$\boxed{2.2}$ 10/4/24 Linear

Maybe start 2.3

Midterm 10/8

3 Questions

T or F with multiple parts

Solving

Solving & thinking

2.2 Inverse of a matrix

$-7\cdot 2+5\cdot 3$ $-14+15$ $21-21$ $\begin{bmatrix}1&0\\0&\end{bmatrix}$ $-10+10$ $15-$

ex: $A=\begin{bmatrix}2&5\\-3&-7\end{bmatrix}$, $C=\begin{pmatrix}-7&-5\\3&2\end{pmatrix}$

$AC=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ & $CA=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

Ie: A & C are inverses of eachother

If not square, not invertible

Definition: $A\in Mat(n,n)$ is invertible (inv.) if there is some $C\in Mat(n,n)$ such that $AC=I_n$ & $CA=I_n$

We denote such a C as $\boxed{A^{-1}}$

"Determinant (A)"

Theorem: For $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, if $ad-bc\neq 0$, then A is invertible and $A^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$

Theorem: If $A\in Mat(n,n)$ is invertible, then for each $\vec{b}\in\mathbb{R}^n$, $A\vec{x}=\vec{b}$ has the unique sltn. $\vec{x}=A^{-1}\vec{b}$.

ex: $R_\varphi=\begin{pmatrix}\cos\varphi&-\sin\varphi\\\sin\varphi&\cos\varphi\end{pmatrix}$ ~ What is the determinant of $(R_\varphi)=\cos^2\varphi+\sin^2\varphi=1$

counter-clockwise

Thus: $R_\varphi^{-1}=\begin{pmatrix}\cos\varphi&\sin\varphi\\-\sin\varphi&\cos\varphi\end{pmatrix}$ $R_\varphi\cdot R_\varphi^{-1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

clockwise

Thus: $R_\varphi\vec{x}=\vec{b}\Rightarrow\vec{x}=R_\varphi^{-1}\vec{b}$ is the solution.

ex: Solve $\left\{\begin{matrix}3x_1+4x_2=3\\5x_1+6x_2=7\end{matrix}\right\}\Longleftrightarrow\begin{pmatrix}3&4\\5&6\end{pmatrix}\vec{x}=\begin{pmatrix}3\\7\end{pmatrix}$ $18-20$

$\left.\begin{matrix}\det(A)=18-20\\=-2\end{matrix}\right\}\Rightarrow A^{-1}=\frac{-1}{2}\begin{pmatrix}6&-4\\-5&3\end{pmatrix}\rightarrow\begin{pmatrix}6&-4\\-5&3\end{pmatrix}\frac{-1}{2}$

$(\varnothing)\Rightarrow\vec{x}=A^{-1}\begin{pmatrix}3\\7\end{pmatrix}=\frac{-1}{2}\begin{pmatrix}6&-4\\-5&3\end{pmatrix}\begin{pmatrix}3\\7\end{pmatrix}=\frac{-1}{2}\begin{pmatrix}-10\\6\end{pmatrix}=\begin{pmatrix}5\\-3\end{pmatrix}$

$18-28$ $-15+21$ $\begin{pmatrix}-10\\6\end{pmatrix}\frac{-1}{2}$ $\begin{pmatrix}5\\-3\end{pmatrix}$

Theorem: (a) If A is invertible, then $(A^{-1})^{-1} = A$

(b) If A & B are invertible, then AB is invertible & $(AB)^{-1} = B^{-1}A^{-1}$

(c) If A is invertible, then A^T is inv & $(A^T)^{-1} = (A^{-1})^T$

★ A^T is transpose which is making rows columns & columns rows

Note: Generally, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

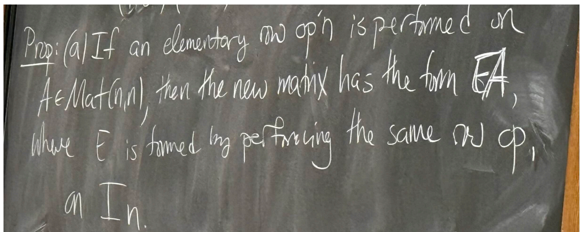
Def: An elementary matrix is one obtained by a single row operation from the identity matrix.

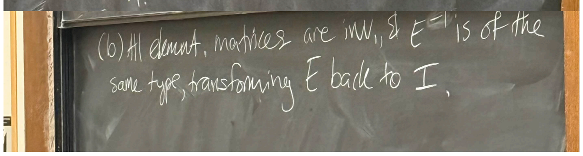
ex: $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ take $2R_1$ and subtract from R_1 $\xrightarrow{R_2 - 2R_1}$ $E_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

$E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$E_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c-2a & d-2b \end{pmatrix}$

$E_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$, $E_3 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 2c & 2d \end{pmatrix}$





Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix}$

Thm: $A \in Mat(n,n)$ is invertible $\Longleftrightarrow$ A is row equivalent to I_n. In that case, the sequence of row operations reducing A to I_n transforms I_n to A^{-1}.

Algorithm: Row Reduce $(A \;\; I_n)$. If a is row equiv. to I, then $(A \; I_n)$ is row equiv. to $(I \;\; A^{-1})$

ex: $(A\,I) = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 4r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right) \cdots \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & \frac{1}{2} \end{array}\right)$

$\boxed{\begin{matrix} -\frac{9}{2} & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & \frac{1}{2} \end{matrix}} = A^{-1}$

everything you do on the left, you do to the right

2.3 Properties of Inv. Matrices

The inverse matrix theorem: Let A be $n \times n$. TFAE (The following are equivalent):

(i) A is inv.

(ii) A is RE to I_n

(iii) A has n pivots

(iv) $A\vec{x} = \vec{0}$ has only the trivial solution

(v) The columns of A are LI

(vi) $\vec{x} \overset{T}{\longleftrightarrow} A\vec{x}$ is 1-1.

(vii) $\vec{x} \overset{T}{\longleftrightarrow} A\vec{x}$ is onto

(viii) $A\vec{x} = \vec{b}$ always has a solution

(ix) There are $n \times n$ C & D such that $CA = I_n = AD$.

(x) A's columns span $\mathbb{R}^n$

(xi) A^T is invertible

Reggie Professor Problems

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13. $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$

$A_{23} = 2 \cdot \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} + 4 \cdot \begin{vmatrix} 3 & 4 & -8 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + 4 \begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix}$

$A_{23} = 2 \cdot \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}$ $\quad A_{22} = 3 \cdot \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}$

$-1 \cdot \begin{vmatrix} 4 & -5 \\ 5 & -3 \end{vmatrix}$ $\quad 2 \cdot \begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix}$ $\quad 8 - 15 = -7 \times 2$

$-12 + 25$

$13 + -14$

$(13 - 14) = -1$

$-1 \times -2 \cdot 3$ $\quad 6$

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