2.20 Show that if G is a connected graph that is not regular, then G contains adjacent vertices u and v such that deg(u) $\neq$ deg(v).

Proof. Since G is not regular, there exist vertices x and y with $\deg(x) \neq \deg(y)$. Because G is connected, there is a path $P : x = v_0, v_1, \ldots, v_k = y$ between them.

Case 1: (x and y are adjacent) If $k = 1$, then x and y are adjacent and we are done.

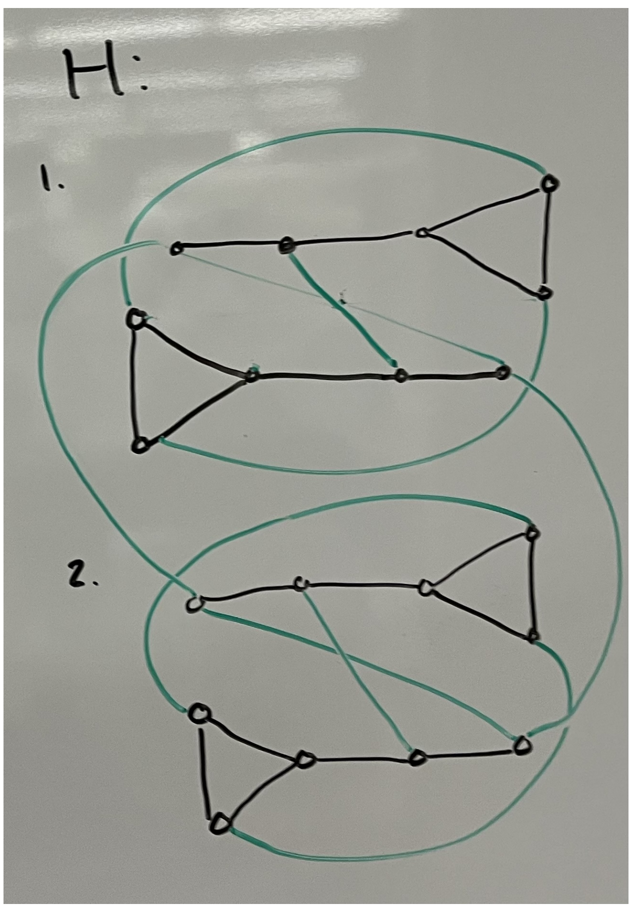
Case 2: (x and y are not adjacent) If $k \geq 2$, consider the sequence of degrees

$$\deg(v_0), \deg(v_1), \ldots, \deg(v_k).$$

This sequence begins with $\deg(v_0) = \deg(x)$ and ends with $\deg(v_k) = \deg(y)$, which are different. Therefore, there must exist some index i with $0 \leq i < k$ such that $\deg(v_i) \neq \deg(v_{i+1})$. Since v_i and v_{i+1} are adjacent in G, this gives the desired pair of adjacent vertices of unequal degree.

Thus, in all cases G contains adjacent vertices u, v with $\deg(u) \neq \deg(v)$. □

2.22



Theorem 2.7: $r \geq \delta$(G). Here, r = 3 and δ(G) = 3, so the theorem holds. Order = 20

2.26

a Show that a graph G is regular if and only if its complement is regular.

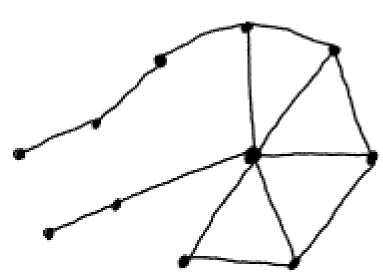
Proof. ($\Rightarrow$) Suppose G is r-regular. Then every vertex has degree r. The degree of a vertex v in $\overline{G}$ is given by deg(v) = n - 1 - deg(v) in G, where n is the order of G. Since every vertex in G has degree r, every vertex in $\overline{G}$ has degree n - 1 - r. Therefore, $\overline{G}$ is (n - 1 - r)-regular.
($\Leftarrow$) Suppose $\overline{G}$ is s-regular. Then every vertex has degree s. The degree of a vertex v in G is given by deg(v) = n - 1 - deg(v) in $\overline{G}$, where n is the order of G. Since every vertex in $\overline{G}$ has degree s, every vertex in G has degree n - 1 - s. Therefore, G is (n - 1 - s)-regular.
Thus, a graph G is regular if and only if its complement is regular. □

b Show that if G and $\overline{G}$ are r-regular for some nonnegative integer r, then G has odd order.

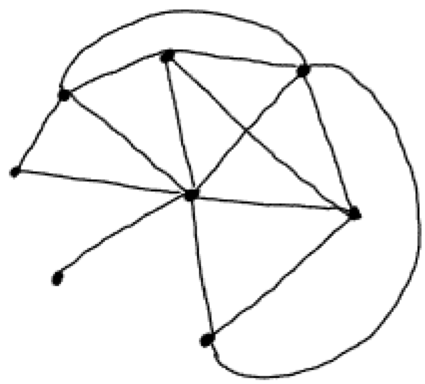
Proof. Let the order of G be n. Then the degree of each vertex in $\overline{G}$ is n - 1 - r. Since $\overline{G}$ is r-regular, we have n - 1 - r = r. Therefore, n - 1 = 2r, and n = 2r + 1. Since r is a nonnegative integer, 2r is even, and adding 1 gives an odd number. So, n is odd, and G has odd order. □

2.32 Determine if the following sequences are graphic. If so, construct a graph.

b S_1: 6,3,3,3,3,2,2,2,2,1,1
S_2: 2,2,2,2,2,2,1,1,1,1
S_3: 2,2,2,1,1,1,1,1,1
S_4: 1,1,1,1,1,1,1,1. Here, we have an even number of 1s, and can recognize this is graphic.



d S_1: 7,5,4,4,4,3,2,1
S_2: 4,3,3,3,2,1,0
S_3: 2,2,2,1,1
S_4: 1,1,1,1. Here, we have an even number of 1s, and can recognize this is graphic.



2.33 Prove that for every integer x with $0 \leq x \leq 5$, the sequence x,1,2,3,5,5 is not graphical.

Proof. The sum of all degrees must be even. Here, the sum is x + 16. So x must be even. Therfore, $x \neq 1, 3, 5$. That leaves 0, 2, and 4. We can use Theorem 2.10 to show each case is not graphical.
x = 0: S_1: 1,2,3,5,5 or 5,5,3,2,1. This is not graphical because there is a vertex of degree 5, but only 4 other vertices.
x = 2: S_1: 2,1,2,3,5,5 or 5,5,3,2,2,1
S_2: 4,3,1,1,0. This is not graphical because there is a vertex of degree 4, but only 3 other vertices.
x = 4:
S_1: 4,1,2,3,5,5 or 5,5,4,3,2,1
S_2: 4,3,2,1,0. This is not graphical because there is a vertex of degree 4, but only 3 other vertices.
Therefore, for every integer x with $0 \leq x \leq 5$, the sequence x,1,2,3,5,5 is not graphical.
□