3 Basic Waves (J&S 16)

Waves can be found in many different areas of physics, including electromagnetic fields, pressure variations in a gas, solids and stretched strings. We will consider many of these as examples of waves as we go along. However, a general definition of a wave is rather hard. Waves do not require a medium to propagate (e.g. electromagnetic waves) though many waves propagate only through a material (e.g. sound). Waves do not need to be periodic (though often are periodic). They involve the transfer of energy (though standing waves appear to violate this definition) and consist of a disturbance which propagates in time and space. We will now derive the equation for the simplest wave propagation.

3.1 Stretched string

Let us consider a string with mass per unit length μ under a tension T. We will investigate what happens when it is displaced slightly perpendicular to its length, and derive a *wave equation* which will turn out to be quite general. So that we know where we're going, here is the wave equation in one dimension:

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}, \tag{72}$$

where c is the speed of the wave. We will now derive this for small displacements of our string. The set up we will consider is illustrated

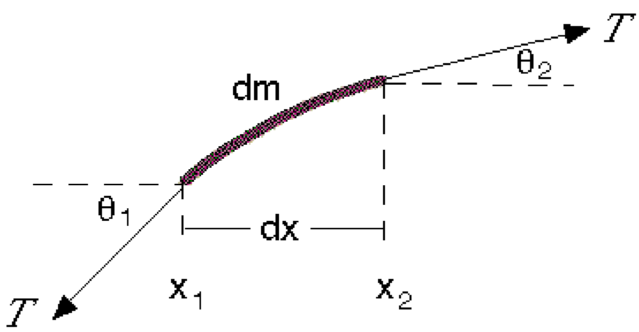

Figure 6: A stretched string displaced away from the horizontal

We assume that the vertical displacement of the string is given by ψ but that it lies initially along the x axis (which will give us a *transverse* wave: one where the displacement is in a different direction to the propagation). We are interested in the forces acting on the string given the displacement; the force can be written, using Newton's third law, as:

$$F = ma = \mu\Delta x\frac{\partial^2\psi}{\partial t^2} \tag{73}$$

What force will act on the displaced string ? It will be a component of the tension, which we will need to resolve into components along x and along the displacement, as the string is no longer purely along x. For small displacements we could write, by inspection:

$$F = T\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x} - T\left(\frac{\partial\psi}{\partial x}\right)_x \tag{74}$$

where we are assuming that the tension does not vary with position (if it did, we would have a rather more complex situation involving *dispersion* which we will discuss briefly later in the course). Another way to arrive at this formula is to resolve the tension geometrically, which gives:

$$F_\psi = T\tan(\theta_{x+\Delta x}) - T\tan(\theta_x) \tag{75}$$

But it is fairly easy to show that $\sin\theta \simeq \tan\theta$ for small values of θ (an approximation which has an error which is third order in θ – i.e. $\sin\theta - \tan\theta = \frac{1}{2}\theta^3$), and we can write $\tan\theta = d\psi/dx$, which just gives us Eq. (74) again. Now we equate these two forces:

$$T\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x} - T\left(\frac{\partial\psi}{\partial x}\right)_x = \mu\Delta x\frac{\partial^2\psi}{\partial t^2} \tag{76}$$

If we recall the formal definition of a derivative, we can write:

$$\frac{\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial\psi}{\partial x}\right)_{x}}{\Delta x} = \frac{\partial^2\psi}{\partial x^2} \text{ as } \Delta x \to 0 \tag{77}$$

If we rearrange Eq. (76) by dividing through by Δx and μ and substituting in Eq. (77), then we find:

$$\frac{T\left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x} - T\left(\frac{\partial\psi}{\partial x}\right)_{x}}{\Delta x} = \mu\frac{\partial^2\psi}{\partial t^2} \tag{78}$$

$$\frac{T}{\mu}\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial t^2} \tag{79}$$

which has the form of a wave equation with velocity $\sqrt{T/\mu}$. A quick check on dimensions (T is a force with dimensions mass (M) $\times$ length (L) $\times$ time$^-2$ (T^{-2}) while μ is mass per unit length (ML^{-1}) so $\sqrt{T/\mu}$ has dimensions LT^{-1} which is a velocity) suggests that this is reasonable. Experience with elastic bands or strings suggests that the note produced gets higher with increasing tension or decreasing mass density: if the note is proportional to the velocity then our formula makes physical sense (we will see later that this is indeed what we'd expect).

What about the signs ? If the string is pulled up then the second derivative with x will be negative (we'll have a maximum) which means that there is a downward force, as we'd expect (and require for sensible motion). What assumptions have we made to get to this point, and how might they break down ?

- Small displacements of the string
- T and μ are independent of position and displacement (this might change for large displacements)
- Resolved force requires a small angle between x and $x + \Delta x$

So provided we don't make too large a displacement of the string, our derivation will hold. We will find that the same equation (though with different displacement variable, and different physical quantities contributing to the velocity) appears in many areas, such as:

- Sound (pressure waves in a gas or solid)
- Plasma (variation in electron density)
- Electromagnetic waves (fields vary as waves)
- Electrical circuits (electrical charge varies with time and position)

The equation can be written as a single term operating on ψ:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0 \tag{80}$$

where the operator in brackets is called the d'Alembertian (named after the French mathematician and physicist d'Alembert).

3.2 Solving the wave equation

Now that we have an equation which is obeyed by the displacement of a stretched string, we need to look for solutions. This will also allow us to understand the general behaviour of wave equations. We can start by noticing that we need the second time derivative to equal the second position derivative multiplied by c^2. This implies that we need the *same* functional form for x and t but with a multipicative constant (and dimensionally a constant with LT^{-1} will convert from time to distance). If we combine the time and position variables into a single variable $x - ct$ (or $x + ct$), we will be able to write $\psi(x, t) = f(x - ct)$ or $\psi(x, t) = g(x + ct)$. We want to check that these will satisfy the wave equation, so we write $u = x - ct$ and use the chain rule:

$$\begin{aligned}
u &= x - vt && (81)\\
\psi &= f(u) && (82)\\
\frac{\partial\psi}{\partial x} &= \frac{\partial\psi}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial\psi}{\partial u} && (83)\\
\frac{\partial\psi}{\partial t} &= \frac{\partial\psi}{\partial u}\frac{\partial u}{\partial t} = -c\frac{\partial\psi}{\partial u} && (84)\\
\frac{\partial^2\psi}{\partial x^2} &= \frac{\partial^2\psi}{\partial u^2}\frac{\partial u}{\partial x} = \frac{\partial^2\psi}{\partial u^2} && (85)\\
\frac{\partial^2\psi}{\partial t^2} &= \frac{\partial^2\psi}{\partial u^2}\frac{\partial u}{\partial t} = c^2\frac{\partial^2\psi}{\partial u^2} && (86)\\
\Rightarrow \frac{\partial^2\psi}{\partial t^2} &= c^2\frac{\partial^2\psi}{\partial x^2} && (87)
\end{aligned}$$

So this will satisfy the wave equation. The same result can be found for $\psi(x, t) = g(x + ct)$ (though in the second line the factor of c is positive), so using the fact that the wave equation is linear, we write:

$$\psi(x, t) = f(x - ct) + g(x + ct) \tag{88}$$

This is, it turns out, the most general form of the solution for the wave equation. It's interesting to note that we can rewrite the wave equation itself in terms of differentials with respect to u and v (noting, for instance, that $\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}$) to give:

$$\frac{\partial^2\psi}{\partial u\partial v} = 0 \tag{89}$$

The most general solution for this equation is $\psi = f(u) + g(v)$ which is, or course, just Eq. (88) again. Yet another way of rewriting the wave equation is this:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\psi = 0 \tag{90}$$

which is again solved by Eq. (88).

The second time derivative (and knowledge of the physical world) might lead us to think of a sinusoidal or periodic function as a solution, but to use this we will need a dimensionless argument. So we divide u or v by some characteristic length; in the case of a periodic function the distance between repeats, which we call the *wavelength* λ seems most appropriate. Notice that rescaling u and v by a constant ($u \to Cu$) does *not* change the wave velocity or the wave equation. Let's go one step further, and incorporate the factor of 2π that we know is the period of sinusoidal functionals, giving:

$$\psi(x, t) = f(\frac{2\pi}{\lambda}x - \frac{2\pi c}{\lambda}t) + g(\frac{2\pi}{\lambda}x + \frac{2\pi}{\lambda c}t) \tag{91}$$

We often define two new variables:

- $k = 2\pi/\lambda$ the wavenumber (we will see later that it is actually a vector quantity, the *wavevector*)
- $\omega = 2\pi c/\lambda = 2\pi/T = ck$ the angular frequency (which is 2π divided by the period, T, or 2π multiplied by the frequency $\nu = 1/T$)

We can now write:

$$\psi(x, t) = f(kx - \omega t) + g(kx + \omega t) \tag{92}$$

where the quantity $kx \pm \omega t$ is known as the *phase.*

So, if we have a periodic solution with period T in time and distance λ between peaks (or troughs), when we advance the position x by λ *or* the time by T the phase increases by 2π, which is what is required for periodic wave motion. If we look at a fixed point in space, then the time between peaks passing that point is $\lambda/c = T$. A summary of these different quantities:

- c or v is the velocity of the wave (the speed at which peaks or troughs move—we will encounter other velocities later)
- λ is the wavelength, the distance between peaks (or troughs) in a periodic function
- f or ν is the frequency, the rate at which peaks or troughs pass a given point for a periodic function $\nu = \frac{c}{\lambda}$

- ω is the *angular* frequency $\omega = 2\pi\nu = \frac{2\pi c}{\lambda} = kc$
- T is the period which is the time taken to perform one repeat in a periodic function (i.e. from peak to peak or trough to trough) $T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{\lambda}{c}$
- k is the wave number (the magnitude of the wave vector $\mathbf{k}$ which we will encounter later) $k = \frac{2\pi}{\lambda} = \omega/c$
- ϕ is an initial, constant phase

All of these different parameters can be used in writing a wave, so long as the appropriate conversion factors are used. Let's list a number of different ways that a periodic sinusoidal wave could be written:

$$
\begin{aligned}
\psi(x,t) &= A\cos(kx - \omega t + \phi) && (93)\\
&= Re\left[Ae^{i(kx-\omega t+\phi)}\right] && (94)\\
&= Re\left[ae^{i(kx-\omega t)}\right] && (95)\\
&\quad \left(a = Ae^{i\phi}\right) && (96)\\
\psi(x,t) &= A\cos\left(\frac{2\pi}{\lambda}(x-ct)+\phi\right) = A\cos\left(2\pi(\frac{x}{\lambda}-\nu t)+\phi\right) && (97)
\end{aligned}
$$

Remember that, when we use the complex exponential form (which is almost always the easiest form to use) we have to take the real part when considering the physical quantity we're measuring.