4.8 Prove that if every vertex of a graph G has degree at least 2, then G contains a cycle.

Proof. Assume G does not contain a cycle. If G is connected and acyclic, then G must be a tree. If G is disconnected then it must be a forest. In either case, there exists at least one leaf vertex (a vertex of degree 1). This contradicts the claim that every vertex has degree at least 2. Therefore, G must contain a cycle. □

4.18 A certain tree T of order n contains only vertices of degree 1 or 3. Show that T has exactly $\dfrac{n-2}{2}$ vertices of degree 3.

Proof. x + 3y = 2(n-1)
x + 3y = 2(x+y-1)
x + 3y = 2x + 2y - 2
y = x - 2
n = x + y = x + x - 2 = 2x - 2
n + 2 = 2x
x = (n + 2)/2
y = (n + 2)/2 - 2 = (n - 2)/2
So, T has exactly (n-2)/2 vertices of degree 3. □

4.22 Let T be a tree of order n. Show that the size of the complement of $overline T$ of T is the same as the size of $K_n - 1$.

Proof. A complete graph K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. A tree T with n vertices has $n-1$ edges.

The complement $\overline{T}$ contains every edge of K_n that is not in T, so

$$|E(\overline{T})| = \binom{n}{2} - (n-1) = \frac{n(n-1)}{2} - (n-1).$$

Simplifying:

$$|E(\overline{T})| = (n-1)\left(\frac{n}{2} - 1\right) = \frac{(n-1)(n-2)}{2} = \binom{n-1}{2}.$$

Thus, $\overline{T}$ has the same number of edges as K_{n-1}. □

5.4 Prove that if v is a cut vertex of G, then v is not a cut vertex of the complement of G.

Proof. Assume v is a cut vertex of $\overline{G}$. After removing v from G, G contains at least two components. Let them be U and W such that V(U) $\neq$ V(W). Let x $\in$ V(U) and y $\in$ V(W). Since v is a cut vertex of G, there is no edge between x and y in G. However, in $\overline{G}$, there must be an edge between x and y. Because E(k) - E(G) = E($\overline{G}$), where k is the complete graph. So, removing v from $\overline{G}$ does not disconnect the graph. Therefore, the assumption is false and v is not a cut vertex of $\overline{G}$. □

5.6 Prove that a 3-regular graph G has a cut vertex if and only if G has a bridge.

Proof. ($\Rightarrow$) Let G be a 3-regular graph and suppose v is a cut vertex of G. $G - v$ has at least two components, let them be $H_1, \ldots, H_k$ with $k \geq 2$. Every component H_i contains at least one vertex adjacent to v since the graph is connected. So the 3 neighbors of v are distributed among the components. Because v has degree 3, we must have $2 \leq k \leq 3$. In either case there is at least one component H_j that contains exactly one neighbour u of v: for $k = 2$ the neighbour counts are 1 and 2, and for $k = 3$ they are $1, 1, 1$. Let $e = uv$ be the unique edge joining v to H_j. Any path from a vertex of H_j to a vertex outside H_j must pass through v, and thus must use the edge uv. So deleting e disconnects H_j from the rest of the graph, so e is a bridge of G. Therefore a cut vertex in a 3-regular graph must have a bridge.

($\Leftarrow$) Assume G does not have a cut-vertex. But G has a bridge, e. G-e is disconnected and contains two components. e must be incident to two vertices, u and v. Since G is 3-regular, both u and v must have two other edges incident to them. Since removing e disconnects G, removing u or v would have the same effect. So either u or v must be a cut vertex. Since G is 3-regular, we can say there is a cut vertex in G by Theorem 5.1. This contradicts the assumption that G does not have a cut vertex. Therefore, if G has a bridge, then G must have a cut vertex. □