4.4 Coordinate Systems

Side note: $[R^{\ell}_{\theta}]_{\{\vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3\}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Thm: Let $\mathcal{B} = \{\vec{b}_1 \cdots \vec{b}_n\}$ be a basis for V. Then $\forall \vec{x} \in V$, there is a unique set of scalars $c_1 \cdots c_n$ s.t. $\vec{x} = c_1\vec{b}_1 + \cdots c_n\vec{b}_n$.

Def: Let $\mathcal{B}$ be a basis for V. The $\mathcal{B}$-coordinates of $\vec{x}$ with weights $c_1, \ldots, c_n$ for which (+) holds is $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

ex: Consider $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$: $\vec{b}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix}1\\2\end{pmatrix}$

Let $\vec{x}$ be s.t. $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix}2\\3\end{pmatrix}$ "Find $\vec{x}$"

$\vec{x} = 2\vec{b}_1 + 3\vec{b}_2 = 2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix}$

Punchline: $\vec{x} = [\vec{x}]_{\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}}$ in practice

ex: Let $\vec{b}_1 = \begin{pmatrix}2\\1\end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix}-1\\1\end{pmatrix}$, & $\vec{x} = \begin{pmatrix}4\\5\end{pmatrix}$. For $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$

Find $[\vec{x}]_{\mathcal{B}}$. Need: $c_1 c_2 \in \mathbb{R}$ s.t. $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$ $\Leftrightarrow$ $\begin{pmatrix}4\\5\end{pmatrix} = c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix} \to \underbrace{\begin{pmatrix}2 & -1\\1 & 1\end{pmatrix}}_{P_{\mathcal{B}}}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$

$P_{\mathcal{B}}^{-1} = \begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3}\end{pmatrix}$

$\Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix} = P_{\mathcal{B}}^{-1}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} = [\vec{x}]_{\mathcal{B}}$

Def: Given $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ $P_{\mathcal{B}} := (\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_n)$ is the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{E}$

Fact: $\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$ & $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x}$

Thm: Let $\mathcal{B} = \{\vec{b}_1, \ldots \vec{b}_n\}$ be a basis for V. Then the cord mapping $B: V \to \mathbb{R}^n$ def. as $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a 1-1 & onto Linear transformation

Def: A linear transformation $T: V \to W$ which is 1-1 & onto is called isomorphism $\Leftrightarrow$ Similar Form

ex: Let $\mathcal{B} = \{1, t, t^2, t^3\}$. $\mathcal{B}$ is a basis for $\mathbb{P}_3$. Send $\vec{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \xmapsto{\sim} [\vec{p}]_{\mathcal{B}} = \begin{pmatrix}a_0\\a_1\\a_2\\a_3\end{pmatrix}$

Thus: $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$

Idea: $\frac{d}{dt} p(t) = a_1 + 2a_2 t + 3a_3 t^2 \xmapsto{\sim} [\vec{p}(t)]_{\mathcal{B}} = \begin{pmatrix}a_1\\2a_2\\3a_3\\0\end{pmatrix}$ Dig: $\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix} = D_t$ (Derivative)