CH 5.1 eigenvectors & eigenvalues

ex: $A\vec{x} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} = 2\vec{x}$!

Def: An eigenvector for $A \in Mat(n,n)$ is $\vec{x} \neq \vec{0} \in \mathbb{R}^n$ s.t. $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$ called the eigenvalue.

ex: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ has no eigen values in $\mathbb{R}^2$

ex: $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$A\vec{u} = \begin{pmatrix} -24 \\ 20 \end{pmatrix} = \boxed{-4\vec{u}}$

$A\vec{v} = \begin{pmatrix} -9 \\ 11 \end{pmatrix}$ theres no λ s.t $\begin{pmatrix} -9 \\ 11 \end{pmatrix} = \lambda\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Q: is $\lambda = 7$ an eigenvalue of A $\quad A\vec{x} = 7\vec{x} \iff A\vec{x} - 7\vec{x} = 0 \iff (A - 7I)\vec{x} = 0$

Subtract 7 from original matrix

Augment: $\left(\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right) \xrightarrow{\text{Reduce}} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \vec{x} = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ solves $A\vec{x} = 7\vec{x}$!

So $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an evec for the eval $\lambda = 7$

Note: Say you have $A\vec{x} = \lambda\vec{x}$. Then $A(\lambda\vec{x}) = \lambda A\vec{x} = \lambda^2\vec{x}$.

Idea: every vector on the line $\vec{x} = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an evec for A w/ eval $\lambda = 7$

num of free variables is dimension of solution, ex: 1 free variable creates a line

Def: The eigen space of A corresponding to λ is $E(A,\lambda) := Nul(A - \lambda I)$

ex: Above, $E(A,7) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. $\lambda = 2$ is an eigenvalue of A, find a basis for $E(A,2)$

$(A - 2I \,\vdots\, \vec{0})$

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right] \xrightarrow{\text{Reduce}} \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \vec{x} = \begin{pmatrix} \frac{x_2}{2} \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \Rightarrow E(A,2) = \text{Span}\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Thm: The eigenvalues of a triangular matrix are the entries on its diagonal

Proof sketch: $A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$

I.e., $(A - \lambda I \,\vdots\, \vec{0})$ has a nonzero solution iff at least one of the diagonal entries $(a_{ii} - \lambda = 0)$

Thm: 0 is an eval of A iff A is not invertible

Thm: If $\{\vec{v}_1, \dots, \vec{v}_r\}$ are evecs of A corresponding to distinct evals $\{\lambda_1, \dots, \lambda_r\}$

then the set $\{\vec{v}_1 \dots \vec{v}_r\}$ is LI

5.2 The Characteristic Equation

ex: Find the evals of $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} \Leftrightarrow$ find all λ s.t. $(A - \lambda I)\vec{x} = \vec{0}$ has non-$\vec{0}$ solutions

$\Leftrightarrow$ Find all λ s.t. $A - \lambda I$ is not inv.

$\Leftrightarrow$ Find all λ s.t. $\boxed{\det(A - \lambda I) = 0}$

Compute: $\det\begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix}$ $\;-(2-\lambda)(6+\lambda) - 9 = -(12 + 2\lambda - 6\lambda - \lambda^2) - 9 = \lambda^2 + 4\lambda - 21 = 0$

Fact: λ is an eval of A iff $\det(A - \lambda I) = 0$

Nomenclature: $P_A(\lambda) := \det(A - \lambda I)$ is A's Characteristic Polynomial

$(\lambda + 7)(\lambda - 3) = 0$

$P_A(\lambda) = \det(A - \lambda I) = 0$ is A's characteristic equation

$\lambda = -7$, $\lambda = 3$

ex: $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \det(A - \lambda I) = (5-\lambda)^2 (3-\lambda)(1-\lambda).$

important, there are 2