Finish 4.5 Maybe Start CH5

Thm: If V has a basis $\{\vec{b}_1, \dots, \vec{b}_n\}$, then any set in V w/ more than n vectors is LD

Thm: If a basis of V has n vectors, then every basis of V has n vectors

Def: If V is spanned by a finite set of vectors, we say that V is finite dimensional or (Fin.dim.). In this case, the dimension of V, $\dim V$, is the $\#\{\text{basis vectors}\}$

ex: $\dim\{\vec{0}\} = 0$ Linearly Dependent, so no basis ex: $\dim \mathbb{R}^n = n$. $\dim \mathbb{P}_n = n+1$

2 dim in 4 dim

ex: Let $H = \text{Span}\left\{\begin{pmatrix}1\\0\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\} \subset \mathbb{R}^4$, $\dim H = 2$

Thm: Let $H \subseteq V$ be a subspace, where V is Fin dim. Then any LI set in H can be extended to a basis of V, & $\dim H \leq \dim V$ because V is a subspace of itself. Important

Basis Thm: Let $\dim V = p \geq 1$. Any LI set in V w/ p elements is a basis for V.

Def: $\text{rank}(A) := \dim \text{Col}(A)$, $\text{nullity}(A) := \dim \text{Nul}(A)$

ex: $A \overset{RREF}{\sim} \begin{pmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{pmatrix}$ → Rank$(A) = 2$ → Nullity$(A) = 1$ → Note: Rank(A) + Nullity$(A) = \#\{\text{Columns of } A\}$

Rank-Nullity Thm: For $A \in \text{Mat}(m,n)$, $\text{rank}(A) + \text{nullity}(A) = n$

ex: Let $A \in \text{Mat}(17,17)$. If num pivots of $A = 10$, how many basis vectors does $\text{Nul}(A)$ have?

$\#\{\text{Pivots of } A\} = \dim \text{Col}(A) = \text{rank}(A) = 10$, Thus, $10 + \dim \text{Nul}(A) = 17 \implies \text{nullity}(A) = 7$

if A is Inv Col space is whole thing

If inv. null space is 0

Invertible Matrix Theorem Continued: TFAE: (n+1) $\text{Col} A = \mathbb{R}^n$, (n+2) $\text{rank}(A) = n$, (n+3) $\text{nullity}(A) = 0$, (n+4) $\text{Nul}(A) = \{\vec{0}\}$

$\boxed{4.6}$ Change of Bases

ex: Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ & $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for V, s.t. $\vec{b}_1 = 4\vec{c}_1 + \vec{c}_2$ & $\vec{b}_2 = -6\vec{c}_1 + \vec{c}_2$

Let $\vec{x} = 3\vec{b}_1 + \vec{b}_2$: i.e. $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Find $[\vec{x}]_{\mathcal{C}}$, First, $[\vec{x}]_{\mathcal{C}} = [3\vec{b}_1 + \vec{b}_2]_{\mathcal{C}} \to 3[\vec{b}_1]_{\mathcal{C}} + [\vec{b}_2]_{\mathcal{C}}$

equivalently, $[\vec{x}]_{\mathcal{C}} = \left([\vec{b}_1]_{\mathcal{C}} \; [\vec{b}_2]_{\mathcal{C}}\right)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ Take inverse to write $\mathcal{C}$ in $\mathcal{B}$ coordinates

Thm: For Bases $\mathcal{B} = \{\vec{b}_i\}_{i=1}^{n}$ & $\mathcal{C} = \{\vec{c}_i\}_{i=1}^{n}$, of V, there is a unique matrix $\boxed{\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}}$ s.t.

$[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{x}]_{\mathcal{B}}$ Corollary $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P^{-1}}$ And: $\left(\vec{c}_1 \ldots \vec{c}_n \,\vdots\, \vec{b}_1 \ldots \vec{b}_n\right) \underset{\sim}{\text{Reduce}} \left(I_n \,\vdots\, \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)$

ex: $\mathcal{B} = \left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}\right\}$ & $\mathcal{C} = \left\{\begin{pmatrix} -7 \\ 9 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \end{pmatrix}\right\}$ Find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$ & $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$

Augment: $\left(\begin{array}{cc:cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right) \underset{\sim}{\text{RREF}} \left(\begin{array}{cc:cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right)$ implies $\Rightarrow \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$

$(\ldots)^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P^{-1}}$