Exercise Sheet 5

The field k is assumed to be algebraically closed.

Exercise 1 (Irreducible components of $Z(F)$ and $Z_+(G)$)

Let $F(T_1, \ldots, T_n) \in k[T_1, \ldots, T_n]$ be a non-constant polynomial.

1. Show that the minimal prime ideals over (F) are principal.
2. Show that the irreducible components of $Z(F) \subseteq \mathbb{A}^n(k)$ are exactly the $Z(P)$'s where P runs through the irreducible factors of F.

Let $G(T_0, \ldots, T_n) \in k[T_0, \ldots, T_n]$ be a non-zero homogeneous polynomial.

3. Using the decomposition into homogeneous components, show that the irreducible factors of G are homogeneous.
4. Show that the irreducible components of $Z_+(G) \subset \mathbb{P}^n(k)$ are exactly the $Z_+(P)$'s where P runs through the irreducible factors of G.

Solution.

1. Let $\mathfrak{p}$ be a minimal prime ideal over (F). Since $(F) \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is prime, there exists an irreducible factor P of F such that $P \in \mathfrak{p}$. So $(F) \subseteq (P) \subseteq \mathfrak{p}$ and since P is irreducible, (P) is a prime ideal ($k[T_0, \ldots, T_n]$ is a UFD). By minimality $(P) = \mathfrak{p}$. We have proved that $\mathfrak{p}$ is principal.
2. Let $F = P_1 \cdots P_m$ be a decomposition in irreducible factors, then for any i the ideal (P_i) is prime, and minimal over (F). Moreover the previous part shows that all minimal prime ideal over (F) are of this form. For any i, taking zero loci, we obtain $Z(F) \supseteq Z(P_i)$. All $Z(P_i)$'s are irreducible (since the (P_i)'s are prime) moreover, suppose that $Z(P_i) \subseteq Z \subseteq Z(F)$, for some Z closed and irreducible. Then there exists $I = \sqrt{I}$ prime such that $Z = Z(I)$ and $(F) \subseteq \sqrt{(F)} = I(Z(F)) \subseteq I(Z(I)) \subseteq I(Z(P_i)) = (P_i)$. This implies that $I = I(Z(I)) = (P_i)$ by minimality.
3. We prove it by contradiction. Let $G = P_1 \cdots P_m$ be a decomposition in irreducible factors and suppose one of those factors is not homogeneous. Up to reordering, we can suppose that this factor is P_1. We can decompose P_1 into homogeneous components: $P_1 = Q_0 + Q_1 + \cdots + Q_d$, where $d > 1$ is the degree of P_1, each Q_j has degree j and $Q_0 + \cdots + Q_{d+1} \neq 0$ (since we suppose that P_1 is not homogeneneous). Then
$$G = P_1 \cdots P_m = (Q_0 + Q_1 + \cdots + Q_d) \cdot P_2 \cdots P_m = (Q_0 + Q_1 + \cdots + Q_{d-1}) \cdot (P_2 \cdots P_m) + Q_d \cdot (P_2 \cdots P_m)$$
and since $\deg((Q_0 + Q_1 + \cdots + Q_{d-1}) \cdot (P_2 \cdots P_m)) < \deg(Q_d \cdot P_2 \cdots P_m)$, and G is homogeneous, we deduce that $(Q_0 + Q_1 + \cdots + Q_{d-1}) \cdot P_2 \cdots P_m = 0$. This implies that $Q_0 + Q_1 + \cdots + Q_{d-1} = 0$. Contradiction.
4. By definition $Z_+(G) = \{p \in \mathbb{P}^n(k) \mid G(p) = 0\}$, but we will consider $Z(G) = \{p \in \mathbb{A}^{n+1}(k) \mid G(p) = 0\}$ instead: (G) is a homogenous ideal, so $Z(G) = \pi^{-1} Z_+(G) \cup \{0\}$, where $\pi \colon \mathbb{A}^{n+1}(k) \setminus \{0\} \to \mathbb{P}^n(k)$ is the canonical projection. We know by the previous parts of the exercise that the irreducible components of $Z(G)$ are the $Z(P_i)$'s where the P_i's are the irreducible factors of G. Moreover the P_i's are homogenous, so $Z(P_i) = \pi^{-1} Z_+(P_i) \cup \{0\}$. So, in conclusion:
$$\pi^{-1} Z_+(G) = Z(G) \setminus \{0\} = \bigcup_i Z(P_i) \setminus \{0\} = \bigcup_i \pi^{-1} Z_+(P_i) = \pi^{-1}(\bigcup_i Z_+(P_i))$$
Since π is surjective, we obtain $Z_+(G) = \bigcup_i Z_+(P_i)$.
Clearly the $Z_+(P_i)$'s are irreducible (since the (P_i)'s are prime) and they are exactly the irreducible components of $Z_+(G)$: suppose that $Z_+(P_i)$ is contained in an irreducible component $Z_+(I)$ of $Z_+(G)$, where I is homogeneous, then $(G) \subseteq I \subseteq (P_i)$, which implies by minimality that $I = (P_i)$.

Exercise 2 (Dimension of affine subvarieties) Let X be an integral affine algebraic variety of dimension n and let $f_1, \dots, f_s \in A(X)$. Show that every irreducible component of $Z(f_1, \dots, f_s)$ has dimension $\geqslant n - s$

Solution. We prove it by induction on s.
If $s = 1$ we can assume that f_1 is not invertible, otherwise $Z(f_1) = \emptyset$, and its set of irreducible components is empty. Also we can assume that $f_1 \neq 0$, so we conclude by Proposition 1.3.24.
To prove the induction step, let Z be an irreducible component of $Z(f_1, \dots, f_s)$. We can take an irreducible component of $Z(f_1)$ containing Z and apply the induction hypothesis.

Exercise 3 (Parameters of affine subvarieties) Let X be an integral affine algebraic variety of dimension n and let Z be an integral affine algebraic subvariety of dimension $r < n$. Show that for every s such that $1 \leqslant s \leqslant n - r$, there exist $f_1, \dots, f_s \in A(X)$ such that:

- $Z \subseteq Z(f_1, \dots, f_s)$;
- All irreducible components of $Z(f_1, \dots, f_s)$ have dimension $n - s$.

Deduce that Z is an irreducible component of $Z(f_1, \dots, f_{n-r})$.

Solution. We prove it by induction on s.
For $s = 1$ we take a non-zero and non-invertible $f \in I(Z)$, which exists, since $Z \subsetneq X$. We can also assume that f is irreducible, passing to an irreducible factor. By construction $Z \subseteq Z(f)$ and the dimension statement is a consequence of Proposition 1.3.24.
For the induction step, assume that there exists $f_1, \dots, f_{s-1} \in A(X)$ verifying the two properties. In particular the irreducible components $Z_1, \dots, Z_m$ of $Z(f_1, \dots, f_{s-1})$ have dimension $n-s+1$. Since $s-1 < r$, none of the components $Z_1, \dots, Z_m$ is contained in Z, so $I(Z) \not\subseteq I(Z_i)$ for all i. This implies (check!!!) that $I(Z) \not\subseteq \bigcup_i I(Z_i)$. So we can choose $f_s \in I(Z) \setminus \bigcup_i I(Z_i)$. By construction, $Z \subseteq Z(f_1, \dots, f_s)$ and any irreducible component Y of $Z(f_1, \dots, f_s)$ has dimension $\geqslant n - s$. To conclude, we observe that $Y \subsetneq Z_i$ for some i (the inclusion is strict, since $f_s \notin I(Z_i)$). This implies that $\dim Y \leqslant n - s$ and concludes the proof.

Exercise 4 (Dimension of fibres) Let $f \colon X \to Y$ be a dominant morphism of integral algebraic varieties and let y be a point of Y. Show that every irreducible component of the fibre $f^{-1}(y)$ has dimension at least $\dim X - \dim Y$. (Hint: you may want to use previous exercises)

Solution. First, we can assume that X and Y are affine (same reduction as in the proof of Exercise 5, Sheet 4). By the previous exercise, there exist $f_1, \dots, f_{\dim Y} \in A(Y)$ such that $Z(f_1, \dots, f_{\dim Y})$ is finite and $y \in Z(f_1, \dots, f_{\dim Y})$. After replacing Y by an open affine subset, we can assume that $\{y\} = Z(f_1, \dots, f_{\dim Y})$. Define $g_i := \phi(f_i)$, where $\phi \colon A(Y) \to A(X)$ is the associated homomorphism of rings. Then (check!!!) $f^{-1}(y) = Z(g_1, \dots, g_{\dim Y})$. Exercise 2 implies that every irreducible component of $f^{-1}(y)$ has dimension $\geqslant \dim X - \dim Y$, which proves the statement.