

- 6.2 Let G_1 and G_2 be two Eulerian graphs with no vertex in common. Let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Let G be the graph obtained from $G_1 \cup G_2$ by adding the edge v_1v_2 . What can be said about G ?

Since G_1 and G_2 are Eulerian, every vertex in each graph has even degree. If we add an additional edge v_1v_2 to form G , then the degrees of v_1 and v_2 each increase by 1, making them odd. Therefore, G has exactly two vertices of odd degree and cannot be Eulerian.

- 6.6 Let G be a connected regular graph that is not Eulerian. Prove that if \overline{G} is connected, then \overline{G} is Eulerian.

Proof. □

- 6.8 (a) Show that every nontrivial graph G has a closed spanning walk that contains every edge of G exactly twice.

- 6.10 Let G be a 6-regular graph of order 10 and let $u, v \in V(G)$. Prove $G - v$ and $G - u - v$ are all Hamiltonian.

Proof. Let G be a 6-regular graph of order 10, and let $u, v \in V(G)$. In $G - v$, the order is 9. Since G is 6-regular, removing v decreases the degree of each of its 6 neighbors by 1, so every vertex in $G - v$ has degree at least 5. Thus $\delta(G - v) \geq 5$. By Dirac's Theorem, if a graph H of order $n \geq 3$ satisfies $\delta(H) \geq n/2$, then H is Hamiltonian. Here, $n = 9$ and $\delta(G - v) = 5 \geq 9/2$, so $G - v$ is Hamiltonian. Now consider $G - u - v$. The order is 8. Removing u and v decreases the degree of each vertex by at most 2 (if the vertex was adjacent to both u and v), so $\delta(G - u - v) \geq 6 - 2 = 4$. Since $4 = 8/2$, by Dirac's Theorem, $G - u - v$ is also Hamiltonian. Therefore, both $G - v$ and $G - u - v$ are Hamiltonian □

- 6.12 Let G be a 3-regular graph of order 12 and H a 4-regular graph of order 11.

- (a) Is $G + H$ Eulerian?

Yes. Originally in G , $\deg = 3$. In $G + H$, each vertex in G gains an edge to all 11 vertices in H , so $\deg = 3 + 11 = 14$.

In H , $\deg = 4$, and in $G + H$ each vertex in H gains an edge to all 12 vertices in G , so $\deg = 4 + 12 = 16$.

Since all vertices have even degree, $G + H$ satisfies the condition for being Eulerian.

- (b) Is $G + H$ Hamiltonian?

Yes. By corollary 6.7, a graph of order $n \geq 3$ is Hamiltonian if $\delta(G) \geq \frac{n}{2}$.

Here, $n = 12 + 11 = 23$ and $\delta(G + H) = 14$, since $\min(14, 16) = 14$.

Because $14 \geq \frac{23}{2}$, the condition holds, so $G + H$ is Hamiltonian