Linear 9/27/24

1. 1.8 rapid Recap
2. 1.9

1.8 recap

Moral: given a matrix A, $\in (m,n)$ we can define a function | mapping | transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ as $T(\vec{x}) = A\vec{x}$

Prop: such a linear transformation satisfies: ① $T(\vec{0}) = \vec{0}$.

② For $\vec{v}, \vec{w} \in \mathbb{R}^n$ & $c, d \in \mathbb{R}$,

$$T(c\vec{v} + d\vec{w}) = cT(\vec{v}) + dT(\vec{w})$$

$\boxed{1.9}$ The matrix of a LT

Recall: $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, etc., $\boxed{\text{Standard basis vectors}}$

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a LT. Then there is a unique (!) A, such that, $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. In particular, $A \in Mat(m,n)$ whose j^{th} column is the vector $T(\vec{e}_j)$;

$$A = (T(\vec{e}_1)\, T(\vec{e}_2) \cdots\cdots T(\vec{e}_n))$$

Note: A is the "standard matrix" for T.

ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be rotated $\measuredangle$ $0 \le \varphi < 2\pi$

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$ & $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \Longrightarrow$ I.e., T's standard matrix is $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

Def: $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto if for each $\vec{b} \in \mathbb{R}^m$, there is an $\vec{x} \in \mathbb{R}^n$, such that, $T(\vec{x}) = \vec{b}$

ex: Projection!

$T: \mathbb{R}^3 \to \mathbb{R}^2$ defined as $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is onto. (Think like a Photo)

ex: $T\begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

other way: $\begin{pmatrix} a \\ b \end{pmatrix} = T\begin{pmatrix} a \\ b \\ x \end{pmatrix}$

↑ free variable

Def: $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each $\vec{b} \in \mathbb{R}^m$ has at most one $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$

ex: Any rotation of $\mathbb{R}^2$ is one to one

Fact: $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{x}) = \vec{x}$ is both one to one & onto

ex: Is $T(\vec{x}) = \begin{pmatrix} \boxed{1} & -4 & 8 & 1 & 1 \\ 0 & \boxed{2} & -1 & 3 \\ 0 & 0 & 0 & \boxed{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ Q's 1-1? NO onto? Yes

$\square$ = pivot

Free ⇓ Not 1-1, bc many vectors in $\mathbb{R}^4$ will map to the same one in $\mathbb{R}^3$

BC there are 3 pivot positions & 3 rows, T is onto!

Meditate on this

Theorem: $T: \mathbb{R}^n \to \mathbb{R}^m$ is one to one, if & only if $T(\vec{x}) = \vec{0}$ having only the solution $\vec{x} = \vec{0}$

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a LT with a standard matrix, A.

Then (a) T is onto $\Longleftrightarrow$ the columns of A span $\mathbb{R}^m$.

(b) T is one to one $\Longleftrightarrow$ the columns of A are Linearly Independent

$\boxed{2.1}$ Matrix Operations

Def: (1) For $A \in \text{Matrix}(m,n)$, "a_{ij}" denotes the entry of A in its i^{th} row & j^{th} column.

(2) The diagonal entries are a_{11}, a_{22}, a_{33}, etc

$$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \leftarrow \text{Diagonal Matrix}$$

Fact: If $A, B \in \text{Mat}(m,n)$, then $A+B$ has entries $a_{ij} + b_{ij}$.

ex: $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+1 \\ -1-1 & 2+3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 5 \end{pmatrix}$

scaling: $7\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$ $\begin{pmatrix} 7 & 0 \\ -7 & 14 \end{pmatrix}$