More Math CH 4 10/18/24

Recall: $\mathbb{P}^n = \{\text{polynomials with } \mathbb{R} \text{ coeffs } \& \deg \le n\}$ is a vector space

Moreover, $\mathbb{P}_n \cong \mathbb{R}^{n+1}$

$\vec{p}(t) = p_0 + p_1 t + \dots p_n t^n \Longleftrightarrow \vec{p} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$

ex: Let $C(\mathbb{R}) := \{\text{continuous functions on } \mathbb{R}\}$ $f(t) = t^2 + 2t$, $g(t) = \sin t$ etc

$f(t) + g(t) = \cancel{t^2 + 2t + \sin t} \in C(\mathbb{R})$

Fact: $C(\mathbb{R})$ is a vector space (And it's infinite dimensional)

Note: $\mathbb{P}_n \subset C(\mathbb{R})$; "$\mathbb{P}_n$ is a subset of $C(\mathbb{R})$"

Def: A subspace of a vector sp. V is a subset $H \subset V$ s.t. ① $\vec{0} \in H$

② if $\vec{u}, \vec{v} \in H$, then $\vec{u} + \vec{v} \in H$. ③ $\forall c \in \mathbb{R}$, $c\vec{u} \in H$

ex: $\mathbb{P}_n \subset C(\mathbb{R})$ is a subspace $\forall n \ge 0$. Let $\mathbb{P} := \{\text{all } \mathbb{R}\text{-coeff Polys}\}$. $\mathbb{P}_n \subset \mathbb{P}$ is a subspace

RMK: $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ in "naive" sense! $\mathbb{R}^2 = \left\{\begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{R}\right\} \cong H := \left\{\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in \mathbb{R}\right\}$

then: $H \subset \mathbb{R}^3$ is a Subspace (must be 0 to remain subspace)

(must have 0 vector to be subspace)

Prop: For $\vec{v}_1, \vec{v}_2 \in V$, $\operatorname{span}\{\vec{v}_1, \vec{v}_2\} =: H$ is a subspace of V.

Proof: ① $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$, so $0 \in H$ ✓

② Let $\vec{u}, \vec{w} \in H$, thus there is a choice of coefficients s.t. $\vec{u} = c_1 v_1 + c_2 v_2$ & $\vec{w} = a_1 v_1 + a_2 v_2$, Compute: $\vec{u} + \vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + a_1\vec{v}_1 + a_2\vec{v}_2 = (c_1 + a_1)\vec{v}_1 + (c_2 + a_2)\vec{v} \in H$, ✓

③ $\alpha\vec{u} = \alpha(c_1\vec{v}_1 + c_2\vec{v}_2) = (\alpha c_1)\vec{v}_1 + (\alpha c_2)\vec{v}_2 \in H$ ✓

Thm: If $\vec{v}_1, \dots, \vec{v}_p \in V$, then $\operatorname{span}\{\vec{v}_1 \dots \vec{v}_p\}$ is a subspace of V

ex: Let $H = \left\{\begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} : a, b \in \mathbb{R}\right\}$ is a subspace of $\mathbb{R}^4$

$\begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ b \\ 0 \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ Thus: $H = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\} \Rightarrow H$ is a subspace of $\mathbb{R}^4$

4.2 Null Spaces and Column spaces

Def: The <u>null space</u> of $A \in Mat(m,n)$ is

$$Nul(A) := \{\vec{x} \in \mathbb{B}^n : A\vec{x} = \vec{0}\}.$$

Thm: For such $A \in Mat(m,n)$, $Nul(A)$ is a subspace of $\mathbb{B}^n$

<u>Proof 1</u>: $A \overset{\mathbb{B}^n}{\overset{\in}{\vec{0}}} = \vec{0} \in \mathbb{B}^m$ (2). Let $\vec{u}, \vec{v} \in Nul(A)$. <u>Then</u>: $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0}$, so $\vec{u}+\vec{v} \in Null(A)$ ✓

(3) For $c \in \mathbb{B}$ & $\vec{u} \in Nul(A)$, $A(c\vec{u}) = cA\vec{u} = \vec{0}$,
so $c\vec{u} \in Null(A)$ ✓