6.3 Orthogonal Projections

recap: Given $\vec{y}, \vec{u} \in \mathbb{R}^n$, $(y - \text{proj}_{\vec{u}}\,\vec{y}) \cdot \vec{u} = 0$

note: $\boxed{\vec{y} = \hat{y} + \vec{z}}$, where $\hat{y} = \text{proj}_{\vec{u}}\,\vec{y}$ & $\vec{z} \perp \hat{y}$.

Idea: $\vec{y}$ can be projected onto larger subspaces!

Thm: Let $W \subset \mathbb{R}^n$ be a subspace, then each $\vec{y} \in \mathbb{R}^n$ (not necessarily W) has the form $\rightarrow$

$\vec{y} = \hat{y} + \vec{z}$, where $\hat{y} \in W$ & $\vec{z} \in W^{\perp}$

In fact, if $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthogonal basis of W, we get $\hat{y} = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}_1\cdot\vec{u}_1}\cdot\vec{u}_1 + \dots \dfrac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}$

In this case, $\vec{z} = \vec{y} - \hat{y}$

Def: $\hat{y}$ or $\text{proj}_W\,\vec{y}$ is the projection of $\vec{y}$ onto W

ex: Let $\vec{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, let $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

☆ Orthogonality means implies Linear Independence

$u_1 \cdot u_2 = 0$
$-4 + 5 - 1 = 0$
orthogonal ✓

Thus: $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal Basis (OB) of $W := \text{Span}\{\vec{u}_1, \vec{u}_2\}$

2. We just compute

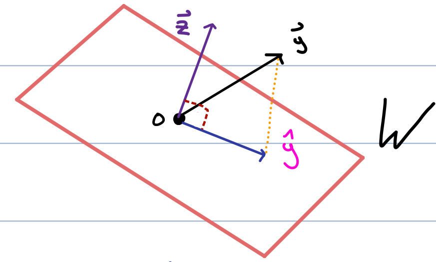
$\hat{y} = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}_1\cdot\vec{u}_1}\cdot\vec{u}_1 + \dots \dfrac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}$

$$\frac{\vec{y}\cdot\vec{u}_1 = 9}{\vec{u}_1\cdot\vec{u}_1 = 30}\cdot\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = \begin{bmatrix} 3/5 \\ 3/2 \\ -3/10 \end{bmatrix}$$

$$\frac{\vec{y}\cdot\vec{u}_2 = 3}{u_2\cdot u_2 = 6}\cdot\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 3/5 \\ 3/2 \\ -3/10 \end{bmatrix} + \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ \frac{1}{5} \end{bmatrix}$$

Thus: $\vec{z} = \vec{y} - \hat{y}$, $\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-2/5\\2\\1/5\end{pmatrix} = \begin{pmatrix}7/5\\0\\14/5\end{pmatrix} = \vec{z}$



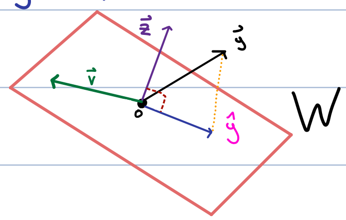
Thus: $\vec{y} = \begin{pmatrix}-2/5\\2\\1/5\end{pmatrix} + \begin{pmatrix}7/5\\0\\14/5\end{pmatrix}$, where these vectors are orthogonal

Prop: If $\vec{y} \in W$, then $\text{proj}_W \vec{y} = \vec{y}$

Note: Projection onto a subspace is a Linear Transformation

(Not tested on)

st Approx Thm: Let $W \subset \mathbb{R}^n$ be a subspace & $\vec{y} \in \mathbb{R}^n$. Then $\hat{y}$ is the "closest" vector in W to $\vec{y}$: i.e., if $\vec{v} \in W$ (distinct from $\hat{y}$), then $\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$



6.4 | Gram-Schmidt Process

Q: Is there a way to make a basis orthogonal? Yes

ex: $\vec{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ Clearly not orthogonal

↪ $S = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is LI so it's a basis of Span $S =: W < \mathbb{R}^4$, a subspace

By design, $\vec{v}_2 \perp \vec{v}_1$

1. Let $\vec{v}_1 = \vec{x}_1$ & span$\{\vec{v}_1\} =: W_1$
2. Let $\vec{v}_2 \overset{\text{-ish}}{=} \vec{x}_2 - \text{proj}_{W_1} \vec{x}_2$

$\begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot v_1 \rightarrow \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \begin{bmatrix}3/4\\3/4\\3/4\\3/4\end{bmatrix} = \begin{bmatrix}-3/4\\1/4\\1/4\\1/4\end{bmatrix}$ or $\frac{1}{4}\boxed{\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}} = \vec{v}_2$

Then: $W_2 := \text{span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-3\\1\\1\\1\end{pmatrix}\right\}$

3. Let $\vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2} \vec{x}_3$

$= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \dfrac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dfrac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \cdot \vec{v}_2$

$\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \dfrac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} : \dfrac{2}{3} \times \vec{v}_2 = \begin{bmatrix}0\\2/3\\2/3\\2/3\end{bmatrix}$

$\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\2/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}0\\-2/3\\1/3\\1/3\end{bmatrix}$ or $\frac{1}{3}\begin{bmatrix}0\\-2\\1\\1\end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ By construction, $\{v_1, v_2, v_3\}$ is orthogonal & a basis for $W := \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Thm: (Gram-Schmidt Process): Given any basis $\{\vec{x}_1, \dots, \vec{x}_p\}$ of $W \subset \mathbb{R}^n$, define

$v_1 = x_1$, $v_2 = x_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} v_1$, $v_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} v_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \cdot v_2 \dots\dots$ $\vec{v}_p = \vec{x}_p - \sum_{i=1}^{p-1} \frac{\vec{x}_p \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} \vec{v}_i$.

Then: $\{\vec{v}_1, \dots, \vec{v}_p\}$ is an orthogonal Basis for W.

Correlary: In this case, $\left\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \dots, \frac{\vec{v}_p}{\|\vec{v}_p\|}\right\}$ is an ONB for W.

6.5) Least Squares

recall: For $W \subset \mathbb{R}^n$ & $\vec{y} \in \mathbb{R}^n$, $\hat{y} := \text{Proj}_W \vec{y}$ is the closest point in W to $\vec{y}$: i.e., $\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$ for any $\vec{v} \in W$

Def: For $A \in Mat(m,n)$ & $\vec{b} \in \mathbb{R}^m$, a least-squares solution (LSS) of $A\vec{x} = \vec{b}$ is $\hat{x} \in \mathbb{R}^n$ s.t. $\|\vec{b} - A\hat{x}\| < \|\vec{b} - A\vec{x}\|$ $\forall \vec{x} = \mathbb{R}^n$

Idea: Consider $\hat{b} := \text{Proj}_{Col(A)}(\vec{b}) \Rightarrow A\vec{x} = \hat{b}$ is consistent i.e., there's some $\hat{x} \in \mathbb{R}^n$ s.t. $A\hat{x} = \hat{b} \rightsquigarrow$ This $\hat{x}$ is the LSS!

Definition of Dot Product

By orthogonal decomposition, $\vec{b} - \hat{b}$ is orthogonal to Col $A \Rightarrow \vec{b} - A\hat{x}$ is orthogonal to each column vector $\vec{a}_j$. $\vec{a}_j^T(\vec{b} - A\hat{x})$. Equiv., $A^T(\vec{b} - A\hat{x}) = 0 \iff A^T A\hat{x} = A^T\vec{b}$

Thm: The set of least-square solutions of $A\vec{x}=\vec{b}$ is the set of solutions to $\boxed{A^TA\vec{x}=A^T\vec{b}.}$

recall: The set of least-square solutions of $A\vec{x}=\vec{b}$ is the set of solutions to $\boxed{A^TA\vec{x}=A^T\vec{b}.}$

"The normal equations"

ex: Find least-squares solution to $A\vec{x}=\vec{b}$, $A=\begin{bmatrix}4&0\\0&2\\1&1\end{bmatrix}$ & $b=\begin{bmatrix}2\\0\\11\end{bmatrix}$

want A^TA & $A^T\vec{b}$ $\begin{pmatrix}4&0&1\\0&2&1\end{pmatrix}\cdot\begin{bmatrix}4&0\\0&2\\1&1\end{bmatrix}=\begin{bmatrix}17&1\\1&5\end{bmatrix}=A^TA$ $\quad\begin{bmatrix}4&0&1\\0&2&1\end{bmatrix}\cdot\begin{bmatrix}2\\0\\11\end{bmatrix}=\begin{bmatrix}19\\11\end{bmatrix}=A^T\vec{b}$

now solve $\begin{bmatrix}17&1\\1&5\end{bmatrix}\hat{x}=\begin{bmatrix}19\\11\end{bmatrix}\xrightarrow[\text{DIY}]{}\hat{x}=\begin{bmatrix}17&1\\1&5\end{bmatrix}^{-1}\begin{bmatrix}19\\11\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

mult by inverse

6.7 Inner Product Spaces

Def: Inner product on V is a function $\langle\,,\rangle: V\times V\to\mathbb{R}$ s.t. for $\vec{u},\vec{v},\vec{w}\in V$ & $c\in\mathbb{R}$.

(1) $\langle\vec{u},\vec{v}\rangle=\langle\vec{v},\vec{u}\rangle$ (2) $\langle\vec{u}+\vec{v},\vec{w}\rangle=\langle\vec{u},\vec{w}\rangle+\langle\vec{v},\vec{w}\rangle$ (3) $\langle c\vec{u},v\rangle=c\langle\vec{u},\vec{v}\rangle$

(4) $\langle\vec{u},\vec{u}\rangle\geq 0$, and $\langle\vec{u},\vec{u}\rangle=0\iff\vec{u}=\vec{0}$. we say: $(V,\langle,\rangle)$ is an inner product space

ex: $(\mathbb{R}^n,\langle,\rangle=\cdot)$ is an IPS

ex: Consider $\mathcal{H}:=C([0,\pi])$: differentiable functions on $[0,\pi]$

consider: $\langle f,g\rangle=\int_0^\pi f(x)g(x)\,dx$

(1) $\langle f,g\rangle=\langle g,f\rangle$ ✓ (2) $\langle f+g,h\rangle=\int_0^\pi (f(x)+g(x))h(x)\,dx=\int_0^\pi f(x)h(x)\,dx+\int_0^\pi g(x)h(x)=\langle f,h\rangle+\langle g,h\rangle$

(3) is familiar $=\int_0^\pi c\cdot f(x)g(x)\,dx=c\int_0^\pi f(x)g(x)\,dx$ (4) $\langle f,f\rangle=\int_0^\pi f(x)^2\,dx\geq 0$

Thm: $\mathcal{H}$ with $\langle f,g\rangle=\int_0^\pi f(x)g(x)\,dx$ is an IPS

Basis of $\mathcal{H}$ is infinite

Q: can we find a basis for $\mathcal{H}$?

Consider: $\langle\cos x,\sin x\rangle=\int_0^\pi\cos x\sin x\,dx=0$ I.e., $\cos x\perp\sin x$ orthogonal

This is true for $\cos(nx)$ & $\sin(nx)$

Moreover: $\left\langle \sqrt{\frac{2}{\pi}}\cos(nx), \sqrt{\frac{2}{\pi}}\cos(mx)\right\rangle = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$

Normalized

Why? $\langle \cos(nx), \cos(nx)\rangle = \frac{\pi}{2}$

Fact: The set $\left\{\sqrt{\frac{2}{\pi}}\cos(nx), \sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}$.

In particular: If $f \in \mathcal{H}$, we can decompose it as $f(x) = \sum_{n=0}^{\infty} a_n\left(\sqrt{\frac{2}{\pi}}\cos(nx)\right) + b_n\left(\sqrt{\frac{2}{\pi}}\sin(nx)\right)$ Foirer Expansion

Consider: $\Delta := \frac{-d^2}{dx^2}$, the "Laplacian". Compute: $\Delta\cos(nx) = \frac{-d}{dx}(-n\sin(nx)) \to \frac{d^2}{dx^2} = n^2\cos(nx) \implies \cos(nx)$ is an eigen vector for Δ with eigenvalue n^2

7.1 Symmetric Matrices

Def: A matrix is symmetric if $A = A^T$ ex: $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ ex: $A = \begin{pmatrix} a_{11} & b & c \\ b & a_{22} & d \\ c & d & a_{33} \end{pmatrix}$

Fact: Any symmetric matrix is square

Thm: If A is symmetric, the evectors corresponding to distinct eigenvalues are orthogonal

Proof: Let $A\vec{v}_1 = \lambda\vec{v}_1$ & $A\vec{v}_2 = \lambda\vec{v}_2$ where $\lambda_1 = \lambda_2$

Then: $\lambda(\vec{v}_1 \cdot \vec{v}_2) = (\lambda_1 v_1)\cdot\vec{v}_2 = (A\vec{v}_1)\cdot\vec{v}_2$, by def $(A\vec{v}_1)^T v_2 = \vec{v}_1^T(A^T \cdot \vec{v}_2) = \vec{v}_1 \cdot A\vec{v}_2 = \vec{v}_1 \cdot \lambda_2\vec{v}_2$

$= \lambda_2(\vec{v}_1\cdot\vec{v}_2) \iff (\lambda_1 - \lambda_2)(\vec{v}_1\cdot\vec{v}_2) = 0$. Thus, $\vec{v}_1 \cdot \vec{v}_2 = 0$. $\square$

Def: $A \in Mat(n,n)$ is orthogonally diagonalizable if $\exists$ orth. matrix P s.t. $A = PDP^{-1}$ for D Diagonal

(equiv:, $AP = PD$)

Spectral Theorem: If $A \in (n,n)$ is symmetrical, then:

(a) A has n eigenvalues (counting multiplicities)

(b) $\dim(E(A,\lambda)) = $ (multiplicity of λ in $P_A(\lambda)$)

(c) The eigenspaces of A are mutually orthogonal

(d) A is orthogonally diagonable

7.2 - ish | Quadratic Forms

ex: Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Compute $\vec{x}^T A \vec{x} = \vec{x} \cdot A\vec{x}$ for $A^1 = \begin{pmatrix} 4 & 0 \\ 3 & 0 \end{pmatrix}$ & $A^2 = \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix}$

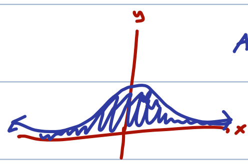
(1) $\vec{x}^T A x = (x_1, x_2)\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} = (x_1, x_2)\begin{pmatrix} 4x_1 \\ 3x_2 \end{pmatrix} = 4x_1^2 + 3x_2^2$

(2) $\vec{x}^T \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix} \vec{x} = (x_1, x_2)\begin{pmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{pmatrix} = 3x_1^2 - 4x_1x_2 + 7x_2^2$

Def: a quadratic form is a function $Q: \mathbb{R}^n \to \mathbb{R}$ of the form $Q(\vec{x}) = \vec{x}^T A \vec{x}$ for A symmetric

A is positive definite if $\vec{x}^T A \vec{x} \geq 0 \quad \forall \vec{x} \in \mathbb{R}^n$

Fact: $\displaystyle\int_{-\infty}^{\infty} \underbrace{e^{-\frac{1}{2}x^2}}_{g(x)} dx = \sqrt{2\pi}$

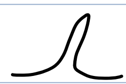


Area under curve $= \sqrt{2\pi}$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = \ldots = 2\pi \Rightarrow I = \sqrt{2\pi}$$

Note: $\displaystyle\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1$

Similarity, $\displaystyle\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx \underset{a>0}{=} \sqrt{\frac{2\pi}{a}}$

Large A:  **small a:** 

How: $\displaystyle\iint_{\mathbb{R}^2} \exp\left[-\frac{1}{2}(ax^2+by^2)\right] dx\,dy = \frac{2\pi}{\sqrt{ab}}$ $\quad a, b > 0$

Note: $a, b > 0$ & $ax^2 + by^2 = (x, y)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$= \vec{x}^T A \vec{x}$ for $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, symmetrical + pos def. $\left(\frac{2\pi}{\sqrt{ab}} = \frac{2\pi}{\sqrt{\det A}} \right)$

Thm: For $A \in Mat(n,n)$ symm + pos. def., $\frac{1}{(2\pi)^{n/2}} \int \cdots \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \vec{x} A \vec{x}\right) d\vec{x} = \frac{1}{\sqrt{\det A}}$

Guess: "C" $\int_{V(\infty \dim)} \exp\left(-\frac{1}{2} \langle \vec{v}, A\vec{v} \rangle\right) \mathcal{D}\vec{v} := \frac{1}{\sqrt{\det A}}$