2.1

Practice Problems

1. Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?

2. Let A be a 4×4 matrix and let $\mathbf{x}$ be a vector in $\mathbb{R}^4$. What is the fastest way to compute $A^2\mathbf{x}$? Count the multiplications.

3. Suppose A is an $m \times n$ matrix, all of whose rows are identical. Suppose B is an $n \times p$ matrix, all of whose columns are identical. What can be said about the entries in AB?

$(Ax)^T = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ $\quad 5\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 3\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \to \begin{bmatrix} -4 & 2 \end{bmatrix}$

$x^T A^T = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix}$ $\quad 5\cdot 1 + -3\cdot 3 \quad -2\,5 + 4\cdot 3$

$x \cdot x^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$

$x^T \cdot x = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 34 \end{bmatrix}$

$A^T x^T = 2 \times 2 \neq 1 \times 2$ so undefined

Fastest way to compute A^2x is to square A, then multiply by x

$m \times n * n \times p$ the entries fit in an $m \times p$ matrix and all are same

2.2

Practice Problems

1. Use determinants to determine which of the following matrices are invertible.

a. $\begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix}$

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

3. If A is an invertible matrix, prove that $5A$ is an invertible matrix.

$6(3) - -9(2) = 18 + 18 = 36$ ✓ Invertible

$4(5) + 9(0) = 20$ ✓ invertible

$6(6) - -9(-6) = 36 - 36 = 0$ Not invertible

$A_{11} = 1\begin{bmatrix} 5 & 6 \\ -4 & 5 \end{bmatrix} + 2\begin{bmatrix} -1 & 6 \\ 5 & 5 \end{bmatrix} - 1\begin{bmatrix} -1 & 5 \\ 5 & -4 \end{bmatrix}$

$25 - 6(-4)$ $-5 - 30$ $4 - 25 = -21$

$49 - 70 + 21 = 0$ non invertible

det(5·A)
"

A is invertible, so det $\neq 0$. det of $5A$ is just $5^n \cdot \det A$ and still Inv

2.3

Practice Problems

1. Determine if $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix}$ is invertible.
2. Suppose that for a certain $n \times n$ matrix A, statement (g) of the Invertible Matrix Theorem is *not* true. What can you say about equations of the form $A\mathbf{x} = \mathbf{b}$?
3. Suppose that A and B are $n \times n$ matrices and the equation $AB\mathbf{x} = \mathbf{0}$ has a nontrivial solution. What can you say about the matrix AB?

1. Non invertible, rows & columns are same

inconsistent for atleast one b in $\mathbb{R}^n$

not invertible

3.1

Practice Problem

Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$.

$A_{13} = 2 \begin{bmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{bmatrix}$

$A_{21} = +5 \begin{bmatrix} 3 & -4 \\ 5 & -6 \end{bmatrix}$

$-18 - -4(5)$

$-18 + 20 = 2 \cdot 5 \cdot 2 = \boxed{20}$

3.2

Practice Problems

1. Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$ in as few steps as possible.

2. Use a determinant to decide if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, when

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

3. Let A be an $n \times n$ matrix such that $A^2 = I$. Show that $\det A = \pm 1$.

$R_2 - 2R_1$: $\begin{pmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{pmatrix}$ $R_3 + 3R_1$ $\begin{pmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{pmatrix}$ R_1 & R_4 are identical & non invertible

det = 0

$\begin{pmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{pmatrix} \begin{matrix} 5 & -3 \\ -7 & 3 \\ 9 & -5 \end{matrix}$

$(75 + 189 + 70) - (54) - (175) - (105)$

$189 + 145 = 334$

$334 - 54 = 280$

$280 - 175 = 105$

$(105 - 105) = 0$ LD

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$1 + 0 \quad 1 \quad 0 \; 0 \; 0 = 0$

$0 + 1 \quad 0 \quad 1$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{1}$

4.1

Practice Problems

1. Show that the set H of all points in $\mathbb{R}^2$ of the form $(3s, 2+5s)$ is not a vector space, by showing that it is not closed under scalar multiplication. (Find a specific vector $\mathbf{u}$ in H and a scalar c such that $c\mathbf{u}$ is not in H.)
2. Let $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, where $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V. Show that $\mathbf{v}_k$ is in W for $1 \le k \le p$. [*Hint:* First write an equation that shows that $\mathbf{v}_1$ is in W. Then adjust your notation for the general case.]
3. An $n \times n$ matrix A is said to be symmetric if $A^T = A$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times 3}$, the vector space of 3×3 matrices.

$$V_K = 0v_1 + \dots + 0v_{K-1} + 1v_K + 0v_{K+1} + \dots + 0v_p$$

4.2

closed under scalar mult & vect add & 0 vector

Practice Problems

1. Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b - c = 0 \right\}$. Show in two different ways that W is a subspace of $\mathbb{R}^3$. (Use two theorems.)
2. Let $A = \begin{bmatrix} 7 & -3 & 5 \\ -4 & 1 & -5 \\ -5 & 2 & -4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$. Suppose you know that the equations $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{x} = \mathbf{w}$ are both consistent. What can you say about the equation $A\mathbf{x} = \mathbf{v} + \mathbf{w}$?
3. Let A be an $n \times n$ matrix. If Col A = Nul A, show that Nul $A^2 = \mathbb{R}^n$.

$a = 3b + c$

$$\begin{bmatrix} 3b+c \\ b \\ c \end{bmatrix} \quad b\underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + c\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$

$W = \text{Span}\{V_1 \; v_2\}$

Both in col space

Nul $A = \mathbb{R}^n$

so Nul$A^2 = \mathbb{R}^n$

4.3

Practice Problems

1. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace W spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$. Then every vector in H is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

not in H somehow

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H? NO

4. Let V and W be vector spaces, let $T : V \to W$ and $U : V \to W$ be linear transformations, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a basis for V. If $T(\mathbf{v}_j) = U(\mathbf{v}_j)$ for every value of j between 1 and p, show that $T(\mathbf{x}) = U(\mathbf{x})$ for every vector $\mathbf{x}$ in V.

1. $$\begin{bmatrix} 1 & -2 \\ -2 & 7 \\ 3 & -9 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 3 & -9 \end{bmatrix} \xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & -3 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

Not a basis for $\mathbb{R}^3$, but a basis for span $\{v_1\ v_2\} \in \mathbb{R}^3$

2. $$\begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \xrightarrow{R_2+3R_1} \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 4 & -1 & 3 & 9 \end{bmatrix} \xrightarrow{R_3-4R_1} \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix}$$

$\downarrow$ $R_2/4$ & $R_3/5$

$$\begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3+R_2} \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & -5 & -1 & 5 \end{bmatrix}$$

Col A = W

$\{v_1\ v_2\}$ = basis for W

4.4

Practice Problems

1. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.
   a. Show that the set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$.
   b. Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.
   c. Write the equation that relates $\mathbf{x}$ in $\mathbb{R}^3$ to $[\mathbf{x}]_{\mathcal{B}}$.
   d. Find $[\mathbf{x}]_{\mathcal{B}}$, for the $\mathbf{x}$ given above.
2. The set $\mathcal{B} = \{1 + t, 1 + t^2, t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 6 + 3t - t^2$ relative to $\mathcal{B}$.

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

Change of coordinates matrix

$$c_1 - 3c_2 + 3c_3 = 0$$
$$4c_2 - 6c_3 = 0$$
$$3c_3 = 0 \qquad c_2 = 0$$

$$\begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

$$c_3 = 0 \qquad c_2 = 0$$

$$c_1 = 0$$

$$\begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}$$

$$6(1+t) + 3(1+t^2) + -1(t+t^2)$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 6 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

How to solve every problem

1. Matrix multiplication: ▭ ▯ $A_i \cdot B_j$ is position of product in $A \cdot B$ matrix

Transpose: swap rows & columns

2. Inverse: If 2x2 → $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, if > 2x2, Augment w identity matrix & reduce

Use determinant to easily determine invertibility

3.
Determinants: Cofactor expansion, Product of main diagonal if ↑ or ↓ triangular

Could try sarrus rule to verify, remember alternating, can check based on pstn.

4. Null space: Augment matrix with 0, write in terms of free variables after reducing, Coefficients is null space

or checking if a vector is in the null space, multiply by it, and if result is 0's, in null space

5. Col space: Reduce to RREF, identify location of pivots, write Col A = Span { vectors of pivot positions from original matrix }

Row space same but rows

6. Basis: Similar to col space, reduce, find location of pivots, Basis of Col A = span { vectors of pivot positions from original matrix }

Make sure LI, can be reduced if a basis vector is an LC of another

7. Coordinate systems: Mostly coefficients, Augment basis vectors with with x & solve for scalars, tells coordinates

8. Rank Nullity: $\text{Rank}(A) = \underbrace{\dim \text{Col}}_{\text{num pivots columns}}$, $\text{Nullity}(A) = \underbrace{\dim \text{Nul}}_{\text{num free vars}}$

$\text{Rank} + \text{Nullity} = n$, or num columns in A

9. Change of basis: $[\, c_1 \; c_2 \,\vdots\, b_1 \; b_2 \,] \sim [\, I \,\vdots\, \underset{C \leftarrow B}{P} \,]$

c & b or switch if $\underset{B \leftarrow C}{P}$, left most in equation is in left of matrix

look at HW 7

look At Notes

Skim T&F, Skim textbook

$T: P_2 \to \mathbb{R}^2 \qquad T(p) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$

a. Show that T is a linear transformation

b. Find a polynomial that spans the basis

c. Describe the range of the kernel