Finish 5.2, start 5.3

Recall: $A = \begin{bmatrix} 5 & -2 & 6 & 7 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightsquigarrow P_A(\lambda) = (5-\lambda)^2(3-\lambda)(-1-\lambda)$

Say that $\lambda = 5$ has multiplicity "$2 = \text{mult}(\lambda)$"

Def: A & $B \in Mat(n,n)$ are similar if there $\exists$ invertible matrix P s.t. $P^{-1}AP = B$

equivalently, $A = PBP^{-1}$

The operation $A \longmapsto P^{-1}AP$ is a <u>similarity transformation</u>

Thm: If $A, B \in Mat(n,n)$ are similar then they have equiv characteristic polynomials $P_A(\lambda) = P_B(\lambda)$

namely, A & B have the same evals

Proof: if $B = P^{-1}AP$ for some inv. P, then $B - \lambda I = P^{-1}AP - \lambda \overset{I}{\overbrace{P^{-1}P}} = P^{-1}AP - P^{-1}\lambda P = P^{-1}(A - \lambda I)P$

$P_B(\lambda) = \det(B - \lambda I) = \det(P^{-1}(A-\lambda I)P) = \det(P^{-1})\underbrace{\det(A-\lambda I)}_{=P_A(\lambda)}\det(P)$ cancel because $\boxed{\det(P) = \frac{1}{\det(P^{-1})}}$ or $\det(P)\cdot\det(P^{-1}) = 1$

rotation matrix @ $\frac{\pi}{2}$

ex: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 \rightsquigarrow \lambda = \pm i$

5.3 Diagonalization

ex: $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \rightarrow D^2 = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} \rightsquigarrow D^k = \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix} \; \forall \; k \geq 1$

ex: Let $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$, then $A = PDP^{-1}$, where $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ & $P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ (DI ys) <u>Note</u>: $P^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

Can we compute A^k? $A^k = (PDP^{-1})^k$, $A^2 = (PD\underbrace{P^{-1})(P}_{I}DP^{-1})$, so PD^2P^{-1}, <u>Repeat</u>: $A^k = PD^kP^{-1}$

$- A^k = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$ so $= A^k = PD^kP^{-1}$

$\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2\cdot 5^k & 5^k \\ -3^k & -3^k \end{pmatrix} = \begin{pmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ -2\cdot 5^k + 3^k & -5^k + 2\cdot 3^k \end{pmatrix}$

ex: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Def: $A \in Mat(n,n)$ is diagonalizable (diagonalizable) if A is similar to a diag. Matrix.

Thm: $A \in Mat(n,n)$ is diagable iff A has n LI EVectors iff [For $A = PDP^{-1}$, the column vectors of P are the LI evectors of A & the entries of D are the eigen values of A] Strong Statement

Corrolary: A is diagable $\iff$ its eigenvectors form a basis for $\mathbb{R}^n$. "Eigenbasis"

ex: If possible, diagonalize $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$

Steps: Compute $\det(A - \lambda I)$ so $\begin{pmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix} = -\lambda^3 - 3\lambda^2 + 4 \to = -(\lambda-1)(\lambda+2)^2$

ie E Vals of A are $\lambda = 1, -2$ twice

Step 2; Solve $E(A,1)$ & $E(A,-2)$

Augment Solve $A - (-1I)$ $A - (-2I)$

$E(A,1) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$ & $E(A,2) = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$

EVecs

EVecs, has 2 unique LI EVecs due to multiplicity of –

Check: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is LI (it is)

Step 3: $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ shows $(\lambda+2)^2$, $P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow A = PDP^{-1}$