10/22 Agenda

Finish 4.2, ~~Start 4.3~~, Quiz in a week

ex: Let H be the set of $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4$ s.t. $a-2b+5c=d$ & $c-a=b$ $\Longrightarrow$ H is the solution set of $\begin{cases} a-2b+5c-d=0 \\ -a-b+c=0 \end{cases}$

I.e $H = \mathrm{Null}\left(\begin{pmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}\right)$ s.t. H is a subspace of $\mathbb{R}^4$ because 4 columns

Idea: $A\vec{x} = \vec{0}$ to explicitly describe $Nul(A)$

ex: Find spanning set for $Nul(A)$, where $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & 4 & 5 & 8 & -4 \end{bmatrix}$ Set up $[A \vdots 0]$ Row reduce $\rightsquigarrow$ $\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & \vdots & 0 \\ 0 & 0 & 1 & 2 & -2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

Remove free bc technically not vector without it

Solution set to $A\vec{x} = \vec{0}$

$\vec{x} = \begin{pmatrix} x_1 = 2x_2 + x_4 - 3x_5 \\ x_2 \text{ free} \\ x_3 = -2x_4 + 2x_5 \\ x_4 \text{ free} \\ x_5 \text{ free} \end{pmatrix} = x_2\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

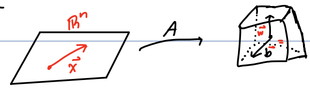
$\Longrightarrow Nul(A) = \mathrm{Span}\{\vec{u}, \vec{v}, \vec{w}\}$ 3 dimensional in $\mathbb{R}^5$

Def: Let $A \in Mat(m,n)$ have the form $A = (\vec{a}_1, \vec{a}_2 \dots \vec{a}_n)$, The column space of A is $Col(A)$: $\mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$

Thm: For $A \in Mat(m,n)$, the column space is a subspace of $\mathbb{R}^m$

ex: $\underset{\substack{\| \\ \mathbb{R}^2}}{Col}\left(\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}\right) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) \to span\{\vec{e_1}, \vec{e_2}\}$ bc $\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\vec{e}_1 + 2\vec{e}_2$

Fact: $Col(A) = \{\vec{b} \in \mathbb{R}^m : \vec{b} = A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n\}$



$A\vec{x}$ is in $col(A)$

Q: If A is $n \times n$ & invertible, what is $col(A)$? $\mathbb{R}^n$, Linearly Independent in $\mathbb{R}^n$, so $\in \mathbb{R}^n$

Q: what about $Nul(A)$? $\vec{0}$ because of homogeneous equation

ex: Find a matrix A s.t. $W = \left\{\begin{pmatrix} 6a-b \\ -a+b \\ -7a \end{pmatrix} : a, b \in \mathbb{R}\right\} = col(A)$

$W = \left\{a\begin{pmatrix} 6 \\ -1 \\ -7 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : a, b \in \mathbb{R}\right\} = \mathrm{Span}\left\{\begin{pmatrix} 6 \\ -1 \\ -7 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\} = Col = \left(\begin{pmatrix} 6 & -1 \\ -1 & 1 \\ -7 & 0 \end{pmatrix}\right) = Col\left(\begin{pmatrix} 6 & 0 & -1 & 2 & 12 \\ -1 & 0 & 1 & -2 & -2 \\ 7 & 0 & 0 & 0 & -14 \end{pmatrix}\right)$

$\mathbb{R}^2 \to \mathbb{R}^3$ making it uglier LD but dif matrices $\mathbb{R}^5 \to \mathbb{R}^3$

Prop: For $A \in Mat(m,n)$, $col(A) = \mathbb{R}^n \Longleftrightarrow A\vec{x} = b$ has a solution $\forall \vec{b} \in \mathbb{R}^m$.

ex: $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & 8 & 6 \end{bmatrix}$ Compute $Nul(A)$ & $Col(A)$ row reduce $\rightsquigarrow$ $\begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $Nul(A) = \left\{\vec{x} = \begin{pmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{pmatrix} : x_3 \in \mathbb{R}\right\}$

Thus: $Nul(A) = \mathrm{Span}\{\langle -9, 5, 1, 0 \rangle\}$

$Col(A) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^3$

Idea: Null space "counts" free variables, Column space "counts" pivot positions