1.22 Let G be a disconnected graph. By Theorem 1.11, $\overline{G}$ is connected. Prove that if u and v are any two vertices of $\overline{G}$, then $\text{diam}(\overline{G}) \leq 2$.

Proof. Let G be a disconnected graph with vertex set V and let $\overline{G}$ be its complement. We show that for any $u, v \in V$ the distance $d_{\overline{G}}(u, v) \leq 2$, whence $\text{diam}(\overline{G}) \leq 2$.

If $u = v$ then $d_{\overline{G}}(u, v) = 0 \leq 2$, so assume $u \neq v$.

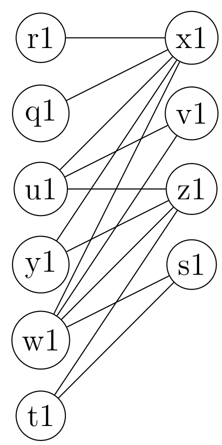
There are two cases.

Case 1: u and v lie in different connected components of G. Then u and v are not adjacent in G, so the edge uv belongs to $\overline{G}$. Hence $d_{\overline{G}}(u, v) = 1 \leq 2$.

Case 2: u and v lie in the same connected component of G. Since G is disconnected, there exists a vertex $w \in V$ that lies in a component different from the component containing u (and thus different from the component containing v). Therefore neither uw nor vw is an edge of G, so both uw and vw are edges of $\overline{G}$. Thus there is a path $u - w - v$ of length 2 in $\overline{G}$, so $d_{\overline{G}}(u, v) \leq 2$.

In every possible case we have $d_{\overline{G}}(u, v) \leq 2$. Since u and v were arbitrary vertices of $\overline{G}$, it follows that $\text{diam}(\overline{G}) \leq 2$. □

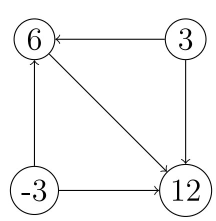
1.24 Determine whether the graphs G_1 and G_2 are bipartite. If a graph is bipartite, then redraw it indicating the partite sets; if not, then give an explanation as to why the graph is not bipartite.



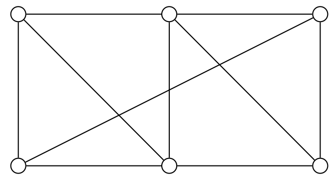
1. G_1 is bipartite.

2. G_2 is not bipartite because it contains at least one odd cycle of order 5.

1.33 A diagraph D has vertex set -3, 3, 6, 12 and $i, j \in D$ if $i \neq j$ and j is a multiple of i.



2.4 Give an example of a Graph G of order 6 and size 10 such that ∂(G) = 3 and Δ(G) = 4.



2.6 Prove that if a graph of order $3n$ ($n \geq 1$) has n vertices of each of the degrees

n - 1, n, and n +1, then n is even.