3.3 Energy in Waves

What is the energy stored in the wave as it travels along the stretched string ? We will have to think about energy *per unit length*. The kinetic energy (transverse to the string) is just given as usual, though we use a mass per unit length:

$$T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu\left(\frac{\partial\psi}{\partial t}\right)^2 = \frac{1}{2}\mu A^2\omega^2\sin^2(kx-\omega t) \tag{98}$$

where we have assumed that the wave is $\psi = A\cos(kx-\omega t)$. The potential energy will come from the stretching of the string and doing work against the tension in the string. How much has a small segment Δx been stretched ? At a point x, we can write the length of the string (using Pythagoras' theorem) as:

$$\begin{aligned}
\Delta l &= \sqrt{\Delta x^2 + \left(\Delta x\frac{\partial\psi}{\partial x}\right)^2} && (99)\\
&= \Delta x\left(1+\left(\frac{\partial\psi}{\partial x}\right)^2\right)^{\frac{1}{2}} && (100)\\
&\simeq \Delta x\left(1+\frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2\right) && (101)
\end{aligned}$$

So the *change* in length is approximately $\frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2$ and the work done against the tension to extend it (and hence the potential energy stored) is:

$$U = \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2 = \frac{1}{2}TA^2k^2\sin^2(kx-\omega t) \tag{102}$$

where we have assumed the same form for the wave as in the kinetic energy in Eq. (98). At this point, we will introduce the *impedance* for waves which we already saw for oscillators. It is a measure of the resistance that a wave encounters (and can be thought of as a generalisation of the idea of resistance which occurs in circuits). We will derive it properly in Sec. 4.1, but until then we will just quote the result. For the stretched string, it is defined as:

$$Z_0 = \sqrt{T\mu} \tag{103}$$

We will shortly see that it is very similar to a *drag* term as we've seen for harmonic oscillators. If we recall that the speed of the wave along the string is given by $c = \sqrt{T/\mu}$ then we see that $\mu = Z_0/c$ and $T = Z_0 c$. So we can write the total energy per unit length (or energy density) in terms of the speed and impedance only:

$$\begin{aligned}
E(x,t) &= T(x,t) + U(x,t) && (104)\\
&= \frac{1}{2}\mu\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2 && (105)\\
&= \frac{1}{2}\frac{Z_0}{c}\left[\left(\frac{\partial\psi}{\partial t}\right)^2 + c^2\left(\frac{\partial\psi}{\partial x}\right)^2\right] && (106)
\end{aligned}$$

This expression is actually true for any disturbance which satisfies the wave equation.

What is the *rate* at which energy moves along the wave (in other words the power delivered along the string) ? The force at any point x due to the string to its left is approximately $-T(\partial\psi/\partial x)$, and the rate at which work is done is:

$$P(z,t) = -T\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t} = -Z_0 c\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t} \tag{107}$$

We will return to this expression in the next section when we consider travelling waves.

3.4 Travelling Waves

Notice that the individual solutions $\psi_- = f(kx-\omega t)$ and $\psi_+ = g(kx+\omega t)$ are both good solutions for the wave equation, and we can use the fact the the equation is *linear* to add them together (this also gives a valid solution, which is the most general solution):

$$\psi = a\psi_- + b\psi_+ \tag{108}$$

This is called the *principle of superposition* and leads to interference effects.

Each of the solutions $\psi_- = f(kx - \omega t + \phi)$ and $\psi_+ = g(kx + \omega t + \phi)$ are called *travelling waves.* The direction of travel is set by the relative sign of x and t: if the *relative* sign is negative (e.g. $kx - \omega t$) then the wave travels in the *positive* x direction; if it is positive (e.g. $kx + \omega t$) then the wave travels in the *negative* x direction.

We can understand this by considering a particular shape at $t = 0$, which will be given by $f(kx)$. Notice that this is an *arbitrary* function - it doesn't have to be periodic. Then at a time t later than this, if we increase x by $\frac{\omega}{k}t$ we will find:

$$\psi(x + \frac{\omega}{k}t, t) = f(k[x + \frac{\omega}{k}t] - \omega t) = f(kx + \omega t - \omega t) = f(kx) = \psi(x, 0) \tag{109}$$

So the shape of the wave is *the same as it was at $t = 0$ but translated a distance $\frac{\omega}{k}t$ along the x-axis.* We have already written that $\omega/k = v$, so we see that the wave is translated (or travels) a distance vt along the axis. This is why the solution is called a travelling wave. We can generalise away from a form with k and ω by writing $f(x \pm ct)$ with the knowledge that we may need to scale the argument by an appropriate constant to get the right units. Now think about differentiating this function with respect to t and x. We will find that:

$$\frac{\partial \psi}{\partial t} \mp c\frac{\partial \psi}{\partial x} = 0 \tag{110}$$

This equation is satisfied by functions of the form $\psi(x, t) = f(x \pm ct)$ which are travelling waves. Notice the close connection with the factorisation of the wave equation we gave above in Eq. (90). If either bracket in that equation is zero then the wave equation will be satisfied - a condition which can also be written in Eq. (110). Note that this is not an exact mapping: all waves satisfy the wave equation, but not all waves satisfy the travelling wave equation.

If we return to the energy density and power formulae, we see that we can eliminate one of the differentials, so that:

$$E(x,t) \quad = cZ_0\left(\tfrac{\partial \psi}{\partial x}\right)^2 \quad = \frac{Z_0}{c}\left(\frac{\partial \psi}{\partial t}\right)^2 \tag{111}$$

$$P(x,t) \quad = -Z_0 c^2\left(\tfrac{\partial \psi}{\partial x}\right)^2 \quad = -Z_0\left(\frac{\partial \psi}{\partial t}\right)^2 = -cE(x,t) \tag{112}$$

So the instantaneous kinetic and potential energy densities are *equal* at any point on a string under tension carrying a travelling wave. The power is given by the energy density multiplied by the speed of wave propagation and travels in the direction of wave propagation.

4 Transverse Waves (J&S 16, 18)

Transverse waves are waves where the displacement of the medium (e.g. the string) is perpendicular (or transverse) to the direction of wave propagation. So, for instance, the string is displaced in the y direction while the wave travels in the x direction.

4.1 Driving a wave

Until now, we have not thought about how waves start and finish: we have assumed (implicitly) that they are infinite in time and space. However, this is not how real systems work: if a wave is to continue to propagate along a string (or down an optical fibre, say) then there must be some driving force. Let's consider a string stretched to a length L with ends at $x = 0$ and $x = L$. We'll place a driving mechanism of some kind at $x = 0$.

What must the driving mechanism do ? It needs to produce a force which balances the transverse component of the tension in the string at $x = 0$. This is a driving force:

$$F_D = -T\left(\frac{\partial \psi}{\partial x}\right)_{x=0} = \frac{T}{c}\left(\frac{\partial \psi}{\partial t}\right)_{x=0} = Z_0\left(\frac{\partial \psi}{\partial t}\right)_{x=0} \tag{113}$$

We have used the travelling wave equation, Eq. (110), to simplify. The force at any instant (or instantaneous force) must be proportional to the transverse velocity of the string—which closely resembles a standard drag force in a damped harmonic oscillator. Why do we have this drag ? It arises from the energy being transported by the way: if we want to keep a constant wave motion going, we must put energy into the system. Notice that the constant of proportionality is the characteristic impedance, Eq. (103). It is a function of the system only (in this case the tension and mass density) and does not depend on the type of motion or its frequency.

4.2 Terminating a wave

We have just seen that creating or driving a wave along a stretched string (or, in fact, in any medium) requires energy to be put in to the system. But what about the other end of the string ? What happens there ? Let's think about a *finite* string, and how we can make it resemble an infinite string. We know what force we need at $x = 0$ to create the wave. At the end of the string, $x = L$, we need a transverse force to balance tension in the string; if we match this tension then it will be as if the end was not there. This condition can be written:

$$F_L = T\left(\frac{\partial\psi}{\partial x}\right)_{x=L} = Z_0\left(\frac{\partial\psi}{\partial t}\right)_{x=L} \tag{114}$$

So if we have some form of drag to absorb the energy being transmitted along the string, then the wave will propagate along as if the string were infinite (assuming that we have set up the driver as described above). This idea is known as *impedance matching* and is important in many areas, particularly electromagnetism (where we must terminate, say, a power line or an aerial correctly). Remember that this all came about because we have a *finite* string (or medium in general) which we want to send a wave down, with the medium behaving *as if it were infinite*. This means that we must put energy in at one end, and take it out at the other.

If we do *not* provide the right force at the end, then something different will happen. In fact, these ideas can be extended to consider *boundaries* between different media (e.g. tying a light string to a rope, or light going from air into water or glass). The two media will have different impedances (in the case of the string and rope, probably different mass densities even if they're under the same tension) and something interesting will happen at the boundary.

4.3 Reflection and Transmission

Imagine a string under tension, with one end tied to a solid object (like a wall) and the other end free to be driven. If we send a pulse down the string (for example by moving the free end up and then down rapidly once), it will propagate along the string as a travelling wave. What will happen when it reaches the solid wall ? Intuition or experience tells us that when a wave reaches a solid object (i.e. an object with very large impedance) it tends to be reflected. Now a reflected wave in one dimension travels in the *opposite* direction to the incoming wave, so we will need *both* solutions for the wave equation (i.e. $\psi = \psi_i + \psi_r = f(x-ct) + g(x+ct)$, where ψ_i is the incoming or *incident* wave and ψ_r is the reflected wave). We can also deduce this need for both waves from a more general situation we'll see below.

Now, we need to work out the relationship between the two components of the wave, and we'll do this by thinking about the *boundary conditions* on the wave. Much of the work done in physics involves working out appropriate boundary conditions, and solutions of differential equations given appropriate boundary conditions. We will assume that the solid wall has an infinite impedance, so there can be no propagation of the wave (the velocity will be zero). The tension *along* the string is provided by the wall, but the tension *transverse* to the string at the wall must be zero (otherwise the wall would move up and down). We can therefore write that:

$$T\left(\frac{\partial\psi_i}{\partial x}\right)_{x=L} + T\left(\frac{\partial\psi_r}{\partial x}\right)_{x=L} = 0 \tag{115}$$

$$\Rightarrow T\left(\frac{\partial\psi_i}{\partial x}\right)_{x=L} = -T\left(\frac{\partial\psi_r}{\partial x}\right)_{x=L} \tag{116}$$

$$\left(\frac{\partial\psi_i}{\partial x}\right)_{x=L} = -\left(\frac{\partial\psi_r}{\partial x}\right)_{x=L} \tag{117}$$

So there must be a 180° phase change on reflection; this is another way of saying that the reflected wave has the opposite sign to the incoming wave. But what happens now if, instead of a solid wall, we have *another* string with a different mass per unit length ? To be clear, we will assume that we have two strings of lengths L_1 and L_2 joined at $x = 0$ held under tension T. The first string has mass density μ_1, impedance $Z_1 = \sqrt{T\mu_1}$ and has a driver at its free end ($x = -L_1$). The second string has mass density μ_2, impedance $Z_2 = \sqrt{T\mu_2}$ and has perfect termination (i.e. a drag term with impedance Z_2) at $x = L_2$.

Now consider an incident wave, starting at $x = -L_1$ and moving towards the joining point at $x = 0$. As it propagates, there is a transverse force on the string given by:

$$-T\left(\frac{\partial\psi}{\partial x}\right) = Z\left(\frac{\partial\psi}{\partial t}\right) \tag{118}$$

At the joining point, $x = 0$, we must have the following condition:

$$Z_1\left(\frac{\partial\psi}{\partial t}\right)_{x=-\delta} = Z_2\left(\frac{\partial\psi}{\partial t}\right)_{x=\delta}, \tag{119}$$

as $\delta \to 0$. This is how we can deduce that there must be both incident and reflected waves on the first piece of string: it is the only way to balance the boundary condition when $Z_1 \neq Z_2$. So we now assume that $\psi = \psi_i + \psi_r, x \leq 0$.

At the joining point, the drag exerted by the second string will be given by its impedance multiplied by the transverse velocity of the string:

$$F_{\text{drag}} = Z_2 \frac{\partial}{\partial t} (\psi_i + \psi_r) = Z_2 \left(\frac{\partial \psi_i}{\partial t} + \frac{\partial \psi_r}{\partial t} \right) \tag{120}$$

This force will be balanced by the transverse force from the string:

$$-T\frac{\partial \psi}{\partial x} = -T \left(\frac{\partial \psi_i}{\partial x} + \frac{\partial \psi_r}{\partial x} \right) = Z_1 \left(\frac{\partial \psi_i}{\partial t} - \frac{\partial \psi_r}{\partial t} \right) \tag{121}$$

Note that the change of sign between the partial derivatives comes when we use the travelling wave equation to relate the spatial and time derivatives: the incoming wave gives a minus sign (which cancels the minus sign outside the bracket) while the reflected wave gives a plus sign. We can now derive a condition relating the incident and reflected waves, by equating the two forces:

$$\begin{aligned}
Z_1 \left(\frac{\partial \psi_i}{\partial t} - \frac{\partial \psi_r}{\partial t} \right) &= Z_2 \left(\frac{\partial \psi_i}{\partial t} + \frac{\partial \psi_r}{\partial t} \right) && (122)\\
(Z_1 - Z_2) \left(\frac{\partial \psi_i}{\partial t} \right) &= (Z_1 + Z_2) \left(\frac{\partial \psi_r}{\partial t} \right) && (123)\\
\left(\frac{\partial \psi_r}{\partial t} \right) &= \frac{Z_1 - Z_2}{Z_1 + Z_2} \left(\frac{\partial \psi_i}{\partial t} \right) && (124)
\end{aligned}$$

If we integrate both sides with respect to time, we find that:

$$\psi_r(0,t) = \frac{Z_1 - Z_2}{Z_1 + Z_2} \psi_i(0,t) \tag{125}$$

This gives the relation between ψ_i and ψ_r at the joining point of the two strings. As both waves are travelling waves, we can relate the value at one time and place to the value at another time and place:

$$\begin{aligned}
\psi_i(-l, t - l/c) &= \psi_i(0,t) && (126)\\
\psi_r(-l, t + l/c) &= \psi_r(0,t) = R\psi_i(0,t) = R\psi_i(-l, t - l/c) && (127)\\
R &= \frac{Z_1 - Z_2}{Z_1 + Z_2} && (128)
\end{aligned}$$

In other words, the reflected wave at $x = -l$ and $t = t + l/c$ is the same as the incident wave at the same position but at a time $2l/c$ in the past, and scaled by R, which we call the *reflection coefficient*. Notice that the time delay is just the time it takes for the wave to travel along the string and back again.

We can also say something about the wave in the second string. We must have the following boundary condition:

$$\psi_i(0,t) + \psi_r(0,t) = \psi_t(0,t), \tag{129}$$

where ψ_t is the transmitted wave. This must be so, otherwise there will be a discontinuity in the string. So the transmitted wave is related to the incident wave by the following formula:

$$\begin{aligned}
\psi_t(0,t) &= \psi_i(0,t) + R\psi_i(0,t) && (130)\\
&= (1 + R)\psi_i(0,t) = T\psi_i(0,t) && (131)\\
T &= 1 + R = \frac{2Z_1}{Z_1 + Z_2} && (132)
\end{aligned}$$

where T is called the *transmission coefficient.* Notice that the reflection coefficient will take values between -1 and 1, while the transmission coefficient will take values between 0 and 2:

$$\begin{aligned}
-1 &\leq R \leq 1 && (133)\\
0 &\leq T \leq 2 && (134)
\end{aligned}$$

If there is a single string terminated with a very large impedance (a solid wall, as before, so $Z_2 \gg Z_1$) then we will have $R = -Z_2/Z_2 = -1$ and we get the phase change we saw before (and no transmission). If the string is terminated with a very small impedance (a free end, for instance, so that $Z_1 \gg Z_2$ then we will have $R = Z_1/Z_1 = 1$ and no phase change in the reflected wave. When there are two strings, then in the limit of the second string having negligible impedance the transmitted wave will have twice the amplitude of the incident wave, and the reflected wave will have the same height as the incident wave. When the two impedances are matched, it will be as if there was only one string ($R = 0, T = 1$).

The phase velocities and wavelengths on either side of an interface between two media will be different, but the frequencies will be the same, provided that the interface has no mechanism for driving waves. We can understand by thinking about the effect of the interface on the second medium: it will act as a harmonic driving force at frequency ω. Consider a join between two strings of different mass per unit length (but under the same tension) to be specific. Then if we send a wave with frequency ω down the first string, all points on the string will oscillate harmonically with frequency ω. This must be true of the junction, which will then excite waves of the same frequency in the second string. So we have:

$$\begin{aligned}
\omega_1 &= \omega_2 = \omega && (135)\\
\nu_1 &= \nu_2 = \frac{\omega}{2\pi} && (136)\\
c_1 &= \sqrt{\frac{T}{\mu_1}} && (137)\\
k_1 &= \frac{\omega_1}{c_1} = \frac{\omega}{c_1} && (138)\\
\lambda_1 &= \frac{c_1}{\nu} && (139)\\
c_2 &= \sqrt{\frac{T}{\mu_2}} && (140)\\
k_2 &= \frac{\omega_2}{c_2} = \frac{\omega}{c_2} && (141)\\
\lambda_2 &= \frac{c_2}{\nu} && (142)\\
& && (143)
\end{aligned}$$

Later in the course we will encounter situations where the frequency and wavelength are not so simply related (called dispersive systems) but even for these materials frequencies are conserved across boundaries.

We can summarise the results on driving waves, terminating, and behaviour at interfaces as follows:

- If we want to drive a wave along a semi-infinite string, then the driver must supply a force proportional to the *transverse* velocity of the end of the string it is driving. The constant of proportionality is known as the *characteristic impedance*.
- If we want to terminate a finite, driven string so that the same wave motion as would be found on an infinite string is supported, it must be terminated with an impedance equal to the characteristic impedance of the string.
- If a string (or any medium) is not terminated with the correct impedance then reflection as well as transmission will occur at the interface according to the reflection and transmission coefficients given above in Eq. (128) and Eq. (132).