→ Exam 1 Review:

1. Suppose that $A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix}$.

$\in \mathbb{R}^2$ $\quad \in \mathbb{R}^3$ $\quad \in \mathbb{R}^3$

(a) Compute $A\vec{u}$, if possible.

not possible, not enough columns in A

$\vec{u}$ needs to be $\in \mathbb{R}^3$, not $\mathbb{R}^2$

$$\begin{bmatrix} 3 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad 2\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} ? \\ \, \end{bmatrix}$$

(b) Compute $A\vec{x}$, if possible.

$$\begin{bmatrix} 3 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \begin{matrix} 4\cdot 3 & 1\cdot 1 \\ 4\cdot 0 & -1\cdot 1 \\ 4\cdot 1 & 2\cdot 1 \end{matrix} \quad \begin{matrix} 12 & 1 \\ 3 & -1 \\ 4 & 2 \end{matrix} \quad \begin{pmatrix} 12 & 1 \\ 0 & -1 \\ 4 & 2 \end{pmatrix}$$

$$4 + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ -1 \\ 6 \end{bmatrix}$$

* can do dot product or purple way

(c) Represent $3\vec{u} - 4\vec{v}$ as a matrix-vector product.

$$3\vec{u} - 4\vec{v} = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$3\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - 4\begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} - \begin{bmatrix} -4 \\ 12 \\ 24 \end{bmatrix} \qquad \begin{bmatrix} 10 \\ -9 \\ -15 \end{bmatrix}$$

2. Find the solution set to the system of eq'ns:

$$
\begin{aligned}
2x_1 + 4x_2 - 2x_3 + 2x_4 &= 4 \\
2x_1 + x_2 + 4x_3 + 2x_4 &= 1 \\
4x_1 + 6x_2 + x_3 + 2x_4 &= 1
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c} 2 & 4 & -2 & 2 & 4 \\ 2 & 1 & 4 & 2 & 1 \\ 4 & 6 & 1 & 2 & 1 \end{array}\right]
\xrightarrow[R_3 \to -2R_1 + R_3]{R_2 - R_1}
\left[\begin{array}{cccc|c} 2 & 4 & -2 & 2 & 4 \\ 0 & -3 & 6 & 0 & -3 \\ 0 & -2 & 5 & -2 & -7 \end{array}\right]
\xrightarrow{R_3 \to -\frac{2}{3}R_2 + R_3}
\left[\begin{array}{cccc|c} 2 & 4 & -2 & 2 & 4 \\ 0 & -3 & 6 & 0 & -3 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right]
$$

$$
\xrightarrow[-6R_3 + R_2 \to R_2]{2R_3 + r_1 \to R_1}
\left[\begin{array}{cccc|c} 2 & 4 & 0 & -2 & -6 \\ 0 & -3 & 0 & 12 & 27 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right]
\xrightarrow{-\frac{1}{3}R_2}
\left[\begin{array}{cccc|c} 2 & 4 & 0 & -2 & -6 \\ 0 & 1 & 0 & -4 & -9 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right]
$$

$R_2 - 6R_3$

$$
\xrightarrow{R_1 - 4R_2}
\left[\begin{array}{cccc|c} 2 & 0 & 0 & 14 & 30 \\ 0 & 1 & 0 & -4 & -9 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right]
\xrightarrow{R_1/2}
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 7 & 15 \\ 0 & 1 & 0 & -4 & -9 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right]
$$

(a) Find the reduced echelon form of the augmented matrix, call it $[A|\vec{b}]$.

(b) Write set of all solutions in vector form.

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \vec{x} = \begin{bmatrix} 15 \\ -9 \\ 5 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 4 \\ 2 \\ 1 \end{bmatrix}
$$

x_4 is free

$x_1 = 15 - 7x_4$

$x_2 = -9 + 4x_4$

$x_3 = -5 + 2x_4$

(c) If the system is consistent, use your solution to write $\vec{b}$ as a linear combination of the column vectors of A.

Note: may be more than one correct answer.

$$
\begin{bmatrix} 2 & 4 & -2 & 2 \\ 2 & 1 & 4 & 2 \\ 4 & 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}
$$

$$
x_1 \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}
$$

(a) Find the reduced echelon form of the augmented matrix, call it $[A|\vec{b}]$.

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 2 & 4 \\ 2 & 1 & 4 & 2 & 1 \\ 4 & 6 & 1 & 2 & 1 \end{array}\right]$$

Reduces to

$$\begin{bmatrix} 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

equivalent system:

$x_1 - 3x_2 - 2x_4 = 0$

$x_3 - x_4 = 0$

$x_5 = 0$

General form

$x_1 = 3x_2 + 2x_4$

$x_2 = \text{free}$

$x_3 = x_4$

$x_4 = \text{free}$

$x_5 = 0$

vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Note: All linear combinations of $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$$\text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) Write the set of all solutions in vector form.

(c) If the system is consistent, use your solution to write $\vec{b}$ as a linear combination of the column vectors of A.

Note: may be more than one correct answer.

System of linear Eq'ns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

"row picture"

$$[A \mid \vec{b}\,] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

"column picture"

$$A\vec{x} = \vec{b}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

matrix-vector product:

$A\vec{x} = \vec{b}$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

⇾ Test for Consistency: (The following are equivalent)

1. $A\vec{x} = \vec{b}$ is consistent.
2. There is no row of the form $[0 \cdots 0 \mid d]$, $d \neq 0$, in echelon form.
3. The vector $\vec{b}$ is in $\text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$.
4. The vector $\vec{b}$ is a linear combination of the column vectors of A.

3. Consider a homogeneous system of linear equations with corresponding matrix

$$A = \begin{bmatrix} 2 & -6 & 0 & -4 & 2 \\ -3 & 9 & 1 & 5 & 2 \\ 1 & -3 & 1 & -3 & -1 \\ -2 & 6 & 4 & 0 & 4 \end{bmatrix}$$ that reduces to $$R = \begin{bmatrix} 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Write the solution set to the system as a span of a set of vectors.

Reduces to $$\begin{bmatrix} 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

equivalent system:

$x_1 - 3x_2 - 2x_4 = 0$

$x_3 - x_4 = 0$

$x_5 = 0$

General form

$x_1 = 3x_2 + 2x_4$

$x_2 =$ free

$x_3 = x_4$

$x_4 =$ free

$x_5 = 0$

vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

note: All linear combinations of $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$$\text{Span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

4. Is the following set of vectors S linearly dependent?

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Defn: Linearly dependent if one of the vectors can be written as a LC of the others.

That is L.D. $\iff A\vec{x} = \vec{0}$ has more than the solution $\vec{x} = \vec{0}$

LD if $x_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ has a solution where x_1, x_2, x_3 are not all zero

Then $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \frac{-x_2}{x_1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \left(-\frac{x_3}{x_1}\right) \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$

Linearly Independent $\iff$ the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

$$[A \mid \vec{0}] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ -1 & 0 & -2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{\text{EROs}} [R \mid \vec{0}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 = 0$
$x_3 = 0$

The only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}_3$

No, the set is linearly dependent

5. Does the set of vectors S generate $\mathbb{R}^4$?

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} \right\}$$

No, these do not generate $\mathbb{R}^4$

Note: span$\{\vec{v}_1\}$ is a line in $\mathbb{R}^4$ through the origin and the point $(1, 1, -1, 1)$

span$\{v_1, v_2\}$ is a 2-d plane in $\mathbb{R}^4$

span S is a 3-d subspace in $\mathbb{R}^4$

need 4 linearly independent vectors to generate $\mathbb{R}^4$

6. Does the following set of vectors S generate $\mathbb{R}^3$?

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \underbrace{\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}}_{\text{echelon form}}$$

The only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

Thus, they are LI. 3 LI. column vectors in $\mathbb{R}^3$ generate $\mathbb{R}^3$

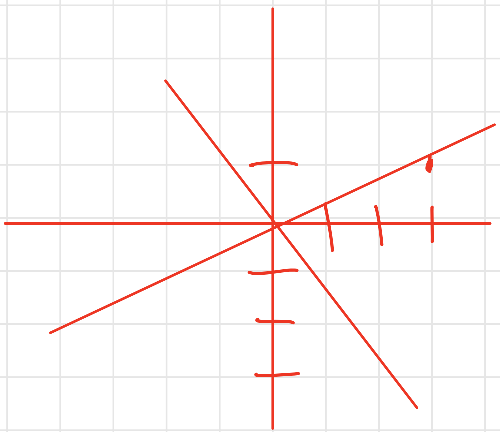
In general, n L.I. vectors in $\mathbb{R}^n$ generate $\mathbb{R}^n$

7. Suppose that $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}$.

Is the equation $A\vec{x} = \vec{b}$ consistent for every $\vec{b} \in \mathbb{R}^2$?

$A\vec{x} = \vec{b}$: $\begin{bmatrix} 3 & 2 & 1 & b_1 \\ 1 & -4 & 5 & b_2 \end{bmatrix} \Longleftrightarrow$ Does $x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

have a slтn for every b_1 & b_2



note $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ generate $\mathbb{R}^2$

yes As $\vec{a}_1$ and $\vec{a}_2$ generate $\mathbb{R}^2$.

every vector in $\mathbb{R}^2$ can be written as

a LC of them

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \to B = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$$

8. Suppose that $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & -3 & 7 \\ -1 & 1 & 2 & -3 \end{bmatrix}$.

Is the equation $A\vec{x} = \vec{b}$ consistent for every $\vec{b} \in \mathbb{R}^3$?

$A \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -6 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ in echelon form

equivalent system: $x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

need pivot in every row to be consistent

no sltn when $b \neq 0$, need 3 L.I vectors

NO it's not

9. The following set of vectors is a linearly dependent set. Write some vector in the set as a linear combination of the others:

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

There exists $x_1\ x_2\ x_3$ not all zero that

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \\ 5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \rightarrow R \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_3 = 0$

$x_2 - 2x_3 = 0$

$x_1 = -3x_3$

$x_2 = 2x_3$

x_3 is free

An example solution: $x_3 = 1$ then $x_1 = -3$ & $x_2 = 2$

$$\begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

10. Let $S = \left\{ \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ -9 \end{bmatrix}, \begin{bmatrix} -6 \\ 8 \\ 12 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ -9 \end{bmatrix} \right\}$.

(a) Does S generate $\mathbb{R}^3$?

$$\begin{bmatrix} 0 & 0 & 3 & -6 & 6 \\ 3 & -6 & -7 & 8 & -5 \\ 3 & -6 & -9 & 12 & -9 \end{bmatrix} \xrightarrow{\text{EROs}} R = \begin{bmatrix} 1 & -2 & 0 & -2 & 3 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

No, it does not

Look up Linear Independent relation to dimension generated

(b) Describe the set Span S geometrically.

2 dimensional plane in $\mathbb{R}^3$

(c) Find a subset of S with the same span as S that is as small as possible.

$$\left\{ \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \left\{ \begin{matrix} -6 \\ 8 \\ 12 \end{matrix} \right\} \right.$$

11. Determine all values of r and s for which the system

$$-x_1 + 3x_2 = s$$
$$4x_1 + rx_2 = -8$$

has (a) no solution, (b) infinitely many sol'ns, (c) exactly one solution.

$$\begin{bmatrix} -1 & 3 & s \\ 4 & r & -8 \end{bmatrix} \xrightarrow{R_2+4R_1} \begin{bmatrix} -1 & 3 & s \\ 0 & r+12 & 4s-8 \end{bmatrix}$$

a. no sltn $\Longleftrightarrow$ a row of the $[0 \; 0 \,|\, d]$ where $d \neq 0$

$r+12=0$ & $4s-8 \neq 0$

$r=-12$ and $s \neq 2$

b. ∞ Solutions when there is a free variable, but consistent system

$\Longleftrightarrow$ $r+12=0$ and $4s-8=0$

$r=-12$ $s=2$

c. Exactly one sltn when so when $r+12 \neq 0$

$r \neq -12$

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right]$$

12. Determine all values of r for which $\vec{v} \in \operatorname{Span} S$ where

$$S = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\} \text{ and } \vec{v} = \begin{bmatrix} r \\ 4 \\ 0 \end{bmatrix}.$$

$\vec{v} = \operatorname{span} S \iff \vec{v}$ is a LC of the vectors in S.

$\iff A\vec{x} = \vec{v}$ has a solution

$$\begin{bmatrix} -1 & 2 & r \\ 1 & -3 & 4 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} -1 & 2 & r \\ 0 & -1 & r+4 \\ 0 & 3 & r \end{bmatrix} \xrightarrow{R_3+3R_2} \begin{bmatrix} -1 & 2 & r \\ 0 & -1 & r+4 \\ 0 & 0 & 4r+12 \end{bmatrix}$$

$r = 3$

13. Determine the values for r for which S is linearly dependent:

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} r \\ 0 \end{bmatrix} \right\}$$

L.D. for all real numbers

because $p > n$

more columns than rows

14. Determine the values for r for which S is linearly dependent:

$$S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ r \end{bmatrix} \right\}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & r \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+2R_1} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 2 & r+1 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & r-7 \end{bmatrix}$$

L.D. if $r-7=0$, $r=7$

15. Let T be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}-3\\1\end{bmatrix},\quad T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right)=\begin{bmatrix}4\\-1\end{bmatrix},\quad T\left(\begin{bmatrix}0\\0\\2\end{bmatrix}\right)=\begin{bmatrix}-6\\3\end{bmatrix}.$$

(a) Find the domain and codomain of T.

Domain of $T = \mathbb{R}^3$ & codomain $(T) = \mathbb{R}^2$ $\quad T: \mathbb{R}^3 \to \mathbb{R}^2$

(b) Find the standard matrix A of T.

$$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & T(\vec{e_3}) \end{bmatrix} = \begin{bmatrix} -3 & 4 & -6 \\ 1 & -1 & 3 \end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}0\\0\\2\end{bmatrix}\right) = T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\0\\2\end{bmatrix}\right) =$$

$$\begin{bmatrix}4\\-1\end{bmatrix} - \begin{bmatrix}-6\\3\end{bmatrix} = \begin{bmatrix}10\\-4\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\left(\tfrac{1}{2}\begin{bmatrix}0\\0\\2\end{bmatrix}\right) = \tfrac{1}{2}T\left(\begin{bmatrix}0\\0\\2\end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix}-6\\3\end{bmatrix} = \begin{bmatrix}-3\\ \tfrac{3}{2}\end{bmatrix}$$

(c) Find the image of the vector $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ under T.

$$T\left(\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right) = T\left(2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= 2T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + 3T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} -3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 10 \\ -4 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 30 \\ -13 \end{bmatrix}$$

(d) Is the range of T equal to the codomain of T?

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} -3 & 10 & -6 \\ 1 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 10 \\ -4 \end{bmatrix} + x_3\begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

yes, As the column vectors of A, generate $\mathbb{R}^2$.