6.2 Let G_1 and G_2 be two Eulerian graphs with no vertex in common. Let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Let G be the graph obtained from $G_1 \cup G_2$ by adding the edge v_1v_2. What can be said about G?

Since G_1 and G_2 are Eulerian, every vertex in each graph has even degree. If we add an additional edge v_1v_2 to form G, then the degrees of v_1 and v_2 each increase by 1, making them odd. Therefore, G has exactly two vertices of odd degree and cannot be Eulerian.

6.6 Let G be a connected regular graph that is not eulerian. Prove that if $\overline{G}$ is connected, then $\overline{G}$ is eulerian.

Proof. Since G is r-regular and not eulerian, r must be odd. By the first theorem, the sum of degrees must be even. Therefore r * n, where n is the order, must be even. Since r is odd, n must be even. Then, in $\overline{G}$:

$$\deg_{\overline{G}}(v) = (n-1) - \deg_G(v)$$

which is: (n - 1) - r

We have an even - odd - odd, which is even. Therefore, $\overline{G}$ is connected and has even degrees, which satisfies the conditions to be eulerian. □

6.8 (a) Show that every nontrivial graph G has a closed spanning walk that contains every edge of G exactly twice.

Proof. Let G be a connected graph. We can form multigraph H by replacing every edge of G with two parallel copies. Then each vertex, v, of H has degree $\deg_H(v) = 2\deg_G(v)$, so all vertices of H have even degree. H has an Eulerian circuit that traverses each edge of H exactly once (due to doubled edges), starting and ending at v. The same circuit must exist in G, but each edge is traversed twice. So, G has a closed spanning walk that contains every edge exactly twice. □

6.10 Let G be a 6-regular graph of order 10 and let u, $v \in$ V(G). Prove G - v and G - u - v are all Hamiltonian.

Proof. Let G be a 6-regular graph of order 10, and let $u, v \in V(G)$. In $G-v$, the order is 9. Since G is 6-regular, removing v decreases the degree of each of its 6 neighbors by 1, so every vertex in $G-v$ has degree at least 5. Thus $\delta(G-v) \geq 5$. By corollary 6.7, if a graph H of order $n \geq 3$ satisfies $\delta(H) \geq n/2$, then H is Hamiltonian. Here, $n = 9$ and $\delta(G-v) = 5 \geq 9/2$, so $G-v$ is Hamiltonian. Now consider $G-u-v$. The order is 8. Removing u and v decreases the degree of each vertex by at most 2 (if the vertex was adjacent to both u and v), so $\delta(G-u-v) \geq 6-2 = 4$. Since $4 = 8/2$, by corollary 6.7, $G-u-v$ is also Hamiltonian. Therefore, both $G-v$ and $G-u-v$ are Hamiltonian. □

6.12 Let G be a 3-regular graph of order 12 and H a 4-regular graph of order 11.

(a) Is G + H Eulerian?
Yes. Originally in G, deg $= 3$. In $G+H$, each vertex in G gains an edge to all 11 vertices in H, so deg $= 3 + 11 = 14$.
In H, deg $= 4$, and in $G+H$ each vertex in H gains an edge to all 12 vertices in G, so deg $= 4 + 12 = 16$.
Since all vertices have even degree, $G+H$ satisfies the condition for being Eulerian.

(b) Is G + H Hamiltonian?
Yes. By corollary 6.7, a graph of order $n \geq 3$ is Hamiltonian if $\delta(G) \geq \frac{n}{2}$.
Here, $n = 12 + 11 = 23$ and $\delta(G+H) = 14$, since $\min(14, 16) = 14$.
Because $14 \geq \frac{23}{2}$, the condition holds, so $G+H$ is Hamiltonian