COMP2008 Theory III. Assessed coursework: First-Order Logic, Graphs and Games

Robin Hirsch

December 7, 2009

This coursework is compulsory and counts for about 2% of the total marks for COMP2008. Write your answers on paper and hand them in at the departmental office by midday on Thursday 14th January 2010.

Let $\mathcal{L}$ be a first-order language with no constants, no function symbols and two binary predicate symbols, $=$ and E. The structures we will use will be graphs G and we will interpret E as the set of all edges. So if A is a variable assignment then $G, A \models E(x, y)$ if and only if there is an edge from $A(x)$ to $A(y)$ in G.

For each of the following graph properties, write down an $\mathcal{L}$-formula that expresses it.

1. The graph is irreflexive.

   6 marks

2. The graph is symmetric.

   6 marks

3. The graph is transitive.

   6 marks

4. There are no isolated points (points not incident with any edge) in the graph.

   6 marks

5. The maximum "out-degree" of any node (i.e. the maximum number of edges going out from a node) is 2.

   6 marks

OK, that was just to warm up. Let $n \in \mathbb{N}$. Consider the following game $G^k_n(G, H)$, played on two graphs (G, H). There are two players, A and E (sometimes called "Adam" and "Eve", sometimes "Abelard" and "Eloise". Actually they have lots of names, but a common convention is that A is male and E is female.) There are n rounds in this game. Each player has a pile of k pebbles which initially are not in play. A's pebbles are $p_1, p_2, \ldots, p_k$ and E's pebbles are $q_1, q_2, \ldots, q_k$. In the first round A places one of his pebbles (say p_i) on some node of graph G and E must place q_i on a node of H. In the second round A can either pick up p_i and place it on any node of G (in which case E must pick up q_i and place it on any node of H) or he can put a new pebble p_j on some node of G (in this case E must place q_j on a node of H). The game continues like this for n rounds. In each round A either picks up one of the pebbles in play or uses a new pebble and places it on a node of G and E does the corresponding move on H. If at any stage there are pebbles p_i, p_j (for some $i, j \leq k$) with an edge from the position of p_i to the position of p_j in G but there is no edge from the node occupied by q_i to the node occupied by q_j in H then A wins the game. Or if there is no edge from p_i to p_j but there is an edge from q_i to q_j then A wins. Otherwise, if A does not win in any of the n rounds, E wins. There are no draws.

E has a winning strategy in the game $G^k_n(G, H)$ if, no matter what moves A makes, she has a way of playing that guarantees she will win.

6. For any $n \in \mathbb{N}$, let K_n be the complete, reflexive graph on n nodes (i.e. there is an edge between every pair of nodes) and let I_n be the complete irreflexive graph on n nodes (i.e. there is an edge between every pair of distinct nodes, but no reflexive edges). For which of the following games does E have a winning strategy?

   (a) $G^3_3(I_4, I_3)$.

   (b) $G^4_4(I_4, I_3)$.

   (c) $G^4_3(I_4, I_3)$.

   (d) $G^3_9(I_4, I_3)$.

   (e) $G^4_4(I_3, I_4)$.

16 marks

7. Define two graphs G, H such that E has a winning strategy in $G^2_7(G, H)$ but she has no winning strategy in $G^2_8(G, H)$.

7 marks

Now some logic.

8. Write down an $\mathcal{L}$-sentence ϕ_1 such that for any graph G,

$$G \models \phi_1 \Leftrightarrow E \text{ has a w.s. in } G_2^2(I_2, G)$$

Write down an $\mathcal{L}$-sentence ϕ_2 such that for any graph G,

$$G \models \phi_2 \Leftrightarrow E \text{ has a w.s. in } G_2^2(K_2, G)$$

10 marks

The *quantifier depth* of a formla ϕ is written $qd(\phi)$ and it is defined by structured formula induction, as follows. We also define the *universal* and *existential* formulas. Let ϕ, ψ be arbitrary formulas and let x be an arbitrary variable.

- An atomic formula has quantifier depth zero. Atomic formulas are universal and existential.
- For a negated formula $\neg\phi$ we have $qd(\neg\phi) = qd(\phi)$. If ϕ is universal then $\neg\phi$ is existential. If ϕ is existential then $\neg\phi$ is universal.
- For any binary connective $\circ$ we have $qd(\phi \circ \psi) = max(qd(\phi), qd(\psi))$. If ϕ and ψ are both universal (existential) then $(\phi \wedge \psi)$ and $(\phi \vee \psi)$ are also universal (respectively, existential). But note that if ϕ is existential but not universal and ψ is universal but not existential, then $(\phi \wedge \psi)$ and $(\phi \vee \psi)$ are neither existential nor universal. If ϕ is universal and ψ is existential then $(\phi \rightarrow \psi)$ is existential. If ϕ is existential and ψ is universal then $(\phi \rightarrow \psi)$ is universal.
- Finally, $qd(\exists x\phi) = qd(\forall x\phi) = 1 + qd(\phi)$. If ϕ is universal then $\forall x\phi$ is also universal. If ϕ is existential then $\exists x\phi$ is existential.

8. Write down the quantifier depth of each of the following formulas. Also, state if the formula is universal, existential, both or neither.

   (a) $(P(x) \rightarrow Q(y))$

   (b) $(\forall x P(x) \rightarrow \exists y Q(x, y))$

   (c) $(\neg P(x) \vee \forall x \exists y (P(x) \rightarrow Q(x, y)))$

   (d) $\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$

15 marks

9. Suppose that G, H are two graphs and for every existential sentence ϕ of quantifier depth n or less if $G \models \phi$ then $H \models \phi$. This tells you that E has a winning strategy in which game?

12 marks

10. Suppose G, H are two graphs and E has a winning strategy in $G_n^k(G, H)$. Which kinds of sentences ϕ have the property that

$$G \models \phi \Rightarrow H \models \phi?$$

10 marks

Total=100 marks

Optional extra question.

11. Let $G_n'^k(G, H)$ be an n-round, k-pebble game like $G_n^k(G, H)$ but with the following modification. Player A is allowed to place his pebbles on either graph. So in any round he can either place a new pebble on either graph, in which case E must place her pebble on the other graph, or he can pick up any one of his pebbles in play and place it on either graph, in this case E must pick up the corresponding pebble and place it on the other graph. The definition of the winner of the game is the same as for $G_n^k(G, H)$. Suppose the following are equivalent:

    - E has a w.s. in $G_n'^k(G, H)$
    - $G \models \phi$ iff $H \models \phi$

    For which type of formula ϕ are these two statements equivalent?

10 marks