1.1.9 Show that the summation identity

$$1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

holds for all $n \in \mathbb{N}$.

Proof. In the case of $n = 1$, we have the left hand side of (1) is 1, while the right hand side is $\frac{1(2)(3)}{6} = 1$. So (1) holds for $n = 1$.
Assume (1) holds for some $k \in \mathbb{N}$. We will now show that under this assumption, the identity holds for $k + 1$. We see that

$$\begin{aligned}
1 + 4 + 9 + \cdots + (k+1)^2 &= 1 + 4 + 9 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2, \text{ by assumption} \\
&= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= (k+1)\left[\frac{k(2k+1) + 6(k+1)}{6}\right] \\
&= (k+1) \cdot \frac{2k^2 + 7k + 6}{6} \\
&= (k+1) \cdot \frac{(k+2)(2k+3)}{6} \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6}.
\end{aligned}$$

Therefore, by the Principle of Mathematical Induction, (1) holds for all $n \in \mathbb{N}$. □

3.2.3 Given that $p \nmid n$ or $p \not| n$ for all primes $p \leq \sqrt[3]{n}$