Contents

Chapter 1

Algebraic varieties

Algebraic geometry studies sets of solutions of polynomial systems, e.g. for a fix $n \geq 1$, sets of $(x, y, z) \in \mathbb{Z}^3$ such that $x^n + y^n = z^n$; or $(x, y) \in \mathbb{Q}^2$ such that $y^2 = x^3 + 1$; or $(x, y, z, u) \in \mathbb{C}^4$ s.t. $x^2 + y^2 + z^2 + u^2 = xy - zu = 0$. Algebraic geometry has a long history (see J. Dieudonné, Cours de géomètrie algébrique, vol. 1, Presse Universitaire de France). Tremendous developments were made in 1950's by Serre (Faisceaux algébriques cohérents...), and then by Grothendieck and his school (theory of schemes, EGA, SGA...) and al. The present course is an introduction to this "modern" algebraic geometry.

Convention. All rings considered are commutative with a unit.

1.1 Algebraic sets, regular functions and morphisms

We try to give a first definition of affine algebraic varieties.

1.1.1 Affine algebraic sets

We start with the very basic aspect of varieties: algebraic sets. In this (set-theoretical) point of view, we deal with equations. In all this section, k is a fixed **field**. For any positive integer n, $k[T_1, ..., T_n]$ denotes the polynomial ring in n variables and with coefficients in k (it is k if $n = 0$).

Definition 1.1.1. *The* affine n-space over k: $\mathbb{A}^n(k)$ *(or loosely k^n) is just the Cartesian product k^n. The affine line is $\mathbb{A}^1(k)$, the affine plane is $\mathbb{A}^2(k)$.*

Definition 1.1.2. *Let k be a field. Let S be a subset of $k[T_1, \dots, T_n]$. Let*

$$Z(S) := \{x \in \mathbb{A}^n(k) : P(x) = 0, \text{ for all } P \in S\}.$$

We call any set of this form algebraic set.

Remark 1.1.3. *Let S be a subset of $k[T_1, ..., T_n]$ and let I be the ideal of $k[T_1, ..., T_n]$ generated by S. Then $Z(S) = Z(I)$. So every algebraic set is defined by an ideal. In the following we always suppose that S is an ideal.*

Now let's play with algebraic sets in $\mathbb{A}^n(k)$.

Proposition 1.1.4. *Let $\{Z_\alpha = Z(I_\alpha)\}_\alpha$ be a collection of algebraic sets in $\mathbb{A}^n(k)$.*

(1) *If $J \supseteq I$, $Z(J) \subseteq Z(I)$ (note that we change the inclusion directions).*

(2) *$Z(0) = \mathbb{A}^n(k)$, $Z(1) = \emptyset$.*

(3) *$\cap_\alpha Z_\alpha = Z(\sum_\alpha I_\alpha)$, so it is algebraic.*

(4) *Any* finite *union $\cup_{i=1}^n Z(I_i)$ is algebraic and equal to $Z(\cap_{i=1}^n I_i) = Z(I_1 \cdots I_n)$.*

So there exists a unique topology on $\mathbb{A}^n(k)$ whose closed sets are the sets of the form $Z(I)$, with I ideals of $k[T_1, \dots, T_n]$.

Proof.

1. It is clear.
2. It is clear.
3. The inclusion $\supseteq$ follows from 1.. For the other inclusion we remark that if $z \in \cap_\alpha Z_\alpha$ then $f(z) = 0$ for any $f \in I_\alpha$, for any α. So $f(z) =$ for any $f \in \sum_\alpha I_\alpha)$.
4. By 1. we get that $\cup_{i=1}^n Z(I_i) \subseteq Z(\cap_{i=1}^n I_i) \subseteq Z(I_1 \cdots I_n)$. Now we prove the inclusion $Z(I_1 \cdots I_n)$. Let us suppose, without loss of generality, that $z \in Z(I_1 \cdots I_n)$ but $z \notin Z(I_i)$ for $i = 1, \dots, n-1$ and we will porve that $z \in Z(I_n)$. This means that, for $i = 1, \dots, n-1$, there exist $f_i \in I_i$ such that $f_i(z) \neq 0$. Now let $f \in I_n$ then from one hand $(f_1 \cdots f_{n-1} f)(z) = 0$ since $z \in Z(I_1 \cdots I_n)$, on the other hand $(f_1 \cdots f_{n-1} f)(z) = (f_1 \cdots f_{n-1})(z) f(z)$, with $(f_1 \cdots f_{n-1})(z) \neq 0$, therefore $f(z) = 0$ and then $z \in Z(I_n)$.

□

Remark 1.1.5. *We observe that $Z(I) = Z(\sqrt{I})$, since $P(x) = 0$ if and only if $P^n(x) = P(x)^n = 0$ for some n. Similarly we have that $I(Z)$ is a radical ideal.*

Definition 1.1.6. *If Z is a subset of $\mathbb{A}^n(k)$ we set*

$$I(Z) := \{P \in k[X_1, \dots, X_n] : P(x) = 0, \text{ for all } x \in Z\}.$$

It is an ideal of $k[T_1, \dots, T_n]$

Lemma 1.1.7.

1. *For any pair of affine algebraic subsets Z, W, $Z \subseteq W$ if and only if $I(W) \subseteq I(Z)$.*
2. *For any family Z_λ of subsets of k^n we have $\cap_\lambda I(Z_\lambda) = I(\cup_\lambda Z_\lambda)$.*

Proof. Both are clear.

□

Remark 1.1.8. *Let $Z = Z(I)$. It is clear that $\sqrt{I} \subseteq I(Z(I))$ and $Z \subseteq Z(I(Z))$.*

But in general $\sqrt{I} \neq I(Z(I))$. Over a non algebraically closed field k, Z could by empty with $I \neq k[T_0, \dots, T_n]$ However, $Z(I(Z)) = Z$ since $\sqrt{I} \subseteq I(Z)$ implies, by the previous Proposition, that $Z(I(Z)) \subseteq Z = Z(\sqrt{I})$.

For any ring A, we denote by $\operatorname{Spm} A$ the set of the maximal ideals of A, call it the *maximal spectrum* of A. Next we will describe the maximal spectrum of the polynomial rings.

Let A be a ring. We denote by $\operatorname{Spm} A$ the set of all maximal ideals of A. We call it the *maximal spectrum* of A. By convention the unit ideal is not prime. And $\operatorname{Spm}\{0\} = \emptyset$.

We will now endow $\operatorname{Spm} A$ with a topological structure. For any ideal I of A we set $V(I) = \{\mathfrak{p} \in \operatorname{Spm} A : I \subseteq \mathfrak{p}\}$. These sets will be the closed sets in the Zariski topology of $\operatorname{Spm} A$. And, if $f \in A$, we sets $D(f) = \operatorname{Spm} A \setminus V(f)$.

We prove now that they form a family of closed sets.

Lemma 1.1.9. *Let A be a ring.*

1. *For any pair I, J of ideals of A if $I \subseteq J$ we have $V(J) \subseteq V(I)$.*
2. *$V(\{0\}) = \operatorname{Spm} A$ and $V(A) = \emptyset$.*
3. *For any family I_λ of ideals of A we have $\cap_\lambda V(I_\lambda) = V(\sum_\lambda I_\lambda)$.*
4. *For any finite collections of ideals of A we have $\cup_{i=1}^n V(I_i) = V(\cap_{i=1}^n I_i) = V(I_1 \cdots I_n)$.*

So there exists a unique topology on $\operatorname{Spm} A$ whose closed sets are the sets of the form $V(I)$, with I ideals of A. And the open sets $D(f)$ constitutes a basis of open sets for this topology.

Proof.
1. This is clear.
2. This is clear.
3. The inclusion $\supseteq$ is clear. We now prove the opposite inclusion. Let $\mathfrak{p}$ be a prime ideal which contains I_λ for any λ. Then it contains $\sum_\lambda I_\lambda$ since this one is the smallest ideal which contains all I_λ.
4. It is enough to prove that $\cup_{i=1}^n V(I_i) = V(I_1 \dots I_n)$ since

$$\cup_{i=1}^n V(I_i) \subseteq V(\cap_{i=1}^n I_i) \subseteq V(I_1 \cdots I_n),$$

given that $I_1 \cdots I_n \subseteq \cap_{i=1}^n$ and $\cap_{i=1}^n$ is contained in I_i for any $i = 1, \dots, n$. Now prove the inclusion $V(I_1 \cdots I_n) \subseteq \cup_{i=1}^n V(I_i)$. We suppose that a prime ideal $\mathfrak{p}$ contains $I_1 \cdots I_n$. With no loss of generality we can suppose that it does not contain I_i, for $i = 1, \dots, n-1$. This means that there exists $a_i \in I_i$, for $i = 1, \ldots, n-1$ such that $a_i \notin \mathfrak{p}$. Let $a \in J$. Then $a_1 \cdots a_{n-1} a \in I_1 \cdots I_n \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal and $a_i \notin \mathfrak{p}$ for $i = 1, \ldots, n-1$, then $a \in \mathfrak{p}$. Since this is true for any $a \in J$ then we have that $J \subseteq \mathfrak{p}$.

Now it is clear, by definition, the existence of the topology demanded. For the last sentence we remark that if $U = \operatorname{Spm} A \setminus V(I)$, with I an ideal of A generated by a family of elements $\{f_\lambda\}$ then, by (3), we have that $V(I) = \cap_\lambda V(f_\lambda)$, so $U = \cup_\lambda D(f_\lambda)$. This proves the statement. □

Lemma 1.1.10. *Let k be a field and let $x = (x_1, \ldots, x_n) \in k^n$. Then $\mathfrak{m}_x := (X_1 - x_1, \ldots, X_n - x_n)$ is maximal and we have an injection $\psi : k^n \mapsto \mathbb{A}^n_k$ given by $x \mapsto \mathfrak{m}_x$.*

Moreover if I is an ideal of $k[X_1, \ldots, X_n]$ then $x \in Z(I)$ if and only if $\mathfrak{m}_x \in V(I)$. So the algebraic sets are the closed sets for the topology over k^n induced by the Zariski topology.

Proof. We consider the evaluation map

$$\varphi_x : k[X_1, \ldots, X_n] \to k$$

given by $X_i \mapsto x_i$. We now prove that its kernel is $\mathfrak{m}_x$. Let $Q(Y_1, \ldots, Y_n) = P(Y_1 + x_1, \ldots, P_n + x_n) \in k[Y_1, \ldots, Y_n]$. Then

$$Q = Q(0, \ldots, 0) + \tilde{Q}$$

with $\tilde{Q} \in (Y_1, \ldots, Y_n)$. Since $P(X_1, \ldots, X_n) = Q(X_1 - x_1, \ldots, X_n - x_n)$, this implies that

$$P = P(x_1, \ldots, x_n) + \tilde{P}$$

with $\tilde{P} \in \mathfrak{m}_x$. Then $P \in \ker \varphi$ if and only if $P \in (X_1 - x_1, \ldots, X_n - x_n)$. So this ideal is maximal. Clearly ψ is injective.

Finally $x \in Z(I)$ means $I \subseteq \ker \varphi_x = \mathfrak{m}$. So $x \in Z(I)$ if and only if $\mathfrak{m}_x$. Therefore for any ideal I of $k[X_1, \ldots, X_n]$ we have $\psi(Z(I)) = V(I) \cap \psi(k^n)$ which means that the induced topology on k^n by Zariski topology is given by algebraic sets. □

If by the previous lemma we identify $k^n \subseteq \mathbb{A}^n(k)$ we have simply that $Z(I) = V(I) \cap k^n$.

Theorem 1.1.11 (Nullstellensatz)**.** *Let k be an algebraically closed closed field. Let I be an ideal of $k[T_1, ..., T_n]$. Then $I(Z(I)) = \sqrt{I}$. In other word, if $F \in k[T_1, ..., T_n]$ vanishes at the commun zeros of I, then some positive power of F belongs to I.*

Corollary 1.1.12. *If k is algebraically closed and $P \in k[T_1, \ldots, T_n]$ then $Z(P) = \emptyset$ if and only if $P \in k^*$.*

Corollary 1.1.13. *If k is algebraically close d we have $Z(I) \subseteq Z(J)$ if and only if $J \subseteq \sqrt{I}$.*

1.1.2 Proof of Nullstellensatz

We will use the Noether normalization lemma which has its own interest.

Definition 1.1.14. *Let A be a (commuatative unitary) ring. An A-*algebra *B is a ring endowed with a ring homomorphism $A \to B$. A* morphism of A-algebras *is a ring homomorphism compatible with the structure of A-algebra.*

We say that B is an A algebra of finite type over A *(or finitely generated) if there exists a surjective A-algebras morphism $A[T_1, \dots, T_n] \to B$.*

Here we are going to consider the case of a finitely generated algebra A over a field k, i.e. $A = k[X_1, \dots, X_n]/I$ for some n and some ideal I of $k[X_1, \dots, X_n]$. If $I = 0$ then we call $\mathrm{Spm}\,(k[X_1, \dots, X_n])$ the space of dimension n, $\mathbb{A}^n_k$. We first begin with some definitions.

Definition 1.1.15. *We say that a morphism of rings $\phi : A \to B$ is integral if for any element $b \in B$ there exists a monic polynomial $\sum_i a_i T^i$ such that $\sum_i \phi(a_i) b^i = 0$. And we say that the ϕ is finite if B is finite as A-module.*

We remark that ϕ is finite if and only if it is integral and B is a finitely generated A-algebra.

Proposition 1.1.16 (Emmy Noether's normalization)**.** *Let A be a finitely generated algebra over a field a k. Then there exists an integer d and an injection $k[Y_1, \dots, Y_d] \to A$ of k-algebras such that A is finite as $k[Y_1, \dots, Y_d]$-module.*

Proof. We proceed by induction on n. There is nothing to show if $n = 0$. Suppose $n \geq 1$ and that the lemma holds for $n-1$ variables. We can suppose $I \neq 0$. Let $P \in I$ be non-zero. We will first change variables to make P unitary in T_n.

Let $m_1, \dots, m_{n-1} \geq 1$ be positive integers and consider the automorphism

$$\sigma : k[T_1, \dots, T_n] \to k[T_1, \dots, T_n]$$

of k-algebras defined by $\sigma(T_i) = T_i + T_n^{m_i}$ for $i \leq n-1$ and $\sigma(T_n) = T_n$. It is enough to prove the lemma for the ideal $\sigma(I)$. We have

$$\sigma(P) = P(T_1 + T_n^{m_1}, \dots, T_{n-1} + T_n^{m_{n-1}}, T_n).$$

Claim: we can find the m_i in such a way that

$$\sigma(P) = cT_n^r + e_{r-1}(T_1, \dots, T_{n-1})T_n^{r-1} + \cdots, \quad c \in k^*, r \geq 1.$$

Thus we can suppose I contains a monic polynomial $P \in k[T_1, \dots, T_{n-1}][T_n]$. Consider the homomorphism of k-algebras

$$k[T_1, \dots, T_{n-1}]/(I \cap k[T_1, \dots, T_{n-1}]) \to k[T_1, \dots, T_n]/I.$$

It is injective, finite type and integral, thus finite. We conclude by induction on n

So, now, we prove the claim. Let $P = \sum_{\mathbf{l}\in\mathbb{N}^n} a_{\mathbf{l}}\mathbf{X}^{\mathbf{l}}$ be a non-zero element of I. We recall that if $\mathbf{l} = (l_1, \dots, l_n)$ then $\mathbf{X}^{\mathbf{l}}$ means $X_1^{l_1} \cdots X_n{}^{l_n}$. Let $\mathbf{m} = (m_1, \dots, m_{n-1}, 1) \in (\mathbb{N} \setminus \{0\})^n$. So we have that

$$\sigma(P) = \sum_{\mathbf{l}\in\mathbb{N}^n} a_{\mathbf{l}} X_n^{\mathbf{l}\cdot\mathbf{m}} + g(T_1, \dots, T_{n-1}, T_n)$$

where g is a polynomial which does not contain monomials purely in T_n and such that $\deg_{T_n} g < \max\{\mathbf{l}\cdot\mathbf{m} | a_{\mathbf{l}} \neq 0\}$

Let $\tilde{l}$ be the maximum $\mathbf{l}$, for the lexicographic order, such that $a_{\mathbf{l}} \neq 0$. Now we take $\mathbf{m} = (m_1, \dots, m_{n-1}, 1)$ such that the scalar product $\mathbf{m}\cdot\tilde{\mathbf{l}}$ is larger than $\mathbf{m}\cdot\mathbf{l}$ for any $\mathbf{l}$ such that $a_{\mathbf{l}} \neq 0$. To find $\mathbf{m}$ we can, for instance, proceed as follows. We set $\mathbf{m} = (d^{n-i})_{i=1,\dots,n}$, with $d \in \mathbb{N}$. Then for any $\mathbf{l}$ we have that

$$\mathbf{m}\cdot\mathbf{l} = \sum_{i=1}^{n} d^{n-i} l_i.$$

So, since $\sum_{i=1}^n T^{n-i}\tilde{l}_i$ grows to infinity faster then $\sum_{i=1}^n T^{n-i} l_i$, with $a_{\mathbf{l}} \neq 0$, it is clear that if we take $d >> 0$ we have that $e = \mathbf{m}\cdot\tilde{\mathbf{l}}$ is the larger one. So we have that

$$\sigma(P) = a_{\tilde{\mathbf{l}}} T_n^e + \sum_{i=0}^{e-1} Q_i(T_1, \dots, T_{n-1}) T_n^i.$$

as wanted.

□

Remark 1.1.17. *We will see later how to interpret this geometrically.*

Corollary 1.1.18. *Let A be a non-zero finitely generated algebra over a field k. Let $\mathfrak{m}$ be a maximal ideal. Then the field $A/\mathfrak{m}$ is a finite algebraic extension of k. In particular if k is algebraically closed then $A/\mathfrak{m} = k$.*

Proof. Since $A/\mathfrak{m}$ is a finitely generated algebra over k then by the previous Proposition there exists an positive integer r and a finite injective k-algebras morphism $\phi : k[Y_1, \dots, Y_r] \to A/\mathfrak{m}$. Now since $A/\mathfrak{m}$ is a field then $k[Y_1, \dots, Y_r]$ is a field by the following Lemma, which implies that $r = 0$.

□

Lemma 1.1.19. *Let $\phi : A \to B$ be an integral injection of integral domains. Then A is a field if and only if B is a field.* [1]

Proof. If part. Since B is a field then for any non-zero $a \in A$ we have $\phi(a)$ is invertible. Now ϕ is integral so there exist $a_i \in A$ for $i = 1, \dots, n$, with $a_n = 1$, such that $\sum_{i=0}^n \phi(a_i)\frac{1}{\phi(a)^i} = 0$. If we multiply by $\phi(a)^n$ we obtain

$$1 = \phi(-\sum_{i=0}^{n-1} a_i a^{n-i})$$

[1] If B is a field then automatically A is an integral domain.

which says that a is invertible in B since ϕ is injective.

Only if part. Let $b \in B \setminus 0$. Then there exist $a_i \in A$ for $i = 1, \ldots, n$, with $a_n = 1$, such that $\sum_{i=0}^{n} \phi(a_i) b^i = 0$. We chose n minimal. Since B is an integral domain and n minimal we have that $\phi(a_0) \neq 0$. but a_0 is invertible, since different from zero, then we have that

$$1 = -\frac{\sum_{i=1}^{n} \phi(a_i) b^i}{\phi(a_0)},$$

which implies that b is invertible.

□

Exercise 1.1.1. *Let k be a finite field. Then k^n with Zariski topology is a finite topological space where any point is closed. So the topology is discrete. This shows that in general it is not enough to work with algebraic sets since we lose informations. For instance we have that $(\mathbb{F}_p)^n$ is homeomorphic to $\mathbb{F}_{p^n}$ for any n.*

Lemma 1.1.20. *Let A be a non-zero finitely generated algebra over a field k and let I be an ideal of A. Then*

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{m} \in V(I), \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}.$$

In particular $I \mapsto V(I)$ gives a bijection, inclusion reversing, between radical ideals and closed subsets of $\operatorname{Spm} A$.

Proof. We can suppose that $I = 0$ up to pass to quotient by I. The inclusion $\subseteq$ is clear. Let f be an element of $\bigcap_{\mathfrak{m} \in \operatorname{Spm} A} \mathfrak{m}$. we prove it is nilpotent. We suppose it isn't. Then we have that A_f is a non-zero ring. Then it has a maximal ideal $\mathfrak{m}$, by Zorn's lemma. Let $\varrho : A \to A_f$ the localization map. Let us consider $\varrho^{-1}(\mathfrak{m})$. Now we remark that $A_f = A[T]/(fT - 1)$ is a finitely generated algebra over k. So, by 1.1.18, we have that $A_f/\mathfrak{m}$ is finite algebraic extension of k. Since we have the following injections

$$k \to A/\varrho^{-1}(\mathfrak{m}) \to A_f/\mathfrak{m}$$

and the composition is a finite algebraic extension of fields then in particular $A/\varrho^{-1}(\mathfrak{m}) \to A_f/\mathfrak{m}$ is an integral extension. Then by Lemma 1.1.19 we have that $A/\varrho^{-1}(\mathfrak{m})$ is a field and so $\varrho^{-1}(\mathfrak{m})$ is maximal. But it does not contain f, which is absurd.

The last part is immediate since we can reconstruct I by the maximal ideal which contain it.

□

Example 1.1.21. *This is false in general. For instance take R is a discrete valuation ring. Then $\sqrt{(0)} = (0)$ and there is only one maximal ideal, which is not zero.*

Corollary 1.1.22. *Let k be an algebraically closed field Then the application $\psi : k^n \to \mathbb{A}_k^n$ given by $x = (x_1, \ldots, x_n) \mapsto \mathfrak{m}_x := (X_1 - x_1, \ldots, X_n - x_n)$ is an homeomorphism.*

Proof. It is enough to prove that it is surjective. Let $\mathfrak{m}$ be a maximal ideal of $k[X_1, \ldots, X_n]$. Then $k[X_1, \ldots, X_n]/\mathfrak{m} \simeq k$ by Corollary 1.1.18 and the fact that k is algebraically closed. The quotient map is necessarily the evaluation map $P \mapsto P(x_1, \ldots, x_n)$ in a point $(x_1, \ldots, x_n)$. So $\mathfrak{m} = \mathfrak{m}_x = \psi(x)$.

□

Remark 1.1.23. *The above result is false if k is not algebraically closed. For instance if $k = \mathbb{R}$ and $I = (T^2 + 1) \subseteq \mathbb{R}[T]$ then $Z(I) = \emptyset$ but $V(I)$ is a closed point of* $\operatorname{Spm} \mathbb{R}[T]$.

Proposition 1.1.24 (Nullstellensatz)**.** *Let k be an algebraically closed field. For any ideal I of $k[X_1, \ldots, X_n]$ we have that*

$$I(Z(I)) = \sqrt{I}.$$

Proof. If $x = (x_1, \ldots, x_n) \in k^n$ then $I(\{x\}) = \{f \in k[X_1, \ldots, X_n] | f(x) = 0\} = \mathfrak{m}_x$. So

$$I(Z(I)) = I(\bigcup_{x \in Z(I)} \{x\}) = \bigcap_{x \in Z(I)} I(\{x\}) = \bigcap_{x \in Z(I)} \mathfrak{m}_x \overset{*}{=} \bigcap_{\substack{\mathfrak{m} \in V(I) \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m} = \sqrt{I}.$$

where the equality $*$ follows by the fact that any maximal ideals is of the form $\mathfrak{m}_x$, and the last equality follows by the previous Lemma. □

Remark 1.1.25. *From the Nullstellensatz it follows for instance that, over an algebraicaly closed field, if $I \neq A$ then $Z(I) \neq \emptyset$. Indeed $Z(I) = \emptyset$ implies $\sqrt{I} = I(Z(I)) = I(\emptyset) = A$.*

Corollary 1.1.26. *Let k be an algebraically closed field then the application $Z \mapsto I(Z)$ gives a bijection, inclusion reversing, between the algebraic sets of k^n and radical ideals of $k[X_1, \ldots, X_n]$ whose inverse is $I \mapsto Z(I)$.*

Proof. By Nullstellensatz we have that if I is radical then

$$I(Z(I)) = \sqrt{I} = I.$$

Moreover if Z is an algebraic set then $Z(I(Z)) = Z$. Indeed $Z = Z(I)$ for some ideal I. Then

$$Z(I(Z(I)) = Z(\sqrt{I}) = Z(I).$$

□

Let us introduce some notation. Let $Z = Z(I) \subseteq \mathbb{A}^n(k)$ with I radicial. Let

$$A(Z) = k[T_1, \ldots, T_n]/I.$$

For any $f \in A(Z)$, let $\tilde{f} \in k[T_1, \ldots, T_n]$ be an arbitrary lifting of f. For any $z \in Z$, denote by

$$f(z) = \tilde{f}(z) \in k.$$

This is independent on the choice of the lifting $\tilde{f}$. Denote by

$$D(f) = D(\tilde{f}) \cap Z = \{z \in Z \mid f(z) \neq 0\}.$$

This open subset of Z is independent on the choice of $\tilde{f}$ and is called a *principal open subset of Z*.

If J is an ideal of $A(Z)$, denote

$$Z(J) = Z(\tilde{J}) \cap Z = \{z \in Z \mid f(z) = 0 \; \forall z \in J\}$$

where the ideal $\tilde{J}$ is the preimage of J in $k[T_1, \ldots, T_n]$. Corollary 1.1.13 can be easily generalized to the following :

Corollary 1.1.27. *In Z, we have*

1. $Z(J_1) \subseteq Z(J_2)$ *if and only if* $J_2 \subseteq \sqrt{J_1}$;
2. $D(f) \subseteq D(g)$ *if and only if there exists* $N \geq 1$ *and* $h \in A$ *such that* $f^N = gh$.

Lemma 1.1.28. *Let Z be an algebraic set.*

(1) *The familly of principal open subsets $\{D(f) \cap Z\}_{f \in A}$ is a basis of topology for Z.*

(2) *Let U be an open subset of Z. Then the topological space U is quasi-compact.*

Proof. (1) Any open subset U of Z is the complement of some $Z(J) \subseteq Z$ for some ideal $J \subseteq A := A(Z)$. We have $Z(J) = \cap_{f \in J} Z(f)$, so $U = \cup_{f \in J} D(f)$. Note that as A is noetherian, J is generated by finitely many $f_1, \ldots, f_m$, and then U is covered by finitely many principal open subsets $D(f_1), \ldots, D(f_m)$.

(2) Let $U = \cup_j U_j$ be an open covering, by (1) we can refine U_j into a covering by principal open subsets and restrict ourselves to the case when each U_j is a principal open subset $D(g_j)$. Thus

$$U = \cup_j D(g_j).$$

This means that $X \setminus U = \cap_j Z(g_j) = Z(\sum_j g_j A)$. As A is noetherian, $\sum_j g_j A$ is generated by finitely many g_j's : $g_{j_1}, \ldots, g_{j_r}$. This implies that

$$X \setminus U = Z(\sum_{q \leq r} g_{j_q} A) = \cap_{q \leq r} Z(g_{j_q})$$

and $U = \cup_{q \leq r} D(g_{j_q})$. This proves the quasi-compactness. $\square$

1.1.3 Topology and irreducible components

Remark 1.1.29. *The Zariski topology on $\mathbb{A}^n(k)$ is very different from the usual (say metric) topologies. It has rather few closed subsets and is far from being (T_1-)separated. When k is a topological field ($\mathbb{C}$, $\mathbb{C}_p$ etc...), we can use the stronger product topology on $\mathbb{A}^n(k)$. This being said, in general, the Zariski topology contains enough information for everyday's life. To go further Grothendieck introduced étale topology, flat topology etc...*

Example 1.1.30. *Closed subsets of $\mathbb{A}^1(k)$: all finite subsets and $\mathbb{A}^1(k)$ itself.*

Definition 1.1.31. *We want to decompose a topological space into small pieces.* Irreducible topological spaces*: X is* irreducible *if it is non-empty and can not be the union of two strict closed subsets. A closed subset of X is said to be irreducible if it is irreducible for the induced topology.* Irreducible components *$\{X_i\}_i$ of X : these are irreducible closed subsets, not strictly contained in another irreducible closed subset, and such that $X = \cup_i X_i$.*

Remark 1.1.32. *Again, usual topological spaces are not irreducible.*

Proposition 1.1.33. *An affine algebraic set Z is irreducible if and only if $I(Z)$ is a prime ideal. If k is algebraically closed then $Z(I)$ is irreducible if and only if $\sqrt{I}$ is prime.*

Exercise 1.1.2.
1. *Prove that a topological space X is irreducible if and only if any non-empty open subset is dense. Or equivalently any two non-empty open subsets have non-empty intersection.*
2. *Prove that if $Z \subseteq X$ is irreducible then $\overline{Z}$ is also irreducible.*

Using Zorn Lemma one sees that any $x \in X$ belongs to a maximal connected. and moreover by previous exercise we have that irreducible components are closed.

Proof. Let us suppose that Z is irreducible and let $f, g \in k[T_1, \dots, T_n]$ such that $fg \in I(Z)$. Since $Z(I(Z)) = Z$ then $Z \subseteq Z((fg)) = Z((f)) \cup Z((g))$. But Z is irreducible so we can suppose that $Z \subseteq Z((f))$ then $f \in I(Z)$.

Converesely let us suppose that $I(Z)$ is prime and $Z = Z_1 \cup Z_2$ union of two nonempty affine algebraic sets such that $Z_i \neq Z$ for $i = 1, 2$. Then $I(Z) = I(Z_1) \cap I(Z_2)$. Let $f_i \in I(Z_i) \setminus I(Z)$, for $i = 1, 2$. This is possibile since $Z_i \subsetneq Z$ implies $I(Z) \subsetneq I(Z_i)$.

If k is algebraically closed then $\sqrt{I} = I(Z(I))$ so the result follows by the first part. □

Remark 1.1.34. *The second part is not true in both direction if k is not algebraically closed. Take for instance $Z = Z((0)) \in \mathbb{A}^1(\mathbb{F}_p)$ or $Z((X(X^2 + 1)))$ in $\mathbb{A}^1(\mathbb{R})$.*

Corollary 1.1.35. *If k is algebraically closed $\mathbb{A}^n(k)$ is irreducible.*

Lemma 1.1.36. *Let A be a noetherian ring. Then any radical ideal is a finite interesection of prime ideals.*

Proof. Consider the set of radical ideals which are not finite intersections of prime ideals and choose a maximal element I (use noetherianity). Then I is not prime. Choose $a, b \in A \setminus I$ such that $ab \in I$. Consider $\sqrt{I + aA}$ and $\sqrt{I + bA}$. They are both finite intersections of prime ideals and their intersection is I. Therefore I is also a finite intersection of prime ideals, contradiction ! $\square$

Proposition 1.1.37. *Let k be algebraically closed. Let Z be an algebraic set in $\mathbb{A}^n(k)$. Then there are unique irreducible algebraic sets $Z_1, ..., Z_m$ such that $Z = Z_1 \cup ... \cup Z_m$, and that there is no inclusion relation between the Z_i's. The Z_i's are the irreducible components of Z.*

Proof. The uniqueness is easy. Applying Lemma 1.1.36 to $A = k[T_1, \ldots, T_n]$ and $I(Z)$, we see (*need Nullstellensatz*) that Z is a finite union of irreducible closed subsets. Now remove those which are not minimal and we are done. $\square$

Exercise 1.1.3. *Some properties of irreducible topological spaces:*

1. *$X \neq \emptyset$ is irreducible if and only if any non-empty open subset is dense in X.*

2. *If X is irreducible, then any non-empty open subset of X is irreducible.*

3. *If $Y \subset X$, then Y is irreducible and only if its closure $\overline{Y}$ in X is irreducible.*

4. *Let $X = \cup_i U_i$ be an open covering. Then Y (supposed to be non-empty) is irreducible if and only if for all i, $Y \cap U_i$ is either empty or irreducible.*

Exercise 1.1.4.

1. *Show that the intersection of an algebraic set in $\mathbb{A}^n(k)$ with a principal open subset can be naturally identified with an algebraic set in $\mathbb{A}^{n+1}(k)$.*

2. *Give a natural structure of algebraic sets to $Sl_n(k)$ and $Gl_n(k)$.*

3. *Finite subsets of $\mathbb{A}^n(k)$ are algebraic.*

4. *Let Z be an algebraic set in $\mathbb{A}^2(k)$ and let L be a line in $\mathbb{A}^2(k)$. Show that either we have either $L \subseteq Z$ or $L \cap Z$ is finite. Show the similar statement in the projective case.*

5. *Is $\{(x, y) \in \mathbb{C}^2 | \sin x = y^2\}$ an algebraic subset in the affine plane ?*

6. *Show that $\{(a^2, ab, b^2) \in \mathbb{P}^2(k) \mid (a, b) \in \mathbb{P}^1(k)\}$ is a projective algebraic set.*

7. *Describe the image of the map*
$$\mathbb{A}^2(k) \to \mathbb{A}^2(k), \quad (x, y) \mapsto (x, xy)$$
and show that the image is not open nor closed in $\mathbb{A}^2(k)$.

8. *Let k be endowed with the Zariski topology. Show that the Zariski topology is strictly finer (has more open subsets) on k^2 than the product topology.*

1.1.4 Projective algebraic sets

In algebraic geometry, there is a class of varieties who share a lot of properties with compact complex manifolds, they are called proper algebraic varieties. A special kind of proper varieties are projective varieties.

Definition 1.1.38. *A polynomial $f \in k[T_0, ..., T_n]$ is homogeneous of degree d if $f = \sum a_{i_0 \dots i_n} T_0^{i_0} \cdots T_n^{i_n}$ with $a_{i_0 \dots i_n} \neq 0$ only if $i_0 + \cdots + i_n = d$. Any $f \in k[X_0, \dots, X_n]$ has a unique expression $f = f_0 + f_1 + \cdots + f_N$ in which f_d is homogeneous of degree d for each $d = 0, 1, \dots, N$.*

Proposition 1.1.39. *If f is homogeneous of degree d then $f(\lambda T_0, \dots, \lambda T_n) = \lambda^d f(T_0, \dots, T_n)$ for all $\lambda \in k$; if k is an infinite field then the converse also holds.*

Definition 1.1.40. *An ideal $I \subseteq k[X_0, \dots, X_n]$ is homogeneous if for all $f \in I$, the homogeneous decomposition $f = f_0 + f_1 + \cdots + f_N$ of f satisfies $f_i \in I$ for all i.*

This is equivalent to ask that I is generated by homogenous polynomials.

Let $\mathbb{P}^n(k)$ be the n-dimensional projective space over a field k Then $f \in k[T_0, \dots, T_n]$ is not a function on $\mathbb{P}^n(k)$: by definition, $\mathbb{P}^n(k) = kn+1 \setminus \{0\} / \simeq$, where $\simeq$ is the equivalence relation given by $(t_0, \dots, t_n) \simeq (\lambda t_0, \dots, \lambda t_n)$ for $\lambda \in k \setminus \{0\}$; f is a function on k^{n+1}. We denote $\pi : k^{n+1} \setminus \{0\} \to \mathbb{P}^n(k)$. Nevertheless, for $p \in \mathbb{P}^n(k)$, the condition $f(p) = 0$ is well defined provided that f is homogeneous: suppose $p = (t_0 : \cdots : t_n)$, so that $(t_0, \dots, t_n)$ is a representative in $k^{n+1} \setminus \{0\}$ of the equivalence class of p. Then since $f(\lambda t_0, \dots, \lambda t_n) = \lambda^d f(t_0, \dots, t_n) = 0$, the condition $f(p) = 0$ is independent of the choice of representative.

Definition 1.1.41. *Let I be an homogenous ideal of $k[T_0, \dots, T_n]$. We define $Z_+(I) = \{p \in \mathbb{P}^n(k) : f(p) = 0 \text{ for all } f \in I\}$. If k is infinite and $Z \subseteq \mathbb{P}^n(k)$, we define $I_+(Z) = I(\pi^{-1} Z)$. It is an homogenous ideal.*

We have that Z_+ and I_+ satisfies the same conditions of Z and I. We have Zariski topology in which closed subsets are algebraic closed subset of $\mathbb{P}^n(k)$.

We have a projective version of Nullstellensatz.

Theorem 1.1.42. *Assume that k is an algebraically closed field. Let J be an homogenous ideal. Then*

1. *$Z_+(J) = \emptyset$ if and only if $\sqrt{J} \supseteq (T_0, \dots, T_n)$ (the so called* irrelevant ideal*);*
2. *If $Z_+(J) \neq \emptyset$ then $I_+(Z_+(J)) = \sqrt{J}$.*

Proof. For a homogeneous ideal $J \subseteq k[T_0, ..., T_n]$, we consider the affine algebraic set $Z(J) \subseteq \mathbb{A}^{n+1}(k)$. Then, since J is homogeneous, $Z(J)$ is empty or $Z(J) = \pi^{-1} Z_+(J) \cup \{0\}$ Hence $Z_+(J) = \emptyset$ if and only if $Z(J) \subseteq \{0\}$ if and only if $\sqrt{J} \supseteq (T_0, \dots, T_n)$, where the last implication uses the affine Nullstellensatz. Also, if $Z_+(J) \neq \emptyset$ then $f \in I_+(Z_+(J))$ if and only if $f \in I(Z(J)) = \sqrt{J}$. $\square$

Proposition 1.1.43. *Let k be an algebraically closed field. Let I, J be proper homogeneous ideals of $k[T_0, \dots, T_n]$. Then $Z_+(I) \subseteq Z_+(J)$ if and only if $J \subseteq \sqrt{I}$.*

Corollary 1.1.44. *$Z_+(I)$ is irreducible if and only if $\sqrt{I}$ is prime and different from the irrelevant ideal $\mathfrak{m}_0 = (T_0, \dots, T_n) \subset k[T_0, \dots, T_n]$.*

Definition 1.1.45. *Let f be an homogenous polynomial we define $D_+(f) = \{p \in \mathbb{P}^n(k) : f(p) \neq 0\} = \mathbb{P}^n(k) \setminus Z_+((f))$.*

We have $\mathbb{P}^n(k) = \cup_i D_+(T_i)$. Moreover, as in the affine case, the $D_+(F)$ form a basis for the Zariski topology.

Proposition 1.1.46. *For any $0 \leq i \leq n$, $D_+(T_i)$ is homeomorphic to $\mathbb{A}^n(k)$.*

Proof. Let us consider the function

$$\phi : \mathbb{A}^n(k) \to D_+(T_i)$$

given by

$$(x_1, \dots, x_n) \mapsto [x_1 : \dots : 1 : \dots : x_n]$$

where the 1 is at the i-th place. It is bijective and, for any homogenous polynomial $F(T_0, \dots, T_n)$, $\phi^{-1}(D_+(F)) = D(f)$ where $f(S_1, \dots, S_n) = F(S_1, \dots, 1, \dots, S_n)$ and, for any polynomial $f(S_1, \dots, S_n)$, $\phi(D(f)) = D_+(F)$ where $F(T_0, \dots, T_n) = T_i^{\deg f}(f(T_0/T_i, \dots, T_n/T_i))$. □

1.1.5 Regular functions

Let k be an algebraically closed field and let $Z = Z(I) \subseteq \mathbb{A}^n_k$ be some algebraic set with I radicial. We want to define the set of the regular functions $\mathcal{O}_Z(U)$ on any open subset U of Z. This is a particular subset of k^U. By *functions* we means maps with values in k.

There are some reasonable requirements.

(1) The restriction to Z of any polynomial function $\in k[T_1, \dots, T_n]$ should be regular on Z. So we should have a map $k[T_1, \dots, T_n] \to \mathcal{O}_Z(Z)$ given by restriction. This map factors naturally into an injective map $k[T_1, \dots, T_n]/I \to \mathcal{O}_Z(Z)$.

(2) Let $z \in Z$. Let $F \in k[T_1, \dots, T_n]$ such that $F(z) \neq 0$. Then $F(z') \neq 0$ for any z' in the open neighborhood $D(F) \cap Z$ of z in Z. So $1/F$ should be regarded as a regular function. Combining with (1), we see that $G/F \in k(T_1, \dots, T_n)$ should be considered as regular in an open neighborhood of z. Again, this makes sense modulo I, so the classe of G/F in $k[T_1, \dots, T_n, 1/F]/I)$ should be regular at z.

(3) Note that $D(F) \cap Z$ is naturally an algebraic subset in k^{n+1} with variables $k[T_1, \dots, T_n, S]$ and the relation $FS - 1 = 0$. So (1) implies that the class of S should be a regular function on $D(F) \cap Z$ and this coincides with (2).

Definition 1.1.47. *Let $Z = Z(I)$ be an affine algebraic set in $\mathbb{A}^n(k)$ with I radicial. Let U be an open subset of Z. A* regular function *on U is a map $f : U \to k$ such that for any $z \in U$, there exist $F, G \in k[T_1, \dots, T_n]$ such that $F(z) \neq 0$ and $f(z') = G(z')/F(z')$ for all z' in some open neighborhood of z contained in $D(F) \cap U$.*

We will denote by $\mathcal{O}_Z(U)$ the k-algebra of regular functions on U. We observe that $\mathcal{O}_Z(U)$ is naturally a k-algebra and if V is an open subset in U, then the restriction of maps takes elements of $\mathcal{O}_Z(U)$ to elements of $\mathcal{O}_Z(V)$. Note that in general $\mathcal{O}_Z(U)$ is not of finite type over k.

We have a natural map $k[T_1, \dots, T_n] \to \mathcal{O}_Z(Z)$ whose kernel is exactly I (radicial). So we get a natural injective homomorphism of k-algebras

$$A(Z) := k[T_1, \dots, T_n]/I \to \mathcal{O}_Z(Z).$$

Proposition 1.1.48. *Let $Z = Z(I) \subseteq \mathbb{A}^n(k)$ be an algebraic set defined by a radicial ideal I.*

(1) *Let $U \subseteq Z$ be an open subset. Then any regular function $U \to k$ is continuous for the Zariski topology.*

(2) *For any $f \in A(Z)$, the natural map $A(Z) \to \mathcal{O}_Z(D(f))$ induces an isomorphism $A(Z)_f \to \mathcal{O}_Z(D(f))$, where $A(Z)_f = A(Z)[S]/(fS - 1)$ and the kernel is $\{g \in A(Z) | fg = 0\}$*

(3) *The natural map $A(Z) \to \mathcal{O}_Z(Z)$ above is an isomorphism.*

Proof. (1) Let h be a regular function on U. Let $z \in U$. Then there exists $z \in V \subseteq U$ and $F, G \in k[T_1, \dots, T_n]$ such that $V \subseteq D(F)$ and $h = G/F$ on V. It is enough to show that the function $h_0 = G/F$ is continuous on $D(f)$ (where f is the image of F is $A(Z)$). The proper closed subsets of k are finite, so it is enough to show that $h_0^{-1}(\lambda)$ is closed in $D(f)$ for any $\lambda \in k$. But this set is nothing but $\{z \in D(f) \mid (g - \lambda f)(z) = 0\}$. So it is closed in $D(f)$.

(2) The statement about the kernel is clear. The above injection allows us to view f as a regular function on Z. Its restriction on $D(f)$ never vanishes. If we look at the definition of regular functions, we see that $1/f : D(f) \to k$, $z \mapsto 1/f(z)$ is a regular function on $D(f)$. Consider the homomorphism of k-algebras $A(Z)[S] \to \mathcal{O}_Z(D(f))$ which takes S to $1/f$. Then it factors through the quotient $A(Z)[S]/(fS - 1) \to \mathcal{O}_Z(D(f))$. Now let $g \in \mathcal{O}_Z(D(f))$ and let us take $D(f) = \cup_{i=1}^n D(f_i)$ such that $g_{|D(f_i)} = g_i/f_i$. The last condition implies that $gf_i^2 = g_i f_i$ in $\mathcal{O}_Z(Z)$. Now $D(f) = \cup_{i=1}^n D(f_i^2)$ and therefore there exists r such that $f^r = \sum_i^n a_i f_i^2$ therefore $gf^r = \sum_i^n a_i g_i f_i$ which means that $g \in A[Z]_f$.

(3) This follows from 2 with $f = 1$.

□

Example 1.1.49. *1. If $Z = \mathbb{A}_k^n$ and $g \in k[T_1, \dots, T_n]$, then $\mathcal{O}_Z(D(g)) = k[T_1, \dots, T_n, 1/g] = k[T_1, \dots, T_n, S]/(gS - 1)$. This also corresponds to the fact $D(g)$ is an algebraic set in k^{n+1}.*

2. If $Z = \mathbb{A}_k^2$ and $U = Z \backslash \{(0,0)\}$. Then the restriction $\mathcal{O}_Z(Z) = k[T_1, T_2] \to \mathcal{O}_Z(U)$ is an isomorphism.

1.1.6 Morphisms of affine algebraic sets

The topological spaces $\mathbb{A}^1(k)$ and $Z(y^2-x^3) \subset \mathbb{A}^2(k)$ are homeomorphic by the map $t \mapsto (t^2, t^3)$. But visually they are different. The second has a singular point (a cusp). What make them different k and $Z(y^2 - x^3) \subset k^2$? The answer is that they are not isomorphic as algebraic sets. To give a sense to this claim, we have to define morphisms of algebraic sets. There are some natural requirements:

(1) $\mathrm{Mor}(Z, \mathbb{A}^1(k)) = \mathcal{O}_Z(Z)$;

(2) $\mathrm{Mor}(Z, \mathbb{A}^m(k)) = \mathcal{O}_Z(Z)^n = \mathrm{Hom}_{k-\mathrm{alg.}}(k[S_1, \dots, S_m], \mathcal{O}_Z(Z))$; Note that for $f : Z \to \mathbb{A}^m(k)$, the corresponding homomorphism $\varphi : k[S_1, \dots, S_m] \to \mathcal{O}_Z(Z)$ takes S_i to f_i the i-th coordinate of f.

(3) If $Z' = Z(J) \subseteq \mathbb{A}^m(k)$, then $f(Z) \subseteq Z'$ if and only if $J \subseteq \ker\varphi$. So $\mathrm{Mor}(Z, Z') = \mathrm{Hom}_{k-\mathrm{alg.}}(\mathcal{O}_{Z'}(Z'), \mathcal{O}_Z(Z))$;

Definition 1.1.50. *A morphism of affine algebraic sets*

$$f : Z = Z(I) \to Y = Z(J), \quad I \subseteq k[T_1, \dots, T_n],\ J \subseteq k[S_1, \dots, S_m]$$

(with I, J radicial) is given by

$$f(z) = (F_1(z), \dots, F_m(z))$$

for some $F_1, \dots, F_m \in k[T_1, \dots, T_n]$ such that $f(z) \in Y$, for any $z \in Z$. So f is just the restriction to $Z(I)$ of a polynomial function $k^n \to k^m$.

It is clear that the composition of morphisms of algebraic sets (when applicable) is again a morphism of algebraic set and the identity on Z is a morphism. So we have the notion of isomorphisms of algebraic sets.

An bijective morphism of algebraic sets, even when it is a homeomorphism, is not necessarily an isomorphism.

Let f be a morphism as above. We are going to associate a homomorphism of k-algebras $\varphi : A(Y) \to A(Z)$ as follows. Let $\Phi : k[S_1, \dots, S_m] \to k[T_1, \dots, T_n]$ be the unique homomorphism of k-algebras defined by $\Phi(S_j) = F_j(T_1, \dots, T_n)$ for all $j \leq m$. Let $z \in \mathbb{A}^n(k)$ and let $P(S_1, \dots, S_m) \in k[S_1, \dots, S_m]$. Then

$$P(f(z)) = P(F_1(z), \dots, F_m(z)) = P(F_1, \dots, F_m)(z) = \Phi(P)(z) \tag{1.1}$$

(one can write $P = \sum \alpha_v S_1^{\nu_1} ... S_m^{\nu_m}$ and check directly the above equality). Thus for all $z \in Z$ and for all $P \in J$, we have $\Phi(P)(z) = 0$. Therefore $\Phi(J) \subseteq I$ and Φ induces a unique commutative diagram

$$\begin{tikzcd} k[S_1, \dots, S_m] \arrow[r, "\Phi"] \arrow[d] & k[T_1, \dots, T_n] \arrow[d] \\ A(Y) \arrow[r, "\varphi"] & A(Z) \end{tikzcd}$$

with a homomorphism of k-algebras φ. Moreover for any $h \in A(Y)$ we have

$$h(f(z)) = \varphi(h)(z), \quad \forall z \in Z \tag{1.2}$$

In other words, $h \circ f = \varphi(h)$ in $A(Z)$.

Proposition 1.1.51. *The functor from the category of affine algebraic sets to the category of reduced k-algebra of finite type given by $Z \to A(Z)$, is anti-equivalence of category.*

Proof. Let $\varphi : k[S_1, \ldots, S_m]/J \to k[T_1, \ldots, T_n]/I$ be a k-algebra homomorphism. Let $F_j \in k[T_1, \ldots, T_n]$ be a lifting of $\varphi(\bar{S}_j)$. Define $f : Z \to \mathbb{A}_k^m$ by $f(z) = (F_1(z), \ldots, F_m(z))$. Let us prove $f(Z) \subseteq Y$ (this will imply that f is a morphism from Z to Y). The map φ fits in a commutative diagram

$$\begin{tikzcd}
k[S_1, \ldots, S_m] \arrow[r, "\Phi"] \arrow[d] & k[T_1, \ldots, T_n] \arrow[d] \\
A(Y) \arrow[r, "\varphi"] & A(Z)
\end{tikzcd}$$

For any $P \in J$, we have

$$P(f(z)) = P(F_1(z), \ldots, F_m(z)) = P(F_1, \ldots, F_m)(z) = \Phi(P)(z) = 0.$$

So $f(z) \in Y$ and $f \in \mathrm{Mor}(Z, Y)$. It is straightforward to check that $\varphi \mapsto f$ is the inverse construction of the construction $f \mapsto \varphi$. $\square$

Corollary 1.1.52. *Two algebraic sets Y, Z are isomorphic if and only if the k-algebras $A(Z)$ and $A(Y)$ are isomorphic.*

Example 1.1.53. *The map $t \mapsto (t^2, t^3)$ induces a homeomorphism*

$$\mathbb{A}^1(k) \to Z := Z(y^2 - x^3) \subset \mathbb{A}^2(k).$$

But the two algebraic sets are not isomorphic because the $A(Z)$ is not a principal ideal domain (the ideal generated by x, y in $A(Z)$ is not principal).

Any algebraic set Z induces a reduced k-algebra of finite type $A(Z)$. Conversely, any reduced k-algebra of finite type A is isomorphic to $A(Z)$ for some (not unique) Z. Moreover, morphisms of algebraic sets correspond to homomorphisms of k-algebras. So, up to isomorphisms, algebraic sets are equivalent to reduced k-algebras of finite type.

1.1.7 Exercises

Exercise 1.1.5. *Let $A = k[T_0, \ldots, T_n]$ and denote by A_d the vector space of homogeneous polynomials of degree d.*

1. *Show that an ideal I of A is homogeneous if and only if $I = \oplus_{d \geq 0} I \cap A_d$.*

2. *If I is homogeneous, show that $\sqrt{I}$ is homogeneous.*

3. *If I, J are homogeneous, show that $I+J, IJ$ and $I \cap J$ are homogeneous.*

4. *Let I be an ideal, let $I^h = \oplus_{d \geq 0} I \cap A_d$. Show that I^h is the homogeneous ideal generated by the homogeneous elements of I. Show that if I is prime then so is I^h.*

5. *If I is homogeneous, show that any prime ideal over I (prime ideal minimal among those containing I) is homogeneous and that I is a finite intersection of homogeneous prime ideals.*

Exercise 1.1.6. *Show that*

$$\mathbb{P}^1(k) \to \mathbb{P}^2(k), \quad (u; v) \mapsto (u^2; uv; v^2)$$

is well-defined an is a homeomorphism onto $Z_+(xz - y^2)$.

Exercise 1.1.7. *Find the irreducible components of $Z(xy - z^2, xz - x) \subset \mathbb{A}^3(k)$ and that of $Z_+(xy - z^2, xz - xt) \subset \mathbb{P}^3(k)$.*

Exercise 1.1.8. *Let $\phi : A \to B$ be a finite homomorphism (of k-algebras of finite type, but this does not matter). Let $\mathfrak{m}$ be a maximal ideal of A. Then the set of maximal ideals of B containing $\phi(\mathfrak{m})$ is finite.*

Exercise 1.1.9. *Let $f : X \to Y$ be a continuous map of topological spaces. Suppose X is non-empty.*

1. *X irreducible implies $f(X)$ is irreducible and X is connected.*

2. *X is irreducible iff any non-empty open subset of X is dense in X. Then any non-empty open subset of X is irreducible.*

3. *If Z is a subspace of X. Then Z is irreducible iff $\overline{Z}$ is irreducible.*

4. *if X, Y are irreducible, then the product topological space $X \times Y$ is irreducible.*

Exercise 1.1.10. *Let $P, Q \in k[x]$ be separable of different degrees. Show that $k[x, 1/P]$ is not isomorphic to $k[x, 1/Q]$.*

Exercise 1.1.11. *Show that any finite subset of $\mathbb{A}^2(k)$ is defined by two equations.*

Exercise 1.1.12. *Points of $\mathbb{P}^n(k)$. Consider the set $\operatorname{Proj} k[T_0, \ldots, T_n]$ of homogeneous prime ideals, different from $(T_0, \ldots, T_n)$, and maximal for this property.*

1. *Let $(a_0, \ldots, a_n) \in \mathbb{P}^n(k)$. Show that $(a_i T_j - a_j T_i)_{i,j} \in \operatorname{Proj} k[T]$ (consider the quotient ring).*

2. *Let $f \in k[T_0, \ldots, T_n]$ be homogeneous. Show that $f(a_0, \ldots, a_n) = 0$ if and only if $f \in (a_i T_j - a_j T_i)_{0 \leq i,j \leq n}$.*

3. *Show the above processus defines an injective map $\mathbb{P}^n(k) \to \operatorname{Proj} k[T]$.*

4. *Show the above map is surjective (localize at some T_i).*

1.2 Abstract algebraic varieties

1.2.1 Ringed topological spaces

Definition 1.2.1. *Let k be field. A ringed topological space X over k consists of a topological space $|X|$ endowed with a sheaf $\mathcal{O}_X$ of k-algebras. We denote it $(|X|, \mathcal{O}_X)$ and $\mathcal{O}_X$ is called the structure sheaf.*

A ringed topological space is called locally ringed topological space if for any $x \in |X|$ the stalk $\mathcal{O}_{X,x}$ is a local k-algebra.

Definition 1.2.2. *A morphism $(|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$ of ringed topological spaces is a pair $(|f|, f^{\#})$ of a continuous function $|f| : |X| \to |Y|$ and of a morphism of sheaves $f^{\#} : \mathcal{O}_Y \to |f|_* \mathcal{O}_X$.*

A morphism of locally ringed spaces is a morphism $f = (|f|, f^{\#})$ of ringed spaces such that for any $x \in X$, if $y = |f|(x)$, the natural induced morphism

$$f_y^{\#} : \mathcal{O}_{Y,y} \to (|f|_* \mathcal{O}_X)_y \to \mathcal{O}_{X,x}$$

is a morphism of local rings, i.e. $f_y^{\#}(\mathfrak{m}_y) \subseteq \mathfrak{m}_x$, where $\mathfrak{m}_y$ and $\mathfrak{m}_x$ are, respectively, the maximal ideals of $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$.

So we have the category of locally ringed topological spaces which we note by **LocRingSp**. Of course the same is true for ringed spaces. The composition is defined as follows. Let $f = (|f|, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g = (|g|, g^{\#}) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ two morphisms of locally ringed spaces. Then we define the composition as

$$(|g| \circ |f|, g_* f^{\#} \circ g^{\#}) : (|X|, \mathcal{O}_X) \to (|Z|, \mathcal{O}_Z).$$

If $X = (|X|, \mathcal{O}_X)$ is a ringed topological space over k, then any open subset U of X inherits a structure of ringed topological space, the sheaf $\mathcal{O}_U := \mathcal{O}_X|_U$ being defined by $\mathcal{O}_U(V) = \mathcal{O}_X(V)$ for all $V \subseteq U$. An open subset U is always endowed with this sheaf of functions unless the contrary is stated. A morphism $f : X \to Y$ such that $f(X) = V$ is open and $f : X \to V$ is an isomorphism is called an *open immersion.*

Example 1.2.3. *Let X be an affine algebraic set and let $\mathcal{O}_X(U)$ be the set of the regular functions on U. Then $(X, \mathcal{O}_X)$ is a ringed topological space over k.*

Proposition 1.2.4. *Let k be algebraically closed. The natural functor, given by $Z \mapsto (Z, \mathcal{O}_Z)$, from the category affine algebraic sets to the category of locally ringed space is an equivalence of categories.*

Proof. Let $f : X \to Y$ be a morphism of algebraic sets. Let V be an open subset of $Y \subseteq \mathbb{A}^m(k)$ and let $s \in \mathcal{O}_Y(V)$. We have to show that $s \circ f \in \mathcal{O}_X(f^{-1}(V))$. Let $\varphi : A(Y) \to A(X)$ be the homomorphism of k-algebras corresponding to f, induced by some $\Phi : k[S_1, \ldots, S_m] \to k[T_1, \ldots, T_n]$. Let $x_0 \in f^{-1}(V)$. By definition, there exists an open neighborhood $f(x_0) \in V_0 \subseteq V$ and $G, H \in$

$k[S_1, \dots, S_m]$ such that $H(y) \neq 0$ for all $y \in V_0$ and that $s(y) = G(y)/H(y)$. For all $x \in f^{-1}(V_0)$, we have $\Phi(H)(x) = H(f(x)) \neq 0$ and

$$s(f(x)) = G(f(x))/H(f(x)) = \Phi(G)(x)/\Phi(H)(x)$$

by the relation (1.1). So $s \circ f$ is defined by a rational function without pole in a neighborhood of x_0 and $s \circ f \in \mathcal{O}_X(f^{-1}(V))$. Therefore f is a morphism of locally ringed topological spaces.

Let $(|f|, f^\sharp) : X \to Y$ be a morphism of locally ringed topological spaces. The composition with the inclusion $Y \subseteq \mathbb{A}^m(k)$ is a morphism of locally ringed topological spaces and it is enough to show that $f : X \to \mathbb{A}^m(k)$ is a defined by a polynomial map. Write $X = Z(I) \subseteq \mathbb{A}^n(k)$ and $S_1, \dots, S_m$ be the coordinate functions on $\mathbb{A}^m(k)$. First of all, since it is a morphism of locally ringed space then $f^\sharp(g) = g \circ f$ for any $g \in k[T_1, \dots, T_n]$.

Then

$$f_j := f^\sharp(S_j) = S_j \circ f \in A(X) = k[T_1, \dots, T_n]/I$$

for all $j = 1, \dots, m$. Let $F_j \in k[T_1, \dots, T_n]$ be a lifting of f_j. For any $x \in X \subseteq \mathbb{A}^n(k) = k^n$ and for any $j \leq m$ we have

$$F_j(x) = f_j(x) = (S_j \circ f)(x) = S_j(f(x))$$

is the j-th coordinate of $f(x) \in \mathbb{A}^m(k)$. So the polynomial map $(F_1, \dots, F_m) : \mathbb{A}^n(k) \to \mathbb{A}^m(k)$ coincides with f on X and the proposition is proved. $\square$

Definition 1.2.5. *An* affine algebraic variety over k *is a ringed topological space over k isomorphic to $(X, \mathcal{O}_X)$ for some affine algebraic set X over k.*

Note that in Hartshorne, an affine variety is required to be irreducible. Here we do not impose this condition.

Lemma 1.2.6. *Let X be an algebraic set. Let $f \in A(X)$. Then $(D(f), \mathcal{O}_X|_{D(f)})$ is an affine variety.*

Proof. Write $X = Z(I) \subseteq \mathbb{A}^n(k)$, $F \in k[T_1, \dots, T_n]$ a lifting of f and let $Z = Z(I, FS - 1) \subseteq \mathbb{A}^{n+1}(k)$. The projection $\mathbb{A}^{n+1}(k) \to \mathbb{A}^n(k)$ to the n-th first coordinates induces a morphisms of algebraic sets $p : Z \to \mathbb{A}^n(k)$. For points $(t, s) \in \mathbb{A}^{n+1}(k)$ with $t \in \mathbb{A}^n(k)$, we have $(t, s) \in Z$ if and only if $t \in X$ and $sf(t) = 1$. This implies easily that $p(Z) = D(f)$. We are going to prove that p is an isomorphism of ringed topological spaces $Z \to D(f)$.

First it is clear that $p : Z \to D(f)$ is bijective. Let us show it is a homeomorphism.It is enough to show that p is open, or that it maps a principal open subset onto a principal open subset. Let $G(\underline{T}, S) \in k[T_1, \dots, T_n, S]$. Using Euclidean division by $FS - 1 \in K[\underline{T}][S]$, we get a relation

$$F^N G(\underline{T}, S) = (FS - 1)H(\underline{T}, S) + R(\underline{T})$$

for some $N \geq 1$. So $(t, s) \in D(G) \cap Z$ if and only if $t \in X$ and $R(t) \neq 0$, thus $p(D(G) \cap Z) = D(R) \cap X$. This implies that p is an open map, hence a homeomorphism.

To show that p is an isomorphism, we need to show that for any open subset V of X contained in $D(f)$, the composition with p induces an isomorphism $\mathcal{O}_X(V) \to \mathcal{O}_Z(p^{-1}(V))$ of k-algebras. Again one can restrict to the case when $V = D(r)$ for some $r \in A(X)$. Let $R \in k[\underline{T}]$ be a lifting of r. Then $p^{-1}(D(r)) = D(R) \cap Z = D(r')$ where $r' \in A(Z)$ is the image of r by $\varphi : A(X) \to A(Z)$. The composition with p induces a commutative diagram of homomorphisms of k-algebras

$$\begin{tikzcd} A(X) \arrow[r, "\varphi"] \arrow[d] & A(Z) \arrow[d] \\ \mathcal{O}_X(D(r)) \arrow[r, "\rho"] & \mathcal{O}_Z(D(r')) \end{tikzcd}$$

where the vertical arrows are the restriction maps (use Proposition 1.2.4). It remains to show that ρ is an isomorphism. The above commutative diagram can be explicitely described as

$$\begin{tikzcd} A(X) \arrow[r, "\varphi"] \arrow[d] & A(X)[S]/(fS-1) = A(Z) \arrow[d] \\ A(X)[W]/(rW-1) \arrow[r, "\rho"] & A(Z)[U]/(rU-1) \end{tikzcd}$$

where φ and the vertical arrows are the canonical ones. Let w, u be the respective images of W and U in the quotients. We have $rw = 1$, so $r\rho(w) = 1$ and $\rho(w) = u$. Then ρ can be interpreted as the natural map

$$A(X)[W]/(rW-1) \to (A(X)[W]/(rW-1))[S]/(fS-1).$$

The condition $D(r) \subseteq D(f)$ in X implies that $r^N = fg$ for some $N \geq 1$ and some $g \in A(X)$ (see Corollary 1.1.27(2)). This implies that f is invertible in $A(X)[W]/(rW-1)$. We can now apply the next lemma to conclude. $\square$

Lemma 1.2.7. *Let A be a ring and let $f \in A$. If f is invertible, then the canonical homomorphism $A \to A[S]/(fS-1)$ is an isomorphism.*

Proof. Indeed, $(fS-1) = (S - f^{-1})$ as ideals and $A \to A[S]/(S - f^{-1})$ is an isomorphism. $\square$

Definition 1.2.8. *An* algebraic variety over k *is a locally ringed topological space $(X, \mathcal{O}_X)$ over k such that there is a finite open covering $X = \cup_i X_i$ where $(X_i, \mathcal{O}_X|_{X_i})$ is an affine algebraic variety for all i.*

Remark 1.2.9. *Equivalently, an algebraic variety over $k$is a locally ringed topological space $(X, \mathcal{O}_X)$ over k such that $|X|$ is quasi compact and there is an open covering $X = \cup_i X_i$ where $(X_i, \mathcal{O}_X|_{X_i})$ is an affine algebraic variety for all i.*

Proposition 1.2.10. *If X is affine, then any open subset $U \subseteq X$ is an algebraic variety.*

Proof. It is enough to show that for any principal open subset $D(f) \subseteq X$, the space $(D(f), \mathcal{O}_X|_{D(f)})$ is affine. But this is just Lemma 1.2.6. $\square$

Remark 1.2.11. *In general, U is not an affine variety.*

Corollary 1.2.12. *Let X be an algebraic variety, let U be an open subset. Then U is an algebraic variety.*

Proposition 1.2.13. *Let k be algebraically closed. For any algebraic variety X and an affine algebraic variety Y we have a natural bijection*

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

In particular $\mathcal{O}_X(X) = \mathrm{Hom}(X, \mathbb{A}^1(k))$.

1.2.2 Constructing algebraic varieties by glueing

This is the method of charts and altas.

Let $X_1, \dots, X_n$ be algebraic varieties over k. Suppose we are given for any pair $i, j \leq n$, an open subset X_{ij} of X_i and an isomorphism (of algebraic varieties) $f_{ij} : X_{ij} \to X_{ji}$ such that the diagram

$$\begin{tikzcd} X_{ij} \cap X_{iq} \arrow[rr, "f_{ij}|_{X_{ij}\cap X_{iq}}"] \arrow[dr, "f_{iq}"'] & & X_{ji} \cap X_{jq} \\ & X_{qi} \cap X_{qj} \arrow[ur, "f_{qj}"'] & \end{tikzcd}$$

is commutative for all $i, j, q \leq n$. Then there exist an algebraic variety X and open immersion $f_i : X_i \to X$ such that $X = \cup_i f_i(X_i)$ and

$$f_i|_{X_{ij}} = f_j|_{X_{ji}} \circ f_{ij}, \quad i, j \leq n.$$

Moreover, if $X', f_1', \dots, f_n'$ are another tuple satisfying the same conditions, then there exists a unique isomorphism $f : X \to X'$ such that $f_i = f \circ f_i'$ for all $i \leq n$.

Example 1.2.14. *Let $X_1 = \mathbb{A}^1(k)$ with coordinate function t and $X_2 = \mathbb{A}^1(k)$ with coordinate function s, $X_{12} = D(t)$ and $X_{21} = D(s)$. Then there are two different isomorphisms*

$$f_{12} : X_{12} \to X_{21}, \quad s \mapsto t;$$

$$g_{12} : X_{12} \to X_{21}, \quad s \mapsto 1/t.$$

The glueing with the first isomorphism gives rise to the "line with a doubled point" and the second gives $\mathbb{P}^1(k)$.

1.2.3 Subvarieties

Definition 1.2.15. *Let $(X, \mathcal{O}_X)$ be an algebraic variety. An* open subvariety *if an open subset U of X endowed with the sheaf of regular functions $\mathcal{O}_X|_U$.*

An open subset U of X is called affine open subset *if the corresponding open subvariety is affine.*

An affine open covering *of X is an open covering of X by affine open subsets.*

Let Z be a closed subset of X. Define a presheaf $\mathcal{O}_Z$ on Z as follows:

Let $V \subseteq Z$ be an open subset of Z. A function $f : V \to k$ belongs to $\mathcal{O}_Z(V)$ if for any $z \in V$, there exists an open neighborhood U of z in X and $g \in \mathcal{O}_X(U)$ such that $U \cap Z \subseteq V$ and $f(z') = g(z')$ for all $z' \in U \cap Z$.

Lemma 1.2.16. *Let Z be a closed subset of X. Then $\mathcal{O}_Z$ is a sheaf and $(Z, \mathcal{O}_Z)$ is an algebraic variety. If X is affine, then so is Z.*

Proof. It is clearly a sheaf. Now suppose X is affine $Z(I) \subseteq \mathbb{A}^n(k)$ and Z is given by $Z = Z(J) \cap X \subseteq \mathbb{A}^n(k)$. Then it is easy to check that $\mathcal{O}_Z$ is the sheaf of regular functions on the algebraic set $Z(I, J) \subseteq \mathbb{A}^n(k)$. □

Definition 1.2.17. *A* closed subvariety of X *is a closed subset Z of X endowed with the sheaf $\mathcal{O}_Z$ defined as above.*

We now define projective varieties as closed subvarieties of $\mathbb{P}^n(k)$. Let X be a projective algebraic set in $\mathbb{P}^n(k)$ defined by a radicial homogeneous ideal $I \subseteq k[T_0, \ldots, T_n]$. Let us define a sheaf $\mathcal{O}_X$ on X. Let U be an open subset of X. Let $f : U \to k$ be a function on U. We will say that f is a regular function if for any $x \in U$, there exist an open neighborhood $x \in V \subseteq U$ and homogeneous polynomials $F(T_0, \ldots, T_n), G(T_0, \ldots, T_n)$ of the same degree such that $G(t) \neq 0$ and $f(t) = F(t)/G(t)$ for all $t \in V$. This defines a sheaf of k-valued regular functions $\mathcal{O}_X$ on X and $(X, \mathcal{O}_X)$ is a locally ringed space.

Let $B = k[T_0, \ldots, T_n]/I$. Let $P \in k[T_0, \ldots, T_n]$ be homogeneous of degree $d \geq 0$ and let $p \in B$ be the image of P in B (such a p is called a *homogeneous element of B of degree p*). We denote by

$$k[T_0, \ldots, T_n]_{(P)} = \{\frac{Q}{P^r} \in k(T_0, \ldots, T_n) \ \mid \ r \geq 0,\ Q \in k_{rd}[T_0, \ldots, T_n]\},$$

(this is a k-algebra of finite type, generated by M/P for monomials M of degree d),

$$I_{(P)} = \{\frac{Q}{P^r} \in k[T_0, \ldots, T_n]_{(P)} \ \mid \ Q \in I \cap k_{rd}[T_0, \ldots, T_n]\}.$$

This is an ideal of $k[T_0, \ldots, T_n]_{(P)}$, and

$$B_{(p)} = k[T_0, \ldots, T_n]_{(P)}/I_{(P)}.$$

Remark 1.2.18. *The above $B_{(p)}$ is independent of the lifting $P \in k_d[T_0, \ldots, T_n]$ of p as well as the open subset*

$$D_+(p) := D_+(P) \cap X,$$

called a principal open subset of X*.*

Similarly to Proposition 1.1.48, we have

Proposition 1.2.19. *Let X be a closed subset of $\mathbb{P}^n(k)$ defined by a radicial homogeneous ideal I.*

(1) *Let U be an open subset of X. Then any $f \in \mathcal{O}_Z(U)$ is continuous (when $k = \mathbb{A}^1(k)$ is endowed with the Zariski topology).*

(2) *If $P \in k[T_0, \ldots, T_n]$ is homogeneous of degree $d \geq 0$, p is the image of P in B and $U = X \cap D_+(P)$, then*

$$\mathcal{O}_X(U) = B_{(p)}.$$

(3) *If $p \in B$ is homogeneous of degree ≥ 1, then $D_+(p)$ is an affine open subvariety with $A(D_+(p)) = B_{(p)}$.*

Proof. 1. This follows by point 3) and by the affine case, since X can be covered by affine open subvarieties.

2. We have a natural injective map

$$B_{(p)} \to \mathcal{O}_X(U).$$

The statement about the kernel is clear. The above injection allows us to view f as a regular function on Z.

We have to prove that it is surjective. Let $g \in \mathcal{O}_X(U)$ and let us take $D_+(p) = \cup_{i=1}^n D(f_i)$ such that $g_{|D_+(f_i)} = G_i/F_i$, with F_i and G_i homogenous of the same degree. The last condition implies that $gF_i^2 = G_iF_i$ in $\mathcal{O}_X(X)$. Now $D_+(p) = \cup_{i=1}^n D_+(f_i^2)$ and therefore there exists r such that $p^r = \sum_i^n a_i f_i^2$. We can suppose that $\deg a_i f_i^2 = r \deg p$ therefore $gp^r = \sum_i^n a_i g_i f_i$ which means that $g \in B_{(p)}$.

3. Let $d = \deg P$. Let us consider the morphism $\mathbb{P}_k^n \to \mathbb{P}_k^N$ given by

$$\mu : [x_0, \ldots, x_n] \mapsto [M_0, \ldots, M_N]$$

where the M_i are the monomials of degree $\deg P$ ($N = \binom{n+d}{n} - 1$) and $\mu^{\#}$ is defined by composition (verify that it is well defined). Then μ is an isomorphism between $\mathbb{P}_k^n$ and $Z_+(I)$ where I is the kernel of the map $k[T_0, \ldots, T_M] \to k[S_0, \ldots, S_n]$ given by $T_i \mapsto M_i$ (Exercise!).

Then $\mu(D_+(P)) = D_+(H) \cap \mu(\mathbb{P}_k^n))$ where H is an homogenous polynomial of degree 1 given by the coefficients of P). It's enough to prove that $D_+(H)$ is affine since $\mu(\mathbb{P}_k^n))$ is closed. Let us suppose that the polynomial H 'contains' T_0 then we can consider the isomorphism

$$\mathbb{P}_k^n \to \mathbb{P}_k^n$$

given by $[t_0, \ldots, t_n] \times [H, t_1, \ldots, t_n]$ whixhe sends $D_+(H)$ on $D_+(T_0)$. So it is enough to prove that $D_+(T_i)$ is an isomorphic to $\mathbb{A}_k^n$.

We proved that the application $|\phi_i| : D_+(T_i) \to \mathbb{A}^n(k)$ given by $[x_0 : x_1 : \cdots : x_n] \mapsto (x_0/x_i, \dots, \widehat{x_i/x_i}, x_n/x_i)$ is an homeomorphism. Moreover for any open subset U of $\mathbb{A}^n(k)$, the morphism $\phi_i^{\#}(U) : \mathcal{O}_{\mathbb{A}^n_k}(U) \to \mathcal{O}_{D_+(T_i)}(\phi_i^{-1}(U))$ given by $f \mapsto f \circ |\phi_i|$ is well defined, as it is easy to verify using the definition of regular functions. The inverse of $(|\phi_i|, \phi_i^{\#})$ is $(|\psi_i|, \psi_i^{\#})$ where $|\psi_i| : \mathbb{A}^n(k) \to D_+(T_i)$ is given by $(x_1, \dots, x_n) \mapsto (x_1, \dots, 1, \dots, x_n)$ and, for any V open subset of $D_+(T_i)$, $\psi_i^{\#} : \mathcal{O}_{D_+(T_i)}(V) \to \mathcal{O}_{\mathbb{A}^n(k)}(\psi_i^{-1}(V))$ is given by $g \mapsto g \circ |\psi_i|$. It is easy to check that it is well defined, homogeneizing locally. This also proves that if X is a closed subset of $\mathbb{P}^n_k$ then $X \cap D_+(T_i)$ is isomorphic to a an affine closed subset of X and $\mathcal{O}_X(X \cap D_+(T_i)) = B_{(t_i)}$.

□

Notation 1.2.20. *The projective space will also be denoted by $\mathbb{P}^n_k$ when we want to insist in the fact that this is a ringed topological space. The same for $\mathbb{A}^n_k$.*

For any arbitrary radicial homogeneous ideal I, we find

Corollary 1.2.21. *$X = Z_+(I)$ endowed with the sheaf $\mathcal{O}_X$ is a closed*

Definition 1.2.22. *A* projective variety *is a ringed topological space isomorphic to a projective algebraic set X endowed with the sheaf of regular functions $\mathcal{O}_X$ as above.*

Remark 1.2.23. *Projective varieties X share lot of properties with compact complex manifolds. For example, the algebra of regular functions on X are locally constant.*

Definition 1.2.24. *A* quasi-projective variety *is a ringed topological space isomorphic to an open subvariety of a projective variety.*

Proposition 1.2.25. *Affine algebraic varieties are quasi-projective.*

Remark 1.2.26. *Upon to 1950's (when abstract algebraic varieties are defined), algebraic geometers were only concerned with quasi-projective varieties. Of course, there are non-quasiprojective varieties, e.g. the line with a doubled point (Example 1.2.14) and also more sophisticate ones.*

1.3 Dimension of varieties

We want to define the first numerical invariant of an algebraic variety X, its dimension. Let X be any topological space. The *Krull dimension* of X is to the supremum of integers $n \leq +\infty$ such that there exists a strictly increasing chain

$$Z_0 \subset Z_1 \subset ... \subset Z_n$$

of irreducible closed subsets (Definition 1.1.31) of X. By convention $\dim \emptyset = -\infty$.

Example 1.3.1. *If X is a discrete (i.e. all subsets are open) and non-empty topological space, then* $\dim X = 0$ *because the only irreducible subsets are singletons.*

Example 1.3.2. *If X is a Hausdorff topological space, then* $\dim X \leq 0$. *Indeed, if Z is an irreducible subset of X, then Z is a singleton (if $z_1 \neq z_2$ are points of Z, there exists open subsets $V_1 \ni z_1$, $V_2 \ni z_2$ in Z with $V_1 \cap V_2 = \emptyset$, hence V_1 is a non-empty open subset, but is not dense in Z). So* $\dim X = 0$ *if X is non-empty.*

Recall that an irreducible component of a topological space X is an irreducible subset Z of X, maximal for the inclusion. As $\overline{Z}$ is also irreducible (Exercise 1.1.9(3)), we have necessarily $Z = \overline{Z}$, so Z is closed. Using Zorn's lemma, we see easily that any irreducible subset of X is contained in an irreducible component.

If $X \neq \emptyset$, as $\{x\}$ is irreducible for any $x \in X$, there is at least one chain of length ≥ 0, so then $\dim X \geq 0$.

Proposition 1.3.3. *Let X be a topological space.*

(1) *If Y is a subset of X with the induced topology, then* $\dim Y \leq \dim X$.

(2) *If X is irreducible of finite dimension and $Y \subseteq X$, then $Y = X$ if and only if Y is closed of dimension* $\dim X$.

(3) *Let $\{X_i\}_i$ be the irreducible components of X. Then* $\dim X = \sup_i \dim X_i$.

(4) *If $\{U_i\}_i$ is a covering of X by open subsets, then* $\dim X = \sup_i \dim U_i$.

Proof. (1) If $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ is a chain of irreducible closed subsets of Y, then their closures $\overline{Z}_i$ in X is an increasing sequence of irreducible closed subsets of X. As $\overline{Z}_i \cap Y = Z_i$, the sequence is strictly increasing. Thus $\dim Y \leq \dim X$.

(2) - (3) are straightforward from the definition.

(4) If $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ is a chain of closed irreducible subsets of X, Z_0 meets one of the X_i's. Then the intersection of the Z_j's with this X_i is a chain of irreducible closed subsets in X_i. This implies the statement □

The *dimension* of an algebraic variety is the Krull dimension of the underlying topological space.

Example 1.3.4. *As expected,* $\dim \mathbb{A}^1_k = \dim \mathbb{P}^1_k = 1$. *Actually, using Proposition 1.1.33, we see that irreducible closed subsets of $\mathbb{A}^1_k$ are $\mathbb{A}^1_k$ and singletons $\{x\}$, $x \in \mathbb{A}^1_k$. So the maximal length of chains of irreducible closed subsets in $\mathbb{A}^1_k$ is* 1. *Hence* $\dim \mathbb{A}^1_k = 1$. *As $\mathbb{P}^1_k$ is union of two open subsets $D_+(T_0)$, $D_+(T_1)$ of dimension* 1*, it is also of dimension* 1 *by Prop. 1.3.3. By an exercise we did before (irreducible closed subsets of $\mathbb{A}^2(k)$ are points, principal closed subsets defined by irreducible polynomials and $\mathbb{A}^2(k)$).*

Definition 1.3.5. *Let R be a (commutative unitary) ring. The* Krull dimension *of R is the supremum of the lengths of chains of prime ideals of R:*

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n.$$

Example 1.3.6. *If $R = \mathbb{C}[T_1, T_2, \dots,]$ (infinitely many variables), then for all $n \geq 1$, the ideal $\mathfrak{p}_n := (T_1, \dots, T_n)$ is prime because $R/\mathfrak{p}_n$ is a domain. As $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots$ is an infinite chain of prime ideals of R, we have $\dim R = +\infty$.*

Remark 1.3.7. *1. Let I be an ideal of R and let $\pi : R \to R/I$ be the canonical surjection. Then there is a canonical bijection between the set of the prime ideals $\mathfrak{p}$ of R containing I and that of R/I, which is given by $\mathfrak{p} \mapsto \overline{\mathfrak{p}} = \mathfrak{p}/I$. The inverse correspondence is given by $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$.*

2. As a consequence, and because all prime ideals contain the nilradical $\sqrt{0}$ of R, we have $\dim R = \dim(R/\sqrt{0})$.

3. If X is an affine algebraic set. Then by Proposition 1.1.33

$$\dim X = \dim A(X).$$

If $A \to B$ is finite, and $b \in B$, how to find a monic polynomial in $A[T]$ vanishing at b ? Fix a system of generators $e_1, \dots, e_n$ of B over A. Then we can write a matrix of the A-linear map $[b] : B \to B$, $y \mapsto by$ in the given system of generators. The characteristic polynomial of this matrix then vanishes at b (in B) by Cayley-Hamilton theorem.

Proposition 1.3.8. *Let $\varphi : A \to B$ be a finite ring homomorphism.*

(1) *We have $\dim B \leq \dim A$.*

(2) *The preimage of a prime ideal is maximal if and only if the ideal is maximal.*

(3) *If φ is moreover injective, then for any prime ideal $\mathfrak{p}$ of A, there exists a prime ideal $\mathfrak{q}$ of B such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.*

(4) *Under the hypothesis of (2), we have $\dim B = \dim A$.*

Proof. (1) Let $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1$ be two prime ideals of B. Let us show that $\varphi^{-1}(\mathfrak{q}_0) \subsetneq \varphi^{-1}(\mathfrak{q}_1)$. This will imply the inequality on the dimensions. The homomorphism φ induces by factorisation an injective and finite homomorphism $A/\varphi^{-1}(\mathfrak{q}_0) \to B/\mathfrak{q}_0$. Replacing A by $A/\varphi^{-1}(\mathfrak{q}_0)$ and B by $B/\mathfrak{q}_0$, we may assume that B is an integral domain and that $\mathfrak{q}_0 = 0$. We must then show that $\varphi^{-1}(\mathfrak{q}_1) \neq 0$. Let $b \in \mathfrak{q}_1$ be non-zero. Let $b^n + \varphi(a_{n-1})b^{n-1} + \cdots + \varphi(a_0) = 0$ be an integral equation for b over A, of minimal degree n. Then $\varphi(a_0) = b(-\varphi(a_1) - \varphi(a_2)b - \cdots - b^{n-1}) \in \mathfrak{q}_1 \setminus \{0\}$ and $a_0 \in \varphi^{-1}(\mathfrak{q}_1) \setminus \{0\}$.

(2)Let $\mathfrak{p}$ a prime ideal of B. Then we have an injective inetgral morphism of integral domains $A/\varphi^{-1}(\mathfrak{p}) \to B/\mathfrak{p}$. Therefor $\mathfrak{p}$ is maximal if and only if $\varphi^{-1}(\mathfrak{p})$ is maximal.

(3) Now suppose that φ is injective.

$A_\mathfrak{p} \to B \otimes_A A_\mathfrak{p}$ is integral. and since ϕ is injective then $T = (\phi(A \setminus \mathfrak{p}))$ and $B \otimes_A A_\mathfrak{p} = T^{-1}B \neq 0$.

(4) Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1$ be prime ideals of A. Let $\mathfrak{q}_0 \subset B$ be a prime ideal such that $\varphi^{-1}(\mathfrak{q}_0) = \mathfrak{p}_0$. By considering the injective integral homomorphism $A/\mathfrak{p}_0 \hookrightarrow$

$B/\mathfrak{q}_0$, we obtain a prime ideal $\mathfrak{q}_1$ of B such that $\varphi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$ and $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1$. By repeatedly applying this result to any finite chain of prime ideals of A, we get $\dim A \leq \dim B$. □

Remark 1.3.9. *Any quotient homomorphism* $A := k[T_1, \ldots, T_n] \to B := A/I$ *is finite, so* $\dim B = \dim Z(I) < \dim \mathbb{A}^n(k) = \dim A$ *if* $I \neq 0$. *Therefore the equality of dimensions does not holds in general if we do not assume injectivity of* φ.

Corollary 1.3.10. *Let* K *be any field. Let* $n \geq 0$ *and let* $F \in K[T_1, \ldots, T_n] \backslash K$.

(1) $\dim K[T_1, \ldots, T_n] = n$.

(2) $\dim K[T_1, \ldots, T_n]/(F) = \dim K[T_1, \ldots, T_{n-1}]$.

Proof. (2) In the course of the proof of Noether's normalization lemma (**??**), we showed that up to a K-automorphism of $K[T_1, \ldots, T_{n+1}]$, we can suppose that F is monic and of positive degree in T_{n+1}. Then the K-homomorphism

$$K[T_1, \ldots, T_n] \to K[T_1, \ldots, T_{n+1}]/(F), \quad T_i \mapsto \overline{T}_i$$

is finite and injective. We then apply the above proposition.

(1) Note that

$$\{0\} \subset (T_1) \subset (T_1, T_2) \subset \cdots \subset (T_1, \ldots, T_n)$$

is a chain of prime ideals of length n. So $\dim K[T_1, \ldots, T_n] \geq n$.

We proceed by induction on n to prove the converse inequality. The property is true for $n \leq 1$ (Example 1.3.4). Suppose it is true for $n - 1$. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots. \subset \mathfrak{p}_m$ be a chain of prime ideals of $K[T_1, ..., T_n]$. We want to show that $m \leq n$. Let $P \in \mathfrak{p}_1$ be non-zero. The images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ is a chain of prime ideals in $K[T_1, \ldots, T_n]/(P)$, so $m - 1 \leq \dim K[T_1, \ldots, T_n]/(P) = \dim K[T_1, \ldots, T_{n-1}] = n - 1$ By (2) we just proved and the induction hypothesis. Therefore $m \leq n$ and the corollary is proved. □

Corollary 1.3.11. *Algebraic varieties have finite dimensions.*

Proof. Any algebraic variety is a finite union of affine open subvariety, and any affine variety has finite dimension by the above corollary and Proposition 1.3.3. □

Corollary 1.3.12. *Let* $P \in k[T_1, \ldots, T_n] \backslash k$. *Then* $\dim \mathbb{A}^n(k) = n$ *and* $\dim Z(P) = n - 1$.

1.3.1 Function fields

Definition 1.3.13. *Let* k *be a field. A* field of finite type over k *or* a function field over k *is a field extension* K/k *such that there exists an sub-*k*-algebra* A *of* K*, of finite type over* k *with* $K = \mathrm{Frac}(A)$.

If $A = k[t_1, \ldots, t_n]$ *(this means that* A *is a quotient of* $k[T_1, \ldots, T_n]$ *and we denote by* t_i *the class of* T_i *in the quotient), then* $k(t_1, \ldots, t_n)$ *is by definition* $\mathrm{Frac}(A)$.

Applying Noether's normalization lemma to A, we see that K is then a finite extension of a field of rational functions $k(T_1, \ldots, T_d)$ (the T_i's are variables).

Definition 1.3.14. *Let K/k be a field extension. A familly $\{t_1, \ldots, t_n\}$ of elements of K are said* algebraically independent *if for any non-zero $F(T_1, \ldots, T_n) \in k[T_1, \ldots, T_n]$, we have $F(t_1, \ldots, t_n) \neq 0$. This is equivalent to saying that the k-homomorphism*

$$k[T_1, \ldots, T_n] \to K, \quad T_i \mapsto t_i$$

is injective. Such a k-homomorphism induces a homomorphism of k-extensions $k(T_1, \ldots, T_n) \to K$.

If moreover K is algebraic over $k(t_1, \ldots, t_n)$, then $\{t_1, \ldots, t_n\}$ is called a transcendence base *of K over k.*

Lemma 1.3.15. *A field extension K/k is of finite type if and only if there are algebraically independent elements $t_1, \ldots, t_n \in K$ such that K is a finite extension of $k(t_1, \ldots, t_n) := \mathrm{Frac}(k[t_1, \ldots, t_n]) \simeq k(T_1, \ldots, T_n)$.*

Proposition 1.3.16. *Let K/k be an extension of finite type. If K is algebraic over $k(t_1, ..., t_n)$ and over $k(s_1, ..., s_m)$, then $n = m$.*

Proof. Suppose $m > n$. Write a polynomial dependence of s_1 over $k(t_1, \ldots, t_n)$. At least one variable, say t_1, appears in the relation. Then t_1 is algebraic over $s_1, t_2, \ldots, t_n$. Hence K is algebraic over $k(s_1, t_2, \ldots, t_n)$. In particular we have an algebraic relation of s_2 over this field. In this relation, at least one of the t_i's, $i \geq 2$, appears because s_1, s_2 are algebraically independent. We can suppose that t_2 appears. Then t_2 (hence K) is algebraic over $k(s_1, s_2, t_3, \ldots, t_n)$. Repeating the same argument we find that K (hence s_{n+1}) is algebraic over $k(s_1, s_2, \ldots, s_n)$, contradiction. □

Definition 1.3.17. *The cardinality n of a transcendence base in K is called the* transcendence degree *of K over k. We denote it by $\mathrm{degtr}_k K$. We say that K is a* function field in n variables*. A finite extension has $\mathrm{degtr}_k K = 0$.*

Definition 1.3.18. *An algebraic variety X is* integral *if its underlying topological space is irreducible.*

Lemma 1.3.19. *X is an integral variety if and only if for any non-empty open subset U of X, the ring $\mathcal{O}_X(U)$ is an integral domain.*

Proof. Let us suppose that X is integral. Then U is integral for any open subset U of X. So we can suppose $U = X$ and we prove that $\mathcal{O}_X(X)$ is integral. Let $f, g \in \mathcal{O}_X(X)$ such that $fg = 0$. Let W be an open subset of X and let $Z(f) := \{x \in X | f(x) = 0\}$ and $Z(g) := \{x \in W | g(x) = 0\}$, whic are closed since f and g are continuous. Then $X = Z(f) \cup Z(g)$. Since X is irreducible then $f = 0$ or $g = 0$.

Conversely, let us suppose that there exists U, V open subsets of X such that $U \cap V = \emptyset$. Since $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$, then it is not integral. □

Lemma 1.3.20. *Let X be an integral algebraic variety. Let $V \subseteq U$ open subsets of X with $V \neq \emptyset$. Then the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is injective. If moreover U, V are affine, then* $\mathrm{Frac}(\mathcal{O}_X(U)) \to \mathrm{Frac}(\mathcal{O}_X(V))$ *is an isomorphism.*

Proof. Let $f \in \mathcal{O}_X(U)$ such that $f_{|V} = 0$. Then $V \subseteq Z(f) \subseteq U$. Since X is irreducible then V is dense in X so $Z(f) = 0$ and then $f = 0$. If U and V are affine we take a principal oben subset W of U contained in V. So

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V) \to \mathcal{O}_X(W)$$

are two injections. So it is enough to prove that the morphism $\mathrm{Frac}(\mathcal{O}_X(U)) \to \mathrm{Frac}(\mathcal{O}_X(W))$ is an isomorphism. So we reduce to the case where U is affine and V is a principal open subset $D(f)$ with $f \in \mathcal{O}_X(U)$. In this case we have an inclusion

$$\mathcal{O}_X(U) \to \mathcal{O}_X(U)_f$$

and it is clear that we have an isomorphism passing to the fraction field.

$\square$

Definition 1.3.21. *Let X be an integral algebraic variety. The* field of rational functions of X *is the inductive limit*

$$k(X) = \varinjlim_{U} \mathcal{O}_X(U)$$

where U runs through the non-empty open subsets of X. So the elements of $k(X)$ are regular functions on some non-empty open subsets of X.

Proposition 1.3.22. *Let X be an integral algebraic variety. Let U_0 be a non-empty affine open subset of X. Then $k(X)$ is canonically isomorphic to the field of fractions* $\mathrm{Frac}(\mathcal{O}_X(U_0))$. *If U is a non-empty open subset, then $k(X) \simeq k(U)$.*

Proof. Let U be any non-empty open subset of X. Let $V = D(f)$ be a non-empty principal open subset of U_0 contained in $U \cap U_0$. Then we have canonical maps

$$\mathcal{O}_X(U) \to \mathcal{O}_X(U \cap U_0) \to \mathcal{O}_X(V) = \mathcal{O}_X(U_0)_f \subseteq \mathrm{Frac}(\mathcal{O}_X(U_0)).$$

This gives a canonical injective map $k(X) \to \mathrm{Frac}(\mathcal{O}_X(U_0))$. This map is surjective because any element $f/g \in \mathrm{Frac}(\mathcal{O}_X(U_0))$ is regular over the non-empty principal open subset $D(g)$ of U_0.

For a given non-empty open subset U, there exists a non-empty affine open subset $U_0 \subseteq U$. So $k(X) \simeq k(U_0) = \mathrm{Frac}(\mathcal{O}_X(U_0)) \simeq k(U)$. $\square$

Next we will relate the dimension of an integral variety X to a numerical invariant of its field of rational functions, the transcendental degree. We need some terminology in field extensions.

Proposition 1.3.23. *Let X be an integral algebraic variety. Then*

$$\dim X = \operatorname{degtr}_k k(X).$$

If U is a non-empty open subset of X, then $\dim U = \dim X$*.*

Proof. First suppose that X is affine. By Noether's normalization lemma, there is an injective finite homomorphism $k[T_1, ..., T_n] \to A(X)$. So on the one hand, $\dim X = n$ by Proposition 1.3.8, and on the other hand, $k(X) = \operatorname{Frac}(A)$ is finite over $k(T_1, ..., T_n)$, thus $\operatorname{degtr}_k k(X) = n = \dim X$. In general, let $X = \cup_i U_i$ be an affine covering of X. Then

$$\dim X = \max_i \dim U_i = \max_i \operatorname{degtr}_k k(U_i) = \max_i \operatorname{degtr}_k k(X) = \operatorname{degtr}_k k(X).$$

Finally, $k(U) = k(X)$, so $\dim U = \dim X$. □

Proposition 1.3.24. *If X is an integral algebraic variety and $f \in \mathcal{O}_X(X)$ is non-zero and non-invertible, then* $\dim Z(f) = \dim X - 1$*.*

Proof. As X is irreducible and $Z(f) \neq X$ (because $f \neq 0$), we have $\dim Z(f) \leq \dim X - 1$. There exists an affine open subset U of X such that $f|_U$ is non-invertible. It is enough to show that $\dim Z(f|_U) \geq \dim U - 1$ because $\dim Z(f) \geq \dim Z(f|_U)$ and $\dim U = \dim X$. So we can suppose X is affine and integral.

Let $k[T_1, \ldots, T_n] \to A = A(X)$ be an injective finite homomorphism. Then $\dim X = n$. We have a finite injective k-homomorphism

$$k[T_1, \ldots, T_n]/(fA \cap k[T_1, \ldots, T_n]) \to A/fA.$$

It is enough to show that the left-hand side term has dimension $n - 1$ because $\dim Z(f) = \dim A(Z(f)) = \dim(A/\sqrt{fA}) = \dim(A/fA)$.

Let F be the norm of f in the extension $\operatorname{Frac}(A)/k(T_1, \ldots, T_n)$. This is some power of the constant term of the monic minimal polynomial of f over $k(T_1, \ldots, T_n)$. As f is integral over $k[T_1, \ldots, T_n]$ and the latter is a UFD, it is known that the minimal polynomial of f has coefficients in $k[T_1, \ldots, T_n]$. In particular $F \in k[T_1, \ldots, T_n]$. We then have

$$fA \cap k[T_1, \ldots, T_n] \subseteq \sqrt{(F)} \subseteq \sqrt{fA \cap k[T_1, \ldots, T_n]}.$$

The first inclusion comes frome the fact that if $x = fa \in fA \cap k[T_1, \ldots, T_n]$ then $N(x) = x^r = N(f)N(a) = FN(a)$ where r is the degree of the extension. So it is enough to show that $\dim k[T_1, \ldots, T_n]/(F) = n - 1$. But this is just Corollary 1.3.12. □

Exercise 1.3.1. *Under the hypothesis of Proposition 1.3.24, show that each irreducible component of $Z(f)$ has dimension $n - 1$.*

Corollary 1.3.25. *Let $n \geq 1$. Then $\dim \mathbb{P}^n(k) = n$. Any hypersurface $Z_+(F)$ of $\mathbb{P}^n(k)$ with F non-constant homogenous polynomial has dimension $n - 1$.*

Proof. Use Proposition 1.3.3 and Corollary 1.3.10. □

1.4 Product

Let X, Y be two algebraic varieties over k (algebraically closed field). We would like to endow the set $X \times Y$ with the structure of an algebraic variety. This means we need to define a topology and a sheaf of regular functions. Of course, the structure on $\mathbb{A}^n(k) \times \mathbb{A}^m(k)$ should be that of $\mathbb{A}^{n+m}(k)$, so the topology must not be the product topology (Exercise 1.1.4 (8)) and the problem is more subtle than we could imagine at first.

If $X = Z(I) \subseteq \mathbb{A}^n(k)$ and $Y = Z(J) \subseteq \mathbb{A}^m(k)$ are algebraic sets, then the natural structure on $X \times Y$ is $Z(I, J) \subseteq \mathbb{A}^{n+m}(k)$. If X, Y are abstract affine varieties, we have some isomorphisms $f : X \simeq Z(I)$, $g : Y \simeq Z(J)$ and we transfer the algebraic variety structure on $Z(I) \times Z(J)$ to $X \times Y$ via the bijection $f \times g : X \times Y \to Z(I) \times Z(J)$. We can see that this structure is independent of the choice of $Z(I)$ and $Z(J)$. But for the moment this does not matter.

Definition 1.4.1. *Let X, Y be algebraic varieties over k. The* product of X and Y over k *is an algebraic variety Z with morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$ such that for any algebraic variety W over k, $f : W \to X$ and $g : X \to Y$ morphisms there exists a unique morphism $\phi : W \to Z$ such that $p_X \circ \phi = f$ and $p_Y \circ \phi = g$.*

When such a structure exists, it is called the product variety*. We should keep in mind that the topology of the product variety will not be the product topology.*

Remark 1.4.2. *If the product exists it is unique up to a unique isomorphism and it is denoted by $X \times Y$.*

Lemma 1.4.3. *For any algebraic variety X over an algebraic closed field there is a natural bijection between $|X|$ and $Hom(\mathrm{Spm}(k), X)$.*

Lemma 1.4.4. *If the product of two algebraic varoeties X and Y exist the set $|X \times Y|$ is $|X| \times |Y|$ and p_X and p_Y are, as set functions, the projections.*

Proof. This follows by previous Lemma. □

Lemma 1.4.5. *Let X, Y be affine varieties. Then $X \times Y$ with the above structure of affine algebraic variety is the product variety.*

Proof. This follows by Proposition 1.2.13.

□

Lemma 1.4.6. *Let X, Y be algebraic varieties. Suppose that the product variety $X \times Y$ exists. Then for any open subsets $U \subseteq X$, $V \subseteq Y$, $p_X^{-1}(U) \cap p_Y^{-1}(V)$ is open in $X \times Y$ and the structure of open subvariety on $p_X^{-1}(U) \cap p_Y^{-1}(V)$ induced by that of $X \times Y$ is the product variety of U and V.*

Proof. Let p, q be the projection maps to X and Y. Then $p^{-1}(U) \cap q^{-1}(V)$ is open in $X \times Y$. The projection maps from $U \times V$ to U and V are morphisms and the universal property is straighforward. □

Proposition 1.4.7. *The product $X \times Y$ exists as algebraic variety. When X, Y are both affine, then $X \times Y$ is affine.*

Proof. First suppose that Y is affine. Cover X affine open subsets $\{X_i\}_i$. Then as a set $X \times Y$ is the union of the $X_i \times Y$'s. The intersections $(X_i \times Y) \cap (X_j \times Y) = (X_i \cap X_j) \times Y$ are open in both $X_i \times Y$ and $X_j \times Y$, and has a unique structure of product variety. The varieties $X_i \times Y$'s then glue together to give a structure of algebraic variety on $X \times Y$ (use the method of §1.2.2, or directly: a subset will be open if its intersection with each $X_i \times Y$ is open and a function on W will be regular if its restriction to each $W \cap (X_i \times Y)$ is regular).

In the general case, cover Y by affine open subsets $\{Y_j\}_j$, and glue the product varieties $X \times Y_j$ as above. □

Example 1.4.8. *The product variety $\mathbb{P}^n(k) \times \mathbb{A}^m(k)$. We can endow*

$$k[T_0, \ldots, T_n, S_1, \ldots, S_m] = A[T_0, \ldots, T_n], \quad \textit{where } A = k[S_1, \ldots, S_m],$$

with a graduation: homogeneous elements of degree d are A-linear combinations of monimials in the T_i's. Let I be a homogeneous ideal of $A[T_0, \ldots, T_n]$. Define

$$Z_+(I) = \{(x, y) \in \mathbb{P}^n(k) \times \mathbb{A}^m(k) \mid P(x, y) = 0, \forall P \in I\}.$$

For any $y \in \mathbb{A}^m(k)$, $P(T_0, \ldots, T_n, y)$ is homogeneous with coefficients in k, so the condition $P(x, y) = 0$ makes sense in $\mathbb{P}^n(k) \times \mathbb{A}^m(k)$. It is easy to see that there is a unique topology for which the closed subsets are the $Z_+(I)$'s. So on $\mathbb{P}^n(k) \times \mathbb{A}^m(k)$ we have two topologies. They are actually the same. This is because for each $i \leq n$, the induced topologies on $D_+(T_i) \times \mathbb{A}^m(k)$ are the same (exercise).

Definition 1.4.9. *A morphism $f : X \to Y$ is a* closed immersion *if $f(X)$ is closed in Y and $X \to f(X)$ (the closed subvariety) is an isomorphism.*

Lemma 1.4.10. *Let $f : X \to Y$ be a morphism of affine varieties. Then f is a closed immersion if and only if the corresponding homomorphism $A(Y) \to A(X)$ is surjective.*

Proof. Suppose first that f is a closed immersion. If $Y = Z(I) \subseteq \mathbb{A}^n(k)$ and $f(X) = Z(J) \cap Z(I)$, then

$$A(Y) = k[T_1, \ldots, T_n]/I, \quad A(f(X)) = k[T_1, \ldots, T_n]/\sqrt{(I, J)}.$$

So $A(Y) \to A(f(X)) \simeq A(X)$ is surjective.

Conversely suppose that $\varphi : A(Y) \to A(X)$ is surjective. It factors as

$$A(Y) \twoheadrightarrow A(Y)/\ker = A(Z(\ker \varphi)) \simeq A(X).$$

So f induces an isomorphism $X \to Z(\ker \varphi)$. □

Exercise 1.4.1. *Let X, Y be algebraic varieties. Let $X' \subseteq X$, $Y' \subseteq Y$ be closed subsets. Show that $X' \times Y'$ is isomorphic to a closed subvariety of $X \times Y$.*

Exercise 1.4.2. *Let $f : X \to Y$ be a morphism of algebraic varieties and let $Y = \cup_i Y_i$ be an open covering. Show that f is a closed immersion if and only if for all i, $f^{-1}(Y_i) \to Y_i$ is a closed immersion.*

Exercise 1.4.3. *Let X, Y be algebraic varieties. Let Z be a closed or open subvariety of Y, let $f : X \to Z$ be a map. Show that f is a morphism of algebraic varieties if and only if the composition $X \to Y$ of f with the inclusion $Z \to Y$ is a morphism.*

Proposition 1.4.11 (Segre embedding)**.** *Let $n, m \geq 1$. Then there exists a closed immersion $\mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^{nm+n+m}(k)$.*

Proof. Let us write $\mathbb{P}^n(k) = \mathrm{Proj}\, k[T_0, ..., T_n]$, $\mathbb{P}^m(k) = \mathrm{Proj}\, k[S_0, ..., S_m]$, and

$$\mathbb{P}^{nm+n+m}(k) = \mathrm{Proj}\, k[U_{ij}]_{0 \leq i \leq n, 0 \leq j \leq m}.$$

The map $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^{nm+n+m}(k)$ given by

$$((t_0, ..., t_n), (s_0, ..., s_m)) \mapsto (t_i s_j)_{0 \leq i \leq n, 0 \leq j \leq m}$$

is well defined. For any pair $i \leq n, j \leq m$, we have

$$f^{-1}(D_+(U_{ij})) = D_+(T_i) \times D_+(S_j)$$

and f is clearly a morphism $D_+(U_{ij}) \to D_+(T_i) \times D_+(S_j)$. The corresponding homomorphism of k-algebras is (i, j fixed and i', j' vary)

$$U_{i'j'}/U_{ij} \mapsto (T_{i'} T_i^{-1})(S_{j'} S_j^{-1})$$

which is surjective. Therefore

$$f : f^{-1}(D_+(U_{ij})) \to D_+(U_{ij})$$

is a closed immersion. This implies that f is a closed immersion. □

Remark 1.4.12. *Any closed subset Z of $\mathbb{P}^n(k) \times \mathbb{A}^m(k)$ is the pre-image by*

$$\mathbb{P}^n(k) \times \mathbb{A}^m(k) \simeq \mathbb{P}^n(k) \times D_+(S_0) \to \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^{nm+n+m}(k)$$

of a closed subset $Z_+(J)$ of $\mathbb{P}^{nm+n+m}(k)$. If $P(U_{ij})_{i,j} \in J$ is homogeneous of degree d, then $P(T_i S_j)_{i,j} \in k[S_0, S_1, \dots, S_m][T_0, \dots, T_n]$ is homogeneous of degree d in the variables $T_0, \dots, T_n$ and

$$Z = \{(t_0, \dots, t_n, s_1, \dots, s_m) \in \mathbb{P}^n(k) \times \mathbb{A}^m(k) \mid P(t_i s_j)_{i,j} = 0, \forall \textit{ homog. } P \in J\}$$

(with $s_0 = 1$). If I is the ideal of $k[S_1, \dots, S_m][T_0, \dots, T_n]$ generated by the $P(T_i S_j)_{i,j}$ with $P \in J$ homogeneous and $S_0 = 1$. Then $Z = Z_+(I)$ as in Example 1.4.8. This gives a different proof of the statement in 1.4.8.

Corollary 1.4.13. *The product of projective (resp. quasi-projective) varieties is projective (resp. quasi-projective).*

Proof. Use Proposition 1.4.11, Exercise 1.4.1 and Lemma 1.4.6. □

Proposition 1.4.14. *Let X, Y be varieties of respective dimensions n and m. Then* $\dim(X \times Y) = n + m$.

Proof. We can suppose that X and Y are affine. By Noether's Normalization Lemma **??**, there are injective finite homomorphisms $k[T_1, ..., T_n] \to A$ and $k[S_1, ..., S_m] \to B$. This induces a finite injective homomorphism

$$k[T_1, ..., T_n, S_1, ..., S_m] = k[T_1, ..., T_n] \otimes_k k[S_1, ..., S_m] \to A \otimes_k B.$$

Therefore $\dim X \times Y = \dim(A \otimes_k B) = n + m = \dim X + \dim Y$. □

Exercise 1.4.4. *Let X be a topological space.*

1. *If X is a finite union of irreducible closed subsets $X_1, \ldots, X_n$, show that any irreducible component of X is equal to some X_i.*
2. *If a topological space X is covered by finitely many irreducible subsets, show that X has finitely many irreducible components.*
3. *Suppose X has finitely many irreducible components $F_1, \ldots, F_m$. Let U be a non-empty open subset of X. Show that the irreducible components of U are exactly the non-empty intersections $F_i \cap U$.*
4. *If X is an algebraic variety, show that X has finitely many irreducible components.*

Exercise 1.4.5. *Let X be an algebraic variety over k. Then the following properties are equivalent:*

(i) *X is integral;*

(ii) *X is connected and $X = \cup_i X_i$ with X_i affine and $\mathcal{O}_X(X_i)$ integral.*

(iii) *Any non-empty affine open subset of X is integral.*

Exercise 1.4.6. *One-dimensional connected closed subvarieties of $\mathbb{A}(k)^2$ and of $\mathbb{P}^2(k)$ are defined by one equation.*

Exercise 1.4.7. $V_+(f_1) \cap \cdots \cap V_+(f_n) \neq \emptyset$ *if f_i homogeneous non-constant in* $k[T_0, \ldots, T_n]$.

Exercise 1.4.8. *Show that $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$ and $\mathbb{P}^2(k)$ are not isomorphic to each other.*

Exercise 1.4.9. *Let X be an algebraic variety. Show that* $\dim X = 0$ *if and only if X is a finite set.*

Exercise 1.4.10. *Let $\phi : k[T_0, \ldots, T_n] \to k[S_0, \ldots, S_m]/J$ be a homogeneous homomorphism (there exists $r \geq 1$ such that the image of any homogeneous element of degree d has degree rd). Show that ϕ induces a morphism of algebraic varieties*

$$Z_+(J) \setminus Z_+(\phi(T_0, \ldots, T_n)) \to \mathbb{P}^n.$$

Exercise 1.4.11. *Show that* $k[X,Y]/(X^3+Y^3+1) = k[X]\oplus yk[X]\oplus y^2k[X]$. *Let* E *be the projective curve defined by* $X^3+Y^3+Z^3=0$. *Show that* $\mathcal{O}_E(E)=k$.

Exercise 1.4.12. *Let* $f : X \to Y$ *be a morphism of integral varieties over* k. *We say that* f *is* dominant *if* $f(X)$ *is dense in* Y. *Show that* f *then induces an injective homomorphism* $k(Y)\to k(X)$, *and* $\dim Y \le \dim X$.

We say that f *is* birational *if* f *is dominant and if* $k(Y)\to k(X)$ *is an isomorphism. Show that this is equivalent to say that there are dense open subsets* U *of* X *and* V *of* Y *such that* $f|_U : U \to V$ *is an isomorphism.*

Exercise 1.4.13. *Let* $f : X \to Y$ *be a morphism with* X *irreducible. Show that the Zariski closure* $\overline{f(X)}$ *of* $f(X)$ *is irreducible and* $\dim \overline{f(X)} \le \dim X$.

1.5 Proper algebraic varieties

A topological space X is separated if and only if the diagonal $\Delta(X) = \{(x,x)\in X\times X \mid x\in X\}$ is closed in $X\times X$ for the product topology.

Definition 1.5.1. *Let* X *be an algebraic variety over* k. *The diagonal map* $\Delta : X \to X\times X$, $x\mapsto (x,x)$ *is a morphism. The image of* Δ *is called the* diagonal *of* $X\times X$. *When* $\Delta(X)$ *is a closed, we say that* X *is* separated.

The condition to be separated replaces the classical condition to be Hausdorff.

Lemma 1.5.2. *Let* X *be an algebraic variety and let* Y *be a an algebraic separated variety. If* $f,g : X\to Y$ *are two morphisms which coincides over an open dense subset of* X *then they are equal.*

Exercise 1.5.1. *Show that affine varieties are separated. Show that open subvarieties and closed subvarieties of a separated variety are separated.*

Exercise 1.5.2. *If* X *is separated, then the diagonal map is a closed immersion.*

Exercise 1.5.3. *Show that* X *is separated if and only if there exists an affine covering* $\{X_i\}_i$ *of* X *such that for all* i,j, $X_i\cap X_j$ *is affine, and the canonical homomorphism* $\mathcal{O}_X(X_i)\otimes_k \mathcal{O}_X(X_j)\to \mathcal{O}_X(X_i\cap X_j)$ *such that* $f_i\otimes f_j \mapsto f_i|_{X_i\cap X_j} f_j|_{X_i\cap X_j}$ *is surjective. Show that products of separated varieties are separated. Show that projective varieties are separated.*

Definition 1.5.3. *An algebraic variety* X *over* k *is said to be* proper *(or* complete *in the old terminology) if it is separated and if for any algebraic variety* Y *over* k, *the second projection* $X\times Y\to Y$ *is a closed map (i.e. image of a closed subset is a closed subset).*

Closed subvarieties of proper varieties are proper.

Let X be a topological space. If X is compact, then it is separated and for any topological space Y, the projection $X\times Y\to Y$ is a closed map. Thus we see that proper algebraic varieties are analogue to compact topological spaces. We know that on a compact complex manifold, global holomorphic functions are all constant. For proper algebraic varieties we have the same statement.

Proposition 1.5.4. *Projective varieties are proper.*

Proof. It is enough to show that $X = \mathbb{P}^n(k)$ is proper. The separatedness is easy (Exercise 1.5.3). Let Y be any variety and cover it by affine open subsets $\{Y_i\}_i$. The projection $q : X \times Y \to Y$ is closed if $X \times Y_i \to Y_i$ is closed for all i. So we can suppose that Y_i is affine and even equal to some affine space $\mathbb{A}^m(k)$. So let $Z = Z_+(I)$ be a closed subset of $\mathbb{P}^n(k) \times \mathbb{A}^m(k)$ (Example 1.4.8). We want to show that its image $q(Z)$ under the projection to $\mathbb{A}^m(k)$ is closed.

Let $y = (a_1, \dots, a_m) \in \mathbb{A}^m(k)$. Let $\mathfrak{m}_y$ be the maximal ideal $(S_1 - a_1, \dots, S_m - a_m)$ of $k[S_1, \dots, S_m]$ corresponding to y. Denote by I_y the ideal of $k[T_0, \dots, T_n] = (k[S_1, \dots, S_m]/\mathfrak{m}_y)[T_0, \dots, T_n]$ given by the image of I in the quotient. The image of $P(T_0, \dots, T_n, S_1, \dots, S_m)$ is simply $P(T_0, \dots, T_n, a_1, \dots, a_m)$. Then

$$q^{-1}(y) \cap Z_+(I) = Z_+(I_y) \times \{y\} \subseteq \mathbb{P}^n(k) \times \{y\}.$$

In particular,

$$q^{-1}(y) \cap Z_+(I) = \emptyset \Longleftrightarrow (T_0, \dots, T_n)^N \subseteq I_y \subseteq k[T_0, \dots, T_n] \text{ for some } N \geq 1$$

by Proposition 1.1.43. The latter is equivalent to the inclusion

$$(T_0, \dots, T_n)^N \subseteq I + \mathfrak{m}_y k[T_0, \dots, T_n, S_1, \dots, S_m]$$

in $k[T_0, \dots, T_n, S_1, \dots, S_m]$.

Let $y_0 \in \mathbb{A}^m(k) \setminus q(Z)$. Then

$$(T_0, \dots, T_n)^N \subseteq I + \mathfrak{m}_{y_0} k[T_0, \dots, T_n, S_1, \dots, S_m]$$

for some $N \geq 1$. Considering homogeneous elements of degree N in this inclusion, we get

$$B_N \subseteq I_N + \mathfrak{m}_{y_0} B_N,$$

(where B_N is the set of the homogeneous elements of degree N and $I_N = I \cap B_N$) or, equivalently, $B_N/I_N = \mathfrak{m}_{y_0}(B_N/I_N)$. By Nakayama's lemma 1.5.5, this implies that there exists $P \in k[S_1, \dots, S_m] \setminus \mathfrak{m}_{y_0}$ such that $PB_N \subseteq I_N$, so $P(S_1, \dots, S_m) \cdot (T_0, \dots, T_n)^N \subseteq I^N$. We then have $q^{-1}(D(P)) \cap Z = \emptyset$. □

Lemma 1.5.5. *Let A be a ring with a proper ideal I. Let M be an A-module of finite type such that $M = IM$. Then there exists $f \in 1 + I$ such that $fM = 0$.*

Proof. Let $e_1, \dots, e_n$ be a system of generators of M. There exist elements $\{y_{ij}\}_{1 \leq i,j \leq n}$ of I such that

$$e_n = \sum_{1 \leq j \leq n} y_{ij} e_j, \quad 1 \leq i \leq n$$

. Denote by V the matrix with the entries y_{ij} and by

$$\chi_V(T) = \det(T\mathrm{I}_n - V) \in A[T]$$

the characteristic polynomial of V. By Cayley-Hamilton we have $\chi_V(V) = 0$. But since V represents the identity map of M we have that $\chi_V(\mathrm{Id}_M) = 0 \in \mathrm{End}(M)$. On the other hand, the endomorphism $\chi_V(\mathrm{Id}_M)$ is nothing but the multiplication by $f = \chi_V(1) = \det(\mathrm{I}_n - V)$. So $fM = 0$. As $f \in 1 + I$, we are done. □

Definition 1.5.6. *Let $f : X \to Y$ be a morphism of algebraic varieties over k. Consider the morphism $f \times \mathrm{Id}_Y : X \times Y \to Y \times Y$. We denote by Γ_f the subset $(f \times \mathrm{Id}_Y)^{-1}(\Delta(Y))$.*

If Y is separated, then Γ_f is a closed subset of $X \times Y$. It is called the graph of f. *It is endowed with the structure of a closed subvariety of $X \times Y$.*

Lemma 1.5.7. *Let $f : X \to Y$ be a morphism with Y separated. Then the morphism $(\mathrm{Id}_X, f) : X \to X \times Y$ is a closed immersion with image Γ_f*

Corollary 1.5.8. *Let $f : X \to Y$ be a morphism with Y separated. If X is proper, then f a closed map.*

Proof. The composition $(\mathrm{Id}_X, f) : X \to X \times Y$ and $q : X \times Y \to Y$ is equal to f. Since (Id_X, f) induces a homeomorphism from X onto Γ_f by the previous lemma, we can factorize f as $X \simeq \Gamma_f$, $q : \Gamma_f \to Y$. As the second map is closed because $q : X \times Y \to Y$ is closed, we see that f is closed. □

Proposition 1.5.9. *Let X be a connected proper algebraic variety over k. Then $\mathcal{O}_X(X) = k$.*

Proof. Let $f \in \mathcal{O}_X(X)$. It is a morphism $f : X \to \mathbb{A}^1(k)$. By Corollary 1.5.8, $f(X)$ is closed. It is connected because X is connected. Hence $f(X)$ is either $\mathbb{A}^1(k)$ or one point. By composing f with an open immersion $\mathbb{A}^1(k) \to \mathbb{P}^1(k)$, and by applying Corollary 1.5.8 with $Y = \mathbb{P}^1(k)$, we see that $f(X)$ is closed in $\mathbb{P}^1(k)$, so it is different from $\mathbb{A}^1(k)$. Therefore $f(X)$ is just one point in $\mathbb{A}^1(k)$ and f is constant. □

Exercise 1.5.4. *By Proposition 1.5.9, we see that an affine algebraic variety X is never proper except it has dimension 0. Prove this fact directly (let $f \in A(X)$ non-invertible and non-constant, consider the closed subset $Z(fT - 1)$ in $X \times \mathbb{A}^1(k)$ and find its image by the second projection).*

1.6 Smooth varieties

1.6.1 Tangent spaces

When we have a planar curve $C : F(x, y) = 0$ in $\mathbb{R}^2$ without singularity, the tangent line to C at a point $(a, b) \in C$ is the line defined by

$$(x-a)\frac{\partial F}{\partial x}(a,b) + (y-b)\frac{\partial F}{\partial y}(a,b) = 0.$$

The curve is non-singular when precisely this equation defines a line.

Definition 1.6.1. *Let $X = Z(I) \subseteq \mathbb{A}^n(k)$ be an algebraic set, let*

$$p = (a_1, \ldots, a_n) \in X.$$

Then the Zariski tangent space *of X at p is the vector space*

$$T_{X,p} := \{(t_1, ..., t_n) \in k^n \mid \sum_{1\le i\le n} t_i \frac{\partial F}{\partial T_i}(p) = 0,\ \forall F \in I\}$$

Now we want to relate the Zariski tangent space to a purely algebraic objet. Let E be the vector space k^n. Let $p \in \mathbb{A}^n(k)$. We have a homomorphism

$$D_p : k[T_1, ..., T_n] \to E^\vee$$

defined by $F \mapsto D_pF$ where

$$D_pF : (t_1, ..., t_n) \mapsto \sum_{1\le i\le n} t_i \frac{\partial F}{\partial T_i}(p).$$

So D_pF is nothing but the differential of F at p.

Lemma 1.6.2. *Let $\mathfrak{m}$ be the maximal ideal of $k[T_1, ..., T_n]$ corresponding to $p \in \mathbb{A}^n(k)$. Then the restriction of D_p to $\mathfrak{m}$ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \simeq E^\vee$.*

Proof. Write $p = (a_1, ..., a_n)$. Then $\mathfrak{m} = (T_1-a_1, \ldots, T_n-a_n)$. The isomorphism is easily proven using Taylor expansion of polynomials at p. □

Let $X = Z(I)$ (with radicial I) be a closed subvariety of $\mathbb{A}^n(k)$, let $p \in X(k)$. Denote by $\mathfrak{n}$ (resp. $\mathfrak{m}$) the maximal of $k[T_1, ..., T_n]$ (resp. $A(X) = k[T_1, ..., T_n]/I$) corresponding to the point p. Let D_pI denote the image of I by D_p. Then in E we have

$$T_{X,p} = (D_pI)^\perp := \{v \in E \mid \varphi(v) = 0, \forall\varphi \in D_pI\}.$$

Note that if I is generated by $F_1, ..., F_m$ (as an ideal), then D_pI is generated (as a k-vector space) by $D_pF_1, ..., D_pF_m$ and

$$T_{X,p} = \{(t_1, ..., t_n) \in E \mid \sum_{1\le i\le n} t_i \frac{\partial F_j}{\partial T_i}(p) = 0,\ j = 1, \ldots, m.\}$$

We have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I/(I \cap \mathfrak{n}^2) \arrow[r] \arrow[d, "D_p"] & \mathfrak{n}/\mathfrak{n}^2 \arrow[r] \arrow[d, "D_p"] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D_p I \arrow[r] & E^\vee \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r] & 0
\end{tikzcd}$$

Taking the dual in the second horizontal line, we see that $T_{X,p}$ can be identified canonically to $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. We just proved the following

Proposition 1.6.3. *Let X be an affine variety, let $p \in X$ and let $\mathfrak{m}$ be the maximal ideal of $A(X)$ corresponding to p (this the set of the regular functions on X vanishing at p). Then $T_{X,p} \simeq (\mathfrak{m}/\mathfrak{m}^2)^\vee$. In particular the tangent space is intrinsic (independent of the choice of an embedding in an affine space).*

Lemma 1.6.4. *Let A be a ring and $\mathfrak{m}$ a maximal ideal of A. Then the canonical homomorphism of A-modules $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2 A_\mathfrak{m}$ is an isomorphism.*

Proof. Let $x \in \mathfrak{m}$ such that $sx \in \mathfrak{m}^2$ with $s \in A \setminus \mathfrak{m}$. . Since $A/\mathfrak{m}$ is a field there exists $t \in A$ such that $ts - 1 \in \mathfrak{m}$. Therefore $tsx - x \in \mathfrak{m}^2$ which implies $x \in \mathfrak{m}^2$. so the above application is injective.

Now let $x \in \mathfrak{m}A_\mathfrak{m}$. Then there exists $s \in 1A \setminus \mathfrak{m}$ such that $sx \in \mathfrak{m}$. As above, multipling by t, we get that $tsx - x \in \mathfrak{m}^2 A_\mathfrak{m}$, so the class of x is the image of the class of $tsx \in \mathfrak{m}$. □

Definition 1.6.5. *Let X be an algebraic variety and let $p \in X$. The* Zariski tangent space *$T_{X,p}$ of X at p is the dual, as k-vector space, of*

$$\mathfrak{m}\mathcal{O}_{X,p}/\mathfrak{m}^2\mathcal{O}_{X,p}$$

Remark 1.6.6. *By the above Lemma this correspond to $T_{U,p}$ for any affine neighbohood of p.*

The construction of the tangent space is functorial: if $f : X \to Y$ is a morphism, then we have a canonical k-linear map $T_{f,p} : T_{X,p} \to T_{Y,f(p)}$. Note that if f is a closed (resp. open) immersion, than $T_{f,p}$ is injective (resp. bijective).

Exercise 1.6.1. *Let X be an affine algebraic variety, let I, J be ideals of $A(X)$. Suppose that there exist a covering $X = \cup_i D(f_i)$ such that $I\mathcal{O}_X(D(f_i)) = J\mathcal{O}_X(D(f_i))$ for all i. Show that $I = J$.*

1.6.2 Smooth algebraic varieties

Definition 1.6.7. *We set*

$$\dim_p X = \min \{\dim U \mid U \ni p\},$$

the minimul being taken on the open neighborhoods U of p. This is the dimension of X at p. *If W is an open neighborhood of p, then* $\dim_p W = \dim_p X$. *Note that* $\dim_p X$ *is just the maximum of the dimensions of the irreducible components of X passing through p. In particular, if X is irreducible, then* $\dim_p X = \dim X$ *for all $p \in X$.*

Proposition 1.6.8. *Let X be an algebraic variety over k and let $p \in X$. Then $\dim T_{X,p} \geq \dim_p X$.*

Proof. We will proceed by induction on $\dim T_{X,p}$. Replacing X by an open neighborhood of p, we can suppose X is affine and $\dim_p X = \dim X$. Let $\mathfrak{m}$ be the maximal ideal of $A(X)$ corresponding to p. If $\dim T_{X,p} = 0$, then $\mathfrak{m} = \mathfrak{m}^2$. By Nakayama's Lemma 1.5.5, there exists $f \in A(X) \setminus \mathfrak{m}$ such that $f\mathfrak{m} = 0$. Replacing X by the open neighborhood $D(f)$ of p, we get $\mathfrak{m} = 0$ and $A(X) = A(X)/\mathfrak{m} = k$. Therefore $\dim_p X = 0$.

Suppose that $\dim T_{X,p} \geq 1$. Let Z be an irreducible component of X passing through p and having dimension $\dim_p X$. Then $\dim T_{X,p} \geq \dim T_{Z,p}$ and $\dim_p X = \dim_p Z$. So we can suppose that X is irreducible. Suppose $\dim_p X \neq 0$. There exists $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. Consider $Y = Z(f) = Z(\sqrt{fA(X)})$. Then $\dim T_{Y,p} \leq \dim T_{X,p} - 1$. Every irreducible component of Y has dimension $\dim X - 1$ (Proposition 1.3.24). Therefore $\dim_p Y = \dim_p X - 1$. By induction hypothesis we have

$$\dim T_{X,p} \geq \dim T_{Y,p} + 1 \geq \dim_p Y + 1 = \dim_p X$$

and we are done. □

Definition 1.6.9. *We say that X is* smooth *(or nonsinguar or regular) at p if $\dim_p X = \dim T_{X,p}$. In this case we also say that p is a regular or smooth point of X. The variety X is called smooth or nonsingular or regular if it is smooth at all its points.*

Example 1.6.10. *Affine and projective spaces $\mathbb{A}^n(k), \mathbb{P}^n(k)$ are smooth. The plane curve $Z(y^2 - x^3)$ is singular (i.e. not regular) only at the origin.*

Let $X = Z(I) \subseteq \mathbb{A}^n(k)$, $p \in X$ and let $J(X)_p$ denote the matrix

$$(\partial F_j / \partial T_i(p))_{1 \leq i \leq n, 1 \leq j \leq m}$$

where $F_1, \dots, F_m$ is a system of generators of I. When $E^\vee$ is identified to E via the canonical basis, $D_p I \subseteq E^\vee$ is the vector subspace generated by the column vectors of $J(X)_p$. Hence

$$\dim T_{X,p} = \dim (D_p I)^\perp = n - \operatorname{rank} J(X)_p.$$

This implies the

Proposition 1.6.11 (Jacobian criterion)**.** *Let X be as above. Let $p \in X$. Then X is smooth at p if $\operatorname{rank} J(X)_p = n - \dim_p X$, and is singular at p if $\operatorname{rank} J(X)_p < n - \dim_p X$.*

Corollary 1.6.12. *Let $X = Z(F)$ be a hypersurface in $\mathbb{A}^n(k)$ with F without multiple factors. Let $p \in X$. Then X is smooth at p if at least one partial derivative $\partial F / \partial T_i(p)$ at p is non-zero. It is singular at p if all $\partial F / \partial T_i(p)$ are null.*

Proof. The defining ideal of X is (F) and we have $\dim_p X = n - 1$. $\square$

Exercise 1.6.2. *Let X be an algebraic variety over k. Show that the set of the smooth points (the* smooth locus*) of X is open in X.*

Proposition 1.6.13. *Let $k(X)$ be a function field over k. Then $k(X)$ is a finite separable extension of some $k(T_1, \dots, T_n)$.*

Proof. We have a presentation of $k(X)$ as a finite extension of $K = k(T_1, \dots, T_n)$. If $k(X)/K$ is separable, there is nothing to prove. Let us suppose that $k(X)$ is not equal to the separable closure L of K in $k(X)$. Then k has positive characteristic p. Let $\theta \in k(X) \setminus L$ be such that $\theta^p \in L$. We will show that $L[\theta]$ is finite and separable over a purely transcendental extension. This will imply the proposition by decomposing $k(X)/L$ into a sequence of purely inseparable extensions of degree p.

Let $H(S) = S^r + f_{r-1}S^{r-1} + \cdots + f_0 \in K[S]$ be the minimal polynomial of $\theta^p \in L$ over K. Then at least one $f_i \notin k(T_1^p, \dots, T_n^p)$. Indeed, in the opposite case, $H(S^p) = Q(S)^p$ for some $Q(S) \in K[S]$ and Q would be separable. We can therefore assume, for example, that a power of T_1 prime to p appears in one of the f_i. It follows that T_1 is algebraic and separable over $k(\theta, T_2, \dots, T_n)$. As $L[\theta]$ is finite separable over $k(\theta, T_1, T_2, \dots, T_n) = K[\theta]$, it is also finite separable over $k(\theta, T_2, \dots, T_n)$. Finally, this last extension is purely transcendental because its transcendence degree over k is equal to that of $L[\theta]$, which is n. $\square$

Let R be an integral domain with field of fractions K. Let

$$H(S) = S^r + f_{r-1}S^{r-1} + \cdots + f_0 \in R[S]$$

be a monic polynomial. Its *discriminant* Δ is an element of R. For any ring homomorphism $\phi : R \to F$ to a field F, the polynomial $\phi(H)(S) := S^r + \sum_{i \le r-1} \phi(f_i)S^i \in F[S]$ has common zero with its derivative $\phi(H)'(S)$ if and only if $\phi(\Delta) = 0$. Moreover, $H(S)$ (viewed as an element of $K[S]$) is separable if and only if $\Delta \neq 0$.

Lemma 1.6.14. *Let $H(S) = S^r + f_{r-1}S^{r-1} + \cdots + f_0 \in k[T_1, \dots, T_n][S]$ be a separable polynomial in S (viewed in $k(T_1, \dots, T_n)[S]$). Let $\Delta \in k[T_1, \dots, T_n]$ be its discriminant. Let X be the affine variety $Z(H(S)) \subseteq \mathbb{A}^{n+1}(k)$. Then $X \cap D(\Delta)$ is non-empty and consists in smooth points of X.*

Proof. A point $(q, s) \in \mathbb{A}^n(k) \times \mathbb{A}^1(k) = \mathbb{A}^{n+1}(k)$ belongs to X if and only if s is a root of $S^r + \sum_{i \le r-1} f_i(q)S^i$.

Let $q \in D(\Delta) \neq \emptyset$. Let $\phi : k[T_1, \dots, T_n] \to k$, $F \mapsto F(q)$ be the evaluation map at q. Let $s \in k$ be any root of $\phi(H)(S) = S^r + \sum_{i \le r-1} f_i(q)S^i$. Then $p = (q, s) \in X$. The Jacobian matrix of X at p is

$$\mathrm{Jac}(X)_p = \left(\frac{\partial H}{\partial T_1}(p), \cdots, \frac{\partial H}{\partial T_n}(p), \frac{\partial H}{\partial S}(p) \right).$$

The discriminant of $\phi(H)(S)$ is equal to $\phi(\Delta) = \Delta(q) \neq 0$. As $\phi(H)(s) = 0$, we have $\partial H/\partial S(p) = \phi(H)'(s) \neq 0$. Hence the Jacobian matrix has rank $1 = (n + 1) - \dim_p X$ and X is smooth at p. $\square$

Proposition 1.6.15. *Let X be an algebraic variety. The smooth locus of X is dense in X. In particular, it is non-empty if X is non-empty.*

Proof. We have to show that any non-empty open subset of X contains a smooth point. As any such subset contains an irreducible affine open subset, it is enough to show that any integral affine variety contains a smooth point. Denote again by X such a variety.

By Proposition 1.6.13, $k(X)$ is finite separable over $k(T_1, \dots, T_n)$. So $k(X) = k(T_1, \dots, T_n)[S]/(S^r + f_{r-1}S^{r-1} + \dots + f_0)$ with $f_i \in k(T_1, \dots, T_n)$ and $H(S) := S^r + f_{r-1}S^{r-1} + \dots + f_0$ irreducible separable as polynomial in S. Multiplying S by a suitable non-zero element of $k[T_1, \dots, T_n]$, we can suppose that the f_i's are in fact polynomials. By Gauss's theorem $H(S)$ is irreducible in $k[T_1, \dots, T_n, S]$. So X is birational to the integral variety $W = Z(H(S)) \subset \mathbb{A}^{n+1}(k)$. By Exercise 1.4.12, X contains a dense open subset isomorphic to a dense open subset W_0 of W. As the smooth points of W contains a dense open subset by 1.6.14, W_0 (hence X) has at least one smooth point. □

1.6.3 Local structure of smooth algebraic varieties

Theorem 1.6.16. *Let X be a smooth algebraic variety. Let $p \in X$.*

(1) *There is only one irreducible component of X passing through p. In otherwords, if X is connected then it is integral.*

(2) *Let U be a connected affine open subset of X. Then $\mathcal{O}_X(U)$ is integrally closed. In particular if $\dim U = 1$, then $\mathcal{O}_X(U)$ is a Dedekind domain.*

(3) *The local ring $\mathcal{O}_{X,p}$ is a UFD.*

Proposition 1.6.17. *Let $p \in X$ and $d = \dim_p X$. Then X is smooth at p if and only if for any small enough affine open neighborhood U of p, the maximal ideal of $\mathcal{O}_X(U)$ corresponding to p can be generated by d elements.*

Proof. The if part comes essentially from the definition of $T_{X,p}$. To prove the converse, we can suppose X is affine. Let $\mathfrak{m}$ be the maximal ideal of $A(X)$ corresponding to the smooth point p. Then $\mathfrak{m}/\mathfrak{m}^2$ is generated by d elements $e_1, \dots, e_d \in \mathfrak{m}$. Since $m = (e_1, \dots, e_d) + \mathfrak{m}^2$ then $m/(e_1, \dots, e_n) = \mathfrak{m}(\mathfrak{m}/e_1, \dots, e_n)$. By Nakayama's lemma, there exists $f \in A(X) \setminus \mathfrak{m}$ such that $f\mathfrak{m} \subseteq \sum_{1 \le i \le d} e_i A(X)$. Let us take $U = D(f)$ the ideal $\mathfrak{m}\mathcal{O}_X(U) = \mathfrak{m}A(X)_f$ is generated by $e_1|_U, \dots, e_d|_U$. As $\mathfrak{m}\mathcal{O}_X(U)$ is the maximal ideal of $\mathcal{O}_X(U)$ corresponding to $p \in U$ the proposition is proved. □

Corollary 1.6.18. *Let X be an affine connected curve over k. Then X is smooth if and only if $A(X)$ is a Dedekind domain.*

Proof. It is enough to remark that a Dedekind is a notherian integrally closed domain of dimension 1. And, equivalently it is a noetherian ring whose localization at any maximal ideal is a discrete valuation ring. □

Exercise 1.6.3. *Give the analogue of Jacobian criterion for closed subvarieties of $\mathbb{P}^n_k$. Suppose k has characteristic 0. Show that the set of singular points of a hypersurface $V_+(F)$ is equal to $\cap_{0\leq i\leq n}V_+(\partial F/\partial T_i)$.*

Exercise 1.6.4. *Let $X = V_+(I)$ be a closed subvariety of $\mathbb{P}^n(k)$. Let $H = V_+(L)$ be a linear subspace of $\mathbb{P}^n(k)$ (i.e. L is a homogeneous polynomial of degree 1). Consider $Y = V_+(I + Lk[T_0, ..., T_n])$. Set-theoretically we have $Y = X \cap H$. Show that for any $y \in X$, we have $T_{Y,y} = T_{X,y} \cap H$ viewed in $T_{\mathbb{P}^n(k),y}$. Suppose that X is irreducible, smooth not contained in H. Show that Y is smooth if and only if H does not contain $T_{X,y}$ (viewed in $T_{\mathbb{P}^n(k),y}$) for all $y \in Y$.*

Definition 1.6.19. *Let $p \in X$ be a smooth point. The set of regular functions $f_1, \dots, f_r \in A(X)$ is called a* system of parameters *of X at p if they all vanish at p and if their images in $T^\vee_{X,p}$ is free over k (equivalently can be completed into a basis). If $r = \dim_p X$, then the system is called a complete system of parameters.*

Take an affine open neighborhood $U \ni p$ and let $\mathfrak{m}$ be the corresponding maximal ideal in $\mathcal{O}_X(U)$. Then by definition the image of $f \in A(X)$ in $T^\vee_{X,p} = \mathfrak{m}/\mathfrak{m}^2$ is the class of $f|_U \in \mathfrak{m}$ in the quotient.

Remark 1.6.20. *A morphism $f : X \to Y$ of differential manifolds such that $T_{f,p}$ is an isomorphism is a local diffeomorphism. Similar statement for complex manifolds holds as well. But this fails for algebraic varieties because the Zariksi topology has not enough open subsets. For example, if we consider the smooth affine curve $C : y^2 = x^3 + 1$ in characteristic 0, then $A(C)$ is not a PID as we saw in exercise. Let $f : C \to \mathbb{A}^1(k)$ be a morphism as in the above proposition. It will not be a local isomorphism. Indeed, it was locally an isomorphism at some point, then it is a birational morphism. By Corollary 2.1.11, f would be an open immersion, and $A(C)$ would a PID. Contradiction.*

A morphism of smooth algebraic varieties with bijective $T_{f,p}$ is called étale. *There is a notion of* étale topology *introduced by Grothendieck to remedy to this situation. It behaves as the complex topology in some sense.*

Exercise 1.6.5. *Let $f(T) \in k[T] \setminus k$. Let $X = Z(S^2 - f(T)) \subset \mathbb{A}^2(k)$. Find the singular points of X. Pay special attention when k has characteristic 2.*

Exercise 1.6.6. *Let k be of characteristic 0 and let $n \geq 1$. Show that the Fermat curve $Z_+(x^n + y^n + z^n) \subset \mathbb{P}^2(k)$ is irreducible and smooth.*

Exercise 1.6.7. *Let X be a subvariety of $\mathbb{P}^n(k)$. Show that $\dim T_{X,p} \leq n$ for any $p \in X$.*

Exercise 1.6.8. *Let X be an algebraic variety, let $p \in X$ be a smooth point and let $f \in A(X)$. Suppose that in some affine open neighborhood $U \ni p$, we have $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ for the maximal ideal of $\mathcal{O}_X(U)$ corresponding to $p \in U$.*

1. *Show that if U is small enough and integral, then $f\mathcal{O}_X(U)_\mathfrak{m}$ is a prime ideal and $f\mathcal{O}_X(U)$ is radicial.*

2. *Show that $Z(f)$ is smooth at p.*

Exercise 1.6.9. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Suppose that $T_{X,p}$ is surjective for some $p \in X$. Show that the closed subvariety $f^{-1}(f(p)) \subseteq X$ is smooth at p.*

Exercise 1.6.10. *Suppose X is integral. Show that a morphism f as in Proposition* **??** *is necessarily dominant and $k(\mathbb{A}^n(k)) \to k(X)$ is finite separable.*

1.6.4 Normal algebraic varieties

Definition 1.6.21. *An integral algebraic variety X is called* normal *if for any affine open subset U of X, $\mathcal{O}_X(U)$ is an integrally closed domain.*

Remark 1.6.22. *By Theorem 1.6.16, integral smooth algebraic varieties are normal, and the converse is true for integral curves by Corollary 1.6.18.*

Example 1.6.23. *Let $X = Z(y^2 - f(x)) \subset \mathbb{A}^2(k)$ in characteristic different from 2. Then X is normal if and only if $f(x)$ has no multiple factors. If $f(x) = f_1(x)f_2(x)^2$ with $f_1(x)$ separable, then $X' := Z(y^2 - f_1(x))$ is normal and birational to X. A birational morphism $X' \to X$ is given by $(x, y) \mapsto (x, yf_2(x))$.*

Proposition 1.6.24. *Let X be an integral affine variety such that $A(X)$ is integrally closed. Then X is normal.*

Proof. Let U be a non-empty open affine subset of X. We have to show that $\mathcal{O}_X(U)$ is integrally closed. Let $\alpha \in \mathrm{Frac}(\mathcal{O}_X(U)) \subseteq k(X)$ be integral over $\mathcal{O}_X(U)$. For any principal open subset $D(f)$ of X contained in U, α is integral over $\mathcal{O}_X(D(f)) = A(X)[1/f] \supseteq \mathcal{O}_X(U)$. So there exists $N \geq 1$ such that $f^N\alpha$ is integral over $A(X)$. Thus $f^N\alpha \in A(X)$ and $\alpha \in \mathcal{O}_X(D(f))$. Now cover U by principal open subsets $\{D(f_i)\}_i$ of X. Then $\alpha \in \mathcal{O}_X(D(f_i))$ for all i, hence $\alpha \in \mathcal{O}_X(U)$. $\square$

Definition 1.6.25. *A morphism $f : X \to Y$ of algebraic varieties is* affine *if for any affine open subset V of Y, $f^{-1}(V)$ is affine. It is* finite *if it is affine and if for any affine open subset V of Y, the homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ of k-algebras is finite.*

One can show that if there exists an affine covering $\{V_i\}_i$ of Y such that for all i, $f^{-1}(V_i)$ is affine (resp. and if $\mathcal{O}_X(f^{-1}(V_i))$ is finite over $\mathcal{O}_Y(V_i)$), then f is affine (resp. finite).

Definition 1.6.26. *Let X be an integral algebraic variety over a field k. The* normalization of X *is a normal algebraic variety X' together with a finite birational morphism $X' \to X$.*

Let $\phi : k(X) \to L$ be a finite homomorphism of extensions of k. The normalization of X in L *is a finite surjective morphism $\pi : X' \to X$ such that $k(X') = L$, X' is normal and the homomorphism $k(X) \to k(X') = L$ induced by π is equal to ϕ. The* normalization of X *is by definition the normalization of X in $k(X)$.*

Proposition 1.6.27. *Let A be an integral k-algebra of finite type with field of fractions K. Let B be the integral closure of A in some finite extension L of K. Then* $\mathrm{Frac}(B) = L$ *and B is finite over A.*

Proof. Let $\lambda \in L$. Then there exists $a \in A$ non-zero such that $a\lambda$ is integral over A, hence $a\lambda \in B$. In particular $\lambda \in \mathrm{Frac}(B)$.

To prove the finiteness of B over A, we first supppose that L is separable over K and A is integrally closed. Let us consider the trace form $L \times L \to K$, $(x, y) \mapsto \mathrm{Tr}_{L/K}(xy)$. This is a non-degenerate bilinear form because L/K is separable. Let $\{e_1, \dots, e_n\}$ be a basis of L/K made up of elements of B. There then exists a basis $\{e_1^*, \dots, e_n^*\} \subset L$ dual to $\{e_1, \dots, e_n\}$ (i.e., $\mathrm{Tr}_{L/K}(e_i e_j^*) = \delta_{ij}$). Let $b \in B$. We have $b = \sum_j \lambda_j e_j^*$ with $\lambda_j \in K$. It follows that $\lambda_j = \mathrm{Tr}_{L/K}(be_j) \in B \cap K = A$ (we need the hypothesis A integrally closed to insure that $\mathrm{Tr}_{L/K}(B) \subseteq A$). Consequently, B is a sub-A-module of $\sum_j Ae_j^*$, and is therefore finite over A.

In the general case, let $k[T_1, \dots, T_n] \to A$ be a finite injective homomorphism (Noether's normalization). Then the integral closure of A in L is also the integral closure of $k[T_1, \dots, T_n]$ in L. So we can suppose $A = k[T_1, \dots, T_n]$. There exists a finite extension of L (e.g., the normal closure of L/K) that is separable over a purely inseparable extension K' of K. Let A' be the integral closure of A in K'. Then B is the integral closure of A' in the extension L/K'. By the previous case, it suffices to show that A' is finite over A. We may assume that $\mathrm{char}(k) = p > 0$. Let $\{e_j\}_j$ be a finite system of generators of K' over K. There exists a $q = p^r$, $r \in \mathbb{N}$, such that $e_j^q \in K$ for every j. Then $K' \subseteq k(S_1, \dots, S_n)$, where $S_i = T_i^{1/q}$ in some algebraic closure of K. Now $k[S_1, \dots, S_n]$ is integrally closed and finite over A, so $A' \subseteq k[S_1, \dots, S_n]$ and is therefore finite over A. □

Proposition 1.6.28. *The normalization of X in L exists and is unique up to isomorphisms.*

Proof. If the normalization $\pi : X' \to X$ exists, then for any affine open subset U of X, $\pi^{-1}(U)$ must be affine and $\mathcal{O}_{X'}(\pi^{-1}(U))$ must be the integral closure of $\mathcal{O}_X(U)$ in L. This determines $\pi^{-1}(U)$ up to canonical isomorphism. So X' is unique up to isomorphism.

To prove the existence, we start with the case X is affine. Let X' be the affine variety defined by the integral closure of $A(X)$ in L. That X' exists is guaranteed by Proposition 1.6.27. In the general case we cover X by affine open subsets X_i's and glue the normalizations of the X_i's in L. The glueing is possible by the uniqueness of the normalization in L of $X_i \cap X_j$. □

Remark 1.6.29. *Let X be an integral algebraic variety over a field k. We see that there is a way to make X normal (by a finite birational morphism). We can ask whether it is possible to have similar result for smoothness. A morphism $\pi : X' \to X$ is called a desingularization of X if π is birational, proper*[2] *and X' is smooth. By a theorem of Hironaka, if k has characteristic* 0,

[2]A morphism $f : X \to Y$ of algebraic varieties over k is *proper* if for every algebraic variety Z over k, $f \times \mathrm{Id}_Z : X \times Z \to Y \times Z$ is closed, and if for every affine open subset U of Y, $f^{-1}(U)$ is separated over k.

then there always exists a desingularization of X. In characteristic $p > 0$, the result is only known for curves (by normalization), surfaces, and threefolds with some conditions. However, a theorem of de Jong states that a weaker form of desingularization exists. Namely, there exists a smooth variety X' over k and a proper morphism $\pi : X' \to X$ which can be decomposed into a proper birational morphism $X' \to Z$ followed by a finite surjective morphism $Z \to X$.

Exercise 1.6.11. *Let X be an integral variety over k. Then there exists a dense open subset U of X which is normal.*

Chapter 2

Algebraic curves

We fix an algebraically closed field k. All algebraic varieties we consider are over k. An *algebraic curve* over k is a separated algebraic variety pure of dimension 1 over k.

2.1 Curves and their function fields

The aim of this section is to prove that integral proper smooth curves are determined by their fields of rational functions (Theorem 2.1.7).

If X is an integral proper smooth curve, then its fields of rational functions $k(X)$ is a function field of transcendence degree 1. We can ask whether all such functions fields are obtained in this way.

Lemma 2.1.1. *Let $f : X \to Y$ be a finite morphism of algebraic varieties with Y proper, then X is proper.*

Proof. First we prove the separatedness of X. Let

$$X \times_Y X = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\} \subseteq X \times X.$$

As $X \times_Y X = (f \times f)^{-1}(\Delta(Y))$ is closed in $X \times X$, it is enough to show that $\Delta(X)$ is closed in $X \times_Y X$. We cover Y by affine open subsets Y_i's and denote by $X_i = f^{-1}(Y_i)$. Then $X \times_Y X$ is covered by the open subsets $X_i \times_{Y_i} X_i$. Indeed, if $(x, x') \in X \times_Y X$, then $x \in X_i$ for some i. So $f(x') = f(x) \in Y_i$ and $x' \in X_i$. It is enough to show that $\Delta(X) \cap (X_i \times_{Y_i} X_i)$ is closed in $X_i \times_{Y_i} X_i$. The former is nothing but $\Delta(X_i)$. As X_i is affine, $\Delta(X_i)$ is closed in $X_i \times X_i$, hence closed in $X_i \times_{Y_i} X_i$.

Let now Z be an algebraic variety. We have to show that the projection map $X \times Z \to Z$ is closed. We decompose this map into $X \times Z \to Y \times Z \to Z$ and it is enough to show that $X \times Z \to Y \times Z$ is closed. Covering Y and Z by affine open subsets, we are reduced to the case when Y, Z (hence X) are affine and we have to show that $A(Y) \otimes_k A(Z) \to A(X) \otimes_k A(Z)$ is finite. If $A(X) =$

$A(Y)f_1 + \cdots + A(Y)f_n$, then $A(X) \otimes_k A(Z)$ is generated over $A(Y) \otimes_k A(Z)$ by the images of the f_i's. $\square$

Corollary 2.1.2. *Let K be a function field of transcendence degree 1 over k (Definition 1.3.17). Then there exists an integral proper smooth curve X over k such that $K \simeq k(X)$.*

Proof. First take an integral affine curve W such that $k(W) \simeq K$. Embed W into an integral projective curve $\overline{W}$ and then normalize $\overline{W}$. $\square$

Next we can ask whether the X in the above Corollary is unique.

Theorem 2.1.3. *Let U be a smooth algebraic curve, let Y be a proper algebraic variety over k, and let $V \subseteq U$ be an open dense subset. Then any morphism $f : V \to Y$ extends uniquely to a morphism $U \to Y$.*

Proof. The uniqueness comes from the separatedness of Y.

To prove the existence we can suppose U is affine and irreducible, and $V = U \setminus \{p\}$ for some $p \in U$. Let $A = A(U)$ and let $\mathfrak{m}$ be the maximal ideal of A corresponding to p. As $\dim T_{U,p} = 1$, taking U small enough, we can suppose that $\mathfrak{m} = tA$ is principal (Proposition 1.6.17).

Consider the graph $\Gamma_f \subset V \times Y$ of f. The projection $V \times Y \to V$ induces an isomorphism $\Gamma_f \simeq V$ (Lemma 1.5.7). Let Z be the Zariski closure of Γ_f in $U \times Y$. This is an integral separated curve. The projection $g : Z \to U$ is a birational morphism. We are going to show that g is an isomorphism. As $U \times Y \to U$ is closed, $g : Z \to U$ is closed and its image contains V, thus g is surjective. Let $z \in g^{-1}(p)$. Let $B = \mathcal{O}_Z(W)$ for some affine open neighborhood of z. Then g induces injective homomorphisms $A \to B$ and $B \to K := \mathrm{Frac}(A)$. We identify B to its image in K. Any non-zero $b \in B$ can be written as $b = t^n a/u$ with $a, u \in A \setminus tA$ and $n \in \mathbb{Z}$. As $u(z)b(z) = t^n(z)a(z)$, we see that $n \geq 0$ and $b \in A[1/u]$. As B is finitely generated k-algebra, there exists $u_0 \in A \setminus tA$ such that $B \subseteq A[1/u_0]$. So $g^{-1}(D(u_0)) \cap W \to D(u_0)$ is an isomorphism.

Let $\{z_1, \ldots, z_r\} = g^{-1}(p)$. Then there exists $U_0 \ni p$ small enough, and open neighborhoods $W_i \ni z_i$ such that the projection $Z \to U$ induces an isomorphism $g_i : W_i \to U_0$ for all $i \leq r$. If $r \geq 2$, then $g_1^{-1}, g_2^{-1} : U_0 \to Z$ are two distinct morphisms which coincide on V with the inverse of $\Gamma_f \to V$. As Z is separated, this is impossible. Therefore $r = 1$ and g is an isomorphism. The composition of $g^{-1} : U \to Z$ with the projection $U \times Y \to Y$ is a morphism $U \to Y$ which extends f. $\square$

Corollary 2.1.4. *Let X, Y be integral proper smooth curves over k. Suppose there is an isomorphism $f : U \simeq V$ between respective dense open subsets of X, Y (equivalently, $k(X) \simeq k(Y)$ as extensions of k, see Exercise 1.4.12). Then f extends uniquely into an isomorphism $X \simeq Y$.*

Proof. By Theorem 2.1.3, $f : U \to Y$ extends to $f' : X \to Y$ and $f^{-1} : V \to X$ extends to $g' : Y \to X$. The composition $f' \circ g' : X \to X$ is indentity on U, so $f' \circ g' = \mathrm{Id}_X$. Similarly $g' \circ f' = \mathrm{Id}_Y$. So f' is an isomorphism. $\square$

Remark 2.1.5. *The above statement fails in higher dimension. For example* $\mathbb{P}^2(k)$ *and* $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$ *are birational but not isomorphic to each other.*

Corollary 2.1.6. *Let* $f : X \to Y$ *be a morphism of integral proper smooth curves over* k*. Then* f *is either constant or finite and surjective.*

Proof. As $f(X)$ is closed in the irreducible space Y and $\dim Y = 1$, we have either $f(X)$ is one point, in which case f is constant, or $f(X) = Y$. Suppose we are in the second case. Then f is dominant and $k(Y) \to k(X)$ is a finite extension because both function fields are of transcendence degree 1. Let $\pi : X' \to Y$ be the normalization of Y in $k(X)$. Then X' is integral, proper, smooth and birational to X. By Corollary 2.1.4, there exists an isomorphism $g : X \to X'$ corresponding to the identity $k(X') = k(X)$. The morphisms $\pi \circ g$ and f induce the same field extension $k(Y) \to k(X)$. So they are equal (Corollary **??**). Therefore f is finite as π is finite. □

Theorem 2.1.7. *There exists an equivalence of categories between the category of functions fields of transcendence degree* 1 *and the category of integral smooth proper curves (morphisms are non-constant morphisms). More concretely, this means the following:*

1. *For any integral proper smooth curve* X*,* $k(X)$ *is a function field of transcendence degree* 1*. Conversely, any such function field is isomorphic to the field of rational functions of a (unique up to unique isomorphism) proper integral smooth curve.*

2. *Any non-constant morphism* $X \to Y$ *is finite and induces a finite homomorphism of* k*-extensions* $k(Y) \to k(X)$*, and any homomorphism of* k*-extension* $k(Y) \to k(X)$ *is finite and comes from a unique finite morphism* $X \to Y$*.*

Corollary 2.1.8. *Any integral proper smooth curve* X *admits a finite surjective morphism* $X \to \mathbb{P}^1(k)$*.*

Exercise 2.1.1. *Show that Corollaries 2.1.6 and 2.1.8 are true without assuming* X, Y *smooth.*

To finish with this section, we notice the following

Theorem 2.1.9. *Let* X *be an integral smooth proper curve. Then* X *is projective.*

Proof. Let $\{U_i\}_i$ be a finite affine covering of X. Each U_i is an open subvariety of an integral projective curve X_i. Let P be the product $\prod_i X_i$. Then P is a projective variety (Corollary 1.4.13). The canonical morphism $U := \cap_i U_i \to P$ then extends to a morphism $f : X \to P$ by Theorem 2.1.3. Let Z be the image of X. Then Z is closed and irreducible because X is proper and irreducible (Corollary 1.5.8). So Z is an integral projective variety. Denote again by f the morphism $X \to Z$ induced by $f : X \to P$.

We are going to show that Z is smooth. It is enough to show that any point $z \in Z$ admits a normal affine open neighborhood W (Corollary 1.6.18). Then $z = f(x)$ for some $x \in U_i \subseteq X$. The projection $P \to X_i$ induces a morphism $p_i : Z \to X_i$ and we have the commutative diagram

$$\begin{tikzcd} U_i \arrow[r, "f"] \arrow[dr, "g_i"'] & Z \arrow[d, "p_i"] \\ & X_i \end{tikzcd}$$

where g_i is an extension of the inclusion $U \to X_i$ to U_i. By the uniqueness (X_i is separated), g_i is the inclusion $U_i \to X_i$. Let W be an affine open neighborhood of z contained in $f(U_i)$ (note that $f(U_i)$ is open because $\dim Z = 1$ and $Z \setminus f(U_i) \subseteq f(X \setminus U_i)$). Then the commutative diagram

$$\begin{tikzcd} f^{-1}(W) \arrow[r, "f"] \arrow[dr, "g_i"'] & W \arrow[d, "p_i"] \\ & f^{-1}(W) \end{tikzcd}$$

implies that $\mathcal{O}_Z(W) \simeq \mathcal{O}_X(f^{-1}(W))$ is normal. So W is normal (Proposition 1.6.24).

The above commutative diagram implies that $k(Z) \simeq k(X)$ and f is birational. This implies that f is an isomorphism by Corollary 2.1.4 and that X is projective. □

Remark 2.1.10. *One can show that any proper curve is projective and any smooth proper surface is projective.*

Another way to prove Theorem 2.1.9 is to show that if $X \to Y$ is a finite morphism to a projective variety, then X is also projective.

Exercise 2.1.2. *Let X be a (separated) smooth integral curve. Show that X is isomorphic to an open subset of an integral projective smooth curve.*

Corollary 2.1.11. *Let $f : X \to Y$ be a birational morphism of integral smooth curves. Then f is an open immersion.*

Proof. Embed X, Y respectively into proper smooth curves $\overline{X}$ and $\overline{Y}$. Then f extends to a birational morphism $\bar{f} : \overline{X} \to \overline{Y}$ by Theorem 2.1.3. So $f = \bar{f}|_X$ is an open immersion. □

2.2 Divisors

A way to study the algebraic curves is to consider the divisors. They are the analogue of the fractional ideals in a Dedekind domain.

2.2.1 Basic definitions

Let X be an integral smooth curve over k. A *divisor on* X is a formal finite sum $D = \sum_x n_x[x]$ with $n_x \in \mathbb{Z}$. The set $Z^1(X)$ of the divisors on X is the free abelian group generated by the points of X.

There is a special kind of divisors which are given by non-zero rational functions. Let $x \in X$. We define a *valuation* $v_x : k(X)^* \to \mathbb{Z}$ as follows. Take a small enough affine open neighborhood $U \ni x$ so that the maximal ideal $\mathfrak{m} \subset \mathcal{O}_X(U)$ corresponding to x is generated by one element t (which is also called a *uniformizer* or a *local coordinate* at x). Any non-zero element $f \in \mathcal{O}_X(U)$ can be written in a unique way as $f = t^v u$ with $v \in \mathbb{Z}$ and $u = a/b$, $a, b \in \mathcal{O}_X(U) \setminus \mathfrak{m}$. We put $v_x(f) = v$. When $f \in \mathcal{O}_X(U)$, this is just the order of vanishing of f at x. One checks easily that

$$v_x(fg) = v_x(f) + v_x(g), \quad v_x(f+g) \geq \min\{v_x(f), v_x(g)\}$$

if $f + g \neq 0$. By convention we write $v_x(0) = +\infty$.

For those who are familiar with the theory of Dedekind domains, the subring

$$\mathcal{O}_X(U)_{\mathfrak{m}} := \{a/s \in \operatorname{Frac}(\mathcal{O}_X(U)) \mid a \in \mathcal{O}_X(U), s \in \mathcal{O}_X(U) \setminus \mathfrak{m}\}$$

is a discrete valuation ring with field of fractions $k(X)$, and v_x is nothing but the normalized valuation on $k(X)$ associated to this discrete valuation ring. The ring $\mathcal{O}_X(U)_{\mathfrak{m}}$ is also denoted by $\mathcal{O}_{X,x}$, it is independent on the choice of an affine open neighborhood of x.

Let $f \in k(X) \setminus \{0\}$. Then $v_x(f) = 0$ for all $x \in X$ except finitely many of them. Indeed, let U be a dense affine open subset of X. Then $f = a/b$ with $a, b \in \mathcal{O}_X(U)$ and a, b non-zero. Then $v_x(f) = 0$ for all $x \in U \setminus (Z(a) \cup Z(b))$. As $(X \setminus U) \cup (Z(a) \cup Z(b))$ is finite, we are done. We define

$$\operatorname{div}(f) = \sum_{x \in X} v_x(f)[x] \in Z^1(X).$$

Such a divisor is called a *principal divisor*. The set of the principal divisors on X form a subgroup of $Z^1(X)$ because $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$. Two divisors D, E on X are said to be *linearly equivalent* if $D - E$ is a principal divisor. We then write $D \sim E$.

The points $x \in X$ such that $v_x(f) < 0$ (resp. $v_x(f) > 0$) are called *poles of* f (resp. *zeros of* f). The divisors

$$\operatorname{div}(f)_\infty := -\sum_{v_x(f) < 0} v_x(f)[x]$$

and

$$\operatorname{div}(f)_0 := \sum_{v_x(f) > 0} v_x(f)[x]$$

are called respectively *pole divisor* and *zero divisor* of f. We have $\operatorname{div}(f) = \operatorname{div}(f)_0 - \operatorname{div}(f)_\infty$.

2.2.2 Pushforward and pullback

When $f : X \to Y$ is a finite surjective morphism of integral smooth curves, one can pushforward divisors on X to Y:

$$f_*\left(\sum_x n_x[x]\right) := \sum_x n_x[f(x)] = \sum_y \left(\sum_{x\in f^{-1}(y)} n_x\right)[y].$$

This is clearly a homomorphism of groups $Z^1(X) \to Z^1(Y)$. (If f is constant, then by definition $f_*D = 0$ for any $D \in Z^1(X)$).

Now we define the pullback of a divisor on Y to X. Let $y \in Y$. Let $x \in f^{-1}(y)$. Let V be a small affine open neighborhood of y and $U \subseteq f^{-1}(V)$ a small affine open neighborhood of x. Denote by $\phi : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ the homomorphisme induced by f. Let t_y be a local coordinate at y. Then $v_x(\phi(t_y)) = e_{x/y}$ for some integer $e_{x/y} \geq 1$. This integer is called the *ramification index of f at x*. It is just the ramification index of the extension of discrete valuation rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

Lemma 2.2.1. *We have $e_{x/y} = \dim_k \mathcal{O}_X(U)/(\phi(t_y))$ if $U \cap f^{-1}(y) = \{x\}$.*

Definition 2.2.2. *Let $D = \sum_y n_y[y] \in Z^1(Y)$. Then we let*

$$f^*D = \sum_{x\in X} e_{x/f(x)} n_{f(x)}[x] \in Z^1(X).$$

This defines a group homomorphism $Z^1(Y) \to Z^1(X)$.

Proposition 2.2.3. *Let A be a Dedekind domaing, let B be a Dedekind domain containing and finite over A. Let $\mathfrak{m}$ be a maximal ideal of A. There are only finitely many maximal ideals $\mathfrak{m}_1, ..., \mathfrak{m}_r$ of B lying oveer $\mathfrak{m}$. Let e_i be the ramification index of $B_{\mathfrak{m}_i}/A_m$. Then*

$$[\mathrm{Frac}(B) : \mathrm{Frac}(A)] = \sum_{1\leq i\leq r} e_i[B/\mathfrak{m}_i : A/\mathfrak{m}].$$

This formula is better known under the form $n = \sum_i e_i f_i$ for extensions of rings of integers in number fields.

Definition 2.2.4. *Let $f : X \to Y$ be a finite morphism of integral smooth curves over k. We call* degree of f *and denote it by $\deg f$ the degree $[k(X) : k(Y)]$.*

Corollary 2.2.5. *Let $f : X \to Y$ be a finite surjective morphism of integral smooth proper curves. Then for any $D \in Z^1(Y)$, we have*

$$f_*(f^*D) = (\deg f)D.$$

Proof. As f_* and f^* are group homomorphism, it is enough to prove the equality when $D = [y]$ is supported in one point. Then $f^*D = \sum_{x\in f^{-1}(y)} e_{x/y}[x]$ and $f_*f^*D = (\sum_{x\in f^{-1}(y)} e_{x/y})[y] = (\deg f)[y]$ by Proposition 2.2.3. □

Exercise 2.2.1. *Let $f : X \to Y$ be finite and separable (i.e. $k(Y) \to k(X)$ is a separable extension). Show that there are only finitely many ramification points (i.e. $x \in X$ such that $e_{x/f(x)} > 1$).*

2.2.3 Degree of a divisor

Definition 2.2.6. *The* degree of a divisor $D = \sum_x n_x[x]$ *is* $\deg D = \sum_x n_x$. *This defines a group homomorphism*

$$\deg : Z^1(X) \to \mathbb{Z}.$$

Proposition 2.2.7. *Let $f : X \to Y$ be a finite morphism of integral smooth curves over k. Let D be a divisor on Y. Then*

$$\deg f^*D = (\deg f) \deg D.$$

Proof. Note that pushforwarding a divisor does not change its degree. So $\deg f^*D = \deg f_*f^*D = \deg((\deg f)D) = (\deg f)\deg D$ by Corollary 2.2.5. □

Let X be an integral proper smooth curve and let $f \in k(X)^*$. Let U be an affine open subset of X such that $f \in \mathcal{O}_X(U)$. Then $f : U \to k$ is a morphism, and the corresponding homomorphism $\phi : k[T] \to \mathcal{O}_X(U)$ is defined by $\phi(T) = f$. By Theorem 2.1.3, $f : U \to \mathbb{A}^1(k) \subset \mathbb{P}^1(k)$ extends uniquely to a morphism $\tilde{f} : X \to \mathbb{P}^1(k)$. See also Corollary 2.1.8.

Exercise 2.2.2. *Suppose that $f \notin k$. Show that $\tilde{f}^*(\mathrm{div}(T)) = \mathrm{div}(f)$.*

Corollary 2.2.8. *Let X be an integral smooth proper curve over k. Let $f \in k(X)$ be a non-zero rational function. Then* $\deg \mathrm{div}(f) = 0$.

Proof. If $\tilde{f}$ is constant, then $f \in k^*$. Otherwise, $\tilde{f}$ is a finite morphism. With the above notations and by Proposition 2.2.7, we have

$$\deg \mathrm{div}(f) = \deg \tilde{f}^*([0] - [\infty]) = \deg \tilde{f} \deg([0] - [\infty]) = 0.$$

□

Exercise 2.2.3. *We have a partial order on $Z^1(X)$ defined by $\sum_x n_x[x] \geq \sum_x m_x[x]$ if $n_x \geq m_x$ for all x. Let $f \in k(X)^*$. Show that* $\mathrm{div}(f) \geq 0$ *if and only if $f \in k$ (then* $\mathrm{div}(f) = 0$*).*

2.2.4 Picard group

The group $Z^1(X)$ we are interested in does not really bring information on X as this is just a free abelian group of rank $\mathrm{card}(k)$. Similarly to number fields, we have the following definition.

Definition 2.2.9. *Let X be an integral proper smooth curve. The quotient* $\mathrm{Pic}(X)$ *of $Z^1(X)$ by the sous-group of principal divisors is called the* class group *or* Picard group *of X.*

The class group of X can be stratified using the degree. By Corollary 2.2.8, the homomorphism $\deg : Z^1(X) \to \mathbb{Z}$ induces a homomorphism $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$ whose kernel is denoted by $\mathrm{Pic}^0(X)$. What is interesting is that this group has a geometric structure. More precisely, $\mathrm{Pic}^0(X)$ is the set of the points of a projective smooth variety $J(X)$, called the *Jacobian variety of* X. Torelli's theorem says that if there is an isomorphism $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(Y)$ as algebraic varieties (plus a condition of "polarization"), then $X \simeq Y$. The variety $J(X)$ has a structure of commutative group and is similar to a compact Lie group.

Proposition 2.2.10. *The map* $\deg \mathrm{Pic}(\mathbb{P}^1_k) \to \mathbb{Z}$ *is an isomorphism.*

Proof. Let $\mathbb{P}^1(k) = \mathrm{Proj}\, k[T_0, T_1]$, $T = T_1/T_0 \in k(\mathbb{P}^1(k))$. The map deg is clearly surjective. Since $\mathrm{div}(T-a) = [a] - [\infty]$ then, if $D = \sum_i r_i[x_i] \in Z^1(\mathbb{P}^1(k))$ then $D \simeq \deg D[\infty]$, si the map is injective. This implies that the map $\deg \mathrm{Pic}(\mathbb{P}^1(k)) \to \mathbb{Z}$, is an isomorphism. $\square$

Proposition 2.2.11. *Let X be an integral smooth proper curve over k. Suppose that there exist two distinct points $x_0, x_1 \in X$ such that $[x_0] \sim [x_1]$. Then $X \simeq \mathbb{P}^1(k)$.*

Proof. Let $f \in k(X)^*$ be such that $\mathrm{div}(f) = [x_0] - [x_1]$. Then $f \notin k$ because f has a pole. Let $\tilde{f} : X \to \mathbb{P}^1(k)$ be the finite morphism associated to f (see notations from §2.2.3). Write $\mathrm{div}(T) = [0] - [\infty]$. Then $\tilde{f}^*[0] - \tilde{f}^*[\infty] = \tilde{f}^*\mathrm{div}(T) = [x_0] - [x_1]$ and $\tilde{f}^*[0] = [x_0]$. Therefore $\deg \tilde{f} = 1$ (Proposition 2.2.7) and $\tilde{f}$ is an isomorphism (Corollary 2.1.4). $\square$

Corollary 2.2.12. *Let X be an integral smooth proper curve over k. If* $\mathrm{Pic}^0(X) = 0$, *then $X \simeq \mathbb{P}^1(k)$.*

2.3 Riemann-Roch Theorem

We fix an integral smooth proper curve X over a field k. Let D be a divisor on X. We define

$$L(D) = \{f \in k(X)^* \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}$$

Write $D = \sum_{x \in D_0} n_x[x] - \sum_{y \in D_\infty} m_y[y]$ with $n_x, m_y \geq 0$ and $D_0 \cap D_\infty = \emptyset$. Then $L(D)$ is a set of rational functions on X with conditions on zeros and poles : for all $x \in D_0$, f has a zero of order $\geq n_x$ at x; for all $y \in D_\infty$, f has a pole of order $\leq m_y$; and f is regular elsewhere.

We have $L(0) = \mathcal{O}_X(X) = k$.

Lemma 2.3.1. *The set $L(D)$ is a k-vector space.*

Proof. This comes from the fact that $\mathrm{div}(\lambda f) = \mathrm{div}(f)$ if $\lambda \in k^*$ and $f \in k(X)^*$, and $v_x(f+g) \geq \min\{v_x(f), v_x(g)\}$ for all $x \in X$ and $f, g \in k(X)^*$. $\square$

Definition 2.3.2. *We denote by $l(D)$ the dimension* $\dim_k L(D)$. *We have $l(0) = 1$.*

Definition 2.3.3. *We set* $|D| = \{E \in Z^1(X) | E \geq 0, E \sim D\}$

We observe that elements of D are in one-to-one correspondence with $\mathbb{P}(L(D)) = (L(D) \setminus \{0\})/k^*$. The correspondence is given by $[f] \mapsto \operatorname{div}(f) + D$.

Definition 2.3.4. *Let D be a divisor on X. The subset $\{x \in X \mid n_x \neq 0\}$ of X is called the* support of D *and denote by* $\operatorname{Supp} D$.

Proposition 2.3.5. *Let X be an integral smooth proper curve over k.*

(a) *If $D \sim E$, then $l(D) = l(E)$.*

(b) *If* $\deg D < 0$*, then* $l(D) = 0$.

(c) *If $D' \leq D$, then*

$$l(D') \leq l(D) \leq l(D') + \deg(D - D').$$

(d) *If* $\deg D \geq 0$*, then* $l(D) \leq \deg D + 1$.

(e) *If* $\deg D = 0$*, then* $l(D) > 0$ *if and only if* $D \sim 0$.

Proof. (a) Let $f \in k(X)^*$ be such that $D = E + \operatorname{div}(f)$. Then $h \in L(D) \setminus \{0\}$ iff $fh \in L(E) \setminus \{0\}$. Hence the multiplication by f induces an isomorphism $L(D) \simeq L(E)$. Thus $l(D) = l(E)$.

(b) If $f \in k(X)^*$, then $\deg(\operatorname{div}(f) + D) = \deg D < 0$, so $\operatorname{div}(f) + D$ can not be a effective divisor. Therefore $L(D) = 0$.

(c) If $D' < D$, there exists $x \in \operatorname{Supp} D$ such that $D' \geq D - [x]$. So by induction on $\deg D$, it is enough to show that if $x \in \operatorname{Supp} D$, then $l(D) \leq l(D - [x]) + 1$. Let t_x be a local coordinate at x, and let $n_x = v_x(D)$ be the coefficient of $[x]$ in D. If $f \in L(D) \setminus \{0\}$, then $v_x(ft_x^{n_x}) = v_x(f) + n_x \geq 0$, and $ft_x^{n_x} \in$ is regular at x. Let $\rho : L(D) \to k$ be the k-linear map defined by $f \mapsto (ft_x^{n_x})(x)$. Then $\operatorname{Ker}(\rho) = L(D - [x])$. Thus $l(D) \leq l(D - [x]) + 1$.

(d)

If $l(D) = 0$, then the inequality is trivially true. If $l(D) \neq 0$ then $D \simeq D'$, such that $D' \geq 0$. So the result follows by (c).

(e) If $l(D) > 0$, there exists $D' \geq 0$ tel que $D' \sim D$ and $\deg D' = 0$. Therefore $D' = 0$, then $D \sim 0$.

□

Corollary 2.3.6. *The k-vector space $L(D)$ is always finite dimensional.*

In fact the hard question is to find a lower bound for $l(D)$ in terms of $\deg D$.

Example 2.3.7. *Let $X = \mathbb{P}^1(k)$, let D be any divisor of non-negative degree. Then* $l(D) = \deg D + 1$ *because* $D \sim (\deg D)[\infty]$*, and* $L(m[\infty]) = \oplus_{0 \leq i \leq m} kT^i$.

To have an exact expression of $l(D)$, we need

Theorem 2.3.8 (Riemann-Roch)**.** *Let X be an integral smooth proper curve. Then there exist an integer $g \geq 0$ and a divisor K_X on X such that for all divisors D on X, we have*

$$l(D) = \deg D + 1 - g + l(K_X - D). \tag{2.1}$$

Lemma 2.3.9. *We have $\deg K_X = 2g-2$ and $l(K_X) = g$. And the divisor K_X is unique up to linear equivalence.*

Proof. Taking $D = 0$ in (2.1), we have $1 = 1 - g + l(K_X)$. Thus $l(K_X) = g$. Taking $D = K_X$ in (2.1), we find $l(K_X) = \deg K_X + 1 - g + 1$. So $\deg K_X = 2g-2$. If (2.1) is true for a different divisor K'_X, then

$$l(K'_X) = \deg(K'_X) + 1 - g + l(K_X - K'_X)$$

with $l(K'_X) = g$ and $\deg K'_X = 2g-2$. So $\deg(K_X - K'_X) = 0$ and $l(K_X - K'_X) = 1$. Therefore $K_X \sim K'_X$. □

Definition 2.3.10. *The integer $g = l(K_X)$ is called the* genus *of X and K_X is called the* canonical divisor *on X.*

Corollary 2.3.11. *Let X be as in Theorem 2.3.8. Then for any divisor D of degree $\deg D > 2g-2$, we have $l(D) = \deg D + 1 - g$.*

Example 2.3.12. *If $X = \mathbb{P}^1(k)$, as $l(D) = \deg D + 1$ for any D of degree ≥ 0, we find $g = 0$ and $\deg K_X = -2$. So $K_X \sim -2[\infty]$.*

Exercise 2.3.1. *If $f : X \to Y$ is finite, then the pushforward of a principal divisor on X is a principal divisor on Y.*

Theorem 2.3.13. (Riemann-Hurwitz) *Let Let $f : X \to Y$ be a finite morphism of smooth, proper and geometrically connected curves over k. Let d be the degree of f. Suppose that f is separable (i.e. $k(X)/k(Y)$ is a separable extension). Then*

$$2g - 2 \geq d(2g(Y) - 2) + \sum_{x \in X} (e_{x/f(x)} - 1)[k(x) : k],$$

and the equality holds if and only if all ramification points are tamely ramified.

Definition 2.3.14. *An* étale cover *is a finite morphism which is unramified everywhere.*

Proposition 2.3.15. *Let $f : X \to Y$ be an étale cover of smooth, proper, geometrically connected curves over k. If $g(Y) = 0$, then f is an isomorphism. If $g(Y) = 1$, then $g = 1$.*

Proposition 2.3.16. *Let X be a smooth proper geometrically connected curve of genus ≥ 2. Let $f : X \to X$ be a non-constant morphism. Then f is an isomorphism.*

Proposition 2.3.17. (Luroth's theorem) *Let $f : \mathbb{P}^1_k \to Y$ be a non-constant morphism where X is a smooth proper curve over k. Then $Y \simeq \mathbb{P}^1_k$.*

2.4 Proofs of Riemann-Roch and Riemann-Hurwitz Theorems

We will explain where comes the canonical divisor K_X when X is a proper smooth curve.

2.4.1 Kähler differentials

Definition 2.4.1. *Let B a k-algebra Let F be the free B-module generated by the symbols db, $b \in B$. Let E be the submodule of F generated by the elements of the form $d\lambda$, $\lambda \in k$; $d(b_1 + b_2) - db_1 - db_2$, and $d(b_1 b_2) - b_1 db_2 - b_2 db_1$ with $b_i \in B$.*

We define $\Omega^1_{B/k} = F/E$.

Example 2.4.2. $\Omega^1_{k[T]/k} = k[T]dt$.

In the following proposition, we enumerate some properties of modules of differentials.

Exercise 2.4.1. *Let S be a multiplicative subset of B; then $S^{-1}\Omega^1_{B/k} \simeq \Omega^1_{S^{-1}B/k}$.*

Corollary 2.4.3. *Let B be a finitely generated k-algebra or a localization of such an algebra. Then $\Omega^1_{B/k}$ is finitely generated over B.*

Example 2.4.4. *Let $B = k[T_1, \ldots, T_n]$, let $F \in B$, and let $C = B/FB$. Then we have*

$$\Omega^1_{C/k} = (\oplus_{1 \le i \le n} C dT_i)/C dF,$$

with $dF = \sum_i (\partial F/\partial T_i) dT_i$. This is the quotient of a free C-module of rank n by a simple submodule (i.e., generated by one element).

Example 2.4.5. *Let K a field extension of K and let $P(T) \in K[T]$ be an irreducible separable polynomial, let $K' = K[T]/(P(T))$ with P separable. Then $\Omega^1_{K'/k} \simeq \Omega^1_{K,k} \otimes K'$.*

2.4.2 Sheaves of relative differentials

On an algebraic variety over k, we can glue the differential forms on affine open subsets and thus define the sheaf of relative differentials.

Proposition 2.4.6. *Let X be an algebraic variety over a field k. Then there exists a unique sheaf of $\mathcal{O}_X$-modules $\Omega^1_{X/k}$ on X such that for any affine open subset U of X, and any $x \in U$, we have*

$$\Omega^1_{X/k}(U) = \Omega^1_{\mathcal{O}_X(U)/k}, \quad (\Omega^1_{X/k})_x \simeq \Omega^1_{\mathcal{O}_{X,x}/k}.$$

Definition 2.4.7. *Let X be an algebraic variety over a field k. The sheaf $\Omega^1_{X/k}$ is called the* sheaf of relative differentials (or differential forms) of degree 1 of X over k. *If there is no ambiguity, we also denote $\Omega^1_{X/k}$ by Ω^1_X.*

Example 2.4.8. *If $X = \mathbb{A}^n_k$, then $\Omega^1_{X/k} \simeq \mathcal{O}^n_X$.*

Proposition 2.4.9. *Let X be a smooth variety pure of dimension d (i.e. all irreducible components of X are of dimension d). Then $\Omega^1_{X,x}$ is free of rank d on $\mathcal{O}_{X,x}$ for all $x \in X$.*

Proof. Use jacobian criterion : in a neighborhood of x, X is a closed subvariety of $\mathbb{A}^n_k$ defined by $F_{d+1}, ..., F_n \in k[T_1, ..., T_n]$. We can suppose that the matrix jacobian of $(T_1, ..., T_d, F_{d+1}, ..., F_n)$ is invertible in a neighborhood of x. This implies that $dT_1, ..., dT_d$ is a base of $\Omega^1_{X,x}$ over $\mathcal{O}_{X,x}$. □

Corollary 2.4.10. *Let X be as above. Suppose moreover that X is connected. Then $\Omega^1_{k(X)/k}$ has dimension d over $k(X)$.*

Proof. Fix $x \in X$. Then $k(X)$ is a localization of $\mathcal{O}_{X,x}$. So $\Omega^1_{k(X)/k} = S^{-1}\Omega^1_{\mathcal{O}_{X,x}/k}$, where S is the set of regular elements of $\mathcal{O}_{X,x}$. Therefore $\Omega^1_{k(X)/k} \simeq S^{-1}\mathcal{O}^d_{X,x} = k(X)^d$. □

The elements of $\Omega^1_{k(X)/k}$ are called *rational or meromorphic differentials on* X.

Let X be a smooth, proper and connected curves over an algebraically closed field k. Then $\Omega^1_{k(X)/k}$ is one-dimensional $k(X)$-vector space. Let $\omega \neq 0$. Let $x \in X$, let ω_x be a generator of $\Omega^1_{X,x}$. As $\omega \in \Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} k(X)$, we have $\omega = g_x\omega_x$ for some $g_x \in k(X)$. We denote by $v_x(\omega) = v_x(g_x)$ (it does not depends on the choice of ω_x because two generators different by a multiplicative factor in $\mathcal{O}^*_{X,x}$. Let

$$\mathrm{div}\omega = \sum_{x \in X} v_x(df)[x] \in \mathrm{Div}(X).$$

Then the canonical divisor K_X is just divω for some non-zero rational differential ω on X. If ω' is another non-zero rational differential on X, then $\omega' = h\omega$ for some $h \in k(X)^*$, hence $\mathrm{div}\omega' = \mathrm{div}\omega + \mathrm{div}h \sim \mathrm{div}\omega$. Thus (div$\omega$ is unique up to linear equivalence.

Let $h \in k(X)^*$. Then $v_x(h) + v_x(\omega) = \geq 0$ if and only if $h\omega \in \Omega^1_{X,x}$. Therefore, $h \in L(K_X)$ if and only if $h\omega \in \Omega^1_{X,x}$ for all $x \in X$. It can be easily seen that the latter is equivalent to $h\omega \in \Omega^1_{X/k}(X)$. Thus the multiplication by ω induces a k-linear isomorphism $L(K_X) \simeq \Omega^1_{X/k}(X)$.

Remark 2.4.11. *Let X be smooth, pure of dimension d. Let $x \in X$ and let $x_1, ..., x_d$ be a system of generators of $\mathfrak{m}_x\mathcal{O}_{X,x}$. Then $dx_1, ..., dx_d$ is a basis of $\Omega^1_{X,x}$ over $\mathcal{O}_{X,x}$ because it is a system of generators of $\Omega^1_{X,x}$ and the latter is free of rank d.*

2.4.3 Proof of Riemann-Roch Theorem

In the following a curve is a smooth connected projective curve X over a an algebraically field k. First of all for any divisor D over a curve X we define the sheaf $\mathcal{L}(D)$, where for any open subset U of X, $\mathcal{L}(D)(U) = \{f \in k(X) | (\mathrm{div}(f) + D)_{|U} \geq 0\}$.

We observe that $L(D) = H^0(X, \mathcal{L}(D))$. Morever $\mathcal{L}(0) = \mathcal{O}_X$.

We state the following theorems without proof.

Theorem 2.4.12. *Let $\mathcal{F}$ a sheaf of abelian groups over a topological space of dimension n. Then $H^i(X, \mathcal{F}) = 0$ for any $i > n$.*

The following is known as the Serre duality.

Theorem 2.4.13. *For any divisor D over a curve X then $H^1(X, \mathcal{L}(D)) = H^0(X, \mathcal{L}(K_X - D)) = L(K_X - D)$.*

Definition 2.4.14. *For any sheaf $\mathcal{F}$ of abelian groups over a curve X we define* [1] *the Euler-Poincaré characteristic as $\chi(\mathcal{F}) = H^0(X, \mathcal{F}) - H^1(X, \mathcal{F})$.*

Now we can give the proof of Riemann-Roch Theorem First of all we remark that we the above definition and the duality of Serre the Riemann-Roch Theorem can be rephrased as $\chi(\mathcal{L}(D)) - \chi(\mathcal{O}_X) = \deg D$. Of course the result is true for $D = 0$. So it is enough to prove that for any point $p \in X$, $\chi(\mathcal{L}(D + p)) = \chi(\mathcal{L}(D)) + 1$, since any divisor can be obtained by the divisor 0 adding or substracting points. Now we consider the exact sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + p) \to k(p) \to 0$$

where $k(p)$ is the skyscraper sheaf concentrated in p.

Now we get the result since $\chi(\mathcal{L}(D)) + \chi(k(p)) = \chi(\mathcal{L}(D + p))$. Here we use that $k(p) = i_*\mathcal{O}_p$, where i is the closed immersion of p in X and then $H^i(X, k(p)) = H^i(p, \mathcal{O}_p)$.

2.4.4 Proof of Riemann-Hurwitz Theorem

Definition 2.4.15. *Let $f : X \to Y$ be a finite morphism of integral smooth curves. We say that f is* ramified *at $x \in X$ and x is a* ramification point *of f if the extension $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is ramified. The image $f(x)$ of a ramification point is called a* branch point. *The set of ramification points (resp. branch points) is called the* ramification locus *(resp.* branch locus*) of f.*

Proposition 2.4.16. *Let $f : X \to Y$ be a finite morphism of normal connected proper curves over an algebraically closed field. If $k(X)/k(Y)$ is separable, then the ramification locus of f is finite.*

[1] when the cohomology groups are finite dimensional

Proof. Let $A = \mathcal{O}_Y(V)$ be the ring of regular functions on an affine open subset V of Y. Let $U = f^{-1}(V)$ and $B = \mathcal{O}_X(U)$. We have $k(X) = k(Y)[\theta]$ where θ is a root of some irreducible separable polynomial $P(T) \in k(Y)[T]$. As $\mathrm{Frac}(A) = k(Y)$, there exists $a_1 \in A$ non-zero such that $P(T) \in A_{a_1}$. As B is finite over A, $B = \sum_i Ae_i$, $e_i \in k(Y)[h]$. There exists $a_2 \in A$ such that the $e_i \in A_{a_2}[\theta]$. As B_a is the integral closure of A_a in $k(X)$ for any $a \in A \setminus \{0\}$, we see that replacing A by $A_{a_1a_2}$, we can suppose that $B \subseteq A[\theta]$ and θ integral over A. Hence $B = A[\theta]$.

Let $b = P'(\theta) \in B$. It is non-zero because $P(T)$ is separable. Consider the surjective homomorphism $C := A[T, 1/P'(T)]/(P(T)) \to B_b$, $T \mapsto \theta$. Let $\mathfrak{m} \in \mathrm{Spm}\, B_b$ and $\mathfrak{n} = \mathfrak{m} \cap A$. Let $k' = A/\mathfrak{n}$. Then $C/\mathfrak{n}C = k'[T, 1/\bar{P}'(T)]/(\bar{P}(T))$, where $^{-}$ means image in $k'[T]$. We then see that $C/\mathfrak{n}C$ and hence $B/\mathfrak{n}B$ is a finite sum of finite separable extensions of k. Let $\bar{m}$ be the image of $\mathfrak{m}$ in $B/\mathfrak{n}B$. Then $B_\mathfrak{m}/\mathfrak{n}B_\mathfrak{m} = (B/\mathfrak{n}B)_{\bar{\mathfrak{m}}}$ is a finite separable extension of $A/\mathfrak{n}A$. This implies that $A \to B_\mathfrak{m}$ is unramified. □

Proof of Riemann Hurwitz Theorem. It consists in comparing K_X (of degree $2g-2$) and f^*K_Y (of degree $d(2g(Y)-2)$). Since f is separable $\Omega^1_{K(Y)/k} \to \Omega^1_{K(X)/k}$ is injective (note that this is false if f is not separable).

Let $\omega = dh$ be a generator of $\Omega^1_{k(Y)/k}$ with $h \in k(Y) \subseteq k(X)$. We want to compare $\mathrm{div}\omega \in \mathrm{Div}(Y)$ with $\mathrm{div}\omega \in \mathrm{Div}(X)$. Let $x \in X$, $y = f(x)$, let π_x, π_y be respective uniformizing elements of $\mathcal{O}_{X,x}$, $\mathcal{O}_{Y,y}$. Then $d\pi_x$, $d\pi_y$ are respective generators of $\Omega^1_{X,x}$, $\Omega^1_{Y,y}$. We have $\omega = \pi_y^{n_y} u_y d\pi_y$ with $n_y = v_y(dh) \in \mathbb{Z}$, u_y invertible. There exists $a_{x/y} \in \mathcal{O}_{X,x}$ such that $d\pi_y = a_{x/y} d\pi_x$. So $v_x(dh) = e_{x/y}n_y + v_x(a_{x/y})$ and $K_X = f^*K_Y + \sum_x v_x(a_{x/f(x)})[x]$. It remains to compute $v_x(a_{x/y})$.

We have $\pi_y = \pi_x^{e_{x/y}} w_x$ for some $w_x \in \mathcal{O}^*_{X,x}$. Write $dw_x = b_x d\pi_x$ for some $b_x \in \mathcal{O}_{X,x}$. Then

$$d\pi_y = \pi_x^{e_{x/y}-1}(e_{x/y}w_x + \pi_x b_x) d\pi_x$$

Hence $v_x(a_{x/y}) \geq e_{x/y} - 1$ with equality if and only if $e_{x/y}$ is prime to $\mathrm{char}(k)$. □