Exercise Sheet 1

We denote by k an algebraically closed field.

Exercise 1 (Properties of irreducible spaces) Let X be a non-empty topological space. Prove that:

1. X is irreducible if and only if any non-empty open subset is dense in X.
2. If X is irreducible, then any non-empty open subset of X is irreducible.
3. If $Y \subseteq X$ is a subset, then Y is irreducible if and only if its closure $\overline{Y}$ in X is irreducible.

Exercise 2 (Examples of algebraic sets)

1. Show that the intersection of an algebraic set in $\mathbb{A}^n(k)$ with a principal open subset can be naturally identified with an algebraic set in $\mathbb{A}^{n+1}(k)$.
2. Give a natural structure of algebraic sets to $\mathrm{SL}_n(k)$ and $\mathrm{GL}_n(k)$.
3. Show that finite subsets of $\mathbb{A}^n(k)$ are algebraic.
4. Let Z be an algebraic set in $\mathbb{A}^2(k)$ and let L be a line in $\mathbb{A}^2(k)$. Show that either $L \subseteq Z$ or $L \cap Z$ is finite. Show the similar statement in the projective case.
5. Is $\{(x, y) \in \mathbb{C}^2 | \sin x = y^2\}$ an algebraic subset in the affine plane ?
6. Show that $\{[a^2 : ab : b^2] \in \mathbb{P}^2(k) \mid [a : b] \in \mathbb{P}^1(k)\}$ is a projective algebraic set.
7. Describe the image of the map
$$\mathbb{A}^2(k) \to \mathbb{A}^2(k), \quad (x, y) \mapsto (x, xy)$$
and show that the image is not open nor closed in $\mathbb{A}^2(k)$.
8. Let k be endowed with the Zariski topology. Show that the Zariski topology on k^2 is strictly finer (has more open subsets) than the product topology.

Exercise 3 (Irreducible subsets of the plane) Let $X = \mathbb{A}^2(k)$ be endowed with the Zariski topology.

1. Let $f, g \in k[x, y]$ be polynomials with no common factor. Show that the cardinality of $Z(f, g)$ is finite.
2. Let $h \in k[x, y]$ be an irreducible polynomial and Y an infinite algebraic set of X. Prove that if $Y \subseteq Z(h)$, then $I(Y) = \langle h \rangle$.
3. Deduce that the irreducible algebraic subsets of X are:
   - X;
   - $Z(f)$ where $f \in k[x, y]$ is irreducible and $Z(f)$ is an infinite set;
   - $\{(a, b)\}$, with $a, b \in k$.

Exercise 4 (Algebraic set with three irreducible components) Let $X \subset \mathbb{A}^3_k$ be the algebraic subset defined by $V(x^2 - yz, xz - x)$. Show that X is the union of three irreducible components.

Exercise 5 (Noetherianity) A topological space is *noetherian* if it satisfies the descending chain condition (DCC) for closed subsets, i.e. for any sequence $C_1 \supseteq C_2 \supseteq \ldots$ of closed subsets, there is an integer r such that $C_r = C_{r+1} = \cdots$.

1. Show that the following properties are equivalent:

(a) X is noetherian;

(b) every non-empty family of closed subsets has a minimal element;

(c) X satisfies the ascending chain condition (ACC) for open subsets;

(d) every non-empty family of open subsets has a maximal element.

2. Show that a noetherian topological space is *quasi-compact*, i.e. every open cover has a finite subcover.

3. Prove that any subset of a noetherian space is noetherian in the induced topology.

4. Show that a noetherian space which is Hausdorff must be a finite set with the discrete topology.

5. Show that any affine algebraic set endowed with the Zariski topology is noetherian.