4.3, Maybe 4.4, Quiz tuesday

Techically 4.2 definition: A linear transformation $T: V \to W$ is a function that satisfies certain properties (1) $\forall \vec{u}, \vec{v} \in V$, $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

(2) $\forall c \in \mathbb{R}$ & $\forall \vec{u} \in V$, $T(c\vec{u}) = cT(\vec{u})$.

The kernel or null space of the linear transformation of T is $\ker(T) := \{\vec{u} \in V : T(\vec{u}) = \vec{0}\}$ the range of T is $\operatorname{ran}(T) := \{T(\vec{u}) : \vec{u} \in V\}$

Fact: The kernel of T, $\ker(T) \subset V$ is a subspace & $\operatorname{ran}(T) \subset W$ is a subspace

4.3 Bases (Need to know definitions of LI & LD)

Thm: A set $\{\vec{v}_1 \dots \vec{v}_p\}$ with $\vec{v}_1 \neq \vec{0}$, is LD iff some $\vec{v}_j$, $j>1$, is a linear combination of the preceding vectors $\vec{v}_1, \dots, \vec{v}_{j-1}$

ex: $\{\sin(2t), \cos(2t)\} \subset C(\mathbb{R})$. ← LI explanation → $\{\sin(2t), \cos(2t), \sin t \cos t\}$ b/c $\sin(2t) = 2\sin t \cos t$

Def: Let $H \subset V$ be a subspace. $\mathcal{B} := \{\vec{b}_1, \dots, \vec{b}_p\}$ is a basis for H if (i) $\mathcal{B}$ is LI, & (ii) $\operatorname{span}\{\vec{b}_1, \dots, \vec{b}_p\} = H$

ex: $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \subset \mathbb{R}^3$ is a basis for $\mathbb{R}^3$ Called "Standard basis"

Note: A basis for V is a LI set spanning V.

ex: $\left\{\vec{u}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{u}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \vec{u}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ is a LI set ⇒ Basis for $\mathbb{R}^3$!

ex: Let $A \in Mat(n,n)$ be invertible. A's column vectors are LI, so they are a basis for $\mathbb{R}^n$.

Can easily determine if column vectors are a basis for the space by taking determinant since det ≠ 0 means LI

Spanning Set Theorem: Let $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ be a subset of V. & let $H = \operatorname{span} S$. (a) If $\vec{v}_k$ is a lin. combo. of the remaining vectors in S, then $\{S \text{ w/out } \vec{v}_k\}$ spans H

Note: Null set has nothing, not 0

(b) if $H \neq \{\vec{0}\}$, then some subset of S is a basis for H. (Find culprit that's making LD & remove until LI)

ex: $S = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\6\\0\end{pmatrix}\right\} \subset \mathbb{R}^3$. $\operatorname{span} S = \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\6\\0\end{pmatrix}\right\}$

Lin Combos of 3rd vector

Q: how do we get bases for Col(A) & Nul(A)?

For Col(A), find free variable & column it's associated with & remove

ex: $\begin{array}{c} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \end{array}$ ⇝ $Col(B) = \operatorname{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_4\}$ b/c $\vec{b}_3 = 2\vec{b}_1 - \vec{b}_2$ removed $\vec{b}_3$

Note: When A is reduced to B, Col(A) & Col(B) are different (as sets), but $A\vec{x} = \vec{0}$ & $B\vec{x} = \vec{0}$ have same solution sets.

ex: $A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 5 & 5 \\ 2 & 1 & 3 & 2 \\ 5 & 2 & 8 & 8 \end{bmatrix} \\ \begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix} \end{array} \xrightarrow{\text{Reduces to B}} B.$ ↗ b/c $\vec{b}_3 = 2\vec{b}_1 - \vec{b}_2$ $\vec{a}_3 = 2\vec{a}_1 - \vec{a}_2$ → Try at home, row reduce A & see same relationship

⇒ So $Col(A) = \operatorname{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$ & Null spaces are same as sets

Thm: The pivot columns of A are a basis for Col(A)

Def: For $A \in Mat(m,n)$ with row vectors $\vec{r}_1, \vec{r}_2, \vec{r}_m$ (n entries in $\mathbb{R}^n$), $Row(A) = \operatorname{span}\{\vec{r}_1, \dots \vec{r}_m\}$ is the row space of A thm: If A & B are row equivalent then the row space is equal to the row space of B

$$\text{Row}(A) = \text{Row}(B)$$