Imitation Learning for Robotics : Assignment 1

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January 20, 2025

Introduction

Please use the provided Jupyter Notebook for the programming exercises. Submit the filled in Jupyter notebook and a PDF with answers to theory questions. Use the fourth order Runge Kutta method (RK-4) for integration.

1 Exercise 1.1: Modulated Dynamical System

Consider a 2-dimensional linear DS, $\dot{x} = Ax$, with initial condition $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We wish to introduce a modulation matrix M to modify the dynamics as follows:

$$\dot{x} = MAx$$

Given the modulation matrix:

$$M = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

Tasks:

1. Find a diagonal matrix $A = \text{diag}(a_1, a_2)$, with $a_1 \neq a_2$, for which the modulated system converges to the attractor $x^* = \binom{0}{0}$.
2. Compute and plot the path integral of the modulated DS starting from $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Programming Steps: Go to **Exercise 1.1** in the notebook.

- Define the matrix A in the code.
- Implement the vector field computation $\dot{x} = MAx$.
- The provided `plot_ds` function will visualize the streamlines and trajectory.

2 Exercise 1.2: Coupled System Stability

Consider two variables x and y coupled with the following dynamics:

$$\dot{x} = \beta x, \quad \beta \in \mathbb{R}$$
$$\dot{y} = -y + \alpha x, \quad \alpha \in \mathbb{R}$$

Answer the following (Analytical):

1. Does this system have an equilibrium point? What is it?
2. For what values of α and β is the system **stable** at the equilibrium point?
3. For what values of α and β is the system **unstable** at the equilibrium point?

3 Exercise 1.3: Lyapunov Analysis

Consider the following nonlinear DS:

$$\dot{x}_1 = -x_1 + x_1 x_2$$
$$\dot{x}_2 = -x_2$$

Consider a Lyapunov candidate function $V(x) = x_1^2 + x_2^2$.

Tasks (Analytical):

1. Find the equilibrium point(s) of the system.
2. Find a region of attraction and show that the equilibrium point is asymptotically stable using $V(x)$.

4 Exercise 1.4: Pendulum with Friction

Consider the pendulum DS with friction (damping):

$$\ddot{\theta} = -g \sin(\theta) - \lambda \dot{\theta}$$

Tasks:

1. Write down a state space representation using variables $x = (x_1, x_2) = (\theta, \dot{\theta})$.
2. Assuming $x = (0, 0)$ is a Lyapunov-stable equilibrium point, there exists a Lyapunov candidate function $V(x)$ such that:
   - $V(0, 0) = 0$
   - $V(x) > 0, \quad \forall x \neq (0, 0)$

- $\dot{V}(x) \leq 0, \quad \forall x \neq (0,0)$

3. Write a Lyapunov function satisfying these conditions and show that $(0,0)$ is stable analytically.

4. **Programming (Case A):** Compute and plot the path integral of the pendulum DS **without damping** ($\lambda = 0$) for the following initial conditions:

$$x(0) \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.78 \\ 0 \end{pmatrix}, \begin{pmatrix} 2.35 \\ 0 \end{pmatrix}, \begin{pmatrix} 3.14 \\ 1 \end{pmatrix}, \begin{pmatrix} 3.14 \\ 4 \end{pmatrix} \right\}$$

5. **Programming (Case B):** Compute and plot the path integral of the pendulum DS **with damping** ($\lambda = 1.0$) for the same initial conditions.

Programming Steps: Go to **Exercise 1.4** in the notebook.

- Implement the dynamics function inside the loop:

```
# x1_dot = x2
# x2_dot = -g * sin(x1) - damping * x2

```

- The notebook will generate plots for both Case A (Conservative) and Case B (Stable).

Adapted from *Learning Adaptive and Reactive Control for Robots*, Aude Billard, LASA, EPFL
