6.2 | Orthogonal Sets

Def: A set of vectors $\{u_1, \dots u_p\}$ is orthogonal if $\vec{u}_i \cdot \vec{u}_j = 0 \ \forall\, i \neq j$

Thm: If S $\{\vec{u}_1, \dots \vec{u}_p\}$ is orthogonal s.t. $\vec{u} \neq \vec{0}$ for all i, then S is LI (thus S is a Basis for span S)

Def: An orthogonal basis is a basis that's orthogonal

Thm: Let $\{\vec{u}_1 \dots \vec{u}_p\}$ be an orthogonal basis of $W \subset \mathbb{R}^n$. Then for all $\vec{y} \in W$, the weights in $\vec{y} = c_1\vec{u}_1 + \dots c_p\vec{u}_p$

$$\boxed{c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}} \quad \forall j$$

Note: (1) If $\vec{y} \perp \vec{u}_j$, then $c_j = 0$ (2) If $\|\vec{u}_j\| = 1$, then $c_j = \vec{y} \cdot \vec{u}_j$

ex: Let $S = \left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{pmatrix} \right\} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

Steps (1) Check if it's a basis, by augmenting and if $\neq 0$, basis

check orthogonality: $\vec{u}_1 \cdot \vec{u}_2 = -3 + 2 + 1 = 0$, $\vec{u}_2 \cdot \vec{u}_3 = \frac{1}{2} - 4 + \frac{7}{2} = 0$, $\vec{u}_1 \cdot \vec{u}_3 = -\frac{3}{2} - 2 + \frac{7}{2} = 0$

write $\vec{y} = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$

$c_1\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{pmatrix}$, would augment, but since it's orthogonal use $\boxed{c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}}$

$\vec{y} \cdot \vec{u}_1 = 18 + -8 = 11$, $\vec{y} \cdot \vec{u}_2 = -6 + 2 - 8 = -12$, $\vec{y} \cdot \vec{u}_3 = -3 - 2 - 28 = -33$

$\vec{u}_1 \cdot \vec{u}_1 = 9 + 1 + 1 = 11$, $\vec{u}_2 \cdot \vec{u}_2 = 1 + 4 + 1 = 6$, $\vec{u}_3 \cdot \vec{u}_3 = \frac{1}{4} + 4 + \frac{49}{4} = \frac{33}{2}$ so $\vec{y} = \frac{\vec{y}_1 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} + \dots = \boxed{\vec{u}_1 - 2\vec{u}_2 - 2\vec{u}_3}$

Moral: the c_j's quantify how much of $\vec{y}$ "lies along" the $\vec{u}_j$.

Def: Let $\vec{u} \in \mathbb{R}^n$ & let $L = \text{span}\{\vec{u}\}$. Then the orthogonal Projection of $\vec{y}$ onto L is $\text{Proj}_L \vec{y}$ $:= \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$. Sometimes $\text{Proj}_{\vec{u}} \vec{y}$. Basically multiplying $\vec{y}$ by unit vector $\vec{u}$, $\vec{y} \cdot \left\| \frac{\vec{u}}{\|\vec{u}\|} \right\| \frac{\vec{u}}{\|\vec{u}\|}$

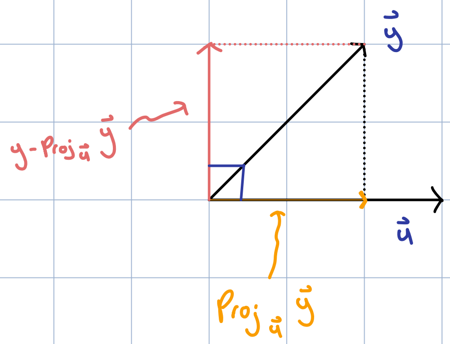
Note: For $\|\vec{u}\| = 1$, $\text{Proj}_L \vec{y} = (\vec{y} \cdot \vec{u})\vec{u}$

ex: $\vec{y} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$ & $\vec{u} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$: compute $\text{Proj}_{\vec{u}} \vec{y} \rightarrow$ $\vec{y} \cdot \vec{u}$: $28 + 12 = 40$ $\quad \vec{u} \cdot \vec{u}$: $16 + 4 = 20$. Thus $\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} = 2$

$\Longrightarrow \text{Proj}_{\vec{u}} \vec{y} = 2\vec{u} = \boxed{\begin{pmatrix} 8 \\ 4 \end{pmatrix}}$ observe: $\vec{u} \cdot (\vec{y} - \text{Proj}_{\vec{u}} \vec{y})$ $\begin{pmatrix} 4 \\ 2 \end{pmatrix} \left(\begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} \right) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 0!$

Geometric

$\downarrow$



i.e, $\vec{u} \cdot (\vec{y} - \text{Proj}_{\vec{u}}\vec{y}) = 0$

Proposition: For any $\vec{y}, \vec{u} \in \mathbb{R}^n$, $\vec{u} \cdot (\vec{y} - \text{Proj}_{\vec{u}}\vec{y}) = 0$, Try to Prove yourself

Def: $\{\vec{u}_1 \dots \vec{u}_p\}$ is orthonormal (ON) if it is orthogonal if $\|\vec{u}_i\| = 1$ $\forall\ 1 \le i \le p$.

ex: $S = \{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$ $\|\vec{u}_1\| = 1 = \|\vec{u}_2\|$. Moreover, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$

$\rightarrow$ $\vec{u}_1$ & $\vec{u}_2$ are $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ & $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ for $\theta = \frac{\pi}{4}$, Thus, rotating $\{\vec{e}_1, \vec{e}_2\}$ by an $\measuredangle\ \theta$ gives an ON set

Fact: Any ON basis for $\mathbb{R}^n$ comes from rotating $\{\vec{e}_1, \dots, \vec{e}_n\}$ (appropriately)

Thm: $U \in \text{Mat}(m,n)$ have ON columns, iff $\Longleftrightarrow U^T \cdot U = I_n$

ex: $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Thm: Let $U \in \text{Mat}(m,n)$ have ON columns & $\vec{x}, \vec{y} \in \mathbb{R}^n$, Then: (a) $\|U\vec{x}\| = \|\vec{x}\|$

(b) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$

(c) $(U\vec{x}) \cdot (U\vec{y}) = 0 \Longleftrightarrow \vec{x} \cdot \vec{y} = 0$

Own: Orthonormality always implies orthogonality between the unit vectors. However, just because 2 vectors are orthogonal, it doesn't mean they both can be normalized

Practice Problems from book

Practice Problems

1. Let $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.
2. Let $\mathbf{y}$ and L be as in Example 3 and Figure 3. Compute the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto L using $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ instead of the $\mathbf{u}$ in Example 3.
3. Let U and $\mathbf{x}$ be as in Example 6, and let $\mathbf{y} = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$. Verify that $U\mathbf{x} \cdot U\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.
4. Let U be an $n \times n$ matrix with orthonormal columns. Show that $\det U = \pm 1$.

1. $\frac{-1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = 0$, so orthogonal ✓

$\sqrt{\left(\frac{-1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{1} = 1$ $\quad$ $\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{1} = 1$ can be normalized ✓

Therefore $\{u_1, u_2\}$ is an orthonormal basis of $\mathbb{R}^2$

2. $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ $\quad \hat{y} = \text{Proj}_{\vec{u}} \vec{y}$ or $\frac{y \cdot u}{u \cdot u} \cdot u$ $\quad$ $y \cdot u = 20$, $u \cdot u = 5$ $\quad 4u = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\quad \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ $\quad \begin{bmatrix} 8 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -8 + 8 = 0$

3. $u = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ $\quad x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ $\quad y = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$ $\quad$ verify $ux \cdot uy = x \cdot y$

$$u \cdot x = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \sqrt{2}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} + 3\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = ux$$

$$u \cdot y = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix} = uy$$

$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -7 \\ 2 \end{pmatrix} = 3 + 7 + 2 = 12 \qquad x \cdot y = \begin{pmatrix} \sqrt{2} \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3\sqrt{2} \\ 6 \end{pmatrix} = -6 + 18 = 12$$

therefore $ux \cdot uy = x \cdot y$

4. Since orthonormal, $U^T \cdot U = I$, so $\det(U^T U) = \det(I)$, ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det = 1$