10/11/24 Linear

recap: $A, B, \in \text{Mat}(m,n)$ can be added : $(A+B)_{ij} = a_{ij} + b_{ij}$

Theorem: Let $A, B,$ & $C$ have the same size & let $r, s, \in \mathbb{R}$. Then (i) $A+B = B+A$

(ii) $A+(B+C) = (A+B)+C$. (iii) $r(A+B) = rA + rB$ (iv) $(r+s)A = rA + sA$

(v) $r(sA) = (rs)A$. (vi) $A + 0 = A$

Try proof: Hint $A+B = (\vec{a}_1 \dots \vec{a}_n) + (\vec{b}_1 \dots \vec{b}_n) = (\vec{a}_1 + \vec{b}_1 \dots \vec{a}_n + \vec{b}_n)$, etc

## Next Multiplication!

Let $A \in \text{Mat}(m,n)$ & $B \in \text{Mat}(n,p)$, & $\vec{x} \in \mathbb{R}^p$ Then....

$B\vec{x} = x_1\vec{b}_1 + \dots + x_p\vec{b}_p$ is in $\mathbb{R}^n$. $\Rightarrow A(B\vec{x}) = A(x_1\vec{b}_1 + \dots + x_p\vec{b}_p)$

$= x_1 A\vec{b}_1 + \dots + x_p A\vec{b}_p$

Definition: The matrix product of $A \in \text{Mat}(m,n)$ with $B \in \text{Mat}(n,p)$ is

$$AB = (A\vec{b}_1, A\vec{b}_2 \cdots A\vec{b}_p).$$

ex: $A = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}$ $AB = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{pmatrix}$

$(4 \cdot 2) + (1 \cdot 3)$

$\begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \end{pmatrix}$ .... Do the rest $\begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 \\ -9 \end{pmatrix}$

Note: $A\vec{x}$ is itself matrix multiplication

ex: Let $A$ be $3 \times 5$ & $B$ be $5 \times 2$

Columns

$AB \left(\vdots \cdots\right)\left(\vdots \vdots\right) = \left(\vdots \vdots\right)$

Rows

Question: Why is $BA$ not defined? "The middle must match."

Row. Column rule: $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}+b_{2j} + \dots + a_{in}b_{nj}$, where $n =$ # (columns of $A$).

ex: $A = \begin{pmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{pmatrix}$ → what is the 2nd row of $AB$

Row 2 $(A)\, B = (-1 \;\; 3 \;\; -4)\begin{pmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{pmatrix} = (5 \;\; 1)$

$-4+21-12$
$17-12 = 5$

$6+3-8$
$9-8 = 1$

Fact: $\text{row}_i(AB) = \text{row}_i(A)B$.

Fact: $A \in Mat(n,n)$, then $A^k = AAA \dots A$ is defined

Theorem: Let $A$ be $m \times n$ & $B, C$ be products defined

a. $A(BC) = (AB)C$ b. $A(B+C) = AB + AC$

c. $(B+C)A = BA + CA$ d. $r(AB) = (rA)B = A(rB)$

e. $I_m A = (AI_n) = A$ $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Remarks:

(1) In most cases, $\boxed{AB \neq BA}$ important

$5 \cdot 2 + 1 \cdot 4$
$= 14$

$2 \cdot 3 + (4 \cdot \cdot 2)$
$6 \;\; -8$

ex: $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} \Rightarrow AB = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$ so $AB \neq BA$

$BA = \begin{bmatrix} 10 & 2 \\ 29 & -6 \end{bmatrix}$

2: No cancellation laws: ie, $AB = AC$ Does not imply that $B = C$

3. It's possible to have $AB = 0$, where $A \neq 0$ & $B \neq 0$

ex: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Definition: The transpose of $A \in Mat(m,n)$ is $A^T \in Mat(n,m)$ defined by swapping the rows & columns

ex: $A = \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & 6 & -7 & 0 \end{bmatrix} \rightsquigarrow A^T = \begin{pmatrix} 1 & 0 \\ 2 & 6 \\ 3 & -7 \\ 9 & 0 \end{pmatrix}$

<u>Theorem</u>: (a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$

d $(AB)^T = B^T A^T$