1.22 Let $G$ be a disconnected graph. By Theorem 1.11, $\overline{G}$ is connected. Prove that if $u$ and $v$ are any two vertices of $\overline{G}$, then $\text{diam}(\overline{G}) \leq 2$.

*Proof.* Let $G$ be a disconnected graph with vertex set $V$ and let $\overline{G}$ be its complement. We show that for any $u, v \in V$ the distance $d_{\overline{G}}(u, v) \leq 2$, whence $\text{diam}(\overline{G}) \leq 2$.

If $u = v$ then $d_{\overline{G}}(u, v) = 0 \leq 2$, so assume $u \neq v$.

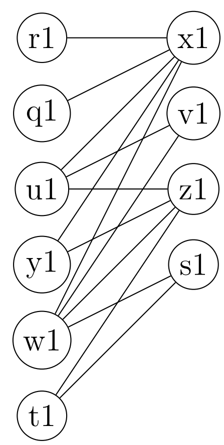
There are two cases.

*Case 1: $u$ and $v$ lie in different connected components of $G$.* Then $u$ and $v$ are not adjacent in $G$, so the edge $uv$ belongs to $\overline{G}$. Hence $d_{\overline{G}}(u, v) = 1 \leq 2$.

*Case 2: $u$ and $v$ lie in the same connected component of $G$.* Since $G$ is disconnected, there exists a vertex $w \in V$ that lies in a component different from the component containing $u$ (and thus different from the component containing $v$). Therefore neither $uw$ nor $vw$ is an edge of $G$, so both $uw$ and $vw$ are edges of $\overline{G}$. Thus there is a path $u - w - v$ of length 2 in $\overline{G}$, so $d_{\overline{G}}(u, v) \leq 2$.

In every possible case we have $d_{\overline{G}}(u, v) \leq 2$. Since $u$ and $v$ were arbitrary vertices of $\overline{G}$, it follows that $\text{diam}(\overline{G}) \leq 2$. □

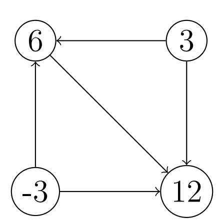
1.24 Determine whether the graphs $G_1$ and $G_2$ are bipartite. If a graph is bipartite, then redraw it indicating the partite sets; if not, then give an explanation as to why the graph is not bipartite.



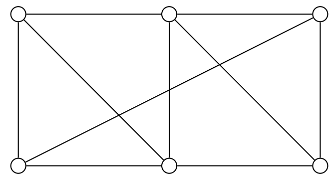

1. $G_1$ is bipartite.

2. $G_2$ is not bipartite because it contains at least one odd cycle of order 5.

1.33 A diagraph D has vertex set -3, 3, 6, 12 and $i, j \in D$ if $i \neq j$ and j is a multiple of i.



2.4 Give an example of a Graph $G$ of order 6 and size 10 such that $\partial$(G) = 3 and $\Delta$(G) = 4.



2.6 Prove that if a graph of order $3n$ ($n \geq 1$) has $n$ vertices of each of the degrees

$n$ - 1, $n$, and $n$ +1, then $n$ is even.

*Proof.* Let $G$ be a graph of order $3n$ with $n$ vertices of each of the degrees $n-1$, $n$, and $n+1$. By Theorem 2.1, the sum of degrees of a graph is $2m$, where $m$ is the size of $G$. $2m$ is even, so $3n^2$ must be even. Since 3 is odd, $n^2$ must be even, and thus $n$ is even. □