4.5 Going beyond one dimension

So far we have only considered waves in one dimension: the wave variable ψ depends only on x (and of course t). But of course there's nothing special about the x direction, and we could just as easily have chosen y or z. This suggests that we should consider three dimensions when thinking about wave motion:

$$\psi(x, y, z, t) = \psi(\mathbf{r}, t) \tag{172}$$

But if the variable (e.g. displacement on a string) can depend on all three spatial dimensions, then we must also generalise the wave equation to include differentials with respect to all three dimensions. This is done by replacing $\partial^2/\partial x^2$ with ∇^2:

$$\nabla^2\psi(\mathbf{r}, t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{r}, t) \tag{173}$$

Of course we now have to generalise the solutions of the wave equation. Coming back to our solution where we wrote $\psi(x, t) = e^{i(kx-\omega t)}$, it's quite easy to see that this actually describes a plane (hence the term *plane wave*). For any given value of the phase $kx - \omega t$ all points in the $y - z$ plane passing through x will share this phase. But mathematically, this plane is defined by:

$$\mathbf{k}\cdot\mathbf{r} = constant \tag{174}$$

$$\mathbf{k} = (k, 0, 0) \tag{175}$$

with the position vector given as usual by $\mathbf{r} = (x, y, z)$. So we see that a wave which propagates along $(x, 0, 0)$ can be written in terms of a *wavevector* $\mathbf{k} = (k, 0, 0)$ with magnitude k. We can generalise this to write:

$$\psi(\mathbf{r}, t) = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{176}$$

which travels in the direction $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ (using a unit vector to give the direction) and has wavelength $\lambda = 2\pi/|\mathbf{k}|$. Now the Laplacian ∇^2 has a simple form in Cartesian coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{177}$$

Now we can substitute Eq. (176) into Eq. (173) which gives the following condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \tag{178}$$

But as $k_x^2 + k_y^2 + k_z^2 = |\mathbf{k}|^2$ we can write:

$$|\mathbf{k}| = \frac{\omega}{c} \tag{179}$$

though it's quite common to write $k = |\mathbf{k}|$.

Another aspect of three dimensional waves which is important to understand is spherical waves. Point sources will emit spherical waves (that waves which expand outwards as spheres with an amplitude which decreases as $1/r$ or faster). When we are far from the source, these can be treated as plane waves (just as locally the surface of the earth appears to be planar). If the source is at $\mathbf{r}_0$ and the observer is at $\mathbf{r}$ then the direction of the wave locally is $(\mathbf{r} - \mathbf{r}_0)/|\mathbf{r} - \mathbf{r}_0|$. This approximation is valid when the following condition is fulfilled: the length scale over which we consider the wave (say Δr) is much smaller than the distance from the source, r: $\Delta r \ll r$. When this is true, the variation in amplitude from the $1/r$ decrease will be negligible, and locally the wave will appear flat.

5 Longitudinal Waves (J&S 17)

So far, we have only considered a wave where the disturbance is *transverse* (or perpendicular) to the direction of propagation of the wave. However, this is not required; in fact, waves where the disturbance is *along* the direction of propagation (known as *longitudinal* waves) are rather common. The physics of the situation is no different–we can still think of the individual points of the medium as oscillating harmonically, but instead of the oscillations lying perpendicular to the direction of wave travel, they are parallel.

Examples of longitudinal waves include sound waves (illustrated in Fig. 9), elastic waves in solids and p-waves in the earth's crust caused by earthquakes. The same wave equation governs longitudinal waves and transverse waves, though the physical processes responsible for the restoring force are different. We will start by thinking about elastic waves in a rod, and then consider sound waves.

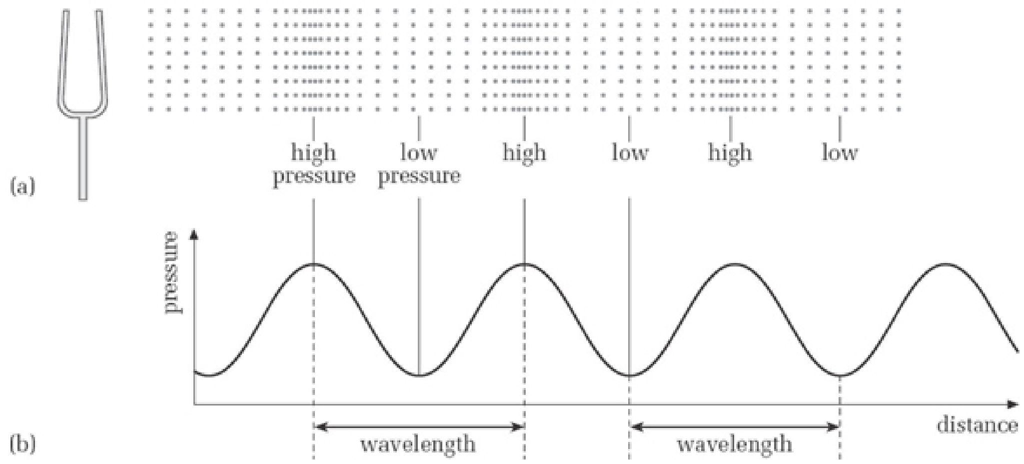

Figure 9: An illustration of a sound wave, showing the variation of pressure schematically (top) and as a graph (bottom).

5.1 Elastic Waves

We will consider small distortions of a rod (or a stretched string) which involve local compressions and extensions of the rod along its axis. We will think about a short segment lying between x_0 and $x_0 + \Delta x$ which is bounded by two planes. Each plane will undergo harmonic oscillations, though not necessarily in phase. We could, for instance, write for the location of the first plane as $x = x_0 + \psi \cos(kx_0 - \omega t)$. In general, we will allow a point on the rod to move from x_0 to $x_0 + \psi$ and $x_0 + \Delta x$ to $x_0 + \Delta x + \psi + \Delta\psi$, with $\Delta\psi \ll \Delta x$. So as a wave passes along the rod, the short segment will be both *shifted* and *stretched or compressed*[2].

There are various material parameters which allow us to think about compression and expansion. Young's modulus is a measure of the stiffness of the material (we will use Y though E is often used). For a rod with cross-section A and Young's modulus Y, when a force F is applied along the length, the rod extends by a fraction F/AY. We can define the local *strain* on the element as the change in length (or $\Delta\psi$) divided by the length (or Δx): $\epsilon = \Delta\psi/\Delta x$. The local *stress* is defined as the average force per unit area (which is of course equivalent to a pressure): $\sigma = F/A$; the Young's modulus is thus defined as the ratio of stress to strain. We can write for the element:

$$\sigma = Y\epsilon \tag{180}$$

$$\frac{F}{A} = Y\frac{\Delta\psi}{\Delta x} \to Y\frac{\partial\psi}{\partial x} \text{ as } \Delta z \to 0 \tag{181}$$

But we will have different forces at the two ends of the element: F and $F + \Delta F$ at x and $x + \Delta x$. We can expand ΔF:

$$F + \Delta F = F + \frac{\partial F}{\partial x}\Delta x \tag{182}$$

The *excess* force on the element will be $F + \Delta F - F$. If we assume that the rod has density ρ then the mass of the element will be $\rho A\Delta x$, and we can write the net force on the element as:

$$\rho A\Delta x\frac{\partial^2\psi}{\partial t^2} = \Delta F = \frac{\partial F}{\partial x}\Delta x \tag{183}$$

$$\rho A\frac{\partial^2\psi}{\partial t^2} = \frac{\partial}{\partial x}\left(AY\frac{\partial\psi}{\partial x}\right) \tag{184}$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2\psi}{\partial x^2} \tag{185}$$

This is, of course, just the wave equation with a velocity given by $\sqrt{Y/\rho}$. Notice that the variable which undergoes wave motion is now the *displacement* of a small segment of the rod along its axis, which will give rise to waves of compression and expansion. We can estimate the speed of these waves for steel which has $Y = 2 \times 10^{11}$Pa and $\rho = 8,000$ kg m^{-3} giving $c = 5,000$ ms^{-1}. This is a typical wave speed in a solid. But contrast this to the speed along a stretched piano wire in Section 4.4.2 which was ~ 280ms^{-1}, or a factor of 20 smaller. To achieve a transverse wave speed the same as the longitudinal speed, we would need a tension

[2]For a real material, the cross-section will also change a small amount, but we will neglect this.

divided by the cross-sectional area equal to the Young's modulus (which would pull the wire well out of the linear, elastic regime).

But this derivation is really only valid for a thin rod where we can neglect the change in cross-section. Elastic waves in a solid can consist of longitudinal waves pressure waves (as we've just described) and transverse *shear* waves. Elastic constants are more complex than we have considered: forces applied along different axes will lead to different expansions. If we consider an infinite bulk solid, then the freedom to expand or contract perpendicular to the applied force is removed, which increases the elastic constants and thus the wave speeds. There are a number of different elastic moduli (or elastic constants) which relate to compressions along individual axes or several axes, but this is beyond the scope of this course.

We can also look at the characteristic impedance for a rod, starting with the equation for the stress force, Eq. (181). The longitudinal waves will be travelling waves, so we can substitute $\frac{1}{c}\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial x}$ and write:

$$\frac{F}{A} = \frac{Y}{c}\frac{\partial\psi}{\partial t} \tag{186}$$

$$F = \frac{AY}{c}\frac{\partial\psi}{\partial t} = A\sqrt{\rho Y}\frac{\partial\psi}{\partial t} \tag{187}$$

which gives us a formula for the characteristic impedance (defined as above as the force divided by the velocity):

$$Z_0 = A\sqrt{\rho Y} \tag{188}$$

Note that this now depends on the *cross-sectional area* of the rod. For a steel wire with area 1mm^2, using the values above, we find a characteristic impedance around 40 kg/s for *longitudinal* travelling waves.

5.2 Waves in a Fluid

Instead of Young's modulus for a fluid we must consider a different elastic modulus, the Bulk modulus; the difference arises because a fluid cannot support shear stresses The bulk modulus measures how easily a fluid can be compressed, and is defined as the change in pressure divided by the fractional change in volume. So if a fluid at ambient pressure P with volume V is compressed (or expanded) to $P + dP$ with resulting volume $V + dV$, we can write:

$$B = -\frac{dP}{dV/V} = -V\frac{dP}{dV} \tag{189}$$

where the minus sign ensures that increasing pressure results in decreased volume. Note that this equation is sometimes written in terms of the *compressibility* of the gas which is just $\kappa = 1/B$.

We can now derive the equation of motion for an element of a fluid, which will follow the derivation for longitudinal waves in a rod rather closely (see Sec. 5.1). Let's assume (for simplicity) that we have a pipe with cross-section A filled with a fluid with density ρ and bulk modulus B. As before, we will consider an element of the fluid lying between x and $x+\Delta x$ which will be disturbed so that it lies between $x+\psi$ and $x+\Delta x+\psi+\Delta\psi$. We can write the change in the *thickness* of the element:

$$d\psi = \frac{\partial\psi}{\partial x}dx \tag{190}$$

This corresponds to a change in volume of $Ad\psi$, so that the fractional change in volume of the element is:

$$\frac{dV}{V} = \frac{Ad\psi}{Adx} = \frac{\partial\psi}{\partial x} \tag{191}$$

Now, we're looking for the force which acts on the element; this will be given by the change in pressure multiplied by the area. But we have to be a little careful about what pressure we're examining. We are interested in the *change* in pressure away from the ambient pressure P which we will label p. Now we have just found the fractional change in volume (or volume strain), and can use the bulk modulus to relate this to the pressure change:

$$B = -V\frac{dP}{dV} \tag{192}$$

$$\Rightarrow dP = -B\frac{dV}{V} \tag{193}$$

$$\Rightarrow p = -B\frac{\partial\psi}{\partial x} \tag{194}$$

where we have substituted Eq. (191) in the last line. But the difference in pressure across the element is given by:

$$p(x+dx) - p(x) = \left(p(x) + \frac{\partial p}{\partial x}dx\right) - p(x) = \frac{\partial p}{\partial x}dx \tag{195}$$

We can now relate this to the acceleration experienced by the element:

$$-\frac{\partial p}{\partial x}Adx = \rho A dx \frac{\partial^2 \psi}{\partial t^2} \tag{196}$$

$$-\frac{\partial}{\partial x}\left(-B\frac{\partial \psi}{\partial x}\right) = \rho \frac{\partial^2 \psi}{\partial t^2} \tag{197}$$

$$\frac{B}{\rho}\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial t^2} \tag{198}$$

which is, again, a wave equation with velocity $c = \sqrt{B/\rho}$. This derivation has assumed nothing specific about the fluid - in particular, we have not said whether it is a liquid or a gas.

It is important to understand that, for small pressure variations, the same wave equation is obeyed by both the displacement (which we have just demonstrated) and the pressure change. From Eq. (194) we have that $p \propto \partial\psi/\partial z$. It must thus satisfy the same equation as ψ, so the pressure change will satisfy the wave equation as well.

Speed of sound in air is about 340 m/s. Impedance (by analogy to elastic waves in rod) is AB/c or $A\sqrt{B\rho}$. The acoustic impedance is sometimes defined as the characteristic impedance per unit area or $\sqrt{B\rho}$.

5.3 Standing Waves in a Fluid

We can have standing longitudinal waves as well as standing transverse waves (which we first saw in Sec. 4.4). Many musical instruments use these waves as the basis for generating sound. However, the boundary conditions are rather different. If we consider a pipe of air, and apply an oscillator at one end of the pipe this will generate an *antinode* of displacement at the end where the air is driven. If the other end is open then it will also be an antinode of displacement. Notice that the pressure, which we've shown obeys a wave equation which is the same as the equation for the displacement, is $\pi/2$ out of phase with the displacement, so that a pipe open at both ends will have pressure *nodes*. In terms of displacement, the first four modes for a pipe open are shown in Fig. 10.

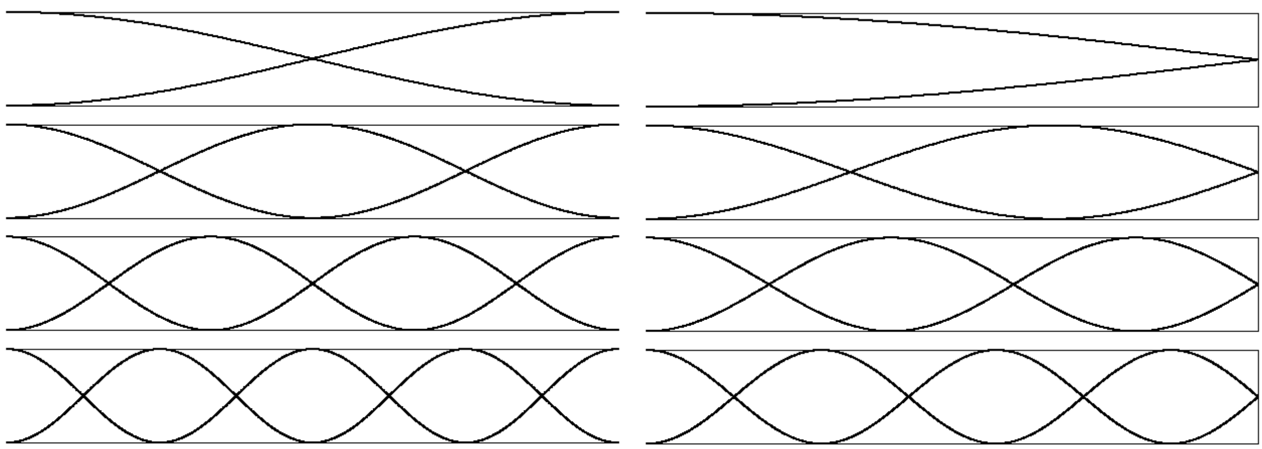

Figure 10: (Left) Standing waves for the first four modes in an open pipe. (Right) Standing waves for the first four modes in a pipe with one end closed.

A pipe which is open at both ends will carry all harmonics of the fundamental frequency. The fundamental mode will have a wavelength equal to twice the length of the pipe (as was the case for a stretched string), so that we can write:

$$\lambda_n = \frac{2L}{n}, n = 1, 2, 3, \ldots \tag{199}$$

$$f_n = \frac{c}{\lambda_n} = \frac{nc}{2L}, n = 1, 2, 3, \ldots \tag{200}$$

By contrast, a pipe which is *closed* at one end has a displacement node enforced there, which means that the allowed standing waves are odd multiples of half-wavelengths; the first four modes for a pipe with one closed end are shown in Fig. 10. The fundamental mode will have a wavelength which is *four* times the length of the column, and it is not possible to fit even harmonics (which will have either nodes or antinodes at both ends) onto the boundary conditions. In this case, we can write:

$$\lambda_n = \frac{4L}{n}, n = 1, 3, 5, \ldots \quad (201)$$

$$f_n = \frac{c}{\lambda_n} = \frac{nc}{4L}, n = 1, 3, 5, \ldots \quad (202)$$

Pipe organs (as often found in churches and concert halls) label their different sets of pipes (or *stops*) by the timbre and the effective length of the longest pipe. The standard set of pipes are open at both ends, and are labelled as 8ft (eight feet or about 2.4m for the longest pipe). Given the speed of sound in air as 330 m/s, what is the frequency of the longest pipe, which corresponds to the lowest note ? How long a pipe is needed to produce the same pitch if one end is closed ?

The frequency is found as $f = c/\lambda$. Now for a pipe open at both ends, the fundamental wavelength is $2 \times L = 4.8m$. This implies that the lowest note has a frequency of $330/4.8 = 69$Hz. If the pipes were closed at one end, then the fundamental wavelength is $4 \times L$, so we would need a pipe half the length (i.e. 4 feet or about 1.2m long). Organs which have to fit into small spaces will often use a stopped pipe (i.e. one which is closed at one end) to produce the sound of an eight foot open pipe with a four foot closed pipe.