2.4 Combining oscillations (J&S 18.7)

Instead of combining a single driving force with a damped response, what happens if we have an oscillator with *two* driving forces ? The full equation looks like this:

$$m\frac{\partial^2\psi}{\partial t^2} = -s\psi - b\frac{\partial\psi}{\partial t} + F_1\cos(\omega_1 t + \phi_1) + F_2\cos(\omega_2 t + \phi_2), \tag{31}$$

We know that the SHO equation is *linear*, so we can try solving this problem by summing the steady state solutions arising from each driving force independently. We will consider three specific cases (where it is easier to understand the behaviour) with the general case more complex:

1. Same amplitude and frequency ($F_1 = F_2 = F$, $\omega_1 = \omega_2 = \omega$), different phase ($\phi_1 \neq \phi_2$)
2. Same frequency ($\omega_1 = \omega_2$), different phase and amplitude ($F_1 \neq F_2, \phi_1 \neq \phi_2$)
3. Same amplitude and phase ($F_1 = F_2 = F, \phi_1 = \phi_2$), different frequency ($\omega_1 \neq \omega_2$)

In all cases, we assume that we can write the displacement as a superposition of the responses to the individual oscillations:

$$\psi = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2) \tag{32}$$

Same amplitude and frequency For simplicity, we will set $\phi_1 = 0$ and set $\phi_2 = \phi$. Then we can perform the following manipulations, using the complex exponential form for simplicity:

$$\begin{aligned}
\psi &= Ae^{i\omega t} + Ae^{i(\omega t+\phi)} && (33)\\
&= Ae^{i\omega t}\left(1 + e^{i\phi}\right) && (34)\\
&= Ae^{i\omega t}\left(e^{i\phi/2}e^{-i\phi/2} + e^{i\phi/2}e^{i\phi/2}\right) && (35)\\
&= Ae^{i\omega t}e^{i\phi/2}\left(e^{-i\phi/2} + e^{i\phi/2}\right) && (36)\\
&= 2Ae^{i(\omega t+\phi/2)}\cos(\phi/2) = 2A\cos(\phi/2)e^{i(\omega t+\phi/2)} && (37)
\end{aligned}$$

where the identification of $\cos(\phi/2)$ follows from De Moivre's theorem. So the *resultant* oscillation has magnitude $2A\cos(\phi/2)$ and phase $\phi/2$. The same result can be found using trigonometry and a phasor diagram, as illustrated in Fig. 2(a).

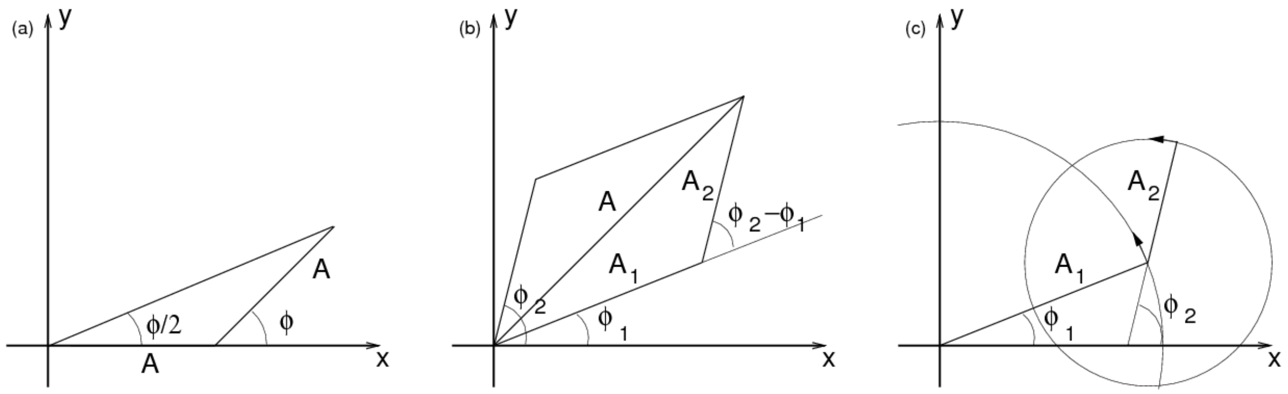

Figure 2: (a) Phasors with the *same* frequency and amplitude. (b) Phasors with *same* frequencies. The resultant phasor A is made from the two driving phasors with amplitudes A_1 and A_2 and phases ϕ_1 and ϕ_2. (c) Phasors with different frequencies and amplitudes (see Fig. 4 for path of tip).

Same frequency We can write a solution as before, by using superposition:

$$\psi = A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2) = A\cos(\omega t + \theta) \tag{38}$$

This is illustrated in a phasor diagram in Fig. 2(b). It is just an oscillation with frequency ω but a total amplitude and phase which depend on the amplitudes and phases of the two driving forces. If, for instance, the phase difference is $\phi_1 - \phi_2 = \pi$ then the two driving forces are out-of-phase and we will get destructive interference. The use of phasors allows a simple visualisation of the resultant. Using trigonometry from the phasor diagram (or the cosine rule), we can write the amplitude of the resultant as:

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1) \tag{39}$$

Now we can see that the amplitude will vary between $A_1 + A_2$ if $\phi_1 = \phi_2$ and $|A_1 - A_2|$ if $|\phi_1 - \phi_2| = \pi$. If the driving forces are *in phase* then we have the maximum amplitude, while if they are in *antiphase* we have the minimum amplitude.

The phase is also found using trigonometry; by projecting onto the real and imaginary axes, we can write:

$$\tan\theta = \frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2} \tag{40}$$

We can get the same result using complex notation. Start by finding that, for two general complex numbers z_1 and z_2:

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1+z_2)(z_1+z_2)^\star = (z_1+z_2)(z_1^\star + z_2^\star) && (41)\\
&= |z_1|^2 + |z_2|^2 + (z_1z_2^\star + z_1^\star z_2) && (42)\\
&= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1z_2^\star) && (43)
\end{aligned}$$

Now we have $z_1 = A_1e^{i(\omega t+\phi_1)}$ and $z_2 = A_2e^{i(\omega t+\phi_2)}$ with A_1 and A_2 real. This gives:

$$\begin{aligned}
|z_1 + z_2|^2 &= |z_1|^2 + |z_2|^2 + 2\Re(z_1z_2^\star) && (44)\\
&= A_1^2 + A_2^2 + 2\mathrm{Re}(A_1e^{i(\omega t+\phi_1)}A_2e^{-i(\omega t+\phi_2)}) && (45)\\
&= A_1^2 + A_2^2 + 2\mathrm{Re}(A_1A_2e^{i(\phi_1-\phi_2)}) && (46)\\
&= A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1) && (47)
\end{aligned}$$

while we can find the phase from:

$$\begin{aligned}
\arg(z_1+z_2) &= \tan^{-1}\left[\frac{\mathrm{Im}(z_1+z_2)}{\mathrm{Re}(z_1+z_2)}\right] && (48)\\
\Rightarrow \theta &= \tan^{-1}\left[\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}\right] && (49)
\end{aligned}$$

Same amplitude and phase: beats For this case, we use the principle of superposition to write:

$$\psi = A\cos\omega_1 t + A\cos\omega_2 t \tag{50}$$

Note that we have assumed that the common phase can be set to zero for simplicity. But we can rearrange this using a standard trigonometrical formula for the sum of two cosines:

$$\begin{aligned}
\psi &= A\left(\cos\omega_1 t + \cos\omega_2 t\right) && (51)\\
&= 2A\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right) && (52)
\end{aligned}$$

Now let's define two *new* frequencies, and rewrite the solution:

$$\begin{aligned}
\omega &= \frac{1}{2}\left(\omega_1+\omega_2\right) && (53)\\
\Delta\omega &= \frac{1}{2}\left(\omega_1-\omega_2\right) && (54)\\
\psi &= 2A\cos(\omega t)\cos(\Delta\omega t) && (55)
\end{aligned}$$

If the two frequencies are relatively close, then the form of the resulting oscillation is rather simple. There is an oscillation at the average frequency (the term $\cos\omega t$) whose amplitude is *modulated* by a slow oscillation at the difference frequency (the term $\cos\Delta\omega t$ - also known as the envelope). This is a phenomenon known as *beats*, and is illustrated in Fig. 3. It is important to understand that, while the frequency of the modulation is $\Delta\omega = \frac{1}{2}|\omega_1 - \omega_2|$, the frequency at which *peaks* of activity occur is $|\omega_1 - \omega_2|$ (or equivalently zeroes of activity). The number of minima per second is $\Delta\omega/\pi$. The perceived effect (say for sound) will be of a sound at the average frequency with its amplitude varying according to the envelope.

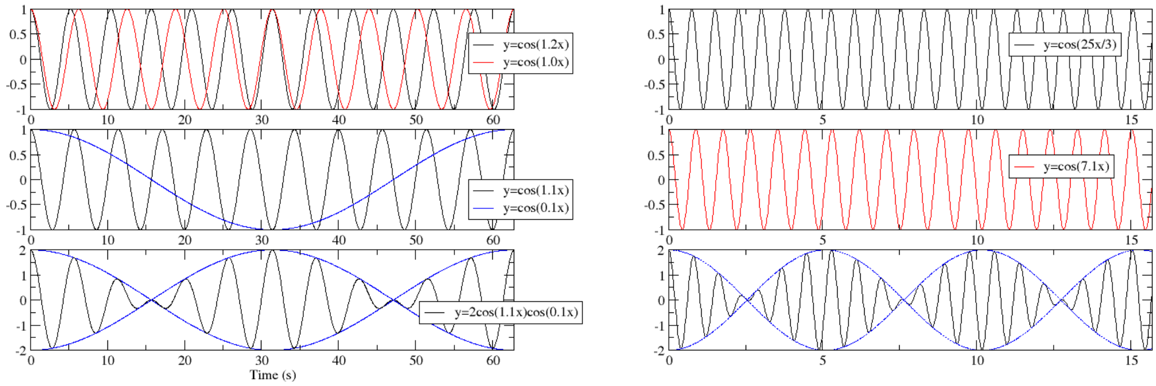

Figure 3: (a) The result of two driving forces on one SHO with different frequencies but the same amplitude. Top: two forces shown together. Middle: the sum and difference oscillations. Bottom: the resultant motion with the envelope superimposed in dashed lines. (b) Two driving forces on one SHO (the frequencies do not have integer relationship). Top: first force. Middle: second force; note that the peaks and troughs of the two forces do not *quite* coincide. Bottom: the resultant motion with the envelope superimposed in dashed lines.

Figure 3(a) shows an illustration of exactly this behaviour for the frequencies $\omega_1 = 1.2s^{-1}$ and $\omega_2 = 1.0s^{-1}$ which gives a resulting motion at frequency $\omega = 1.1s^{-1}$ modulated by an envelope with frequency $\Delta\omega = 0.1s^{-1}$.

The resulting motion will show a true periodicity if the ratio of the frequencies can be written as a ratio of integers (i.e. $\omega_1/\omega_2 = n_1/n_2$). The motion from two frequencies which are almost non-periodic is shown in Fig. 3(b) (I used 25/3 and 7.1 here but we could have made it properly non-periodic if we'd tried harder).

General Case If the amplitudes and frequencies all differ, then the phasor diagram must be dynamic as illustrated in Fig. 2(c), and the tip of the resultant will trace out a shape in time called a *cycloid*. For instance, the path followed by the tip of the resultant vector in the complex plane for different amplitudes and frequencies differing by a factor of two is plotted in Fig. 4.

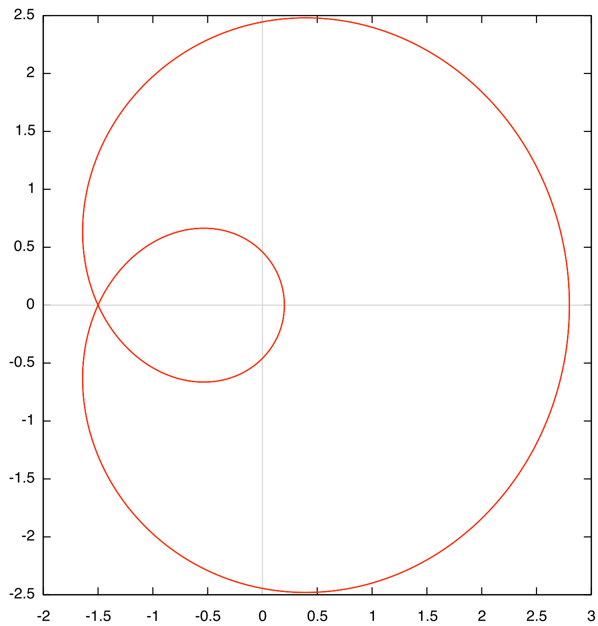

Figure 4: The location of the tip of the resultant vector for the two driving forces $1.5\cos 1.2t$ and $1.3\cos 0.6t$, which draws out a cycloid.

2.5 Normal modes of coupled oscillators

Instead of making the driving force on an oscillator a mysteriously external phenomenon (which we have implicitly done so far: we haven't allowed the oscillator to react back on the driver in the way that, say, a small child on a swing will react with their feet), what happens if it comes from *another* oscillator ? We will tackle this problem for two oscillators in one dimension only, but the generalisation to many oscillators is rather powerful: it can be used as a model of various properties of materials, for instance. This will also introduce the important concept of *normal modes*, though the full power of this concept requires matrices and matrix diagonalisation which will be introduced in the second year.

Let's consider two oscillators joined by a spring of stiffness S (we can imagine, for instance, two pendulums joined by a spring, or two masses connected to opposite walls of a box with springs and joined by a different spring). An example of the set-up is shown in Fig. 5.

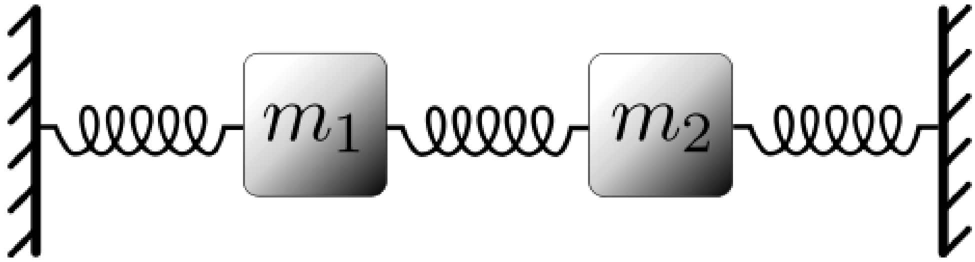

Figure 5: Example of coupled oscillators with masses m_1 and m_2. The spring joining the masses has constant S, while the springs joining the masses to the walls has constant s.

We assume that the masses are the same ($m_1 = m_2 = m$) though this is not a restrictive assumption. Then we can write the equations of motion for the individual oscillators:

$$m\frac{\partial^2\psi_1}{\partial t^2} = -s\psi_1 - S(\psi_1 - \psi_2) \tag{56}$$

$$m\frac{\partial^2\psi_2}{\partial t^2} = -s\psi_2 - S(\psi_2 - \psi_1) \tag{57}$$

We notice two things: first, that these are just the same as equations for driven oscillators; second, that the driving terms *link* the equations of motion. The link makes solving the equations potentially rather more complicated that we'd find for a single oscillator.

If the coupling was not present ($S = 0$) then the two oscillators would move harmonically, with no link between them. Given that this is their fundamental behaviour it's reasonable to look for harmonic solutions to the coupled problem. For this set of equations, we can find these harmonic solutions by noticing that adding or subtracting the equations simplifies them considerably:

$$m\frac{\partial^2}{\partial t^2}(\psi_1 + \psi_2) = -s(\psi_1 + \psi_2) \tag{58}$$

$$m\frac{\partial^2}{\partial t^2}(\psi_1 - \psi_2) = -s(\psi_1 - \psi_2) - 2S(\psi_1 - \psi_2) = -(s + 2S)(\psi_1 - \psi_2) \tag{59}$$

These are just the harmonic equations we were looking for, but we are acting on combinations of the original coordinates (which are more generally known as *normal modes*). We find that there are two solutions:

$$\psi_1 + \psi_2 = A_a \cos(\omega_a t + \phi_a) \tag{60}$$

$$\psi_1 - \psi_2 = A_b \cos(\omega_b t + \phi_b) \tag{61}$$

$$\omega_a = \sqrt{s/m} \tag{62}$$

$$\omega_b = \sqrt{(s + 2S)/m} \tag{63}$$

These can be written either in terms of ψ_1 and ψ_2 alone, or in terms of new coordinates q_a and q_b:

$$\begin{aligned}
\psi_2 &= \psi_1 = \frac{1}{2}A_a\cos(\omega_a t + \phi_a) && (64)\\
-\psi_2 &= \psi_1 = \frac{1}{2}A_b\cos(\omega_b t + \phi_b) && (65)\\
q_a &= (\psi_1 + \psi_2)\sqrt{\frac{m}{2}} && (66)\\
q_b &= (\psi_1 - \psi_2)\sqrt{\frac{m}{2}} && (67)
\end{aligned}$$

The qs are called *mode coordinates* or *normal coordinates* of the system. These are very useful, as they lead to uncoupled equations ($\ddot{q}_1 - \omega_a^2 q_1 = 0$ and equivalent for q_2) and also simple forms for the energies in the system:

$$\begin{aligned}
T &= \frac{1}{2}m\dot{\psi_1}^2 + \frac{1}{2}m\dot{\psi_2}^2 && (68)\\
&= \frac{1}{2}\dot{q_a}^2 + \frac{1}{2}\dot{q_b}^2 && (69)\\
V &= \frac{1}{2}s\psi_1^2 + \frac{1}{2}s\psi_2^2 + \frac{1}{2}S(\psi_1 - \psi_2)^2 && (70)\\
&= \frac{1}{2}\omega_a^2 q_a^2 + \frac{1}{2}\omega_b^2 q_b^2 && (71)
\end{aligned}$$

Notice that the potential in particular contains no cross-terms when we work with the mode coordinates, whereas terms in $\psi_1\psi_2$ will appear with the original system coordinates.