4.4 Standing Waves

So far, we have considered a stretched string with a driver at one end and some form of impedance at the other end. Now we will consider a set-up where the string is *fixed at both ends*; we will assume that there is some means of driving a wave in the string (for instance a guitar pick or a violin bow). This is another example of boundary conditions which we can implement; again, we must have both forms of wave: $\psi = f(x-ct)+g(x+ct)$. If the string has length L then we can write:

$$\begin{aligned} \psi(0,t) &= \psi(L,t) = 0 && (144) \\ \left(\frac{\partial \psi}{\partial t}\right)_{x=0} &= \left(\frac{\partial \psi}{\partial t}\right)_{x=L} = 0 && (145) \end{aligned}$$

Let us consider the general wave solution and assume that we will have a sinusoidal solution of some kind. Then we can write, quite generally,

$$\psi(x,t) = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)} \tag{146}$$

Note that I've written the two so that the time variation shares the same sign and the x variation differs. This is just as general a solution of the wave equation, and makes the maths slightly easier. What can we learn from the boundary conditions ? Let's look at $x = 0$ first.

$$\begin{aligned} \psi(0,t) &= Ae^{i\omega t} + Be^{i\omega t} = 0 && (147) \\ \Rightarrow A &= -B && (148) \\ \psi(x,t) &= Ae^{i\omega t}\left(e^{ikx} - e^{-ikx}\right) && (149) \\ &= 2iAe^{i\omega t}\sin(kx) && (150) \end{aligned}$$

So we've shown something quite important about the form of the wave just from the first condition; notice that the minus sign between the two components is exactly what we'd expect for a wave reflected from a solid wall. Now let's consider $x = L$:

$$\begin{aligned} \psi(L,t) &= 2iAe^{i\omega t}\sin(kL) = 0 && (151) \\ \Rightarrow kL &= n\pi && (152) \end{aligned}$$

where n is an integer, as we know that sin has zeroes for integer multiples of π. So we have a series of solutions for k which will all fit the boundary conditions. We can write:

$$\begin{aligned} k_n &= \frac{n\pi}{L} && (153) \\ \lambda_n &= \frac{2\pi}{k_n} = \frac{2L}{n} && (154) \\ n &= 1, 2, 3, \ldots && (155) \end{aligned}$$

So there are only a certain set of wavelengths that will fit onto the string; this makes sense, as we have imposed a certain length scale on the system, and should expect the resulting solutions to fit into that length. In fact, if we take the idea of normal modes from Sec. 2.5 to its continuous limit, we find that the allowed solutions for the string are the normal modes of the system.

The first five standing waves for a string fixed at both ends are shown in Fig. 8. The patterns along the string are important, and we can define:

- *Nodes* Where the wave amplitude is zero (these points are stationary at all times)
- *Antinodes* Where the wave amplitude is a maximum

There are n maxima or anti-nodes, and $n - 1$ stationary points or nodes for mode n. Since the speed of the wave depends on the material properties of the system (the tension and mass density in a string), the frequencies of the waves must also be arranged in a series which depends on n:

$$\begin{aligned} \omega_n &= \frac{c}{k_n} = \frac{n\pi c}{L} && (156) \\ \nu_n &= \frac{\omega_n}{2\pi} = \frac{nc}{2L} && (157) \end{aligned}$$

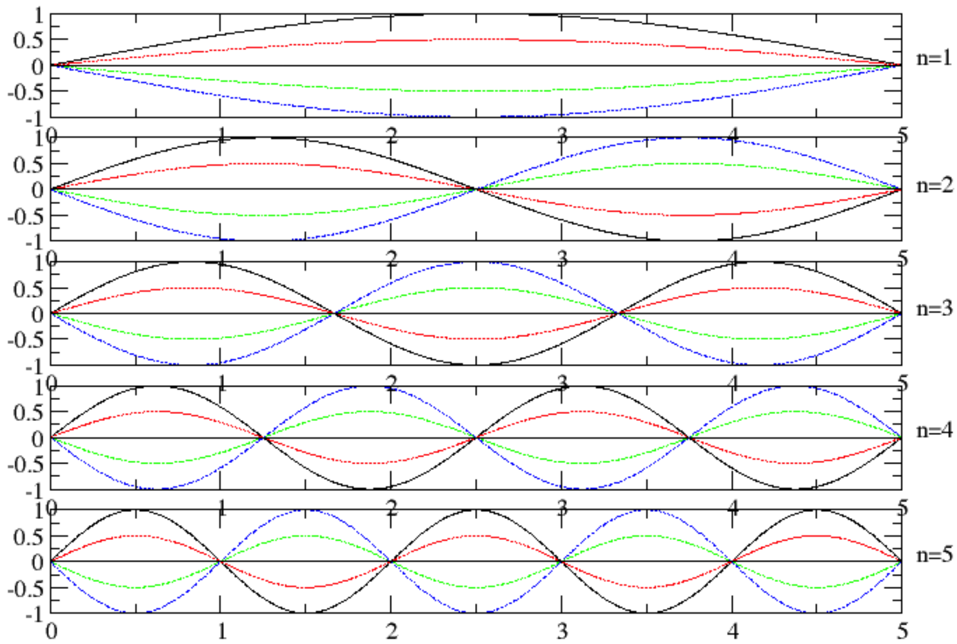

Figure 8: The first five standing waves ($n = 1 - 5$) for a string of length $L = 5$. Waves are displayed at four different times ($\omega t = 0, \omega t = \pi/3, \omega t = 2\pi/3, \omega t = \pi$) to indicate the full range of displacements.

These are the *normal frequencies* which arise because of the boundary conditions imposed on the system. Notice that, for this system (which is rather special), the normal frequencies are integer multiples of the first frequency. In general, the integer multiples of a fundamental frequency are called the *harmonics*.

Notice that the standing waves arise because of *interference* between two waves moving in opposite directions. The formation of standing waves occurs because of *constructive* interference between waves moving in opposite directions.

4.4.1 Resonance

A string fixed at both ends can support a series of different waves; the lowest frequency wave (with longest wavelength) is called the *fundamental* frequency. In the same way, most physical systems will have natural frequencies as we saw in the case of the harmonic oscillator. These frequencies are also known as resonant frequencies, as an excitation applied at this frequency will generate a resonance; as we mentioned briefly in Sec. 2.3, the amplitude of a damped, driven harmonic oscillator will be at a maximum when the driving frequency equals the natural or resonant frequency.

If a complex excitation is applied to an object, causing it to vibrate, it will tend to pick out its resonant frequencies, as these will respond with a large amplitude. A sharp impulse (like a kick or a whack from a stick) delivered to a simple oscillator will give a motion which is somewhat chaotic initially but will settle down to an oscillation at the natural frequency. The impulse consists of many frequencies (which can be investigated with Fourier analysis) but the oscillator will only respond strongly at its resonant frequency. Most objects which vibrate will have multiple resonant frequencies (depending on the boundary conditions which they impose).

4.4.2 Nodes and Antinodes

The wave which is supported on a string fixed at both ends have a very different form to those we used above: the time variation has been separated from the spatial variation:

$$\psi(x,t) = \mathrm{Re}\left[2iAe^{i\omega t}\sin(kx)\right] = -2A\sin(\omega t)\sin(kx) \tag{158}$$

So each point on the string is *in phase* with all other points on the string, and they undergo simple harmonic motion, with the amplitude of the motion depending on the position along the string. The points on the string where $kx = n\pi$ for n an integer will have zero amplitude (as $\sin(n\pi) = 0$); these are the nodes. The points on the string where $kx = (n + \frac{1}{2})\pi$ for n an integer will have maximum amplitude of $2A$.

The positions x of the nodes and antinodes are found by substituting $k = 2\pi/\lambda$ into the expressions given above:

$$
\begin{aligned}
x_{\text{node}} &= \frac{n\pi}{2\pi/\lambda} = n\frac{\lambda}{2} && (159)\\
x_{\text{antinode}} &= \frac{\left(n+\frac{1}{2}\right)\pi}{2\pi/\lambda} = \left(n+\frac{1}{2}\right)\frac{\lambda}{2} && (160)
\end{aligned}
$$

The nodes are separated from each other by half a wavelength (which is easy to see by calculating $x(n = 1) - x(n = 0)$), and the anti-nodes are also separated from each other by half a wavelength (this is by definition: these are the zeroes and extrema of a sinusoidal function and must be separated by half a wavelength). Each node is separated by $\lambda/4$ from the nearest anti-nodes.

As an example, consider a piano string (this is an oversimplification - pianos have two or three strings for each note). The mass per unit length is typically around 0.01 kg/m and the tension is around 800N (equivalent to the gravitational force exerted by the mass of an average person). The velocity of waves on this string will be $\sqrt{800/0.01} = 283$m/s. If the wire is 0.6m long and the ends are fixed, then the lowest allowed wavelength will be 1.2m (with other waves with wavelengths 0.6m and 0.4m also allowed). The frequencies of these waves will be:

$$
\begin{aligned}
f_1 &= \frac{c}{\lambda_1} = 236\text{Hz} && (161)\\
f_2 &= 472\text{Hz} && (162)\\
f_3 &= 708\text{Hz} && (163)
\end{aligned}
$$

The fundamental will be just below middle C (which is about 262 Hz).

4.4.3 Harmonics

The fundamental frequency for an ideal vibrating string will arise from the longest wavelength which can be supported. This is $\lambda_1 = 2L$ for a string of length L. We then write:

$$
\begin{aligned}
f_1 &= \frac{c}{\lambda_1} = \frac{c}{2L} && (164)\\
f_n &= \frac{2}{\lambda_n} = \frac{nc}{2L} && (165)
\end{aligned}
$$

for $n = 1, 2, 3, \ldots$ where $c = \sqrt{T/\mu}$. A *harmonic* is simply a vibration at an integer multiple of the fundamental frequency. Systems with the *same* boundary conditions at both ends (e.g. fixed strings, open air columns etc.) will vibrate with all harmonics of the fundamental frequency, while systems with different boundary conditions (i.e. one end fixed and one free) will vibrate with only the odd harmonics. Most musical instruments will produce some harmonics when the fundamental frequency is excited (and it is also possible to excite the harmonics only, for instance by overblowing a wind instrument or inducing a node on a stringed instrument by touching the string lightly).

4.4.4 Other Boundary Conditions

It is worth noting that there are other ways to set up standing waves. If a continuous wave (a sinusoidal wave, for example) is propagated along a semi-infinite string towards a fixed end, then it will reflect with a phase change of π (or a change in sign). If we take the incoming wave to be $\psi_i = A\cos(kx - \omega t + \phi)$ the reflected wave will be $\psi_r = -A\cos(kx + \omega t + \phi)$. Then we will find:

$$
\begin{aligned}
\psi(x,t) &= \psi_i(x,t) + \psi_r(x,t) && (166)\\
&= A\cos(kx - \omega t) - A\cos(kx + \omega t) && (167)\\
&= -2A\sin(kx)\sin(\omega t + \phi) && (168)
\end{aligned}
$$

which is exactly the form we saw above in Eq. (158).

This suggests that it is worth examining other boundary conditions on waves, to explore the full range of standing waves.

Instead of a fixed end (or a node) we could allow the medium to have an antinode (or free end) at one or both ends. This is quite common when thinking about sound waves and we will consider pipes in detail in Sec. 5.3. For now, let us consider what an antinode at both ends would mean (think of holding a thin steel ruler lightly in the centre and wiggling it up and down: you impose a node at the centre but nothing at the ends).

We will take advantage of our knowledge of the system to suggest the form:

$$\psi(x,t) = 2A\cos(kx)\sin(\omega t) \tag{169}$$

We will again impose the condition that $kx = n\pi$ at $x = 0$ and $x = L$ for n an integer, but this time we are forcing an antinode to be at the boundaries. The allowed series of wavelengths will then be given by:

$$k_n = \frac{n\pi}{L} \tag{170}$$

$$\Rightarrow \lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n} \tag{171}$$

which is the same set of wavelengths (really all we've done to the system is introduce a phase shift of $\pi/2$). We'll consider the effect of a free end and a fixed end when we look at sound waves in pipes.