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Extended Kalman filter

1 Motivation

1.1 Non-linear models

Kalman Filter assumption

KF is applicable only when several assumptions hold

- state transition model is **linear**

$$\mathbf{x}_k = \boldsymbol{A}\ \mathbf{x}_{k-1} + \boldsymbol{B}\ \mathbf{u}_{k-1} + \mathbf{w}_{k-1}$$

- measurement model is **linear**

$$\mathbf{z}_k = \boldsymbol{C}\ \mathbf{x}_k + \mathbf{v}_k$$

- **prior, predicted and posterior states** distributions are **Gaussian** (they can be parametrized by their means and covariances)
- **noises** (in state transition and measurement models) are **Gaussian**

What if the following form cannot be obtained and the system is defined by **non-linear models**?

We can use **Extended Kalman Filter (EKF)** !

Non-linear equations of EKF

The system is defined by **non-linear equations**

- **non-linear** state transition model

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1})$$

  non-linear function f

  relates prior state $\mathbf{x}_{k-1}$ to current state $\mathbf{x}_k$

  includes as parameters control input $\mathbf{u}_{k-1}$ and process noise $\mathbf{w}_{k-1}$

- and/or **non-linear** measurement model

$$\mathbf{z}_k = h(\mathbf{x}_k, \mathbf{v}_k)$$

  non-linear function h

  relates current state $\mathbf{x}_k$ to current measurement $\mathbf{z}_k$

  includes as parameter measurement noise $\mathbf{v}_k$

where

- $\mathbf{w}_{k-1}$ is a previous state of process noise $\mathbf{W}(t_k) \sim N_n(\mathbf{0}, \boldsymbol{Q}_k)$ representing n-dimensional zero-mean white Gaussian noise $\{\mathbf{W}(t)\}$
- $\mathbf{v}_k$ is a current state of measurement noise $\mathbf{V}(t_k) \sim N_m(\mathbf{0}, \boldsymbol{R}_k)$ representing m-dimensional zero-mean white Gaussian noise $\{\mathbf{V}(t)\}$

1.2 Linearization

- **linearization** refers to finding the **linear approximation** to a function at a given point
- linearization of a function is the first order term of its **Taylor expansion** around the point of interest

Linear approximation of 1-variable function

The equation for the linearization of 1-variable function $f(x)$ at point $p(a)$ is:

$$y \approx f(a) + f'(a)(x-a)$$

Linear approximation of 2-variable function

The equation for the linearization of 2-variable function

$f(x, y)$ at a point $p(a, b)$ is:

$$f(x,y) \approx f(a,b) + \left.\frac{\partial f(x,y)}{\partial x}\right|_{a,b}(x-a) + \left.\frac{\partial f(x,y)}{\partial y}\right|_{a,b}(y-b)$$

Linear approximation of n-variable function

The general equation for the linearization of n-variable function $f(\mathbf{x})$ at a point $\mathbf{p}$ is:

$$f(\mathbf{x}) \approx f(\mathbf{p}) + \mathbf{J}_f(\mathbf{p}) \cdot (\mathbf{x} - \mathbf{p}) \qquad \text{where}$$

- $\mathbf{x} = [x_1, x_2, ..., x_n]^T$ is n-dimensional vector of variables
- $\mathbf{p} = [a_1, a_2, ..., a_n]^T$ is n-dimensional linearization point of interest
- $\mathbf{J}_f$ is a **Jacobian matrix**

If $f : \mathbb{R}^n \to \mathbb{R}^m$ then $\mathbf{J}_f$ is an $m \times n$-dimensional **Jacobian matrix**

$$\mathbf{J}_f = \frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \frac{\partial \mathbf{f}}{\partial x_2} & ... & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & ... & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & ... & \frac{\partial f_2}{\partial x_n} \\ ... & ... & ... & ... \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & ... & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

or, component-wise $\mathbf{J}_{i,j} = \frac{\partial f_i}{\partial x_j}$.

2 Extended Kalman Filter

2.1 Theory

Non-linear equations of EKF

The system is defined by **non-linear** equations

- **non-linear** state transition model

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1})$$

- **non-linear** measurement model

$$\mathbf{z}_k = h(\mathbf{x}_k, \mathbf{v}_k)$$

Linearization of dynamical system of EKF

- using the **partial derivatives** of
  - process function $f(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1})$
  - measurement function $h(\mathbf{x}_k, \mathbf{v}_k)$
- we can linearize the estimation around the current estimate

Prediction in EKF

In practice, at each time-step t_k, we do not know

- the individual values of n-dimensional process noise $\mathbf{w}_{k-1}$
- the individual values of m-dimensional measurement noise $\mathbf{v}_k$

We **approximate (predict) the state and measurement vectors without noises**

- $\mathbf{x}_k^- = f(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0})$, n-dimensional approximated state
- $\mathbf{z}_k^- = h(\mathbf{x}_k^-, \mathbf{0})$, m-dimensional approximated measurement

where $\hat{\mathbf{x}}_{k-1}$ is a posterior state estimate at previous time-step t_{k-1}

Remark 1 (Fundamental flaw)**.** ***Prior, predicted and posterior states*** *distributions (after undergoing their respective* ***non-linear*** *transformation) are* ***no longer Normal (Gaussian) distributions****.*

Linearization of EKF

- state transition model
$$\mathbf{x}_k \approx \mathbf{x}_k^- + \boldsymbol{A}_k(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1}) + \boldsymbol{W}_k\ \mathbf{w}_{k-1}$$
- measurement model
$$\mathbf{z}_k \approx \mathbf{z}_k^- + \boldsymbol{C}_k(\ \mathbf{x}_k - \mathbf{x}_k^-) + \boldsymbol{V}_k\ \mathbf{v}_k$$

where

$\mathbf{x}_k$ is **actual** n-dimensional **state** vector (we try estimate)

$\mathbf{z}_k$ is the **actual** m-dimensional **measurement** vector (we have access to)

$\mathbf{x}_k^- = f(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0})$ is a **predicted state (without noise)**

$\mathbf{z}_k^- = h(\mathbf{x}_k^-, \mathbf{0})$ is a **predicted measurement (without noise)**

$\hat{\mathbf{x}}_{k-1}$ is a **posterior state estimate at previous time-step** t_{k-1}

$\mathbf{w}_{k-1}$ is a **previous state of process noise** $\mathbf{W}(t_k) \sim N_n(\mathbf{0}, \boldsymbol{Q}_k)$ representing n-dimensional zero-mean white Gaussian noise $\{\mathbf{W}(t)\}$

$\mathbf{v}_k$ is a **current state of measurement noise** $\mathbf{V}(t_k) \sim N_m(\mathbf{0}, \boldsymbol{R}_k)$ representing m-dimensional zero-mean white Gaussian noise $\{\mathbf{V}(t)\}$

- state transition model
$$\mathbf{x}_k \approx \mathbf{x}_k^- + \boldsymbol{A}_k(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1}) + \boldsymbol{W}_k\ \mathbf{w}_{k-1}$$
- measurement model
$$\mathbf{z}_k \approx \mathbf{z}_k^- + \boldsymbol{C}_k(\ \mathbf{x}_k - \mathbf{x}_k^-) + \boldsymbol{V}_k\ \mathbf{v}_k$$

where

$\boldsymbol{A}_k$ is the Jacobian matrix of partial derivatives of f with respect to $\mathbf{x}$

$\boldsymbol{W}_k$ is the Jacobian matrix of partial derivatives of f with respect to $\mathbf{w}$

$\boldsymbol{C}_k$ is the Jacobian matrix of partial derivatives of h with respect to $\mathbf{x}$

$\boldsymbol{V}_k$ is the Jacobian matrix of partial derivatives of h with respect to $\mathbf{v}$

Remark 2 (Omitted matrix $\boldsymbol{B}$)**.** *Matrix $\boldsymbol{B}$, defining the relation between the state transition and control inputs is omitted, but this relation is represented by process function*

$$\mathbf{x}_k^- = f(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0}).$$

Jacobian matrices of EKF

$\boldsymbol{A}_k$ is the Jacobian matrix of partial derivatives of f with respect to $\mathbf{x}$

$$\boldsymbol{A}_{k_{[i,j]}} = \frac{\partial f_{[i]}}{\partial x_{[j]}}(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0})$$

$\boldsymbol{W}_k$ is the Jacobian matrix of partial derivatives of f with respect to $\mathbf{w}$

$$\boldsymbol{W}_{k_{[i,j]}} = \frac{\partial f_{[i]}}{\partial w_{[j]}}(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0})$$

$\boldsymbol{C}_k$ is the Jacobian matrix of partial derivatives of h with respect to $\mathbf{x}$

$$\boldsymbol{C}_{k_{[i,j]}} = \frac{\partial h_{[i]}}{\partial x_{[j]}}(\mathbf{x}_k^-, \mathbf{0})$$

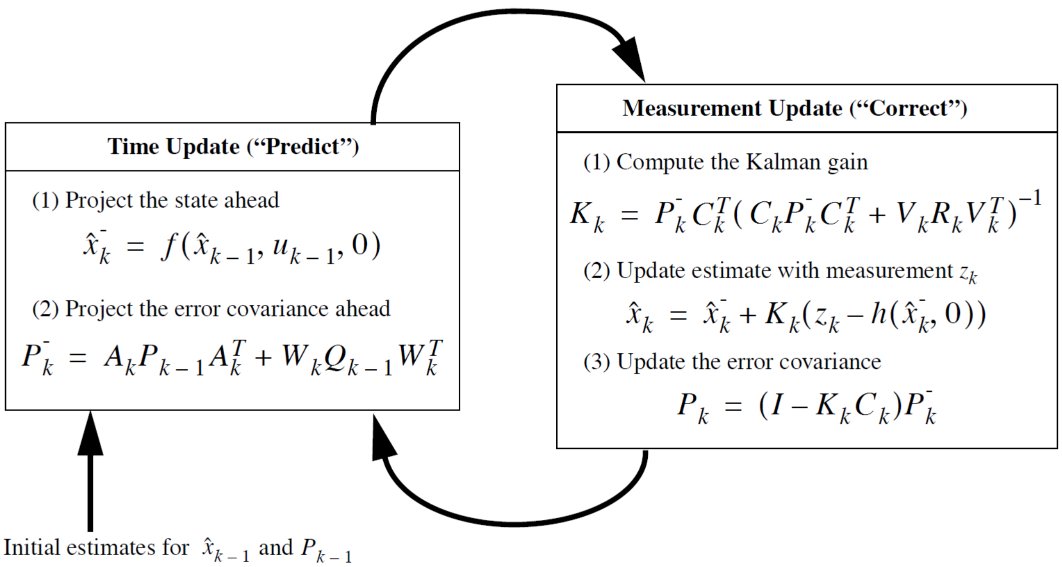
$\boldsymbol{V}_k$ is the Jacobian matrix of partial derivatives of h with respect to $\mathbf{v}$

$$\boldsymbol{V}_{k_{[i,j]}} = \frac{\partial h_{[i]}}{\partial v_{[j]}}(\mathbf{x}_k^-, \mathbf{0})$$

EKF operations

- INITIALIZATION – given
  initial prior state $\hat{\mathbf{x}}_0$ with
  Covariance matrix $\boldsymbol{P}_0$
- PREDICTION – time-step update
  - having prior state $\hat{\mathbf{x}}_{k-1}$ with Covariance matrix $\boldsymbol{P}_{k-1}$
    (and optionally, control inputs $\mathbf{u}_{k-1}$)
  - using **non-linear** state transition model $\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1})$
  - we obtain predicted state $\mathbf{x}_k^- = f(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0})$ with
    Covariance matrix $\boldsymbol{P}_k^- = \boldsymbol{A}_k \boldsymbol{P}_{k-1} \boldsymbol{A}_k^T + \boldsymbol{W}_k \boldsymbol{Q}_{k-1} \boldsymbol{W}_k^T$
- CORRECTION – measurement update
  - having predicted state $\mathbf{x}_k^-$ with Covariance matrix $\boldsymbol{P}_k^-$
  - using **non-linear** measurement model $\mathbf{z}_k = h(\mathbf{x}_k, \mathbf{v}_k)$
  - we can compute Kalman gain
    $\boldsymbol{K}_k = \boldsymbol{P}_k^- \boldsymbol{C}_k^T \left(\boldsymbol{C}_k \boldsymbol{P}_k^- \boldsymbol{C}_k^T + \boldsymbol{V}_k \boldsymbol{R}_k \boldsymbol{V}_k^T \right)^{-1}$
  - and obtain posterior state $\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \boldsymbol{K}_k \left(\mathbf{z}_k - h(\hat{\mathbf{x}}_k^-, \mathbf{0}) \right)$ with
    Covariance matrix $\boldsymbol{P}_k = (\boldsymbol{I} - \boldsymbol{K}_k\, \boldsymbol{C}_k)\, \boldsymbol{P}_k^-$

Extended Kalman filter
Basic formulas



Example 1. *To measure the water level (WL) in Tank* T1 *we use two sensors – a sonar and a float (see at the diagram below). Let vector* $\mathbf{x}_k = [d, \theta]^T$ *represent WL state in Tank* T1 *(distance to the water and float deflection angle) at time-step* t_k*.*

Initial WL prior state in Tank T1 *is* $\hat{\mathbf{x}}_0 = \begin{bmatrix} 1 \\ 45^\circ \end{bmatrix}$ *and* $\boldsymbol{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Moreover,

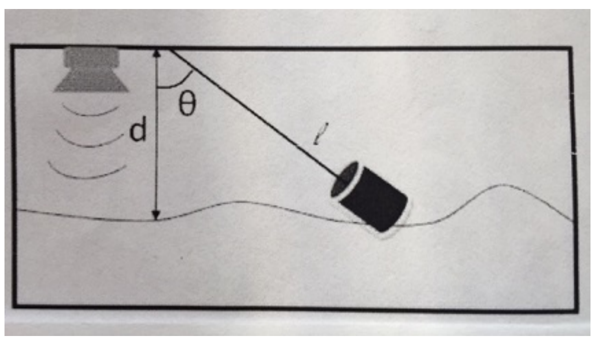
state transition model equals $f(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}) = \begin{bmatrix} l\cos\theta + w_1 \\ \theta + w_2 \end{bmatrix}$ *(***non-linear***)*

where $\mathbf{W}(t_{k-1}) \sim N_2(\mathbf{0},\ \boldsymbol{Q})$ *with* $\boldsymbol{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

measurement model equals $h(\mathbf{x}_k, \mathbf{v}_k) = \begin{bmatrix} \theta + v_1 \end{bmatrix}$ *(it can be* **non-linear***)*

where $\mathbf{V}(t_k) \sim N_1(\mathbf{0},\ \boldsymbol{R})$ *with* $\boldsymbol{R} = \begin{bmatrix} 1 \end{bmatrix}$

- *determine WL state in Tank* T1 *after prediction at time-step* t_1
- *given float deflection angle measurement* $\mathbf{z}_1 = \begin{bmatrix} 42^\circ \end{bmatrix}$,
  determine posterior WL state in Tank T1 *at time-step* t_1



Solution

state vector $\mathbf{x}_k = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d \\ \theta \end{bmatrix}$ where $l = \frac{d}{\cos\theta}$

state transition model $f(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} l\cos\theta + w_1 \\ \theta + w_2 \end{bmatrix}$

measurement model $h(\mathbf{x}_k, \mathbf{v}_k) = \begin{bmatrix} h_1 \end{bmatrix} = \begin{bmatrix} \theta + v_1 \end{bmatrix}$,

Jacobian matrices

$$\boldsymbol{A}_{k_{[i,j]}} = \frac{\partial f_{[i]}}{\partial x_{[j]}}(\hat{\mathbf{x}}_{k-1}, \mathbf{0}) \quad \text{so} \qquad \boldsymbol{A}_k(\hat{\mathbf{x}}_{k-1}, \mathbf{0}) = \begin{bmatrix} 0 & -l\sin\theta \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{W}_{k_{[i,j]}} = \frac{\partial f_{[i]}}{\partial w_{[j]}}(\hat{\mathbf{x}}_{k-1}, \mathbf{0}) \quad \text{so} \qquad \boldsymbol{W}_k = \boldsymbol{W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{C}_{k_{[i,j]}} = \frac{\partial h_{[i]}}{\partial x_{[j]}}(\mathbf{x}_k^-, \mathbf{0}) \quad \text{so} \qquad \boldsymbol{C}_k = \boldsymbol{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\boldsymbol{V}_{k_{[i,j]}} = \frac{\partial h_{[i]}}{\partial v_{[j]}}(\mathbf{x}_k^-, \mathbf{0}) \quad \text{so} \qquad \boldsymbol{V}_k = \boldsymbol{V} = \begin{bmatrix} 1 \end{bmatrix}$$

INITIALIZATION given initial prior state of WL

$\hat{\mathbf{x}}_0 = \begin{bmatrix} 1 \\ 45^\circ \end{bmatrix}$ with Covariance matrix $\boldsymbol{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

PREDICTION

using **non-linear** state transition model $f(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}) = \begin{bmatrix} l\cos\theta + w_1 \\ \theta + w_2 \end{bmatrix}$

where $\mathbf{W}(t_{k-1}) \sim N_2(\mathbf{0},\ \boldsymbol{Q})$ with $\boldsymbol{Q} = \boldsymbol{Q}_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and Jacobian matrices $(-l\sin\theta = -\frac{d}{\cos\theta}\sin\theta = -1\,\mathrm{tg}(45^\circ) = -1)$

$\boldsymbol{A}_1(\hat{\mathbf{x}}_0, \mathbf{0}) = \begin{bmatrix} 0 & -l\sin\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

we obtain predicted state of WL $\mathbf{x}_1^- = f(\hat{\mathbf{x}}_{k-1}, \mathbf{0}) = \begin{bmatrix} 1 \\ 45^\circ \end{bmatrix}$ with

Covariance matrix $\boldsymbol{P}_1^- = \boldsymbol{A}_1\boldsymbol{P}_0\boldsymbol{A}_1^T + \boldsymbol{W}\boldsymbol{Q}\boldsymbol{W}^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

CORRECTION having predicted state of WL

$\mathbf{x}_1^- = \begin{bmatrix} 1 \\ 45^\circ \end{bmatrix}$ with Covariance matrix $\boldsymbol{P}_1^- = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

using measurement model $h(\mathbf{x}_k, \mathbf{v}_k) = \begin{bmatrix} \theta + v_1 \end{bmatrix}$

where $\mathbf{V}(t_k) \sim N_1(\mathbf{0},\ \boldsymbol{R})$ with $\boldsymbol{R} = \boldsymbol{R}_k = \begin{bmatrix} 1 \end{bmatrix}$

and Jacobian matrices

$\boldsymbol{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $\boldsymbol{V} = \begin{bmatrix} 1 \end{bmatrix}$

we obtain predicted measurement $\mathbf{z}_1^- = h(\hat{\mathbf{x}}_1^-, \mathbf{0}) = \begin{bmatrix} \theta + 0 \end{bmatrix} = \begin{bmatrix} 45^\circ \end{bmatrix}$

and Kalman gain $\boldsymbol{K}_1 = \boldsymbol{P}_1^-\boldsymbol{C}^T\left(\boldsymbol{C}\boldsymbol{P}_1^-\boldsymbol{C}^T + \boldsymbol{V}\boldsymbol{R}\boldsymbol{V}^T\right)^{-1} = \begin{bmatrix} -0.33 \\ 0.67 \end{bmatrix}$

moreover, with given float deflection angle measurement $\mathbf{z}_1 = \begin{bmatrix} 42^\circ \end{bmatrix}$

we obtain posterior state of WL $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_1^- + \boldsymbol{K}_1\left(\mathbf{z}_1 - \mathbf{z}_1^-\right) = \begin{bmatrix} 2 \\ 43^\circ \end{bmatrix}$

with Covariance matrix $\boldsymbol{P}_1 = (\boldsymbol{I} - \boldsymbol{K}_1\,\boldsymbol{C})\,\boldsymbol{P}_1^- = \begin{bmatrix} 1.67 & -0.33 \\ -0.33 & 0.67 \end{bmatrix}$

Example 2. *For presented in Example 1 Tank* T1

- *determine WL state after prediction at time-step* t_2
- *given float deflection angle measurement* $\mathbf{z}_2 = \begin{bmatrix} 43^\circ \end{bmatrix}$, *determine posterior WL state at time-step* t_2

Example 3. *For presented in Example 1 and Example 2 Tank* T1

- *determine WL state after prediction at time-step* t_3
- *given float deflection angle measurement* $\mathbf{z}_3 = \begin{bmatrix} 45^\circ \end{bmatrix}$, *determine posterior WL state at time-step* t_3

Example 4. *To measure the water level (WL) in Tank* T2 *we use two sensors – a sonar and a float (see at the diagram in Example 1). Let vector* $\mathbf{x}_k = \begin{bmatrix} d \\ \theta \end{bmatrix}$ *represent WL state in Tank* T2 *(distance to the water and float deflection angle) at time-step* t_k.

Initial WL prior state in Tank T2 *is* $\hat{\mathbf{x}}_0 = \begin{bmatrix} 5 \\ 75^\circ \end{bmatrix}$ *and* $\boldsymbol{P}_0 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Moreover,

non-linear state transition model equals $f(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}) = \begin{bmatrix} l\cos\theta + w_1 \\ \theta + w_2 \end{bmatrix}$

where $\mathbf{W}(t_{k-1}) \sim N_2(\mathbf{0},\ \boldsymbol{Q})$ *with* $\boldsymbol{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

measurement model equals $h(\mathbf{x}_k, \mathbf{v}_k) = \begin{bmatrix} d + v_1 \\ \theta + v_2 \end{bmatrix}$

where $\mathbf{V}(t_k) \sim N_2(\mathbf{0},\ \boldsymbol{R})$ *with* $\boldsymbol{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- *determine WL state in Tank* T2 *after prediction at time-step* t_1
- *given distance to the water and float deflection angle measurements* $\mathbf{z}_1 = \begin{bmatrix} 4.5 \\ 77^\circ \end{bmatrix}$, *determine posterior WL state in Tank* T2 *at time-step* t_1

Solution

state vector $\mathbf{x}_k = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d \\ \theta \end{bmatrix}$ where $l = \frac{d}{\cos\theta}$

state transition model $f(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} l\cos\theta + w_1 \\ \theta + w_2 \end{bmatrix}$

measurement model $h(\mathbf{x}_k, \mathbf{v}_k) = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} d + v_1 \\ \theta + v_2 \end{bmatrix}$,

Jacobian matrices

$\boldsymbol{A}_{k_{[i,j]}} = \dfrac{\partial f_{[i]}}{\partial x_{[j]}}(\hat{\mathbf{x}}_{k-1}, \mathbf{0})$ so $\boldsymbol{A}_k(\hat{\mathbf{x}}_{k-1}, \mathbf{0}) = \begin{bmatrix} 0 & -l\sin\theta \\ 0 & 1 \end{bmatrix}$

$\boldsymbol{W}_{k_{[i,j]}} = \dfrac{\partial f_{[i]}}{\partial w_{[j]}}(\hat{\mathbf{x}}_{k-1}, \mathbf{0})$ so $\boldsymbol{W}_k = \boldsymbol{W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\boldsymbol{C}_{k_{[i,j]}} = \dfrac{\partial h_{[i]}}{\partial x_{[j]}}(\mathbf{x}_k^-, \mathbf{0})$ so $\boldsymbol{C}_k = \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\boldsymbol{V}_{k_{[i,j]}} = \dfrac{\partial h_{[i]}}{\partial v_{[j]}}(\mathbf{x}_k^-, \mathbf{0})$ so $\boldsymbol{V}_k = \boldsymbol{V} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

INITIALIZATION

given initial prior state of WL $\hat{\mathbf{x}}_0 = \begin{bmatrix} 5 \\ 75^\circ \end{bmatrix}$ with Covariance matrix $\boldsymbol{P}_0 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

PREDICTION – at time-step t_1

using non-linear state transition model $f(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}) = \begin{bmatrix} l\cos\theta + w_1 \\ \theta + w_2 \end{bmatrix}$

where $\mathbf{W}(t_{k-1}) \sim N_2(\mathbf{0},\ \boldsymbol{Q})$ with $\boldsymbol{Q} = \boldsymbol{Q}_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and Jacobian matrices $(-l\sin\theta = -\frac{d}{\cos\theta}\sin\theta = -5\,\mathrm{tg}(75°) = -18.66)$

$\boldsymbol{A}_1(\hat{\mathbf{x}}_0, \mathbf{0}) = \begin{bmatrix} 0 & -l\sin\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -18.66 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

we obtain predicted state of WL $\quad \mathbf{x}_1^- = f(\hat{\mathbf{x}}_{k-1}, \mathbf{0}) = \begin{bmatrix} 5 \\ 75° \end{bmatrix}$ with

Covariance matrix $\boldsymbol{P}_1^- = \boldsymbol{A}_1 \boldsymbol{P}_0 \boldsymbol{A}_1^T + \boldsymbol{W}\boldsymbol{Q}\boldsymbol{W}^T = \begin{bmatrix} 697.41 & -37.32 \\ -37.32 & 3 \end{bmatrix}$

CORRECTION – at time-step t_1

having predicted state of WL $\quad \mathbf{x}_1^- = \begin{bmatrix} 5 \\ 75° \end{bmatrix}$ with Covariance matrix $\boldsymbol{P}_1^- = \begin{bmatrix} 697.41 & -37.32 \\ -37.32 & 3 \end{bmatrix}$

using measurement model $h(\mathbf{x}_k, \mathbf{v}_k) = \begin{bmatrix} d + v_1 \\ \theta + v_2 \end{bmatrix}$

where $\mathbf{V}(t_k) \sim N_2(\mathbf{0},\ \boldsymbol{R})$ with $\boldsymbol{R} = \boldsymbol{R}_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and Jacobian matrices $\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\quad \boldsymbol{V} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

we obtain predicted measurement $\mathbf{z}_1^- = h(\hat{\mathbf{x}}_1^-, \mathbf{0}) = \begin{bmatrix} d + 0 \\ \theta + 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 75° \end{bmatrix}$

and Kalman gain $\boldsymbol{K}_1 = \boldsymbol{P}_1^- \boldsymbol{C}^T \left(\boldsymbol{C}\boldsymbol{P}_1^-\boldsymbol{C}^T + \boldsymbol{V}\boldsymbol{R}\boldsymbol{V}^T\right)^{-1} = \begin{bmatrix} 0.9971 & -0.0266 \\ -0.0266 & 0.5014 \end{bmatrix}$

and with given distance to the water and float deflection angle $\mathbf{z}_1 = \begin{bmatrix} 4.5 \\ 77° \end{bmatrix}$

we obtain posterior state of WL $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_1^- + \boldsymbol{K}_1\left(\mathbf{z}_1 - \mathbf{z}_1^-\right)) = \begin{bmatrix} 4.45 \\ 76.02° \end{bmatrix}$

with Covariance matrix $\quad \boldsymbol{P}_1 = (\boldsymbol{I} - \boldsymbol{K}_1\ \boldsymbol{C})\,\boldsymbol{P}_1^- = \begin{bmatrix} 0.9971 & -0.0266 \\ -0.0266 & 0.5014 \end{bmatrix}$

Example 5. *For presented in Example 4 Tank* T2

- *determine WL state after prediction at time-step* t_2
- *given distance to the water and float deflection angle measurements* $\quad \mathbf{z}_2 = \begin{bmatrix} 6 \\ 74° \end{bmatrix}$, *determine posterior WL state at time-step* t_2

Example 6. *For presented in Example 4 and Example 5 Tank* T2

- *determine WL state after prediction at time-step* t_3
- *given distance to the water and float deflection angle measurements* $\quad \mathbf{z}_3 = \begin{bmatrix} 5.5 \\ 73° \end{bmatrix}$, *determine posterior WL state at time-step* t_3