+1-p)d-(n+2)}$. So

$$A^{n+1}(X, n-p) + dA^n(X, n-p) \cong \sum_{j=0}^{n+1} S_{(n-p)d-(n+1)} \frac{\partial F}{\partial z_j} + S_{(n-p)d-(n+2)} F.$$

Definition 22.7. Let $F \in \mathbb{C}[Z_0, \cdots, Z_{n+1}]$. The Jacobian ideal of F is

$$\mathcal{J}(F) := \langle \frac{\partial F}{\partial z_0}, \cdots, \frac{\partial F}{\partial z_{n+1}} \rangle.$$

This jacobian ideal is very important, because it describes the singular locus of X. Important in commutative algebra as well. When F is homogenous, then $\mathcal{J}(F)$ becomes a homogenous ideal with generators of degree $d-1$.

Remark: F homogenous of degree d implies that $F \in \mathcal{J}(F)$ because of Euler formula, $dF = \sum_{j=0}^{n+1} z_j \frac{\partial F}{\partial z_j}$.

Definition 22.8. The Jacobian algebra of F is $R(F) = S/\mathcal{J}(F)$.

The corollary is that

Proposition 22.9. $H_0^{p,n-p} \cong R(F)_{(n+1-p)d-(n+2)}$.

Try to read about Griffiths-Harris residues in higher dimension, actually proves this isomorphism. We'll accept this for now, and do examples.

Example 22.10. Let $X \subseteq \mathbb{P}^2$ be a hyperplane of degree d. These are special among Riemann surfaces, because not every compact Riemann surface can be embedded in $\mathbb{P}^2$. Every smooth projective variety of dimension n can be embedded in $\mathbb{P}^{2n+1}$. So curves can be embedded in $\mathbb{P}^3$ (hartshorne, secants and tangents), but there may be singularities if you project to $\mathbb{P}^2$.

Have $H^1(X, \mathbb{C}) = H^1(\mathbb{P}^2, \mathbb{C}) \oplus H_0^1(X, \mathbb{C}) \cong 0 \oplus H_0^{1,0} \oplus H_0^{0,1}$. Have

$$H_0^{1,0} = R(F)_{d-3}.$$

So in degree $d-3$, this Jacobain ideal is generated by degree $d-1$ polynomials. So doesn't hit anything. So $R(F)_{d-3} = S_{d-3}$. So this has dimension $\frac{(d-1)(d-2)}{2}$. But primitive $H_0^{1,0}$ is $H^{1,0}$ but this is the genus of the curve. But we've obtained the hyperclassical fact of the genus of a curve.

Example 22.11 (hypersurface)**.** Have $X \subseteq \mathbb{P}^3$, $\deg F = d$. We're interested in

$$H^2(X, \mathbb{C}) = H^2(\mathbb{P}^3, \mathbb{C}) \oplus H_0^2(X, \mathbb{C}) = \mathbb{C}[\omega_{FS}] \oplus H_0^{2,0} \oplus H_0^{1,1} \oplus H_0^{0,2}.$$

So $H_0^{2,0}$ and $H_0^{0,2}$ are conjugate to each other. So we need to apply the Griffiths formula for $H_0^{2,0}$ and $H^{1,1}$. By Lefschetz hyperplane theorem, have $H^1(X, \mathbb{C}) \cong H^1(\mathbb{P}^2, mathbbC) = 0$.

Simplest example of K3 surface (2 dimensional calabi yau) is plane quartic. Have $H_0^{2,0} = R(F)_0 = S_0 \cong \mathbb{C}$. Have $H_0^{1,1} = R(F)_4$. Not as simple because you start with $\deg F = 4$, and deg of partials is 3. So can generate something of degree 4. But that's exactly how you do it. Linear combinations of partials give poly of degree 4 in Jacobain ideal. So $H_0^{1,1} = \frac{S_4}{\sum \frac{\partial F}{\partial z_j} S_1}$. The dimension $h_0^{1,1} = \binom{4+3}{3} - 4\binom{1+3}{3} = 19$.

Hodge diamond is:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

This is the Hodge diamond for every K3 surface. The 19 is very relevant. Comes from primitive cohomology, something about intersection cohomology and signature. This hodge diamond tells us that $b_1 = b_3 = 0, b_2 = 22$ for K3 surfaces.

Note if you have some submersion $X \to B$, hodge numbers for fibers is constant. Ehrsmann theorem. Upper semicontinuity. We'll discuss this later at some point.

So if you want to compute hodge numbers for hypersurface of degree d, or anything deformation equivalent to hypersurface of degree d, it's enough to compute hodge number for Fermat hypersurface $X_0^d + \cdots + X_n^d = 0$.

Popa says: for fun, look at quintic threefold in $\mathbb{P}^4$. Quintic threefold, dimension 3 degree 5 in $\mathbb{P}^4$, is the most important calabi yau manifold. Mirror symmetry purposes lol.

Plan for next two days: get Lefschetz decomposition and then hard lefschetz theorem. The Hard Lefschetz theorem is a culmination of everything we've done this semester.

23. 11/29/23: PRIMITIVE COHOMOLOGY, LEFSCHETZ DECOMPOSITION, HARD LEFSCHETZ THEOREM, HODGE-RIEMANN BILINEAR RELATIONS

"It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry."

- Solomon Lefschetz.

We talked about the Hodge decomposition. Today we'll talk about the Lefschetz decomposition for compact Kahler manifolds, which reduces cohomology classes to primitive forms. Underlying the Lefschetz decomposition is a sl_2-representation. This will give us a handle on how the Λ acts on forms. But we'll also be able to get information about the Lefschetz operator, namely the Hard Lefschetz theorem.

Let X be compact Kahler of dimension n. Recall that the Lefschetz operator $L : A^k(X) \to A^{k+2}(X)$ given by $L(\alpha) = \omega \wedge \alpha$. With respect to the metric on forms, L admitted an adjoint $\Lambda : A^{k+2}(X) \to A^k(X)$ where $\Lambda = (-1)^k * L *$.

Lemma 23.1. *$H := [L, \Lambda] = (k - n)Id$ on $A^k(X)$.*

Proof. Proof philosophy is similar to Kahler identities proof in that you reduce to the Euclidean identities and use the fact that a Kahler metric osculates to euclidean metric to order 2. ∎

The exact same theory of elliptic operators/functional analysis which gave the decomposition of $A^k(X)$ as the kernel and image of the laplacian gives us a decomposition

$$A^k(X) = \ker \Lambda \oplus Im(L).$$

Definition 23.2. We say $\alpha \in A^k(X)$ is primitive if $\Lambda\alpha = 0$.

Then for every $\alpha \in A^k(X)$, we can write $\alpha = \alpha_0 + L\beta_0$ where $\alpha_0 \in A^k(X)$ is unique (why?) and primitive, and $\beta_0 \in A^{k-2}(X)$.

We can continue to apply the decomposition so that

$$\beta_0 = \alpha_1 + L\beta_1, \beta_1 = \alpha_2 + L\beta_2, \cdots$$

So $\alpha = \alpha_0 + L\alpha_1 + L^2\alpha_2 + \cdots + L^{\lfloor k/2 \rfloor}\alpha_{\lfloor k/2 \rfloor}$ where the α_j primitive. A priori, only the α_0 is uniquely determined by α, and we're not sure whether the other α_i are uniquely determined. But in fact, all these α_j are uniquely determined by α, and this will be our next goal.

Lemma 23.3. *For $\alpha \in A^{n-\ell}(X)$ is primitive, then for every $k \geq 1$ we have:*

$$\Lambda L^k \alpha = k(\ell - k + 1)L^{k-1}\alpha, \ell \in \mathbb{Z}$$

and if $\ell < 0$, then $\alpha = 0$.

Proof. We have $\Lambda\alpha = 0$ and $H = L\Lambda - \Lambda L$, so $H\alpha = (n - \ell - n)\alpha = -\ell\alpha$. This takes care of the $k = 1$ case. Now induct on k. We gave

$$\Lambda L^{k+1}\alpha = (L\Lambda - H)L^k\alpha = Lk(\ell - k + 1)L^{k-1}\alpha - (-\ell + 2k)L^k\alpha =$$
$$= (k\ell - k^2 + k + \ell - 2k)L^k\alpha = (k + 1)(\ell - k)L^k\alpha.$$

Now we show that if $\ell < 0$, then $\alpha = 0$. Certainly if $\ell < 0$ then $L^{n+1}\alpha = 0$. Then define

$$k_0 := \min\{k | L^k\alpha = 0\}.$$

We'd like to show that $k_0 = 0$. Assume FSOC that $k_0 > 0$. We have

$$\Lambda L^k\alpha = k_0(\ell - k_0 + 1)L^{k_0-1}\alpha.$$

Since $L^k\alpha = 0$, and $\ell - k_0 + 1 \neq 0$, this implies that $L^{k_0-1}\alpha = 0$, contradicting the minimality of k_0. So we must have $k_0 = 0$. $\square$

As a corollary, we obtain the following:

Proposition 23.4. *For nonzero primitive $\alpha \in A^{n-\ell}(X)$:*

$$\alpha, L\alpha, \cdots, L^\ell\alpha \neq 0, \text{ and } L^{\ell+1}\alpha = 0.$$

Proof. From the lemma, we obtain

$$\Lambda L^{\ell+1}\alpha = (\ell+1)(\ell-(\ell+1)+1)L^\ell\alpha = 0 \implies \beta = L^{\ell+1}\alpha \in \ker\Lambda \cap Im(L) = \{0\}.$$

This shows that $L^{\ell+1}\alpha = 0$.

Now we want $\alpha, L\alpha, \cdots, L^\ell\alpha \neq 0$. It's enough to show that $L^\ell\alpha \neq 0$. Well, note that from our formula, we get that

$$\Lambda^\ell L^\ell\alpha = (\ell!)^2\alpha \neq 0,$$

so $L^\ell\alpha$ could not possibly be zero.

$\square$

Proposition 23.5. *For every $\alpha \in A^k(X)$, the decomposition*

$$\alpha = \alpha_0 + L\alpha_1 + \cdots + L^{\lfloor k/2 \rfloor}\alpha_{\lfloor k/2 \rfloor}$$

where α_j primitive in $A^{k-2j}(X)$ is unique. Note that really the decomposition is: $\sum_{\max\{k-n,0\}}^{\lfloor k/2 \rfloor} L^j\alpha_j$.

Proof. First suppose $\alpha = 0$. We want to show $\alpha_j = 0, \forall j$. We have

$$0 = \Lambda\alpha = \Lambda\alpha_0 + \Lambda L\alpha_1 + \cdots.$$

By the lemma, we know that $\Lambda L^k\gamma = k(\ell - k + 1)L^{k-1}\gamma$ where $\gamma \in A^{n-\ell}(X) \implies$

$$0 = \sum_{j=\max\{k-n,0\}}^{\lfloor k/2 \rfloor} j(j+n-k+1)L^{j-1}\alpha_j.$$

This writes $c\alpha_1 = L(--)$ for some nonzero constant c. Then $\alpha_1 \in Im(L) \cap \ker\Lambda = \{0\} \implies \alpha_1 = 0$. Now we're done with α_1, then apply Λ again to get $\alpha_2 = 0$. Inductively, get all $\alpha_j = 0$. $\square$

We've obtained a well-defined decomposition of $A^k(X)$ in terms of $\ker\Lambda$'s and images of L. But note that both of these operator Λ and L preserve harmonic forms, and each cohomology class in $H^k(X, \mathbb{C})$ has a unique harmonic representative.

Theorem 23.6 (Lefschetz Decomposition)**.** *let X be a compact Kahler manifold of dimension n with Kahler form ω induces L and Λ. Then for every k, every cohomology class $\alpha \in H^k(X, \mathbb{C})$ admits a unique decomposition*

$$\alpha = \sum_{j=\max\{k-n,0\}}^{\lfloor k/2 \rfloor} L^j\alpha_j$$

where $\alpha_j \in H^{k-2j}(X,\mathbb{C})$ *are primitive cohomology. In other words,*

$$H^k(X,\mathbb{C}) = H_0^k(X,\mathbb{C}) \oplus LH_0^{k-2}(X,\mathbb{C}) \oplus \cdots$$

Proof. Let $[\alpha] \in H^k(X,\mathbb{C})$. There is a unique harmonic representative α, which admits a decomposition

$$\alpha = \sum_{j=\max\{k-n,0\}}^{\lfloor k/2 \rfloor} L^j \alpha_j.$$

Then note that $\Lambda^{\lfloor k/2 \rfloor}\alpha = c\alpha_{\lfloor k/2 \rfloor}$, so $\Delta\Lambda^{\lfloor k/2 \rfloor}\alpha = \Lambda^{\lfloor k/2 \rfloor}\Delta\alpha = 0 = c\Delta\alpha_{\lfloor k/2 \rfloor}$. Thus, $\alpha_{\lfloor k/2 \rfloor}$ is harmonic. Then note that

$$\Lambda^{\lfloor k/2 \rfloor - 1}\alpha = c\alpha_{\lfloor k/2 \rfloor - 1} + c' L\alpha_{\lfloor k/2 \rfloor},$$

and applying Δ, commutativity of Δ with Λ and L and harmonicity of $\alpha_{\lfloor k/2 \rfloor}$ imply that $\alpha_{\lfloor k/2 \rfloor - 1}$ is harmonic. We can proceed in this fashion, and show that all α_j are harmonic. Thus we obtain well-defined decomposition. □

This is actually a sl_2-representation on $H^*(X)$. More discussion on the Lie theory here: https://mathoverflow.net/questions/14667/intuition-for-primitive-cohomology.

Example 23.7. Recall our example when we studied primitive cohomology of hypersurfaces in $\mathbb{P}^{n+1}$. We had

$$H^n(X,\mathbb{C}) = H^n(\mathbb{P}^{n+1},\mathbb{C}) \oplus H_0^n(X,\mathbb{C}) \cong \mathbb{C}[\omega_{FS}^{\wedge k}] \oplus H_0^n(X,\mathbb{C})$$

where $n = 2k$. The map from $H^n(\mathbb{P}^{n+1},\mathbb{C}) \to H^n(X,\mathbb{C})$ is simply $[\omega_{FS}^{\wedge k}] \mapsto [\omega_{FS}|_X^{\wedge k}]$. Thus, we see that our example from last time is just a special case of the general Lefschetz decomposition. We see every element of $H^n(X,\mathbb{C})$ is in the image of the Lefschetz operator.

The Lefschetz decomposition is compatible with the Hodge decomposition. This is because $L(A^{p,q}) \subseteq A^{p+1,q+1}$, and recalling the behavior of the Hodge $*$-operator with respect to (p,q)-forms, we see $\Lambda(A^{p+1,q+1}) \subseteq A^{p,q}$. This means that if η is primitive, then each $\eta^{p,q}$ is primitive as well. If we have a Lefschetz decomposition,

$$\alpha = \sum_{j=\max\{k-n,0\}}^{\lfloor k/2 \rfloor} L^j \alpha_j,$$

then

$$\alpha^{p,q} = \sum_{j=\max\{k-n,0\}}^{\lfloor k/2 \rfloor} L^j \alpha_j^{p-j,q-j}.$$

$$
\begin{tikzcd}
\alpha^{k,0} \arrow[d, dash] & & & & \\
\alpha_0^{k,0} \arrow[d, dash] & \ddots & & & \\
L\alpha_1^{k,0} \arrow[d, dash] & & \alpha^{p,q} \arrow[d, dash] & & \\
\vdots \arrow[d, dash] & & \alpha_0^{p,q} \arrow[d, dash] & \ddots & \\
L^j\alpha_j^{k,0} & & L\alpha_1^{p,q} \arrow[d, dash] & & \alpha^{0,k} \arrow[d, dash] \\
 & & \vdots \arrow[d, dash] & & \alpha_0^{0,k} \arrow[d, dash] \\
 & & L^j\alpha_j^{p,q} & & L\alpha_1^{0,k} \arrow[d, dash] \\
 & & & & \vdots \arrow[d, dash] \\
 & & & & L^j\alpha_j^{0,k}
\end{tikzcd}
$$

An instant consequence of the Lefschetz decomposition is a very central duality, called the *Hard Lefschetz theorem*. This is the culmination of our entire semester:

Theorem 23.8 (Hard Lefschetz Theorem)**.** *Let X be a compact Kahler manifold of dimension n. Let $k \leq n$. Then*

$$L^{n-k} : H^k(X;\mathbb{C}) \to H^{2n-k}(X;\mathbb{C})$$

is an isomorphism.

Proof. First we verify surjectivity. Let $\alpha \in H^{2n-k}(X;\mathbb{C})$. The lefschetz decomposition yields

$$\alpha = \sum_{j=n-k}^{\lfloor n-\frac{k}{2}\rfloor} L^j \alpha_j,$$

and we see immediately that α is in the image of L^{n-k}. Now we verify injectivity. Let $\alpha \in H^k(X;\mathbb{C})$ such that $L^{n-k}\alpha = 0$. Lefschetz decomposition yields

$$\alpha = \sum_{j=0}^{\lfloor k/2\rfloor} L^j \alpha_j \implies L^{n-k}\alpha = \sum_{j=0}^{\lfloor k/2\rfloor} L^{n-k+j}\alpha_j = 0.$$

Note that since $\alpha_0 \in H^k(X;\mathbb{C})$, then $L^{n-k}\alpha_0 \neq 0$ for nonzero α_0. But if we apply Λ^{n-k}, then we get α_0 is in the image of L, which implies $\alpha_0 = 0$. Similarly, since $\alpha_1 \in H^{k-2}(X;\mathbb{C})$, we have $L^{n-k+1}\alpha_1 \neq 0$ for nonzero α_1. But if we apply another Λ, we get α_1 is in the image of L, so we get $\alpha_1 = 0$. Proceeding in this fashion yields $\alpha_j = 0 \forall j \implies \alpha = 0$. □

The combination of Hodge decomposition and Lefschetz decomposition told us how Λ actually interacts with cohomology classes of forms. But in fact, we also get insight into how the Lefschetz operator acts via the Hard Lefschetz theorem!

Note we knew before that these cohomology groups were isomorphic by Poincare duality. But here the Lefschetz operator also gives an isomorphism. In fact, these will be related up to a sign.

Remark: primitive cohomology is a major reduction. It has something called a polarized Hodge structure. Before that, we need to talk about the Hodge-Riemann bilinear relations.

The Hard Lefschetz theorem told us that $L^{n-k} : H^k(X) \to H^{n-k}(X)$ was an isomorphism. This reminds us of Poincare duality. Note that $L^{n-k}\alpha$ is of type $(n-q, n-p)$. Note that $*\alpha$ is also of this type. This suggests that there is some relationship between powers of the Lefschetz operator and the Hodge $*$-operator.

Lemma 23.9. *For primitive $\alpha \in A^{p,q}(X)$, we have*

$$*\alpha = \frac{(-1)^{\frac{k(k+1)}{2}} i^{p-q}}{(n-k)!} L^{n-k}\alpha.$$

Proof. Reduction to Euclidean metric then calculate. □

It is tempting to believe that maybe on $H^k(X;\mathbb{C})$ there is some relationship between the Hodge $*$ operator and powers of the Lefschetz operator. But this is not generally true. Let X be compact Kahler. Note that for Poincare duality, we simply needed X to be compact, orientable, and smooth. But with the additional structure of X being Kahler, we can decompose $H^k(X;\mathbb{C})$ via Hodge and Lefschetz decomposition, which are compatible. As a result, we can decompose $H^k(X;\mathbb{C})$ into $H_0^{p,q}(X)$. We only know that powers of the Lefschetz operator and the Hodge $*$ operator agree on the primitive (p,q) forms. On the other pieces of the Lefschetz decomposition of $H^{p,q}(X)$, it is unclear, since $*$ and L do not commute.

So the isomorphism maps in Poincare duality and the Hard Lefschetz theorem are quite different, although they agree on primitive cohomologies up to constants.

This relation but difference between powers of the Lefschetz operator and the Hodge $*$-operator leads to the Hodge-Riemann bilinear relations. When X was smooth, compact, orientable, then for we had a metric on smooth k forms $\alpha, \beta \in A^k(X)$ such that

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta.$$

What happens in the setting where, furthermore, X is compact Kahler and we replace $*$ with L^{n-k}?

Definition 23.10. For $0 \leq k \leq n$, define bilinear forms $Q : A^k(X) \times A^k(X) \to \mathbb{R}$ such that

$$Q(\alpha, \beta) = (-1)^{\frac{k(k+1)}{2}} \int_X \omega^{\wedge(n-k)} \alpha \wedge \beta.$$

Note that Q is real-valued because α, β are smooth k-forms here, and the Kahler form ω is also a smooth 2-form.

Note that $Q(\alpha, \beta) = (-1)^k Q(\beta, \alpha)$, so Q is symmetric when k is even and Q is alternating when k is odd. Furthermore, when $\alpha, \beta \in A^{p,q}$ are primitive and $p + q = k$, we have

$$\langle \alpha, \beta \rangle_X = \int_X \alpha \wedge *\overline{\beta} = \frac{(-1)^{k(k+1)/2} i^{q-p}}{(n-k)!} \int_X \alpha \wedge L^{n-k}\overline{\beta} = \frac{i^{q-p}}{(n-k)!} Q(\alpha, \overline{\beta}).$$

Theorem 23.11 (Hodge-Riemann bilinear relations)**.** *Let X be compact Kahler. Let $Q : A^k(X) \times A^k(X) \to \mathbb{R}$ be the bilinear form as defined before, and note we can extend Q $\mathbb{C}$-bilinearly.*

- *Let $p + q = x + y = k$. Then $H^{p,q}(X)$ and $H^{x,y}(X)$ are orthogonal with respect to $Q(\alpha, \beta)$ unless $(x, y) = (q, p)$. Thus, the Hodge decomposition is orthogonal with respect to $Q'(\alpha, \beta) = Q(\alpha, \overline{\beta})$. Note Q' here is sesquilinear and not necessarily hermitian.*
- *(Positivity) Let $0 \neq \alpha \in H^{p,q}$ be primitive. Then $i^{q-p}Q'(\alpha, \alpha) = i^{q-p}Q(\alpha, \overline{\alpha}) > 0$.*

Proof. Let's show that the Hodge decomposition $H^k(X; \mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(X)$ is orthogonal with respect to $Q'(\alpha, \beta) = Q(\alpha, \overline{\beta}) = (-1)^{k(k+1)/2} \int_X L^{n-k}\alpha \wedge \overline{\beta}$. But this follows immediately from keeping track of (p, q) indices on $L^{n-k}\alpha \wedge \overline{\beta}$.

The positivity statement follows immediately from the fact that

$$(n-k)!\langle \alpha, \alpha \rangle_X = i^{q-p} Q(\alpha, \overline{\alpha}) > 0$$

by positivity of $\langle , \rangle_X$. □

The Hodge-Riemann bilinear relations is technically nothing new, but somehow we are expressing it in a way that emphasizes the role of primitive cohomology. It's going to have a lot of consequences and it's so important that we've formalized it. Primitive forms carry something called polarized structures. Next time, we'll show how Hodge Riemann bilinear relation works on curves and surfaces.

Example 23.12. Let X be a Riemann surface (n=1). Then $H^1(X; \mathbb{C}) \cong H^{1,0} \oplus H^{0,1}$. Note that $H^{1,0}$ is primitive. Let $\alpha \in H^{1,0}$. Then $\alpha = [f dz]$. Then we can check positivity by seeing that

$$i^{-1}(-1) \int \alpha \wedge \overline{\alpha} = i \int |f|^2 dz \wedge d\overline{z} = 2 \int |f|^2 dx \wedge dy > 0.$$

Later, we'll see that a consequence of the positivity of $i^{q-p}Q'$ is the Hodge index theorem, a very deep result.

24. 12/4/23: Hodge Index Theorem, Hodge structures, polarizations, end of semester 1

Note: next semester will be modeled after Griffiths and Harris Chapter 1. Today we'll discuss further the Hodge-Riemann bilinear relations and its consequences, specifically polarization.

Let X be a compact Kahler manifold of dimension n. Consider $0 \leq k \leq n$. Have

$$Q : A^k(X) \times A^k(X) \to \mathbb{R}$$

where $Q(\alpha,\beta) = (-1)^{\frac{k(k+1)}{2}} \int_X \omega^{\wedge(n-k)} \wedge \alpha \wedge \beta$. It will be convenient to write $\underline{Q}(\alpha,\beta) := \int_X \omega^{\wedge(n-k)} \wedge \alpha \wedge \beta$. Recall that if $\alpha, \beta \in A^{p,q}(X)$ are primitive and $p+q=k$, then

$$\langle \alpha, \beta \rangle_X = \int_X \alpha \wedge *\overline{\beta} = \frac{i^{q-p}}{(n-k)!} Q(\alpha, \overline{\beta}) = \frac{i^{q-p}}{(n-k)!} Q'(\alpha, \beta)$$

by using the fact that $*\alpha = (-1)^{\frac{k(k-1}{2}} \frac{i^{p-q}}{(n-k)!} L^{n-k}\alpha$ when α is a primitive (p,q) form. Recall the Hodge-Riemann bilinear relations:

- In $H^k(X,\mathbb{C})$, the subspaces $H^{p,q}$ and $H^{p',q'}$ are orthogonal with respect to Q unless $p'=q, q'=p$.
- If $0 \neq \alpha \in H^{p,q}(X)$ primitive, then $i^{p-q}Q(\alpha,\overline{\alpha}) > 0$.

Here is a consequence of the Hodge-Riemann bilinear relations: the Hodge Index theorem for surfaces.

Theorem 24.1 (Hodge Index for Surfaces)**.** *Let X be a compact Kahler surface (n=2). Then the* intersection pairing

$$H^2(X;\mathbb{R}) \times H^2(X;\mathbb{R}) \to \mathbb{R}$$

where $([\alpha],[\beta]) \mapsto \int_X \alpha \wedge \beta$ *has signature*

$$(2h^{2,0}(X)+1, h^{1,1}(X)-1).$$

Proof. The Hodge decomposition yields

$$H^2(X;\mathbb{C}) \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}.$$

Consider the subspace $(H^{2,0} \oplus H^{0,2}) \cap H^2(X;\mathbb{R})$. We see that this must be $\{\alpha^{2,0} + \alpha^{0,2} | \overline{\alpha^{0,2}} = \alpha^{2,0}\}$. Furthermore, we have

$$\int_X \alpha \wedge \alpha = 2\int_X \alpha^{2,0} \wedge \alpha^{0,2} = 2\int_X \alpha^{2,0} \wedge \overline{\alpha^{2,0}} > 0$$

by Hodge-Riemann relations, specifically positivity of $i^{0-2}(-1)^{2(3)/2} \int_X \alpha^{2,0} \wedge \overline{\alpha^{2,0}}$. Then choosing a basis for $H^{2,0}$, we see immediately that we have a contribution of $h^{2,0}(X)$ in the positive signature.

Now we examine $H^{1,1}(X)$. This has Lefschetz decomposition

$$H^{1,1} \cong H_0^{1,1} \oplus \mathbb{C}\omega.$$

Let us first describe $H_0^{1,1}$. We claim that

$$H_0^{1,1} = \{\alpha \in H^{1,1} | \int_X \omega \wedge \alpha = 0.\}$$

- Forward direction: suppose $\Lambda\alpha = 0$. Note that $\Lambda = *L*$, and so $\Lambda\alpha = *L*\alpha = 0$. But note that since α is a primitive (p,q) form, we have $*\alpha = \frac{(-1)^{2(3)/2}}{(2-2)!}(i)^{1-1}L^{2-2}\alpha = -\alpha$. So $\Lambda\alpha = -*L\alpha = 0$, which forces $L\alpha = 0$. So clearly $\int_X \omega \wedge \alpha = 0$.
- Reverse direction: suppose $\alpha \in H^{1,1}$ such that $\int_X \omega \wedge \alpha = 0$. We'd like to show that $\Lambda\alpha = 0$. By Lefschetz decomposition we can write $\alpha = \alpha_0 + C\omega$ where α_0 is primitive. Then $\int_X \omega \wedge (\alpha_0 + C\omega) = C\int \omega^{\wedge 2} = Cvol(X)$, where we use the fact $\int_X \omega \wedge \alpha_0 = 0$ since α_0 is primitive. Note our resultant can only be zero unless $C = 0$, in which case α is primitive.

Note that this also implies that $H_0^{1,1}$ and $\mathbb{C}\omega$ are orthogonal with respect to the intersection pairing, since $\int \omega \wedge \alpha$ where α is primitive is 0.

Note that $\int \omega \wedge \omega = vol(X) > 0$, so we get a positive contribution of $+1$. Note that $\omega \in H^2(X;\mathbb{R})$.

Now we focus on $H_0^{1,1}$. Note we showed that if $\alpha \in H_0^{1,1}$, then $L\alpha = 0$. Then $L\overline{\alpha} = 0$, so $\overline{\alpha} \in H_0^{1,1}$. Now we claim that $\alpha = \overline{\alpha}$, so that $H_0^{1,1} \subset H^2(X;\mathbb{R})$. To see this, note that

$$\langle L(\alpha - \overline{\alpha}), \eta \rangle = 0$$

for every $\eta \in H^{2,2}$ is zero. This is because $H^{2,2}$ has Lefschetz decomposition $L^2 H_0^{0,0}$, so

$$\langle L(\alpha - \overline{\alpha}), L^2\xi \rangle = \langle (\alpha - \overline{\alpha}), \Lambda L^2 \xi \rangle = \langle (\alpha - \overline{\alpha}), L\Lambda L\xi \rangle = 0,$$

noting that $[\Lambda, L] = (n-k)Id$. This proves that $H_0^{1,1} \subset H^2(X;\mathbb{R})$. Furthermore, we have that for $\alpha \in H_0^{1,1}$, noting that $*\alpha = -\alpha$, we have

$$\int \alpha \wedge \alpha = -\int \alpha \wedge (-\alpha) = -\int \alpha \wedge *\alpha = -\int \alpha \wedge *\overline{\alpha} = -\langle \alpha, \alpha \rangle_X < 0.$$

Thus, the signature is

$$(h^{2,0}(X) + 1, h^{1,1}(X) - 1).$$

□

The Hodge Index theorem is a very deep result. We have used both the Lefschetz and Hodge decomposition. A priori, this intersection pairing seems to only be utilizing data from the underlying smooth manifold. But the result is something which is related to the complex structure of the Kahler manifold. Somehow this pairing is just cup product (the $\underline{Q}$ map is related to Poincare dual map on middle cohomology), but something like the hodge numbers pops out. Middle cohomology is special.

More generally, if $n = 2k$, then

$$H^{2k}(X, \mathbb{R}) \times H^{2k}(X, \mathbb{R}) \to \mathbb{R}$$

where $[\alpha], [\beta] \mapsto \int_X \alpha \wedge \beta = \underline{Q}(\alpha, \beta)$ is non-degenerate symmetric bilinear form.

Theorem 24.2. *The signature of X is*

$$sgn(X) := sgn(\underline{Q}) = \sum_{p,q=0}^{2k} (-1)^p h^{p,q}(X).$$

Proof. For example, if X is a surface, then

$$\begin{aligned} sgn(X) = h^{0,0} + h^{0,1} + h^{0,2} - h^{1,0} - h^{1,1} - h^{1,2} \\ + h^{2,0} + h^{2,1} + h^{2,2} = 2 + 2h^{2,0} - h^{1,1}. \end{aligned}$$

The proof uses Lefschetz decomposition + HR bilinear relations + a lot of calculations. Look in Huybrechts page 130, or Griffiths Harris. □

Beyond surfaces, signature doesn't come up a lot. Sometimes try to find finer invariants than the signature for higher dimensional geometry. Nevertheless, the signature is still interesting and useful for the study of surfaces.

Now we will talk a bit about Hodge structures.

Definition 24.3. A pure rational (integral) Hodge structure of weight k is a $\mathbb{Q}$-vector space (free abelian group) $H_{\mathbb{Q}}$ ($H_{\mathbb{Z}}$) together with a decomposition

$$H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{p+q=k} H^{p,q}.$$

satisfying $H^{p,q} = \overline{H^{q,p}}$.

Example 24.4. $V_{\mathbb{Q}}$ vector space such that $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes_{\mathbb{R}} \mathbb{R}$ has an almost complex structure. Then we've seen that $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$.

Example 24.5. If X compact Kahler manifold, then $H^k(X,\mathbb{Z})$ carries an integral Hodge structure of weight k, and $H^k(X,\mathbb{Q})$ carries a rational Hodge structure of weight k. Have hodge decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}.$$

Example 24.6. If Kahler form $[\omega] \in H^2(X,\mathbb{Q})$ then $H^k_0(X,\mathbb{Q})$ carries a hodge structure of weight k. So $H^k_0(X,\mathbb{Q}) = \oplus_{p+q=k} H^{p,q}_0(X)$.

We'll see that next semester, the Kodaira embedding theorem, which says that $[\omega] \in H^2(X,\mathbb{Z}) \iff X$ is smooth projective variety. This is the most important way of distinguishing the projective varieties from all compact Kahler manifolds.

If you move a complex manifold in family (deformation of complex structure), then the $H^{p,q}$ spaces are not well-behaved, they do not vary holomorphically. But the hodge filtration is well-behaved, which is one reason why we study them:

Definition 24.7. If $H_{\mathbb{Q}}$ carries a Hodge structure of weight k, then

$$F^m H_{\mathbb{C}} := \bigoplus_{p \geq m} H^{p,q}.$$

We obtain

$$F^k H_{\mathbb{C}} \subseteq F^{k-1} H_{\mathbb{C}} \subseteq \cdots \subseteq F^0 H_{\mathbb{C}} = H_{\mathbb{C}}.$$

This is called the *Hodge filtration.*

The Hodge filtration recovers the Hodge structure because $H^{p,k-p} = F^p H_{\mathbb{C}} \cap \overline{F^{k-p} H_{\mathbb{C}}}$. This is the beginning of the theory of variations of Hodge structures.

Remark: $H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{k-p+1} H_{\mathbb{C}}}$.

The intersection pairing on surfaces that gave the Hodge index theorem is a polarization – it polarizes the manifold. It's a form on the manifold that lets you intersect cycles.

Definition 24.8. An integral polarized Hodge structure of weight k is a $\mathbb{Z}$-Hodge structure of weight k, together with an "intersection form"

$$\underline{Q} : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$$

such that

- $\underline{Q}$ is symmetric when k is even, and skew-symmetric when k is odd.
- The Hodge decomposition $H_{\mathbb{C}} = \bigoplus H^{p,q}$ is orthogonal with respect to $\underline{Q}_{\mathbb{C}}(\alpha, \overline{\beta})$.
- Have $i^{p-q}(-1)^{\frac{k(k-1)}{2}} \underline{Q}(\alpha,\alpha) > 0$ for all nonzero $\alpha \in H^{p,q}$.

To get a hermitian form on $\underline{Q}_{\mathbb{C}}(\alpha, \overline{\beta})$, sometimes people write $S(\alpha,\beta) = i^k Q_{\mathbb{C}}(\alpha, \underline{\beta})$. So sometimes when people define polarizations on Hodge structures, people will normalize by i^k to get a Hermitian form. Very confusing and something to watch out for. The main example:

Example 24.9. Say X is compact Kahler with $[\omega] \in H^2(X, \mathbb{Q})$ (equivalent to X being smooth projective variety), then take

$$H_{\mathbb{Q}} = H_0^k(X, \mathbb{Q})$$

has Hodge structure of weight k. Then by the Hodge Riemann bilinear relations, have

$$\underline{Q}(\alpha, \beta) = \int_X \omega^{\wedge(n-k)} \wedge \alpha \wedge \beta, Q(\alpha, \beta) = (-1)^{\frac{k(k-1)}{2}} \underline{Q}(\alpha, \beta)$$

so $\underline{Q}(\alpha, \beta)$ gives polarization on $H_{\mathbb{Q}}$ (or again, sometimes confusingly, people will say $S(\alpha, \beta) = i^k \underline{Q}(\alpha, \beta)$ gives polarization).

Note we can define polarization over rationals as well.

Example 24.10. Let $\mathbb{Z}(k)$ denote the unique Hodge structure of weight $-2k$ and dimension 1 where $k \in \mathbb{Z}$. So we have $\mathbb{C}(k) = H^{-k,-k}$.

For example, look at $H^{2k}(\mathbb{P}^n, \mathbb{Z})$ is 1-dimensional and has weight $2k$. So this is the same as $\mathbb{Z}(-k)$.

Suppose we have H, H' two $\mathbb{Z}$-Hodge structures of weights k, ℓ. Then

$$H_{\mathbb{Z}} \otimes_{\mathbb{Z}} H'_{\mathbb{Z}}$$

carries a $\mathbb{Z}$-Hodge structure of weight $k + \ell$. So

$$(H_{\mathbb{Z}} \otimes_{\mathbb{Z}} H'_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H_{\mathbb{C}} \otimes_{\mathbb{C}} H'_{\mathbb{C}}$$
$$= \bigoplus_{p+q=k+\ell} (H \otimes H')^{p,q} = \bigoplus_{p'+p"=p, q'+q"=q} H^{p',q'} \otimes H^{p",q"}.$$

Example 24.11. Let X, X' be compact Kahler manifolds. Then

$$H^k(X \times X', \mathbb{Z}) = \bigoplus_{r+s=k} H^r(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^s(X', \mathbb{Z})$$

.

Example 24.12.

$H^k(X \times \mathbb{P}^1, \mathbb{Z}) = H^k(X, \mathbb{Z}) \oplus H^{k-2}(X, \mathbb{Z}) \otimes H^2(\mathbb{P}^1, \mathbb{Z}) = H^k(X, \mathbb{Z}) \oplus H^{k-2}(X, \mathbb{Z})(-1)$.

The twist is called a "Tate twist."

In general, if H is a hodge structure, then

$$H(k) := H \otimes \mathbb{Z}(k)$$

is called the k-th Tate twist of H. This has enhancements to something called motivic operations. Popa says: wait, replace everything about this Tate twist discussion to $\mathbb{Q}$. Go back here and make edits.

For X smooth projective variety, $[\omega] \in H^2(X, \mathbb{Q})$, for $0 \le k \le n$, the Lefschetz decomposition gives us

$H^k(X, \mathbb{Q}) = H_0^k(X, \mathbb{Q}) \oplus LH_0^{k-2}(X, \mathbb{Q}) \oplus \cdots = H_0^k(X, \mathbb{Q}) \oplus H_0^{k-2}(X, \mathbb{Q})(-1) H_0^{k-4}(X, \mathbb{Q})(-2) \oplus \cdots .3$

Define $S : H_0^k(X, \mathbb{Q}) \times H_0^k(X, \mathbb{Q}) \to \mathbb{Q}$. So $S = S_k \oplus S_{k-2} \oplus S_{k-4} \oplus \cdots$

25. 1/22/24: Semester 2, Weil Divisors, Cartier Divisors, Line Bundles

Last semester, we ended with the Lefschetz decomposition and the Hodge-Riemann bilinear relations. The Hodge-Riemann bilinear relations, in particular, showed that we had polarized Hodge structures on primitive cohomology.

Our next big goal is to prove the Kodaira embedding theorem, which says that a compact Kahler manifold X is projective $\iff$ its Kahler form $[\omega] \in H^2(X;\mathbb{Z})$ is in integral cohomology. In order to work our way towards the Kodaira embedding theorem, we will need to talk about some differential (connections and curvature) and algebrogeometric notions (divisors and line bundles). The next few classes will be devoted to the algebrogeometric notions.

Last semester we saw some holomorphic line bundles $\pi : L \to X$ on a complex manifold X. We defined them by means of trivializations. Letting $\{U_i\}$ be an open cover of X, we had trivializations

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$$

and over overlaps $U_i \cap U_j$, we obtained holomorphic transition functions

$$g_{ij} : \phi_j \circ \phi_i^{-1} : U_i \cap U_j \to \mathbb{C} = GL_1(\mathbb{C}).$$

Specifying L was the same as specifying these U_i and these transitions functions $g_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$ which satisfy the cocyle condition. We saw that we can express all of this in terms of Cech cohomology. We have the Picard group

$$Pic(X) = \{\text{isomorphism classes of line bundles}\}$$

is isomorphic to $H^1(X, \mathcal{O}_X^\times)$. The Picard group is a group under tensor product of line bundles, where the transition functions are just products of the transition functions $g_{ij}g'_{ij}$. The inverse to a line bundle $\mathcal{L} = \{U_i, g_{ij}\}$ is $\mathcal{L}^{-1}$ given by the data of $\{U_i, g_{ij}^{-1}\}$.

This is a sheaf cohomological interpretation of the Picard group of X. Recall also that the data of a vector bundle is equivalent to a locally free sheaf, given by taking its sheaf of sections. Studying vector bundles via their sheaf of sections is often a useful translation.

Example 25.1. The sheaf of sections of a trivial line bundle $X \times \mathbb{C} \to X$ is simply $\mathcal{O}_X$.

The sheaf of sections of the tangent bundle TX on X is the sheaf of vector fields on X.

The sheaf of sections of Ω_X^1 (dual of holomorphic tangent bundle) are sheaves of holomorphic one-forms, and in general, the sheaf of sections of Ω_X^p are sheaves of holomorphic p-forms. The global section of Ω_X^p is $A^{p,0}(X)$.

Example 25.2. If X is a 1 dimensional complex manifold, TX and Ω_X^1 are holomorphic line bundles.

If X is n-dimensional, then $\omega_X = \Omega_X^n = \bigwedge^n \Omega_X$ is called the canonical line bundle. Its sheaf of sections is the sheaf of top holomorphic forms.

The canonical line bundle is called "canonical" for a reason; it is one of the most important things you can understand about a variety.

One can also consider $\omega_X^{\otimes m}$ where $m \in \mathbb{Z}$. In the theory of Riemann surfaces, sections of these are called differentials of the m-th kind.

Example 25.3 (Determinant Bundle)**.** Given a rank r vector bundle E on X, the determinant bundle is defined to be

$$\det E := \bigwedge^r E.$$

It is called the determinant bundle because note that if $\{g_{ij} : U_i \cap U_j \to GL_r(\mathbb{C})\}$ are the transition functions for E, then the transition functions for $\bigwedge^r E$ are

$$\{\det g_{ij} : U_i \cap U_j \to GL_1(\mathbb{C})\}.$$

Recall the exponential exact sequence on a complex manifold X

$$0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

which induces a long exact sequence in cohomology

$$H^1(X, \underline{\mathbb{Z}}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \underline{\mathbb{Z}}_X) \to H^2(X, \mathcal{O}_X) \to \cdots$$

When X is compact (and we assume connected here), $H^0(X, \underline{\mathbb{Z}}_X) \cong H^0(X, \mathcal{O}_X) \cong \mathbb{Z}$, so $H^0(X, \mathcal{O}_X^\times) = 0$. So when X is compact and connected, the above long exact sequence starts with a 0. We've identified that $H^1(X, \mathcal{O}_X^*) = Pic(X)$. The map

$$c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X; \underline{\mathbb{Z}}_X) \cong H^2(X; \mathbb{Z})$$

is called the first Chern class map. Note that if $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, then we immediately have that the first Chern class map c_1 is an isomorphism.

Example 25.4. Consider $\mathbb{P}^n$. This is compact and Kahler via the Fubini-Study metric. Furthermore, $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0$. One can compute this using the standard sheaf cohomology techniques, or by appealing to the Hodge number calculations we did for $\mathbb{P}^n$ and using Dolbeaut's theorem.

From the above discussion, we immediately see that

$$Pic(\mathbb{P}^n) \cong H^2(\mathbb{P}^n; \mathbb{Z}) \cong \mathbb{Z}.$$

This implies that the only line bundles of $\mathbb{P}^n$ are the $\mathcal{O}_{\mathbb{P}^n}(m)$ for $m \in \mathbb{Z}$.

We can also immediately deduce the Picard group of $\mathbb{C}^n$, since $H^1(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \cong H^{0,1}(\mathbb{C}^n) = 0$ by Dolbeaut's theorem and the $\overline{\partial}$-Poincare lemma and the same goes for $H^2(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$.

Let X be a complex manifold of dimension n. Let $Y \subseteq X$ be an analytic hypersurface. Then there exists an open cover $\{U_i\}$ of X such that $Y = X \cap U_j = Z(f_j)$ where $f_j \in \mathcal{O}_X(U_j)$.

We can decompose Y into irreducible components $Y = \bigcup_i Y_i$. Furthermore, $\bigcup_i Y_i$ can be taken to be locally finite, and the Y_i are themselves irreducible hypersurfaces. If X is compact, then this decomposition is finite. This came from our study of the ring of germs of holomorphic functions, where we established via Weierstrass preparation theorem that $\mathcal{O}_{X,x}$ is Noetherian + UFD.

Associating Y with just $\sum Y_i$ is bad because it doesn't distinguished between $z = 0$ and $z^2 = 0$. We want to keep track of multiplicities (which is a first step towards the notion of a scheme). So we include the order of vanishing in the picture.

Definition 25.5. A Weil divisor on X is a locally finite formal linear combination

$$D = \sum a_i Y_i,$$

where Y_i is an irreducible hypersurface and $a_i \in \mathbb{Z}$. If X is compact, this reduces to finite sums. Let $Div(X)$ denote the free abelian group generated under addition. If all $a_i \geq 0$, we say D is effective ($D \geq 0$).

If X is compact, then a Weil divisor $D = \sum a_i Y_i$ must be a finite sum otherwise we could find a point contained in infinitely many of the Y_i with nonzero a_i otherwise we'd contradict compactness.

Let $Y \subset X$ be a hypersurface and $x \in Y$. Suppose that $[Y]_x$ is an irreducible germ, so this germ is the zero set of an irreducible $g \in \mathcal{O}_{X,x}$. Let $f \in K(\mathcal{O}_{X,x})$. Then $f = g^a h$ with $h \in \mathcal{O}^*_{X,x}$. Then $ord_{Y,x}(f) := a \in \mathbb{Z}$.

Remark 25.6. The order $ord_{Y,x}(f)$ does not depend on the defining equation g. Two irreducible functions $g, g' \in \mathcal{O}_{X,x}$ such that g, g' cut out the same analytic germ only differ by a unit in $\mathcal{O}_{X,x}$.

Globally, one defines the order $ord_Y(f)$ of a meromorphic function $f \in K(X)$ along an irreducible hypersurface $Y \subset X$ as $ord_Y(f) = ord_{Y,x}(f)$ for $x \in Y$ such that $[Y]_x$ is an irreducible analytic germ. Such a point $x \in Y$ always exists, e.g. we may choose a regular point $x \in Y_{reg}$ (exercise 1.1.14 Huybrechts). Moreover the definition does not depend on x, by proposition 1.1.35 and the fact that Y_{reg} is connected if Y is irreducible.

Definition 25.7. For $f \in K(X)$, define

$$(f) := \sum ord_Y(f) Y \in Div(X)$$

where the sum is taken over all irreducible hypersurfaces $Y \subset X$. Such a divisor is called principal. A principal divisor is automatically locally finite since each $\mathcal{O}_{X,x}$ is UFD.

These Weil divisors are equivalent to a different perspective on divisors, called Cartier divisors. Consider the short exact sequence

$$0 \to \mathcal{O}_X^\times \to K_X^\times \to K_X^\times/\mathcal{O}_X^\times \to 0$$

where $K_X^\times$ is the sheaf of meromorphic functions and $\mathcal{O}_X^\times$ is the sheaf of invertible functions. Then you pass to cohomology

$$0 \to H^0(X, \mathcal{O}_X^\times) \to H^0(X, K_X^\times) \to H^0(X, \frac{K_X^\times}{\mathcal{O}_X^\times}) \to^\delta H^1(X, \mathcal{O}_X^\times) \to \cdots$$

Note that in each element in $H^0(X, K_X^\times)$ we've associated a principal divisor. The global sections $H^0(X, \frac{K_X^\times}{\mathcal{O}_X^\times})$ are called Cartier divisors. We can also identify them with Weil divisors. Furthermore, $H^1(X, \mathcal{O}_X^\times) \cong Pic(X)$. So there must be some relation between Weil divisors, Cartier divisors, and line bundles.

Let's interpret $H^0(X, \frac{K_X^\times}{\mathcal{O}_X^\times}) = f$. The global sections are equivalent to the data of $\{U_i, f_i \in K^\times(U_i)\}$ such that on $U_i \cap U_j$, we have $f_j f_i^{-1} \in \mathcal{O}_X^\times(U_i \cap U_j)$.

Lemma 25.8. *On a complex manifold X, there exists a natural isomorphism*

$$H^0(X, \frac{K_X^*}{\mathcal{O}_X^*}) \to Div(X).$$

Proof. First we define the map. Suppose we have a Cartier divisor $\{U_i, f_i \in K^\times(U_i)\}$. Then $f_i f_j^{-1} \in \mathcal{O}_X(U_i \cap U_j)^\times$. Note that if Y is an irreducible hypersurface such that $Y \cap U_i \cap U_j \neq \emptyset$, then $f_i f_j^{-1} \in \mathcal{O}_X(U_i \cap U_j)^\times$ implies that $ord_Y(f_i) = ord_Y(f_j)$. Then we can define the map to be

$$\{U_i, f_i \in K^\times(U_i)\} \mapsto \sum_Y ord_Y(f_i) Y$$

where $ord_Y(f_i)$ is taken for $Y \cap U_i \neq \emptyset$. Additivity of the order implies the map is a group homomorphism.

Now suppose we have Weil divisor $D = \sum a_i Y_i$. There exists an open cover $X = \bigcup U_i$ such that $Y_i \cap U_j$ is defined by some $g_{ij} \in \mathcal{O}_X(U_j)$ which is unique up to elements in $\mathcal{O}_X^\times(U_j)$. We can do this since D is locally finite. Then define $f_j := \prod_i g_{ij}^{a_i}$. Since g_{ij} and g_{ik} on $U_j \cap U_k$ define the same zero set, they differ only by elements in $\mathcal{O}^\times(U_j \cap U_k)$. Then these f_j glue together to $f \in H^0(X, K_X^\times / \mathcal{O}_X^\times)$. These two maps are inverses to each other. □

Note everything that we've done relies on $\mathcal{O}_{X,x}$ being UFD.

Remark 25.9. You can study divisors on singular spaces. The simplest type singular thing in $\mathbb{C}^2$ is the cone $Z(X_0^2 + X_1^2 + X_2^2) = 0$. In $\mathbb{P}^2$, this is a smooth conic. In $\mathbb{C}^2$, this is the cone (hourglass) over the smooth conic. If you take Y to be one of the rulings. The claim is that Y is not $f = 0$ for some function f on X in a neighborhoood of the origin (the only singular point). This is not a locally principal divisor. It is codimension 1, hypersurface, but it is singular, and you cannot write it as the vanishing loci around the origin as vanishing of one function, so it is not a Cartier divisor. The local ring $\mathcal{O}_{X,0}$ is not a UFD. However, $2Y$ is the zero locus of some function.

Where a multiple can be localy principal, called Q factorial. When you contract small thing (codimension ≥ 2) into singular things, this is when Q-factorial comes up in birational geometry?

So this double cone is still quite nice. The singularity is not that bad. It is Q-factorial.

Proposition 25.10. *There is a natural homomorphism*

$$Div(X) \to Pic(X)$$

which is identified with $H^0(X, \frac{K_X^\times}{\mathcal{O}_X^\times}) \to H^1(X, \mathcal{O}_X^\times)$.

Proof. Consider $D = \sum a_i Y_i$. This comes from $f \in H^0(X, \frac{K_X^\times}{\mathcal{O}_X^\times})$, so $f_i f_j^{-1} \in \mathcal{O}_X^\times(U_i \cap U_j)$. Define $g_{ij} = f_i f_j^{-1}$. This obviously satisfies a cocycle condition. These give the data of a line bundle $\mathcal{L}(D)$. □

26. 1/24/24: complete linear system

Let X be a complex manifold. We're always assuming its connected. Last time we constructed a map from Weil divisors to Picard group,

$$Div(X) \to Pic(X).$$

Recall that we identified $Div(X) \cong H^0(X, K_X^\times / \mathcal{O}_X^\times)$. If $D = \sum a_i Y_i$ is a Weil divisor, then because it is locally finite, D is locally cut out by a local meromorphic function. We can associate D then with a Cartier divisor $\{U_i, f_i\}$ where $f_i \in K_X^\times(U_i)$, and $f_i f_j^{-1} \in \mathcal{O}_X^\times(U_i \cap U_j)$. Then $D \mapsto \mathcal{O}_X(D)$ where the bundle tranistion functions are $f_i f_j^{-1}$.

Remark 26.1. It is easy to see that $\mathcal{O}_X(D + D') \cong \mathcal{O}_X(D + D') \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$. Furthermore, $\mathcal{O}_X(1) \cong \mathcal{O}_X$, as you can take the global function "1" as defining a Cartier divisor. Furthermore, $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^{-1}$.

Taking the LES of

$$0 \to \mathcal{O}_X^\times \to \mathcal{K}_X^\times \to \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 0$$

we obtain

$$H^0(X, K_X^\times) \to^\alpha H^0(X, K_X^\times/\mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times).$$

Let V denote the image of $H^0(X, K_X^\times)$ in $H^0(X, K_X^\times/\mathcal{O}_X^\times)$. Then note that

$$\frac{H^0(X, K_X^\times/\mathcal{O}_X^\times)}{V} \hookrightarrow H^1(X, \mathcal{O}_X^\times) \cong Pic(X)$$

is an injection. Furthermore, we say that Weil divisors $D_1, D - 2 \subset X$ are *linearly equivalent* if and only if $D_1 - D_2 = (f)$ where $f \in H^0(X, K_X^\times)$. In other words, Weil divisors are linearly equivalent if they differ by a global meromorphic function, i.e. a principal divisor. It is easy to check that

$$\frac{H^0(X, K_X^\times/\mathcal{O}_X^\times)}{V} \cong \frac{Div(X)}{\sim}.$$

Remark 26.2. The map $\frac{Div(X)}{\sim} \hookrightarrow Pic(X)$ is often a strict inclusion. But when X is projective, we have a surjection $Div(X) \to Pic(X)$, and thus the map is an isomorphism. This is due to a result of Serre, you can twist a line bundle to a high enough degree so that it has a global section. Such a global section produces an effective divisor which will map to the line bundle under consideration. However in complex geometry, in general, this map is not necessarily surjective. But key idea here remains: in order to go from line bundles to divisors we need global sections.

Divisors give line bundles, and many divisors give the same line bundle. To go backwards, we need to talk about divisors associated to global sections of line bundles.

Let $\mathcal{L}$ be a line bundle over X, with transition functions $\{g_{ij}\}$. Let $0 \neq s \in H^0(X, \mathcal{L})$ be a nonzero global section. Note that s is equivalent to the data of $s_i \in \mathcal{O}_X(U_i)$ which are compatible under the transition functions. Thus, a global section s gives $\{U_i, s_i\} \in H^0(X, K_X^\times/\mathcal{O}_X^\times)$ a Cartier divisor and thus a Weil divisor D. Furthermore, the Weil divisor D gives back $\mathcal{L}$, i.e. $\mathcal{O}_X(D) \cong \mathcal{L}$.

Example 26.3. A homogeneous polynomial $F \in \mathbb{C}[x_0, \cdots, x_n]$ of degree d is a global section of $\mathcal{O}_{\mathbb{P}^n}(d)$. The Weil divisor associated to F is the hypersurface $Z(F)$. Then $\mathcal{O}_X(Z(F)) \cong \mathcal{O}_{\mathbb{P}^n}(d)$.

Remark 26.4. If you have two line bundles $\mathcal{L}_1, \mathcal{L}_2$, then there is a map on global sections

$$H^0(X, \mathcal{L}_1) \otimes H^0(X, \mathcal{L}_2) \to H^0(X, \mathcal{L}_1 \otimes \mathcal{L}_2).$$

If $(s, s') \in H^0(X, \mathcal{L}_1) \otimes H^0(X, \mathcal{L}_2)$ are locally given by $s_i, s_i' \in \mathcal{O}_X(U_i)$, then we can just locally multiply to get $s_i s_i' \in \mathcal{O}_X(U_i)$ and $\{U_i, s_i s_i'\}$ glues to a global section of $\mathcal{L}_1 \otimes \mathcal{L}_2$.

If D and D' are the effective divisors given by s and s', then $D + D'$ is the effective divisor obtained by $ss' \in H^0(X, \mathcal{L}_1 \otimes \mathcal{L}_2)$.

Note that if $D \subset X$ is an effective divisor and $\mathcal{O}_X(D) \cong \mathcal{L}$, then there exists $0 \neq s \in H^0(X, \mathcal{L})$ which cuts out D. To see this, note that the Cartier divisor associated to D is $\{U_i, f_i\}$ where $f_i \in \mathcal{O}_X(U_i)$. Then this $\{U_i, f_i\}$ is exactly the data of a nonzero global section of $\mathcal{L}$.

Corollary 26.5 (Complete Linear System)**.** *Assume that X is a compact complex manifold. If $s_1 \in H^0(X, \mathcal{L}_1)$ and $s_2 \in H^0(X, \mathcal{L}_2)$ and D_1 and D_2 are the effective divisors associated to s_1 and s_2, then we see that*

$$D_1 \sim D_2 \implies \mathcal{L}_1 \cong \mathcal{L}_2.$$

Furthermore, if $D_1 \sim D_2$, then if D_1 and D_2 correspond to Cartier divisors $\{(U_i, f_{i1})\}$ and $\{(U_i, f_{i2})\}$, then $f_{i1} = g f_{i2}$ where g is a global meromorphic function. But since $f_{i1}, f_{i2} \in \mathcal{O}_X(U_i)$, this implies that g is a holomorphic function on X. But X is compact (and we assume connected), so g must be constant. Thus, $s_1 = \lambda s_2$ for some constant λ. Thus,

$$\{D \text{ effective } | \mathcal{O}_X(D) \cong \mathcal{L}\} \cong \mathbb{P}H^0(X, \mathcal{L}).$$

Often we write $|\mathcal{L}| := \mathbb{P}H^0(X, \mathcal{L})$. Complete linear systems are hugely important to the study of complex projective geometry, where we study shapes inside of projective space. Such shapes always admit embeddings via complete linear systems. This is why the Italian geometers worked so much on complete linear systems.

Remark 26.6. One can also take incomplete linear systems $\mathbb{P}W \subset \mathbb{P}H^0(X, \mathcal{L})$, where $W \subset H^0(X, \mathcal{L})$ is a sub-vector space. For example, a $\mathbb{P}^1 \subset \mathbb{P}H^0(X, \mathcal{L})$ is called a *pencil*. A $\mathbb{P}^2 \subset \mathbb{P}H^0(X, \mathcal{L})$ is called a *net*.

Example 26.7. For $\mathbb{P}^n$, $|\mathcal{O}_{\mathbb{P}^n}(d)| = \mathbb{P}^{\binom{n+d}{d}-1}$.

Eventually we will see that in order to prove the Kodaira embedding theorem, we will start with a manifold of a special type, a line bundle, and a global section. This data will provide an embedding for our compact Kahler manifold via the complete linear system.

Corollary 26.8. *The image of $\frac{Div(X)}{\sim} \hookrightarrow Pic(X)$ is the subgroup generated by line bundles $\mathcal{L}$ with $H^0(X, \mathcal{L}) \neq 0$.*

Proof. First, if $\mathcal{L} \in Pic(X)$ such that $H^0(X, \mathcal{L}) \neq 0$, then this line bundle is clearly in the image of the map, since we can take any nonzero global section $s \in H^0(X, \mathcal{L})$, and the associated divisor D will be such that $\mathcal{O}_X(D) \cong \mathcal{L}$.

Now suppose we have any divisor D. Then we can write

$$D = D_+ - D_-$$

where D_+ and D_- are both effective. Then $\mathcal{O}_X(D) \cong \mathcal{O}_X(D_+) \otimes \mathcal{O}_X(D_-)^{-1}$, and since D_+ and D_- are both effective, both $\mathcal{O}_X(D_+)$ and $\mathcal{O}_X(D_-)$ □

Chapter 2 Section 6 Hartshorne is a good complement to this whole discussion. When $\mathcal{F}$ is an $\mathcal{O}_X$-module, we write $\mathcal{F}(D) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$, and on $\mathbb{P}^n$, we write $\mathcal{F}(m) := \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(m)$. And for $X \subseteq \mathbb{P}^n$ we write $\mathcal{O}_X(m) := \mathcal{O}_{\mathbb{P}^n}(m)|_X$. Serre's theorem on coherent sheaves says that for $m >> 0$ we have $H^0(X, \mathcal{F}(m)) \neq 0$. In general, outside of algebraic geometry, you don't have such a theorem. A line bundle being able to do this an ample bundle, and in differential geometry such a line bundle is called positive (it has positive curvature). We will study this in the next few lectures.

When you have $Y \subseteq X$ an analytic subset of complex manifold X. We have

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

where $\mathcal{I}_Y$ is the sheaf of functions on X that vanish along Y, and $\mathcal{O}_Y$ is the sheaf of functions on Y.

Remark 26.9. If $\mathcal{L}$ is a line bundle, $s \in H^0(X, \mathcal{L})$ is the same as a map $\mathcal{O}_X \to \mathcal{L}$.

If Y is hypersurface, have $s_Y \in H^0(X, \mathcal{O}_X(Y))$. Have $\mathcal{O}_X \to^{s_Y} \to \mathcal{O}_X(Y)$ and have $\mathcal{O}_X(-Y) \to \mathcal{O}_X$, latter locally is given by $\frac{1}{s_i} \to 1$. So $\mathcal{O}_X(-Y) \cong \mathcal{I}_Y$.

If you have D effective divisor on X, you have

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Tensoring by $\mathcal{O}_X(D)$, we get

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0.$$

This $\mathcal{O}_D(D)$ is important because it will be the normal bundle associated to this effective divisor.

Example 26.10 (Divisor on compact Riemann surface)**.** All compact Riemann surfaces are Kahler, and thus they are also projective. An irreducible hypersurface on X is a point. So Weil divisors look like

$$D = \sum_{x \in X \text{ finitely many}} n_i x_i, \text{ where } n_i \in \mathbb{Z}.$$

So we define $\deg D = \sum n_i \in \mathbb{Z}$. So we have a map $\deg : Div(X) \to \mathbb{Z}$. Claim: the degree map factors through linear equivalence. In other words, if $D_1 \sim D_2$ then $\deg D_1 = \deg D_2$. In other words, for meromorphic f on X, show $\deg(f) = 0$. Look at

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) = 0.$$

Since X is projective, we will eventually show that $\frac{Div(X)}{\sim} \cong Pic(X)$. So the deg map extends to $Pic(X)$. We will see that $\deg : Pic(X) \to \mathbb{Z}$ is the same as the first Chern class map $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$. Since the map is surjective, we can take its kernel to obtain

$$0 \to Pic^0(X) \to Pic(X) \to \mathbb{Z} \to 0,$$

where $Pic^0(X)$ is the degree 0 line bundles or "topologically trivial line bundle". We will see that

$$Pic^0(X) \cong \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})} \cong \frac{\mathbb{C}^g}{H^1(X, \mathbb{Z}}.$$

We know from hodge numbers that $H^1(X, \mathcal{O}_X) \cong \mathbb{C}^g$. The new information is $H^1(X, \mathbb{Z})$ embeds as a lattice. So $Pic^0(X)$ is a complex torus. And it is actually projective. Also abelian variety.

This tells you what it means to classify line bundles. There's a discrete life, $\mathbb{Z}$, and a continuous part, the torus $Pic^0(X)$.

Among Riemann surfaces and divisors, $\mathbb{P}^1$ has very special behavior. The defining feature of $\mathbb{P}^1$ among Riemann surfaces is that any two distinct points are linearly equivalent.

Lemma 26.11. *Let X be a compact Riemann surface. Then $X \cong \mathbb{P}^1$ if and only if there exists $x \neq y$ such that $x \sim y$.*

Proof. If $X \cong \mathbb{P}^1$, then any two distinct points are linearly equivalent via the appropriate meromorphic function.

Now suppose there exist two distinct points linearly equivalent. Since x, y are effective divisors, both $\mathcal{O}_X(x)$ and $\mathcal{O}_X(y)$ have nonzero global sections s_x and s_y. Then we can define

$$X \to \mathbb{P}^1 \text{ where } z \mapsto [s_x(z) : s_y(z)],$$

and we see that this map is well-defined because s_x and s_y are not simultaneously zero. Furthermore, $[0 : 1]$ and $[1 : 0]$ are both hit only once at y and x respectively. Thus the map is not constant. Thus the map must be surjective because otherwise the map would map into a copy of $\mathbb{C}$, and the only such maps are constants since X is compact. To show injectivity, we use the following fact: let $f_k = z^k$. Then for any non-constant holomorphic map $f : X \to Y$ between Riemann surfaces where X is connected, there exists $k \geq 1$ such that f locally looks like f_k. Since we have our map is injective on $[0 : 1]$ and $[1 : 0]$, we see that this map must be injective. Thus, we have an isomorphism. □

27. Extra: a note on pullbacks of divisors

Let $f : X \to Y$ be a holomorphic map between connected complex manifolds. Let $D \in Div(Y)$ be a divisor. We'd like to define the pullback $f^*D \in Div(X)$. For this notion to be useful, we'd like for the pullback to respect the isomorphisms between Cartier divisors and Weil divisors. In particular, if $\{U_i, f_i\}$ is the Cartier divisor associated to D, then we would like the following diagram to commute:

$$\begin{tikzcd} D \arrow[r] \arrow[d] & f^*D \arrow[d] \\ \{(U_i, f_i)\} \arrow[r] & \{(f^{-1}(U_i), f_i \circ f)\} \end{tikzcd}$$

Analyzing when this diagram makes sense leads us to *how to define the pullback and when the pullback of a divisor is well-defined.* In particular, f^*D is well-defined when none of the irreducible components of $f(X)$ are contained in the support of D.

Let's see why this must be the case. For clarity, let us suppose D is an irreducible hypersurface so $f_i \in \mathcal{O}_Y(U_i)$, and $f(X)$ is contained in D. Then we see that on $f^{-1}(U_i)$, the function $f_i \circ f$ is entirely zero. So the Cartier divisor of what should be the pullback line bundle is not well-defined.

In general, if no component of $f(X)$ is contained in the support of divisor $D \subset Y$, then we define f^*D in the following way. Let D be an irreducible hypersurface. Let Y_i be the irreducible components of $f^{-1}(D)$. Pick $p \in Y_i$ to be a smooth point. Then around $f(p)$, D is locally cut out by an irreducible g. Then locally around p, $f \circ g$ decomposes into a product $\prod g_i^{\eta_i}$ where g_i are irreducible cutting out Y_i. So we define $f^*D = \sum eta_i Y_i$. In geral, if $D = \sum a_i D_i$ is a divisor, and no component of $f(X)$ is contained in the support of any D_i, then $f^*D = \sum a_i f^*(D_i)$.

Proposition 27.1. *Let $f : X \to Y$ be a holomorphic map of connected complex manifolds. If no irreducible component of $f(X)$ is contained in the support of a* *divisor D, then f^*D is well-defined. In particular,*

$$\mathcal{O}_X(f^*D) \cong f^*\mathcal{O}_Y(D).$$

When f is a dominant morphism (the image is dense), then the pullback of every divisor $D \subset Y$ is well-defined. In this class, we also define the pullback of on Cartier divisors to be $f^*\{(U_i, f_i)\} = \{(f^{-1}(U_i), f_i \circ f)\}$. Note that these pullback maps, when well-defined, respect the group homomorphism structures.

28. 1/29/24: Normal bundle, adjunction formula,

Now that we have talked about line bundles and divisors, we are working towards the canonical divisor. To do this, let's discuss the normal bundle. We start with a complex manifold X, and $Y \subseteq X$ is a submanifold of dimension r.

Lemma 28.1. *Let X be a complex manifold and $Y \subseteq X$ a complex submanifold of codimension r. There exists a canonical inclusion*

$$T_Y \hookrightarrow T_X|_Y.$$

This is an inclusion of vector bundles (rank doesn't drop at a point, not just a morphism of sheaves). This gives rise to a short exact sequence

$$0 \to T_Y \to T_X|_Y \to \mathcal{N}_{Y/X} \to 0,$$

where the normal bundle $\mathcal{N}_{Y/X}$ has rank r.

Proof. We can choose $\{(U_i, \phi_i)\}$ atlas of X which witness Y as a complex submanifold of X, such that $\phi_i(U_i \cap Y) = \phi(U_i) \cap \{z_{m+1} = \cdots = z_n = 0\}$. The transition functions for T_X are

$$g_{ij} = \mathcal{J}(\phi_{ij}) \circ \phi_j.$$

Denote

$$g_{ij}^Y = \mathcal{J}(\phi_{ij}|_Y)\phi_j|_Y$$

so

$$g_{ij}|_Y = \begin{pmatrix} g_{ij}^Y & * \\ 0 & h_{ij} \end{pmatrix}$$

and the h_{ij} provide the transition functions for $\mathcal{N}_{Y/X}$. □

Remark 28.2. In differential geometry, a short exact sequence of smooth vector bundles always splits. In complex geometry, a short exact sequence of holomorphic vector bundles is rarely split. This tangent normal bundle short exact sequence rarely splits.

If X is $\mathbb{P}^n$, the tangent normal bundle SES splits $\iff$ Y is a linear space.

Remark 28.3. If you have a short exact sequence of holomorphic vector bundles

$$0 \to E \to F \to G \to 0$$

where the transition functions for F are

$$f_{ij} = \begin{pmatrix} e_{ij} & * \\ 0 & g_{ij} \end{pmatrix}$$

note that in matrices, we have $\det(f_{ij}) = \det(e_{ij})\det(g_{ij})$. Then

$$\det F \cong \det E \otimes \det G.$$

Corollary 28.4 (Adjunction formula)**.** *Let X be a complex manifold and $Y \subseteq X$ a complex submanifold of codimension r. Then*

$$\omega_Y \cong \omega_X|_Y \otimes \det \mathcal{N}_{Y/X}.$$

Proof. We have

$$\det T_X|_Y \cong \det T_Y \otimes \det \mathcal{N}_{Y/X}.$$

The determinant of a bundle commutes with duals, so we obtain the claim. □

Let's do some explicit examples.

Example 28.5 (Trivial example)**.** When $Pic(X) = \{0\}$, for example $\mathbb{C}^n$, then $\omega_X \cong \mathcal{O}_X$.

Example 28.6. If A is a torus, then we know that

$$T_A \cong \mathcal{O}_A^{\oplus \dim A}$$

because a torus is a parallelizable manifold. You can drag the tangent space around, so the tangent bundle is trivial. Then

$$\omega_A \cong \mathcal{O}_A.$$

Example 28.7. By definition, a Calabi-Yau manifold is a compact complex Kahler manifold with

$$\omega_X \cong \mathcal{O}_X, H^1(X, \mathcal{O}_X) \cong H^{0,1}(X) = 0.$$

A K3 surface is a 2 dimensional Calabi-Yau manifold. Note that a compact complex torus are is not Calabi-Yau. We have $H^1(X, \mathcal{O}_X) \cong g$, the genus g.

Example 28.8. A compact Riemann surface of genus g. So

$$\Omega_C = \omega_C.$$

Then

$$H^0(C, \omega_C) = H^{1,0}(C) \cong \mathbb{C}^g.$$

Thus the canonical bundle of a compact Riemann surface of genus g, is quite massive for g large. Has lots of global sections.

Remark 28.9. Every line bundle on a curve has a degree

$$\deg : Pic(C) \to \mathbb{Z}.$$

It is a fact that $\deg \omega_C = 2g - 2$, and one needs the Riemann-Roch formula plus Serre duality

$$h^0(C, \omega_C) - h^1(C, \omega_C) = \deg \omega_C - g + 1.$$

Serre duality says that $h^1(c, \omega_C) \cong h^0(C, \mathcal{O}_C)$. Later we will prove Riemann roch and serre duality.

Example 28.10 (Canonical bundle of $\mathbb{P}^n$)**.** We have

$$\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

A priori, we already knew that the canonical line bundle had no nonzero global sections and we know all the line bundles on $\mathbb{P}^n$. Here is how we knew this: note

$$H^0(\mathbb{P}^n, \omega_{\mathbb{P}^n}) \cong H^{n,0}(\mathbb{P}^n).$$

If n is odd, then this is immediately zero. If n is even, then $H^n(\mathbb{P}^n) \cong \mathbb{Z}$, and the only nonzero term in the Hodge decomposition is $H^{n,n}$ given by $[\omega_{FS}^{\wedge n}]$, so $H^{n,0} = 0$.

The proof is the standard way of using transition functions in Huybrechts 2.4.3. Another way of calculating this is using the Euler exact sequence.

To prove this, we introduce the famous Euler exact sequence.

Proposition 28.11. *There exists a natural short exact sequence of vector bundles*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \bigoplus^{n+1} \mathcal{O}_{\mathbb{P}^n}(1) \to T_{\mathbb{P}^n} \to 0$$

called the Euler exact sequence. Note the dual is

$$0 \to \Omega_{\mathbb{P}^n} \to \bigoplus_{n+1} \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

A more canonical way of writing the Euler exact sequence is

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to V \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to T_{\mathbb{P}^n} \to 0$$

where $V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

Remark 28.12. The $\mathcal{O}(1)$ are called positive line bundles. They will have a metric with positive curvature (algebraically this means ample). The quotient of anything positive is positive. This immediately implies that the tangent bundle is an ample vector bundle. Mori proved the following conjecture of various people: any tangent bundle is ample, then that manifold is $\mathbb{P}^n$. That meant introducing methods of existence rational curves on manifolds, reduction via mod p, and other things in Mori theory.. is what got him the fields medal.

Sections of canonical bundles have to do with Hodge theory, but sections of pluricanonical bundles don't. Note

$$\omega_{\mathbb{P}^n}^{\otimes m} \cong \mathcal{O}_{\mathbb{P}^n}(-m(n+1)) \implies h^0(\mathbb{P}^n, \omega_{\mathbb{P}^n}^{\otimes m}) = 0.$$

We say then that $K(\mathbb{P}^n) = -\infty$, the Kodaira dimension of $\mathbb{P}^n$. A lot of the work in complex geometry depends on understanding pluricanonical bundles. But pluricanonical bundles don't say anything about Hodge theory because of they don't have a meaning in terms of forms. Yum Tong Siu works a lot with pluricanonical bundles, like introducing metrics. This is a big problem in complex geometry.

Let us now show that $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. Consider the canonical projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Consider distinguished affine $U_0 = \{(Z_0 : \cdots : Z_n) | z_0 \neq 0\}$. We have a map

$$\pi : U_0 \to \mathbb{C}^n$$

via the obvious way where $Z_j \mapsto z_j = \frac{Z_j}{Z_0}$ for $j \neq 0$.

Recall that if $f : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic, where the coordinates of the domain is $y_1, \cdots, y_n$ and the coordinates of $\mathbb{C}^m$ are $w_1, \cdots, w_m$, then

$$df(\frac{\partial}{\partial y_j}) = \sum_{k=1}^{m} \frac{\partial f_k}{\partial y_j} \frac{\partial}{\partial w_k}.$$

We have

$$d\pi(\frac{\partial}{\partial z_i} = \frac{1}{z_0} \frac{\partial}{\partial z_i}$$

$$d\pi(\frac{\partial}{\partial z_0}) = \sum_{i=1}^{n} -\frac{z_i}{z_0^2} \frac{\partial}{\partial z_i}$$

and each of these vector fields descend to $\mathbb{P}^n$ since the function coefficients of the vectors are homogeneous. We also have a formula on $\mathbb{P}^n$

$$z_0 \frac{\partial}{\partial z_0} + \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} = 0.$$

where we identify $\frac{\partial}{\partial z_i}$ with $d\pi(\frac{\partial}{\partial z_i})$. We have $T_x\mathbb{P}^n$ is generated by $\frac{\partial}{\partial z_0}, \cdots, \frac{\partial}{\partial z_n}$ with relation

$$\sum_{i=0}^{n} z_i \frac{\partial}{\partial z_i} = 0.$$

Recall that the global sections of $\mathcal{O}_{\mathbb{P}^n}(1)$ are the homogeneous polynomials of degree 1. Then the map

$$\bigoplus^{n+1} \mathcal{O}_{\mathbb{P}^n}(1) \to T_{\mathbb{P}^n}$$

is given by $(s_0, \cdots, s_n) \mapsto \sum s_i \frac{\partial}{\partial z_i}$. This map is surjective. The kernel of this map is

$$\mathcal{O}_{\mathbb{P}^n} \to \bigoplus^{n+1} \mathcal{O}_{\mathbb{P}^n}(1)$$

where $1 \mapsto (z_0, \cdots, z_n)$, which is in the kernel by the Euler relation.

note we also have exactness on global sections:

$$0 \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \to H^0(\mathbb{P}^n, \bigoplus^{n+1} \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) \to 0.$$

Remark 28.13. Given SES of holomorphic vector bundles

$$0 \to E \to F \to L \to 0,$$

we have

$$0 \to \bigwedge^p E \to \bigwedge^p F \to \bigwedge^{p-1} E \otimes L \to 0,$$

for every $p \geq 1$.

If we apply this to dual of Euler exact sequence, and apply the above remark, we get the Koszul complex

$$\begin{tikzcd}
\cdots & \bigoplus \mathcal{O}_{\mathbb{P}^n}(-3) \arrow[rr] \arrow[dr] & & \bigoplus \mathcal{O}_{\mathbb{P}^n}(-2) \arrow[rr] \arrow[dr] & & V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \arrow[r] & \mathcal{O}_{\mathbb{P}^n} \arrow[r] & 0 \\
\Omega^3_{\mathbb{P}^n} \arrow[ur] \arrow[dr] & & \Omega^2_{\mathbb{P}^n} \arrow[ur] \arrow[dr] & & \Omega_{\mathbb{P}^n} \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Something about global section of $H^0(\mathcal{O}) \to \bigoplus H^0\mathcal{O}_{\mathbb{P}^n}(1)$. Something about locally Koszul complex of $(X_0, \cdots, X_n) \subset \mathbb{C}[X_0, \cdots, X_n]$.

Example 28.14. Let $Y \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d.

Proposition 28.15. *If $Y \subseteq X$ is a smooth hypersurface and $L = \mathcal{O}_X(Y)$, then*

$$\mathcal{N}_{Y/X} \cong L|_Y$$

and $\omega_Y = \omega_X|_Y \otimes L|_Y$.

Corollary 28.16. *If $Y \subseteq \mathbb{P}^n$ is a hypersurface of degree d, then*

$$\omega_Y \cong \mathcal{O}_Y(d-n-1) := \mathcal{O}_{\mathbb{P}^n}(d-n-1)|_Y.$$

Now we prove that the 1st + adjunction implies 2nd.

1st: $\phi_i : U_i \to \mathbb{C}^n$, where $\phi_i(U_i \cap Y) = \phi_i(U_i) \cap (z_n = 0)$. And $\phi_{ij} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ where $\phi_{ij} = (\phi_{ij}^1, \cdots, \phi_{ij}^n)$ and $\phi_{ij}^n(z_1,, z_{n-1}, 0) = 0$, so $\phi_{ij}^n = z_n h(z_1, \cdots, z_n)$ where h is holomorphic.

We have transition functions for $T_X|_Y$

$$\mathcal{J}(\phi_{ij})|_Y \circ \phi_j|_Y = \begin{pmatrix} \mathcal{J}(\phi_{ij}|_Y) \circ \phi_j|_Y & * \\ 0 & \frac{\partial \phi_{ij}^n}{\partial z_n}|_Y \circ \phi_j|_Y \end{pmatrix}$$

and the transition functions for $\mathcal{N}_{Y/X}$ are

$$g_{ij} = \frac{\partial \phi_{ij}^n}{\partial z_n}|_Y \circ \phi_j|_Y$$

so there exists global section $s \in H^0(X, \mathcal{O}_X(Y))$ such that $Z(s) = Y$, where s is locally given by $s_j \in \mathcal{O}_X(U_j)$ and the transition function of $\mathcal{O}_X(Y)$ is $h_{ij} = s_i/s_j$. Since $Y = (z_n = 0)$, in fact $s_i = \phi_i^n$, where $\phi_i = (\phi_i^1, \cdots, \phi_i^n)$.

So at x s.t. $\phi_j(x) = (z_1, \cdots, z_{n-1}, 0)$

$$h_{ij}(x) = \frac{\phi_i^n}{\phi_j^n}(x) = \frac{(\phi_{ij} \circ \phi_j)^n}{\phi_j^n}(x) = \frac{\phi_{ij}^n}{z_n} \circ \phi_j(x) = h(z_1, \cdots, z_{n-1}, 0) = \frac{\partial \phi_{ij}^n}{\partial z_n}(z_1, \cdots, z_{n-1}, 0) = \frac{\partial \phi_{ij}^n}{\partial z_n} \circ \phi_j(x).$$

29. 1/31/24: CONNECTIONS

If you have a hypersurface $Y \subseteq X$, we obtain the normal bundle $\mathcal{N}_{Y/X} \cong \mathcal{O}_X(Y)|_X =: \mathcal{O}_Y(Y)$. And we had

$$\omega_Y \cong \omega_X|_Y \otimes \mathcal{O}_Y(Y) = \omega_X(Y)|_Y.$$

As a corollary we had:

Corollary 29.1. *For $Y \subseteq \mathbb{P}^n$ of degree d, then $\omega_Y \cong \mathcal{O}_Y(d-n-1)$, since $\mathcal{O}_{\mathbb{P}^n}(Y) \cong \mathcal{O}_{\mathbb{P}^n}(d)$.*

For Riemann surfaces, we have:

- Genus $g = 0 \iff X = \mathbb{P}^1$, and $\omega_X = \mathcal{O}_{\mathbb{P}^1}(-2) < 0$.
- Genus $g = 1 \iff X$ is elliptic, $\omega_X \cong \mathcal{O}_X$.
- Genus $g \geq 2 \iff h^0(X, \omega_X) = g$ and $\deg(\omega_X) = 2g - 2 > 0$.

Now for Hypersurfaces in $\mathbb{P}^n$, we have:

- If $d \leq n$, then $\omega_X < 0$ (Fano variety)
- $d = n+1$, then $\omega_X \cong \mathcal{O}_X$ (Calabi-Yau)
- $d \geq n+2$ then $\omega_X = \mathcal{O}_X(> 0)$ (general type)
-

In both cases, we see there's an initial case where there is negativity (positive curvature), then there are cases where theres triviality (flat curvature), and then there is a general positive case. This classification illuminates the importance of the canonical line bundle.

In general for arbitrary varieties, there is much more that goes between Calabi-Yau and general type. For example, if you have an elliptic curve times a genus 2 curve, this would not yield a clean (Fano, Calabi-Yau, general type).

Remark 29.2. If $Z \subseteq X$ is a submanifold of codimension r, then

$$\mathcal{N}_{Z/X} \cong (\frac{\mathcal{I}_Z}{I_Z^2})^\vee.$$

Proof. Some remarks: note that since $Z \subseteq X$ is smooth submanifold, its local ring $\mathcal{J}_{Z,x}$ at each $x \in X$ is a regular local ring. It is generated by exactly r elements.

For example, we knew that if $Z = Y$ is a smooth hypersurface, then $\mathcal{I}_Y \cong \mathcal{O}_X(-Y)$ then $M/IM \cong M \otimes_R R/I$. So

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \cong \mathcal{I}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}_Y = \mathcal{O}_X(-Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y = O_Y(-Y).$$

Note this submanifold is a local complete intersection. □

Definition 29.3. A complete intersection of codimension r in X is a submanifold Z such that $codim_X Z = r$ and $Z = Y_1 \cap \cdots \cap Y_r$, the Y_i are smooth hypersurfaces in X.

Example 29.4. The twisted cubic, which is the image of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ under the degree 3 Veronese map is a complete intersection of three quadrics in $\mathbb{P}^3$.

Proposition 29.5. *Let $Z \subseteq \mathbb{P}^n$ be a complete intersection of hypersurfaces of degree $d_1, \cdots, d_r$. Then $\omega_X \cong \mathcal{O}_Z(d_1 + \cdots + d_r - n - 1)$.*

Example 29.6. The most basic example of a K3 surface is a degree 4 hypersurface in $\mathbb{P}^3$. Its canonical bundle is

$$\omega_S \cong \mathcal{O}_S(4 - 3 - 1) = \mathcal{O}_S,$$

and we computed that $h^1\mathcal{O}_S = 0$. So we have finally justified that this is a K3 surface.

But we have more examples of K3 surfaces now. We can take a complete intersection $S \subseteq \mathbb{P}^5$ of type $(2, 3)$, where we take two quadrics and intersect them transversely (so their intersection is smooth). Then $\omega_S \cong \mathcal{O}_{\mathbb{P}^5}(2+3-4-1) \cong \mathcal{O}_S$.

You can take a complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^5$, and you get $\omega_S = \mathcal{O}_S(6 - 5 - 1) = \mathcal{O}_S$.

Remark 29.7. If $X = Y \times Z$ is a product of complex manifolds, then

$$T_X \cong \pi_1^* T_Y \otimes \pi_2^* T_Z.$$

Then using $\det(E \oplus F) = \det E \otimes \det F$,

$$\omega_X \cong \pi_1^*\omega_Y \otimes \pi_2^*\omega_X =: \omega_Y \omega_X$$

insert box product latex symbol.

$$H^0(X, \omega_X^{\otimes m}) \cong H^0(Y, \omega_Y^{\otimes m}) \otimes H^0(Z, \omega_Z^{\otimes m}).$$

Convince yourself that we have Kunneth formula for box products. So for example for $Y \times \mathbb{P}^n$,

$$h^0(\omega_X^{\otimes m}) = 0$$

for $m \geq 1$. So from the point of view of pluricanonical forms, these sorts of products

It is often the most important task for a variety to calculate their canonical and normal bundles. Study moduli spaces, and computing their canonical bundles is one of the most important things you can do, and is extremely hard. If you've heard of the moduli space of curves of genus g, this is not quite smooth but also smooth, Mumford and Harris showed that in most cases it is positive. These are all very deep reasons. Even curves in projective space, its often hard to tell what their normal bundle is.

Anyway, recall that we're moving towards the Kodaira embedding theorem. We will talk next about connections on vector bundles.

Let X be a complex manifold of dimension n. Let $\pi : E \to X$ be a holomorphic vector bundle of rank r. (sometimes we can just take it to be complex, but let's just assume its holomorphic for now). If $\{U_i\}_{i \in I}$ open cover of X which provide local trivializations ϕ_i for $\pi : E \to X$. Let

$$g_{ij} := \phi_j \circ \phi_i^{-1} : U_i \cap U_j \to GL_r(\mathbb{C})$$

denote the transition functions. Recall that a section $s : X \to E$ is equivalent to the data of local $s_i : U_i \to \mathbb{C}^r$ such that $g_{ij}(s_j) = s_i$. We would now like to differentiate sections of vector bundles. It makes sense to take local functions, and glue them according to the transition function data. Similarly, it makes sense to take differential forms, and glue them along the transition function data.

Definition 29.8. Let X be a complex manifold and $E \to X$ a rank r holomorphic (can be taken to be just complex) vector bundle. Then for $U \subset X$,

$$A^k(U, E)$$

is defined to be sections of $E \otimes \bigwedge^k T^*X$ over U.

Then $s \in A^k(X, E)$ is equivalent to the data of $s_i = (s_{i,1}, \cdots, s_{i,r}) \in A^k(U_i)^{\oplus r}$ such that over $U_i \cap U_j$,

$$g_{ij} \begin{pmatrix} s_{i,1} \\ \vdots \\ s_{i,r} \end{pmatrix} = \begin{pmatrix} s_{j,1} \\ \vdots \\ s_{j,r} \end{pmatrix},$$

where $g_{ij} : U_i \cap U_j \to GL_r(\mathbb{C})$ is the transition function data for E.

Holomorphic forms are special, because you can still talk about the $\overline{\partial}$ operator.

Definition 29.9. Define

$$\overline{\partial} : A(X, E) \to A^1(X, E)$$

in the following way. Suppose $s \in A(X, E)$. This is locally $s_i : U_i \to \mathbb{C}^r$, where

$$s_i = (s_{i,1}, \cdots, s_{i,r}).$$

Then

$$\overline{\partial} s_i := (\overline{\partial} s_{i,1}, \cdots, \overline{\partial} s_{i,r}).$$

Note $g_{ij} s_j = s_i$ and by the chain rule, we have

$$\overline{\partial s_i} = \overline{\partial}(g_{ij} s_i) = \overline{\partial} g_{ij} s_i + g_{ij} \overline{\partial s_j} = g_{ij} \overline{\partial} s_j$$

since g_{ij} are holomorphic. So these glue to $\overline{\partial} s$.

You could try to define a twisted Hodge theory, but you can't define a d or ∂ similarly. We have a generalization of $\overline{\partial}$ because the transition functions are holomorphic.

Definition 29.10. A connection on E is

$$\nabla : \Gamma(X, T_X) \times A(X, E) \to A(X, E)$$

where $(\xi, s) \mapsto \nabla_\xi s$, the covariant derivative of s along ξ, such that

(1) ∇ is $A(X)$-linear in ξ and additive in s
(2) (Leibniz rule) $\nabla_\xi(fs) = \xi(f)s + f\nabla_\xi s$, for $f \in A(X)$.

This is not an algebrogeometric object. This is not $\mathcal{O}_X$-linear. This is only $\mathbb{C}$-linear. We also obtain $\nabla : A(X,E) \to A^1(X,E)$, and

$$\nabla : T_x \otimes A^0(E) \to A^0(E)$$

is a sheaf homomorphism. In algebraic geometry, a connection is $\nabla : E \to E \otimes \Omega^1_X$

There is a collection of 1-forms that describe the connection. In differential geometry, you think in terms of frames. In a local trivialization, you take the standard basis sections $(e_1, \cdots, e_r) : U \to U \times \mathbb{C}^r$ and this is the "frame" for the bundle. You can apply the connection to the frames

$$\nabla_\xi e_j = \sum_{k=1}^{r} \theta_{jk}(\xi) e_k$$

where $\theta_{jk} \in A^1(U)$. So the connection locally is determined by these 1-forms $(\theta_{jk})_{j,k}$.

There is a Levita-Cevita connection which is useful for determining when a metric is Kahler. There is another connection here that we want to talk about, which is the Chern connection. Recall a hermitian inner product is a hermitian form ($\mathbb{C}$-lienar in first factor, $h(v,w) = \overline{h(w,v)}$) that is positive definite so $h(v,v) > 0 \iff v \neq 0$.

Definition 29.11. A Hermitian metric on a vector bundle E is a collection of hermitian inner products $h_x : E_x \times E_x \to \mathbb{C}$ which varies smoothly with x. Locally, if you have local trivialization U, and you take local frame

$$e = (e_1, \cdots, e_r) : U \to U \times \mathbb{C}^r$$

then h is determined by

$$h_{jk} = h(e_j, e_k) : U \to \mathbb{C}$$

where $h_{jk} = h_{kj}$ and (h_{jk}) is positive definite at each $x \in U$.

Last semester, we talked about a hermitian metric on a complex manifold. This was really a hermitian metric on its holomorphic tangent bundle.

Remember that we had $T_{\mathbb{C}}(X) = T'X \oplus T"X$. So after complexifying the tangent bundle, the connection on TX decomposes into

$$\nabla' : T'X \times A^0(E) \to A^0(E)$$

$$\nabla" : T"X \times A^0(E) \to A^0(E)$$

the ∇' goes into $A^{1,0}(E)$? And $\nabla"$ goes into $A^{0,1}(E)$?

Proposition 29.12. *Suppose E is a holomorphic vector bundle on X with hermitian metric h. Then there exists a unique connection ∇ such that*

(1) ∇ is compatible with the metric h, i.e.

$$d(h(s_1, s_2))(\xi) = h(\nabla_\xi s_1, s_2) + h(s_1, \nabla_\xi s_2)$$

where $d(h(s_1,s_2)) = \xi h(s_1,s_2)$, and ξ vector field and $s_1, s_2 \in A(X,E)$.

(2) ∇ is compatible with the complex structure i.e. $\nabla" = \overline{\partial}$, i.e.

$$\nabla"_\xi s = (\overline{\partial} s)(\xi), \forall \xi, \forall s.$$

Next time, we will show that $\theta = \partial h h^{-1}$, where θ is the matrix of $(\theta_{jk}) =$ and ∂h is the matrix of h_{jk}. We will show $\nabla = \nabla' + \overline{\partial}$, and ∇' will be our "∂" and ∇ will be our "d". On trivial bundles, $E = X \times \mathbb{C} \to X$, our ∇ will be exactly d and ∇' will be exactly ∂. We will then try to do a version of Hodge theory with these new operators.

30. Extra: Connections

Let's take a step back and think about what we did when we did Hodge theory on a real manifold. We were primarily concerned with the bundles $\bigwedge^k T^*X$, and the Riemannian metric on our manifold induced a metric on differential forms. Secretly, what we were really doing was specifying a vector bundle metric on TX which naturally induced metrics on T^*X and $\bigwedge^k T^*X$.

Furthermore, we worked quite a lot with the exterior derivative. Really, the exterior derivative is a $\mathbb{R}$-linear map (not $\mathcal{O}_X$-linear) of sheaves

$$\Omega_X^k \to \Omega_X^{k+1}.$$

We can generalize what we've done here to arbitrary vector bundles. Given a complex (not necessarily holomorphic) vector bundle E of rank r over X, where X is at least a smooth manifold, we have the sheaf of sections of E. These provide local functions on X which are vector-valued in E. The first question to ask when trying to "twist" Hodge theory is: how do we differentiate sections of E?

Let $\mathcal{A}^0(E)$ denote the sheaf of sections of E. Let $\mathcal{A}^k(E)$ denote the sheaf of sections of $E \otimes \bigwedge^k T^*E$. Note that if $g_{ij} : U_i \cap U_j \to GL_r(\mathbb{C})$ are the transition function data for E, then a global section $\omega \in H^0\mathcal{A}^k(E)$ is locally

$$\omega_i = (\omega_{i,1}, \cdots, \omega_{i,r}) \in A^k(U_i)^{\oplus r}$$

such that

$$g_{ij} \begin{pmatrix} \omega_{j,1} \\ \vdots \\ \omega_{j,r} \end{pmatrix} = \begin{pmatrix} \omega_{i,1} \\ \vdots \\ \omega_{i,r} \end{pmatrix}.$$

So we see that sections in $\mathcal{A}^k(E)$ are k-forms with coefficients in E, i.e. are vector valued in E. The proper generalization for the exterior derivative in order to differentiate sections of E is that of a connection.

Definition 30.1. A connection ∇ on a complex vector bundle $\pi : E \to X$ is a $\mathbb{C}$-linear sheaf homomorphism

$$\mathcal{A}^0(E) \to \mathcal{A}^1(E)$$

satisfying the Leibniz rule:

$$\nabla(f \cdot s) = df \cdot s + f \cdot \nabla(s)$$

for all local sections s of E and local functions f on X.

Let us unpackage what a connection ∇ is locally. Given a local trivialization $\pi^{-1}(U) \cong U \times \mathbb{C}^r$ of E, we can take the local frame $e_1, \cdots, e_r$. Note e_j is the constant function on U which maps to the j-th standard unit vector in $\mathbb{C}^r$. Any local section f of E can be written as $\sum f_j \cdot e_j$, so

$$\nabla(\sum f_j \cdot e_j) = \sum \nabla(f_j \cdot e_j) = \sum df_j \cdot e_j + f_j \cdot \nabla(e_j).$$

Then ∇ is determined by how it maps the local frame, $\nabla(e_j)$. We have

$$\nabla(e_j) = \sum a_{jk} e_k$$

where θ_{jk} are smooth 1-forms on U. Then locally, ∇ is determined by the matrix $A = (a_{jk})_{1 \le j,k \le r}$. If we write $f = \sum f_j \cdot e_j$ as a column vector, we see that

$$\nabla(f) = (d + A) \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}.$$

So locally, ∇ acts via $d + A$ with respect to the standard frame.

Proposition 30.2. *Given two connections ∇, ∇' on E, we have*

$$\nabla - \nabla' \in H^0 \mathcal{A}^1(End(E)).$$

Proof. Note that given any local section s of E, we have $(\nabla - \nabla')(s)$ is a local section of $E \otimes T^*X$. Furthermore,

$$(\nabla - \nabla')(f \cdot s) = f \cdot (\nabla - \nabla')(s)$$

for local function f on X. This implies the claim. □

Furthermore, if ∇ is a connection on E and $a \in H^0\mathcal{A}^1(End(E))$, then $\nabla' = \nabla + a$ is also a connection. For local section s of E, we have

$$\nabla'(s) = \nabla(s) + a(s),$$

which is a local section of $E \otimes T^*X$. Furthermore,

$$\nabla'(f \cdot s) = \nabla(f \cdot s) + a(f \cdot s) = df \cdot s + f \cdot \nabla(s) + f \cdot a(s) = df \cdot s + f[\nabla'(s)]$$

so ∇' satisfies the Leibniz rule.

Proposition 30.3. *The space of connections on E is naturally an affine space over $H^0\mathcal{A}^1(End(E))$.*

Let ∇_1 and ∇_2 be connections on E_1 and E_2. Then

- ∇ is a connection on $E_1 \oplus E_2$, where if $s = s_1 + s_2$ is a local section, then
$$\nabla(s) = \nabla_1(s_1) + \nabla_2(s_2).$$
- ∇ is a connection on $E_1 \otimes E_2$ where if $s_1 \otimes s_2$ is a local section, then
$$\nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla(s_2).$$
- ∇ is a connection on $Hom(E_1, E_2)$, where if ϕ is a local section, then $\nabla(\phi)$ is local section of $Hom(E_1, E_2) \otimes T^*X$ so that for local section s_1 of E_1,
$$\nabla(\phi)(s_1) = \nabla_2\phi(s_1) - \phi\nabla_1(s_1).$$
In particular, given connection ∇ on E, we naturally have connection ∇^* on $E^* = Hom(E, \mathcal{O}_X)$ where
$$\nabla^*(\phi)(s) = d\phi(s) - \phi\nabla(s).$$
for local section ϕ of $E^* \otimes T^*X$ and local section s of E.
- Given smooth $f : Y \to X$, a connection ∇ on vector bundle E over X naturally induces a connection $f^*\nabla$ on f^*E over Y (observe what happens locally on $f^{-1}(U) \to U$).

Now that we've discussed some general theory of connections, let's make it more interesting. In the Hodge theory that we did last semester, we saw that the exterior derivative crucially interacted with the choice of metric. Here we generalize compatability of hermitian metrics on complex bundles, and connections.

Recall that a hermitian metric on a complex vector bundle $E \to X$ on smooth manifold X is a hermitian inner product $h_p : E_x \times E_x \to \mathbb{C}$ which varies smoothly with p. In other words, given local trivialization

$$\phi : \pi^{-1}(U) \to U \times \mathbb{C}^r,$$

we have $h_x(\phi^{-1}(-), \phi^{-1}(-))$ is a hermitian inner product on $\mathbb{C}^r$ varying smoothly with x. Thus, if $h_{ij}(x) = h_x(\phi^{-1}(e_i), \phi^{-1}(e_j))$, then (h_{ij}) should be a smooth function $U \to GL_r(\mathbb{C})$ that is positive-definite everywhere.

Definition 30.4. Let (E, h) be a hermitian bundle over X. We say a connection ∇ on E is compatible with h if for all local sections s_1, s_2,

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2),$$

where we abuse notation so $h(s \otimes \omega, s_2) := h(s, s_2)\omega$ and $h(s_1, s \otimes \omega) = h(s_1, s)\overline{\omega}$.

We know that if $a \in H^0\mathcal{A}^1(End(E))$, then $\nabla' = \nabla + a$ defines another connection on E. Then ∇' is also compatible with the hermitian metric $\iff$ $h(as_1, s_2) + h(s_1, as_2) = 0$ for all local sections s_1, s_2 of E.

Proposition 30.5. *Let (E, h) be a hermitian bundle over X. Let $End(E, h)$ denote the subsheaf of $End(E)$ such that*

$$h(as_1, s_2) + h(s_1, as_2) = 0$$

for all appropriate local sections s_1, s_2. Then the space of all connections over E compatible with h is naturally n affine space over

$$H^0\mathcal{A}^1(End(E, h)).$$

Now recall that in the Hodge theory we did last semester, our hermitian metric decomposed, and the exterior derivative was separated into the $\partial, \overline{\partial}$ operators. We discuss a generalization of this idea.

Let E be a holomorphic vector bundle over complex manifold X. Note that the operator

$$\overline{\partial} : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$$

still remains a valid $\mathbb{C}$-linear sheaf homomorphism. This is because if we have local sections $\omega_j \in A^k(U_i)^{\oplus r}$ such that $g_{ij}\omega_j = \omega_i$ over $U_i \cap U_j$, then note

$$\overline{\partial} g_{ij}\omega_j = (\overline{\partial} g_{ij})\omega_j + g_{ij}\overline{\partial}\omega_j = g_{ij}\overline{\partial}\omega_j.$$

So because the transition functions are holomorphic, the $\overline{\partial}$-operator remains well-defined for twisted k-forms on a holomorphic vector bundle.

Note that the sheaf $E \otimes \Omega^1_X$ decomposes into $E \otimes A^{1,0}_X \oplus E \otimes A^{0,1}_X$, since $T_{\mathbb{C}}X \cong T^{1,0} \oplus T^{0,1}$, since X is a complex manifold. Then any connection

$$\nabla : E \to E \otimes \Omega^1_X \cong E \otimes A^{1,0} \bigoplus E \otimes A^{0,1}_X$$

splits into $\nabla = \nabla^{1,0} + \nabla^{0,1}$. We see that

$$\nabla^{1,0}(f \cdot s) = \partial f \cdot s + f \cdot \nabla^{1,0}(s) \text{ and } \nabla^{0,1}(f \cdot s) = \overline{\partial} f \cdot s + f \cdot \nabla^{0,1}(s),$$

for local section s of E and local function f on X.

Definition 30.6. Let E be a holomorphic vector bundle over complex manifold X. A connection ∇ on E is compatible with the holomorphic structure if

$$\nabla^{0,1} = \overline{\partial}.$$

Proposition 30.7. *The space of connections on E compatible with the holomorphic structure is an affine space over*

$$H^0\mathcal{A}^{1,0}(End(E)).$$

Given a hermitian holomorphic vector bundle (E, h), we arrive to the **Chern connection**.

Theorem 30.8 (Chern connection)**.** *There is a unique connection ∇ on E compatible with its hermitian metric and holomorphic structure.*

Proof. First we prove uniqueness. Verifying compatibility with the hermitian and holomorphic structure is a purely local problem, so we can assume E is the trivial bundle $X \times \mathbb{C}^r$.

Let $e_1, \cdots, e_r$ denote the standard frames. Note that we can write ∇ as $d + A$, where A is a matrix of 1-forms with respect to the standard frame. Let $H = (h_{ij})$ where $h_{ij} = h(e_i, e_j)$. Since ∇ is compatible with h, we have

$$dh(e_i, e_j) = h(\nabla e_i, e_j) + h(e_i, \nabla e_j) = h(\sum \omega_{ik} e_k, e_j) + h(e_i, \sum \omega_{j\ell} e_\ell),$$

which implies that

$$dH = A^t H + H\overline{A}.$$

Note $\nabla^{0,1} = \overline{\partial}$, and since $\nabla = d + A = \partial + \overline{\partial} + A$, we must have A is a matrix of (1,0) forms. Then

$$\overline{\partial} H = H\overline{A} \implies A = \overline{H}^{-1}\overline{\overline{\partial} H},$$

thus A, which determines ∇, is uniquely determined by the metric h.

An alternative proof of this fact is that, note if we have two such ∇, ∇' which are compatible with the holomorphic struture and hermitian metric, then

$$a = \nabla - \nabla' \in H^0\mathcal{A}^{1,0}(E) \cap H^0\mathcal{A}^1(End(E, h)).$$

But note that this intersection is 0. Why? Because then

$$h(as, t) + h(a, st) = 0,$$

the first term is a $(1, 0)$ form and the second is necessarily a $(0, 1)$ form. So both must be zero, so $a = 0$.

This proof also shows existence. Locally, we must have that

$$A = (\overline{H})^{-1}\partial\overline{H}.$$

If we restrict to a $U_i \cap U_j$, under the two identifications what A could be over $U_i \cap U_j$, note that the $\overline{H}$ pull back to each other. The operator $\overline{\partial}$ also commutes with pullback. So we get agreement on overlaps, and thus these local connections glue to a global connection. $\square$

31. Extra: Curvature

The notion of a connection generalized the exterior derivative so that we could differentiate sections of vector bundles. However, given a connection ∇ on E, we do not necessarily have $\nabla^2 = 0$, i.e. ∇ may not be a differential. The obstruction of a connection from being a differential is measured by its curvature.

Before we discuss curvature, let us extend connections to higher forms. Given a connection

$$\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E),$$

we define

$$\nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$$

by $\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \otimes \nabla(s)$ where ω is a local k-form and s is a section of E. This gives a $\mathbb{C}$-linear morphism of sheaves. It is also well-defined, as repeated use of Leibniz rule shows

$$\nabla(f\omega \otimes s) = df\omega \otimes s + (-1)^k f\omega \otimes \nabla(s) = fd\omega \otimes s + df \wedge \omega \otimes s + (-1)^k f\omega \otimes \nabla(s)$$

$$= d\omega \otimes fs + (-1)^k \omega \otimes \nabla(f \cdot s) = \nabla(\omega \otimes f \cdot s)$$

for local function f. Repeated use of the Leibniz rule shows there is an extended Leibniz rule. Given local ℓ-form ξ and ω is local section in $\mathcal{A}^k(E)$, we have

$$\nabla(\xi \wedge \omega) = d(\xi) \wedge \omega + (-1)^\ell \xi \wedge \nabla(\omega).$$

Definition 31.1. The curvature F_∇ of a connection ∇ on E is defined to be

$$F_\nabla := \nabla \circ \nabla : \mathcal{A}^0(E) \to \mathcal{A}^2(E).$$

Note that if $F_\nabla = 0$, then $\nabla \circ \nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+2}(E)$ will be zero for every k. This can be seen from:

$$\nabla \circ \nabla(\omega \otimes s) = \nabla(d\omega \otimes s + (-1)^k \omega \wedge \nabla(s)) = (-1)^{k+1} d\omega \wedge \nabla(s) + (-1)^k \nabla(\omega \wedge \nabla(s))$$

$$= (-1)^{k+1} d\omega \wedge \nabla(s) + (-1)^k d\omega \wedge \nabla(s) + (-1)^{2k} \omega \wedge F_\nabla(s) = 0.$$

Proposition 31.2. $F_\nabla \in H^0\mathcal{A}^2(End(E))$.

Proof. This can be seen from observing that

$$F_\nabla(f \cdot s) = \nabla \circ \nabla(f \cdot s) = \nabla(df \otimes s + f \cdot \nabla(s)) = -df \wedge \nabla(s) + df \wedge \nabla(s) + f \cdot F_\nabla(s) = f \cdot F_\nabla(s)$$

for all local functions f and local sections s of E. $\square$

Let's examine the curvature of connections on trivial bundle $X \times \mathbb{C}^r$. If the connection is trivial, then the connection is just given by d. Then the curvature is $d^2 = 0$.

Otherwise, the connection is given by $d + A$, where $A = (a_{jk})$ where $\nabla(e_k) = \sum a_{jk} e_j$. Then

$$F_\nabla(s) = (d + A)(ds + As) = d(As) + A(ds) + AAs$$

$$= (dA)s - A(ds) + A(ds) + AAs \implies F_\nabla = dA + A \wedge A,$$

where $A \wedge A$ is like matrix multiplication, except you're wedging instead of taking products of entries.

32. 2/5/24: first Chern class and curvature of Chern connection

Recall that on a complex manifold X, the exponential exact sequence

$$0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$$

yielded a map

$$H^1(X, \mathcal{O}_X^\times) \to H^2(X, \underline{\mathbb{Z}}_X)$$

which we call the first Chern class map. Since X is locally contractible, $H^2(X, \underline{\mathbb{Z}}_X) \cong H^2(X; \mathbb{Z})$, and the former can be identified with $Pic(X)$. Thus, we are interested in the map

$$c_1 : Pic(X) \to H^2(X; \mathbb{Z}).$$

We will now work to relate the cohomology class $c_1(\mathcal{L})$ to concrete geometry about $\mathcal{L}$ and X, namely the cohomology class of a curvature form.

Recall that any holomorphic vector bundle admits a hermitian metric (take hermitian metric on local trivilizations combined with partition of unity). Let $\mathcal{L} \in Pic(X)$. It admits a hermitian metric h. Let ∇ denote the Chern connection with respect to $(\mathcal{L}, h)$. Then locally, ∇ looks like $d + A$ where $A = \overline{H}^{-1}\partial\overline{H}$, but note that H is a single real-valued function, so

$$A = \frac{\partial h}{h} = \partial \log h.$$

Then the curvature F_∇ of ∇ is locally

$$dA + A \wedge A = d\partial \log h = \overline{\partial}\partial \log h = -\partial\overline{\partial} \log h.$$

Proposition 32.1. *Let $\mathcal{L}$ be holomorphic line bundle over complex manifold X. Let h be a hermitian metric. Let ∇ be the Chern connection of (L, h), and let F_∇ be the curvature of the Chern connection. Then*

1. *$F_\nabla \in A^2(X)$*
2. *$[F_\nabla] \in H^{1,1}(X)$ is well-defined, i.e. does not depend on the choice of metric h.*
3. *If $F_{\nabla'}$ is the curvature for Chern connection on $\mathcal{L}^{-1}$, then $[F_{\nabla]}] = -[F_\nabla]$.*
4. *If we have a tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$, then*

$$[F_{\nabla_1 \otimes \nabla_2}] = [F_{\nabla_1}] + [F_{\nabla_2}].$$

Proof. (1) We have $F_\nabla \in H^0\mathcal{A}^2(End(\mathcal{L}))$, and since $\mathcal{L}$ is a line bundle, then $\mathcal{L}^\vee \cong \mathcal{L}^{-1}$. And $End(\mathcal{L}) \cong \mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{O}_X$.

(2) Note since F_∇ is locally $-\partial\overline{\partial}h$, we see that it gives a class in $H^{1,1}(X)$ (with respect to either ∂ or $\overline{\partial}$, of course these coincide by conjugating when X is compact via the Hodge theory we did last semester). Furthermore, if h' is another metric, then we have $h' = \psi h$ where $\psi : X \to \mathbb{R}_+$, so

$$\overline{\partial}\partial \log h' = \overline{\partial}\partial \log \psi h = \overline{\partial}\partial \log \psi + \overline{\partial}\partial \log h$$

so the Dolbeaut cohomology class is well-defined.

(3) Suppose the hermitian metric for $\mathcal{L}$ is h. We also let h denote the local function $h(e, e)$, where e is the unit frame in the local trivialization. There is an induced hermitian metric for $\mathcal{L}^\vee$, and on local trivialization, $h^\vee = \frac{1}{h}$. Note then that

$$\overline{\partial}\partial \log h = -\overline{\partial}\partial \log \frac{1}{h}.$$

(4) Hermitian metrics on $\mathcal{L}_1, \mathcal{L}_2$ induce hermitian metric on $\mathcal{L}_1 \otimes \mathcal{L}_2$ given by the obvious product. So if locally the hermitian matrices with respect to unit frames is h_1, h_2, then it is $h_1 h_2$ for $\mathcal{L}_1 \otimes \mathcal{L}_2$. Then we see
$$\overline{\partial}\partial \log h_1 h_2 = \overline{\partial}\partial \log h_1 + \overline{\partial}\partial \log h_2.$$

$\square$

A priori, we have that $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$, and curvature form $[F_L] \in H^{1,1}(X) \subseteq H^2(X, \mathbb{C})$. It turns out that we have:

Theorem 32.2. *Let $\mathcal{L}$ be holomorphic line bundle over complex manifold X. Let $[F_{\mathcal{L}}] \in A^{1,1}(X)$ denote the curvature $(1,1)$-form associated to $\mathcal{L}$. Then*
$$c_1(\mathcal{L}) = \frac{i}{2\pi}[F_{\mathcal{L}}].$$

Proof. Here is the main idea of the proof. We are first going to interpret $c_1(\mathcal{L})$ in terms of its Cech cohomology data from the transition functions of $\mathcal{L}$. Then we will pass through the identification of sheaf cohomology of $\underline{\mathbb{Z}}_X$ to deRham cohomology to interpret $c_1(\mathcal{L})$ as a 2-form. Then we will relate this 2-form to the curvature form and see they agree up to $\frac{i}{2\pi}$.

Let's first get started on interpreting $c_1(\mathcal{L})$ via Čech cohomology. Choose $\underline{U} = \{U_\alpha\}$ open cover which provide local trivializations of $\mathcal{L}$, and such that the U_α are contractible. Denoting the transition functions as $g_{\alpha\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)^\times$, note we have $\{g_{\alpha\beta}\} \in \check{H}^1(\underline{U}, \mathcal{O}_X^\times)$.

Note that in the exponential exact sequence
$$0 \to \underline{\mathbb{Z}}_{U_\alpha \cap U_\beta} \to \mathcal{O}_{U_\alpha \cap U_\beta} \to \mathcal{O}^\times_{U_\alpha \cap U_\beta} \to 0$$
since $U_\alpha \cap U_\beta$ is contractible, we have
$$H^1(U_\alpha \cap U_\beta, \underline{\mathbb{Z}}_{U_\alpha \cap U_\beta}) \cong H^1(U_\alpha \cap U_\beta, \mathbb{Z}) = 0.$$
So for each $g_{\alpha\beta}$, there exist $f_{\alpha\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)$ such that
$$e^{2\pi i f_{\alpha\beta}} = g_{\alpha\beta}.$$
Note that the cocycle condition tells us that
$$g_{\alpha\beta} = g_{\alpha\gamma} g_{\gamma\beta},$$
so
$$c_{\alpha\beta\gamma} = f_{\gamma\beta} - f_{\alpha\beta} + f_{\alpha\gamma} \in \mathbb{Z}.$$
Then we have the $\{c_{\alpha\gamma\beta}\} \in H^2(\underline{U}, \underline{\mathbb{Z}}_X) \cong H^2(X, \mathbb{Z})$, since $\underline{\mathbb{Z}}_X$ is acyclic over $\underline{U}$, and this cocyle data is identified with $c_1(\mathcal{L})$.

Let us now interpret $c_1(\mathcal{L}) = \{c_{\alpha\gamma\beta}\}$ via deRham's theorem. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then define
$$\phi_\beta = \sum_\alpha \rho_\alpha df_{\alpha\beta}.$$
Then note that $\phi_\beta - \phi_\gamma = \sum_\alpha \rho_\alpha [df_{\alpha\beta} - df_{\alpha\gamma}] = \sum_\alpha \rho_\alpha df_{\gamma\beta} = df_{\gamma\beta}$. Then $d\phi_\beta = d\phi_\gamma$. Then the $\{d\phi_\beta\}_\beta$ glue to a global $\omega \in H^2(X)$, and we have the identification
$$c_1(\mathcal{L}) = \{c_{\alpha\gamma\beta}\} \in H^2(X, \mathbb{Z}) \mapsto \omega \in H^2_{dR}(X)$$
via deRham's theorem.

Now we use this differential forms interpretation of $c_1(\mathcal{L})$ to relate it to the curvature form. Suppose on U_α we have $h_\alpha = h(e_\alpha, e_\alpha)$, where e_α is the unit frame over U_α. Then $h(g_{\alpha\beta}e_\alpha, g_{\alpha\beta}e_\alpha) = |g_{\alpha\beta}|^2 h_\alpha$, so this implies that $h_\beta = |g_{\alpha\beta}|^2 h_\alpha$. Then

$$\partial \log h_\beta = \partial \log |g_{\alpha\beta}|^2 + \partial \log h_\alpha,$$

and note $\partial \log |g_{\alpha\beta}|^2 = \partial \log g_{\alpha\beta}\overline{g_{\alpha\beta}} = \frac{\partial g_{\alpha\beta}}{g_{\alpha\beta}} = \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} = d \log g_{\alpha\beta}$, so

$$\overline{\partial}\partial \log h_\beta = \overline{\partial} 2\pi i df_{\alpha\beta} + \overline{\partial}\partial \log h_\alpha,$$

so

$$2\pi i \phi_\beta = \sum_\alpha \rho_\alpha (2\pi i) df_{\alpha\beta} = \sum_\alpha \rho_\alpha [\partial \log h_\beta - \partial \log h_\alpha] = \partial \log h_\beta - \sum_\alpha \partial \log h_\alpha,$$

so

$$d\phi_\beta = \frac{i}{2\pi} d\partial \log h_\beta - \frac{i}{2\pi} d \sum_\alpha \rho_\alpha \partial \log h_\alpha$$

$$\implies d\phi_\beta = \frac{i}{2\pi} \overline{\partial}\partial \log h_\beta - \frac{i}{2\pi} d \sum_\alpha \rho_\alpha \partial \log h_\alpha$$

which implies that

$$c_1(\mathcal{L}) = [\omega] = [F_L].$$

□

Note that we have the $H^{2,0}$ term hanging around, but note that $(H^{2,0} \oplus H^{0,2}) \cap H^2(X;\mathbb{Z}) = H^{0,2} \cap H^2(X;\mathbb{Z})$, because real forms are invariant under complex conjugation.

Looking forwards, we are going to prove the Lefschetz (1,1) theorem, Kodaira vanishing theorem, and the weak Lefschetz theorem. In Huybrechts, he denotes

$$H^{1,1}(X,\mathbb{Z}) := H^{1,1}(X) \cap H^2(X,\mathbb{Z}) \subseteq H^2(X,\mathbb{C})$$

to be the Hodge classes. What we have shown is that the Chern classes of line bundles are Hodge classes. In general, Hodge classes are elements of $H^{p,p}(X) \cap H^{2p}(X,\mathbb{Z}) \subseteq H^{2p}(X,\mathbb{C})$. Often when we produce Hodge classes, there will be a common theme: we will be dealing with (higher) Chern classes, looking at divisors, etc. So the Hoge classes we construct often come from some "concrete" geometry. The Hodge conjecture posits that this is always the case; that the Hodge classes actually all come from "geometry." The Lefschetz (1,1) theorem is that every Hodge class of type (1,1) on a compact Kahler manifold actually comes from geometry. So the Hodge conjecture is true in the case of $(1,1)$ on compact Kahler.

33. 2/7/24: Lefschetz (1,1), Hodge conjecture, twisted Dolbeaut, positivity

Today we will talk about the Lefschetz (1,1) theorem. Lefschetz originally proved this in a very different way, first in low dimensions. It should also be thought of as a lemma, since it will be immediate from the work we've done.

Theorem 33.1 (Lefschetz (1,1)). *Let X be a compact Kahler manifold. Then every integral Hodge class of type (1,1)*

$$H^{1,1}(X,\mathbb{Z}) := H^{1,1}(X) \cap H^2(X;\mathbb{Z})$$

comes from the first Chern class of a line bundle on X.

Proof. Taking sheaf cohomology on the exponential exact sequence, we obtain

$$\cdots \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \underline{\mathbb{Z}}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

The claim is that the following diagram commutes:

$$\begin{tikzcd}
H^1(X, \mathcal{O}_X) \arrow[r] & H^2(X, \underline{\mathbb{Z}}) \arrow[r] \arrow[d, hook] & H^2(X, \mathcal{O}_X) \cong H^{0,2}(X) \\
& H^2(X, \mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X) \arrow[ur, "\pi"] &
\end{tikzcd}$$

where π is given by projection. First, suppose we show that this diagram commutes. Then the theorem follows immediately. Because if we have a integral Hodge class of type (1,1), it will be in the kernel of the map, and thus must be in the image of $H^1(X, \mathcal{O}_X^*) \to H^2(X, \underline{\mathbb{Z}})$.

Let us now show commutativity of the diagram. Note that $\underline{\mathbb{Z}}$ is acyclic on contractible cover of X, and so is $\mathcal{O}_X$ via the Dolbeaut theorem. Thus, we will think of the sheaf cohomology elements as Čech cocycles. Showing commutativity of this diagram will amount to showing explicitly the isomorphisms from Cech cohomology to deRham cohomology.

First, what is the map from $\check{H}^2(X, \underline{\mathbb{Z}}) \to H^2_{dR}(X, \mathbb{Z})$? Suppose we have a cover $\{U_\alpha\}$ of contractible opens whose intersections are also contractible (this is possible because X is paracompact). Then suppose we have Cech cocyle

$$\{c_{\alpha\beta\eta_1}\} \in \check{H}^2(X, \underline{\mathbb{Z}}).$$

So each $c_{\alpha\beta\eta_1}$ is a constant integer function on $U_{\alpha\beta\eta_1}$. Here is how we get a 2-form. Take a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. Then take

$$\sum_\alpha \rho_\alpha c_{\alpha\beta\eta_1},$$

where the ρ_α are restricted to $U_{\alpha\beta\eta_1}$. This is well-defined over $U_{\beta\eta_1}$. Taking the exterior derivative yields

$$\sum_\alpha d\rho_\alpha \wedge c_{\alpha\beta\eta_1}.$$

defined over $U_{\beta\eta_1}$. Now consider

$$\sum_{\alpha,\beta} \rho_\beta d\rho_\alpha \wedge c_{\alpha\beta\eta_1},$$

and this is defined over U_{η_1}. Then take the exterior derivative

$$\sum_{\alpha,\beta} d\rho_\beta \wedge d\rho_\alpha \wedge c_{\alpha\beta\eta_1},$$

then define

$$\omega = \sum_{\alpha,\beta,\eta_1} \rho_{\eta_1} d\rho_\beta \wedge d\rho_\alpha \wedge c_{\alpha\beta\eta_1}.$$

This is the global 2-form in $H^2_{dR}(X, \mathbb{Z})$ corresponding to $\{c_{\alpha\beta\eta_1}\}$. To see that it is closed, note that the differential is

$$d\omega = \sum_{\alpha\beta,\eta_1} d\rho_{\eta_1} \wedge d\rho_\beta \wedge d\rho_\alpha \wedge c_{\alpha\beta\eta_1}.$$

Furthermore, over $U_{\alpha\beta\eta_1\eta_2}$, we have

$$c_{\beta\eta_1\eta_2} - c_{\alpha\eta_1\eta_2} + c_{\alpha\beta\eta_2} - c_{\alpha\beta\eta_1}.$$

So we have

$$c_{\beta\eta_1\eta_2} - \sum_\alpha \rho_\alpha c_{\alpha\eta_1\eta_2} + \sum_\alpha c_{\alpha\beta\eta_2} - \sum_\alpha c_{\alpha\beta\eta_1} \implies 0 = -\sum_\alpha d\rho_\alpha \wedge c_{\alpha\eta_1\eta_2} + \sum_\alpha d\rho_\alpha \wedge c_{\alpha\beta\eta_2} - \sum_\alpha d\rho_\alpha \wedge c_{\alpha\beta\eta_1}$$

and

$$0 = -\sum_\alpha d\rho_\alpha \wedge c_{\alpha\eta_1\eta_2} + \sum_{\alpha,\beta} \rho_\beta d\rho_\alpha \wedge c_{\alpha\beta\eta_2} - \sum_{\alpha,\beta} \rho_\beta d\rho_\alpha \wedge c_{\alpha\beta\eta_1}$$

so we have

$$0 = \sum_{\alpha,\beta} d\rho_\beta \wedge d\rho_\alpha \wedge c_{\alpha\beta\eta_2} - \sum_{\alpha,\beta} d\rho_\beta \wedge d\rho_\alpha \wedge c_{\alpha\beta\eta_1}$$

then

$$0 = \sum_{\alpha,\beta} d\rho_\beta \wedge d\rho_\alpha \wedge c_{\alpha\beta\eta_2} - \sum_{\alpha,\beta,\eta_1} \rho_{\eta_1} d\rho_\beta \wedge d\rho_\alpha \wedge c_{\alpha\beta\eta_1} \implies 0 = \sum_{\alpha,\beta,\eta_1} d\rho_{\eta_1} \wedge d\rho_\beta \wedge d\rho_\alpha \wedge c_{\alpha\beta\eta_1}$$

Now what is the map $H^2(X, \mathcal{O}_X) \cong H^{0,2}(X)$? If we have Cech cocyle $\{f_{\alpha\beta\eta}\}$, then by the same idea as the above, the $(0,2)$ form associated to it is

$$\sum_{\alpha,\beta,\eta} \rho_\eta \overline{\partial}\rho_\beta \wedge \overline{\partial}\rho_\alpha \wedge f_{\alpha\beta\eta}.$$

Now that we have unpackaged the Cech-to-deRham maps, commutativity follows easily.

The map $H^2(X, \underline{\mathbb{Z}}) \to H^2(X, \mathcal{O}_X)$ sends $[\{c_{\alpha\beta\eta}\}] \mapsto [\{c_{\alpha\beta\eta}\}]$. The form in $H^2_{dR}(X, \mathbb{Z})$ is given by

$$\sum_{\alpha\beta\eta} \rho_\eta d\rho_\beta \wedge d\rho_\alpha \wedge c_{\alpha\beta\eta} = \sum_{\alpha\beta\eta} \rho_\eta (\partial + \overline{\partial})\rho_\beta \wedge (\partial + \overline{\partial})\rho_\alpha \wedge c_{\alpha\beta\eta},$$

and projecting to $H^{0,2}(X)$ yields

$$\sum_{\alpha\beta\eta} \rho_\eta \overline{\partial}\rho_\beta \wedge \overline{\partial}\rho_\alpha \wedge c_{\alpha\beta\eta},$$

thereby proving commutativity.

A nice reference for general analytic sheaf cohomology technology, deRham-Weil isomorphism, and for getting the explicit map from deRham to Cech can be found in Demailley's notes: https://www-fourier.ujf-grenoble.fr/~demailly/analytic_geometry_2019/sheaves_cech_cohomology.pdf. □

Remark 33.2. So for projective varieties, which are compact Kahler, every integral Hodge (1,1) class comes from geometry. It comes from a first Chern class of a line bundle, and thus comes from a hypersurface (apriori, Weil divisors moduli linear equivalence injects to Picard group, but is surjection for projective manfiolds). Furthermore, there's a story with poincare duality. The first Chern class $c_1(\mathcal{L}) = \eta_D \in H^2(X, \mathbb{Z})$, and by Poincare duality, we have this η_D maps to $[D] \in H_{2n-2}(X, \mathbb{Z})$. There's a lot of analysis involved here, like estimates of integrals. Can find it in Griffths and Harris page 141 or Huybrechts 4.4.13.

Have to develop a stokes theorem for complex varieties, look at regular sets, do estimates of integrals, etc.

And note $H_{2n-2}(X,\mathbb{Z}) \cong H^{2n-2}(X,\mathbb{Z})^\vee$. So we are looking at dual on singular cohomology. It is obtained by taking

$$\omega \mapsto \int_D \omega|_D.$$

Remark 33.3. There's a duality to this Lefschetz (1,1) theorem when $\omega \in H^{1,1}(X,\mathbb{Q})$ (which we will see later holds $\iff$ X is projective).

Recall the Hard Lefschetz theorem: we have an isomorphism

$$L^{n-2} : H^2(X;\mathbb{C}) \to H^{2n-2}(X;\mathbb{C}).$$

In particular, this restricts to an isomorphism

$$H^{1,1}(X) \to H^{n-1,n-1}(X).$$

Furthermore, if the Kahler form $\omega \in H^{1,1}(X,\mathbb{Q})$, then we also have a restriction

$$H^{1,1}(X,\mathbb{Q}) \to H^{n-1,n-1}(X,\mathbb{Q}).$$

Thus, if we have a nice theorem like all the cohomology classes of $H^{1,1}(X,\mathbb{Q})$ come from geometry, then the same is true for $H^{n-1,n-1}(X,\mathbb{Q})$.

We arrive to the most important conjecture in algebraic geometry.

Theorem 33.4 (Hodge conjecture)**.** *If X is a smooth projective manifold, then every class in*

$$H^{p,p}(X,\mathbb{Q}) := H^{p,p}(X) \cap H^{2p}(X,\mathbb{Q}) \subseteq H^{2p}(X,\mathbb{C})$$

is analytic, i.e. it is (the Poincare dual of) a $\mathbb{Q}$-linear combination of classes $[Z]$, where $Z \subseteq X$ is an analytic subvariety of X.

Note that Lefschetz (1,1) proves Hodge conjecture over (1,1). But Lefschetz (1,1) is actually stronger because its over integral cohomology and more generally deals with compact Kahler manifolds. So the nature of the problem is quite different. The Hodge conjecture if false for integral cohomology. Atiyah and Hirzebruch proved it was false over $\mathbb{Z}$. If X is not smooth projective, then the Hodge conjecture is definitely not true – there are examples where there are no analytic subvarieties, you can't do any geometry on it.

Say $Z \subseteq X$ is a closed submanifold of codimension p, where X has dimension n. We have a map

$$\phi_Z : H^{2n-2p}(X,\mathbb{C}) \to \mathbb{C}$$

which sends a harmonic representative of a class ω to $\int_Z \omega|_Z$. So

$$\phi_Z \in H^{2n-2p}(X,\mathbb{C})^\vee \cong H^{2p}(X,\mathbb{C})$$

and the claim is that Poincare duality gives you a map $H_{2n-2p}(X,\mathbb{Z}) \to H^{2p}(X,\mathbb{C}) \cong H^{2n-2p}(X,\mathbb{C})^\vee$ where

$$[Z] \mapsto \phi_Z.$$

We have

$$\int_Z \omega|_Z = \int_X \eta_Z \wedge \omega =: \int_X \alpha \wedge \omega$$

where α is the harmonic representative of η_Z, and where we claim that $[\alpha] = \eta_Z \in H^{p,p}(X)$. note $\int_Z \omega|_Z = \int_Z \omega^{n-p,n-p}|_Z = \int_X \alpha \wedge \omega^{n-p,n-p} = \int_X \alpha^{p,p} \wedge \omega^{n-p,n-p} = \int_X \alpha^{p,p} \wedge \omega$. But non-degeneracy, perfect pairing, gives you $\alpha^{p,p} = \alpha$.

Another approach: you an use resolution of singularities. Singular algebraic varieties always have a birational map to something nonsingular. So instead of defining a class of analytic subset, you could define the class of a smooth variety, together with a map to X. Birational map.

Example 33.5. If $X = \mathbb{P}^n$, and let $L = \mathcal{O}_{\mathbb{P}^n}(-1) \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$. Note L has hermitian metric induced from the Euclidean metric since it is a subbundle of the trivial bundle.

Taking trivialization over $U_0 = (z_0 \neq 0)$, we have a section s_0 trivializing this bundle restriction, which is

$$s_0(1 : z_1 : \cdots : z_n) = (1, z_1, \cdots, z_n).$$

So $h_0 = h(s_0, s_0) = 1 + |z_1|^2 + \cdots + |z_n|^2$. We know how to compute the curvature and first Chern class:

$$-\frac{i}{2\pi}\partial\overline{\partial}\log(1 + |z_1|^2 + \cdots + |z_n|^2) = -\omega_{FS}|_{U_0}.$$

But we've seen this before! This is the Fubini-Study metric. So we have

$$c_1(\mathcal{O}_{\mathbb{P}^n}(-1)) = -[\omega_{FS}] \in H^{1,1}(\mathbb{P}^n).$$

Thus,

$$c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = [\omega_{FS}].$$

Note ω_{FS} is a positive form. We will later say then that $\mathcal{O}_{\mathbb{P}^n}(1)$ is a positive line bundle.

Example 33.6. Let (X, h) be a hermitian complex manifold. Then we can obtain a local description of the curvature associated to its canonical line bundle ω_X in terms of h. Note h is a metric on the holomorphic tangent bundle $T'X$.

Locally, we can write $h_{jk} = h(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k})$, and let $H = (h_{jk})$.

The metric naturally induces a metric on $\Omega_X := (T'X)^*$. Locally, we can write this as $h^{jk} = h(dz_j, dz_k)$. $(h^{jk}) = H^{-1}$. And thus the metric on $\bigwedge^n \Omega_X = \omega_X$ is $h(dz_1 \wedge \cdots \wedge dz_n, dz_1 \wedge \cdots \wedge dz_n) = \det H^{-1} = \frac{1}{\det H}$.

Then the curvature of the Chern connection on ω_X is

$$-\partial\overline{\partial}\log\frac{1}{\det H} = \partial\overline{\partial}\log\det H.$$

To see $(h^{jk}) = H^{-1}$, begin by noting that

$$dz_j = h(-, \sum_t c_{jt}\frac{\partial}{\partial z_t})$$

and

$$dz_j(\frac{\partial}{\partial z_\ell}) = h(\frac{\partial}{\partial z_\ell}, \sum_t c_{jt}\frac{\partial}{\partial z_t}) = \sum_t \overline{c_{jt}} h_{\ell t} = \delta_{j\ell}$$

then follow your nose.

We are working towards the Kodaira vanishing and Kodaira embedding theorem. Let us briefly discuss Harmonic theory and Hodge theorem in this generalized context. The proofs are all the same in the classical setting, but at least we'll mention the statements.

Let's fix a line bundle $\mathcal{L}$ on a compact complex manifold X. We can always define

$$\overline{\partial} : A^{p,q}(\mathcal{L}) \to A^{p,q+1}(\mathcal{L}).$$

where $\overline{\partial}^2 = 0$. So we can define a generalized Dolbeaut cohomology with coefficients in line bundles.

Definition 33.7. The Dolbeaut cohomology groups with coefficients in $\mathcal{L}$ are

$$H^{p,q}(X, \mathcal{L}) := \frac{\ker \overline{\partial}}{Im(\overline{\partial})}.$$

This is a twisted version of ordinary Dolbeaut cohomology $H^{p,q}(X) = H^{p,q}(X, \mathcal{O}_X)$. Doing this sheaf-theoretically, instead of just on global sections, gives us the Dolbeaut complex

$$0 \to \mathcal{A}^{p,0}(\mathcal{L}) \to^{\overline{\partial}} \mathcal{A}^{p,1}(\mathcal{L}) \to^{\overline{\partial}} \mathcal{A}^{p,2}(\mathcal{L}) \to^{\overline{\partial}} \cdots$$

Proposition 33.8. *(1) $\mathcal{A}^{p,*}(\mathcal{L})$ is a resolution of $\Omega_X^p \otimes \mathcal{L}$.*
(2) $\mathcal{A}^{p,q}(\mathcal{L})$ is a fine sheaf.

These are just as we did in the untwisted version. So we have the twisted Dolbeaut theorem:

$$H^q(X, \Omega_X^p \otimes \mathcal{L}) \cong H^{p,q}(X, \mathcal{L}).$$

These will be our main objects of study in the near future. Big story in algebraic geometry, you consider canonical line bundle, and you twist by a positive line bundle, then a lot of these twisted Dolbeaut cohomologies vanish.

Now we have Hodge theory as well. Fix hermitian metric h on X, and you pick some metric $h_{\mathcal{L}}$ on $\mathcal{L}$. Then you get a hermitian inner product on $A^{p,q}(X, \mathcal{L})$ and recall this was done by integration. So we have

$$\langle \omega_1 \otimes s_1, \omega_2 \otimes s_2 \rangle := \int_X h(w_1, w_2) h_L(s_1, s_2) \cdot vol(g)$$

for every $\omega_i \in A^{p,q}(X)$ and $s_i \in \mathcal{A}^0(X, \mathcal{L})$. In particular, we can define $\overline{\partial}^*$ to be the adjoint of the $\overline{\partial}$ operator with respect to $(,)_{\mathcal{L}}$. Then we can construct the Laplace operator

$$\overline{\Box} := \overline{\partial \partial}^* + \overline{\partial}^* \overline{\partial}$$

Laplace operator that is self adjoint on $A^{p,q}(X, \mathcal{L})$. Satisfies the same theory of ellptic operator, in particular we have the kernel and image decomposition of forms. So

$$\mathcal{H}^{p,q}(X, \mathcal{L}) = \ker \overline{\Box} = \text{ harmonic p,q forms with values in } \mathcal{L}$$

We thought of harmonic forms as minimal representatives of cohomology classes with respect to metric, and the same story is true here. There is also a twisted Hodge decomposition, but it only works for line bundles with zero chern class which implies it admits a holomorphic connection. So such line bundles which admit twisted Hodge decomposition live in $Pic^0(X)$.

Now we will talk about positive line bundles and Kodaira vanishing theorem.

Definition 33.9. A line bundle $\mathcal{L}$ on complex manifold X is positive if $c_1(\mathcal{L})$ can be represented by a closed $(1, 1)$ form ω with positive definite associated hermitian form. We say ω is positive.

In other words, note we can represent ω locally as

$$\frac{i}{2} \sum_{jk} h_{jk} dz_j \wedge d\overline{z_k}$$

where $H = (h_{jk})$ is positive definite Hermitian metric. Recall that we have ω is closed.

This automatically implies that the existence of a positive line bundle on X implies X is Kahler, since that $c_1(\mathcal{L})$ has associated hermitian metric and this gives a hermitian metric on X.

Example 33.10. First, you can take $X = \mathbb{P}^n$, and $L = \mathcal{O}_{\mathbb{P}^n}(1)$, and we said

$$c_1(\mathcal{L}) = [\omega_{FS}]$$

and we know this is a positive (1,1) form. So $\mathcal{O}_{\mathbb{P}^n}(1)$ is positive line bundle. But if $Y \subseteq \mathbb{P}^n$, then you can take

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)|_Y,$$

and the first Chern class of this will be the restriction of the fubini-study (1,1) form to Y. So the restriction of any positive line bundle to Y will be positive.

In fact, what we are going to prove is that this is really all the examples. This contains all of projective algebraic geometry.

Lemma 33.11. *Suppose X is a complex manifold. Then a line bundle $\mathcal{L}$ is positive $\iff$ $\mathcal{L}^{\otimes m}$ is positive for every $m \geq 1$.*

Proof. Suppose $\mathcal{L}$ is positive. Then the 1,1 form $c_1(\mathcal{L})$ is d-closed and has associated hermitian metric which is positive-definite. Note $c_1(\mathcal{L}^{\otimes m})$ is $mc_1(\mathcal{L})$. Then if locally $c_1(\mathcal{L})$ is $\frac{i}{2}h_{jk}dz_j \wedge d\overline{z_k}$, then locally $c_1(\mathcal{L}^{\otimes m})$ is $\frac{i}{2}mh_{jk}dz_J \wedge d\overline{z_k}$. Then if (h_{ij}) is positive definite, certainly (mh_{ij}) is positive definite as well, since $m \geq 1$.

The reverse implication is trivially true. $\square$

There will be some positive line bundles that do not give us a map to projective space, but up to some high enough power they do give us maps. This is the difference between ample and very ample.

Theorem 33.12 (Kodaira-Akiguki-Nakano Vanishing)**.** *If $\mathcal{L}$ is a positive line bundle on compact complex X of dimension n, then*

$$H^q(X, \Omega_X^p \otimes \mathcal{L}) = 0$$

for every $p + q > n$.

Thus, when $p = n : H^q(X, \omega_X \otimes \mathcal{L}) = 0$ for all $q > 0$. This is called Kodaira vanishing. For preparation, we need the generalized Kahler identities.

Lemma 33.13. *If $\mathcal{L}$ is positive line bundle on compact Kahler manifold with $c_1(\mathcal{L}) = [\omega]$, where ω is closed positive $(1,1)$ form (associated to Kahler metric h). Then there exists a (essentially unique) hermitian metric $h_{\mathcal{L}}$ on $\mathcal{L}$ such that $\omega = \frac{i}{2\pi}\Omega_{\mathcal{L}}$*

This will be an application of the $\partial\overline{\partial}$-lemma.

34. Serre duality, some twisted Hodge theory

Let (X, h) be a compact complex manifold with a hermitian metric, i.e. a hermitian metric on its holomorphic tangent bundle $T'X$. Let E be a holomorphic vector bundle over X with hermitian metric h_E. In this section we will twist some of the classical Hodge theory we did last semester. In particular, we will prove a Hodge isomorphism nd prove Serre duality.

Recall that $\overline{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ is well-defined. Our first step is to define a metric on each $\mathcal{A}^{p,q}(E)$. This metric should look like

$$\langle \alpha_1 \otimes s_1, \alpha_2 \otimes s_2 \rangle = \int_X h(\alpha_1, \alpha_2) h(s_1, s_2) vol(g).$$

But in untwisted Hodge theory, to define adjoints of operators we needed the Hodge star operator $*$. So we should also define Hodge $*$ operator in this twisted setting, and the aforementioned metric of forms guides us to the correct definition. The inner terms of the integral has a $h(\alpha_1, \alpha_2) vol(g)$, so we should expect to use the usual Hodge $*$ operator on differential forms. But we also have the $h(s_1, s_2)$ term, so we need to define a twisted Hodge $*$ operator which makes this term naturally fall out.

Definition 34.1. Define

$$\overline{*_E} : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E^*)$$

to be the $\mathbb{C}$-antilinear morphism of sheaves such that

$$\overline{*_E}(\alpha_1 \otimes s_1) = *(\overline{\alpha_1}) \otimes h_E(-, s_1)$$

for local (p, q) form α_1 and s_1 a local section of E.

We see that $(\alpha_1 \otimes s_1) \wedge \overline{*_E}(\alpha_2 \otimes s_2) = \alpha_1 \wedge * \overline{\alpha_2} \cdot h(s_1, s_2) = h(\alpha_1, \alpha_2) h_E(s_1, s_2) vol(g)$. Thus,

$$\langle \alpha_1 \otimes s_1, \alpha_2 \otimes s_2 \rangle = \int_X h(\alpha_1, \alpha_2) h_E(s_1, s_2) vol(g) = \int_X (\alpha_1 \otimes s_1) \wedge \overline{*_E}(\alpha_2 \otimes s_2).$$

Furthermore, note that $\overline{*}_E \overline{*}_E(\alpha \otimes s) = \overline{*}_E * \overline{\alpha} \otimes h(-, s) = ** \alpha \otimes s = (-1)^{p+q} \alpha \otimes s \implies \overline{*}_E \overline{*}_E = (-1)^{p+q}$. With $\overline{*_E}$, we can define the adjoint to $\overline{\partial}_E$.

Proposition 34.2. *The operator*

$$\overline{\partial}_E^* := -\overline{*_E}\, \overline{\partial}_{E^*} \overline{*_E} : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q-1}(E)$$

is adjoint to $\overline{\partial}_E$ *on global sections.*

Proof. For $\alpha \in A^{p,q}(X, E), \beta \in \mathcal{A}^{p,q+1}(E)$,

$$\begin{aligned} \langle \alpha, \overline{\partial}_E^* \beta \rangle &= \int_X \alpha \wedge \overline{*}_E \overline{\partial}_E^* \beta = -\int_X \alpha \wedge \overline{*}_E \overline{*}_E \overline{\partial}_{E^*} \overline{*}_E \beta \\ &= (-1)(-1)^{p+q} \int_X \alpha \wedge \overline{\partial}_{E^*} \overline{*_E} \beta. \end{aligned}$$

But note that by Stokes' theorem and Leibniz rule,

$$0 = \int_X d(\alpha \wedge \overline{*_E} \beta) = \int_X \overline{\partial}(\alpha \wedge \overline{*}_E \beta) = \int_X \overline{\partial}_E \alpha \wedge \overline{*}_E \beta + (-1)^{p+q} \alpha \wedge \overline{\partial}_{E^*} \overline{*}_E \beta.$$

So we have

$$\langle \alpha, \overline{\partial}_E^* \beta \rangle = \int_X \overline{\partial}_E \alpha \wedge \overline{*}_E \beta = \langle \overline{\partial}_E \alpha, \beta \rangle.$$

$\square$

Note that when $E \cong \mathcal{O}_X$, this adjoint restricts to $- * \partial *$. This is because $-\overline{*}\overline{\partial}\overline{*}(\alpha) = -\overline{*}\overline{\partial}\overline{*\alpha} = -\overline{*}\overline{\partial * \alpha} = - * \partial * \alpha$.

Definition 34.3. Define the Laplacian to be

$$\overline{\square}_E := \overline{\partial}_E^* \overline{\partial}_E + \overline{\partial}_E \overline{\partial}_E^* : \mathcal{A}^{p,q}(X,E) \to \mathcal{A}^{p,q}(X,E),$$

and the $\overline{\partial}_E$ harmonic forms to be

$$\mathcal{H}^{p,q}(X,E) := \ker \overline{\square}_E.$$

Note that the Laplacian is self-adjoint, and α is a $\overline{\partial}_E$ harmonic form $\iff$ $\overline{\partial}_E \alpha, \overline{\partial}_E^* \alpha = 0$. By the usual theory of adjoint operators, $\mathcal{A}^{p,q}(X,E)$ admits a direct sum decomposition in terms of kernel and images, and $\mathcal{H}^{p,q}(X,E)$ is finite dimensional $\mathbb{C}$-vector space. Note that

$$\overline{*_E} : \mathcal{H}^{p,q}(X,E) \to \mathcal{H}^{n-p,n-q}(X,E^*)$$

is a $\mathbb{C}$-antilinear isomorphism. But it is not $\mathbb{C}$-linear and depends on the metrics h, h_E and is thus not ideal.

Theorem 34.4 (Hodge isomorphism)**.** *We have an isomorphism*

$$\mathcal{H}^{p,q}(X,E) \cong H^{p,q}(X,E).$$

Just like in the untwisted Hodge theory, a $\overline{\partial}_E$ harmonic form is the representative of its cohomology class with minimal norm with respect to the metric on forms.

Proof. First, note that each $\alpha \in \mathcal{H}^{p,q}(X,E)$ gives an element $[\alpha] \in H^{p,q}(X,E)$, since $\overline{\square}_E \alpha = 0 \implies \overline{\partial}_E \alpha = 0$. Now we show injectivity. Suppose α, β are harmonic such that

$$\alpha - \beta = \overline{\partial}_E \eta.$$

But note $\overline{\partial}_E^*(\alpha - \beta) = 0$. But $\ker \overline{\partial}_E^* \cap Im(\overline{\partial}_E) = 0$, so $\alpha - \beta = 0$. Now we show surjectivity. Suppose $[\alpha] \in H^{p,q}(X,E)$. Since $\alpha \in \mathcal{A}^{p,q}(X,E)$, we can write

$$\alpha = \omega + \overline{\square}_E \eta = \omega + \overline{\partial}_E^* \overline{\partial}_E \eta + \overline{\partial}_E \overline{\partial}_E^* \eta,$$

then applying $\overline{\partial}_E$, we have $\overline{\partial}_E \overline{\partial}_E^* \overline{\partial}_E \eta = 0$, which forces $\overline{\partial}_E^* \overline{\partial}_E \eta = 0$. Thus,

$$\alpha = \omega + + \overline{\partial}_E \overline{\partial}_E^* \eta,$$

where ω is harmonic. Thus we have surjectivity. □

Note that we also have the following:

Proposition 34.5. *We have the following isomorphism*

$$H^{p,q}(X,E) \cong H^q(X, \Omega_X^p \otimes E).$$

Proof. Recall that because the sheaves $\mathcal{A}_X^{p,q}$ admit partition of unity they are acyclic, and we have a resolution

$$0 \to \Omega_X^p \to \mathcal{A}_X^{p,0} \to \mathcal{A}_X^{p,1} \to \cdots$$

and exactness follows from the $\overline{\partial}$-Poincare lemma. Now if we tensor by E, note that $\mathcal{A}_X^{p,q} \otimes E$ still admits partition of unity and thus is acyclic, and the resolution remains a resolution. Thus, we have an acyclic resolution

$$0 \to \Omega_X^p \otimes E \to \mathcal{A}_X^{p,*} \otimes E$$

which implies the claim. □

Theorem 34.6. *We have a non-degenerate bilinear pairing*

$$H^{p,q}(X,E) \otimes H^{n-p,n-q}(X,E^*) \to \mathbb{C}$$

via $(\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$, *where wedging is given by wedging forms and pairings of sections of* E, E^*.

Proof. The pairing is well-defined on cohomology classes by using the Hodge isomorphism and defining the pairing on harmonic representatives.

The pairing is non-degenerate since for any α, $\overline{*_E}\alpha \in H^{n-p,n-q}(X,E^*)$ (can use harmonicity to see this easily), and the pairing of $\alpha, \overline{*_E}\alpha$ gives $||\alpha||$ with respect to the norm on (p,q) forms with coefficients in E, and this is zero $\iff$ $\alpha = 0$. □

Corollary 34.7 (Serre duality)**.** *We have isomorphisms*

$$H^{p,q}(X,E) \cong H^{n-p,n-q}(X,E^*)^\vee$$

$$H^q(X,\Omega_X^p \otimes E) \cong H^{n-q}(X,\Omega_X^{n-p} \otimes E^*)^\vee.$$

Recall that we had an isomorphism via $\overline{*}_E$, but it was $\mathbb{C}$-antilinear and it also depended on the metrics. Thus, Serre duality is superior. Serre duality (together with Hirzebruch-Riemann-Roch and Kodaira vanishing) is one of the most useful tools to control the cohomology of holomorphic vector bundles.

One could wonder whether we could also generalize the Hodge decomposition to the twisted setting, given that we have generalized to $\overline{\partial}_E$ harmonic forms. Recall the connections were generalizations of the exterior derivative. At least in the case of line bundles, we can do a twisted Hodge decomposition for line bundles whose Chern connection has zero curvature, i.e. for line bundles with first Chern class 0. Recall that this is because the curvature measures failure of the connection to be a differential. Thus, for example, if we are working over $\mathbb{P}^n$, there are no twisted Hodge decompositions with respect to nontrivial line bundles, since all the nontrivial line bundles over $\mathbb{P}^n$ have nonzero Chern class.

Remark 34.8. There is some generalization of Hodge decomposition w.r.t constant sheaves to perverse sheaves. Work of Beilinson.

35. 2/12/24: Kodaira-Akizuki-Nakano Vanishing

Today we are going to prove the Kodaira vanishing theorem. In order to do this, we need a few ingredients, namely generalized Kahler identities. Let us work towards the generalized Kahler identities.

Let $(\mathcal{L},h)$ be a hermitian vector bundle on X a complex manifold. Let ∇ be the Chern connection. So ∇ decomposes into $\nabla = \nabla' + \overline{\partial}$.

Proposition 35.1. *We have equality of operators*

$$\nabla'\overline{\partial} + \overline{\partial}\nabla' = \Theta_{\mathcal{L}}$$

as morphisms of sheaves $\mathcal{A}^{p,q}(\mathcal{L}) \to \mathcal{A}^{p+1,q+1}(\mathcal{L})$, *where* $\Theta_{\mathcal{L}}$ *denotes the operator of wedging by the curvature form of* ∇.

Proof. Let $\alpha \otimes s$ be a local section of $\mathcal{A}^{p,q}(\mathcal{L})$, where s is a local frame for $\mathcal{L}$ and α is a local (p,q) form. Then note $\overline{\partial}(\alpha \otimes s) = \overline{\partial}\alpha \otimes s$, and

$$\nabla'(\alpha \otimes s) = \partial\alpha \otimes s + (-1)^{p+q}\alpha \wedge \nabla'(s) = \partial\alpha \otimes s + A_{\mathcal{L}} \wedge \alpha \otimes s.$$

Then we find

$$(\nabla'\overline{\partial}+\overline{\partial}\nabla')(\alpha\otimes s)=\nabla'(\overline{\partial}\alpha\otimes s)+\overline{\partial}(\partial\alpha\otimes s+A_{\mathcal{L}}\wedge\alpha\otimes s)$$
$$=\partial\overline{\partial}\alpha\otimes s+A_{\mathcal{L}}\wedge\overline{\partial}\alpha\otimes s+\overline{\partial}\partial\alpha\otimes s+\overline{\partial}(A_{\mathcal{L}}\wedge\alpha)\otimes s$$

and noting that $\overline{\partial}(A_{\mathcal{L}}\wedge\alpha)=\overline{\partial}A_{\mathcal{L}}\wedge\alpha-A_{\mathcal{L}}\wedge\overline{\partial}\alpha$, we finally have

$$=\overline{\partial}A_{\mathcal{L}}\wedge\alpha\otimes s=\Theta_{\mathcal{L}}\wedge\alpha\otimes s.$$

□

Note that when $\mathcal{L}$ is the trivial line bundle with the standard hermitian metric, the Chern connection is simply the usual exterior derivative, and we have $\partial\overline{\partial}+\overline{\partial}\partial=0$, and note the curvature of the exterior derivative is indeed 0. So this is unpacking more of what the obstruction measured by curvature is. Note that a less illuminating but quicker way to verify this equality of operators is to note that the curvature is $\nabla^2=(\nabla'+\overline{\partial})(\nabla'+\overline{\partial})$, and this immediately follows once you verify that $\nabla'^2=0$.

Recall that we had a hermitian metric on $\mathcal{A}^{p,q}(X,\mathcal{L})$ in terms of the metric on X and the metric on $\mathcal{L}$. We can use this to also define the adjoint of ∇'. Assume now that X is compact.

Proposition 35.2. *The adjoint of ∇' with respect to the metric on $\mathcal{A}^{p,q}(X,\mathcal{L})$ is*

$$\nabla'^{*}=\partial^*=-*\overline{\partial}*.$$

Proof. Let $\alpha\otimes s\in\mathcal{A}^{p,q}(\mathcal{L})$, and $\beta\otimes s\in\mathcal{A}^{p+1,q}(\mathcal{L})$, where locally s is a local frame for $\mathcal{L}$. Then

$$\langle\nabla'(\alpha\otimes s),\beta\otimes s\rangle=\langle\partial\alpha\otimes s+A_{\mathcal{L}}\wedge\alpha\otimes s,\beta\otimes s\rangle=\int_X h(\partial\alpha+A_{\mathcal{L}}\wedge\alpha,\beta)h_{\mathcal{L}}(s,s)vol(g)$$
$$=\int_X h(h_{\mathcal{L}}(s,s)\partial\alpha+h_{\mathcal{L}}(s,s)A_{\mathcal{L}}\wedge\alpha,\beta)vol(g)=\int_X h(\partial h_{\mathcal{L}}(s,s)\alpha,\beta)vol(g)$$
$$=\int_X h(h_{\mathcal{L}}(s,s)\alpha,\partial^*\beta)vol(g)=\langle\alpha\otimes s,\partial^*\beta\otimes s\rangle.$$

□

Remark 35.3. In general, the formal adjoint of ∇' is $-\overline{*}_{E^*}\nabla'_{E^*}\overline{*}_E$. But as demonstrated, when E is a line bundle this becomes a linear operator on forms.

Now suppose $\mathcal{L}$ is a positive line bundle on compact X. Thus, $c_1(\mathcal{L})=[\omega]$, where $\omega\in A^{1,1}(X)$ provides a Kahler metric for X. We define the Lefschetz operator

$$L:\mathcal{A}^k(\mathcal{L})\to\mathcal{A}^{k+2}(\mathcal{L})$$

to send $\alpha\otimes s\mapsto\omega\wedge\alpha\otimes s$. Thus we see that the twisted Lefschetz operator is essentially the same as the

Proposition 35.4. *The adjoint to L with respect to the metric on $A^k(\mathcal{L})$ is Λ, where you simply wedge by the adjoint of the untwisted Lefschetz operator.*

Proof. We have

$$\langle L(\alpha\otimes s),\beta\otimes s'\rangle=\langle L\alpha\otimes s,\beta\otimes s'\rangle=\int_X h(L\alpha,\beta)h_{\mathcal{L}}(s,s')vol(g)$$
$$=\int_X h(\alpha,\Lambda\beta)h_{\mathcal{L}}(s,s')vol(g)=\langle\alpha\otimes s,\Lambda\beta\otimes s'\rangle.$$

□

We are almost ready to show the generalized Kahler identities. We just need one more very useful lemma, which will relate a Kahler form to the curvature form on the nose.

Lemma 35.5. *Let $\mathcal{L}$ be a positive line bundle on complex manifold X, such that $c_1(\mathcal{L}) = [\omega]$. Then there exists (an essentialy unique) hermitian metric $h_{\mathcal{L}}$ such that the curvature form of the induced Chern connection is*

$$\omega = \frac{i}{2\pi}\Theta_{\mathcal{L}}.$$

Proof. Let h_0 be a hermitian metric on $\mathcal{L}$, and let Θ_0 be the corresponding curvature form. We know that

$$c_1(\mathcal{L}) = \frac{i}{2\pi}[\Theta_0],$$

thus there exist $\overline{\partial}$-exact form η such that $\omega - \frac{i}{2\pi}\Theta_0 = \eta$. Since d kills the left hand side, we have $\partial\eta = 0$, and $\overline{\partial}\eta = 0$. Since X is Kahler, by lemma 21.5, we have $\eta = \frac{i}{2\pi}\partial\overline{\partial}\psi$. Thus,

$$\begin{aligned}\omega &= \frac{i}{2\pi}(-\partial\overline{\partial}\log h_0 + \partial\overline{\partial}\psi)\\ &= -\frac{i}{2\pi}(\partial\overline{\partial}\log h_0 e^{-\psi}),\end{aligned}$$

thus our new metric is $h = h_0 e^{-\psi}$. Note ψ is a positive real-valued function. This is because ω is a real form, and so is $\frac{i}{2\pi}\Theta_0$, so this forces ψ to be real as well, so we have no issues exponentiating it. □

Remark 35.6. This formula $h_{\mathcal{L}} = h_0 e^{-\psi}$ is the start of a whole area of study called singular hermitian metrics on line bundles, and leads to the world multiplier ideals. This is a domain which Yum Tong Siu is a master of. If ψ is a pluri-subharmonic form this leads to a pseudoeffective cone. There is a nice survey per Demailley called "singular hermitian metrics on line bundles" that is worth looking at.

Note that before, in proving the Lefschetz (1,1) theorem, we had equality on the level of cohomology classes. But this lemma is about the existence of a metric which gives equality directly on the nose. This will be useful for our generalized Kahler identities.

Proposition 35.7 (Generalized Kahler Identities)**.** *Let X be a compact complex manifold, and $\mathcal{L}$ a positive line bundle, with $c_1(\mathcal{L}) = [\omega]$. Fix $h_{\mathcal{L}}$ in the sense of lemma 35.5. Then:*

(1) *We have* $\nabla'^*\overline{\partial}^* + \overline{\partial}^*\nabla'^* = 2\pi i\Lambda$
(2) $[\Lambda, \overline{\partial}] = -i\nabla'^* = \partial^*$

Proof. We have

(1) follows from noting that we proved

$$\nabla'\overline{\partial} + \overline{\partial}\nabla' = \Theta_{\mathcal{L}}$$

and $\Theta_{\mathcal{L}} = -2\pi i\omega$ by lemma 35.5, so taking adjoints, we have

$$\nabla'^*\overline{\partial}^* + \overline{\partial}^*\nabla'^* = 2\pi i\Lambda.$$

(2) follows from the fact the ordinary Kahler identities; all the operators mentioned here simply act by wedging by forms and references none of the twisting structure of $\mathcal{L}$.

$\square$

Theorem 35.8 (Kodaira-Akizuki-Nakano Vanishing). *Let $\mathcal{L}$ be a positive line bundle on compact complex X. Then*

$$H^{p,q}(X, \mathcal{L}) = 0$$

for all $p + q > n = \dim X$.

Proof. Since $\mathcal{L}$ is positive, we have $c_1(\mathcal{L}) = [\omega]$ where ω is a Kahler (1,1) form. Note that we have

$$H^{p,q}(X, \mathcal{L}) \cong \mathcal{H}^{p,q}(X, \mathcal{L}).$$

Then it suffices to show that every harmonic form $\alpha \in \mathcal{H}^{p,q}(X, \mathcal{L})$ is zero when $p + q > n$. To show this, we will show that α is a primitive form, i.e. that $\Lambda\alpha = 0$. This will force $\alpha = 0$, since there are no primitive forms of degree $> n$. We have

$$||\Lambda\alpha||^2 = \frac{i}{2\pi}\langle \Lambda\alpha, (\partial^*\overline{\partial}^* + \overline{\partial}^*\partial^*)\alpha\rangle$$

by proposition 35.7. Since α is harmonic, we have $\overline{\partial}\alpha, \overline{\partial}^*\alpha = 0$. So this simplifies to

$$\frac{i}{2\pi}\langle \Lambda\alpha, \overline{\partial}^*\partial^*\alpha\rangle = \frac{i}{2\pi}\langle \overline{\partial}\Lambda\alpha, \partial^*\alpha\rangle.$$

By proposition 35.7, this becomes

$$\frac{i}{2\pi}\langle i\partial^*\alpha, \partial^*\alpha\rangle = -\frac{1}{2\pi}||\partial^*\alpha||^2 \leq 0.$$

Thus, we must have $\Lambda\alpha = 0$. Since $p + q > n$, this forces $\alpha = 0$. $\square$

Kodaiara-Akizuki-Nakano vanishing led to all sorts of developments in algebraic geometry. It is a very important theorem. Note it was important here that $\mathcal{L}$ was a line bundle. A lot of what we used implicitly relied on working with a line bundle.

In algebraic geometry, there is a more general statement called Le Poitier vanishing. It states that $H^q(X, \omega_X \otimes E) = 0$ when E is an "ample" vector bundle and $q \geq rk(E)$. But in differential geometry, a generalization becomes commplicated. There are many generalizations of what positivity could mean for higher rank vector bundles, such as Griffths positivity or Nakano positivity, and they each have their advantages and disadvantages. Griffths positivity is briefly discussed in Huybrechts.

The combination of Kodaira vanishing with Serre duality is very potent.

Corollary 35.9. *If $\mathcal{L}$ is a positive line bundle, then*

$$H^q(X, \Omega_X^p \otimes \mathcal{L}^\vee) = 0$$

for all $p + q < n$. In particular, if $p = 0$, then

$$H^q(X, \mathcal{L}^\vee) = 0$$

for all $q < n$.

Example 35.10. We can use this to deduce one of the most important cohomology calculations in algebraic geometry: the cohomology of line bundles over $\mathbb{P}^n$.

Let $d \geq 0$. First we show that $H^i(\mathcal{O}_{\mathbb{P}^n}(-d)) = 0$ for $i < n$. Note that this is immediate by the previous corollary, since $\mathcal{O}_{\mathbb{P}^n}(d)$ is positive. Furthermore, we already know what $H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ looks like. Then suppose $0 < i < n$. Then

$$H^i(\mathcal{O}_{\mathbb{P}^n}(d)) \cong H^{n-i}(\mathcal{O}_{\mathbb{P}^n}(-d-n-1)) = 0.$$

Then for $i = n$, by Serre duality we have non-degenerate pairing for any d:

$$H^n(\mathcal{O}_{\mathbb{P}^n}(d)) \times H^0(\mathcal{O}_{\mathbb{P}^n}(-d-n-1)) \to \mathbb{C}$$

Of course, this proof is bad because these facts are true over any algebraically closed field. And using Kodaira vanishing is a lot of hard work – there are easier proofs (see Hartshorne).

A technique that Hartshorne does not follow: Serre has an amazing paper called FAC: Faisceaux Alebriques Coherents. He explains why it's true there. The point is this: you take the cohomology of all them together $\bigoplus_m H^i \mathcal{O}_{\mathbb{P}^n}(m)$, and this forms a module over H^0. So this is a graded module over the graded ring, which is a polynomial ring. What Serre shows, is that the cohomology of these guys is just an inductive limit of the cohomology of Koszul complexes. Koszul complexes are associated to sequences of elements in a ring. When you take $x_0, \cdots, x_n$.. then take $x_0^2, \cdots, x_n^2$, ... and $x_0^m \cdots x_n^m$... no matter what power you pick, these form a regular sequence. One of the most famous results in homological agebra, Koszul complex to regular sequence is acyclic.

Next time we will prove the weak Lefschetz theorem, as a consequence of Kodaira vanishing. A weak Lefschetz theorem is a statement about the topology of algebraic varieties. If you have $D \subseteq X$ is a hypersurface, such that the associated line bundle $\mathcal{O}_X(D)$ is positive, then you get an isomorphism

$$H^i(X, \mathbb{C}) \to H^i(D, \mathbb{C}).$$

which is an injection for $i \leq n-2$, and a isomorphism for $i = n-1$. This theorem is actually stronger, it is true over $\mathbb{Z}$. This is proven in Math 571 p-adic/motivic integration? The nicest way is to prove a vanishing theorem for singular cohomology of affine varieties using Morse theory, then you can use Weak Lefschetz to prove Kodaira embedding. Very illuminating approach. But we are going to prove it in the reverse direction.

Some people thought about weak lefschetz theorem, and weren't satisfied. They thought it should also hold over characteristic p. And this is due to Deligne-Illusie. And they showed Kodaira vanishing can be proven via reduction mod p. But it is false in general for characteristic p.

36. 2/14/24: weak lefschetz theorem, Cartan-Serre vanishing theorems

As a consequence of the Kodaira embedding theorem, we will discuss the weak Lefschetz theorem. Informally, this says that under suitable hypotheses, the shape of a codimension 1 subvariety is determined by the shape of its ambient space.

Theorem 36.1. *Let X be a compact complex manifold of dimension n, and $D \subseteq X$ is a smooth hypersurface such that $\mathcal{O}_X(D)$ is positive. Then the natural restrictions*

$$H^i(X; \mathbb{C}) \to H^i(D; \mathbb{C})$$

are isomorphisms for $i \leq n-2$ and is injective for $i = n-1$.

Proof. Note that X is Kahler since it admits a positive line bundle, and so is D as a smooth hypersurface. Then we have Hodge decompositions and the restriction map is compatible with their respective Hodge structures, so it suffices to prove that

$$H^{p,q}(X) \to H^{p,q}(D)$$

is bijective for $p+q \leq n-2$, and injective for $p+q = n-1$. Note that by Dolbeaut's theorem, $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$ and $H^{p,q}(D) \cong H^q(D, \Omega_D^p)$. So we would like to relate Ω_X^p and Ω_D^p. Tensoring the short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \iota_* \mathcal{O}_D \to 0$$

by Ω_X^p, we obtain

$$0 \to \Omega_X^p(-D) \to \Omega_X^p \to \Omega_X^p|_D \to 0.$$

Then taking the conormal sequence

$$0 \to \mathcal{O}_D(-D) \to \Omega_X|_D \to \Omega_D \to 0,$$

then alternating this sequence we get

$$0 \to \Omega_D^{p-1}(-D) \to \Omega_X^p|_D \to \Omega_D^p \to 0.$$

Note that

$$H^q(X, \Omega_X^p(-D)) \cong H^{n-q}(X, \Omega_X^{n-p} \otimes \mathcal{O}_X(D)) = 0$$

for $p+q < n$ by Serre duality and Kodaira vanishing, since $\mathcal{O}_X(D)$ is positive. Applying similar analysis for the second exact sequence, and comparing long exact sequences in sheaf cohomologies, we obtain the claim. $\square$

The true meaning of the weak Lefschetz theorem is about the vanishing of the cohomology of an affine variety. For example, when you have hyperplane $H \subset \mathbb{P}^n$, the complement is an affine variety, namely $\mathbb{C}^n$. The Morse theory proof shows that $H^i(X, \mathbb{Z})$ where X is affine variety of dimension n, is zero when $i > n$. And this theorem about vanishing of cohomology of affine varieties has many generalizations in topology. $\mathbb{Z}$ can be replaced by any locally constant sheaf, and even a perverse sheaf.

In the course of the proof of weak Lefschetz, note that Kodaira embedding gave us weak Lefschetz. But we also see that if we assume weak Lefschetz, we can also provide Kodaira vanishing. This shows a route: if one proves the vanishing of higher cohomology for affine varieties via Morse theory and obtains weak Lefschetz, we can prove Kodaira vanishing.

Remark 36.2. Maybe a downside of the proof is that it doesn't let us understand $\Omega_X^p \to \Omega_D^p$ directly. There is a way to understand this map directly.

Definition 36.3. Let D be smooth of codimension 1. The sheaf of 1-forms with at most poles along D is defined in the following way: choose local coordinates $z_1, \cdots, z_n$ such that $D = (z_n = 0)$. Then

$$\Omega_X^1(\log D)$$

is the $\mathcal{O}_X$ module generated locally by $dz_1, \cdots, dz_{n-1}, \frac{dz_n}{z_n}$. Note that the generators are linearly independent, so we get a locally free sheaf of the same rank as Ω_X. Note we have an injection of sheaves

$$\Omega_X^1 \to \Omega_X^1(\log D).$$

Definition 36.4. $\Omega^p_X(\log D) := \bigwedge^p(\Omega^1_X(\log D))$.

Note that since $\Omega^1_X \to \Omega^1_X(\log D)$ is an injection of sheaves, we have

$$\Omega^p_X \to \Omega^p_X(\log D)$$

is an injection of sheaves as well.

Lemma 36.5. *For every $p \geq 1$, there exist short exact sequences*

- $0 \to \Omega^p_X \to \Omega^p_X(\log D) \to \Omega^{p-1}_D \to 0$ *(residue sequence).*
- 0

Proof. We do the $p = 1$ case for each.

- First, note we have a short exact sequence
$$0 \to \Omega_X \to \Omega_X(\log D) \to^{res} \iota_*\mathcal{O}_D \to 0$$
where locally, the quotient map is given by
$$f_1 dz_1 + \cdots + f_{n-1}dz_{n-1} + f_n \frac{dz_n}{z_n} \mapsto f_n|_D.$$
If $f_n|_D = 0$, then locally $f_n = z_n g_n$, so we see that $f_1dz_1 + \cdots + f_{n-1}dz_{n-1} + f_n\frac{dz_n}{z_n}$ is equal to $f_1dz_1 + \cdots + g_n dz_n$, so we see the kernel is indeed Ω_X.
- Note that there is a natural restriction map $\Omega_X \to \Omega_D$. Locally, the kernel then is simply
$$\langle z_n dz_1, \cdots, z_n dz_{n-1}, \frac{dz_n}{\rangle}.$$
This can be rewritten as
$$z_n \cdot \langle dz_1, \cdots, dz_{n-1}, \frac{dz_n}{z_n}\rangle,$$
which we see corresponds to the local generators of $\mathcal{O}_X(-D)$ and $\Omega_X(\log D)$.

□

There is a version of Kodaira embedding for these logarithmic forms.

Corollary 36.6. $H^q(X, \Omega^p_X(\log D)(-D)) = 0$ *for $p + q < n$ if $\mathcal{O}_X(D)$ is positive*

The story of these logarithmic forms is that they were introduced by Deligne to study the cohomology of certain noncompact varieties. If on X we have D a divisor with simple normal crossings (union of smooth irreducible divisors which all intersect transversely, so that locally at any point D looks like the vanishing loci of $(z_{i_1} \cdots z_{i_k} = 0)$), then one can study the cohomology of $U = X \setminus D$. There turns out to be structure to $H^k(U; \mathbb{C})$. Deligne show that it has a mixed Hodge structure, as opposed to a pure Hodge structure, and its Hodge decomposition can be written as

$$H^k(U; \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega^p_X(\log D)).$$

we did the smooth case here, but often you cannot compatctify something whose boundary is smoething smooth. More often you have situation where you require D not smooth but simple normal crossing. then you can do this.

If locally $D = (z_1 \cdot \cdots \cdot z_r = 0)$ where $r \leq n$, we have $\Omega^1_X(\log D)$ is generated locally by $\frac{dz_1}{z_1}, \cdots, \frac{dz_r}{z_r}, dz_{r+1}, \cdots, dz_n$.

By Hironaka's resolution of singularities, every quasiprojective variety can be compactified to a smooth projective variety whose boundary is a simple normal crossing divisor.

There is a so-called weight filtration on space of logarithmic forms– filters how many denominators you have.

Remark 36.7. You can read about all of the Morse theory proof of weak Lefschetz in Lazarsfeld's book on positivity, or Popa's notes. Vanishing theorems in Lazarsfeld's positivity. The main reference for vanishing theorems, per Popa, is lectures in vanishing theorems by Esnault and Viehweg.

Let's go back to Kodaira vanishing. We have

$$H^i(X, \omega_X \otimes \mathcal{L}) = 0, \forall i > 0$$

if $\mathcal{L}$ is positive. You can wonder if there's a statement that you can make about line bundles in general. There's something that predates Kodaira vanishing: if you allow yourself to twist by sufficiently positive powers, then you can kill the cohomology if anything you want. We always have, very vaguely, "asymptotic vanishing."

Theorem 36.8 (Serre vanishing, or Cartan-Serre Theorem A)**.** *If $\mathcal{L}$ is a positive line bundle on X, E is any vector bundle on X, then there exists m_0 (depending on E) such that for all $m \geq m_0$, we have*

$$H^i(X, E \otimes \mathcal{L}^{\otimes m}) = 0, \forall i > 0.$$

Proof. Fix hermitian metrics h_E and h_L, giving us Chern connections ∇_E and ∇_L. And we choose h_L such that we have equality on the nose $\frac{i}{2\pi}\Theta_L = \omega$ where ω is the Kahler 1,1/Chern class. We then get a hermitian metric on $E \otimes \mathcal{L}^{\otimes m}$ and thus a Chern connection

$$\nabla = \nabla_E \otimes id + id \otimes \nabla_{L^{\otimes m}}$$

and this gives us curvature form

$$\frac{i}{2\pi}\Theta_\nabla = \frac{i}{2\pi}\Theta_E \otimes 1 + m(1 \otimes \omega).$$

Now take any $\alpha \in \mathcal{H}^{p,q}(E \otimes \mathcal{L}^{\otimes m})$ and using

Lemma 36.9. *For any $\alpha \in \mathcal{H}^{p,q}(E)$ we have:*

$$\frac{i}{2\pi}([\Lambda, \Theta_\nabla](\alpha), \alpha) \geq 0.$$

we have

$$0 \leq \frac{i}{2\pi}([\Lambda, \Theta_\nabla](\alpha), \alpha) = \frac{i}{2\pi}([\Lambda, \Theta_E](\alpha), \alpha) + m([\Lambda, L](\alpha), \alpha)$$

$$= \frac{i}{2\pi}([\Lambda, \Theta_E](\alpha), \alpha) + m(n-p-q)||\alpha||^2$$

using usual Kahler identities. Note usual argument in functional analysis, since P continuous operator, $\frac{\langle P(\alpha), \alpha\rangle}{||\alpha||^2} \leq C$ as function on unit disk, so we can write our inequality as

$$\leq m(n-p-q)||\alpha||^2 + C||\alpha||^2.$$

So if $p + q > n$, then for $m >> 0$, the RHS means $\alpha = 0$. $\square$

It's quite useful for theoretical arguments. But the problem is that its hard to identify m_0 for some sheaf. Emphasizes the magic of Kodaira vanishing.

Vanishing theorems are very closely related to the existence of global sections.

Theorem 36.10 (Cartan-Serre Theorem B). *If $\mathcal{L}$ is a positive line bundle, and E is any vector bundle, then there exists m_1 (still depending on E), such that for all $m \geq m_1$, the vector bundle $E \otimes \mathcal{L}^{\otimes m}$ is generated by global sections.*

Another way to say globally generated is surjection $\mathcal{O}_X^{\oplus r} \to E \otimes \mathcal{L}^m \to 0$ for some r. Another way is to say

$$\bigoplus \mathcal{L}^{-m} \to E \to 0.$$

beginning of the theory of free resolutions. see very closed relating to theorem A.

37. 2/21/24: an extraneous fun section of some examples

Today will be an example day! We'll compute the Picard group of various varieties. For example, we already know that

Example 37.1. $Pic(\mathbb{P}^n) = \mathbb{Z}$, $Pic(\mathbb{A}^n) = 0$, and $Pic(\mathbb{P}^m \times \mathbb{P}^n) = \mathbb{Z} \times \mathbb{Z}$. And using the exact sequence

$$\mathbb{Z} \to Pic(X) \to Pic(X \setminus Y) \to 0$$

when Y is codim 1, we have

$$Pic(\mathbb{P}^n \setminus dH) \cong \mathbb{Z}/d\mathbb{Z}.$$

Now we ask, what is the picard group of a hypersurface? This is work due to Grothendieck and Lefschetz. Here's the simple version first.

Theorem 37.2 (Grothendieck-Lefschetz). *Let X be a compact complex manifold, and $Y \subseteq X$ a smooth hypersurface such that $\mathcal{O}_X(Y)$ is positive. Then if $\dim Y \geq 3$, then $Pic(X) \cong Pic(Y)$. If $\dim Y = 2$, we have an injection $Pic(S) \hookrightarrow Pic(Y)$.*

Proof. Note that the inclusion $Y \hookrightarrow X$ induces a morphism of exponential short exact sequences on Y and X. In turn, these induce maps between long exact sequences in cohomology:

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(X, \mathbb{Z}_X) \arrow[r] \arrow[d] & H^1(X, \mathcal{O}_X) \arrow[r] \arrow[d] & H^1(X, \mathcal{O}_X^*) \arrow[r] \arrow[d] & H^2(X, \mathbb{Z}_X) \arrow[r] \arrow[d] & H^2(X, \mathcal{O}_X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^1(Y, \mathbb{Z}_Y) \arrow[r] & H^1(Y, \mathcal{O}_Y) \arrow[r] & H^1(Y, \mathcal{O}_Y^*) \arrow[r] & H^2(Y, \mathbb{Z}_Y) \arrow[r] & H^2(Y, \mathcal{O}_Y) \arrow[r] & \cdots
\end{tikzcd}$$

Note that all the hypotheses are satisfied to apply the weak Lefschetz theorem. When $\dim Y \geq 3$, we see that the first, second, fourth, and fifth maps are isomorphisms, and thus by the five lemma, the map on picard groups is an isomorphism. When $\dim Y \geq 2$, we have that the first two maps are isomorphisms, and the fourth map and fifth maps are injections. By the four lemma applied to the first four maps, since the first map is surjective, and the second and fourth map are injective, we have the map on Picard groups is injective. $\square$

Corollary 37.3. *If X is a hypersurface of $\mathbb{P}^n$, then $Pic(X) \cong \mathbb{Z}$ for $n \geq 4$.*

Then what about hypersurfaces of dimension 2? We are working in $\mathbb{P}^3$. If degree 1 hypersurface, then it is just a $\mathbb{P}^2$. So $Pic(\mathbb{P}^2) \cong \mathbb{Z}$). If it is degree 2 hypersurface, then it is a smooth qaudric in $\mathbb{P}^3$. These are isomrophic to $\mathbb{P}^1 \times \mathbb{P}^1$. You can take the quadratic polynomial, and change of coordinates so that it becomes $xy - uv$. And this picard group is $\mathbb{Z} \times \mathbb{Z}$. Each generator is a family of rulings on the quadric which looks like a curved cylinder. in coordinates, we have

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

where

$$[x_0, x_1], [y_0, y_1] \mapsto [x_0 y_0, x_0 y_1, x_1, y_0, x_1 y_1] = [x : y : u : v]$$

If you fix x_0, x_1, see we get a family of lines. If you fix y_0, y_1, get another family of lines.

Now a degree 2 hypersurface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We claim that $\mathcal{O}_X(1)$ is of type (1,1).

Now consider twisted cubic $C : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, where

$$(s, t) \mapsto (s^3, s^2 t, st^2, t^3].$$

Note twisted cubic is contained in a quadric $Y := V(xu - y^2) \subseteq \mathbb{P}^3$. So $X \cap Y = C \cup L$ where L is the line $[0, 0, y_0, y_1]$. $X \cap Y$ type (2,2) and C is type (2,1) and L is type (0,1). The geometry of divisors on quadrics are very complciated. The divisors are not just rulings. There is also twisted cubic.

Picard group of smooth cubic surface. Fact: any smooth cubic surface X is isomorphic to $X \cong Bl_{6pt}\mathbb{P}^2$ is the blow up of projective plane at 6 general points. The picard group of a blow up $Bl_Z X$.. suppose Z defined by some $f_1, \cdots, f_k$, then $X \mapsto \mathbb{P}^{k-1}$ where $x \mapsto (f_1(x), \cdots, f_k(x)]$. Then you can think of blow up as completion of graph of this map in $X \times \mathbb{P}^{k-1}$.

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & \tilde{X} = Bl_Z(X) \subseteq X \times \mathbb{P}^{k-1} \arrow[d] \arrow[dr] & \\
Z \arrow[r, hook] & X \arrow[r, "x \mapsto {[f_1(x):\cdots:f_k(x)]}"'] & \mathbb{P}^{k-1}
\end{tikzcd}$$

Where $U = \tilde{X} \setminus E \cong X \setminus Z$, then we actually have SES

$$0 \to \mathbb{Z} \to Pic(\tilde{X}) \to Pic(U) \to 0$$

but note $Pic(U) \cong Pic(X)$ since we are assuming we are blowing up along codimesnion ≥ 2, since blow up along codimension 1 doesn't do anything. Where $1 \mapsto [\mathcal{O}_{\tilde{X}}(E)]$.

Lemma 37.4. *When Z is a point, $E \cong \mathbb{P}^{n-1}$. Then Have $\mathcal{O}_{\tilde{X}}(E)|_E = \mathcal{O}_E(E) = \mathcal{O}_E(-1) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.*

If we prove this lemma, then we have injectivity of the above SES because otherwise $\mathcal{O}_{\mathbb{P}^{k-1}} \cong \mathcal{O}_{\tilde{X}}(kE)|_E \cong \mathcal{O}_{\mathbb{P}^{k-1}}(-k)$.

Now assume $X = \mathbb{A}^n$, then

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & Bl_0\mathbb{A}^n \arrow[r, hook] \arrow[d] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[r] & \mathbb{P}^{n-1} \\
0 \arrow[r, hook] & \mathbb{A}^n & &
\end{tikzcd}$$

remember we have $Bl_0\mathbb{A}^n \cong \mathcal{O}_{\mathbb{P}^n}(-1)$. So we want to show injectivity now. Wnat to show

$$\mathcal{O}_{\tilde{X}}(E) = \bigcup_{(\ell,z)}$$

note $\mathcal{O}_{\tilde{X}}(E)$ is sections which vanish, but E is actually $\{(\ell,0)\}$. Ok very confusing here.

Now Rosie talks about curves. We are interested in compact Riemann surfaces, 1 dimensional compact complex manifolds. Topologically, they are classified by their genus. What is the picard group of these? Note in our situation, we always have an isomorphism

$$Div(X)/\sim \cong Pic(C).$$

If we start with $g = 0$, then we have just the Riemann sphere. Remember we had $Pic(\mathbb{P}^1) \cong \mathbb{Z}$. Divisors $\sum n_i[P_i]$ are determined up to principal divisors by $\deg(\sum n_i[P_i]) = \sum n_i$.

If the genus is 1, then the curve is $\mathbb{C}$ modulo some lattice. The lattice is given by some generators $\lambda \cong \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, where $\omega_1/\omega_2 \notin \mathbb{R}$. We can ask if the degree map

$$Pic(\mathbb{C}/\Lambda) \cong Div(\mathbb{C}/\Lambda)/\sim \to^{\deg \mathbb{Z}}$$

is injective. So given $\sum n_i[P_i]$ such that $\sum n_i = 0$. Do we have it is a principal divisor? Suppose we had $[p] - [0] = div(f)$. Well let's calculate the integral

$$\frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}dz$$

in two ways. One is by residue theorem, then you get $p - 0$. But if you use double periodicity, then we have

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

and $\int_{C_1} + \int_{C_3}$ will be integer multiple of ω_2, and $\int_{C_2} + \int_{C_4}$ will be integer multiple of ω_1. So this implies that the integral lies on the lattice. This shows that in $\mathbb{C} \setminus \Lambda$, $p = 0$. Contradiction. So this divisor cannot be a principal divisor of some meromorphic function.

In general, if $\sum n_i[P_i] = div(f)$, we have

$$\frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}dz = \sum n_i P_i$$

by residue theorem, but this integral will also be in the lattice. So this shows that for a divisor to be a principal divisor, we must have $\sum n_i P_i = 0$ on $\mathbb{C}/\Lambda$.. this turns out to be the only obstruction. More precisely, any divisor on a torus whose degree is 0, so $\sum n_i = 0, \sum n_i P_i \in \Lambda$ on $\mathbb{C} \setminus \Lambda$, comes from a principal divisor. So if you have $2[a] - [b] - [c]$ on torus, can take $div(\frac{\sigma(z-a)^2}{\sigma(z-b)\sigma(z-c)}$, and there are fsuch functions on the torus, and they are called the Weierstrass sigma functions. They're important property of this σ is that it is an entire function and σ has simple zeroes at 0 and no other zeroes elsewhere. And as such, it cannot be a function on the torus so not doubly periodic, but it is almost doubly periodic, just need a factor. If this condition of a divisor $\sum n_i P_i = 0$, then $\prod \sigma(z - p_i)^{n_i}$ will be doubly periodic. So the obstructino ot this being doubly periodic is exactly if $\sum n_i = 0, \sum n_i P_i \in \Lambda$.

Then we have that the degree 0 divisors on $\mathbb{C} \setminus \Lambda$ modulo principal divisors, is isomorphic to $\mathbb{C} \setminus \Lambda$, given by $\sum n_i[P_i] \to \sum n_i P_i$. Then remember we had a SES

$$0 \to \mathbb{C}/\Lambda \to Pic(\mathbb{C}/\Lambda) \to^{\deg} \mathbb{Z} \to 0,$$

so quotient of abelian groups.. you get a dimension 1 complex manifold, an abelian variety. This shows that Pic^0 is an abelian variety and admits the structure of a lattice. It turns out that for higher genus g curves, the kernel will be some $\dim g$ complex manifold... something about Abel-Jacobi map.

Question: degree 4 hypersurface in $\mathbb{P}^3$? This is a very nontrivial question.

Theorem 37.5. *A general degree $d \geq 4$ hypersurface in $\mathbb{P}^3$ has picard group isomorphic to $\mathbb{Z}$.*

A paper by Griffiths and Harris: https://link.springer.com/article/10.1007/BF01455794.

38. 2/26/24: on maps to projective space

Recall we are aiming to prove the Kodaira embedding theorem. Very important result that we will later combine with Chow's theorem. Pretty nontrivial, maybe the most complciated, besides the Hodge decomposition, that we will have done.

When we had a compact complex submanifold $X \subseteq \mathbb{P}^n$, then we can automatically deduce that X is Kahler. But if you begin with an arbitrary compact Kahler manifold, its unclear what kinds of spaces you can embed X in. So the main question is

Question 38.1. Given X compact Kahler, when do we have X admits an embedding into projective space?

Now when $X \subseteq \mathbb{P}^n$, not only do we know X is Kahler, but we know it has a positive line bundle $\mathcal{O}_{\mathbb{P}^n}(1)|_X$. In fact, this turns out to be all you need to know.

Theorem 38.1 (Kodaira embedding)**.** *If X compact Kahler manifold has a positive line bundle, then there exists an embedding $X \subseteq \mathbb{P}^n$, some n.*

If X is compact Kahler, by the Lefschetz (1,1) theorem, we know that all integral (1,1) classes come from the first Chern class of a line bundle. So we just need to find a positive integral (1,1) class, and this will correspond to a positive line bundle.

If X is just a compact complex manifold, we know that the first Chern classes of its line bundles are integral (1,1). But not every integral (1,1) class is the first Chern class of a line bundle. But if we found a positive integral (1,1) class, then X is Kahler, and then we have the Lefschetz (1,1) theorem, so the positive integral (1,1) class does correspond to a positive line bundle. Thus, we can actually drop the assumption that X is Kahler. We just need to find a positive integral (1,1) class, but this de-emphasizes the role that the line bundle plays, which is crucial to constructing the embedding.

Remark 38.2. We will also prove that this embedding is algebraic, i.e. defined by finitely many homogeneous polynomials, via Chow's theorem. We will do this later on.

There's a lot we need to build up to this. First, we need to understand maps to $\mathbb{P}^n$. You cannot think of maps to $\mathbb{P}^n$ globally, because the only global functions on $\mathbb{P}^n$ are constants. So we need to think locally, so sections of line bundles are suitable for thinking about maps to projective space.

Consider $f : X \to \mathbb{P}^n$, and look at the pullback $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)|_X$. Recall $\mathcal{O}_{\mathbb{P}^n}(1)$ is trivialized by $(z_i \neq 0)$ with transition functions $\frac{z_j}{z_i} = g_{ij}$. So we can trivialize $\mathcal{L}$ on $f^{-1}(U_i)$, with transition functions $g_{ij} \circ f = \frac{z_j}{z_i} \circ f$. Also have

$$f^* : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1) \to H^0(X, \mathcal{L}), s \mapsto f^*s.$$

Definition 38.3. f is nondegenerate if $f(X)$ is not contained in any hyperplane in $\mathbb{P}^n$.

If $f(X)$ was contained in a hyperplane, then you can just restrict to that hyperplane and cut down the dimension of the target projective space. Typically, people use the language nondegenerate to refer to a subvariety in projective space, but can also use it for the case of maps. This implies we have that f^* is injective $\iff$ f is non-degenerate. This is because the zeroes of a global section of $\mathcal{O}_{\mathbb{P}^n}(1)$ pulled back to vanish everywhere on X, then X would be contained in a hyerplane.

So we are considering nondegenerate f, so f^* is an injection. Conversely, construct $f : X \to \mathbb{P}^n$ from sections of $H^0(X, \mathcal{L})$.

Definition 38.4. Let $\mathcal{L}$ be a line bundle on X, $V \subseteq H^0(X, \mathcal{L})$ linear subspace, with a fixed basis $\langle s_0, \cdots, s_N \rangle$. Define

$$f : X - - > \mathbb{P}^N, \text{ where} x \mapsto [s_0(x) : \cdots : s_N(x)].$$

In general, this is only a rational map, as there could be some x where $s_0(x) = \cdots = s_N(x)$.

Remark 38.5. We have an evaluation map of global sections. We have $H^0(X, \mathcal{L}) \to \mathcal{L}_x \cong \mathcal{O}_{X,x}$. There's a maximal ideal of functions vanishing at the point, so we can also look at the quotient, $\mathcal{O}_{X,x}/m_x \cong \mathbb{C}$. So can look at

$$H^0(X, \mathcal{L}) \to \mathcal{L}_x \to L(x) = \frac{L_x}{m_x \cdot L_x}$$

where $s \mapsto [s]_x = s(x)$.

Definition 38.6. $V \subseteq H^0(X, \mathcal{L})$ is called base point free if for every $x \in X$, there exists $s \in V$ such that $s(x) \neq 0$. This is precisely when we get a map $f : X \to \mathbb{P}^N$.

Geometrically: if V is a linear system that is not base point free, then each $s \in V$ vanish at x, then the divisors corresponding to each $s \in V$, those divisors all share that point x. So this is why we call it a base point. For non base point free linear systems, you can study the base locus. You can study how to get rid of it.

Remark 38.7 (Coordinate free description/Grothendieck convention)**.** Think of $\mathbb{P}(V)$ as the space of hyperplanes in V, same as set of lines through 0 in V^*.

Have $f : X - - > \mathbb{P}(V)$ where $x \mapsto H = \{s | s(x) = 0\}$. Exercise: analyze the relationship between this and the previous definition. This is a more intrinsic map.

Lemma 38.8. *We obtain a 1 to 1 correspondence between*

$$\{ \textit{ nondegenerate holomorphic maps } f : X \to \mathbb{P}^N\}$$

and

$$\{\textit{line bundle } \mathcal{L} + \textit{ base point free } V \subseteq H^0(X, \mathcal{L}) \textit{ with } \dim V = N + 1.\}$$

Definition 38.9. If $V = H^0(X, \mathcal{L})$, we call $\mathbb{P}(V)$ the a complete linear system on X (because geometrically, scalars don't affect vanishing loci).

If V is base point free, we also say that $\mathcal{L}$ is globally generated, or generated by global sections. It means that at each point, evaluation to stalks $H^0(X, \mathcal{L}) \to \mathcal{L}_x$ is surjective for every $x \in X$. This is an accident since $\mathcal{L}$ is 1-dimensional. For line bundles, base point free and globally generated are the same.

Remark 38.10. For higher rank $\mathcal{O}_X$-modules $\mathcal{F}$, we say $\mathcal{F}$ is globally generated if

$$H^0(X, \mathcal{F}) \to \mathcal{F}_x$$

is a surjection for every $x \in X$. You need enough sections to globally generate. At each point, for a locally free sheaf, $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$ so you would need to demonstrate r global sections which are linearly independent locally at x in $\mathcal{F}_x$.

Example 38.11. $\mathcal{O}_{\mathbb{P}^n}(k)$ is base point free, for every $k \geq 0$. Here's the easiest proof. Take a point. There are definitely k distinct lines which avoid the point. Then these k lines give a degree k hypersurface.

Philosophy: sometimes look at very degenerate solutions. Theory of castlenuevo curves often does this, techinque of degeneration.

These line bundles over $\mathbb{P}^n$ give the famous k-th Veronese embeddings

$$\phi_k : \mathbb{P}^n \hookrightarrow \mathbb{P}^N,$$

where $N = \binom{n+k}{k} - 1$. The map can be represented by

$$(x_0 : \cdots : x_n) \mapsto (x_0^k : x_0^{k-1}x_1 : \cdots : x_n^k)$$

where you send coordinates to all monomials of degree k. When $n = 1$, this is the famous rational normal curve.

More precise form of Kodaira embedding.

Theorem 38.12. *Let $\mathcal{L}$ be a positive line bundle on a compact complex manifold X. Then there exists $k_0 \in \mathbb{N}^*$ (depending on $\mathcal{L}$) such that for every $k \geq k_0$, the line bundle $\mathcal{L}^{\otimes k}$ is basepoint free and $\phi_{\mathcal{L}^{\otimes k}} : X \to \mathbb{P}^N$ is an embedding.*

Suppose we fix $x \in X$, M line bundle on X. We have

$$0 \to \mathcal{I}_x \to \mathcal{O}_X \to \mathcal{O}_x \to 0,$$

and we can tensor by M to obtain

$$0 \to M \otimes \mathcal{I}_x \to M \to M_x \to 0$$

and

$$0 \to H^0(X, M \otimes \mathcal{I}_x) \to H^0(X, M) \to M_x \to H^1(X, M \otimes \mathcal{I}_x) \to \cdots$$

Note base point freeness means surjectivity of $H^0(X, M) \to H^0(X, \mathcal{O}_x \otimes M) = M_x$. So in particular, $H^1(X, M \otimes \mathcal{I}_x) = 0$ implies that x is not a base point for $|M|$. This seems like we've reinterpreted our problem into a harder problem, but luckily we have Kodaira vanishing theorem and Serre duality. But we already run into trouble: if $\dim X \geq 2$ is at least 2, then a point is not a divisor, so the ideal sheaf of the point $\mathcal{I}_x$ is definitely not a line bundle, so we already in trouble.

But the hope is: to blow up the point! Want to turn the point into a divisor, play with positivity on divisors... see if you can do something to apply Kodaira vanishing.

Recall blow-ups: we have $\tilde{X} = Bl_x X \to X$, it is isomorphism away from $x \in X$, but over $x \in X$, we have the exceptional divisor $E \cong \mathbb{P}^{n-1}$, where E represents all the lines through x. E is a smooth hypersurface.

Lemma 38.13. $\mathcal{O}_E(-E) := \mathcal{O}_{\tilde{X}}(-E)|_E \cong \mathcal{O}_E(1)(\cong \mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. Statement local on X. We may assume $X = \mathbb{C}^n$, and x is origin. We have

$$\begin{tikzcd} \mathbb{C}^n \times \mathbb{P}^{n-1} \arrow[dr] & Bl_0\mathbb{C}^n \arrow[l, hook'] \arrow[r, "q"'] \arrow[d] & \mathbb{P}^{n-1} \\ & \mathbb{C}^n & \end{tikzcd}$$

and recall $Bl_0\mathbb{C}^n$ was the tautalogical line bundle over $\mathbb{P}^{n-1}$. Note the exceptional divisor E is just the zero section. The result follows from the following exercise: if you have zero section $Z \subseteq L$ where $\pi : L \to X$ is line bundle. The zero section is a submanifold, and $\mathcal{O}_L(-Z) \cong \pi^*\mathcal{L}^{-1}$. □

You took something negative, and all of the sudden you get something positive. How could this be? There's something really special about this exceptional divisor. It doesn't move in a linear system, in some nice deformation class. That is because E is contracted to a point. Things that are contracted are negative. Intersection theory: means $E \cdot E = -1$, a (-1) curve.

Note why we care about $\mathcal{O}_{\tilde{X}}(-E)$. Because $\pi^*\mathcal{I}_x \cong \mathcal{O}_{\tilde{X}}(-E)$. We want to situate ourselves to talk about Kodaira vanishing, so also, whats the canonical line bundle of blow-up, knowing the canonical line bundle of base?

Lemma 38.14. *Say* $\dim X = n$. *Then*

$$\omega_{\tilde{X}} \cong \pi^*\omega_X \otimes \mathcal{O}_{\tilde{X}}((n-1)E).$$

Proof. We have $\pi : \tilde{X} \setminus E \cong X \setminus x$. So $\omega_{\tilde{X}}|_{\tilde{X}\setminus E}$ and $\omega_X|_{X\setminus x}$ agree. Then we should have a formula

$$\omega_{\tilde{X}} \cong \pi^*\omega_X \otimes \mathcal{O}_{\tilde{X}}(kE).$$

Local calculation, we may assume that $X \cong \mathbb{C}^n$, x is 0. Open cover of $\tilde{\mathbb{C}^n}$ given by $V_j = g^{-1}(U_j) \cong \mathbb{C}^n$ where

$$(\underline{x}, \underline{w}) \mapsto (\frac{x_1}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, w_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j}).$$

and $x_i w_j = x_j w_i$. Then map $\pi : V_j \to \pi(V_j)$ sends

$$(y_1, \cdots, y_n) \mapsto (y_j y_1, \cdots, y_j y_{j-1}, y_j, \cdots, y_j y_n).$$

In this chart, E gets identified with $y_j = 0$. Now you have

$$dz_1 \wedge \cdots dz_n = d(y_j y_1) \wedge \cdots \wedge dy_j \wedge \cdots \wedge d(y_j y_n) = y_j^{n-1} d_1 \wedge \cdots \wedge dy_n.$$

Have $dz_1 \wedge \cdots \wedge dz_n$ local generator of ω_X, and $dy_1 \wedge \cdots \wedge dy_n$ local generator of $\omega_{\tilde{X}}$. This y_j^{n-1} is the local generator for $\mathcal{O}_{\tilde{X}}(-(n-1)E)$. □

39. 2/28/24: CRITERIA FOR EMBEDDINGS, KODAIRA EMBEDDING

We want to prove the Kodaira embedding theorem. If compact X Kahler has positive line bundle $\mathcal{L}$, then a sufficietnly higher power of $\mathcal{L}$ will give us an embedding into projective space.

Let M be a line bundle on X, and M is base point free. So the map given by the sections of M

$$\phi_M : X \to \mathbb{P}^N$$

is everywhere defined (and holomorphic). Recall that we had a cohomological criterion for when M was base point free, that for every point $p \in X$, $H^1(X, M \otimes \mathcal{I}_p) = 0$.

So let us first talk about what it conditions allow for a linear system to give an embedding.

Lemma 39.1. *ϕ_M is an embedding if and only if*

(1) ϕ_M is injective

(2) ϕ_M is an immersion, i.e. for every $p \in X$,

$$(d\phi_M)_p : T'_p X \to T'_{\phi_M(p)}\mathbb{P}^N$$

is injective.

Proof. To sketch a proof, note X is compact. So ϕ_M is open. So injection means it is a homeomorphism onto its image.

The implicit function theorem means there exists ϕ^{-1} defined on the image and it is a holomorphism. □

Thse criteria for an embedding are kind of abstract. We want concrete criteria from a line bundle to check whether it is an embedding.

Let $H^0(X, M) = \langle s_0, \cdots, s_N \rangle$. Note $\phi_M(x) := (s_0(x) : \cdots : s_N(x))$. Then

(1) ϕ_M is injective $\iff$ for all $x, y \in X$, we have the vectors $\phi_M(x)$ and $\phi_M(y)$. Then they are linearly independent only if $H^0(M) \to M_x \oplus M_y$ is surjective. Note the image is already at least one-dimensional. But this holds $\iff$ for all x, y distinct, there exists $s \in H^0(M)$ s.t. $s(x) \neq 0$ and $s(y) = 0$. This conditions is called: ϕ_M separates points. Geometric interpretation, any distinct x, y there is divisor through y but not through x.

(2) ϕ_M separates tangent vectors: fix $s_0 \in H^0(M)$ s.t. $s_0(x) \neq 0$. Have $H^0(M) \to M_x \to 0$ surjection (bpf condition). And kernel of this all the sectiosn vanishing at x, so

$$0 \to H^0(M \otimes \mathcal{I}_x) \to H^0(M) \to M_x \to 0.$$

So can choose $s_1, \cdots, s_N$ s.t. $s_i(x) = 0$ for all $i = 1, \cdots, N$. We can always write $s_i = f_i \cdot s_0$ for $f_i \in \mathcal{O}_X(U)$ around x. So

$$(s_0(x) : \cdots : s_N(x)) = (1 : f_1(x) : \cdots : f_N(x)),$$

and $d\phi_{M,x}$ injective $\iff$ $(\frac{\partial f_i}{\partial z_j})_{i=1,\cdots,N, j=1,\cdots,n}$ has rank $n = \dim X$. This is $\iff$ holomorphic tangent space $T^{1,0}_x X$ spanned by $df_1, \cdots, df_N$.

But we want a more intrinsic criterion. Focus on $H^0(X, M \otimes \mathcal{I}_x)$, this is like focusing only on the divisors which go through x. Want to say that for any section $s \in H^0(X, M \otimes \mathcal{I}_x)$, you can define a map s which hits all the

1-forms in $T_x^{1,0}X$.. but we have to be careful about how sections transform and be well-defined. Need to take into account the stalk plus evaluation. The claim is that there is a well-defined map

$$s \to ds_x \otimes s_x.$$

Have s is collection of data (s_α, U_α) where $s_\alpha = g_{\alpha\beta} s_\beta$. Want this map to not depend on s_α. We have

$$ds_\alpha = dg_{\alpha\beta} s_\beta + g_{\alpha\beta} ds_\beta.$$

At x, s_β is zero, so we have

$$ds_{\alpha,x} = g_{\alpha\beta}(x) ds_{\beta,x}.$$

So this map is

$$H^0(X, M \otimes \mathcal{I}_x) \to^{\phi_x} T_x^{1,0} X \otimes M(x)$$

Another way to write is $s_i = f_i s_0$, and $s_i \mapsto df_i(x) \otimes s_0(x)$. So claim is $d\phi_H$ injective at $x \iff \phi_x$ is surjective.

Remark 39.2. In algebraic geometry, we think of $T'_x X = (m_x/m_x^2)^\vee$, and $T_x^{1,0}(X) \cong m_x/m_x^2$ in abstract setting. And this map simply becomes

$$H^0(X, M \otimes \mathcal{I}_x) \to M(x) \otimes m_x/(m_x^2)$$

Always have this sequence

$$0 \to I_x^2 \to I_x \to I_x/I_x^2 \to 0.$$

Then tensor by $\otimes_{\mathcal{O}_X} M$ and pass to section

$$H^0(M \otimes \mathcal{I}_x^2) \to H^0(M \otimes \mathcal{I}_x) \to M_x \otimes_{\mathcal{O}_{x,x}} (m_x/m_x^2) \to H^1(M \otimes m_x^2)$$

where here we identifying loosely $I_x = m_x$, The surjectivity of m_x means sections of M generate the 1-jets of x. TO generate 0-jet is to be base point free, to generate 1-jet, you move up the taylor series. All sections at x, have taylor series. Up to order 1, we hit all the order 1 vanishing. Means we can find divisor through x which does not contain the tangent vector at x.

Chapter 2 section 7. Hartshorne does over any field.

This criterion is still very hard.

Example 39.3. Why does $\mathcal{O}_{\mathbb{P}^n}(k)$ gives an embedding for $k \geq 0$.

So we have this cohomological criterion for checking separation of tangent vectors, and cohomological criterion for checking separation of points. It is these H^1 groups. We could not directly apply Kodaira vanishing since point is not divisor. But if we blow the point up, we get a divisor. Seeing positivity on this blow up is our last and most important step.

Lemma 39.4. *Let $\mathcal{L}$ be a positive line bundle on X, $x \in X$, and $\pi : \tilde{X} = Bl_x X \to X$. If $\tilde{L} := \pi^* L$, then for $k >> 0$, then*

$$\tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)$$

is positive.

Proof. We have $c_1(\mathcal{L}) = [\omega]$, where ω is positive and $\omega = \frac{i}{2\pi}\Omega_L$. On $\tilde{L} = \pi^* L$, have $\pi^*\omega = \frac{i}{2\pi}\pi^*\Omega_L = \frac{i}{2\pi}\Omega_{\tilde{L}}$.

On $\tilde{X} \setminus E \cong X \setminus x$. have $\pi^*\omega = \omega > 0$. On E, $\tilde{L}|_E = \mathcal{O}_E$, $\pi^*\omega \cong 0$.

On $U_1 = \pi^{-1}(D)$, where D neighborhood of x, have

$$Bl_0(D) = U_1 \to^q \mathbb{P}^{n-1}$$

we have $\mathcal{O}_{U_1}(-E) = q^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. This is semipositive, have for all x $h(v, v) \geq 0$. But restriction to E, we get $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$, positive.

Get a metric h_1 on U_1, Θ_1 for $\mathcal{O}_{U_1}(-E)$ by pulling back via q. So

$$\frac{i}{2\pi}\Theta_1 = q^*\omega_{FS}.$$

Now $U_2 := \tilde{X}\backslash E$, have $\mathcal{O}_{\tilde{X}}(-E)$ trivial on U_2, so trivialized by s_E, take $h_2(s_E, s_E) = 1$, glue these two metrics, to get a metric on the whole thing. Standard procedure, consider partition of unity, ρ_1, ρ_2 subordinate to cover of U_1, U_2 of $\tilde{X}$. Define $h_E := \rho_1 h_1 + \rho_2 h_2$, hermitian metric on $\mathcal{O}_{\tilde{X}}(-E)$. Have

$$\Theta_k = \frac{i}{2\pi}\Theta_{\tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)} = k\pi^*\omega + \frac{i}{2\pi}\Omega_E.$$

Claim that this is positive for $k >> 0$. On U_1: Θ_k is positive on some neighbrhood W of E in $\tilde{X}$, and $\tilde{X} \setminus W \subseteq U_2 = \tilde{X} \setminus E$, compact set away from E, positive for $k >> 0$. Note $k\pi^*\omega$ positive on $\tilde{X} \setminus W$, and $\frac{i}{2\pi}\Theta_E$ bounded on $\tilde{X} \setminus W$. □

Remark 39.5. This $\pi^* L$ is never positive, because restriction to E is trivial. The hope that $\tilde{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)$ is positive comes from restriction to E, the latter is positive (?), former is semipositive.. not precise intuition but some hope to believe twisting would give positivity.

Now we prove Kodaira embedding. Let $\mathcal{L}$ be a positive line bundle. We want to show there exists k_0 such that $\forall k \geq k_0$, ϕ_{L^k} gives an embedding. There are three things we need to show:

(1) L^k is base point free for $k >> 0$
(2) L^k separates points for $k >> 0$
(3) L^k separates tangent vectors for $k >> 0$

Base point free:

Proof. Need to show evaluation $H^0 L^k \to L^k_x$ is surjective. Implied by showing $H^1(X, \mathcal{L}^k \otimes I_x) = 0$ for $k >> 0$. You consider the blow up at x, $\tilde{X} \to X$. Have $\pi^*\mathcal{I}_x = \mathcal{O}_{\tilde{X}}(-E)$. So the pull back of $\mathcal{L}^k \otimes I_x$ to $\tilde{X}$ is $\mathcal{L}^k \otimes \mathcal{O}_{\tilde{X}}(-E)$.

Look at diagram:

$$0 \to H^0(X, L^k \otimes \mathcal{I}_x) \longrightarrow H^0(X, \mathcal{L}^k) \longrightarrow \mathcal{L}^k_x \longrightarrow H^1(X, \mathcal{L}^k \otimes \mathcal{I}_x) \to \cdots$$

$$0 \to H^0(\tilde{X}, \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E)) \longrightarrow H^0(\tilde{X}, \tilde{\mathcal{L}}^k) \longrightarrow \tilde{\mathcal{L}}^k|_E \longrightarrow H^1(\tilde{X}, \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E)) \to \cdots$$

The right language for all this is the projection formula. But here we are in a situation so concrete, we don't need to talk about this. The claim is that

$$\pi^* : H^0(X, \mathcal{L}) \to H^0(\tilde{X}, \tilde{\mathcal{L}})$$

is an isomorphism. Injection is clear. They are isomorphism away from $\tilde{X} \setminus E$ so must have injection.

π^* surjective because: if $\tilde{s} \in H^0(\tilde{\mathcal{L}}^k)$, because $\tilde{s}|_{\tilde{X}\setminus E}$ corresponds to section s of $\mathcal{L}^k$ on $X \setminus x$. If $\dim X = 1$, then little x is already a divisor, so the blow up doesn't do anything so the blow up is an isomorphism, so π is identity. If $\dim X \geq 2$, then you can extend s by Hartog's. Same story for

$$\pi^* : H^0(X, \mathcal{L}^k \otimes \mathcal{I} - x) \cong H^0(\tilde{X}, \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E)).$$

$$\begin{array}{ccccccc} 0 \to H^0(X, L^k \otimes \mathcal{I}_x) & \longrightarrow & H^0(X, \mathcal{L}^k) & \longrightarrow & \mathcal{L}^k_x & \longrightarrow & H^1(X, \mathcal{L}^k \otimes \mathcal{I}_x) \to \cdots \\ \downarrow & & \downarrow & & & & \\ 0 \to H^0(\tilde{X}, \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E)) & \longrightarrow & H^0(\tilde{X}, \tilde{\mathcal{L}}^k) & \longrightarrow & H^0(\tilde{\mathcal{L}}^k|_E) & \longrightarrow & H^1(\tilde{X}, \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E)) \to \cdots \end{array}$$

Note $\tilde{L} = \pi^* L$, $\tilde{L}|_E = \mathcal{O}_E$, have $E \to x$ so $H^0(\tilde{\mathcal{L}}^k \otimes \mathcal{O}O_E) \cong \tilde{\mathcal{L}}^k \otimes H^0\mathcal{O}_E$. So the third map is also an isomorphism.

$$\begin{array}{ccccccc} 0 \to H^0(X, L^k \otimes \mathcal{I}_x) & \longrightarrow & H^0(X, \mathcal{L}^k) & \longrightarrow & \mathcal{L}^k_x & \longrightarrow & H^1(X, \mathcal{L}^k \otimes \mathcal{I}_x) \to \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to H^0(\tilde{X}, \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E)) & \longrightarrow & H^0(\tilde{X}, \tilde{\mathcal{L}}^k) & \longrightarrow & H^0(\tilde{\mathcal{L}}^k|_E) & \longrightarrow & H^1(\tilde{X}, \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E)) \to \cdots \end{array}$$

So indeed, it is enough to prove that $H^1(\tilde{X}, \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E))$.

This looks nothing like Kodaira, so typical strategy in higher dimensional geometry, force it to look like Kodaira. Have

$$\tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E) = \omega_{\tilde{X}} \otimes \omega_{\tilde{X}}^{-1} \otimes \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E)$$

and call $M_k = \omega_{\tilde{X}}^{-1} \otimes \tilde{\mathcal{L}}^k \otimes \mathcal{O}_{\tilde{X}}(-E)$. And recall $\omega_{\tilde{X}} \cong \pi^*\omega_X \otimes \mathcal{O}_{\tilde{X}}((n-1)E)$. We would like to say that M_k is positive. So $M_k = \pi^*(L^k \otimes \omega_X^{-1}) \otimes \mathcal{O}_{\tilde{X}}(-nE)$. So homework exercises: since L is positive, there exist ℓ_0, such that $L^{\otimes l} \otimes \omega_X^{-1}$ is positive, on X compact. If N is positive, $\pi^* N$ is semipositive. So can choose sufficiently high power so $L^k \otimes \omega_X^{-1}$ is positive, and choose sufficiently high power of that that makes the whole M_k positive.

There exist m_0 s.t. for $m \geq m_0$, have $\tilde{L}^m \otimes \mathcal{O}_{\tilde{X}}(-E) > 0$, and $\tilde{\mathcal{L}}^{mn} \otimes \mathcal{O}_{\tilde{X}}(-nE) > 0$, and so for $k \geq m_0 n + \ell_0$, have $\pi^*(L^{\ell_0} \otimes \omega_X^{-1}) \otimes \tilde{L}^{k-\ell_0} \otimes \mathcal{O}_{\tilde{X}}(-nE)$. the first half is semipositive, and by the lemma the second half is > 0.

□

40. Extra: Maps to projective space, Kodaira's Embedding, clean writeup

Recall that we want to prove the Kodaira embedding theorem, which says that if X is a compact Kahler manifold with a positive line bundle, then X admits an embedding into projective space. More specifically, if we have a positive line bundle $\mathcal{L}$ for sufficiently high powers $\mathcal{L}^{\otimes m}$, the complete linear system $|\mathcal{L}^{\otimes m}|$ will provide us with an embedding.

Last time we talked about the correspondence between non-degenerate holomorphic maps to projective space, and base point free vector subspaces of $H^0(X, \mathcal{L})$ for some line bundle $\mathcal{L}$. We also provided a cohomological criterion for base point freeness. The first part of today's section will be to discuss the conditions for when a complete linear system is actually an embedding. Then after noting the general proof strategy for Kodaira embedding, we will dive into the proof of Kodaira's embedding theorem.

Suppose we have a line bundle $\mathcal{L}$, and assume $|\mathcal{L}|$ is base point free, so that $|\mathcal{L}|$ actually gives an actual map $X \to \mathbb{P}^N$. When is $|\mathcal{L}|$ an embedding? First, we need the map to be injective. Once we have an injection on image, we would like the image to be a submanifold of the ambient projective space. Thus, the map given by $\mathcal{L}$ should not lose any differential data about X.

Proposition 40.1. *Let X be a compact complex manifold, and $\mathcal{L}$ a line bundle. Let $H^0(X, \mathcal{L}) \cong \mathbb{C}\langle s_0, \cdots, s_N\rangle$. Then*

$$\phi_{\mathcal{L}} : |\mathcal{L}| : X \to \mathbb{P}^N$$

is an embedding if

(1) $|\mathcal{L}|$ is injective
(2) $d\phi_{\mathcal{L}}|_p : T_pX \to T_{\phi_{\mathcal{L}}(p)}\mathbb{P}^N$ is an injection for every $p \in X$.

Proof. By the open mapping theorem, $\phi_{\mathcal{L}}$ is an open map. Then injection implies we have a homeomorphism onto image. Since the differential is injective everywhere, by the complex inverse function theorem, we have an inverse holomorphic map. □

We would like to develop more algebro-geometric interpretations of these criteria and, in particular, have some cohomological criterion. In the case of injectivity, note that for distinct $p, q \in X$, the map $d\phi_{\mathcal{L}}$ sends

$$p \mapsto [s_0(p) : \cdots : s_N(p)], q \mapsto [s_0(q) : \cdots : s_N(q)].$$

Then we see that we have injectivity if and only if

$$H^0(X, \mathcal{L}) \to \mathcal{L}(p) \oplus \mathcal{L}(q)$$

is surjective. A priori, we know that since $|\mathcal{L}|$ is base point free, that this has one-dimensional image. Thus, we have two criteria here. Our map is injective if and only if for every distinct $p, q \in X$, there exist $s \in H^0(X, \mathcal{L})$ such that $s(p) = 0, s(q) \neq 0$. This means that the linear system of divisors "separate points" of X. Furthermore, our map is injective if and only if $H^1(X, \mathcal{I}_{p,q} \otimes \mathcal{L}) = 0$, since tensoring the exact sequence

$$0 \to \mathcal{I}_{p,q} \to \mathcal{O}_X \to \mathcal{O}_p \oplus \mathcal{O}_q \to 0$$

by $\mathcal{L}$ and taking the cohomology LES shows that the vanishing of $H^1(X, \mathcal{I}_{p,q} \otimes \mathcal{L}) = 0$ corresponds to the surjectivity of what we want.

Now let us discuss injectivity of differentials. Note that since $\phi_{\mathcal{L}}$ is base point free, we have that

$$H^0(X, \mathcal{I}_p \otimes \mathcal{L}) \cong \ker[H^0(X, \mathcal{L}) \to \mathcal{L}(p)]$$

has dimension N, and supposing that $s_0(p) \neq 0$, we have $s_1, \cdots, s_N \in H^0(X, \mathcal{I}_p \otimes \mathcal{L})$ is a basis. Then locally around p, we can write $s_j = f_j s_0$, where f_j is a local function of X around p. Then

$$p \mapsto [s_0(p) : \cdots : s_N(p)] = [1 : f_1(p) : \cdots : f_N(p)].$$

Then the differential is injective if and only if $(\frac{\partial f_j}{\partial z_i})_{1 \le i \le n, 1 \le j \le N}$ to have rank $n = \dim X$. Note that $df_1, \cdots, df_N$ are 1-forms at p which are the pull back of the 1-forms $dz_1, \cdots, dz_N$ of $\mathbb{P}^N$ at $\phi_{\mathcal{L}}(p)$ along $\phi_{\mathcal{L}}$. Then we have an injection on tangent spaces $\iff$ the $df_1, \cdots, df_N$ span $A^{1,0}(X)_p$. How do we write this surjection algebrogeometrically? Note that in terms of intrinsic data on a manifold, tangent vectors at p, as directional derivatives, are exactly the space of derivations of functions at p. Since derivations only see the linear terms, the space of derivations is isomorphic to $(m_p/m_p^2)^\vee$. Then $A^{1,0}(X)_p$ is m_p/m_p^2. So the differential is injective at p if and only if

$$H^0(X, \mathcal{I}_p \otimes \mathcal{L}) \to \mathcal{L}(p) \otimes \mathcal{I}_p/\mathcal{I}_p^2$$

is surjective, where locally $s' \mapsto ds'|_p$. To see that this is well-defined, note that $s \in H^0(X, \mathcal{I}_p \otimes \mathcal{L})$ is given by the data of (s_α, U_α) such that $s_\alpha = g_{\alpha\beta}s_\beta$. Then $ds_\alpha|_p = dg_{\alpha\beta}|_p \cdot s_\beta(p) + ds_\beta|_p g_{\alpha\beta}(p) = g_{\alpha\beta}|_p \cdot ds_\beta|_p$. Geometrically, this is "separating tangent vectors:" for any tangent vector at p, there is a divisor through p whose tangent space does not contain that vector.

To obtain a cohomological criterion, we can tensor the short exact sequence

$$0 \to \mathcal{I}_p^2 \to \mathcal{I}_p \to \mathcal{I}_p/\mathcal{I}_p^2 \to 0$$

by $\mathcal{L}$ and take the cohomology long exact sequence. Then our criteria becomes the vanishing of $H^1(X, \mathcal{I}_p^2 \otimes \mathcal{L})$.

Theorem 40.2. *Suppose $|\mathcal{L}|$ is a base point free linear system. Then it is an embedding if it*

- *separates points, i.e. $H^1(X, \mathcal{I}_{p,q} \otimes \mathcal{L}) = 0$ for all distinct $p, q \in X$*
- *and separates tangent vectors, i.e. $H^1(X, \mathcal{I}_p^2) = 0$ for all $p \in X$.*

Example 40.3 (Veronese embeddings)**.** The complete linear systems $|\mathcal{O}_{\mathbb{P}^n}(k)|$ are embeddings for all $k \geq 0$ because...

- they are base point free. Given any point $p \in \mathbb{P}^n$, there is certainly a line which does not go through p. Then that line to the k-th power gives an element of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ which provides the necessary surjection for base point freeness.
- they separate points. Given distinct $p, q \in \mathbb{P}^n$, by an action of PGL_{n+1}, we can assume that $p = [1 : 0 : \cdots : 0]$ and $q = [0 : x_1 : \cdots : x_n]$. For some j, the j-th coordinate of q is nonzero. Applying the inverse of the PGL_{n+1} action gives a global section which vanishes at p but does not vanish at q.
- the differential at each point is injective. Note that the map is given by the monomials of exactly degree k in $n+1$ variables
$$p \mapsto [Z_0^k(p) : Z_0^{k-1}Z_1(p) : \cdots : Z_n^k(p)].$$
WLOG we can assume by a PGL_{n+1} action that $p = [1 : 0 : \cdots : 0]$. If we divide by this, our map becomes
$$p \mapsto [1 : Y_1(p) : \cdots : Y_n^k(p)]$$
where the map spans all monomials of degree $\leq k$ in n variables. The matrix of partials will have rank k, as the monomials of degree 1 with the appropriate partials will contribute the k 1's.

To prove Kodaira's embedding theorem, we will want to use Kodaira vanishing to kill these H^1 obstructions. We immediately run into an issue, however, which is that these ideal sheaves are not line bundles, as point(s) are not divisors when X is not a Riemann surface. The idea is to blow these points up so that we obtain actual divisors, pull back line bundles and twist enough to obtain positivity, and use Kodaira vanishing so that the vanishing of cohomological information at this pull-back/blow-up level implies vanishing of the desired H^1 terms.

Lemma 40.4. *Let X be a compact complex manifold. Let $\tilde{X} = Bl_pX$ be the blow-up of X at some point. Let $\mathcal{L}$ be a positive line bundle on X. Then for $k >> 0$,*

$$\pi^*\mathcal{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)$$

is positive.

Proof. Note that $\pi^*\mathcal{L}^{\otimes k}$ can never be positive, only semi-positive. Why? Well, over $\tilde{X} \setminus E$ it is positive. But over E, the curvature is 0. But note that $\mathcal{O}_{\tilde{X}}(-E) \cong \mathcal{O}_E(1) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ which is positive. So here's what we'll do:

Let U_1 be an open set of E which is the preimage of a neighborhood of $p \in X$ where the blow-up takes place. So the projection $U_1 \to \mathbb{P}^{n-1}$ is that of the tautological line bundle. Note that we obtain a hermitian metric h_1 on the pullback of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ onto U_1, where it is positive over E.

Now let $U_2 = \tilde{X} \setminus E$. Note that $\mathcal{O}_{\tilde{X}}(-E)$ over U_2 is trivial. Then over U_2, we can take a standard frame and so we have a constant hermitian metric $h_2(e, e) = 1$.

Taking a partition of unity ρ_1, ρ_2 subordinate to $\{U_1, U_2\}$, we can glue to get a hermitian metric $h = \rho_1 h_1 + \rho_2 h_2$ on $\tilde{X}$ for the line bundle $\mathcal{O}_{\tilde{X}}(-E)$. Over E, this metric is positive. It will remain positive on some open $W \supseteq E$ by continuity. More specifically, if you think about what happens when you pullback a line bundle, you will see that its semipositive, and whether its not strictly positive depends on when the differential is not injective. But the differential being injective is an open condition, and in these circumstances, the pullback is positive. So we can find such an open $W \supseteq E$.

Then if we consider $\pi^*\mathcal{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)$, the curvature over W will be positive, since the curvature contributed by $\pi^*\mathcal{L}^{\otimes k}$ will be semipositive, while the curvature contributed by $\mathcal{O}_{\tilde{X}}(-E)$ will be positive.

Now consider the curvature over $\tilde{X} \backslash W$. Note this is compact. So the curvature of $\pi^*\mathcal{L}^{\otimes k}$ will be positive, scaled by k, and its unclear what the curvature of $\mathcal{O}_{\tilde{X}}(-E)$ will be over $\tilde{X} \setminus W$. But since this domain is compact, we have bounded quantities. Then for $k >> 0$, we get positivity on this region. □

We now prove Kodaira's embedding theorem.

Theorem 40.5. *Let X be a compact complex manifold. Let $\mathcal{L}$ be a positive line bundle. Then for $k >> 0$,*

$$|\mathcal{L}^{\otimes k}| : X \to \mathbb{P}^N$$

is an embedding.

Proof. We need to show that for $k >> 0$, the complete linear system is base point free, separates points, and separates tangent vectors. First we show **base point freeness**.

We need to show that for $k >> 0$, $H^0(X, \mathcal{L}^{\otimes k}) \to \mathcal{L}_p^{\otimes k}$ is surjective for every p. Consider the blow up at p, $\tilde{X} \to X$. Then we have morphism of long exact sequences

$$\begin{array}{ccccccccc}
0 \to H^0(X, \mathcal{I}_p \otimes \mathcal{L}^{\otimes k}) & \longrightarrow & H^0(X, \mathcal{L}^{\otimes k}) & \longrightarrow & H^0(X, \mathcal{L}^{\otimes k}(p)) & \longrightarrow & H^1(X, \mathcal{I}_p \otimes \mathcal{L}^{\otimes k}) \to \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-E) \otimes \pi^*\mathcal{L}^{\otimes k}) & \to & H^0(\tilde{X}, \pi^*\mathcal{L}^{\otimes k}) & \to & H^0(\tilde{X}, \pi^*\mathcal{L}^{\otimes k} \otimes \mathcal{O}_E) & \to & H^1(X, \mathcal{O}_{\tilde{X}}(-E) \otimes \pi^*\mathcal{L}^{\otimes k}) \to \cdots
\end{array}$$

Really the right language to use here is projection formulas, but our setting here is concrete enough that this is all clear.

The first, lefthand-most, map is clearly an isomorphism. The second map is also an isomorphism. Injection is clear. To see surjection, note if $\dim X = 1$, then blowing up doesn't do anything and we're done. So suppose $\dim X \geq 2$, surjection follows from Hartog's. The third map is also clearly an isomorphism. Then if we prove the vanishing of

$$H^1(X, \mathcal{O}_{\tilde{X}}(-E) \otimes \pi^* \mathcal{L}^{\otimes k})$$

for $k >> 0$ then we're done with base point free. So we situate ourselves in a Kodaira-vanishing situation. Consider

$$\omega_{\tilde{X}} \otimes \omega_{\tilde{X}}^{-1} \otimes \pi^* \mathcal{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E).$$

Note that $\omega_{\tilde{X}} \cong \pi^* \omega_X \otimes \mathcal{O}_X((n-1)E)$. Then $\omega_{\tilde{X}}^{-1} \cong \pi^* \omega_X^{-1} \otimes \mathcal{O}_X(-(n-1)E)$. Then our line bundle becomes

$$\omega_{\tilde{X}} \otimes \pi^* \omega_X^{-1} \otimes \pi^* \mathcal{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-nE).$$

Since X is compact, for $k >> 0$ we have $\omega_X^{-1} \otimes \mathcal{L}^{\otimes k}$ is positive. Then $\pi^* \omega_X^{-1} \otimes \pi^* \mathcal{L}^{\otimes k'}$ is semi-positive for $k' >> 0$. And for $k'' >> 0$, $\pi^* \mathcal{L}^{\otimes k''} \otimes \mathcal{O}_{\tilde{X}}(-nE)$ is positive. Then for $k >> 0$, we see that

$$\pi^* \omega_X^{-1} \otimes \pi^* \mathcal{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-nE)$$

is positive. Then by Kodaira vanishing, since $\tilde{X}$ is compact Kahler,

$$H^1(\tilde{X}, \omega_{\tilde{X}} \otimes \pi^* \omega_X^{-1} \otimes \pi^* \mathcal{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-nE)) = 0.$$

Thus, for $p \in X$, there exists k_0 such that for $k \geq k_0$, we have

$$H^0(X, \mathcal{L}^{\otimes k}) \to \mathcal{L}^{\otimes k}(p)$$

is surjective. To conclude base point freeness, we need to show that such a k_0 exists which does not depend on $p \in X$. Before we do this, we will show similar vanishing which imply separation of points and tangent vectors for choices of points. Afterwards, we will show that bounds can be found which do not depend on choice of point(s).

Let us now address **separation of points.** Let $p, q \in X$ be distinct. Then we are interested in showing that $H^0(X, \mathcal{L}^{\otimes k}) \to \mathcal{L}^k(p) \oplus \mathcal{L}^k(q)$ is a surjection for $k >> 0$. Again, we obtain a morphism of long exact sequences

$$\begin{array}{ccccccc}
0 \to H^0(X, \mathcal{I}_{p,q} \otimes \mathcal{L}^{\otimes k}) & \longrightarrow & H^0(X, \mathcal{L}^{\otimes k}) & \to & H^0(X, \mathcal{L}^{\otimes k}(p) \oplus \mathcal{L}^{\otimes k}(q)) & \longrightarrow & H^1(X, \mathcal{I}_{p,q} \otimes \mathcal{L}^{\otimes k}) \to \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-E) \otimes \pi^* \mathcal{L}^{\otimes k}) & \to & H^0(\tilde{X}, \pi^* \mathcal{L}^{\otimes k}) & \longrightarrow & H^0(\tilde{X}, \pi^* \mathcal{L}^{\otimes k} \otimes \mathcal{O}_E) & \longrightarrow & H^1(X, \mathcal{O}_{\tilde{X}}(-E) \otimes \pi^* \mathcal{L}^{\otimes k})
\end{array}$$

where $\tilde{X}$ is the blow up of X at p and q. Again, the first three maps are isomorphisms, so to prove our desired surjectivity, it suffices to show vanishing of

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-E) \otimes \tilde{L}^{\otimes k})$$

for $k >> 0$. Note $E = E_p + E_q$, where E_p, E_q are the exceptional divisors over p and q.

Again, we set ourselves up to exploit Kodaira's vanishing theorem. We have

$$\mathcal{O}_{\tilde{X}}(-E) \otimes \tilde{L}^{\otimes k}) \cong \omega_{\tilde{X}} \otimes \omega_{\tilde{X}}^{-1} \otimes \mathcal{O}_{\tilde{X}}(-E) \otimes \tilde{L}^{\otimes k})$$
$$\cong \omega_{\tilde{X}} \otimes \pi^* \omega_X^{-1} \otimes \mathcal{O}_{\tilde{X}}(-nE) \otimes \tilde{L}^{\otimes k}.$$

Again, t can be taken large to be large enough so that $\omega_X^{-1} \otimes \tilde{L}^t$ is positive on X, so the pullback is semipositive. And s can be taken large enough so that $\mathcal{O}_{\tilde{X}}(-nE) \otimes \tilde{L}^{\otimes s}$ is positive. The proof of such positivity is essentially the same as lemma 40.4, with the appropriate changes made to account for two exceptional divisors. Thus, by Kodaira vanishing, we have the desired vanishing of the H^1 term.

Now we address **separation of tangent vectors**. We would like to show surjectivity of
$$H^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{I}_p) \to \mathcal{L}_p^{\otimes k} \otimes \mathcal{I}_p/\mathcal{I}_p^2$$
for $k >> 0$. We have the map of long exact sequences
$$\begin{array}{ccccccc} 0 \to H^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{I}(-2p)) & \longrightarrow & H^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{I}(-p)) & \longrightarrow & H^0(X, \mathcal{L}^{\otimes k} \otimes \frac{\mathcal{I}(-p)}{\mathcal{I}(-p)^2}) & \longrightarrow & H^1(X, \mathcal{L}^{\otimes k} \otimes \mathcal{I}(-2p)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \to H^0(\tilde{X}, \tilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-2E)) & \to & H^0(\tilde{X}, \tilde{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-E)) & \to & H^0(\tilde{X}, \tilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_E(-E)) & \to & H^1(\tilde{X}, \mathcal{L}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-2E \end{array}$$
The first two maps are clearly an isomorphism. To see why the third map is an isomorphism, note
$$H^0(X, \mathcal{L}^{\otimes k} \otimes \frac{\mathcal{I}(-p)}{\mathcal{I}(-p)^2}) \cong \mathcal{L}^{\otimes k}(p) \otimes m_p/m_p^2$$
and note $\tilde{\mathcal{L}}$ over E is a trivial line bundle with fiber $\mathcal{L}^{\otimes k}(p)$, and $\mathcal{O}_E(-E) \cong \mathcal{O}_E(1)$, so their tensor product is simply $\mathcal{L}^{\otimes k}(p) \otimes \mathcal{O}_E(1)$. Then the H^0 is
$$\mathcal{L}^{\otimes k}(p) \otimes H^0(\mathcal{O}_E(1)).$$
Note $\mathcal{O}_E(1) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, so if we think of the lines through p as tangent directions, the global sections are linear forms on the tangent directions of p. Then we see that $H^0(\mathcal{O}_E(1)) \cong m_p/m_p^2$, so we have the desired isomorphism.

To show our desired surjectivity, it suffices to show
$$H^1(\tilde{X}, \tilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\tilde{X}}(-2E)) = 0.$$
An analagous argument with Kodaira vanishing goes through.

It remains to show independence of all these bounds.

- To finish base point freeness, note that we have shown is that for every $p \in X$, there exists k_p such that for $k \geq k_p$, $\mathcal{L}^{\otimes k}$ will have a global section which does not vanish at p. Then this global section will not vanish on a neighborhood of p. Take all such neighborhoods, and since X is compact, we can cover X with finitely many of them. Then define k_0 to be the maximum of such k_p's.
- To finish separation of tangent vectors, note we have shown that for every $p \in X$, there exist k_p such that for $k \geq k_p$, we have surjection $H^0(X, \mathcal{L}^{\otimes k}) \to \mathcal{L}^{\otimes k}(p) \otimes m_p/m_p^2$. This in turn implies that the differential at p is injective. By the inverse function theorem, this implies that the map given by $\mathcal{L}^{\otimes k}$ locally has differential injective, so separates tangent vectors in this neighborhood of p. Then, again, we take such neighborhoods of p, and by compatness, we have finitely many, and then take k to be the maximum of the k_p's.
- To finish separation of points, note we have shown that for every pair of distinct p, q, for $k \geq k_{p,q}$ we have that $|\mathcal{L}^{\otimes k}|$ separates these two points. But note that since we proved separation of tangent vectors, the map for $k >> 0$ has differential injective everywhere, particularly at p, q. So for

$k >> 0$, there is a neighborhood around p and neighbrohood around q such that the map is injective. These will map onto neighborhoods around the images of p, q, so we can shrink so that we obtain neighborhoods of p and q that the map is completely injective on these neighborhoods. Then we can cover $X \times X$ with these neighborhoods, and then by compactness obtain a finite cover, and then find the appropriate bound.

□

41. 3/4/24: corollaries of Kodaira embedding, Fujita's Conjecture, most K3 not projective

At the beginning of class on this day we wrapped up the proof of Kodaira's embedding theorem 40.5. In the course of the proof, we saw the following corollary.

Corollary 41.1. *If X compact Kahler has a positive line bundle, then its blow up at some discrete finite set of points has positive line bundle. So X projective means blow up remains projective.*

So now, we need to show that these constants which give us H^1 vanishing do not depend on $p \in X$. For base point free, means we found a section s such that $s(x) \neq 0$ on p, so $s(x) \neq 0$ on neighborhood of p. Since X is compact, choose finite cover. Take maximum of the k_i.

Same argument holds for separation of tangent vectors. Separation of tangent vectors is an open property.

Separation of points. Separation of tangent vectors implies that differential is injective at every point. By calculus this means the map itself is injective in a neighborhood (inverse function theorem). So we have an small neighborhood of each point where it is injective. So take $X \times X \setminus \Delta$, cover by finitely many $U_{x,y}$'s.

So we have Kodaira's embedding theorem. Now we will prove Chow's theorem: every projective compact compelx manifold is actually defined by global homogeneous polynomials. This is a bit mind-boggling. You started with some that is a priori locally defined by holomorphic functinos. Then one day you find this fact: it is globally defined by homogeneous polynomials. Amazing.

We see so many areas of math come together in Kodaira's embedding. Differential geometry involved in vanishing of H^1. Topology involved in Lefschetz hyperplane theorem.

People try very hard to find effective bounds on this theorem. There are examples, where if you choose the line bundle, k goes to infinity. Problem that is still open, if you take $\mathcal{L}^{\otimes k}$ but tensor with canonical bundle. For k sufficiently large, will dominate $\omega_X \otimes \mathcal{L}^{\otimes k}$. There's a conjecture, Fujita's conjecture.

Proposition 41.2. *When $\mathcal{L}$ is positive, $\omega_X \otimes \mathcal{L}^{\otimes k}$ is base point free for $k \geq n+1$. And it is very ample for $k \geq n+2$.*

Open beyond the case of four-folds, five-folds. Most effective way to embed a variety into projective space. Besides low dimensions, we don't know this. Why can't we do any better? Think about projective space itself. $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. So certainly this $n+1$ bound cannot be better.

Definition 41.3.
(1) $\mathcal{L}$ is called very ample if $|\mathcal{L}|$ gives an embedding
(2) $\mathcal{L}$ is ample if for some k we have $|\mathcal{L}^{\otimes k}|$ is very ample.

Note that $\mathcal{L}$ is positive $\iff$ $\mathcal{L}$ is ample.

Corollary 41.4. *Let X be compact Kahler manifold. Then X is projective $\iff$ $\exists \omega \in A^2(X)$ closed positive (1,1) form such that $[\omega] \in H^2(X, \mathbb{Q})$.*

Proof. If it is projective, then $X \subset \mathbb{P}^N$. Then take $\omega_{FS}|_X$, and this is in $H^2(X, \mathbb{Z})$ so we are done. Other direction, having $[\omega] \in H^2(X, \mathbb{Q})$, multiply by some constant you get something in $H^2(X, \mathbb{Z})$. But all integral (1,1) forms by Lefschetz (1,1) theorem comes from first Chern class of line bundle.

Note you technically want to write $Im(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$. Doesn't necessarily embed. $\square$

Here is a very abstract condition for a compact Kahler manifold to be projective.

Corollary 41.5. *X compact Kahler and $H^{2,0}(X) = 0 \implies X$ is projective.*

Proof. Means there is a Kahler (1,1) form, $[\omega_0] \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$ and $H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} = H^{1,1}$. Rational points are dense in this space. You can approach this with a sequence of rational points, and positivity is an open condition. More precisely, there is a Hermitian form h_0 associated to ω_0, and this puts a hermitian inner product on $H^2(X, \mathbb{C})$, doesn't depend on representative of cohomology class. And $H^2(X, \mathbb{Q}) \subseteq H^2(X, \mathbb{R})$ is dense. $\omega_0 \in H^2(X, \mathbb{R})$ is positive. And X is compact. So $[\omega]$ sufficiently close to $[\omega_0]$ with respect to the hermitian form, we get ω is positive. But it is a $(1,1)$ form. And over $\mathbb{Q}$. This implies X is projective. $\square$

Example 41.6. Any compact Riemann surface X is projective. Since $H^2\mathcal{O}_X = 0$.

If X is Calabi-Yau manifold, of dimension at least 3. Then X is projective. By definition, Calabi-Yau means ω_X is trivial and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$. Sometimes you drop that condition, and you call it weak Calabi-Yau. But this is true definition. Otherwise, you include compact tori into Calabi-Yau. For example, hypersurfaces of degree $n+1$ in $\mathbb{P}^n$ are Calabi-Yau.

In dimension 2, Calabi-Yau is the same as K3 surfaces. Fact: there exists a parameter space M of all K3 surfaces of dimension 20. there exists a parameter space P of projective K3 surfaces of dimension 19. Where P is countable union of analytic hypersurfaces in M.

Example 41.7. Joe Harris style: Take $X \subseteq \mathbb{P}^3$, hypersurfaces of degree 4. What is a quartic in $\mathbb{P}^3$. Count them. So $H^0\mathcal{O}_{\mathbb{P}^3}(4) = \mathbb{C}^{35}$. So $|\mathcal{O}_{\mathbb{P}^3}(4)| = \mathbb{P}^{34}$. Take smooth ones, open set. Take automorphisms of $\mathbb{P}$, this gives you isomorphism classes. Have $PGL_4(\mathbb{C})$ action on $\mathbb{P}^3$. Look at $U/PGL_4(\mathbb{C})$. Projectively equivalent quartics. PGL is dimension $4^2 - 1 = 15 in this case. So \mathrm{U}/\mathrm{PGL}_4(\mathbb{C}) = 19$.

So we counted the dimension of a component of P, that which contains the quartics.

Infitesimal deformation sof X paramterized by $H^1(X, T_X)$. Think of this as tangent space to moduli space. Enough to compute this tangent space to estimate dimension of moduli space. Here we get lucky with Calabi-Yau's. On Calabi-Yau, we know that $\omega_X \cong \mathcal{O}_X$, so bySerre duality, $H^1(X, T_X) \cong H^1(X, \Omega^1_X)^\vee$. This is Hodge number. $H^{1,1}$ of Calabi-Yau is 20. Hodge diamond for every K3 surface. Always looks like $\{1\}, \{0,0\}, \{1,20,1\}, \{0,0\}, \{1\}$.

How do you get Hodge dimaond of every K3 surface? If you accept this parameter space, if you have smooth family, then Hodge numbers are constant. Stronger: Family is C^∞ locally trivial, Betti numbers are constant but Hodge numbers are upper semicontinuity.

You know $h^{1,0} = 0$, and $h^{2,0} = 1$. So everything in Hodge diamond of K3 surface is clear, except middle. So to compute $h^{1,1}$, it's enought o compute $b_2(X)$. Then it sneough to compute topological Euler characteristic. Famous theorem for surfaces, Noether formula which is equivalent to Riemann Roch for surfaces. Which says that the Euler characteristic

$$\chi(\mathcal{O}_X) \cong \frac{1}{12}(c_1(\omega_X)^2 + c_2(\omega_X)).$$

Guass-Bonnet theorem says $c_2(\omega_X)$ is topological Euler characteristic of X. Have $c_1(\omega_X)^2 = 0$. So $\chi_{top} = 12 \cdot \chi(\mathcal{O}_X) = 12[h^0(\mathcal{O}_X) - h^1\mathcal{O}_X + h^2\mathcal{O}_X] = 24$. So $2 + b_2 = \chi_{top} \implies b_2 = 22$, so $h^{1,1} = 20$.

Now if X_0 is projective K3 surface. Then take moduli space of K3 surfaces M. Approach with K3 surfaces X. Have $H^2(X, \mathbb{C}) \cong H^2(X_0, \mathbb{C}) \cong \mathbb{C}^{22}$. Can use the betti number calculation, but can do something better (smooth family, cohomology upper semicontinuity??). Have $H^2(X, \mathbb{C}) \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$. Have Kahler $[\omega_0]$ closed positive 1,1 form in integral $H^{1,1}(X_0) \cap H^2(X_0, \mathbb{Z})$. Have $H^{1,1}(X_0)$, think of it embedded in $H^2(X, \mathbb{C})$ via the isomorphism, then project to $H^{2,0}(X)$. So X_0 is projective if and only if ω_0 belongs to the kernel of this projection. So $H^{1,1}(X_0)$ is 20 dimensional, $H^{2,0}(X)$ is 1 dimensional. So that's where we get the 19 dimensions.

So most K3 surfaces are not projective. For complex tori, it's even worse.

Theorem 41.8 (Riemann's Criterion)**.** *If $T = V/\Lambda$, where $V \cong \mathbb{C}^n$. Compact complex torus. Then T is projective $\iff$ there exists a positive definite Hermitian form $h : V \times V \to \mathbb{C}$ such that $Im(h) : \Lambda \times \Lambda \to \mathbb{Z}$ is integral.*

42. 3/6/24: Chow's theorem

We just finished the Kodaira embedding theorem, and as a corollary, we obtained various conditions for when a compact Kahler manifold is projective, such as existence of positive line bundle and $H^{2,0}(X) = 0$. We also gave a heuristic argument that a general K3 surfaces is not projective. A compact complex torus is also generally not projective.

Proposition 42.1. *We have that a torus $V/da = T$ is projective $\iff$ there exists a hermitian form $h : V \times V \to \mathbb{C}$ such that $Im(h) : \Lambda \times \Lambda \to \mathbb{Z}$ is integral.*

Example 42.2. Examples of compact complex tori that are not projective. Intersection product/cup product, generic hyperplanes through Riemann surface.. homology classes or Poincare dual, cohomology classes. Say $T = \mathbb{C}^2/\Lambda \cong \mathbb{Z}^4$. Have $\Lambda = \mathbb{Z}\langle v_1, \cdots, v_4\rangle$, chosen generically.

If T is projective then there exists $C \subseteq T$ of dimension 1 such that $0 \neq \eta_C = (\int_C \cdot)$.

Say z, w coordinates on $\mathbb{C}^2$, $dz \wedge dw$, descends to T. Holomorphic 2-form, $\omega|_C = 0$.

Have $H_1(T, \mathbb{Z}) \cong \Lambda$. Have $H_2(T, \mathbb{Z}) = \bigwedge^2 \Lambda$. Have $H^1(T, \mathbb{R}) = Hom_{\mathbb{Z}}(\Lambda, \mathbb{R})$ (can replace with $\mathbb{C}$). Have $H^2(T) = \bigwedge^2 H^1 = Hom_{\mathbb{Z}}(\bigwedge^2 \Lambda, *)$. Have $[c] \in H_2(T, \mathbb{Z}) = \mathbb{Z}\langle v_i \wedge v_j\rangle_{i<j}$. Evaluate ω as $v_i \wedge v_j$,

$$dz \wedge dw(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}) = |\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}| = a_1b_2 - a_2b_1.$$

Have $0 = [\omega] \cdot [c]$ is $\mathbb{Z}$-linear combo of the 6 2x2 minors of $\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$ if you have generic choice of v_i.

Cohomology of torus, generic choice of lattice cannot give you a projective one. What we have proved here, is that a generic torus has no curve, there is no geometry. It couldn't be projective because it doesn't have any geometry. Shafaverich, Vol2, Ch VIII, section 1.4. Many interesting examples.

Let's move on to Chow's theorem.

Theorem 42.3. *Let $X \subseteq \mathbb{P}^n$ be an analytic subset. Then there exists homogeneous polynomials $F_1, \cdots, F_m \in \mathbb{C}[X_0, \cdots, X_n]$ such that $X = Z(F_1, \cdots, F_m)$, i.e. X is algebraic (projective variety).*

So something in $\mathbb{P}^n$ a priori defined by local holomorphic functions, is defined actually by global polynomials. Combined with Kodaira embedding, Chow's theorem gives the following corollary.

Corollary 42.4. *If X is a compact complex manifold. Then X has positive ($\iff$ ample) line bundle $\iff$ X is projective variety.*

Before we prove Chow's theorem, we remark the following.

Remark 42.5. Chow's theorem is a special case of a vast generalization which is Serre's GAGA principle. He makes this connection between the two worlds. The Zariski topology is strictly coarser than the analytic topology. To any algebraic variety, of any sheaf of $\mathcal{O}_X$ modules, you can associate to it an analytic topology. Sheaf is an anlytic topology, space is an analytic topology. Functor from one category to another. What Serre proves, based on Chow's theorem, is that if you're variety is embedded in projective space, then your functor is an equivalence of categories.

More precisely, there is an equivalence of categories between coherent algebraic sheaves on $\mathbb{P}^n$ and coherent analytic sheaves on $\mathbb{P}^n$ (and isomorphisms on cohomology).

Everything that we've learned about $\mathbb{P}^n$ has been evidence for this fact. On $\mathbb{P}^n$, analytically, we have

$$Pic(\mathbb{P}^n) \cong \mathbb{Z}\langle \mathcal{O}_{\mathbb{P}^n}(1) \rangle.$$

and these are line bundles are trivialized on $U_i = (z_i \neq 0)$, and the transition functions came from $(z_i/z_j)^k$ which are algebraic. Their global section $H^0(\mathcal{O}_{\mathbb{P}^n}(k))$ being homogeneous polynomials of degree k... these were all pointing to Chow's theorem and more broadly this phenomenon described by Chow's theorem.

There's more to it. Think about the exponential exact sequence. It is inherently analytic, we defined it by taking $\exp(2\pi i)$. But all of the sudden, you can use this in the algebraic world.. over the Zariski topology. When $X = \mathbb{P}^n$, cohomology LES of exponential exact sequence gave

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

and we identified $H^2(X, \mathbb{Z})$ with singular cohomology, but this must be that X is analytic space, otherwise it would not make sense to talk about singular cohomology. But Serre's GAGA theorem says that we can mix algebraic data into this story. We have $H^1(X^{an}, \mathcal{O}_X^{an}) \cong H^1(X^{alg}, \mathcal{O}_{X^{alg}})$ and $H^2(X^{alg}, \mathcal{O}_X^{alg})$. So you can use tools from both worlds and mix them, through the medium of say for example the exponential exact sequence, via Serre's GAGA principle.

References for Serre's GAGA principle: Griffiths-Harris, Chapter 1 Section 3. The main reference for GAGA is J.P. Serre "GAGA," Annals Institut Fonrier 6,

1956. Other treatments for Chow's theorem: book by Mumford, "complex projective varieties," chapter 4, red book of varieties on schemes, chapter 1, section 10.

To prove Chow's thoerem, we will use a result in several complex variables which is a generalization of Hartog's theorem.

Theorem 42.6 (Levi extension theorem)**.** *Let X be a connected complex manifold of dimension n and $Z \subset X$ is an analytic subset of codimension $\geq k+1$. If $Y \subseteq X \setminus Z$ is an analytic subset of codimension k, then $\overline{Y} \subseteq X$ is an analytic subset.*

Proof. Can you come up with a simpler proof? Popa thinks we are using too much. $\square$

We have seen analogs of this. If $Z = \{x\} \hookrightarrow X$. If Y is positive dimensional and subset of $X \setminus \{x\}$, then $Y \subseteq X$ is analytic. We are thinking of the zero locus here of holomorphic functions on $X \setminus \{x\}$, and extending these functions via Hartog's. So this Levi extension theore is a geometric generalization of Hartog's theorem.

Chow. We have $X \subset \mathbb{P}^n$. There is projection map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, and we have $\pi^{-1}(X) \subseteq \mathbb{C}^{n+1} \setminus \{0\}$. It is like a cone over X minus the origin. This will be our Y. Every component of Y is positive-dimensional, because it contains the line through every point of X. So Levi-extension theorem applies to this. It is certainly analytic, because it is decsribed by the same functions that describe X. So this means that the closure of $\overline{Y} = C(X)$ is analytic.

We will take advantage of this cone structure. Look at $0 \in \mathbb{C}^{n+1}$ which is in $C(X)$. Look at $\mathcal{O}_{n+1}$, ring of germs of holomorphic functions at $0 \in \mathbb{C}^{n+1}$. Look at $I \subseteq \mathcal{O}_{n+1}$, ideal of Y at 0 of germs which vanish at origin. For $f \in I$, we can write as a convergent power series around the origin, and we can write $f = \sum_{k \geq 0} f_k$ where f_k homogeneous in the z_i of degree k. Since $f \in I$, they vanish at $0 \in C(X)$. The claim is that if $f \in I$, then each of the f_k is in the ideal. (Note, this is mimicking homogeneous ideals, as done in Vakil).

Claim: $f_k \in I, \forall k$. Choose $x \in \overline{Y} = C(X)$, $\lambda \in \mathbb{C}$ and $0 < |\lambda| << 1$. Continue to have

$$0 = f(\lambda x) = \sum_{k \geq 0} f_k(\lambda x) = \sum_{k \geq 0} \lambda^k f_j(x),$$

now we think of this as a power series in terms of λ. So by the identity principle, the $f_k(x) = 0$ for every k. You can do this for every x. For every $f_k \in I$.

This implies that I is generated by homogeneous polynomials. Since $\mathcal{O}_{n+1}$ is Noetherian, this implies there exists homogeneous $F_1, \cdots, F_k$ such that $\overline{Y} = Z(F_1, \cdots, f_K)$ in a neighborhood of 0. But this is enough, since it is a cone. So all of $\overline{Y}$ is given by vanishing: $C(X)$ is algebraic. But this implies X is algebraic. $\square$

Example 42.7. Examples of varieties. For surfaces:

(1) Ruled surfaces. Fix a vector bundle E of rank r on X. Have $E(x) \cong \mathbb{C}^r$. Then we can define a projectivized vector bundle, $\pi : \mathbb{P}(E) \to X$. Have $\pi^{-1}(x) \cong \mathbb{P}^{r-1} = \mathbb{P}(E(x))$, space of 1-dimensional quotients of $E(x)$, or hyperplanes in $E(x)$. Or lines in $E(x)$. This projectivized bundle is locally trivialized over the same local trivializations of E. Comes with a

distinguished line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$. Have to glue over overlaps. Such that $\mathcal{O}_{\mathbb{P}(E)}(1)|_{\pi^{-1}(x)} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(1)$. New line bundle that doesn't come from X.

But we also have line bundle that comes from X. Have line bundles in $\pi^* Pic(X) \subseteq Pic(\mathbb{P}(E))$. Have line bundles $\pi^* Pic(X) \subseteq Pic(\mathbb{P}(E))$. In fact, $Pic(\mathbb{P}(E)) \cong \mathbb{Z}\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \pi^* Pic(X)$.

Examples: when $E = \mathcal{O}_X^{\oplus r}$, have $\mathbb{P}(E) = X \times \mathbb{P}^{r-1}$. If $E = \mathcal{L}^{\oplus r}$, then $\mathbb{P}(E) \cong \mathbb{P}(F) \iff \exists L \in Pic(X)$ such that $E \cong F \otimes L$.

(2)

Let's discuss more examples of varieties. Fix a vector bundle E of rank r on X. Have $E(x) \cong \mathbb{C}^r$. Then we can define a projectivized vector bundle, $\pi : \mathbb{P}(E) \to X$. Have $\pi^{-1}(x) \cong \mathbb{P}^{r-1} = \mathbb{P}(E(x))$, space of 1-dimensional quotients of $E(x)$, or hyperplanes in $E(x)$. Or lines in $E(x)$. This projectivized bundle is locally trivialized over the same local trivializations of E. Comes with a distinguished line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$. Have to glue over overlaps. Such that $\mathcal{O}_{\mathbb{P}(E)}(1)|_{\pi^{-1}(x)} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(1)$. New line bundle that doesn't come from X.

But we also have line bundle that comes from X. Have line bundles in $\pi^* Pic(X) \subseteq Pic(\mathbb{P}(E))$. Have line bundles $\pi^* Pic(X) \subseteq Pic(\mathbb{P}(E))$. In fact, $Pic(\mathbb{P}(E)) \cong \mathbb{Z}\mathcal{O}_{\mathbb{P}(E)}(1) \oplus \pi^* Pic(X)$.

Examples: when $E = \mathcal{O}_X^{\oplus r}$, have $\mathbb{P}(E) = X \times \mathbb{P}^{r-1}$. If $E = \mathcal{L}^{\oplus r}$, then $\mathbb{P}(E) \cong \mathbb{P}(F) \iff \exists L \in Pic(X)$ such that $E \cong F \otimes L$.

Definition 42.8. A surface X is called a ruled surface if there exists a curve C and a vector bundle E on C such that $X \cong \mathbb{P}(E)$.

Among these, the rational ruled surfaces correspond to when $C = \mathbb{P}^1$. These rational ruled surfaces are birational to $\mathbb{P}^2$. A fact that needs to be proven: these are all the surfaces with Kodaira dimension $K(X) = -\infty$. ($\iff H^0(X, \omega_X^{\otimes m}) = 0, \forall m \geq 1$). no pluricanonical forms. Okay not exactly. If you blow up a ruled surface at a point, its not longer a ruling. But if you allow for finitely many points blown up, these are the birational ruled surfaces, and the true statement is that these are all the surfaces with Kodaira dimension $K(X) = -\infty$.

Theorem 42.9 (Grothendieck's theorem)**.** *All vector bundles over $\mathbb{P}^1$ split into a direct sum of the line bundles.*

These $F_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ are the Hirzebruch surfaces. The F_1 is most special, because it is $Bl_p(\mathbb{P}^2)$. It is the only one that is so called non-minimal. Think about this.

43. 3/18/24: CLASSIFICATION OF SURFACES

Let's do more examples of varieties and surfaces. There's a new homework, mostly designed to show you more examples of interesting surfaces. Last time we said something about rationally ruled surfaces, surfaces of Kodaira dimension $-\infty$. We want to go through examples of surfaces in each step of the classification of projective surfaces.

Here are some examples of surfaces with Kodaira dimension 0. In this case of surfaces, $K(X) = 0$ means: $h^0(X, \omega_X^{\otimes m}) = 1$ for m sufficiently divisible and large (so for sufficiently large multiples of some numebr). So the idea is that this grows like a polynomial m to the power 0. So the typical case of achieving this is $\omega_X = \mathcal{O}_X$.

- K3 surfaces

- Abelian surfaces.

Two more surfaces which have Kodaira dimension 0, but don't have trivial canonical bundle. These are Enriques surfaces and bielliptic surfaces. Enriques surfaces are basically quotients of K3 surfaces by finite groups, and bielliptic surfaces are quotients of abelian surfaces by finite groups. In the case of K3 surfaces, have $\omega_X^{\otimes 2} \cong \mathcal{O}_X$ but $\omega_X \neq \mathcal{O}_X$.

In the case of bielliptic surfaces, these are $E \times F/G$ where E and F are elliptic curves, where G acts on E by translations, G acts on F such that $F/G \cong \mathbb{P}^1$.

Every such surface of Kodiara dimension 0 is birational to one of these 4 surfaces: K3, abelian, Enrique, or bielliptic.

The homework where $S \to \mathbb{P}^1$ where S is K3 surface and general fiber is elliptic curve, this is a surprising example. Suprising only for a moment.

Lemma 43.1. *If X is the quotient of a K3 surface Y by a fixed point free involution, then X is Enriques.*

So if you have Y a K3 surface, and $\sigma : Y \to Y$ an involution so that $\sigma^2 = id$, then $X = Y/\sigma$ is an Enriques surface. Cyclic covering construction, the meaning is to take square roots of line bundles. Is there such a K3 surface? It turns out that a general Enriques surface is constructed by this example.

Take $Q_1, Q_2, Q_3 \in \mathbb{C}[x_0, x_1, x_2]$ quadrics. Then you take 3 other quadrics, Q_1', Q_2', Q_3' also in 3 variables, but in 3 other variables, X_3, X_4, X_5. So you now think of them as quadrics in $\mathbb{P}^5$. These are very singular in $\mathbb{P}^5$. Nevertheless, define $P_i = Q_i + Q_i'$ for $i = 1, \cdots, 3$. Then you take your surface to be the complete intersection of these P_i. So $X = Z(P_1) \cap Z(P_2) \cap Z(P_3)$. Exercise: this is smooth if Q_i, Q_i' are general. Complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^5$. We saw this is one of the few things that give us a K3 surface. So Y is a K3 surface. We know canonical bundle of complete intersections, $2 + 2 + 2 = 6$, so $6 - 5 - 1 = 0$, so canonical bundle is trivial.

If you take the involution $i : \mathbb{P}^5 \to \mathbb{P}^5$, where $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5)$. Then the fixed locus is $Z_1 \cup Z_2$ where $Z_1 = (x_0 = x_1 = x_2 = 0)$ and $Z_2 = (x_3 = x_4 = x_5 = 0)$.

Exercise: if Q_i, Q_i' are general, then $X = Y/\sigma$ does not intersect Z_1 and Z_2. So i preservces Y, so it induces $\sigma; Y \to Y$ where the fixed locus of σ is ϕ.

Think about what this does on the cohomology of X. Cohomology of Y to cohomology of X? Another way to define Enriques surface. Known to be those which have $H^0(X, \omega_X) = 0$ and $H^0(X, \Omega_X^1) = 0$.

For Bielliptic surfaces: $G = \mathbb{Z}/2\mathbb{Z}$. E elliptic curve. It is a quotient of $\mathbb{C}$ by a subgroup. Exercise in your homework: how many 2-torsion points. If you have a 2-torsion point, it can act by translation, and this gives an involution on your elliptic curve, since double composition gives back identity.

Another way to act on elliptic curves, $F \to F$, $x \mapsto -x$. This is a ramified map of curves. Exercise, that $F/G \cong \mathbb{P}^1$. Can always act by G on both factors $E \times F/G$, with these different acts by $G \cong \mathbb{Z}/2\mathbb{Z}$. $E \times F/G \to F/G \cong \mathbb{P}^1$, and $E \times F/GtoE/G = E$. Wikipedia page for classification of surfaces is pretty good. This is the story for Kodaira dimension 0.

For Kodaira dimension 1. Have $h^0(X, \omega_X^{\otimes m}) \sim m$ for m sufficiently large. Simple example is $X = E \times C$ where E is elliptic and C cure for genus $g(C) \geq 2$. Might be too special to take a product. So here's a more general construction.

Take $C \times \mathbb{P}^2$. Take any line bundle $\mathcal{L}$ on C which is base point free. Take $M = p_1^* L \otimes p_2^* \mathcal{O}_{\mathbb{P}^2}(3)$. So M is base point free since each of the individual factors are base point free. Then choose a general element $X \in |M|$ in this linear system. This X will be a surface. In general, there's a delicate theorem called the Bertini theorem. Base point free linear system in dimension at least 2, then general member is smooth. Have $X \subset C \times \mathbb{P}^2$, map under p_1 call it f. Then $X|_{f^{-1}(c)}$ is in $|\mathcal{O}_{\mathbb{P}^1}(3)|$ is a plane cubic. So X is elliptic surface. but many elliptic surface are not in Kodaira dimension 1. Have elliptic fibration, but not kodaira dimesnion 1.

Kodaira dimension $K(X) = 2$, where $h^0(X, \omega_X^{\otimes m}) \sim m^2$. These are X of general type. There are so many, that there is no typical surface of general type. Usually what you do is bound some invariant and then construct the moduli space of general type surfaces with that invariant.

Examples include $X = C_1 \times C_2$ where $g(C_i) \geq 2$. Have $X^d \subseteq \mathbb{P}^3$ for $d \geq 5$. A fact: most subvarieties of an abelian variety (like a torus), are of general type. Subvariety of abelian variety is not general type only if it has fibration over a sub abelian variety. The general abelian variety does not contain any sub abelian variety.

$b_i(\mathbb{P}^2) = 1, 0, 1, 0, \cdots$.. there are fake projective planes with the same betti numbers.

Let's discuss a bit about abelian varieties. These are projective compact complex tori. Emphasis on projective. The general torus is not projective (somewhere we showed this before). Let's review some things that we know.

Fix a torus $X = V/\Lambda$ where $V \cong \mathbb{C}^g$. We have $\Lambda \cong \mathbb{Z}^{2g}$. $\mathbb{C}^g$ is the covering space of the torus, so $H_1(X, \mathbb{Z}) \cong \Lambda$. So $H^1(X, \mathbb{Z}) = \Lambda^*$. If $T'_{x,0} \cong V$. $T'_X \cong \mathcal{O}_X^{\oplus g} \cong V \otimes_{\mathbb{C}} \mathcal{O}_X$. And $\Omega_X^1 \cong V^* \otimes \mathcal{O}_X$. And $H^0(X, \Omega_X^1) \cong V^*$. Think of X as $X \cong \frac{H^{1,0}(X)^*}{H_1(X,\mathbb{Z})}$.

Have $\Lambda^* \hookrightarrow \Lambda^* \otimes_{\mathbb{Z}} \mathbb{R} \hookrightarrow \Lambda^* \otimes_{\mathbb{Z}} \mathbb{C} = H^{1,0}(X) H^{0,1}(X)$, and if we project this to $H^{0,1}(X)$, we have $\Lambda^* \hookrightarrow H^{0,1}(X)$ is an embedding. This $H^{0,1}(X) = \overline{V^*}$. And Λ^* is dual of this lattice. So this inclusion gives the dual torus.

Definition 43.2. The dual torus is defined to be

$$\widehat{X} = \frac{\overline{V^*}}{\Lambda^*}$$

for any torus $X = V/\Lambda$.

And we realize this through the Hodge decomposition. But from the exponential exact sequence, we have surjection on the H^0's by compactness, so

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X^*) \to^{c_1} H^2(X, \mathbb{Z}) \to \cdots$$

where $H^1(X, \mathbb{Z}) = \Lambda^*$ and $H^{0,1} = \overline{V^*}$. Now this discussion inspires us to this for any compact Kahler manifold, but we'll get back to this in a second.

So far we have only used the fact the torus is Kahler. Now we use the projective condition. We've seen that X is projective if and only if there exists hermitian form $H : V \times V \to \mathbb{C}$ such that $\omega = Im(h) : \Lambda \times \Lambda \to \mathbb{Z}$ is integral, alternating, and positive definite.

Now let's talk about some of the most important abelian varieties out there, namely the Picard and Albanese tori.

Start with X, compact Kahler manifold. Look at the exponential exact sequence, have $H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X)$ embeds as a lattice. Note after complexification, $H^1(X,\mathbb{Z})$ becomes $H^1(X;\mathbb{C})$ which has the Hodge decomposition and we see it has twice the dimension of $H^1(X,\mathcal{O}_X)$. Here we are using the fact X is Kahler. So we have

$$Pic^0(X) = \frac{H^{1,0}(X)}{H^1(X,\mathbb{Z})}$$

and this is the Picard variety of X. By the exponential sequence, the Picard variety is the moduli space of line bundles which first Chern class 0.

Remark 43.3. Sometimes these are called topologically trivial line bundles, since topologically c_1 classifies these line bundles.

The dimension of $Pic^0(X)$ is $h^{0,1}(X)$. in fact, tangent space is $H^1(X,\mathcal{O}_X)$.

Lemma 43.4. *X is projective implies $Pic^0(X)$ is projective.*

Proof. X projective $\iff$ $[\omega] \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. Hodge Riemann Bilinear relations, have this quadratic form $Q : H^1(X,\mathbb{Z}) \times H^1(X,\mathbb{Z}) \to \mathbb{Z}$, where $([\alpha],[\beta]) \mapsto \int_X \omega^{\wedge n-1} \wedge \alpha \wedge \beta$. This had all the properties we wanted, nice on primitive form. But all H^1 cohomologies are primitive. So this is a positive definite alternating form. So recall $H^2(X,\mathbb{Z}) = Hom(H_2(-),\mathbb{Z})$, where H_2 is $\bigwedge^2 H_1$. So H_1 hodge Riemann bilinear relation induces a polarization on the Picard torus.

You can also go backwards. Projective tori are the same polarized weight 1 Hodge structures. $\square$

Category of tori is equivalent to category of weight 1 integral Hodge structures. Projective torus is same as polarized weight 1 integral Hodge structures.

By definition, if X is torus, then $Pic^0(X) = \widehat{X}$. We $\widehat{Pic^0(X)} = \frac{\overline{(H^{0,1})^*}}{H^1(X,\mathbb{Z})^*}$. Serre duality, $H^0(X,\Omega^1_X)^* \cong H^n(X,\Omega^{n-1}_X) = H^{n-1,n}(X)$, and $\overline{H^{0,1}}$. Have $\overline{H^{0,1}} = H^{1,0}$ by Poincare duality. And $H_1(X,\mathbb{Z}) \cong H^{2n-1}(X,\mathbb{Z}) \hookrightarrow H^{2n-1}(X,\mathbb{C}) = H^{n-1,n}(X) \oplus H^{n,n-1}(X)$, project to $H^{n-1,n}(X)$. So $\widehat{Pic^0(X)} = \frac{H^0(X,\Omega^1_X)^*}{H_1(X,\mathbb{Z})}$, this is Albanese variety of X.

How to embed as lattice? Each $H_1(X,\mathbb{Z})$, some loop γ. They each give a functional in $H^0(X,\Omega^1_X)^*$ by integration along γ. There is an embedding, $X \to \widehat{Pic^0(X)}$. Fix $x_0 \in X$. You map to

$$\int_{x_0}^{x} \omega_1, \cdots, \int_{x_0}^{x} \omega_g)$$

where $\langle \omega_1, \cdots, \omega_g \rangle = H^0(X,\Omega^1_X)$. Well-defined by $H_1(X,\mathbb{Z})$. This is the famous Abel-Jacobi map.

44. 3/20/24: Abel-Jacobi maps, Albanese varieties, Picard varieties

More careful reading in Griffths and Harris. We pick up from the discussion we had last lecture about the Albanese torus. We defined

$$Pic^0(X) := \frac{H^{0,1}(X)}{/} H^1(X,\mathbb{Z}).$$

Then we look at its dual, $\widehat{Pic^0(X)} = \frac{(\overline{(H^{0,1})^*}}{H^1(X,\mathbb{Z})} = \frac{H^0(X,\Omega^1_X)^*}{H_1(X,\mathbb{Z})}$. Always modulo torsion. So you have an injection of $H_1(X,\mathbb{Z})$ modulo torsion into $H^0(X,\Omega^1_X)^*$. Alternatively, if you have pass through Poincare dual, $H_1(X,\mathbb{Z}) \cong H^{2n-2}(X,\mathbb{Z})$, then you complexify which kills the torsion, then you pass to Hodge decomposition and project to $H^{n-1,n}(X)$.

This map $H_1(X,\mathbb{Z}) \to H^0(X,\Omega^1_X)^*$ can also be defined as taking the map $[\gamma]$ and mapping it to the functional where you integrate on that loop: $\int_\gamma$. And you can show that the image is a lattice. This is in Griffths and Harris. And if $H^0(X,\Omega_X) \cong \mathbb{C}\langle\omega_1,\cdots,\omega_g\rangle$, then you can map

$$\lambda_i \mapsto (\int_{\lambda_i} \omega_1, \cdots, \int_{\lambda_i} \omega_g).$$

These are called the periods of X.

Image is a lattice. The images of the map is linearly independent. So $H_1(X,\mathbb{Z}) \cong \langle\lambda_1,\cdots,\lambda_{2g}\rangle$.

See the section in Griffiths and Harris where they discuss complex tori and abelian varieties.

Anyhow, this is called the dual, Albanese Torus $Alb(X)$ of X. There is this map $X \to Alb(X)$ that is uniquely defined up to translation. You fix a point $x_0 \in X$, and you define a map $X \to Alb(X) = \frac{H^0(X,\Omega^1_X)^*}{H_1(X,\mathbb{Z})}$ by

$$x \mapsto (\int_{x_0}^{x} \omega_1, \cdots, \int_{x_0}^{x} \omega_g).$$

This map is well-defined because different paths give elements in $H_1(X,\mathbb{Z})$ and how it embeds. This is the universal map with respect to abelian varieties. Thats how it is constructed in general in the category of abelian varieties. Look at X, and map $X \to A$ to any abelian variety, you can factor any map through the Albanese map $X \to Alb(X)$. This is how you approach it over other fields and rings.

Remark 44.1. Proof of nonrationality of cubic 3-folds. Something about intermediate Jacobians, which don't exist on curves. They're not necessarily projective, which is a bit of an insurmountable problem. One of the most beautiful proofs, is the proof of the fact that the cubic 3-fold (cubic hypersurface in $\mathbb{P}^4$), associate intermediate Jacobian to this cubic 3-fold. Cubic 3-fold satisfies enough properties that the intermediate jacobian is projective. Turns out after a lot of work, cubic 3-fold is not rational is equivalent to intermediate Jacobian is not Jacobian of curve of genus 5?

Look at lines on cubic 3 fold, parameter space, smooth surface of general type. has 5 forms. Its Albanese variety is exactly te intermediate jacobian o fthe cubic 3 fold. Forced to study Albanese variety of the Fano variety.

Deeper more clever way to attack it? Hasn't work in other dimensions except in dimension 3.

By definition, from $a: X \to Alb(X)$, have $a^*: H^0(A,\omega^1_A) \cong "V^*" \cong H^0(X,\Omega^1_X)$, where $A = Alb(X)$. So have $H_1(X,\mathbb{Z})$ modulo torsion is isomorphic to "Λ" isomorphic to $H_1(A,\mathbb{Z})$. So only useful when the first betti number $0 \neq q(x) = h^{1,0}(X)$, so when $0 \neq b_1(X)$.

Example 44.2. If C smooth projective curve of genus $g \geq 1$, fix basis $\langle \omega_1, \cdots, \omega_g \rangle$ for $H^0(C, \omega_C)$. Fix basis $\langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g \rangle$. These are the loop generators. Have $\alpha_i \alpha_j = 0, \beta_i \beta_j = 0$, and $\alpha_i \beta_j = \delta_{ij}$. basis of homology.

Have this period matrix

$$\begin{pmatrix} \int_{\lambda_1} \omega_1 & \cdots & \int_{\lambda_{2g}} \omega_1 \\ \vdots & \cdots & \vdots \\ \int_{\lambda_1} \omega_g & \cdots & \int_{\lambda_{2g}} \omega_g \end{pmatrix} \in M_{g \times 2g}(\mathbb{C}).$$

The λ's are the same as the α and β's in this case.

The Jacobian of C is called

$$J(C) := Alb(C) = \frac{H^0(C, \omega_C)^*}{H_1(C, \mathbb{C})} = \frac{V}{\Lambda}.$$

Strictly speaking, the jacobian is Albanese, not the degree 0 line bundles. Most natural map is to Albanese. Albanese map in this case is the Abel-Jacobi map.

Have $a : C \to \mathcal{J}(C)$. Fact: a is an embedding.

Rational normal curves in $\mathbb{P}^n$, minimal degree rational curves. Some curves $C \to J(C)$, play a similar role, minimal. But geometry is richer.

Integrating forms over loops $H_1(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}) \to \mathbb{Z}$ is basically Poincare duality. Can choose an adapted basis $\omega_1, \cdots, \omega_g$ such that $\int_{\alpha_i} \omega_j = \delta_{ij}$. Won't be able to do this in general for higher dimension.

So period matrix looks like

$$\Omega = (I_g, Z),$$

where $Z \in M_{g,g}(\mathbb{C})$. Can show that $Z = Z^t$ and $Im(Z) > 0$. Griffths Harris Ch2 Section 2, positive definite. Another way of saying Riemann's criterion for projectivity of torus. This is something about periods. These matrices classify the elliptic curves, but not quite. When you learn about elliptic curves, $E \cong E'$ if and only if there is a matrix in $SL_2(\mathbb{Z})$ such that $A\tau = \tau'$. So these Z are not really classifying elliptic curves. But they classify once you mod out by an action of a discrete subgroup. So elliptic curves, you have the upper half plane parameterizing the upper half plane, space of polarized Hodge structures, divide $SL_2(\mathbb{Z})$, so this is the moduli space of elliptic curves.

Riemann Conditions: $X = V/\Lambda$. complex torus. $\Lambda = H_1(X, \mathbb{Z}) = \mathbb{Z}\langle \lambda_1, \cdots, \lambda_{2g} \rangle$. $V \cong \mathbb{C}^g$, standard coordinates $z_1, \cdots, z_g$. Dualize and have $H^1(X, \mathbb{R})$, gen by $dx_1, \cdots, dx_{2g}$.... but the other side V, give rise to $dz_1, \cdots, dz_g$ and $d\overline{z_1}, \cdots, d\overline{z}_g$.

So X is projective if and only if there exists positive closed 1,1 form $\omega = \frac{i}{2} \sum h_{\alpha\beta} dz_\alpha \wedge d\overline{z_\beta}$ that is also integral, so $h_{\alpha\beta}$ are integers.

WRite $\{dx_i\}$ in terms of the $\{dz_\alpha, d\overline{z_\alpha}\}$. So we have

$$dx_i = \sum \pi_{i\alpha} dz_\alpha + \sum \overline{\pi_{i\alpha}} d\overline{z}_\alpha,$$

where $\pi = (\pi_{i\alpha}) \in M_{2g \times g}(\mathbb{C})$. So the change of basis matrix is $(\pi, \overline{\pi}) = \widetilde{\pi}$. This is a $M_{2g \times 2g}$ matrix. So

$$\omega = \frac{1}{2} \sum q_{ij} dx_i \wedge dx_j.$$

giant calculation to give

$$\frac{1}{2} \sum_{ij} q_{ij} \pi_{i\alpha} \pi_{j\beta} dz_\alpha \wedge dz_\beta + \cdots + d\overline{z_\alpha} \wedge d\overline{z_\beta} + \cdots + dz_\alpha \wedge d\overline{z_\beta}$$

and eventually you get coeficients of $dz_\alpha \wedge dz_\beta$ is parameterized by $\pi^T Q\pi$ where $Q = (q_{ij})$, and for $d\overline{z_\alpha} \wedge d\overline{z_\beta}$ you get $\overline{\pi^T Q\pi}$ and for $dz_\alpha \wedge d\overline{z_\beta}$ youget $\frac{1}{2i}(\pi^T Q\overline{\pi} - \pi^T Q^T\overline{\pi})dz_\alpha \wedge d\overline{z_\beta}$.

So is ω of the type 1,1? This is if and only if $\pi^t Q\pi = 0$. And ω positive definite $\iff -i\pi^T Q^{\overline{\pi}>0}$ positive definite. And ω is integral $\iff Q \in M_{2g\times 2g}(\mathbb{Z})$. So Riemann's conditions: X is projective $\iff \exists Q \in M_{2g\times 2g}(\mathbb{Z})$ which satisfies the above.

The general theory of producing abelian varieties.. this language presented here is absolutely crucial. Period domains.

Conversely: $\Lambda = \mathbb{Z}\langle\lambda_1, \cdots, \lambda_{2g}\rangle$. Have $V = \mathbb{C}\langle e_1, \cdots, e_g\rangle$. Have $\lambda_i = \sum \omega_{\alpha i} e_i$. Have $dz_\alpha = \sum_{i=1}^g \omega_{\alpha i} dx_i$. So the period matrix $\Omega = (\omega_{ai}) \in M_{g\times 2g}(\mathbb{C})$ is dual to π, it is π^{-1}. This is the change of basis matrix in the other direction.

$$\widetilde{\Omega} = \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix}$$

is the matrix taking dz basis to dx basis,

$$(\pi^T Q\pi)^{-1} = \Omega Q^{-1}\Omega^T = 0$$

And

$$-i\Omega Q^{-1}\overline{\Omega} > 0$$

X is projective if and only if there exists $Q \in M_{2g}(\mathbb{Z})$ saistfying those two.

Griffiths and Harris page 304.

Lemma 44.3. *If $Q : \Lambda \times \Lambda \to \mathbb{Z}$ integral, skew-symmetric.. then there exists a basis $\lambda_1, \cdots, \lambda_{2g}$ for Λ such that in this basis, we have*

$$Q = \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix}, \Delta_\delta = \begin{pmatrix} \delta_1 & \cdots & \cdots 0 \\ 0 & \delta_2 & \cdots \\ 0 \cdots & \cdots 0 & \delta_g \end{pmatrix}.$$

So in the previous language, this is the matrix $Q = (q_{ij}$ where $\omega = \frac{1}{2}\sum_{ij} q_{ij} dx_i \wedge dx_j$. So we can write $\omega = \sum_{i=1}^g \delta_i dx_i \wedge dx_{i+g}$. So $\Omega = (\Delta_\delta, Z)$.

Another variant of the Riemann conditions. X is projective if and only if Z is a symmetric matrix, and the imaginary part of Z is positive definite.

Some facts: by the Lefschetz 1,1 theorem, there exists a positive line bundle such that $c_1(L) = [\omega]$ which gives us this polarization. We can compute $h^i(L) = 0$ for $i > 0$ because canonical canonical bundle is trivial on torus. So $\chi(X, L) = h^0(X, L) = \delta_1 \cdots \delta_g$. Elementary form of Hirzebruch-Riemann-Roch on torus. And we have exactly number of global sections of this polarization.

Now you can start studying abelian varieties by type. Most famous abelian varieties.

Happens exactly for curves, by Poincare duality, unimodular pairing. Example of principally polarized abelian variety if $J(C) = Alb(C)$. So the intermediate Jacobian of cubic 3 fold is also prinicpally poalrized abelian variety.

If you are on an abelian surface, you have tow numbers to worry about. To save time, principally polarized abelian variety.. positive line bundle where $h^0(X, L) = 1$. Their divisors are called theta divisors. Very primitive. People wonder what happens with $2L, 3L, \cdots$. Best things happen. $2L$ is base point free, $3L$ is very ample.

Next time we will talk about families of varieties. We are aiming for deformations, local systems, and variations of Hodge structures.

45. 3/25/24: FAMILIES OF COMPLEX MANIFOLDS, KODAIRA SPENCER MAP

Today we'll talk about families of complex manifolds, and in most of our cases, projective varieties. To put this in context, we have seen that a compact Kahler manifold has Hodge theory–in particular, it gives rise to a bunch of Hodge structures for each of the cohomology groups. How these hodge structures vary is quite beautiful and complicated, and is the famous work of Griffiths and other people. Even before that, people studied how manifolds could be deformed in such family, going back to Kodaira. So we'd like to understand some of the beginnings of the story of deformations and variations of Hodge structure.

Definition 45.1. A family of compact complex manifolds is a proper holomorphic submersion

$$\pi : \mathfrak{X} \to B$$

where B is the base and $\mathfrak{X}$ is the total space.

Submersion means that the differential is surjective at every point. Thus by the implicit function theorem, the fibers are all complex submanifolds. Properness guarantees that the fibers are also compact complex submanifolds. For every $t \in B$, we denote $X_t := \pi^{-1}(t)$ to be the compact complex submanifold at t. If $0 \in B$ is the reference point, then X_t can be thought of as "deformations" of X_0.

Example 45.2 (Projective bundle)**.** Given a vector bundle E over B, the projective bundle

$$\pi : \mathbb{P}(E) \to B$$

is a family over B. The fibers are $\mathbb{P}^n$, and moreover π is locally trivial. Thus, there exists an open cover $\{U_i\}$ of B so that $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^n$ are isomorphic over U_i. This is the simplest example of a nontrivial family, and can still be interesting. Locally trivial, but not a product.

This can be extended to a more general setting, namely locally trivial fibrations with fiber F being a compact complex manifold. So a surjection $\pi : \mathfrak{X} \to B$ where there is a local trivialization, such that $\pi^{-1}(U_i) \cong U_i \times F$.

Example 45.3 (Universal hypersurface)**.** In $\mathbb{P}^n$, consider the linear system of all hypersurfaces of degree d $|\mathcal{O}_{\mathbb{P}^n}(d)|$. Of course, some are singular, but we know there's an open set B (which is also Zariski open) corresponding to smooth hypersurfaces of degree d. There is a family

$$\mathfrak{X} \subset B \times \mathbb{P}^n \to B$$

which is called the universal hypersurface, where for every $t \in B$, X_t is the hypersurface corresponding to $t \in B$. Tautological family.

To describe this $\mathfrak{X}$, note that we can choose a basis $\langle F_0,\ cdots, F_N\rangle$ for $H^0(\mathcal{O}_{\mathbb{P}^n}(d))$, where $N = \binom{n+d}{d} - 1$. Then every hypersurface is a linear combination of

$$a_0F_0 + \cdots + a_NF_n,$$

up to scalars. So $\mathfrak{X} \subseteq B \times \mathbb{P}^n \subseteq \mathbb{P}^N \times \mathbb{P}^n$ is described over $[Y_0 : \cdots : Y_N]$, coordinates of a smooth hypersurface in $B \subseteq \mathbb{P}^N$, has fiber $V(Y_0F_0 + \cdots + Y_NF_N) \subseteq \mathbb{P}^n$.

This universal hyperplane family is *not* locally trivial. It is much worse than that actually: it is a family with "maximal variation." We can't make that precise right now but the sense is that the "fibers vary maximally." There are as many isomorphism classes of fibers as parameters.

Theorem 45.4 (Fischer-Grauert)**.** *A family of compact complex manifolds is locally trivial if and only if all fibers are analytically isomorphic (biholomorphic).*

In complex geometry, the situation is extremely rigid. We will see that this is very far from being true in the C^∞ world. If you know the fibers are the same, then you know it has to be some kind of bundle. So in the case of the universal smooth hypersurface family, you know it cannot be locally trivial, because there are smooth hypersurfaces which are not analytically isomorphic.

Theorem 45.5. *If $X, Y \subseteq \mathbb{P}^n$ are irreducible hypersurfaces (allowed to be singular), then $X \cong Y$ if and only if there exists $\Phi \in Aut(\mathbb{P}^n) = PGL_{n+1}(\mathbb{C})$ such that $\Phi(X) = Y$, unless:*

- *X, Y are curves and $\deg X, \deg Y \in \{1, 2\}$*
- *X, Y curves, $\deg X = \deg Y = 3$*
- *X, Y surfaces, $\deg X = \deg Y = 4$*

This is not an easy theorem, but a discussion plus references plus an explanation of why these exceptions holds in a survey by Janos Kollar: arxiv1810.02861. Look around theorem 29? Keyword here is Noether-Lefschetz theorem. Why this cannot happen in higher dimension hypersurfaces, there are no extra line bundles except the ones coming from $\mathbb{P}^n$. For surfaces, there could be, but you do it by a case by case analysis and for curves to establish this.

Example 45.6. Write an elliptic curve Λ as $\langle 1, \tau \rangle \subseteq \mathbb{C}$ where τ is in the upper half plane. This is bad terminology, because this is not(?) the moduli space of elliptic curves. We have

$$\mathfrak{X} \to \mathbb{H}$$

where $\mathfrak{X}$ is the quotient $\mathbb{C} \times \mathbb{H}$ by $\mathbb{Z}^2$ where the action by $\mathbb{Z}^2$ is given by

$$(m, n)(z, z) = (z + (m, n) \begin{pmatrix} z \\ 1 \end{pmatrix}, z).$$

Another story: $X_Z \cong X_{Z'} \iff z' = Az, A \in SL_2(\mathbb{Z})$. There does not exist $\mathfrak{X} \to \mathbb{H}/SL_2(\mathbb{Z})$. Have to talk about stacks. Issue is certain automorphisms, get $\mathbb{P}^1$, not an elliptic curve.

Example 45.7. There exist families (when not submersion) having general smooth fibers, but also some irregular fibers.

If you do something simple like $\mathfrak{X} = V(xy - a) \subseteq \mathbb{C}^3$, project down to the a-coordinate in $\mathbb{C}$, so a is the parameter. Fibers are smooth curves except at $a = 0$. You might like this family though because the fiber is irreducible.

Example 45.8. Hartshorne, Chapter 3, Exercise 9.8.4. Have $\mathfrak{X} \subseteq \mathbb{C}^4 = \mathbb{C} \times \mathbb{C}^3 \to \mathbb{C}$, (variables are parameter a and variables x, y, z) defined parametrically

$$x = t^2 - 1, y = t^3 - t, z = at, t \in \mathbb{C}.$$

So here's a little exercise:

$$\mathfrak{X} = Z(a^2(x + 1) - z^2, ax(x + 1) - yz, xz - ay, y^2 - x^2(x + 1)).$$

So what is this thing for a different from 0? So for $a \neq 0$, write $z' = \frac{1}{a}z$, $t^2 = x+1, t^3 = y+z$. Another linear change of coordinates show that this is the twisted cubic.

For $a = 0$, have $Z(z^2, yz, xz, y^2 - x^2(x+1))$, see all partials vanish at 0, so we see that this is a nodal plane cubic. Note this nodal cubic has nilpotence, which comes from the fact that this nodal intersection doesn't just remember that they came together, but also the tiny infinitesimal direction in which the intersection came from. This nodal singularity is called an embedded point, this non-reduced point.

Example 45.9. There exists examples of submersion where all the X_t fibers are isomorphic for $t \neq 0$, but different from X_0. So $\mathfrak{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{C} \to \mathbb{C}$, such that

$$X_t \cong \mathbb{P}^1 \times \mathbb{P}^1, t \neq 0$$

and

$$X_0 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$$

is the Hirzebruch surface $\mathbb{F}_2$. This is something to think about for moduli spaces. Moduli space is not well-defined. Huybrechts example 6.2.1(iv).

Want to eventually do something with Hodge theory of fibers. So we wonder about the singular cohomology of the fibers.

Theorem 45.10 (Ehresmann)**.** *If $\pi : \mathfrak{X} \to B$ is a proper submersion of C^∞-manifolds, then all the fibers are diffeomorphic. In fact, the family is locally trivial.*

If B is contractible, with base point $o \in B$, then there exists a diffeomorphism $\mathfrak{X} \to B \times X_0$ over B.

Proof. Here's an incomplete sketch. The idea is existence of integral curves for vector fields. Take some fiber and flow.

Work in a neighborhood of point, and restrict to interval, so may assume B is one-dimensional, say $(-\epsilon, 1+\epsilon)$. Since π is a submersion, you know $d\pi_x : T_x\mathfrak{X} \to T_0B$ is surjective, for every $x \to 0$. Want to show X_0 diffeomorphic to X_1. Locally around 0 can lift $\frac{\partial}{\partial t}$ to a vector field around x and can cover X_0 by finitely many U_i until a vvector field v_i lifting $\frac{\partial}{\partial t}$ on U_i. Partition of unity gives lift of $\frac{\partial}{\partial t}$ to a vector field v on $\bigcup U_i$ containing X_0. Since map is proper, can choose U_i such that $X_t \subseteq \bigcup U_i$ for t close to 0. General theory of fibers: can flow X_0 to X_t along v. "Ambient isotopy." ∎

This theorem of Ehresmann tells us that the families we'll study, we will not need to worry about the topology or the smooth structure of the fibers – they will not vary. What will vary, is the complex structure. Thus, from now, we'll think of our family (submersion, proper) as a fixed C^∞ manifold with variations in complex/algebraic structure.

Theorem 45.11 (Voisin, Theorem 9.2.3)**.** *If $\pi : \mathfrak{X} \to B$ is a family and X_0 is Kahler, then there exists a neighborhood U containing 0 such that X_t is Kahler for every $t \in U$.*

Thus, Kahler condition is open in families. We won't worry too much in our case, since the families we'll focus have Kahler fibers.

The Kodaira-Spencer map: note for $x \in X_0$, we have surjective

$$d\pi_x : T_x\mathfrak{X} \to T_0B$$

with kernel $\ker d\pi_x = T_x X_0$. This induces a short exact sequence of vector bundles

$$0 \to T_{X_0} \to T_{\mathfrak{X}}|_{X_0} \to T_0 B \otimes_{\mathbb{C}} \mathcal{O}_{X_0},$$

where $T_0 B \otimes_{\mathbb{C}} \mathcal{O}_{X_0} \cong \mathcal{N}_{X_0/\mathfrak{X}} \to \pi^* T_B|_0$. Passing to cohomology, we obtain a map

$$T_0 B \otimes H^0(X_0, \mathcal{O}_{X_0}) \to H^1(X_0, T_{X_0})$$

which is equivalent to the map

$$T_0 B \to^k H^1(X_0, T_{X_0})$$

which is the famous Kodaira-Spencer map. What should this mean? It means that given this tangent direction $v \in T_0 B$, it should lift to a family of normal (to X_0 in $\mathfrak{X}$) vectors pointing in this direction of v along X_0, and this family should give an element of $H^1(X_0, T_{X_0})$, namely an infinitesimal deformation of X_0.

If you adopt this analytic way of thinking, and think of how we are fixing a C^∞ manifold on the central fiber, this is like a direction in which you deform the complex structure. That is what the first cohomology $H^1(X_0, T_{X_0})$ is trying to parameterize: the infinitesimal deformations. Looking ahead, going to talk about this and obstructions to deformations. Then move on to local systems and flat connections.

local systems, flat connections,

46. 3/27/24: infinitesimal deformations

Today we investigate the Kodaira-Spencer map further. We begin with an analog of a scheme.

Definition 46.1. A complex space is a Hausdorff topological space X, together with a sheaf of rings $\mathcal{O}_X$ such that there exists an open covering $\{U_i\}_{i\in I}$ of X such that

$$(U_i, \mathcal{O}_X|_{U_i}) \cong (Z, \mathcal{O}_U/I), \text{ where} Z \subseteq U \subseteq \mathbb{C}^n$$

such that $Z = V(I)$, where I is some sheaf of ideals in $\mathcal{O}_U$.

So the complex space we have defined here is really getting at some space where locally, it is cut out by some holomorphic functions in $\mathbb{C}^n$. In particular, any ideal is allowed, so nilpotence is allowed. The n can vary between connected components. But here we are just implicitly thinking of a connected complex space.

Example 46.2. We saw last time: the "limit" of a twisted cubic was given by $I = (z^2, yz, xz, y^2 - x^2(x+1)) \subseteq \mathcal{O}_{\mathbb{C}^3}$.

Example 46.3. Look at $(z^2 = 0)$ in $\mathbb{C}$. This is $X = Spec\mathbb{C}[\epsilon]/\epsilon^2$.

If X is a complex space, then $\mathcal{O}_{X,x}$ are local rings which are not necessarily regular anymore. The tangent space to X at x is $T_x X := (m_x/m_{2})^*$.

Remark 46.4. Often at singularities of schemes, people study tangent cones instead of tangent spaces.

Morphism of complex spaces, $f : X \to Y$ is

$$(X, \mathcal{O}_X) \to^{(f,\tilde{f})} (Y, \mathcal{O}_Y)$$

where $\tilde{f} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ map of sheaves. Note this has an adjoint $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Fiber over a point $y \in Y$ is given by $X_y = f^{-1}(y)$, and $\mathcal{O}_X/\tilde{f}^{-1}(\underline{m}_y)$. A scheme-theoretic fiber has more algebra than geomtry.

Example 46.5. $\mathbb{C} \to \mathbb{C}$ via $z \mapsto z^2$. Pairs of points map, except 0 has a double solution. The scheme-theoretic fiber over 0 remembers there is a double point above it.

Definition 46.6. A morphism $\pi : \mathfrak{X} \to B$ of complex spaces which is flat and proper, such that all fibers are compact complex manifolds, is called a smooth family.

A morphism $f : X \to Y$ is called flat if for every $x, \in X$, $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module.

Can think of tensoring as extension by scalars. Intuition for flatness: when are families nice? You'd like to say that they have the same dimension. But even if they have same dimension, if you start computing their degree, if they start jumping around, then we're not very happy. So we want to have the same kind of numerical invariants. These invaraints are encoded in one thing, which is called the Hilbert polynomial. What you want to keep constant is the Hilbert polynomial. Main theorem in Hartshorne on flatness is roughly, family is flat if and only if the Hilbert polynomial is constant on fibers.

Definition 46.7. An infinitesimal deformation (of the first order, as you can divide by ϵ^3 instead of ϵ^2) of a compact complex manifold X_0 is a smooth family

$$\pi : \mathfrak{X} \to Spec\mathbb{C}[\epsilon]/\epsilon^2$$

such that we have a commutative diagram

$$\begin{tikzcd}
X = \pi^{-1}(0) \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\
* \arrow[r, hook] & Spec\mathbb{C}[\epsilon]/\epsilon^2
\end{tikzcd}$$

What is this map $Spec\mathbb{C}[[T]]/(T) = * \hookrightarrow Spec\mathbb{C}[\epsilon]/\epsilon^2$? It is given on functions by

$$0 \to \frac{(T)}{T^2} \to \frac{\mathbb{C}[T]}{(T^2)} \to \frac{\mathbb{C}[T]}{(T)} \to 0.$$

Lemma 46.8. *Let B be a complex space. A morphism $Spec\mathbb{C}[\epsilon]/\epsilon^2 \to B$ is the same as specifying a point $t \in B$ and a vector $v \in T_tB$.*

Proof. Topologically, $Spec\mathbb{C}[\epsilon]/\epsilon^2 \to B$ maps a point to $t \in B$. There is also the map on rings. Have for every $U \subseteq B$, map $\mathcal{O}_B(U) \to \mathbb{C}[T]/(T^2)$. If $T \notin U$, then the map is 0. If $t \in U$, then we have a ring homomorphism $\mathcal{O}_{B,t} \to^{\psi} \mathbb{C}[\epsilon]/\epsilon^2$, where $\underline{m}_t \mapsto \psi(\underline{m}_t) \subseteq (\epsilon)$. In particular, note $\psi(\underline{m}_t^2) \subseteq (\epsilon^2) = 0$, so we get an induced map

$$\frac{\underline{m}_t}{\underline{m}_t^2} \to \mathbb{C}.$$

So this gives you can element $v \in T_tB = (m_t/m_t^2)^*$. $\square$

If you have a family $\mathfrak{X} \to B$, then mapping $Spec\mathbb{C}[\epsilon]/\epsilon^2 \to B$, you can take the cartesian product, and this gives you an infintesimal deformation of the fiber over the point in B that is hit, where the deformation occurs in the direction of the specified tangent vector.

If you want to understand deformations of a complex manifold, i.e. how to put it in a family, you must think about infinitesimal deformations. Infinitesimal deformations of X, you should think of, as the tangent space to the space of deformations of X (hilbert scheme).

The theorem we want to prove is

Theorem 46.9. *The infinitesimal deformations of X, up to isomorphism, is isomorphic to $H^1(X, T_X)^*$.*

Proof. Some intuition. These infintiesmal deformation are parameterized by the short exact sequences. Can get to X or $\mathfrak{X}$ or $\mathfrak{X}$ to X by extension of scalars or restriction. If you started with

$$0 \to \frac{(T)}{T^2} \to \frac{\mathbb{C}[T]}{T^2} \to \frac{\mathbb{C}[T]}{T} \to 0.$$

You can tensor this by $\mathcal{O}_{\mathfrak{X}}$, and since this is flat over the base, we get

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_X \to 0.$$

And note $\mathcal{O}_{\mathfrak{X}} \otimes \mathbb{C}[T]/(T) \cong \mathcal{O}_X$. If you have a sequence of modules over a ring,

$$0 \to M \to N \to P \to 0,$$

these extensions are parameterized by $Ext^1(P, M)$. So we have $Ext^1_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_X, \mathcal{O}_X)$ parameterizes these short exact sequences. You can rewrite Ext groups in many ways, but it is an exercise, that $Ext^1_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_X, \mathcal{O}_X) \cong H^1(X, T_X)$. If you replace the the right hand term by $\mathcal{F}$, have $Ext^1_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{O}_X) \cong H^1(X, T_X \otimes \mathcal{F})$. This is the intuitive algebraic approach. Now we will actually prove it.

Suppose we have

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ * \arrow[r, hook] & Spec\mathbb{C}[\epsilon]/\epsilon^2 \end{tikzcd}$$

Since X is a manifold, there exists an open cover $\{U_i\}$ of X, and $\phi_i : U_i \cong B \subseteq \mathbb{C}^n$. Let $U_{ij} = U_i \cap U_j$. We have $\mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_X \otimes_{\mathbb{C}} \frac{\mathbb{C}[T]}{(T^2)}$. We have $\mathfrak{X}$ is covered by U_i, and we have trivializations $\theta_i : \mathcal{O}_{\mathfrak{X}}|_{U_i} \cong \mathcal{O}_{U_i}[T]/(T^2)$, with transition functions

$$\theta_{ij} : \mathcal{O}_{U_{ij}}[T]/(T^2) \to \mathcal{O}_{U_{ij}}[T]/(T^2).$$

$$\begin{tikzcd} 0 \arrow[r] & T/(T^2) \otimes \mathcal{O}_{U_{ij}} \arrow[r] & \mathcal{O}_{U_{ij}}[T]/(T^2) \arrow[r] \arrow[d, "\theta_{ij}"] & \mathcal{O}_{U_{ij}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T/(T^2) \otimes \mathcal{O}_{U_{ij}} \arrow[r] & \mathcal{O}_{U_{ij}}[T]/(T^2) \arrow[r] & \mathcal{O}_{U_{ij}} \arrow[r] & 0 \end{tikzcd}$$

Can write $\theta_{ij}(f) = f + T\psi_{ij}(f)$ where $\psi_{ij} : \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$. Have θ_{ij} ring homomorphism, and $\theta_{ij}(fg) = fg + T\psi_{ij}(fg)$, and $\theta_{ij}(f)\theta_{ij}(g) = (f + T\psi_{ij}(f))(g + T\psi_{ij}(g))$. So we are left with $\psi_{ij}(fg) = f\psi_{ij}(g) + g\psi_{ij}(f)$. This means $\psi_{ij} \in \Gamma(U_{ij}, T_X)$. Moreover, θ_{ij} should give us a $\mathfrak{X}$ globally, so on triple overlaps we want $\theta_{ij}\theta_{jk}\theta_{ki} = Id$ if and only if $\psi_{ij} + \psi_{jk} + \psi_{ki} = 0$. Have $\{\psi_{ij}\} \in C^1(\underline{U}, T_X)$, get a class $[\psi_{ij}] \in \check{H}^1(U, T_X)$ and by Leray, this gives aan element $[\psi_{ij}] \in H^1(X, T_X)$.

The map is surjective. If I start with ψ_{ij}, we can use them to define θ_{ij} using the same formula, and this gives $\mathcal{O}_{\mathfrak{X}}$.

Furthermore, this map is injective. If $[\psi_{ij}] = [\psi'_{ij}]$, they will differ by $\psi_{ij} = \psi'_{ij} + \psi_i - \psi_j$, and we will use this to define an automorphism of $\mathcal{O}_{\mathfrak{X}}$. So shows injection for isomorphism classes of infinitesimal deformations.

$\square$

Theorem 46.10. *The isomorphism classes of abstract infinitesimal deformations of a complex manifold X are in one to one correspondence with $H^1(X, T_X)$.*

Proof. Suppose we have an abstract infinitesimal deformation, so that we have the following diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ Spec\mathbb{C} & \longrightarrow & Spec\frac{\mathbb{C}[\epsilon]}{\epsilon^2} \end{array}$$

Let $\{U_i\}$ be a chart for the manifold X, and $U_{ij} := U_i \cap U_j$. Then note that we can make the identification

$$\theta_i : \mathcal{O}_{\mathfrak{X}}|_{U_i} \cong \mathcal{O}_{U_i} \frac{[\epsilon]}{\epsilon^2}.$$

Note that this identification happens over $Spec\mathbb{C}\frac{[\epsilon]}{\epsilon^2}$. The identifications $\mathcal{O}_{\mathfrak{X}}$ over U_i restricted to the overlap $U_i \cap U_j$ and the identification over U_j restricted to the overlap $U_i \cap U_j$ provides an isomorphism

$$\theta_{ij} : \mathcal{O}_{U_{ij}} \frac{[\epsilon]}{\epsilon^2} \to \mathcal{O}_{U_{ij}} \frac{[\epsilon]}{\epsilon^2}$$

over $Spec\mathbb{C}\frac{[\epsilon]}{\epsilon^2}$. Since when $\epsilon \mapsto 0$ this map should become identity on $\mathcal{O}_{U_{ij}}$, we can then write

$$\theta_{ij}(f) = f + \epsilon\psi_{ij}(f),$$

and note that $\theta_{ij}(\epsilon) = \epsilon$ since this is a map of $Spec\frac{\mathbb{C}[\epsilon]}{\epsilon^2}$-algebras. We have that

$$\theta_{ij}(fg) = fg + \epsilon\psi_{ij}(fg) = \theta_{ij}(f)\theta_{ij}(g) = fg + \epsilon(f\psi_{ij}(g) + \psi_{jj}(f)g)$$

which implies that $\psi_{ij}(fg) = f\psi_{ij}(g) + \psi_{ij}(f)g$. Thus, ψ_{ij} acts as a derivation on functions of $\mathcal{O}_{U_{ij}}$. This implies that $\psi_{ij} \in \Gamma(U_{ij}, TX)$. Furthermore, note we must have

$$\theta_{ij}\theta_{jk}\theta_{ki} = Id.$$

Then

$$f \mapsto^{\theta_{ij}} f + \epsilon\psi_{ij}(f) \mapsto^{\theta_{jk}} f + \epsilon\psi_{jk}(f) + \epsilon\psi_{ij}(f) \mapsto^{\theta_{ki}} f + \epsilon[\psi_{ij} + \psi_{jk} + \psi_{ki}](f).$$

Thus we see that

$$\theta_{ij}\theta_{jk}\theta_{ki} = Id \implies \psi_{ij} + \psi_{jk} + \psi_{ki} = 0.$$

Then $[\psi_{ij}] \in \check{H}^1(\underline{U}, T_X) \cong H^1(X, T_X)$. This is because T_X is fine over the U_i and thus acyclic over this cover $\underline{U_i}$, so we have the Čech-sheaf cohomology isomorphism by Leray's theorem.

Now we have surjectivity. Why? Because given $[\psi_{ij}] \in \check{H}^1(\underline{U}, T_X)$, we can use this to construct the θ_{ij}, and these identifications patch to a sheaf $\mathcal{O}_{\mathfrak{X}}$ satisfying the desired commutative diagram.

Now we also have injectivity. Why? Suppose we have $\mathfrak{X}$ and $\mathfrak{X}'$. So that

$$\theta_i : \mathcal{O}_{\mathfrak{X}}|_{U_i} \cong \mathcal{O}_{U_i} \frac{[\epsilon]}{(\epsilon^2)}, \text{ and } \theta_j : \mathcal{O}_{\mathfrak{X}}|_{U_j} \cong \mathcal{O}_{U_j} \frac{[\epsilon]}{(\epsilon^2)}$$

and

$$\theta_{ij} : \mathcal{O}_{U_{ij}}\frac{[\epsilon]}{\epsilon^2} \to \mathcal{O}_{U_{ij}}\frac{[\epsilon]}{\epsilon^2}$$

$$\theta'_{ij} : \mathcal{O}_{U_{ij}}\frac{[\epsilon]}{\epsilon^2} \to \mathcal{O}_{U_{ij}}\frac{[\epsilon]}{\epsilon^2}$$

with corresponding ψ_{ij} and ψ'_{ij}. Then if $[\psi_{ij}] = [\psi'_{ij}] \in \check{H}^1(\underline{U}, T_X)$, this implies that $\psi_{ij} = \psi'_{ij} + \psi_j - \psi_i$ for some $\{\psi_i\} \in \check{H}^0(\underline{U}, T_X)$.

Note that these ψ_i define over U_i isomorphisms

$$t_i : \mathcal{O}_{U_i}\frac{[\epsilon]}{\epsilon^2} \to \mathcal{O}_{U_i}\frac{[\epsilon]}{\epsilon^2}$$

where $t_i(f) = f + \epsilon\psi_i(f)$ and $t_i(\epsilon) = \epsilon$. So this is a map of $Spec\mathbb{C}\frac{[\epsilon]}{\epsilon^2}$-algebras which becomes the identity map when $\epsilon \mapsto 0$. Then the following square commutes

$$\begin{array}{ccc}
\mathcal{O}_{U_{ij}}\frac{[\epsilon]}{\epsilon^2} & \xrightarrow{t_i|_{U_{ij}}} & \mathcal{O}_{U_{ij}}\frac{[\epsilon]}{\epsilon^2} \\
{\scriptstyle\theta_{ij}}\downarrow & & \downarrow{\scriptstyle\theta'_{ij}} \\
\mathcal{O}_{U_{ij}}\frac{[\epsilon]}{\epsilon^2} & \xrightarrow[t_j|_{U_{ij}}]{} & \mathcal{O}_{U_{ij}}\frac{[\epsilon]}{\epsilon^2}
\end{array}$$

because $\psi'_{ij} + \psi_i = \psi_{ij} + \psi_j$. Then what this data provides is an automorphism of our infinitesimal deformation

$$\begin{array}{ccccc}
X & \longrightarrow & \mathfrak{X} & \longrightarrow & \mathfrak{X} \\
\downarrow & & \downarrow & & \downarrow \\
Spec\mathbb{C} & \longrightarrow & Spec\mathbb{C}\frac{[\epsilon]}{\epsilon^2} & \longrightarrow & Spec\mathbb{C}\frac{[\epsilon]}{\epsilon^2}
\end{array}$$

, so we have injectivity. □

The purely algebraic intuition for this is that infinitesimal deformations are parameterized by short exact sequences. This H^1 should remind one of an Ext calculation. We can obtain X from $\mathfrak{X}$ or $\mathfrak{X}$ from X by restriction or extension by scalars, respectively. If you start with

$$0 \to \frac{(\epsilon)}{\epsilon^2} \to \mathbb{C}\frac{[\epsilon]}{\epsilon^2} \to^{\epsilon\mapsto 0} \mathbb{C} \to 0,$$

then tensoring by $\mathcal{O}_{\mathfrak{X}}$, since this is flat over the base, we get

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_X \to 0.$$

If you have a sequence of modules over a ring, say

$$0 \to M \to N \to P \to 0,$$

then all such short exact exact sequences are parameterizedby $Ext^1(P, M)$. We have $Ext^1_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_X, \mathcal{O}_X) \cong H^1(X, T_X)$. In general, if you replace the right hand term by $\mathcal{F}$, you have $Ext^1_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{O}_X) \cong H^1(X, T_X \otimes \mathcal{F})$.

These might be called "abstract" deformations, in contrast to a problem of deforming $X \subseteq Y$ inside of Y, which is parameterized by $H^0(X, \mathcal{N}_{X/Y})$. Lots of different types of deformation problems. Can deform space with line bundle, or space with line bundle with some specified global sections. Always have some space of deformations and an obstruction space.

Studying this $H^1(X, T_X)$ can be quite hard. To gain more insight, in the next class we will turn to the analytic point of view on infinitesimal deformations and discuss the Maurer-Cartan equation.

47. Extra: towards Newlander-Nierenberg Huybrechts

By Ehressmann's theorem, given a proper submersion of complex manifolds $\mathfrak{X} \to B$, if we examine the underlying C^∞ structure of this family, then this family is actually locally trivial and every fiber, which is a compact complex manifold, is diffeomorphic to one another. Thus, a deformation of the central fiber X_0 may be thought of as really a deformation of its complex structure. To better understand deformations of commplex structure from the analytic point of view, let us first discuss almost complex structures and integrable complex structures.

Definition 47.1. An almost complex manifold is a differentiable manifold X together with a vector bundle endomorphism

$$I : TX \to TX, \text{ with } I^2 = -Id.$$

The endomorphism I is called the *almost complex structure* on the underlying real manifold. If an almost complex structure exists, then the real dimension of X is even.

Proposition 47.2. *Any complex manifold X admits a natural almost complex structure.*

Proof. Cover X by holomorphic charts U_i. Note at each point T_xU, where we consider the real $2n$ tangent space at x, the almost complex structure on this vector space is given by $\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}$, and $\frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}$. Note that when we complexify T_xU, we get a decomposition

$$T_{x,\mathbb{C}}U = T_x^{1,0}U \oplus T_x^{0,1}U,$$

where $T_x^{1,0}U$ and $T_x^{0,1}U$ are the i and $-i$ eigenspaces of I. Note that $T_x^{1,0}U$ will then have basis

$$\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - iI\frac{\partial}{\partial x_i}) = \frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i})$$

and $T_x^{0,1}$ will have basis

$$\frac{\partial}{\partial \overline{z_i}} = \frac{1}{2}(\frac{\partial}{\partial x_i} + i\mathcal{I}\frac{\partial}{\partial x_i}) = \frac{1}{2}(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}).$$

We see in general, this I extends to $I : TU_i \to TU_i$, and since the transition functions are holomorphic, it extends to an endomorphism $I : TX \to TX$ to the entire tangent bundle, so that $I^2 = -Id$. To see why, note that if we complexify everything, we have this map

$$I_i : TX|_{U_i} \cong U_i \times \{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_i}}\} \to U_i \times \{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_i}}\} \cong TX|_{U_i},$$

and the same goes for

$$I_j : TX|_{U_j} \cong U_j \times \{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_i}}\} \to U_j \times \{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_i}}\} \cong TX|_{U_j}.$$

Now the transition function between U_i and U_j on overlap $U_i \cap U_j$ gives the transition function for TX over $U_i \cap U_j$. This transition will preserve the i-eigenspace and $-i$ eigenspace. Thus, we see that these endomorphisms I_i and I_j over $U_i \cap U_j$ commute, by the diagram

$$\begin{array}{ccc} \frac{\partial}{\partial z_i} & \xrightarrow{I_i} & i\frac{\partial}{\partial z_i} \\ \downarrow & & \downarrow \\ \mathcal{J}(\phi_{ij})\frac{\partial}{\partial z_i} & \xrightarrow[I_j]{} & i\mathcal{J}(\phi_{ij})\frac{\partial}{\partial z_i} \end{array}$$

and the same goes for the $-i$ eigenspace. Then by linearity one easily sees that we have commutativity for the $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$. So the complex structure of a complex manifold X gives a natural almost complex structure on TX.

□

Remark 47.3. Not every real manifold of even dimenion admits an almost complex structure. For example, the four-dimensional sphere.

Now, given a real manifold with complex structure (X, I), if we complexify the tangent bundle of X, then we obtain a decomposition

$$T_{\mathbb{C}}X = T^{1,0}(X) \oplus T^{0,1}(X),$$

where $T^{1,0}(X)$ is the vector bundle of (i)-eigenspaces of I and $T^{0,1}(X)$ is the vector bundle of $(-i)$-eigenspaces of I. Dualizing and wedging provides a decomposition of differential forms:

$$\bigwedge^k (T_{\mathbb{C}}X)^* = \bigoplus_{p+q=k} \bigwedge^p (T^{1,0}X)^* \otimes \bigwedge^q (T^{0,1}X)^*,$$

where we denote the sheaf of sections of these vector bundles as $\mathcal{A}_X^k$ and $\mathcal{A}_X^{p,q}$. Extending the exterior derivative $\mathbb{C}$-linearly yields

$$d : \mathcal{A}_X^k \to \mathcal{A}_X^k.$$

It would be very nice if $d = \partial + \overline{\partial}$, where $\partial : A_X^{p,q} \to \mathcal{A}^{p+1,q}$ and $\overline{\partial} : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1}$. In the familiar situation where we know X is a complex manifold, we know this to be true. But in general, given a real manifold with almost complex structure (X, I), this need not be true.

Proposition 47.4. *Let (X, I) be a real manifold with almost complex structure. The following are equivalent:*

(1) $d = \partial + \overline{\partial}$
(2) For $\alpha \in \mathcal{A}^{1,0}(X)$, $\Pi^{0,2} d\alpha = 0$, where $\Pi^{0,2}$ denotes projection to the $(0, 2)$ component.

Proof. (1) implying (2) is almost immediate. Given $\alpha \in A^{1,0}(X)$, we have $d\alpha \in \mathcal{A}^{2,0}(X) \oplus \mathcal{A}^{1,1}(X)$, so projection to $\mathcal{A}^{0,2}(X)$ is zero.

To see that (2) implies (1), note that $d = \partial + \overline{\partial}$ if and only if for every $\alpha \in \mathcal{A}^{p,q}(X)$, we have $d\alpha \in \mathcal{A}^{p+1,q}(X) \oplus \mathcal{A}^{p,q+1}(X)$. We can locally express α as

$$f\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge \omega'_{j_1} \wedge \cdots \wedge \omega'_{j_q}.$$

By the Leibniz formula, $d\alpha$ is written in terms of $df \in \mathcal{A}^{1,0}(X) \oplus \mathcal{A}^{0,1}(X)$, and $d\omega_{i_t} \in \mathcal{A}^{2,0}(X) \oplus \mathcal{A}^{1,1}(X)$, and $d\omega'_{j_t} = \overline{d\overline{\omega'_{j_t}}} \in \overline{\mathcal{A}^{2,0}(X) \oplus \mathcal{A}^{1,1}(X)} = \mathcal{A}^{1,1}(X) \oplus \mathcal{A}^{0,2}(X)$. We see then that $d\alpha \in \mathcal{A}^{p+1,q}(X) \oplus \mathcal{A}^{p,q+1}(X)$ as desired. □

Definition 47.5. An an integrable complex structure I on a real manifold X is an almost complex structure on X such that proposiiton 47.4 holds.

Thus, one way of understanding an integrable complex structures I is that it is an almost complex structure that the exterior derivative behaves "nicely" with respect to it. Another perspective on integrable complex structures is through vector fields.

Proposition 47.6. *An almost complex structure I is integrable if and only if the Lie bracket of vector fields preserves $T_X^{0,1}$, i.e. $[T_X^{0,1}, T_X^{0,1}] \subseteq T_X^{0,1}$.*

Proof. Here we will use the invariant formula for the exterior derivative. If α is a k-form and $v_1, \cdots, v_{k+1}$ are vector fields, then

$$d\alpha(v_1, \cdots, v_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} v_i \cdot \alpha(v_1, \cdots, \widehat{v_i}, \cdots, v_{k+1})$$

$$+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \alpha([v_i, v_j], v_1, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_{k+1}).$$

In the case of 1-forms, we have the that $d\alpha(v_1, v_2) = v_1\alpha(v_2) - v_2\alpha(v_1) - \alpha([v_1, v_2])$. This can be readily verified:

- we can assume that $\alpha = fdg$ for some smooth functions f, g. Then the left hand side simplifies to

  $$d(fdg)(v_1, v_2) = (df \wedge dg)(v_1, v_2) = v_1(f)v_2(g) - v_2(f)dg(v_1).$$

  The right hand side simplifies to

$$v_1(fdg(v_2)) - v_2(fdg(v_1)) - fdg([v_1, v_2])$$

$$= v_1(f)dg(v_2) + fv_1(dg(v_2)) - v_2(f)dg(v_1) - fv_2(dg(v_1)) - f[v_1v_2 - v_2v_1](g)$$

$$= v_1(f)v_2(g) + fv_1v_2(g) - v_2(f)v_1(g) - fv_2v_1(g) - f[v_1v_2 - v_2v_1](g)$$

$$= v_1(f)v_2(g) - v_2(f)dg(v_1).$$

Now suppose that I is integrable. Let X, Y be any two sections of $T_X^{0,1}$. Then for any $\omega \in \mathcal{A}_X^{1,0}$, we have

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Now since I is integrable, we have $d\omega \in \mathcal{A}_X^{2,0} \oplus \mathcal{A}_X^{1,1}$. Then $d\omega(X, Y) = 0$, and $\omega(Y) = \omega(X) = 0$ as well, so $\omega([X, Y]) = 0$ for all $\omega \in \mathcal{A}_X^{1,0}$, which implies that $[X, Y]$ is again a section of $T_X^{0,1}$.

Now suppose that $[T_X^{0,1}, T_X^{0,1}] \subseteq T_X^{0,1}$. Let X, Y be sections of $T^{0,1}$. Then for every $\omega \in \mathcal{A}_X^{1,0}$, we have $\omega([X, Y]) = 0$, so

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) = 0.$$

This implies that $d\omega \in \mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1}$. □

Corollary 47.7. *If I is an integrable almost complex structure, then $\partial^2 = \overline{\partial}^2 = 0$, and $\partial\overline{\partial} = -\overline{\partial}\partial$. Conversely, if $\overline{\partial}^2 = 0$, then I is integrable.*

Proof. The first claims follow immediately from integrability implying $d = \partial + \overline{\partial}$, combined with bidegree considerations and $d^2 = 0$.

Now suppose $\overline{\partial}^2 = 0$. We'd like to show I is integrable. It suffices to show $[T^{0,1}, T^{0,1}] \subseteq T^{0,1}$. Let X, Y be sections of $T^{0,1}$. Note that if α is a (0,1) form, then $d\alpha(X,Y) = \overline{\partial}\alpha(X,Y)$. Then again, using the formula

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]),$$

setting $\omega = \overline{\partial} f$, we find

$$\begin{aligned} 0 = \overline{\partial}^2 f = d\overline{\partial} f(X,Y) &= X\overline{\partial} f(Y) - Y\overline{\partial} f(X) - \overline{\partial} f([X,Y]) \\ &= Xdf(Y) - Ydf(X) - \overline{\partial} f([X,Y]) \\ &= d^2 f(X,Y) + df([X,Y]) - \overline{\partial} f([X,Y]) = \partial f([X,Y]). \end{aligned}$$

Thus, since $\partial f([X,Y]) = 0$ for all f, this implies $[X,Y]$ is a section of $T^{0,1}$. □

The following highly non-trivial theorem tells us that a real manifold that admits an integrable almost complex structure is secretly a complex manifold. (And it is an exercise that an almost complex structure is induced by at most one complex structure)

Theorem 47.8 (Newlander-Nierenberg)**.** *Any integrable almost complex structure is induced by a complex structure.*

Now that we have discussed complex structures, let us discuss deformations of complex structures.

48. 4/1/24: deforming almost complex structures, Newlander-Nirenberg, Maurer-Cartan equations

So we hae this algebraic characterization of infinitesimal deformations. The infinitesimal way of deforming your manifold, it is $H^1(X, T_X)$. But in general, it can be hard to study this object. There is also an analytic point of view which we should mention. This leads to some famous results in the theory that are analytic in nature.

Let us fix X to be a compact C^∞ manifold. We are going to fix an almost complex structure, which is an endomorphism $\mathcal{J} : TX \to TX$ where $\mathcal{J}^2 = Id$. We can extend this endomorphism to an endomorphism of $T_{\mathbb{C}}X := TX \otimes_{\mathbb{R}} \mathbb{C}$, and we obtain a decomposition $T^{1,0} \oplus T^{0,1}$, where $T^{1,0}$ is the i-eigenspace of $\mathcal{J}_{\mathbb{C}}$, and $T^{90,1}$ is the $-i$ eigenspace of $\mathcal{J}$.

Have $T^{1,0} = \overline{T^{0,1}}$. If X complex manifold, then $T_X \cong T^{1,0}$. There's a famous theorem which tells us whether an almost complex structure actually is a complex structure. Huybrechts has a discussion of this in section 2.6.

Theorem 48.1 (Newlander-Nirenberg)**.** *An almost complex structure $\mathcal{J}$ comes from the structure of a complex manifold on X if and only if $[T^{0,1}, T^{0,1}] \subseteq T^{0,1}$, where $[,]$ is the Lie bracket, where for vector fields X, Y, and a function f, we have $[X,Y](f) = X(Y(f)) - Y(X(f))$.*

Such an almost complex structure is called integrable. There's a lot of things about almost complex structures – they form a pretty nice space, infinite dimensional manifold.. but we can naively just think of a one-paramter family of scuh things which depends differentiably on the one parameter. Say $\mathcal{J}(t)$ is a family of almost complex structures, depending real-analytically on a parameter t), $\mathcal{J}(0) = \mathcal{J}$. Have $T_{\mathbb{C}}X = T_t^{1,0} \oplus T_t^{0,1}$. Have

$$T_{\mathbb{C}}X = T_t^{1,0} \oplus T_t^{0,1},$$

and we keep track of the projection

$$-\psi'(t) : T^{0,1} \hookrightarrow T_{\mathbb{C}}(X) \to T_t^{1,0}.$$

We have that $T^{1,0} \to T_{\mathbb{C}}(X) \to T^{1,0}$ is identity, and thus the map $T^{1,0} \to T_{\mathbb{C}}X \to T_t^{1,0}$ is an isomorphism for small t because open condition of matrix. This induces a map $\phi(t) : T^{0,1} \to T^{1,0}$ which is the composition of

$$T^{0,1} \to^{-\psi'(t)} T_t^{1,0} \to T^{1,0}.$$

For $v \in T^{0,1}$, have $v + \phi(t)(v) \in T_t^{1,0}$ in $T^{1,0} \oplus T^{0,1}$. Conversely, if $\phi(t) : T^{0,1} \to T^{1,0}$ depending on t. Define $T_t^{0,1} = (Id + \phi(t))(T^{0,1}) \subseteq T_{\mathbb{C}}X$. Have $\phi(0) = 0$, right dimension for small t.

Remark 48.2. $\phi(t) \in \Gamma(X, (T^{0,1})^* \otimes T^{1,0}) \implies \phi(t) \in \Gamma(X, A^{0,1}(T_X))$, noting that $Hom(A, B) \cong A^* \otimes B$.

We have $\phi(t)$ depends analytically on t. So we have $\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots$, $\phi_i \in \Gamma(X_0, A^{0,1}(T_X))$. So the question is:

Question 48.1. $\mathcal{J}$ is integrable; when is $\mathcal{J}(t)$ integrable?

By the Newlander-Nirenberg theorem 48.1, we need $[T_t^{0,1}, T_t^{0,1}] \subseteq T_t^{0,1}$. Recall that we have the operator $\overline{\partial} : \mathcal{A}^{0,p}(T_X) \to \mathcal{A}^{0,p+1}(T_X)$, and we have the Lie bracket

$$[,] : \mathcal{A}^{0,p}(T_X) \times \mathcal{A}^{0,p}(T_X) \to \mathcal{A}^{0,p+q}(T_X),$$

where if ω_1, ω_2 are forms and v_1, v_2 are vector fields, then $[\omega_1 \otimes v_1, \omega_2 \otimes v_2] = \omega_1 \wedge \omega_2 \otimes [v_1, v_2]$.

Proposition 48.3. *The integrability condition $[T_t^{0,1}, T_t^{0,1}] \subseteq T_t^{0,1}$ equivalent to the Maurer-Cartan equation*

$$\overline{\partial}\phi + [\phi, \phi] = 0$$

(makes sense as sections of $\mathcal{A}^{0,2}(T_X)$) (we've removed t from the notation, the ϕ is really ϕ_t, always in small neighborhood of origin).

Proof. Let's write this in coordinates. We know what a section of $\mathcal{A}^{0,2}(T_X)$ looks like locally:

$$\phi = \sum_{i,j=1\cdots n} \phi_{ij} d\overline{z_i} \otimes \frac{\partial}{\partial z_j}.$$

Recall $T_t^{0,1} = (Id + \phi)(T^{0,1})$. Assume that $[T_t^{0,1}, T_t^{0,1}] \subseteq T_t^{0,1}$ so we look at

$$[\frac{\partial}{\partial \overline{z_i}} + \phi(\frac{\partial}{\partial \overline{z_i}}), \frac{\partial}{\partial \overline{z_k}} + \phi(\frac{\partial}{\partial \overline{z_k}})].$$

We have $[\frac{\partial}{\partial \overline{z_i}}, \frac{\partial}{\partial \overline{z_k}}] = 0$ for $i \neq k$. And we have $\phi(\frac{\partial}{\partial \overline{z_k}}) = \sum_j \phi_{ij} \frac{\partial}{\partial z_j}$. (So $d\overline{z_i}$ contracts with $\frac{\partial}{\partial \overline{z_i}}$. so we get

$$\sum_\ell [\frac{\partial}{\partial \overline{z_i}}, \phi_{k\ell} \frac{\partial}{\partial z_\ell}] + \sum_j [\phi_{ij} \frac{\partial}{\partial z_j, \frac{\partial}{\partial \overline{z_k}}}] \tag{48.4}$$

$$+\sum_{j,\ell}[\phi_{ij}\frac{\partial}{\partial z_j},\phi_{k\ell}\frac{\partial}{\partial z_\ell}] \tag{48.5}$$

and we have that equation 48.4 can be rewritten as

$$\sum_\ell \frac{\partial \phi_{k\ell}}{\partial \overline{z_i}}\frac{\partial}{\partial z_\ell} - \sum_j \frac{\partial \phi_{ij}}{\partial \overline{z_k}}\frac{\partial}{\partial z_j} = \sum_j (\frac{\partial \phi_{kj}}{\partial \overline{z_i}} - \frac{\partial \phi_{ij}}{\partial \overline{z_k}})\frac{\partial}{\partial z_j}$$

and an exercise shows that this equals

$$= \overline{\partial}\phi(\frac{\partial}{\partial \overline{z_i}}, \frac{\partial}{\partial \overline{z_k}}).$$

And equation 48.5 is

$$[\phi,\phi](\frac{\partial}{\partial \overline{z_i}}, \frac{\partial}{\partial \overline{z_k}}),$$

and

$$[\phi,\phi] = \sum_{i,j,k,\ell}(d\overline{z_i}\wedge d\overline{z_k})\otimes [\phi_{ij}\frac{\partial}{\partial z_j},\phi_{k\ell}\frac{\partial}{\partial z_\ell}].$$

Thus, the conclusion is that integrability implies that

$$(\overline{\partial}\phi + [\phi,\phi])(\frac{\partial}{\partial \overline{z_i}, \frac{\partial}{\partial \overline{z_k}}}) \in T_t^{0,1}.$$

So $\overline{\partial}\phi + [\phi,\phi] \in \mathcal{A}^{0,2}(T^{1,0}\cap T_t^{0,1})$. For t sufficiently small, $T^{1,0}\cap T_t^{0,1} = \{0\}$. So $\overline{\partial}\phi + [\phi,\phi] = 0$. □

So after a long equation, we get an amazingly simply formula. What can you do with this? The idea is that you want to build a deformation.. we have an idea of what a deformation of almost complex structures depending analytically on one parameter, is. We could look at something like

$$\phi = \phi_1 t + \phi_2 t^2 + \cdots$$

and we want to have this rule that $\overline{\partial}\phi + [\phi,\phi] = 0$, and this gives you an infinite system of differential equations. It will force

$$\overline{\partial}\phi_1 = 0,$$

and noting that $[\phi_i t^i, \phi_j t^j] = [\phi_i,\phi_j]t^{i+j}$, we have

$$\overline{\partial}\phi_2 + [\phi_1,\phi_1] = 0,$$

and in general

$$\overline{\partial}\phi_k + \sum_{0<i<k}[_i,\phi_{k-i}] = 0.$$

You have to solve all of these one by one. To solve $\overline{\partial}\phi_1 = 0$ is to solve for a first order order deformation. And solving $\overline{\partial}\phi_2 + [\phi_1,\phi_1] = 0$, gives second order deformation. If you're not able to solve this at some point, then the deformations are obstructed. If you can solve to infinity, then it is unobstructed.

There's a famous theroem of Tian-Tobrov, detailed in Huybrechts, which says that: the deformations of Calabi-Yau's are unobstructed. This is a theorem very important in algebraic geometry, but very much not proven using the repertoire of algebraists.

Let's go back to ϕ_1. We had $\phi_1 \in \Gamma(X, A^{0,1}(T_X))$. To solve $\overline{\partial}\phi_1 = 0$ means we get a class $[\phi_1] \in H^{0,1}(T_X)$, and by the Dolbeaut theorem, a cohoomology class in $H^1(X, T_X)$. Can check that this is same cohomology class as before. So the analytic approach gives us the same approach. To be clear, this $[\phi_1]$ is what we call the Kodaira-Spencer class, of the deformation. Now in this analytic point of view, this Kodaira-Spencer class gives us a one-parameter deformation of the almost complex structure on X.

Proposition 48.6. *If X compact complex manifold, then there exists a 1 to 1 correspondence between first order deformations of complex structure, up to isomorphism, with $H^1(X, T_X)$.*

Let's handwave now: the problem here is that you can have the same deformation which leads to the same complex structure. We want to find a way to equate those that give the same complex structure. This one to one correspondence is only between isomorphism classes/cohomology classes. So this is very fuzzy. We havent' really put any structure on the space of complex structures. But let's handwave: there is a space of almost complex structures, which you can actually turn in to an infinite-dimensional manifold, and you can act on this with diffeomorphisms. So you have $Diff(X) \times AC(X) \to AC(X)$.

Remark 48.7. The integrable complex structures don't really form a nice space, that is very interesting, although it doesn't bother us here.

Have $Diff(X) \times AC(X) \to AC(X)$, have $(F, \mathcal{J}) \mapsto dF \circ \mathcal{J} \circ dF^{-1}$.

Definition 48.8. Two almost complex structures $\mathcal{J}, \mathcal{J}'$ are isomorphic if there exists a diffeomorphism $F : X \to X$ such that $dF \circ \mathcal{J} = \mathcal{J}' \circ dF$.

If you think of it, letting $0 \in Diff(X)$, have $T_0 Diff(X) = \Gamma(X, \mathcal{A}^0(T_X))$. You consider $Aut(X)$, then you wonder.. its an infinite dimensional algebraic group, have $T_0 Aut(X) = H^0(X, T_X)$. The global vector fields, are the tangent space to the automorphism group.

We will talk more about first order second order etc deformations. We won't prove the theorem of Tian, but I should look at this myself. Then we'll move onto local systems.

49. 4/3/24: LAST BIT OF DEFORMATION THEORY

Let $\{F_t\}_{t\in T}$ be a family of diffeomorphisms of X depending analytically on t. So this is going to give a 1-parameter family of complex structures, where $t \mapsto \mathcal{J}(t) = dF_t \circ J \circ dF_t^{-1}$.

Where $T_t^{0,1} = dF_t(T^{0,1})$. Then you can expand so that

$$F(x,t) = x + tF(x) + \cdots$$

where $\frac{dF(x,t)}{dt}|_{t=0} = \sum F_i \frac{\partial}{\partial x_i} \in \Gamma(X, A^0(T_X))$.

Lemma 49.1. $\phi_1 = \overline{\partial}(\sum F_i \frac{\partial}{\partial x}) = \overline{\partial}(\frac{dF}{dt}|_{t_0}) \in \Gamma(X, A^{0,1}(TX))$.

Huybrechts page 260. The class of this operation is zero in cohomology, $[\phi_1] = 0 \in H^1(X, T_X)$. And recall that $H^1(X, T_X)$ parameterizes infinitesimal deformations of X up to isomorphism. Thus, 1st order deformations are in one-to-one correspondence with $\overline{\partial}$-closed elements in $A^{0,1}(T_X)$, and they are isomorphic if and only if they differ by $\overline{\partial}$-exact elements.

We want to see if we extend first order deformations further, which we call integration. We want to see if we can integrate 1-st order deformations $v \in H^1(X, T_X)$.

Does there exist a 1-parameter family $\{\mathcal{J}(t)\}_t$ such that $v = [\phi_1]$ is a Kodaira-Spencer class? Are there obstructions?

Let's go back to the Maurer-Caurtan equations. We want to do this inductively. The idea is that we want

$$\overline{\partial}\phi + [\phi, \phi] = 0,$$

and $\overline{\partial}\phi_1 = 0$, and $\overline{\partial}\phi_2 + [\phi_1, \phi_1] = 0$. Say we found $\phi_1 t + \cdots + \phi_k t^k$. Want to solve $\overline{\partial}\phi_{k+1} + \sum_{0<i<k+1}[\phi_i, \phi_{k+1-i}] = 0$. Have $\overline{\partial}\phi_1 = 0 \implies \overline{\partial}[\phi_1, \phi_1] = 0$. So $[\phi_1, \phi_1] \in H^2(X, T_X)$. If $\overline{\partial}\phi_2 + [\phi_1, \phi_1] = 0$. Must have $[\phi_1, \phi_1] = 0 \in H^2(X, T_X)$, $\phi \in \Gamma(A^{0,1}(T_X))$. Typical obstruction: if $[v, v] \neq 0 \in H^2(X, T_X)$, then cannot be integrated.

Have $\alpha \in A^{0,p}(T_X), \beta \in A^{0,q}(T_X)$ implies $\overline{\partial}[\alpha, \beta] = [\overline{\partial}\alpha, \beta] + (-1)^p[\alpha, \overline{\partial}\beta]$. In particular, this bracket induces maps $H^p(X, T_X) \times H^q(X, T_X) \to H^{p+q}(X, T_X)$.

Proposition 49.2. *If $H^2(X, T_X) = 0$, then any v can be formally integrated.*

Proof. Let's assume that we've found $\phi_1 t + \cdots + \phi_{k-1}t^{k-1}$, and we're looking for ϕ_k, such that $[\phi_1] = v$, satisfying the Maurer-Cartan equations. We want $\overline{\partial}\phi_k = -\sum_{0<i<k}[\phi_{ij}, \phi_{k-i}]$, and we define this to be $\omega_{k-1} \in A^{0,2}(T_X)$. In some sense we want to solve this $\overline{\partial}$ equation. The claim is that

$$\overline{\partial}\omega_{k-1} = 0.$$

If we do this, then $[\omega_{k-1}] \in H^2(X, T_X) = 0$. So there would exist ϕ_k as desired.

So $-\overline{\partial}\omega_{k-1} = \sum_{0<i<k} \overline{\partial}[\phi_i, \phi_{k-i}]$

$$= \sum_{0<i<k} ([\overline{\partial}\phi_i, \phi_{k-i}] - [\phi_i, \overline{\partial}\phi_{k-i}],$$

so inductively this is equal to

$$\sum_{i+j=k, i,j>0} ([\overline{\partial}\phi_i, \phi_j] - [\phi_i, \overline{\partial}\phi_j] = \sum_{p+q+r=k, p,q,r>0} [[\phi_p, \phi_q], \phi_r] - [\phi_p, [\phi_q, \phi_r]].$$

Now it's just an exercise. You have to mimic what you know about vector fields, like the Jacobi identity for Lie brackets, so you get

$$\sum_{p+q+r=k, p,q,r>0} [\phi_q, [\phi_p, \phi_r]] = 0.$$

Something here about $[a, [a, b]] = \frac{1}{2}[[a, a], b], [a, [a, a]] = 0.$ ☐

In this case, we can always find a solution to the Maurer-Cartan equation. We're making a stronger statement here, because we're moving up the chain with just this one condition on the cohomology. When does this cohomology vanishing apply? For smooth projective curves, this is clear, since there is no H^2. Also $X = \mathbb{P}^n$, where we also have $H^1(T_{\mathbb{P}^n}) = 0$, which means it is a rigid shape, no infinitesimal deformations. Applies to K3 surfaces. And more generally, to Calabi-Yau manifolds of dimension n such that $h^{1,n-2}(X) = 0$.

In general, it is hard to compute the cohomology of tangent bundles. One reason is exact sequences. But another reason is that these $H^i(X, T_X)$ have no meaning in Hodge theory and topology. In the case of Calabi-Yau manifolds, applying Serre duality, the ω_X is trivial, so that's why we do understand them better.

Remark 49.3. Fact that $\mathbb{P}^n$ is the only manifold with positive tangent bundle, got Mori the fields medal.

Before thinking about moduli spaces, you may just wonder whether your complex manifold is exhibited in a family.

Definition 49.4. Let $\pi : \mathfrak{X} \to B$ is a family, and the central fiber X_0 is what we're interested in. Say π is a complete deformation if every other family $\pi' : \mathfrak{X}' \to B'$ where central fiber is also X_0, this family is a pullback.

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

The main reference for all this goes back to this book by Kodaira and Spencer. This family could be pretty monstrous and repetitive. You can impose further conditions. If in addition, you have that $f : B' \to B$ is unique, then π is called a universal deformation. If only infinitesimally the family is unique, i.e. if $df_0' : T_0B' \to T_0B$ is unique, then π is called a versal deformation. In general, this B could be very nasty. This B could be a scheme with nilpotence. If you at least assume that it is a variety, so no nilpotence in the local ring, then you have:

Theorem 49.5 (Kodaira-Spencer)**.** *If B is reduced, then π is a complete deformation if and only if its Kodaira-Spencer map $T_0B \to H^1(X,$*

Very hard theorem:

Theorem 49.6 (Kuramishi)**.** *Any compact complex manifold X admits a versal deformation space. In part, the Kodaira spencer map is an isomorphism $k : T_0B \to H^1(X, T_X)$. This B might be singular.*

Moreover, if $H^0(X, T_X) = 0$, then it follows that any versal deformation is universal. What could mess up our space is that there are automorphisms, so taking a quotient and getting rid of automorphisms, make space ugly. But if there are no automorphisms, you have nothing to worry about, which is the meaning of this condition.

If $H^2(X, T_X) = 0$, then X admits a smooth versal deformation. (Say that the deformations of X are unobstructed).

if the deformations are unobstructed, then you might imagine that the versal deformation space B should not be too bad. These are hard theorems, would take a whole course and seminar to prove.

You're in the process of trying ot understadn whether there's a parameter space for manifolds of a given type. It's not clear what to do. It wasn't clear to people for a very long time until Mumford, on how to construct moduli spaces. But surely they kewn what they were looking for; you dont' know how to cosntruct the moduli space, but you know its dimension by this theory. Assuming you have moduli space with reasonably nice properties, it is a parameter space for manifolds. Then if theyre' all there, they it should be complete. Coarse moduli space, its tangent space should be locally the deformation space of the manifold. So when people were looking for moduli space of curves, surely knew that you had to look for a space whose dimension was like $h^1(C, T_C)$. You compute this via Riemann Roch, so by Serre duality, this is $h^0(C, \omega_C^{\otimes 2})$ and $\deg \omega_C^{\otimes 2} = 4g - 4$, and for $g \geq 2$ we have $4g - 4 \geq 2g - 1$, so $h^0(C, \omega_C^{\otimes 2}) = 3g - 3$. So people knew they were looking for a space with this dimension. But how to do it?

What people did was take every curve, and embed it into the same projective space. And now you worry about the deformation space inside that projective space with fixed numerical invariants, and this leads you to the Hilbert scheme. Only after you construct and understand the Hilbert scheme as a parameter space, you understand there's some giant reductive group, and then you quotient and then get to the moduli space of curves. At the end of the day, though, there was never any doubt about what the dimension was.

The deep results are those where we don't have $H^2(X, T_X) = 0$. The Tian-Todorov result says that Calabi-Yau's are unobstructed.

Looking forward to the rest of the course. We are going to talk about local systems and flat connections, then talk about variations of Hodge structures. To keep the discussion in the sam espirit, you can have a family of varieties, wonder whether the complex structure moves. But you can also wonder whether the Hodge structure moves, if its a family of compact Kahler manifolds or projective varieties. So there are some moduli spaces for those, called period domains. What does it mean to study a family from this point of view. Maybe say something about the period map. Naturally has to do with the Kodaira-Spencer map.

50. 4/8/24: local systems, classical Riemann-hilbert

For the remainder of the course, we'll discuss the theory of local systems, and many other things. Representations of fundamental groups, vector bundles with flat connections, et cetera. The classical Riemann-Hilbert correspondence. It is one of the most beautiful combinations of topology, analysis, algebra, and geometry out there. We'll just give a few lectures on it. But this correspondence gives the foundation for the variations of Hodge structures. And we're going to tie this into families of varieties and the Kodaira-Spencer map that we studied.

The general theory of local systems is on topological spaces. Let B be a topological space, and assume it is connected and locally path connected. (Eventually: B will become the base of a family. So it will be a manifold so we'll have all of our topological assumptions when we actually apply this theory to what we care about).

We will discuss the "classical" Riemann-Hilbert correspondence. This is a correspondence between three types of objects.

(1) Local systems of $\mathbb{C}$-vector spaces on B.
(2) Representations of the fundamental group $\pi_1(B, b_0) \to GL_n(\mathbb{C})$
(3) Holomorphic vector bundles with flat connections on B.

In this correspondnece, you must use fundamenetal techniques in algebraic topology, and fundamental existence and uniqueness of solutions of first order diffierential equations with initial conditions. There is a lot of math involved. First, let us discuss what a local system is.

Definition 50.1. A local system $\mathcal{L}$ of rank n on B is a locally constant sheaf with stalk $V \cong \mathbb{C}^n$. In other words, there exists an open cover $\{U_i\}_{i \in I}$ of B such that $\mathcal{L}|_{U_i}$ is the constant sheaf $\underline{V}$. ($\Gamma(U_i, \underline{V}) \cong$ each stalk).

Remark 50.2. Since this is usually not a constant sheaf, we have transition functions

$$U_i \cap U_j \to GL(V)$$

which are constant. So just have some $A_{ij} \in GL(V)$.

Example 50.3 (constant sheaf of square roots)**.** Other examples of constant sheaves are $\underline{\mathbb{Q}}_B$, $\underline{\mathbb{R}}_B$.

Also, Q on $\mathbb{C}^*$, sheaf of square-roots. Solving $w^2 = z$. $Q(U) = \{$ holo $f : U \to \mathbb{C} | 2z\frac{df}{dz} = f\}$. The homework exercise is that this is locally constant. But not constant. One of the first questions you'd ask. From $\phi : \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^2$. Have $\phi_* \underline{\mathbb{C}}_{\mathbb{C}^*} = \underline{\mathbb{C}}_{\mathbb{C}^*} \oplus Q$. Simplest example, where Q arises naturally from basic complex analysis. Notice this sheaf is a sheaf of solutions to a differential equation.

One of our main examples of local systems will be the following.

Example 50.4. Suppose $f : \mathfrak{X} \to B$ is a smooth family of compact complex manifolds. When we have a sheaf $\mathcal{F}$ on $\mathfrak{X}$, you can consider its direct image $f_*\mathcal{F}$. This is the sheaf such that over an open set $U \subset B$, we have $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. The way to think of this, is that $(f_*\mathcal{F})(U) = H^0(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$. The higher derived functors are the functors for higher cohomology. So there is a sheaf that looks like for every $k \geq 0$,

$$R^k f_* \mathcal{F} := \text{ sheaf associated to the presheaf} U \mapsto H^k(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}).$$

This is really the relative setting. For example, if you have $X \to *$, then taking these pushforwards is the same as taking cohomology of $\mathcal{F}$ on X. So the former example generalizes the usual sheaf cohomology... one would get a constant sheaf with value in the k cohomology $H^k(X, \mathcal{F})$.

This language works well with base change. If we have a fiber X_t over $t \in B$, then we have $(R^k f_* \mathcal{F})_t = H^k(X_t, \mathcal{F}|_{X_t})$. So these sheaves are nice to put together the cohomology of your fibers. In particular, we'll be interested in

$$(R^k f_* \underline{\mathbb{C}}_{\mathfrak{X}})_t = H^k(X_t, \mathbb{C}).$$

But we know that $\mathfrak{X} \to B$ is a proper submersion. By Ehressmann's theorem, the fibers are all diffeomorphic, and this family is isotrivial. Thus, we know that taking this constant sheaf $\underline{\mathbb{C}}_{\mathfrak{X}}$ on $\mathfrak{X}$, taking these higher direct images on these constant sheaves will give us locally constant sheaves on B.

So if we take $0 \in B$, and $0 \in U$ contractible neighborhood, we have locally trivial so $f^{-1}(U) \cong U \times X_0$ over U, in the smooth category. Have $f^{-1}(U)$ is deformation retract to X_0, and $H^k(\pi^{-1}(U), \mathbb{C}_{\mathfrak{X}}) \cong H^k(X_0, \mathbb{C})$. So $R^k f_* \underline{\mathbb{C}}_{\mathfrak{X}}$ is the locally constant sheaf with fiber $V = H^k(X_0, \mathbb{C})$.

Now we consider an example involving differential equations.

Example 50.5. Let $U \subset \mathbb{C}^n$ open, coordinate z, and $A : U \to GL_n(\mathbb{C})$ holomorphic map. Consider

$$\frac{du}{dz} = A(z) \cdot u.$$

This has n variables and n unknowns. So this is a first-order holomorphic systeems of n linear differential equations on U. What do we know about solutions to such systems? Fundamental theorem, there is existence and uniqueness of solutions with initial conditions.

Say $\gamma : [0,1] \to U \subseteq \mathbb{C}^n$. Local solutions to the differential equation extend on $\gamma([0,1])$. Conclusion: space of solutions over a simply connected neighborhood is $\cong \mathbb{C}^n$. So the solution sheaf is a local system.

Lemma 50.6. *A local system on* $[0,1]$ *is constant.*

Proof. or each $t \in [0,1]$, there is an open interval $t \in I_t$ such that $\mathcal{L}|_{I_t}$ is constant. Take $J_t \subseteq I_t$ to be the "middle third." By compactness, finitely many of these J_t cover $[0,1]$. Fix n such that the length of $J_t > \frac{1}{n}$. We can do this since there are finitely many of them. This implies that the interval $[\frac{m}{n}, \frac{m+1}{n}]$ is contained in some I_t. Have $\mathcal{L}$ is constanto n each I_t. So $\mathcal{L}$ is consatnt everywhere. □

More generally, every local system on $[0,1] \times [0,1]$ is constant. We can cover this box by boxes of the form $[a_i, a_{i+1}] \times [b_i, b_{i+1}]$.

Now if we fix $b_0 \in B$, and $\gamma : [0,1] \to B$ where $\gamma(0) = b_0$ and $\gamma(1) = b_1$, we have $\gamma^*\mathcal{L}$ is constant. In particular, this implies that

$$\phi_\gamma : \mathcal{L}|_{b_0} \cong \Gamma(\gamma([0,1]), \mathcal{L}) \cong \mathcal{L}_{b_1}.$$

So we have obtained an isomorphism between these points. But when we have two homotopic paths, we would like the isomorphism given between stalk at b_0 and b_1 to be the same isomorphism. It should not depend on the homotopy class of the path.

Lemma 50.7. *If $\gamma_1, \gamma_2 \in [0,1] \to B$ are homotopic, then $\phi\gamma_1 = \phi\gamma_2$.*

Proof. Take a homotopy $H : [0,1] \times [0,1] \to B$ between γ_1, γ_2 so that $H(t,0) = \gamma_1(t)$ and $H(t,1) = \gamma_2(t)$, and $H(0,u) = b_0$ and $H(1,u) = b_1$. Need to perform restrictions along

$$(0,0) \hookrightarrow [0,1] \times 0 \hookrightarrow [0,1] \times [0,1] \hookrightarrow B.$$

$$\begin{tikzcd}
\mathcal{L}_{b_0} & \Gamma([0,1], H^*\mathcal{L}|_{[0,1]\times 0}) \arrow[l] \arrow[r] & \mathcal{L}_{b_1} \\
\Gamma([0,1], H^*\mathcal{L}|_{0\times[0,1]}) \arrow[u] \arrow[d] & \Gamma([0,1]\times[0,1], H^*\mathcal{L}) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \Gamma([0,1], H^*\mathcal{L}|_{1\times[0,1]}) \arrow[u] \arrow[d] \\
\mathcal{L}_{b_0} & \Gamma([0,1], H^*\mathcal{L}|_{[0,1]\times 1}) \arrow[l] \arrow[r] & \mathcal{L}_{b_1}
\end{tikzcd}$$

The top row isomorphisms gives ϕ_{γ_1} and bottom row gives ϕ_{γ_2}. The vertical arrows on the left and right are equalities. □

More generally, we have the following.

Proposition 50.8. *Say B is connected, locally path connected, and simply connected (just think of a sufficiently small open set of a manifold lol). Then every local system on B is constant.*

Proof. Take a point $x \in B$ with open neighborhood U, such that $\mathcal{L}|_U$ is constant. And now consider

$$\begin{tikzcd}
\Gamma(X, \mathcal{L}) \arrow[rr, "ev_x"] \arrow[dr, "res"'] & & \mathcal{L}_x \\
& \Gamma(U, \mathcal{L}|_U) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

we know $\Gamma(U, \mathcal{L}) \to \mathcal{L}_x$ is an isomorphism by assumption. We want to show that the restriction $\rho_{XU} : \Gamma(X, \mathcal{L}) \to \Gamma(U, \mathcal{L})$ is injective and surjective.

Pick some $y \in B$, neighborhood U_y. There's a path $\gamma : [0,1] \to B$ between x and y. We know that the local system is constant on this path. So $\mathcal{L}|_{\gamma([0,1])}$ is constant.

Have $\phi_\gamma : \mathcal{L}_x \to \mathcal{L}_y$. So the section we had on U, same as element of stalk at x, transported to element of stalk at y, same as section over U_y. So around each y_1, we find this open neighborhood. So we can cover our space with such opens. So we can glue to global section if there's compatability on overlaps. So the queestion is, if you have $s_x \to (s_{y1})_z$ and $(s_{y_2})_z$. Go from path $x \to y_1 \to z$ and path $x \to y_2 \to z$, contractability implies we have constancy. Same isomorphism. Consequence of the previous lemma. $\square$

Now let's start the story that we will apply to families. We fix B and fix $b_0 \in B$. Have path γ between b_0 and b_1. We have isomorphism $\phi_\gamma : \mathcal{L}_{b_0} \to L_{b_1}$, which depends only on the homotopy class of γ. Then we obtain a group homomorphism

$$\rho : \pi_1(B, b_0) \to GL(\mathcal{L}_{b_0}).$$

This representaton of the fundamental group is called a monodromy representation. so from a local system of rank r, we obtain this monodromy representation. This ρ is independent of the choice of b_0 up to conjugation. So goes from local systems to representations of the fundamental group. We need to prove that we can go backwards.

Proposition 50.9. *If B is path connected and locally simply connected (e.g. a manifold) (we really want to have a universal cover), then there is a 1 to 1 correspondence (in fact an equivalence of categories) between*

$$\{ \text{ local systems on } B \text{ of rank } n\}$$

and

$$\{ \text{ rep } \pi_1(B, b_0) \to GL_n(\mathbb{C})\}.$$

Proof. We'll sketch the proof more carefully next time. But the key idea is that we want to pass to the universal cover. This universal cover $p : \tilde{B} \to B$ has deck transformations; actions on it by the fundmaental group $\pi_1(B, b_0)$. Locally around b_0, we have neighborhood U such that $p^{-1}(U) = \bigsqcup_{\gamma \in \pi_1(B,b_0)} U_\gamma$ where $p|_{W_j} : W_j \to U$ homeomorphism. We are going to define

$$\Gamma(U, \mathcal{L}) = \{s \in \Gamma(p^{-1}(U), \underline{V}) | \text{ s equivariant w.r.t action } \}.$$

The representation $\rho : \pi_1(B, b_0) \to GL(V)$, the equivariance condition is $s(\gamma(x)) = \rho(\gamma)(s(x))$. We will do this in homework. Next time, we will prove in different way, and use language of principal bundles. $\square$

51. 4/10/22: classical Riemann Hilbert correspondence, principal bundles, flat connections

Proposition 51.1. *If B is path connected and locally simply connected, then we have a one to one correspondence between*

$$\{ \text{ local systems on } B \text{ of rank } n \text{ (with stalk } V) \}$$

and

$$\{ \text{ rep } \pi_1(B, b_0) \to GL_n(\mathbb{C})\}.$$

(In fact, it is an equivalence of categories. It is functorial).

We saw that the map in the forward direction is, given a local system on B of rank n, then we obtain a monodromy representation

$$\rho : \pi_1(B, b_0) \to GL(V)$$

where we took $\gamma \mapsto \phi_\gamma$. If we fix ρ, the homework exericse is the following. We take universal cover $p : \tilde{B} \to B$, and we pick neighborhood $U \subseteq B$ of b_0 such that $p^{-1}(U) = \bigsqcup W_j$ where $W_j \to^p U$ is homeomorphism. And they are permuted by the fundamental group by the so called deck transformations. Then we define

$$\Gamma(U, \mathcal{L}) = \{s \in \Gamma(p^{-1}(U), \underline{V}) | s \text{ equivairant i.e. } s(\gamma(x)) = \rho(\gamma)(s(x)), \gamma \in \pi_1(B, b_0), \forall x \in U\}$$

Check that this local system, gives inverse to monodromy representation. So $\Gamma(X, \mathcal{L}) = V^\rho := \{v \in V | \rho(\gamma)(v) = v, \forall \gamma\}$.

Directly produce a vector bundle associated to this local system. Alternatively, have $p : \tilde{B} \to B$ with action $\pi_1(B, b_0)$ by deck transformations. You can construct a vector bundle of rank n to this. This is sometimes called a principal bundle, with this kind of action, and you can take the vector bundlea ssocaited to thsi principal bundle. Have

$$\mathcal{V} = \tilde{B} \times V / \sim$$

Have $(x, v) \sim (\gamma \cdot x, \rho(\gamma^{-1})(v))$, for every $\gamma \in \pi_1(B, b_0)$. Have an obvious map to the base $\pi : \mathcal{V} \to B$, where you take $[(x, v)] \mapsto x$. This is a vector bundle, trivial over the U described before. Have $\mathcal{V}$ vector bundle associated to principal bundle $p : \tilde{B} \to B$, with group $\pi_1(B, b_0)$ via the representation ρ.

Transition functions are constant and are $\rho(\pi_1(B, b_0)) \subseteq GL(V)$, a discrete subgroup.

We can consider constant sections of $\mathcal{V}$. For $v \in V$, have $\tilde{v} : U \to \mathcal{F}$ where

$$x \mapsto [(p_j^{-1}(x), v)],$$

have $\mathcal{L} \subseteq \mathcal{V}$ subsheaf generated by constant sections. Note $p_j : W_j \to U$. Actually it turns out that $\mathcal{V} \cong \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_B$.

Wnat to describe some of these sections of $\mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_B$. Find frames for these vector bundles $\mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_B$ which have a chance to extend over punctures. Something related to the theory of extensions of variations of Hodge structures.

The simplest space we can consider where this theory is nontrivial is a punctured disk. Simplest space that is not simply connected.

Example 51.2. Let $B = \Delta^* = \{z \in \mathbb{C} | 0 < |z| < 1\}$. We take the fundamental group to be

$$\pi_1(\Delta^*, \frac{1}{2}) \cong \mathbb{Z}\langle \gamma_0 \rangle$$

where we take γ_0 is the loop of radius $\frac{1}{2}$ around the origin. Consider the representation

$$\rho : \pi_1(\Delta^*, \frac{1}{2}) \to LG_2(\mathbb{C})$$

where

$$k \mapsto \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

Now what's the universal cover of Δ^*? Have map from the upper half plane

$$\mathbb{H} \to \Delta^*, z \mapsto e^{2\pi i z}.$$

Whats the action of $\pi_1(\Delta^*, \frac{1}{2})$? It corresponds to taking a point z in the upper half plane, and moving it to $z-1$. Upper half plane with these vertical columns, space at the integers.

$$\begin{array}{ccc} \mathbb{H}\times\mathbb{C}^2 & \longrightarrow & \mathcal{V} = \mathbb{H}\times\mathbb{C}^2/\sim \\ \downarrow & & \downarrow \\ \mathbb{H} & \longrightarrow & \Delta^* \end{array}$$

Over contractible open sets $U \subseteq \Delta^*$ have constant sections, $\tilde{v} : U \to \mathcal{V}$ and $x \mapsto [(p_j^{-1}(x), v)]$. Let

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, N^2 = 0.$$

Then $\rho(\gamma_0) = I_2 + N$. Since this matrix is nilpotent, we can do a little trick and add all of its powers, so that

$$\rho(\gamma_0) = I_2 + N + \frac{N^2}{2!} + \frac{N^3}{3!} + \cdots = e^N,$$

and

$$\rho(\gamma_0^{-1}) = I_2 - N \cdots = e^{-N}.$$

We'll see soon why it is convenient to think of it this way. I want to fix $v \in \mathbb{C}^2$. Define

$$\tilde{v} : \Delta^* \to \mathcal{V}, w \mapsto (\frac{\log w}{2\pi i}, e^{\frac{\log w}{2\pi i}N} \cdot v).$$

Let's first see that this is well-defined. Say z such that $e^{2\pi i z} = w$. If you go around γ_0, in universal cover you get z goes to $z-1$, so we need to check that in the vector bundle $\mathcal{V}$, we get the same thing. Get $(z-1, e^{(z-1)N} \cdot v)$. Want to check that

$$(z-1, e^{(z-1)N} v) = (\gamma_0(z), \rho(\gamma_0^{-1})(e^{zN} \cdot v).$$

Have $e^{(z-1)N} = e^{-N}e^{zN}v$, and $e^{-N} = \rho(\gamma_0^{-1})$. On U as above, have $\tilde{v}(\omega) = (e^{\frac{\log w}{2\pi i}N})\widehat{v}(w)$. Can check if $\widehat{v_i}$ local frames for $\mathcal{L}$. And the $\tilde{v}_i$ give a frame for the vector bundle $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_X$.

People want to find how to extend these vector bundles over punctures. With more meaning than that – particularly want to extend notion of variation of Hodge structure over puncture. Want to find some good extension. This was invented by Deligne: consider sections of this form, nilpotent matrices, good behavior near origin, that can be extended over the puncture and produce these vector bundles.. generating sections...

The story is complicated. This is one of the simplest examples one can show. Even the study of rank 1 local systems is a pretty delicate thing, has a very rich study. But already in higher rank, you see it is what it is.

We want to recognize these vector bundles in a formal way. Let us discuss vector bundles with flat connection.

Let $E \to B$ be a holomorphic vector bundle of rank r. A holomorphic connection on E is a map

$$\nabla : E \to \Omega_B^1 \otimes E,$$

such that for every $U \subseteq B$ open, have

$$\nabla : \Gamma(U, E) \to \Gamma(U, \Omega_B^1) \otimes \Gamma(U, E)$$

satisfies

- ∇ is $\mathbb{C}$-linear
- $\nabla(fs) = df \otimes s + f\nabla s$, for every $f \in \mathcal{O}O_B(U), s \in E(U)$.

Remark 51.3. $df = \partial f + \overline{\partial f} = \partial f$, but $\partial\overline{\partial f} = -\overline{\partial}\partial f = 0$, so there's no problem with this definition.

Locally, we have that

$$\nabla s_j = \sum_{i=1}^{r} \theta_{ij} \otimes s_i,$$

where s_i local frame for $\Gamma(U, E)$ and the θ_{ij} are holomorphic one-forms. Let $\Theta = (\theta_{ij})$.

Remark 51.4. Can also think of it as

$$\nabla : T_B \otimes E \to E,$$

where locally on U we have

$$\nabla_\xi(\sum_{i=1}^{r} f_i s_i) = \sum_{i=1}^{r}(\xi(f_i) + \sum_{j=1}^{r} f_J \theta_{ij}(\xi))s_i.$$

Recall calculation for hermitian connections. if you have another local frame $\{s_i'\}_{i=\overline{1r}}$, can write

$$s_i' = \sum_{j=1}^{r} g_{ij} s_j, \text{ and } g_{ij} \in \mathcal{O}_B(U).$$

If we let $g = (g_{ij})$, and define $dg = (dg_{ij})$, then

$$\theta' = g^{-1}dg + g^{-1}\theta g.$$

We also have $\nabla : \Omega_B^1 \otimes E \to \Omega_B^2 \otimes E$ and $\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge ds$. We considered

$$\Theta = \nabla \circ \nabla : E \to \Omega_B^2 \otimes E,$$

and this is the connection. We can think of Θ as a map $\mathcal{O}_B \to \Omega_B^2 \otimes E \otimes E^\vee$, so the connection can be considered as an element

$$\Theta \in H^0(B, \Omega_B^2 \otimes End(E)).$$

Can write

$$\nabla^2 s_j = \sum \Theta_{ij} \otimes s_j, \text{ where } \Theta_{ij} = d\theta_{ij} - \sum_{k=1}^{r} \theta_{ik} \wedge \theta_{kj}.$$

In other words, $\Theta = d\theta - \theta \wedge \theta$. The transformation formula was $\Theta' = g^{-1}\Theta g$.

Let's keep a few things in mind. When we discussed hermitian connections, we defined them on holomorphic vector bundles, but we were looking at smooth sections, and we had this decomposition into this $\nabla_E + \overline{\partial}$. Complex budnles with hermitian connections are not necessarily bundles with holomorphic connections. There's a fact that you can read in Huybrechts, where he introduces Atiyah classes, where:

Proposition 51.5. *A holomorphic vector bundle admits a holomorphic connection* $\iff$ *admits a hermitian metric for which the curvature satisfies* $[\Theta_E] = 0 \in H^1(X, \Omega_X^1 \otimes End(E))$.

Bundle has holomorphic connections if and only if the Atiyah class is zero. On the other hand, for any bundle, the Atiyah class is equal to the class of the curvature form. People don't erally know holomorphic vector bundles that don't have flat holomorphic connection, it is actually a conjecture from ages ago whether in fact a holomorphic vector bundle has to have this flat thing in the sense of holomorphic connections. It's a puzzle. Everything in geometry that comes up naturally, is a bundle with flat holomorphic connection. Because of what we're going to say next:

Definition 51.6. Fix (E, ∇) holomorphic.

(1) A section $s \in \Gamma(U, E)$ is called flat (for ∇) if $\nabla s = 0$.
(2) The connection itself ∇ is called flat if there exists a trivializing cover $\{U_i\}_{i \in I}$ trivializing cover for E and a frame $(s_i^1, \cdots, s_i^r)$ for every i, such that s_i^k are all flat sections of E.

Next time we'll discuss that the second item is equivalent to the curvature being 0. That is usually how people define flat connections. A holomorphic bundle with holomorphic connection is called flat if its curvature is identically 0. The equivalence of these definitions is something called the Frobenius theorem in PDEs.

What's the idea here? A function is constant if and only if its derivative is zero. That's the starting point. We want a replacement of bundles and sections. When is a section constant? It corresponds to the connection being 0. Locally constant sections are the ones satisfying $\nabla s = 0$. In the case of flat connections, you can find enough flat sections so that it generates everything, i.e. you can find frames of "locally constant sections."

Proposition 51.7. *There is a one to one correspondence (in fact, again, an equivalence of categories) between*

$$\{ \text{ vector bundles } \mathcal{V} \text{ with flat connection } \nabla\}$$

and

$$\{ \text{ local systems on } B \text{ of rank } r \ \}.$$

In our previous discussion of local systems to representations, we saw this vector bundle appear. But now we need to worry about the flat connection part.

Let's say something about the easy implication, next time we'll deal with the harder implication which has more content. If $\mathcal{L}$ is local system then we get vector bundle $\mathcal{V} := \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_X$. So this means there is a frame $\{s_i\}$ of locally constant sections. Have $s = \sum_{i=1}^r f_i s_i$, so we define $\nabla s = \sum_{i=1}^r df_i \otimes s_i$. Now let's see why this is a flat connection.

The definition implies that θ_{ij} is trivial $= 0$. More trivially, $\nabla s_i = d(1) \otimes s_i = 0$. So $\nabla s_i = 0$. But its characterizied by the matrix of coefficients, so $\theta_{ij} = 0$. And maybe if you worry about well-definedness, even if you look at $\theta' = g^{-1} dg + g^{-1} \theta g$, but of course we didn't really need to worry about that.

Let's do this in a very down to earth way. If you apply the curvature

$$\nabla^2 s = \nabla^2 (\sum f_i s_i) = \sum_i (\nabla^2 (f_i s_i)) = \sum_i (df_i \otimes s_i - df_i \wedge \nabla s_i),$$

and $\nabla s_i = 0$ and $df_i = 0$. So $\mathcal{V} := \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_X$ is the bundle and we have just defined the flat connection. What are the flat sections? We have $\nabla(\sum f_i s_i) = 0$, so this means $df_i = 0$, so f_i constant. So the flat connections are those sections you can express in terms of s_i with constant f_i. So the kernel is $\ker \nabla = \mathcal{L}$. The flat sections gives you exactly the local system.

52. 4/15/22: classical Riemann Hilbert, Gauss-Manin connection

Recall we are trying to identify vector bundles with flat connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_B$$

to be in 1 to 1 correspondencew with

$$\{ \text{ local systems L on B of rank r}$$

and we saw last time that given a local system $\mathcal{L}$, we can send it to the vector bundle

$$\mathcal{V} = \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_X$$

and $\nabla(\sum f_i s_i) = \sum df_i \otimes s_i$ gives a flat connection.

Now we want to start with a vector bundle of flat connection (V, ∇). We should use the flat connection to get a local system. Requiring that a section be flat is a system of PDEs. The question is: can you solve this system, existence and uniqueness in a neighborhood. Sometimes people say integrable connection instead of flat connection – we'll find out why in the course of the discussion.

Definition 52.1. $\mathcal{L}(U) := \{s \in \mathcal{V}(U) | \nabla s = 0\}$, sometimes written as $\ker \nabla$. It is a subsheaf of $\mathcal{V}$, gives a local system.

Fix $U \subseteq B$ open, such that $\mathcal{V}|_U \cong^{\phi} \mathcal{O}_U^{\oplus r}$ by a local frame $\{s_1, \cdots, s_r\}$. And for every $s \in \Gamma(U, \mathcal{V})$, we have $s = \sum_{j=1}^r f_j s_j$ where $f_j \in \mathcal{O}_X(U)$. And

$$\nabla(f_j s_j) = df_j \otimes s_j + f_J \nabla s_j = df_j \otimes s_j + f_j \sum_{i=1} \theta_{ij} \otimes s_i.$$

In other words,

$$\nabla s = \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix} + \Theta \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}.$$

Thus we have that $df_i + \sum_{j=1}^r \theta_{ij} f_j$ is coefficient of s_i. If we fix local coordinates $z_1, \cdots, z_n$ on U, then you have

$$\theta_{ij} = \sum_{k=1}^n \Gamma^j_{ij}(z) dz_k$$

so that $df_i + \sum_j \theta_{ij} f_i = \sum_{k=1}^n \frac{\partial f_i}{\partial z_k} dz_k + \sum_{j=1}^r (\sum_{k=1}^n \Gamma^k_{ij} dz_k) f_j$.

So the conclusion is that on U, the condition $\nabla s = 0$ if and only if

$$\frac{\partial f_i}{\partial z_k} + \sum_{j=1}^r \Gamma^k_{ij} f_j = 0,$$

for every $i = 1, \cdots, r$ and $k = 1, \cdots, n$. This is the system of linear PDEs. Looking for sections that are flat, they are given by this system of PDEs. Why are these sections locally constant on, say, contractible opens?

Integrability condition:

$$\sum_{i,k} (\frac{\partial f_i}{\partial z_k} + \sum_j \Gamma^k_{ij} f_j) dz_k \otimes s_i,$$

what is the condition $\nabla \circ \nabla = 0$? The exercise is that

$$(\nabla \circ \nabla)(s_k) = \sum_\ell (\sum_{i<j} R^\ell_{ijk} dz_i \wedge dz_j) \otimes s_\ell.$$

where

$$R^\ell_{ijk} = \sum_s (\Gamma^s_{jk}\Gamma^\ell_{is} - \Gamma^s_{ik}\Gamma^\ell_{js}) + (\frac{\partial \Gamma^\ell_{jk}}{\partial z_i} - \frac{\partial \Gamma^\ell_{ik}}{\partial z_j}),$$

which are coefficients of the curvature form. So we see that integrability $\nabla^2 = 0 \iff R^\ell_{ijk} = 0$ for all i, j, k, ℓ. Here comes the big result:

Theorem 52.2 (Frobenius theorem)**.** *this condition $\iff$ there exists a unique solution to system of PDEs in the proof, with initial conditions.*

Proof. Details can be found in: Brian Conrad, "Classical motivation for Riemann-Hilbert correspondence." In one dimension, you can always integrate the solution.. you can extend the solutions along an interval. We talked about this before in local systems. That comes for free, a solution to ODEs, in one dimension.

Now the question becomes equivalent... you have this flow, many vector fields.. and can you find a submanifold such that the vector fields span the tangent space.

why does this sound like integrability for an almost complex structure to be complex structure? Integrability condition is the same as saying, $[W, W] \subseteq W$. Something about $R^\ell_{ijk} = 0 \iff [W, W] \subseteq W$. In general, called something a foiliation. These submanifolds are the leaves of a foiliation. □

This classical Riemann-Hilbert correspondence, as the name suggests, is classical but is already very complicated. It gets even more complicated once we introduce variations of Hodge structures.

Let's go back to our source for back examples. Consider family $\pi : \mathfrak{X} \to B$, proper submersion.

For every $k \geq 0$, let $\mathcal{L} = R^k \pi_* \underline{\mathbb{C}}_x$, local system with fiber $V \cong H^k(X_0, \mathbb{C})$, trivial over contractible $U \subseteq B$. We also get monodromy representation

$$\rho : \pi_1(B, 0) \to GL(V)$$

and we take $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_B$ with the Gauss-Manin connection $\nabla = \nabla_{GM}$.

The Gauss-Manin connection has a differential geometric explanation, comes from Cartan-Lie formula. Going to use it to show the so called period map is holomorphic.

Let's give a concrete formula for ∇_{GM}. Assume $B = U$, contractible for simplicity. Have $s \in \Gamma(B, \mathcal{V})$, where $s(t) \in H^k(X_t, \mathbb{C})$ for every $t \in B$. Such that there exists $\omega \in A^k(\mathfrak{X})$ such that $\omega|_{X_t} = \omega_{X_t}$ closed, such that $[\omega_{X_t}] = s(t)$.

Say $u \in T_0 B$. Then we can lift it to a vector upstairs. Take any lift v on $\mathfrak{X}$. The formula for the Gauss-Manin connection is

$$(\nabla_u \omega)_t = [\iota_v d\omega|_{X_t}] \in H^k(X_t, \mathbb{C}).$$

A reference for this is in Voisin, page 231-232. The contraction of vector field ξ is $\iota_\xi(\omega_1 \wedge_1 \cdots \wedge_k) = \omega_1(\xi)\omega_2 \wedge \cdots \wedge \omega_k$.

Semicontinuity theorem. Let $\pi : \mathfrak{X} \to B$ as above. Let E be a vector bundle on $\mathfrak{X}$, and $E_t := E|_{X_t}$. And $t \in B$, let's fix an integer $i \geq 0$. Here's a theorem:

Theorem 52.3. *The function $\phi : B \to \mathbb{N}$, where $\phi(t) = \dim_{\mathbb{C}} H^i(X_t, E_t)$ is upper semicontinuous. Meaning, $\phi(t) \le \phi(0)$ for t in a neighborhood of 0. Meaning, the value of $\phi(t)$ an only go up in a proper closed subset.*

Remark 52.4. The theorem holds more generally for any coherent sheaf on $\mathfrak{X}$ which is flat over B (either algebraic or analytic category). In algebraic case, Hartshorne Chapter 3 Section 12.

In the Kahler case, however, it follows from a general result in PDEs. Basically, you can argue like this: put Hermitian metrics on $\mathfrak{X}, E$... get metrics on X_t, E_t... get Laplace operators which vary smoothly $\overline{\Box}_t$ (for $\overline{\partial}$) on $\mathcal{A}^{0,i}_{X_t}(E_t)$. Get $H^i(X_t, E_t) \cong \mathcal{H}^{0,i}_{\overline{\Box}_t}(E_t)$. The theorem from PDEs we use can be found in Wells, Chapter 4, Theorem 4.13: we can send $t \mapsto \dim \ker \overline{\Box}_t$ and this is uppersemicontinuous. This is true for any C^∞ family of elliptic operators of same degree.

Example 52.5. Take trivial line bundle $\mathcal{O}_{\mathfrak{X}}$ on $\mathfrak{X} \to B$. Then you are looking at $t \mapsto H^i(X_t, \mathcal{O}_{X_t})$. And we see that this is already interesting. We didn't know this function might be well-behaved a priori in a family.

We might be interested in $H^i(X_t, \Omega^p_{X_t})$. Can take $E = \Omega^p_{\mathfrak{X}}/\pi^*\Omega^p_B$, and this gives you the bundle of relative p-forms $\Omega^p_{\mathfrak{X}}/B$. And

$$\Omega^p_{\mathfrak{X}/B}|_{X_t} = \Omega^p_{X_t}.$$

We quotiented because if you have p-form on base, lifts to be constant on fibers.

Then the upper-semicontinuity theorem gives us that

$$t \mapsto h^{p,q}(X_t) := \dim_{\mathbb{C}} H^q(X_t, \Omega^p_{X_t})$$

is upper semi-continuous for every p, q.

Corollary 52.6. *$t \mapsto h^{p,q}(X)$ is constant.*

Proof. Apply Ehressman's theorem, H^n is same. Use Hodge decomposition, implies Hodge numbers must be the same. $\square$

If you want this property for family of singularities, there are serious obstructions. But if you have duBois singularities... in minimal model program, allow duBois singularities, so you still have this constant Hodge numbers for flat family allowing for singular fibers.

Remember the $h^0(X_t, \omega^{\otimes m}_{X_t})$, the pluricanonical forms, loses Hodge theoretic meaning. Siu's deformation invariance of plurigenera, proves it is constant.

What we have proved is that the Hodge filtrations have the same dimensions. $t \mapsto F^p H^k(X_t, \mathbb{C})$. But they are not the same. Each can be thought of as sitting inside $H^k(X_0, \mathbb{C})$, and they vary as vector spaces. And so this gives you the period map of the family:

$$t \mapsto \mathbb{G}(f_p, V).$$

It is a result of Griffiths that it is holomorphic. If you look atht ederivative, you can identify wiht the Gauss-Manin connection, and Cartan formula will show it vanishes on type $(1, 0)$ vector fields, and thus holomorphic?

53. 4/17/24: Gauss-Manin, Griffths tranversality, Variations of Hodge structures

Let us recollect some facts about Grassmannians. We define

$$G(k,n) = \{ \text{ k dimensional linear subspaces in} \mathbb{C}^n \}.$$

We might also write $G(k,V)$ for k-dimensional linear subspaces of vector space V.

Proposition 53.1. *$G(k,n)$ is a compact complex manifold of dimension $k(n-k)$.*

For $k=1$, we get $G(1,n) \cong \mathbb{P}^{n-1}$. Here is a brief sketch for finding an affine coordinate patch for the Grassmannian. Fix $W \subseteq V$ of dimension k, and we can write $V = W \oplus W'$. Let $G_W = \{U \in G(k,V) | U \cap W' = \{0\}\}$. Then

$$G_W \cong Hom_{\mathbb{C}}(W, V/W) \cong \mathbb{C}^{k(n-k)}.$$

Proposition 53.2. *We have an identification*

$$T_W G = Hom_{\mathbb{C}}(W, V/W).$$

Proof. A useful way to think of this: a tangent vector $v \in T_W G$ is the same as a map $Spec \frac{\mathbb{C}[\epsilon]}{\epsilon^2} \to G$ where on closed points $0 \mapsto [W] \in G$. Then if you take some basis $W = \langle s_1, \cdots, s_k \rangle$, then a tangent vector is associated to $\tilde{s_i} = s_i + \epsilon t_i$. And so $\phi_V : W \to V/W$ is where $s_i \mapsto t_i$.

Imagine you have this W and W' orthogonal complement. A tangent vector is like infiniteismally deforming W, so it is like ϵ nudging the basis vectors for W. □

Proposition 53.3. *On G, there exists a tautological subbundle*

$$0 \to S \to V \otimes \mathcal{O}_G \to Q \to 0.$$

Where over the point $[W] \in G$, the fibers are exactly $W \hookrightarrow V \to V/W$.

Consider $\pi : \mathfrak{X} \to B$, family of compact Kahler manifolds. Fix $k \geq 0$, and construct $\mathcal{L} = R^k \pi_* \mathbb{C}_{\mathfrak{X}}$, $\mathcal{V} = \ \otimes_{\mathbb{C}} \mathcal{O}_X$ with the Gauss-Manin connection ∇_{GM}. Then for every $t \in B$, there is a decreasing Hodge filtration

$$F^p H^k(X_t, \mathbb{C}) \subseteq H^k(X_t, \mathbb{C}) = H^k(X_0, \mathbb{C}) = V,$$

and $F^p H^k(X_t, \mathbb{C}) = \bigoplus_{r+s=k, r \geq p} H^{r,s}(X-t)$. We know that the H^k are fixed by Ehressman's theorem. But these filtration spaces need not be fixed. Last time, we saw that all $h^{r,s}(X_t)$ are constant in t. So we know that

$$f_p := \dim_{\mathbb{C}} F^p H^k(X_t, \mathbb{C})$$

is constant in t. Take $B (= U)$ to be contractible. This gives us a mapping

$$P_p : U \to G(f_p, V)$$

where $V = H^k(X_0, \mathbb{C})$. This is part of the period map of $\pi : \mathfrak{X} \to B$. We know P_p is C^∞, but is it holomorphic? Indeed it is, due to the observation of Philip Griffths.

Recall our main players: we have this map $P_p : B \to G(f_p, V)$. We have

$$P_p^* S = F^p \mathcal{V} \subset \mathcal{V} = P_p^*(V \mathcal{O}_G).$$

where fiber at t of $F^p\mathcal{V}$ is $F^p H^k(X_t, \mathbb{C})$. Note the theorem will imply that $F^p \mathcal{V}$ is a holomorphic subbundle ("hodge filtration on" $\mathcal{V}$). The proof we wil also see the most nontrivial aspect of the Gauss-manin connection, namely Griffths transversality.

Theorem 53.4. *P_p is holomorphic*

Proof. We know that P_p is C^∞. We want to show it is holomorphic. Take $t \in B$, have $(T_tB)_\mathbb{C}$ is complexified tangent space. We want to show that the map

$$dP_{p,t} : (T_tB)_\mathbb{C} \to T_{F^pH^k(X_t,\mathbb{C})}G(f_p, H^k(X;\mathbb{C})$$

satisfies $dP_{p,t}(T^{0,1}) = 0$. This map is

$$dP_{p,t} : (T_tB)_\mathbb{C} \to Hom_\mathbb{C}(F^pH^k(X_t,\mathbb{C}), \frac{H^k(X_t,\mathbb{C}}{F^pH^k(X_t,\mathbb{C})},$$

and $dP_{p,t}(u)(s_t)$ where $s_t \in F^pH^k(X_t,\mathbb{C})$.

We shrunk the picture so much that we have a global form on the total space that restricts to s_t on this fiber and gives us a family of cohomology classes on each fiber. in other words, s_t comes from a section of the bundle $\mathcal{V}$. Fix a section $\tilde{s} : U \to F^p\mathcal{V}$ be a section such that $\tilde{s}(t) = s_t$.

Thinking of Gauss-Manin section as covariant derivative, $\nabla : TB \times \mathcal{V} \to \mathcal{V}$. Have

$$dP_{p,t}(u)(s_t) = \nabla_u(\tilde{s})(t) \mod F^pH^k(X_t,\mathbb{C}).$$

The Cartan-Lie formula tells us that there is $\omega \in A^k(\mathfrak{X})$ such that $\omega|_{X_t}$ is closed and $[\omega|_{X_t}] = \tilde{s}(t)$. For $u \in T_tB$, choose any $v \in T_\mathfrak{X}|_{X_t}$ such that $d\pi(v) = u$. Have $T_\mathfrak{X}|_{X_t} = T_{X_t} \oplus N$.

We have that

$$\nabla_u\tilde{s} = (\nabla_u\omega)(t) = [i_v(d\omega)|_{X_t}] \in H^k(X_t,\mathbb{C})$$

So the Gauss Manin connection is a mechanism for differentiating cohomology classes. In fact, can take $\omega \in F^pA^k(\mathfrak{X})$.

This equation holds for every $u \in (T_bB)_\mathbb{C}$. Now take $u \in T^{0,1}_{B,t}$. Lift v is of type $(0,1)$ along X_t. Have $d\omega \in F^pA^{k+1}(\mathfrak{X})$ and $\iota_v(d\omega) \in F^pA^k(\mathfrak{X})$. So in $T_{[]}G$ is 0.

Summary: looking at this thing, evaluates elements in $F^pH^k(X_t,\mathbb{C})$.. look at (0,1).. preserved by all the operations, so thats why it goes 0. □

The next thing that follows from this calculation is Griffths transversality. Here is a corollary of the proof. Then $dP_p : T_tB \to Hom(F^pH^k, \frac{H^k}{F^pH^k})$ actually takes values in $F^{p-1}H^k/F^pH^k$. Now $u \in T^{1,0}$, have $\iota_v(d\omega) \in F^{p-1}H^k$.

Reformulations:

(1) $F^p\mathcal{V}$ forms a decreasing sequence of holomorphic subbundles of $\mathcal{V}$
(2) The Gauss-Manin connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_B$ has the property $\nabla(F^p\mathcal{V}) \subseteq F^{p-1}\mathcal{V} \otimes \Omega^1_B$.

This is music to the ears of an algebraic geometer. up to now, we basically had these connections with this Leibniz property and $\mathbb{C}$-linearity. But because of this Griffths tranversality property, we can bring it into the world of complex geometry or algebraic geometry. WHen you have a filtration, its natrual to pass to the associated graded of the filtration. So you can consider

$$gr^p_F\mathcal{V} := \frac{F^p\mathcal{V}}{F^{p+1}\mathcal{V}}$$

Get an induced map $\nabla(F^{p+1}\mathcal{V}) \subseteq F^p\mathcal{V} \otimes \Omega^1_B$. Get an induced map.

$$\overline{\nabla} : gr^p_F\mathcal{V} \to gr^{p-1}_F\mathcal{V} \otimes \Omega^1_B.$$

Have $\overline{\nabla}(f \cdot s) = \overline{\nabla(fs)} = \overline{df \otimes s + f\nabla s}$. But $s \in F^p\mathcal{V}$. So the class of $df \otimes s$ is zero. So this equals $\overline{f\nabla(s)} = f\overline{\nabla}(\overline{s})$. So something only $\mathbb{C}$-linear becomes an actual linear over functions. So this $\overline{\nabla}$ is $\mathcal{O}_B$-linear now. So you can actually study this in the realm of algebraic geometry. And we know that

$$(gr_F^p\mathcal{V})(t) = \frac{F^pH^k(X_t,\mathbb{C})}{F^{p+1}H^k(X_t,\mathbb{C})} = H^{p,q}(X_t).$$

Sometimes these $gr_F^p\mathcal{V}$ are called Hodge bundles, call them $\mathbb{H}^{p,q}$. We have these maps

$$\theta_{p,q} : \mathbb{H}^{p,q} \to \mathbb{H}^{p+1,q-1} \otimes \Omega_B^1$$

where $\theta^2 = 0$ and $\mathcal{O}_B$-linear. These are sometimes called Higgs bundles, where people drop the context of variation of Hodge structures and just study these bundles and try to form moduli spaces of them.

Just to recapitulate.. when you have this family of compact complex manifolds... the Higgs operation at each point, its something thats really suprrising when you see it for the first time. For each t, we get that

$$\overline{\nabla}_t : H^q(X_t, \Omega^p_{X_t}) \to H^{q+1}(X_t, \Omega^{p-1}_{X_t}) \otimes \Omega^1_{B,t}.$$

The map drops the degree of the form as a consequence of Griffths tranversality. Would you be able to come up with this map if you knew algebraic operations on sheaves? Play around with this idea. The collection of all these maps are called the infinitesimal variation of Hodge structures at t.

In the last lecture, we'll discuss the derivative of period map as a combination of infinitesimal variations of Hodges structures and the Kodaira spencer map.

Variations of Hodge strucures (VHS). We want to formalize the picture that we explained over the last few lectures. The picture came from geometry, but the picture is in fact completely abstract.

Definition 53.5. Let B be a complex manifold. An integral VHS of weight k on B is a local system of $\mathbb{Z}$-modules, with associated vector bundle $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_B$ with flat connection and a decreasing filtration satisfying two properties:

(1) $\mathcal{V} = F^p\mathcal{V} \cap \overline{F^{k-p+1}\mathcal{V}}$ as C^∞-bundles (conjugate in $\mathcal{L}_{\mathbb{R}} = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}$).
(2) $\nabla(F^p\mathcal{V}) \subseteq F^{p-1}\mathcal{V} \otimes \Omega^1_B$, for every p.

Remark 53.6. Collection of integral Hodge structures of weight k at p, plus Griffths transversality.

Can also define $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ variations of Hodge structure of weight k.

We say that $(\mathcal{L}, F^p\mathcal{V}, \nabla)$ is polarized if there exists a non-degenerate bilinear form

$$Q : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}$$

of parity $(-1)^k$ such that for every $t \in B$,

$$Q_t : V_t \times V_t \to \mathbb{Z}$$

is a polarization of weight k on the Hodge structure V_t.

These Hodge bundles collection the Hodge spaces of teh variation of each fiber. Consider the extremal example, where we go so deep in the filtration we only have one space. So looking at $F^p = \bigoplus_{r \geq p} H^{r,s}(X_t)$, can look at $F^n = H^{n,0}(X_t) = H^0(X_t, \omega_{X_t})$. So one of the Hodge spaces is sections of the canonical bundle. We put them together in a vector bundle. Each fiber is this vector space. So $\mathbb{H}^{n,0}$ is a vector bundle, where each stalk is global sections of canonical bundle. So in fact

$$\mathbb{H}^{n,0} = \pi_*\omega_{\mathfrak{X}/B}.$$

The Hodge bundles all have the interpretation

$$\mathbb{H}^{p,q} = R^q\pi_*\Omega^p_{\mathfrak{X}/B}.$$

This is how Hodge theory applied in the classification of algebraic varieties and higher dimensional geometry. Through pheonena like this. This is how Hodge thoery makes an appearance in the classification of algebraic varieties. non-algebraic information can be mixed to say something about these relative sheaves of differentials.

54. 4/22/24: EXAMPLES

Let's do some examples with the theory we've developed. Consider $\mathbb{C}\setminus\{0\}$. The sheaf where over $U \subset \mathbb{C}^*$ we have

$$Q_\lambda(U) = \{g : U \to \mathbb{C} | z\frac{dg}{dz} = \lambda g\}$$

is in fact a local system of rank 1. The fiber is isomorphic to $\mathbb{C}$. Locally, all such g look like $g = Cz^\lambda$.

Proposition 54.1. *(1) $Q_\lambda \otimes Q_\mu \cong Q_{\lambda+\mu}$.*
(2) $Q_\lambda \cong Q_\mu \iff \lambda - \mu \in \mathbb{Z}$.

Proof. (1) Note that

$$z\frac{dg_\lambda g_\mu}{dz} = z\frac{dg_\lambda}{dz}g\mu + zg_\lambda\frac{dg_\mu}{dz}.$$

(2) In the reverse drection, have $g_\lambda \mapsto z^{u-\lambda}g_\lambda$.In the forward direction, use monodromy. Take γ loop around origin starting at 1 and ending at 1. Then γ acts on $Q_{\lambda,1}$. Have $\gamma : [0,1] \to \mathbb{C}^*$ where $t \mapsto e^{2\pi it}$. Have

$$Q_{\lambda,\gamma(0)} \cong (\gamma^*Q_\lambda)_0 \cong \Gamma(\gamma^*Q_\lambda) \to (\gamma^*Q_\lambda)_1 \cong Q_{\lambda,\gamma(1)}.$$

If you consider the loop γ, and draw neighborhoods U_i which go around to cover γ, in each U_i you have $g = C_i \exp^{2\pi it\lambda}$. Value of intersection is $C_i = C_{i+1}$ for every i. So $C\exp(2\pi 0\lambda)$. γ acts on G_λ by multiplying by $\exp(2\pi i\lambda)$. Look at fundamental representation, have $\exp(2\pi t) : \mathbb{C} \to \mathbb{C}^*$, and $\pi_1(\mathbb{C}^*) \to GL(1,\mathbb{C}) \cong \mathbb{C}^*$. So $M_1(\mathbb{C}^*) \cong \mathbb{C}/\mathbb{Z}$.

□

Now we discuss the Legendre family. This is a family of elliptic curves $\{E_\lambda\}$, where E_λ is defined by the equation

$$y^2 - x(x-1)(x-\lambda).$$

This is a family of elliptic curves over $\mathbb{P}^1$. When $\lambda \neq 0, 1, \infty$, E_λ is a smooth elliptic curve. When $\lambda = 0, 1, \infty$, then E_λ is singular.

If you consider this as a family of smooth elliptic curves over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, you can study the monodromy around the punctured points.

Consider a sufficiently small loop τ around 0, so τ does not go around 1 or ∞. Here we will study monodromy action on E_ϵ where ϵ is very small. Since E_λ is a smooth elliptic curve, it is isomorphic to a donut.

Now you have a 2 to 1 map from $E_\lambda \to \mathbb{P}^1$ where if we are locally thinking of E_λ as being defined by the equation $Y^2 - x(x-1)(x-\lambda)$, then the 2 to 1 map is given by $(x,y) \mapsto x$. This is a ramified map over $0, 1, \lambda, \infty$.

Have $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \cong \mathbb{Z}^2$. Have $T_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $T_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1. \end{pmatrix}$ These generate the representation

$$\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \to GL(2,\mathbb{C}).$$

Now we move onto another example where we work with period maps. For a family of Kahler manifolds $\pi : \mathfrak{X} \to B$, we can define a period map, where if $U \subset B$ is a contractible open subset of the base, we obtain a map

$$P_p : U \to Gr(f_p, V)$$

where $V = H^k(X_t; \mathbb{C})$ and $f_p = \dim F^p H^k(X_t, \mathbb{C})$. Our goal today is to understand what this map actually looks like in the previous example of the Legendre family.

Why is it called the period map? Because the period of a pendulum swinging is related to this map. This is also related to the periods of a compact complex torus (elliptic curve of genus 1). If you have an elliptic curve described by $y^2 = x(x-1)(x-\lambda)$, how do you compute its periods? The periods are given by

$$\tau_1(\lambda) = \int_\delta \omega, \text{ and } \tau_2(\lambda) = \int_r \omega,$$

where $\gamma, r \in H_1(X, \mathbb{Z})$ are loop generators, and we can take ω to be the holomorphic differential $\frac{dx}{y}$.

You can map from th efundamental parallelogram to $E_\lambda \subset \mathbb{P}^2$ by the Weierstrass polynomials, where locally we get $z \mapsto [1 : p(z) : p'(z)]$ where $x = p(z), y = p'(z)$. We see that $\frac{dx}{y} = \frac{p'(z)dz}{p'(z)} = dz$. This is why we recover the periods being

$$\tau_1(\lambda) = \int_0^{\tau_1} dz = \int_\delta \omega, \tau_2(\lambda) = \int_0^{\tau_2} dz = \int_r \omega.$$

This integral

$$\int \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

are elliptic integrals which are related to the period of a pendulum.

Now we can consider the map $\lambda \mapsto (\tau_1(\lambda) : \tau_2(\lambda)) \in \mathbb{P}^1$.

Example 54.2. When $\lambda = -1$, we have the E_{-1} defined by $y^2 = x^3 - x$. We have

$$\tau_1(\lambda) = \int_\delta \frac{dx}{\sqrt{x^3 - x}} = \frac{-i\Gamma(\frac{1}{4})^2}{\sqrt{2\pi}}, \tau_2(\lambda) = \int_r \frac{dx}{\sqrt{x^3-x}} = \frac{\Gamma(\frac{1}{4})^2}{\sqrt{2\pi}},$$

so $-1 \mapsto [1 : i]$.

Let's interpret this map in terms of the abstract period map. Note that we had a holomorphic 1-form $\omega \in H^{1,0} \subset H^1(E_\lambda)$. And if we pick δ, r as our basis for $H_1(E_\lambda, \mathbb{Z})$, then we get a dual basis δ^*, r^* so that

$$\omega = (\int_\delta \omega)\omega^* + (\int_r \omega) r^* = \tau_1 \delta^* + \tau_2 r^*.$$

So we have λ mapsto the 1-dimensional space spanned by ω_λ, a subspace of $H^1(E_\lambda)$. So away from the points $0, 1, \lambda$ we see we have a map

$$\lambda \mapsto [\int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} : \int_r \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}].$$

If you define this on contractible, no ambiguity between choice of isomorphism. But globally, on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ there are monodromy actions just as we saw before. Gets multiplied by those T_0, T_1. Integrating over cycles, value of τ_1, τ_2 will get multiplied.

So you can define a global period map defined away from the singular fibers, where

$$\mathbb{P}^1 \setminus \{0, 1, \infty\} \to \frac{\mathbb{H}}{\Gamma(2)},$$

where $\Gamma(2)$ is generated by $T_0, T_1 \in GL_2(\mathbb{C})$. What happens when you approach $0, 1, \infty$? At singular fibers, this singular cohomology will degenerate, so $H^1(E_\lambda)$ will be 1 dimensional. You can still study the limiting behavior.. way to extend the Hodge structure to singular stuff.

55. 4/24/24: last day, what comes next

Professor Popa wants to spend the last day introducing us to the next step. What people study. When it comes to families of varieties, but more generaly in Hodge theory. One of the most important notions in hodge theory, is just formalizing what we've been discussing for a while. That is the notion of variation of Hodge structures (VHS).

Definition 55.1. Let B be a complex manifold. An integral VHS of weight k on B is a local system of free $\mathbb{Z}$-modules, with associated vector bundle $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O}_B$ with flat connection ∇, plus a decreasing filtration

$$\mathcal{V} = F^0\mathcal{V} \supseteq F^1\mathcal{V} \supseteq \cdots \supseteq F^k\mathcal{V}$$

by holomorphic subbundles, satisfying

(1) $\mathcal{V} = F^p\mathcal{V} \oplus \overline{F^{k-p+1}\mathcal{V}}$ as C^∞ bundles
(2) $\nabla(F^p\mathcal{V}) \subseteq F^{p-1}\mathcal{V} \otimes \Omega^1_B$, for every p.

Remark 55.2. At each $t \in B$, the VHS defines a Hodge structure of weight k :

$$V \otimes_{\mathbb{Z}} \mathbb{C} = V_t = FV_t \supseteq F^1V_t \supseteq \cdots \supseteq F^kV_t \supseteq 0,$$

such that $F^pV_t \oplus \overline{F^{k-p+1}V_t} = V_t$.

Remark 55.3. We can also talk about $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ variations of Hodge structures.

We say $\mathcal{V}$ is polarized if there eixsts a nondegenerate bilinear form

$$Q : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}$$

of parity $(-1)^k$, such that at every $t \in B$,

$$Q_t : V_t \times V_t \to \mathbb{Z}$$

is a polarization on the Hodge structure mentioned in the previous remark. Its a family of polarizations. It can actually be thought of more abstractly as a morphism of Hodge structures.

So you see, that what happens here.. this has content. We made a distinction that Kahler manifolds, cohomology come with polarization. But that's not necessarily rational. The polarization being rational was equivalent to being projective.

The summary of our course. If $\pi : \mathfrak{X} \to B$ is a family of compact Kahler manifolds (or smooth projective) manifolds, then

$$\underline{V} = (\mathcal{L} = R^k\pi_*\underline{\mathbb{Z}}_X, \mathcal{V}, F^p\mathcal{V}, \nabla_{GM})$$

is a $\mathbb{Z}$ variation of Hodge structure on B, polarized if fibers are projective, or $\mathbb{R}$-polarized if fibers are Kahler.

So what do people do next? In general, it is very rare for arbitrary maps $\pi : \mathfrak{X} \to B$ to have always smooth fibers. What happens by generic smoothness or Sard's theorem, is that there exists an open set $U \subseteq B$ over which π is a submersion. So you get a variation of Hodge structure on U.Nowadays, you can extend to B if we replace the VHS by filtered D_B-modules with $\mathbb{Z}$ or $\mathbb{Z}$ structures (called Hodge modules).

The vector bundle $(\mathcal{V}, \nabla)$ on U is replaced by D-module on B. It's exactly what we studied, but we don't require it to be a vector bundle anymore. Just a sheaf. There's still a connection operation

$$\nabla : M \to M \otimes \Omega^1_B$$

but M is not necessarily a vector bundle, it can acquire singularities at the boundary. You can filter $F^p\mathcal{V}$, and the local systems $\mathcal{L}$ are replaced with something called perverse sheaves (they are not actually sheaves).

The fundamental theorem we discussed was that there was a Riemann-Hilbert correspondence between $\mathcal{L}$ local systems and $(\mathcal{V}, \nabla)$. A more advanced Riemann Hilbert correspondence is the following:

$$\{ \text{ regular, holonomic D-modules } \}$$

is in one to one correspondence with

$$\{ \text{ perverse sheaves } \}.$$

The more contemporary version of the Riemann Hilbert correspondence, is a correspondence between the derived category of constructible sheaves and the derived category of regular holonomic D-modules.

Back to $\pi : \mathfrak{X} \to B$ smooth family. Let $\underline{b} = (b_1, \cdots, b_k)$, decreasing sequence of integers. Let $b_p = \dim_{\mathbb{C}} F^pV_t$. For any such $\underline{b}$, and fixed vector space V, we defined the flag variety, a generalization of the Grassmannian, like

$$F_{\underline{b}}(V) := \{(W_1, \cdots, W_k) | V \supseteq W_1 \supseteq W_2 \supseteq \cdots \supseteq W_k, \dim W_j = b_j\}.$$

Grassmannian was just the case of one linear subspace. So $F_{\underline{b}}(V) \subseteq \prod G(b_j, V)$ is a complex submanifold. Incidence correspondence, and so well-behaved under the various projections, that it is a manifold. Here is a fact:

$$T_{\underline{W}}F_{\underline{b}}(V) = \{(u_1, \cdots, u_k) \in \prod_{j=1}^k Hom(W_j, V/W_j) | u_j|_{W_{j+1}} = u_{j+1} \mod W_j\}$$

$(0 \to W_j/W_{j+1} \to V/W_{j+1} \to V/W_j \to 0)$.

If B is contractible, get holomorphic period map. Have

$$P : B \to F_{\underline{b}}(V), t \mapsto (F^p V_t)_{p=1,\cdots,k}.$$

image contained in open set $D \subseteq F_{\underline{b}}(V)$, where $W_p \oplus \overline{W_{p+1}} = V$, called the unpolarized period domain. Understand linear algebra objects, then pull them back. Studying these is like a linearization of problem of studying families of varieties. Has its advantages and disadvantages. Varieties rather different but same Hodge structures, so they aren't distinguished in the image in the period domain.

Let's focus on when you get something substantial in the period domain. Fix p: the derivative of the period map $dP_{p,t}$,

$$dP_{p,t} : T_t B \to Hom(\frac{F^p H^k(X_t, \mathbb{C})}{F^{p+1} H^k(X_t, \mathbb{C})}, \frac{F^{p-1} H^k}{F^p H^k})$$

note the decrease in degree was essentially the content of Griffths transversality, and this derivative is essentially the Gauss-Manin connection. So this map is

$$dP_{p,t} : T_t B \to Hom_{\mathbb{C}}(H^{p,q}(X_t), H^{p-1,q+1}(X_t)).$$

By the Dolbeaut theorem, they are cohomology groups for bundles of forms. We have this looks like $T_t B \to (H^{p,q})^\vee \otimes H^{p-1,q+1}$ and so we see this looks like Gauss Manin connection

$$\overline{\nabla} : H^{p,q}(X_t) \to H^{p-1,q+1}(X_t) \otimes \Omega^1_{B,t}.$$

But recall that by the Dolbeaut theorem that $H^{p,q}(X_t) \cong H^q(X_t, \Omega^p_{X_t})$. Want to rewrite the period map in terms of this interpretation. This is Voisin theorem 10.21. Proposition of Griffths, which does the following. Looks at hte formula of Kodaira Spencer map, and the formula we established from Guass Manin connection and Lie-Cartan formula, combines them, and establishes that $dP_{p,t}$ can be identified with something very basic. So the theorem is that the derivative of the period map can be identified as

$$dP_{p,t} : T_t B \to^{\phi} H^1(X_t, T_{X_t}) \to^{\psi} Hom(H^q(X_t, \Omega^p_{X_t}), H^{q-1}(X_t, \Omega^{p-1}_{X_t})).$$

where ϕ is Kodaira-Spencer map, and ψ is given by

$$\psi : H^1(X_t, T_{X_t}) \times H^q(X_t, \Omega^p_{X_t}) \to^{\text{cup product}} H^{q+1}(X_t, T_{X_t} \otimes \Omega^p_{X_t}) \to H^{q+1}(X_t, \Omega^{p-1}_{X_t}),$$

where the first map is basically cup product (thinking of forms, you're wedging), and the last map is contraction.

So to understand derivative of period map, you need to understand Kodaira spencer map, and cup product pairings of forms. It's hard, but its very algebrogeometric stuff, things that we typically do.

The big problems here.

(1) The Torelli problem. whether the (usually polarized) period map is injective. family of manifolds, just by looking at their polarized hodge structures, can you recover the manifold.
(2) Infinitesimal torelli problem: whether the period map is an immersion (i.e. $dP_{p,t}$ is injective).

The infinitesimal torelli problem can be solved. The first case is that of Calabi Yau manifolds. Deformations of Calabi-Yau's are unobstructed for a fixed calabi-yau X, and there exists locally universal deformation space, i.e.

$$\begin{array}{ccc} \mathfrak{X} & \longleftarrow & X \\ \downarrow & & \downarrow \\ B & \longleftarrow & 0 \end{array}$$

sucht taht kodaira spencer map $T_tB \to H^1(X_t, T_{X_t})$ isomorphism for every $t \in B$. (Tian-Todorov theorem, proved in Huybrechts using analytic methods like Maurer-Cartan).

Suppose we consider the subspace $F^pH^n(X, \mathbb{C}) \subseteq H^n(X, \mathbb{C}) = V$. Suppose X be an n-dimensional Calabi Yau. So $H^{n,0} = H^0(X, \mathcal{O}_X) \cong \mathbb{C}$. So period map P_n goes from $B \to G(1, V) = \mathbb{P}(V)$. Have $dP_{n,t} : T_tB \cong H^1(X_t, T_{X_t}) \to Hom(F^nH^n, F^{n-1}H^n/F^nH^n) = Hom(H^{n,0}(X), H^{n-1,1}(X))$. Have

$$\psi : H^1(X, T_X) \to Hom(H^0(X, \omega_X), H^1(X, \Omega_X^{n-1})),$$

and ω_X is trivial, so $\bigwedge^n \Omega_X^1 = \mathcal{O}_X$, so just choose a form $\alpha \in \bigwedge^n \Omega_X^1$ that is nowhere vanishing. This establishes an isomorphism between

$$T_X \cong \bigwedge^{n-1} \Omega_X^1 = \Omega_X^{n-1}.$$

so basically $H^1(X, T_X) \otimes H^0(X, \omega_X) \cong H^1(X, T_X) \otimes \mathbb{C} \to^{\psi} H^1(X, \Omega_X^{n-1})$, so $\psi(a)(\alpha) = \alpha$ contracted by a. This is one of the most important theorems known for Calabi-Yaus. There is a famous torelli theorem for K3 surfaces. impose extra structure on cohomology in order to get back your K3 surface. Not just hodge structure, plus some bilinear form.

For curves. Let X be smooth projective curve of genus $g \geq 1$. Again, the input here is that the deformations are unobstructed, because $H^2(T_C) = 0$. There exists locally universal family, $k : T_tB \cong H^1(C_t, T_{C_t})$. Have $F^1H^1(X, \mathbb{C}) \subseteq H^1(X, \mathbb{C}) = V$, and $F^1H^1(X, \mathbb{C}) = H^{1,0}(\mathbb{C}) \cong H^0(C, \omega_C) \cong \mathbb{C}^g$. Have period map

$$P_1 : B \to G(g, V), dP_{1,t} : T_tB \cong H^1(C_t, T_{C_t}) \to Hom(H^{1,0}(C_t), H^{0,1}(C_t)).$$

Serre duality implies $H^1(C, T_C) \cong H^0(C, \omega_C^{\otimes 2})$. Have $H^1(C, \mathcal{O}_C) \cong H^0(C, \omega_C)^\vee$ by Serre duality. And note that

$$\psi : H^1(C, T_C) \to H^0(C, \omega_C)^\vee \otimes H^1(C, \mathcal{O}_C).$$

So lemma: ψ is the dual of the multiplication map $H^0\omega_C \otimes H^0\omega_C \to H^0\omega_C^{\otimes 2}$, where $(s, t) \mapsto st$.

Most famous open conjecture about curves, is Green's conjecture. Embedding of curves via canonical embedding. Known for general curves, but otherwise no. This lemma is the first step towards Green's conjecture.

Theorem 55.4 (Max Noether)**.** *ϕ surjective if and only if C is not hyperelliptic (i.e. there does not 2 to 1 map from C to $\mathbb{P}^1$))*

The non hyperelliptic are the "general curves." A corollary of this is that if C is nonhypereliptic, then $dP_{p,t}$ is injective. So we have infinitesimal torelli at such curves.

Voisin expert in torelli problems, studied for hypersurfaces.