9/17/24

Finish 1.4 & 1.5

Recap, $A \in$ Matrix(m,n) is a function which takes $\vec{x} \in \mathbb{R}^n$ & spits out $A\vec{x} \in \mathbb{R}^m$.

Theorem: If $A \in$ Matrix(m,n) of the form $(\vec{a}_1\ \vec{a}_2 \dots \vec{a}_n)$ & $\vec{b} \in \mathbb{R}^m$, then $A\vec{x} = \vec{b}$ has the same solution set as the linear system with Augmented matrix $(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\ \vec{b})$.

Correlary: $A\vec{x} = \vec{b}$ has a solution If & only If $A\vec{x} = \vec{b}$
$\Longleftrightarrow \vec{b} \in \text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$.

Reminder: Span$\{a_1, a_2, \dots a_n\}$ is the set of all linear combinations of those vectors

ex: $A = \begin{pmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ 3 & -2 & -7 \end{pmatrix}$ & $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, when is $A\vec{x} = b$ consistent

$\left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ 3 & -2 & -7 & b_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 - 3b - \frac{1}{2}b_2 - 2b_1 \end{array}\right)$

$(b_3 - \frac{1}{2}b_2 - 5b_1)\ (+)$

Answer: whenever $(+) = 0$

Theorem: Let $A \in$ Mat(m,n), The following are equivalent:

a. For all $b \in \mathbb{R}^m$, $A\vec{x} = b$ has a solution.

b. Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of A's column vectors.

c. A column's vectors span $\in \mathbb{R}^m$.

d. A has a pivot position in every row

Shortcut: For $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, their dot product is $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 + \dots u_n \cdot v_n$

Then if we write $A = \begin{pmatrix} R_1 \\ \vdots \\ R_m \end{pmatrix}$, Then $A\vec{x} = \begin{pmatrix} \vec{R}_1 \cdot \vec{x} \\ \vec{R}_2 \cdot \vec{x} \\ \vec{R}_3 \cdot \vec{x} \end{pmatrix}$

ex: $\begin{pmatrix} 1 & -1 & 4 \\ 0 & 0 & 6 \\ 2 & 3 & 9 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \\ 25 \end{pmatrix}$

3 colums = 3 Rows ✓

DO AT Home: Multiply out like Shown before

ex: $\begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ ✓ num of col = num of rows

$2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} -2 \\ 5 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Very important

Thm: If $A \in Mat(m,n)$, $\vec{u}, \vec{v}$ are in $\mathbb{R}^n$, & $c \in \mathbb{R}$, then

(a) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$. (B) $A(c\vec{u}) = c(A\vec{u})$

$\uparrow$ constant

READ & Understand Proof

1.5. Solution sets of LS

Definition: A liner system is called homogenous if it's vector equation is of the form $A\vec{x} = \vec{0}$

Remark: $A\vec{x} = 0$ Always has the trivial solution, $\vec{x} = \vec{0}$, but we must endeavor to find a nontrivial answer too?

Fact: $A\vec{x} = \vec{0}$ has a nontrivial solution if & only if you have a free variable

ex: $\underbrace{\begin{pmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & 8 & 0 \end{pmatrix}}_{A} \xrightarrow[R_3 - 2R_1]{R_2 + R_1} \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} x_1 = \frac{4}{3}x_2 \\ x_2 = 0 \\ x_3 \text{ is free} \end{cases}$

$\vec{x} = x_3\begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{pmatrix}$

FACT: If a LS has 2 free vars, then the solution set looks like

$\vec{x} = s\vec{u} + t\vec{v}$, where s, t are the free vars ex: Describe solution set of $A\vec{x} = \vec{b}$ for $A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}$, $\vec{b}\begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \vec{x} = \begin{pmatrix} -1 + \frac{4}{3}x_3 \\ 2 + 0x_3 \\ 0 + x_3 \end{pmatrix} \to$$

$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}$$

Note: Sltn to $A\vec{x} = \vec{b}$ has the form $\boxed{\vec{x} = \vec{p} + t\vec{v}}$.

Thus: Solutions to $A\vec{x} = \vec{b}$ are obtained by computing solutions to $A\vec{x} = \vec{0}$ & adding the vector $\vec{p}$.