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Thm: $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is LD if $p>n$.

Pf: let $A = (\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_p)$, an $n \times p$ matrix.

If $p>n$ is greater than n, there must be a free variable! □

ex: $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \end{pmatrix} \right\}$ is LD.

Thm: If $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ contains the zero vector, then S is LD.

Pf: By rearranging the list, we may assume $\vec{v}_1 = 0$. $\Rightarrow 1\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_p = 0$. □

$\boxed{1.8}$ Linear Transformations ⊛

recall: $A \in Mat(m,n)$ is a "machine" with inputs $\vec{x} \in \mathbb{R}^n$ & outputs $A\vec{x} \in \mathbb{R}^m$.

Def: A transformation (or mapping) $T: \mathbb{R}^n \to \mathbb{R}^m$ is a rule assigning to each $\vec{x} \in \mathbb{R}^n$ a vector $T(\vec{x}) \in \mathbb{R}^m$. $\mathbb{R}^n$ is the domain & $\mathbb{R}^m$ is the codomain. $T(\vec{x})$ is the image of $\vec{x}$ & the set of all $T(\vec{x})$ is the range (or image) of T.

Idea: for $A \in Mat(m,n)$, $T(\vec{x}) = A\vec{x}$ is a fcn.

ex: let $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$.

let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be $T(\vec{x}) = A\vec{x}$: i.e.,

$$T(\vec{x}) = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{pmatrix}.$$

Then: $T(\vec{u}) = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$.

Q: Is there $\vec{x} \in \mathbb{R}^2$ s.t. $T(\vec{x}) = \vec{b}$?

$\hookrightarrow$ Solve: $\begin{pmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{pmatrix}$. $(\dagger)$

$\Longrightarrow \vec{x} = \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix}$ is such a vector.

Note: $\vec{x}$ is the only such vector, b/c $(\dagger)$ has no free vars!

Q: Is there $\vec{x} \in \mathbb{R}^2$ s.t. $T(\vec{x}) = \vec{c}$? No!

$\hookrightarrow$ Solve: $\begin{pmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{pmatrix}$.

Bad row

ex: let $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ be $\pi(x) := Ax$, where

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $\rightsquigarrow A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$: i.e. π is a projection of all of $\mathbb{R}^3$ onto the the (x_1, x_2)-plane in $\mathbb{R}^3$!

Def: T is a linear transformation (LT) if

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for $\vec{u}, \vec{v}$ in Domain (T) &

(ii) $T(c\vec{u}) = cT(\vec{u})$ for any scalar $c \in \mathbb{R}$.

Prop: If T is a LT, then: (i) $T(\vec{0}) = \vec{0}$, &

(ii) $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$.

ex: let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be $T(\vec{x}) := r\vec{x}$ for $r \in \mathbb{R}$.

Note: $r\vec{x} = \begin{pmatrix} rx_1 \\ rx_2 \\ rx_3 \end{pmatrix} = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

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ex: let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be $T(x) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$\hookrightarrow T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ & $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

DRAW A PICTURE; PLAY AROUND!