1. (a) Note that, for two complex numbers z_1 and z_2,

$$|z_1 + z_2|^2 = (z_1 + z_2)(z_1 + z_2)^\star = (z_1 + z_2)(z_1^\star + z_2^\star) \quad (1)$$
$$= |z_1|^2 + |z_2|^2 + (z_1 z_2^\star + z_1^\star z_2) \quad (2)$$
$$= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1 z_2^\star) \quad (3)$$

[1]

Now we have $z_1 = A_1 e^{i(\omega t + \phi_1)}$ and $z_2 = A_2 e^{i(\omega t + \phi_2)}$ with A_1 and A_2 real. This gives:

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\Re(z_1 z_2^\star) \quad (4)$$
$$= A_1^2 + A_2^2 + 2\mathrm{Re}(A_1 e^{i(\omega t+\phi_1)} A_2 e^{-i(\omega t + \phi_2)}) \quad (5)$$
$$= A_1^2 + A_2^2 + 2\mathrm{Re}(A_1 A_2 e^{i(\phi_1 - \phi_2)}) \quad (6)$$
$$= A_1^2 + A_2^2 + 2A_1 A_2 \cos(\phi_2 - \phi_1) \quad (7)$$

while we can find the phase from: [2]

$$\arg(z_1 + z_2) = \tan^{-1}\left[\frac{\mathrm{Im}(z_1 + z_2)}{\mathrm{Re}(z_1 + z_2)}\right] \quad (8)$$
$$\Rightarrow \theta = \tan^{-1}\left[\frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2}\right] \quad (9)$$

[1]

(b) The diagrams are shown in Fig. 1. One mark for each diagram and one each for phase and amplitude.

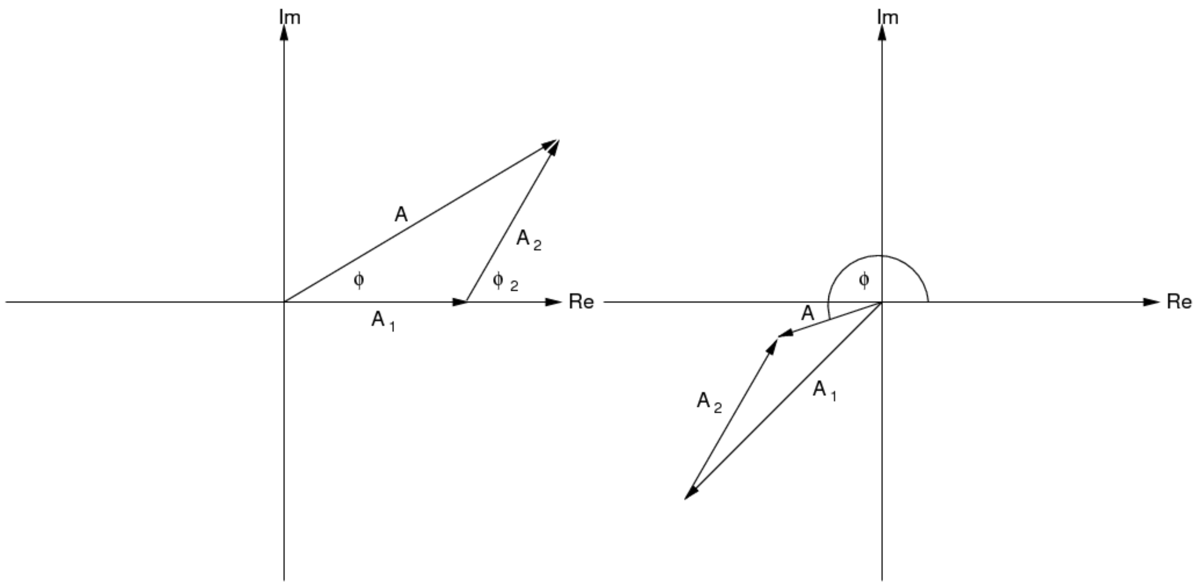

Figure 1: Phasor diagrams for question 1(b)

i. We write:

$$A^2 = 4 + 4 + 8\cos(\pi/3) = 12.000 \quad (10)$$
$$\Rightarrow A = \sqrt{12} = 3.464 \quad (11)$$
$$\tan\phi = \frac{2\sin 0 + 2\sin\pi/3}{2\cos 0 + 2\cos\pi/3} = \frac{\sqrt{3}}{1.5} = 0.577 \quad (12)$$
$$\Rightarrow \phi = \pi/6 \quad (13)$$

(though note that we have a standard formula in the notes which gives $\phi = \phi_2/2$ and $A = 2A\cos(\phi_2/2)$ for equal amplitudes which is also an acceptable route, and should give the same answer).

ii.

$$\begin{aligned}
A^2 &= 9 + 4 + 12\cos(11\pi/12) = 1.409 && (14)\\
\Rightarrow A &= \sqrt{1.409} = 1.187 && (15)\\
\tan\phi &= \frac{3\sin 5\pi/4 + 2\sin\pi/3}{3\cos 5\pi/4 + 2\cos\pi/3} = \frac{\sqrt{3}}{1.5} = 0.347 && (16)\\
\Rightarrow \phi &= \pi + 0.334 = 1.106\pi && (17)
\end{aligned}$$

where we must add π to the phase because of the relative signs of the phasors and the periodicity of tan. A quick check with the phasor diagram shows that this is needed.

(c) We must convert degrees to radians; I find it easier to change the sin to a cos using a phase shift of $\pi/2$ (so $\sin(x) = \cos(x - \pi/2)$. Then we can combine the sum of two cosines into a product. So:

$$\begin{aligned}
27^\circ &= 0.15\pi && (18)\\
121^\circ &= 2\pi/3 && (19)\\
7.5\left(\cos(6.28t + 0.15\pi) - \sin(6.20t - 2\pi/3)\right) &= 7.5\left(\cos(6.28t + 0.15\pi) - \cos(6.20t - 7\pi/6)\right) && (20)\\
&= 15\cos(6.24t - 0.508\pi)\cos(0.04t + 0.658\pi) && (21)
\end{aligned}$$

where we have used a standard rule for cosines. So the frequency of the net motion will be $6.24/2\pi$ s^{-1} or 0.99 s^{-1} and the time between successive beats will be 78.54s (we take $2\pi/\Delta\omega$, with $\Delta\omega = 0.08\text{s}^{-1}$). Note that the time between beats is given by $\omega_1 - \omega_2$ not $(\omega_1 - \omega_2)/2$ - we are interested in the amplitude modulation, not the wave motion. [2] [1] [1]

2. (a) For the oscillator, we have $\omega_0 = \sqrt{s/m} = 60\text{s}^{-1}$ and $\gamma = b/2m = 25$ s^{-1}. We will use the formulae:

$$\begin{aligned}
A &= \frac{F_0}{m}\left(\frac{1}{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}\right)^{\frac{1}{2}} && (22)\\
\tan\phi &= \frac{2\gamma\omega}{\omega^2 - \omega_0^2} && (23)
\end{aligned}$$

Then we have:

i. $A = 0.101, \phi = -0.113$ [2]

ii. $A = 0.074, \phi = -2.182$ [2]

iii. $A = 0.000565, \phi = -3.079$ [2]

(b) i. We know that we can solve the coupled harmonic oscillators with the following quantities defined in lectures:

$$\begin{aligned}
q_a &= (\psi_1 + \psi_2)\sqrt{\frac{m}{2}} = A_a\cos(\omega_a t + \phi_a) && (24)\\
q_b &= (\psi_1 - \psi_2)\sqrt{\frac{m}{2}} = A_b\cos(\omega_b t + \phi_b) && (25)
\end{aligned}$$

where $\omega_a = \sqrt{s/m}$ and $\omega_b = \sqrt{(2K + s)/m}$. If we set $t = 0$ and the initial velocities to zero, then we have:

$$\begin{aligned}
q_a &= A_a\cos(\phi_a) = \frac{\sqrt{m}A_0}{2} && (26)\\
q_a &= A_a\cos(\phi_b) = \frac{\sqrt{m}A_0}{2} && (27)\\
\dot{q}_a &= A_a\sin(\phi_a) = 0 && (28)\\
\dot{q}_b &= A_b\sin(\phi_b) = 0 && (29)
\end{aligned}$$

That is enough information to tell us that $\phi_a = \phi_b = 0$ and that $A_a = A_b = \sqrt{m}A_0/2$. We notice that *both* modes are equally excited. Then we can recover ψ_1 and ψ_2 from the definitions above ($\psi_1 = (q_a + q_b)/\sqrt{2m}$ and $\psi_2 = (q_a - q_b)/\sqrt{2m}$) and write: [2] [2]

$$\begin{aligned}
\psi_1 &= \frac{A_0}{\sqrt{2}}\left(\cos(\omega_a t) + \cos(\omega_b t)\right) && (30)\\
\psi_2 &= \frac{A_0}{\sqrt{2}}\left(\cos(\omega_a t) - \cos(\omega_b t)\right) && (31)
\end{aligned}$$

as required.

ii. We start by finding $\omega_a = \sqrt{81/10} = 2.846\text{s}^{-1}$ and $\omega_b = \sqrt{121/10} = 3.479\text{s}^{-1}$. Then we notice that we can rewrite $\cos(\omega_a t) + \cos(\omega_b t) = 2\cos(\omega t)\cos(\Delta\omega t)$, with $\omega = (\omega_a + \omega_b)/2$ and $\Delta\omega = (\omega_a - \omega_b)/2 = 0.317\text{s}^{-1}$. Then the amplitude of ψ_1 will be zero when $\Delta\omega t = \pi/2$ which gives $t = \pi/(2 \times 0.317) = 4.96$s. The amplitude of ψ_2 will be at a maximum with value $\sqrt{2}A_0$ (because we can write $\cos(\omega_a t) - \cos(\omega_b t) = 2\sin(\omega t)\sin(\Delta\omega t)$). The type of motion is *beats*. [2] [1] [1]

3. (a) First, note that the wavelength is $\lambda = c/\nu = 350/500 = 0.7$m.
   i. We have $\pi/3 = 2\pi\Delta x/\lambda$ so $\Delta x = \lambda/6 = 0.1167$m. [2]
   ii. We have $\Delta\phi = 2\pi\nu\Delta t = \pi$ in radians. [2]
   (b) We have from lectures that $Z_0 = \sqrt{T\mu}$ and $c = \sqrt{T/\mu}$ so $T = Z_0 c = 90$ N and $\mu = Z_0/c = 0.1$kg/m. [2]

4. (a) The key quantities are the characteristic impedances. We have from lectures that $Z_0 = \sqrt{T\mu}$ which will be 4.472 kg/s and 6.325 kg/s for the two strings. Then the reflected amplitude will be $(4.472 - 6.325)/(4.472 + 6.325) = -0.172$ (and a consistency check tells us that it's OK to have a negative sign - this indicates the phase reversal we'll get going from lower impedance to higher). The transmitted amplitude is $2 \times 4.472/(4.472 + 6.325) = 0.828$. [2] [2]
   (b) We want the transmitted amplitude, which will be given by $2 \times 75/(75 + 120) = 0.769$ so the signal will have amplitude 76.9 μV. [2]