9/10 Linear

$\boxed{1.2}$ Row Reduction & Echelon Form

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$4x_1 - 8x_2 + 12x_3 = 1$$

$$\begin{pmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{pmatrix} \xrightarrow[\text{Reduction}]{\text{Row}} \begin{pmatrix} \boxed{2} & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 0 & 0 & 0 & 15 \end{pmatrix}$$

Def: A matrix is in echelon form if all non-zero rows are above any zero rows;

(2) Each leading entry of a row is in a column to the left of the leading entry of the row below

(3) All entries in a column below a leading entry are 0

A matrix in echelon form or (REF) if:

(4) The leading entry in every non-zero row is a 1;

⑤ Each leading 1, is the ONLY non-zero entry of it's column

ex: $\begin{pmatrix} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix} \leftarrow$ RREF

Def: A Pivot point in a matrix is a location in A corresponding to a leading 1 A's REF. A pivot column is a column containing a pivot posn.

ex: $\left(\begin{array}{ccccc:c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & 9 & 12 & -9 & 6 & 15 \end{array}\right)$

$\xrightarrow[R_3]{R_1}$ $\left(\begin{array}{ccccc:c} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right)$

$\downarrow R_2 - R_1$

$\left(\begin{array}{ccccc:c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right)$

$\frac{R_1}{3}$ & $\frac{R_1}{2}$

$\downarrow$

$$\begin{pmatrix} \boxed{1} & -3 & 4 & -3 & 2 & 5 \\ 0 & \boxed{1} & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

$$\Big\downarrow R_3 - 3R_2$$

$$\begin{pmatrix} \boxed{1} & -3 & 4 & -3 & 2 & 5 \\ 0 & \boxed{1} & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{pmatrix}$$

$$R_1 + 3R_2$$

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$x_1 = -24 + 2x_3 - 3x_4$

$x_2 = -7 + 2x_3 - 2x_4$

$x_5 = 4$

x_3 and x_4 are free variables

You can make them whatever you want

x_1 & x_2 are constrained by x_3 & x_4

however there are ∞ many solutions, parameterized by x_3 & x_4

x_1 & x_2 & x_5 are basic variables

Moral of the story, there are ∞-ly many solutions and are parameterized by x_3 & x_4

Question: For uniqueness of a sltn, what should your REF look like.

$$\begin{matrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \end{matrix}$$

(# of pivot pstns = to # of colums = # of rows)

Consistency means at least 1 solution

Theorem: A linear is consistent if & only if the echelon form of it's augmented matrix has no row of the form $(0\,0 \ldots 0 \;\; b)$ for $b \neq 0$,

If the linear system is consistent, then either: 1. It has no free variables → solution is unique, or 2 It has free variables ⟹ solution is not unique