6.1 Inner products

Def: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ Then their <u>inner product</u> or <u>dot product</u> $\vec{u} \cdot \vec{v}$ or $\langle \vec{u}, \vec{v} \rangle$

is: $\vec{u} \cdot \vec{v} := \vec{u}^T v = (u_1, u_2 \cdots u_n)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ u_n \end{pmatrix} = u_1v_1 + u_2v_2 + \dots u_nv_n = \sum_{i=1}^{n} u_iv_i$

ex: $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 2\cdot 3 + \cancel{1\cdot 0} + \cancel{0\cdot 1} = 6$ **Note:** $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 6$

Thm: Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ & $c \in \mathbb{R}^n$, then

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

(c) $c(\vec{v}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

(d) $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0}$

Def: The <u>length</u> (or norm or magnitude) of $\vec{v}$ is $\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots v_n^2}$

equivalently, $\|\vec{v}\| = v_1^2 + \dots + v_n^2$

Fact: $\|c\vec{v}\| = |c|\|\vec{v}\|$, if want to take out scalar/constant, put in absolute value

Def: $\vec{u} \in \mathbb{R}^n$ s.t. $\|\vec{u}\| = 1$ is called a <u>unit vector</u>

ex: $\vec{v} = (1, -2, 2, 0) \rightsquigarrow \|\vec{v}\| = \sqrt{1^2 + (-2)^2 + (2)^2} = 3 \to \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, 0\right)$

Note: $\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \frac{1}{\cancel{\|\vec{v}\|}} \cancel{\|\vec{v}\|} = 1$ Checking that $\|u\| = 1$, compute $\|u\|^2$

Def: $\frac{\vec{v}}{\|\vec{v}\|}$ $(= \vec{u})$ is the <u>normalization</u> of $\vec{v} \in \mathbb{R}^n$ $\frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 0 = 1$

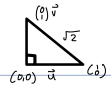
Note: $\text{span}\{\vec{v}\} = \text{span}\left\{\frac{\vec{v}}{\|\vec{v}\|}\right\}$

Def: For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the distance from $\vec{u}$ to $\vec{v}$ is $\text{dist}(\vec{u}, \vec{v})$ or $d(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\| =$

$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2}$ $\text{dist}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$

Def: $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$

ex: $\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = 0$, as expected

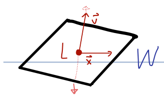
Note: dist$(\vec{u}+\vec{v})$ 

Note: dist$(\vec{u},\vec{v})^2 = \|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}+\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v}$

dist$(\vec{u}+\vec{v}) = \|\vec{u}\| + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$

Thm (Pythagoras): $\vec{u}$ and $\vec{v}$ are orthogonal. $\iff \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

Proof: **ex:** for $W := \{\vec{x} \in \mathbb{R}^3 : x_3 = 0\}$. Consider the line $L := \text{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ Let $\vec{x} \in W$ & $\vec{v} \in L$



Then: $\vec{x}\cdot\vec{v} = 0$

$W = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} \ni x_1\vec{e}_1 + x_2\vec{e}_2 = x$. In this case, W & L are orthogonal

This complements (definition) Notation: $W^{\perp} = L$, $L^{\perp} = W$

Prop: (1) $\vec{v} \in W^{\perp} \iff \vec{v}\cdot\vec{w} = 0 \ \forall \vec{w} \in W$

(2) If $W \subset \mathbb{R}^n$ is a subspace of dimension K, then $W^{\perp} \subset \mathbb{R}^n$ is a subspace of dimension $n-K$

Thm: $A \in \text{Mat}(m,n)$. **Then:**

(1) $(\text{Row}\,A)^{\perp} = \text{Nul}\,A$. (2) $(\text{Col}\,A)^{\perp} = \text{Nul}(A^T)$

Diy: 3x3 & 2x3 cases