2 Simple Harmonic Motion (J&S 15)

We will start by reviewing simple harmonic motion (SHM) as it contains many of the important concepts that we will meet in waves. The most general form of the equation for a simple harmonic oscillator (SHO) including damping and driving forces can be written:

$$m\frac{\partial^2\psi}{\partial t^2} = -s\psi - b\frac{\partial\psi}{\partial t} + F_0\cos\omega t, \tag{1}$$

where ψ is the displacement, m is the mass of the oscillator, s is the constant giving the restoring force (e.g. a spring constant) sometimes known as the *stiffness*, b is the damping coefficient or *resistance* and F_0 gives the magnitude of the driving force (which has *angular* frequency ω).

2.1 SHM and Circular Motion (J&S 15.1–15.4)

If we set the damping and driving coefficients to zero, we recover the original, SHM equation:

$$m\frac{\partial^2\psi}{\partial t^2} = -s\psi, \tag{2}$$

which can be shown to be solved using sinusoidal motion:

$$\psi = A\sin\omega_0 t + B\cos\omega_0 t = C\cos\left(\omega t + \phi\right), \tag{3}$$

where $\omega_0 = \sqrt{s/m}$, ϕ is a constant phase and $A = -C\sin\phi$ and $B = C\cos\phi$. As you have seen in both PHAS1245 (Mathematical Methods I) and PHAS1247 (Classical Mechanics), we can use De Moivre's theorem to write the sinusoidal terms as a complex exponential ($e^{i\theta} = \cos\theta + i\sin\theta$ where $i = \sqrt{-1}$), giving:

$$\psi = Re\left[Ae^{i(\omega_0 t+\phi)}\right] = Re\left[De^{i\omega_0 t}\right] \tag{4}$$

We often call the argument of a trigonometric function the *phase*, and ϕ is often called the phase difference when there is more than one oscillation. Now that we have the representation of the oscillation in terms of the complex exponential (or the sum of a cos and sin) we can see that there is an immediate link with circular motion: the amplitude of the oscillation is just the projection of circular motion onto the x-axis (or any other axis that is chosen). (Note that from now on I will not write the need to take the real part of ψ explicitly, but will assume it.) All these ideas are illustrated in Fig. 1(a).

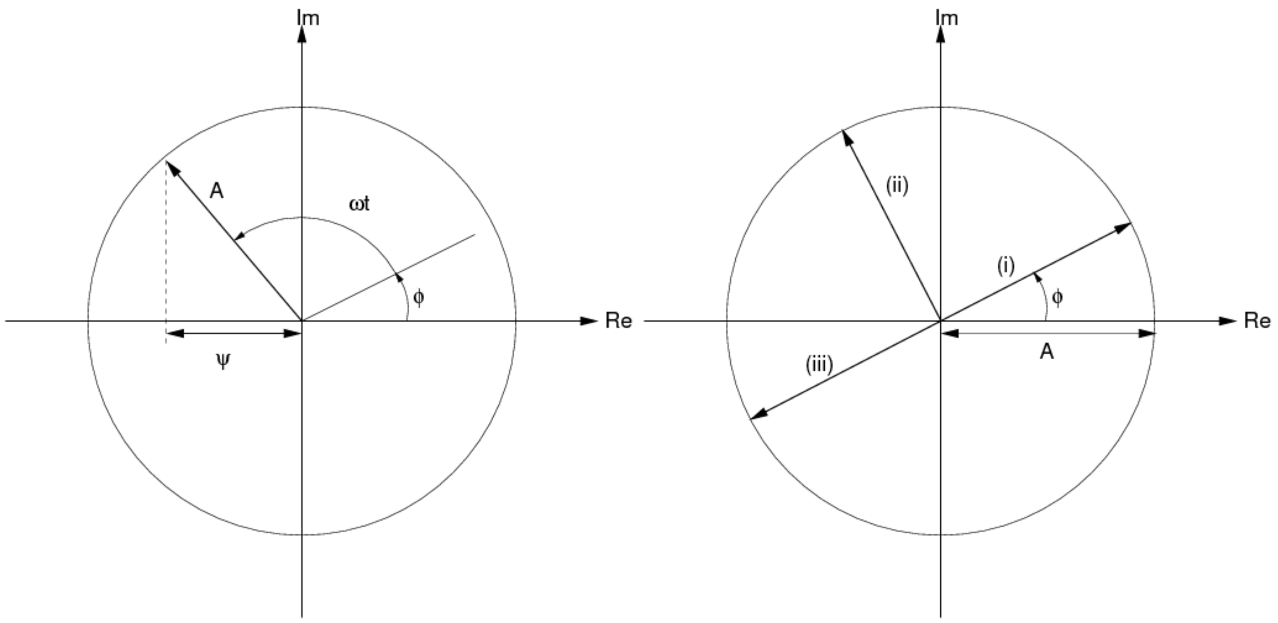

Figure 1: (a) Relation between circular motion of the vector ψ and simple harmonic motion. (b) The phase relations between displacement (i), velocity (ii) and acceleration (iii) for a simple harmonic oscillator. Note that the amplitudes are A, ωA and $\omega^2 A$ and are not to scale.

The representation of a simple harmonic oscillator at a single point in time in the complex plane (e.g. Fig. 1(a)) is often called a *phasor* diagram, and the arrow representing the amplitude and phase of the oscillator relative to a fixed time or phase is called a *phasor*; note that for a phasor we just need the arrow, not the associated circle. We will see later that we can combine two oscillations using phasors (or using complex arithmetic - the two are completely equivalent).

A simple way to think about phase differences is to consider the velocity and acceleration of the oscillator:

$$\begin{aligned}
\psi &= Ae^{i(\omega_0 t+\phi)} && (5)\\
\dot{\psi} &= \frac{\partial \psi}{\partial t} = i\omega A e^{i(\omega_0 t+\phi)} = i\omega\psi && (6)\\
\ddot{\psi} &= \frac{\partial^2 \psi}{\partial t^2} = -\omega^2 A e^{i(\omega_0 t+\phi)} = -\omega^2\psi && (7)
\end{aligned}$$

Notice that both of these quantities vary harmonically and with the same frequency as the oscillation, though with different amplitudes; more importantly, however, note that the velocity is a factor of i different to the dispacement, while the acceleration is a factor of -1 different. These are easier to understand when we write them in complex exponential form: $i = e^{i\pi/2}$ and $-1 = e^{i\pi}$. So we can write:

$$\begin{aligned}
\psi &= Ae^{i(\omega_0 t+\phi)} && (8)\\
\dot{\psi} &= \omega A e^{i(\omega_0 t+\phi+\pi/2)} = \omega\psi e^{i\pi/2} && (9)\\
\ddot{\psi} &= \omega^2 A e^{i(\omega_0 t+\phi+\pi)} = \omega^2\psi e^{i\pi} && (10)
\end{aligned}$$

We see that the velocity leads the displacement by a phase factor of $\pi/2$ and the acceleration leads the velocity by a phase factor of $\pi/2$. This phase relationship is shown in Fig. 1(b).

The simple harmonic oscillator, as with all mechanical systems, has two forms of energy: kinetic and potential. If we have W as the total energy and T as kinetic and V as potential, we can write:

$$\begin{aligned}
T &= \frac{1}{2} m\dot{\psi}^2 && (11)\\
V &= \frac{1}{2} k\psi^2 && (12)\\
W &= T + V && (13)
\end{aligned}$$

The form of the potential energy follows directly from the form of the force (and is why the motion is called harmonic). As there is no form of dissipation in the motion, we can assert that the total energy does not change with time, just exchanging between potential (at a maximum when the displacement is at a maximum and the velocity is at a minimum) and kinetic (at a maximum when the displacement is at a minimum). They are out of phase - as we expect from the phase differences between displacement and velocity.

As the total energy is constant, we can write:

$$\begin{aligned}
\frac{dW}{dt} = \frac{dT}{dt} + \frac{dV}{dt} &= 0 && (14)\\
\Rightarrow m\dot{\psi}\ddot{\psi} + k\psi\dot{\psi} &= 0 && (15)\\
\Rightarrow m\ddot{\psi} &= -k\psi && (16)
\end{aligned}$$

which is of course just the original equation for simple harmonic motion.

2.2 Damping oscillations (J&S 15.6)

Restoring the damping term changes the equation and its solution. We will use $\gamma = b/2m$ in the equations below:

$$m\frac{\partial^2 \psi}{\partial t^2} = -s\psi - b\frac{\partial \psi}{\partial t} \qquad (17)$$

$$\psi = \begin{cases} Ae^{-\gamma t}e^{i\omega t} & \gamma < \omega_0 \\ \quad \omega = \sqrt{\omega_0^2 - \gamma^2} & \\ Ae^{-\mu_+ t} + Be^{-\mu_- t} & \gamma > \omega_0 \\ \quad \mu_\pm = \gamma \mp \sqrt{\gamma^2 - \omega_0^2} & \\ A(1+\omega_0 t)e^{-\omega_0 t} & \gamma = \omega_0 \end{cases} \qquad (18)$$

The cases for damped harmonic motion correspond to light, heavy and critical damping respectively. The total energy is *not* conserved in this case (contrast the undamped oscillator) as the damping force opposes the motion at all times. We write the energy as before, and find the change with time:

$$
\begin{aligned}
W &= \frac{1}{2}m\dot{\psi}^2 + \frac{1}{2}s\psi^2 & (19)\\
\frac{dW}{dt} &= \frac{dW}{d\dot{\psi}}\frac{d\dot{\psi}}{dt} + \frac{dW}{d\psi}\frac{d\psi}{dt} = (m\ddot{\psi} + s\psi)\dot{\psi} & (20)\\
&= -b\dot{\psi}^2 & (21)
\end{aligned}
$$

where the last line comes from using Eq. (17). Notice that the change in energy is always less than zero (unless $\dot{\psi} = 0$) and so the total energy of the system decreases with time (as we would expect).

2.3 Driving oscillations (J&S 15.7)

Now we will introduce a driving term to the system. As we have a harmonic oscillator, it will make sense to use a harmonic driving term (though we could, for instance, use impulses at regular or irregular intervals). Note that the driving term will have an angular frequency ω which is *different* to the natural frequency of the system, $\omega_0 = \sqrt{s/m}$. Also remember that $\omega = 2\pi\nu$ where ν is the frequency. We will retain the damping term to give a damped, driven oscillator.

$$
\begin{aligned}
m\frac{\partial^2\psi}{\partial t^2} &= -s\psi - b\frac{\partial\psi}{\partial t} + F_0\cos\omega t & (22)\\
\psi &= A\cos(\omega t + \phi) & (23)\\
A &= \frac{F_0}{m}\left(\frac{1}{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}\right)^{\frac{1}{2}} & (24)\\
\tan\phi &= \frac{2\gamma\omega}{\omega^2 - \omega_0^2} & (25)
\end{aligned}
$$

The response of the system (including both the phase and the amplitude) depends strongly on the frequency; we can consider three regimes (though there isn't really time to do this in depth).

1. When ω is small (i.e. $\omega \ll \omega_0$), the amplitude can be shown to be $A \simeq F_0/m\omega_0^2 = F_0/s$ and the motion is dominated by the spring constant (*stiffness controlled*).
2. When ω is large (i.e. $\omega \gg \omega_0$), then the amplitude is $A \simeq F_0/m\omega^2$ and the motion is dominated by the mass (*mass controlled*).
3. When $\omega \sim \omega_0$, we are at *resonance*, and the response is controlled by the drag term γ (also known *resistance limited*).

Power adsorption Let's think a little about the power *absorbed* by the oscillator; to that, we must consider the work done *against* the drag. If the displacement changes from ψ to $\psi + \Delta\psi$ then the work done is $-F_d\Delta\psi$, where $F_d = -b\dot{\psi}$. If that takes a time Δt then the rate of work is $-F_d(\Delta\psi/\Delta t)$ which tends to $-F_d\dot{\psi}$ as $\Delta t \to 0$. So the instantaneous power adsorption becomes:

$$
P = -F_d\dot{\psi} = b\dot{\psi}^2 \qquad (26)
$$

We are not actually interested in the instantaneous power, but the time averaged power, which for a harmonic force can be shown to be[1]:

$$
\begin{aligned}
\langle P\rangle &= b\langle\dot{\psi}^2\rangle = \frac{1}{2}b\omega^2A^2 & (27)\\
&= \frac{F_0^2}{m^2}\frac{m\gamma\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2} & (28)
\end{aligned}
$$

where we have substituted $b = 2m\gamma$. This has a maximum value of $F_0^2/(2m\gamma)$ when $\omega = \omega_0$. Note that this is inversely proportional to the resistive force.

Finally we introduce the idea of *impedance* which is a measure of the resistance to the motion of the oscillator. It is defined as $Z(\omega) = F_0/|\dot{\psi}| = b + i(m\omega - s/\omega)$. At resonance, $Z(\omega) = b$ in the resonance region only departs very slightly from this value. Away from resonance, it includes a phase lag or lead, which reflects the relation of velocity to the driving force. We will encounter impedance again with waves.

[1]The time average of $\cos^2\omega t$ is $\frac{1}{2}$ which can be seen by considering the time average of $\cos^2\omega t + \sin^2\omega t$.

Transients Note that this is a *steady state* solution: there is a transient behaviour at the beginning of the oscillation which takes the form of the solution in Eq. (23) added to a solution of the homogenous equation (i.e. the equation with no driving term - Eq. (17) or earlier). So a real system will respond at its natural frequency, but damping will cause that solution to die off, while the driven oscillation persists into the steady state:

$$\psi(t) = A_s \cos(\omega t + \phi) + A_t e^{-\gamma t} \cos(\omega_0 t + \phi_t) \tag{29}$$

where the subscripts s and t stand for *steady state* and *transient* respectively. For lightly damped system near resonance we can approximate and find that $A_t \simeq A_s$ and $\phi_t \simeq \phi_s + \pi$, giving:

$$\psi(t) = A_s \left(\cos(\omega t + \phi) - e^{-\gamma t} \cos(\omega_0 t + \phi_s)\right) \tag{30}$$

We have used the principle of superposition in finding this solution: as the oscillator equation is *linear* in ψ we can always create a new solution for the wave equation as a linear combination of existing solutions. (If the basic equation was $\partial^2\psi/\partial t^2 = -\omega_0^2\psi^2$ say then this would not be possible.) If we switch off the driving force, then we will again have transient behaviour—which will be just a damped harmonic oscillator. Think of a child on a swing: the parent pushing provides the damping force, which is switched off when the parent gets tired. Fortunately for all concerned, the swing comes to rest after some time (though we expect that the swing will be lightly damped).

We have introduced **two** important concepts here:

- Superposition of solutions giving a resultant shape to the oscillations
- The idea that, in *all* real physical processes, there is a beginning, a steady state (the middle if you like) and an end.

These ideas will both come back when we study waves.