Année universitaire 2024-2025
Master 2 de mathématiques
Théorie des nombres - Feuille 1

Exercise 1

Let K be a finite field. Prove that the only absolute value on K is the trivial one.

Exercise 2

a. Let $(K, |\ |)$ be a valued field. Show that $|\ |^s$ is an absolute value on K for all $s \in]0,1]$.

b. What are the $s \in \mathbb{R}_+^*$ such that $|\ |_\infty^s$ is an absolute value on $\mathbb{Q}$?

Exercise 3 : incompleteness of $\mathbb{Q}$

Let $(K, |\ |)$ be a complete valued field such that $|\ |$ is non-trivial. Using Baire's theorem, prove that K is uncountable. Deduce that $\mathbb{Q}$ is non-complete for any of its non-trivial absolute values.

Exercise 4

Let $(K, |\ |)$ be a valued field.

a. Show that if $|\ |$ is ultrametric on a subfield of K, then it is ultrametric on K.

b. Deduce that if $\mathrm{char}(K) \neq 0$, then every absolute value on K is ultrametric.

Exercise 5

Let K be a field and v be a non-trivial valuation on K. Prove that the following are equivalent :
(a) $\mathcal{O}_v$ is a principal ideal domain ;
(b) $\mathcal{O}_v$ is noetherian ;
(c) the ideal $\mathcal{M}_v$ is principal ;
(d) $v(K^*)$ is a discrete subgroup of $\mathbb{R}$.

Exercise 6

Let $(K, |\ |)$ be an ultrametric valued field.

a. Let $r \in]0, 1]$. Show that the open ball $B(1, r)$ is a subgroup of K^*.

b. Prove that each open ball in K is closed.

c. Deduce that the connected components of K are its points.

Exercise 7

Let $(K, |\ |)$ be a valued field. Let $(\widehat{K}, |\ |)$ be a complete valued field and $\iota : K \to \widehat{K}$ be a morphism of valued fields such that $\iota(K)$ is dense in $\widehat{K}$. Show that if $(L, |\ |')$ is a complete valued field and $f : K \to L$ a morphism of valued fields, then there exists a unique morphism of valued fields $\widehat{f} : \widehat{K} \to L$ such that $f = \widehat{f} \circ \iota$.

Exercise 8

Let $(K, |\ |)$ be an ultrametric locally compact valued field such that $|\ |$ isn't trivial. Prove that its residue field is finite and that $|K^*|$ is a discrete subgroup of $\mathbb{R}_+^*$.

Année universitaire 2024-2025
Master 2 de mathématiques
Théorie des nombres - Feuille 2

Exercise 1

Let $(K, |\ |)$ be an ultrametric valued field. Prove that $|K^*| = |\widehat{K}^*|$ and that the residue fields of K and of $\widehat{K}$ are isomorphic.

Exercise 2

Let p be a prime number. Show that the groups $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ and $\mathbb{Q}_p/\mathbb{Z}_p$ are isomorphic.

Exercise 3

Find examples of two complete ultrametric valued fields whose respective residue fields and value groups are isomorphic, but which aren't isomorphic as fields.

Exercise 4

Let $(K, |\ |)$ be a valued field such that $|\ |$ is non-trivial. Let V be a K-vector space. Prove that two norms $\|\ \|_1, \|\ \|_2$ on V are equivalent if and only if there exist constants c, c' in $\mathbb{R}_+^*$ such that $\forall v \in V\ c\|v\|_1 \leqslant \|v\|_2 \leqslant c'\|v\|_1$.

Exercise 5

Let K be a field endowed with the trivial absolute value $|\ |_0$.

a. Let $\rho \in]0, 1[$; put $\|f\|_\rho = \rho^{v_X(f)}$ for all $f \in K[[K]]$. Check that $\|\ \|_\rho$ is a norm on the K-vector space $K[[X]]$.

b. Let $(r, s) \in \mathbb{R}^2$ be such that $0 < r < s < 1$. Show that $\|\ \|_r$ and $\|\ \|_s$ are equivalent, but that there doesn't exist $c \in \mathbb{R}_+^*$ such that $\|\ \|_s \leqslant c\|\ \|_r$.

Exercise 6

Let $(V, \|\ \|_1)$ and $(W, \|\ \|_2)$ be normed vector spaces over a complete valued field $(K, |\ |)$. Assume that V is finite dimensional.

a. Prove that any K-linear map $f : V \to W$ is continuous.

b. Show that any surjective K-linear map $f : V \to W$ is open.

Exercise 7

α

Let $(K, |\ |)$ be a complete ultrametric valued field and $P \in \mathcal{O}_K[X]$ be a monic polynomial. Prove that if P is irreducible in $\mathcal{O}_K[X]$, its image in $\kappa_K[X]$ is the power of an irreducible polynomial.

Exercise 8

α

Consider $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$ as a $\mathbb{Q}$-vector space, endowed with the restriction $\|\ \|$ of the usual absolute value $|\ |_\infty$. Put $\|x+y\sqrt{2}\|' = \max(|x|_\infty, |y|_\infty)$ for all $(x, y) \in \mathbb{Q}^2$. Show that the norms $\|\ \|$ and $\|\ \|'$ aren't equivalent on $\mathbb{Q}(\sqrt{2})$.

Exercise 9

Let A be a local noetherian ring such that its maximal ideal $\mathcal{M}$ is generated by a non-nilpotent element π. The goal of this exercise is to prove that A is a principal ideal domain.

a. Verify that $I = \{x \in A \mid \exists m \in \mathbb{N}\ \pi^m x = 0\}$ is an ideal of A and that there exists $N \in \mathbb{N}$ such that $\pi^N I = \{0\}$.

b. Show that $\bigcap_{n\in\mathbb{N}} \mathcal{M}^n = \{0\}$. *Hint :* if $y = \pi^n x_n$ for every $n \in \mathbb{N}$, then the sequence $(I + x_n A)_{n\in\mathbb{N}}$ is ascending.

c. Prove that any element $y \in A \setminus \{0\}$ can be written in a unique way $y = \pi^{v(y)} u$ with $v(y) \in \mathbb{N}$ and $u \in A^\times$, and that A is an integral domain.

d. Conclude.

Année universitaire 2024-2025
Master 2 de mathématiques
Théorie des nombres - Feuille 3

Exercise 1

Let L be a finite extension of a field K, $x \in L$, and M be a finite extension of L. Show that $\mathrm{N}_{M/K}(x) = \mathrm{N}_{L/K}(x)^{[M:L]}$.

Exercise 2

Let p be a prime number.

a. Let $n \in \mathbb{N}$ and $n = \sum_{i=0}^{r} a_i p^i$ (with $a_i \in [[0, p-1]]$ for all i) be its writing in base p. Prove that $v_p(n!) = \dfrac{n - s_n}{p-1}$, where $s_n = \sum_{i=0}^{r} a_i$ (sum of the digits).

b. Let $x \in \mathbb{C}_p$. Deduce that the series $\sum_{n \geqslant 0} \dfrac{x^n}{n!}$ converges if and only if $v_p(x) > \dfrac{1}{p-1}$.

Exercise 3

Let p be a prime number.

a. Let $u \in \mathbb{Q}_p^*$. Show that the following are equivalent :
(a) $u \in \mathbb{Z}_p^\times$;
(b) u^{p-1} is a n-th power in $\mathbb{Q}_p$ for infinitely many $n \in \mathbb{N}^*$.

b. Prove that the only field endomorphism of $\mathbb{Q}_p$ is $\mathrm{Id}_{\mathbb{Q}_p}$. .

Exercise 4

Let p be a prime number. Show that $\mathbb{Q}_p^{*2} = \{x^2 \ ; \ x \in \mathbb{Q}_p^*\}$ is open in $\mathbb{Q}_p$.

Exercise 5

Let p be a prime number. Prove that $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid \exists y \in \mathbb{Q}_p \ y^2 = 1 + p^3 x^4\}$.

Exercise 6

$|.|_p = (\frac{1}{p})^{v_p}$

Let $(K, | \, |)$ be an ultrametric valued field, and $\rho \in \mathbb{R}_+^*$. If $P = a_0 + a_1 X + \cdots + a_n X^n \in K[X]$, put $|P|_\rho = \max_{i \in [[0,n]]} |a_i| \rho^i$. Check that $| \, |_\rho$ extends into an absolute value on $K(X)$. When are two such absolute values equivalent ?

↳ jamais car sinon ~~ex~~ $\exists s > 0$ tq... et on considère $P = X \Rightarrow \rho = 1$

Exercise 7

Show that $| \, |_\infty$ is the unique absolute value on $\mathbb{C}$ that extends the usual absolute value of $\mathbb{R}$.

Exercise 8 : Ostrowski

Let $(K, | \, |)$ be a complete valued field such that $|2| = 2$. Assume that there exists $i \in K$ such that $i^2 = -1$.

a. Construct a morphism $(\mathbb{C}, | \, |_\infty) \to (K, | \, |)$ of valued fields. We identify $\mathbb{C}$ with its image in K.

Let $a \in K$ and consider the map $f : \begin{array}{ccc} \mathbb{C} & \to & \mathbb{R}_+ \\ z & \mapsto & |z - a| \end{array}$. Let $r = \inf_{\mathbb{C}} f$.

b. Verify that $f^{-1}(r)$ is closed, bounded and non-empty.

c. Prove that if $r > 0$ and $\gamma_0 \in f^{-1}(r)$, then $B(\gamma_0, r) \subset f^{-1}(r)$. *Hint* : if $|\gamma - \gamma_0|_\infty < r$, consider $(\gamma_0 - a)^n - (\gamma_0 - \gamma)^n$.

d. Conclude that $K = \mathbb{C}$.

Année universitaire 2024-2025
Master 2 de mathématiques
Théorie des nombres - Feuille 4

Exercise 1

Let A be an integral domain and $\alpha \in A \setminus \{0\}$. Assume that $A[\alpha^{-1}]$ is integrally closed and that $A/\alpha A$ is reduced. Prove that A is integrally closed.

Exercise 2

Let B be a commutative ring and A be a subring of B such that B is integral over A.

a. Assume that B is an integral domain. Show that A is a field if and only if B is a field.

b. Let $\mathcal{Q} \subset B$ be a prime ideal. Deduce that $\mathcal{Q} \cap A$ is a maximal ideal in A if and only if $\mathcal{Q}$ is maximal in B.

c. Let $\mathcal{Q}_1 \subset \mathcal{Q}_2$ be prime ideals in B such that $\mathcal{Q}_1 \cap A = \mathcal{Q}_2 \cap A$. Prove that $\mathcal{Q}_1 = \mathcal{Q}_2$.

Exercise 3

Let $(K, |\;|)$ be a complete ultrametric valued field. Let $P = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \in K[X]$ be irreducible. Show that $|a_i|^n \leqslant |a_n|^i |a_0|^{n-i}$ for every $i \in [[0, n]]$ (*hint :* consider the splitting field of P over K). Deduce that $|P|_{\mathrm{G}} = \max(|a_0|, |a_n|)$.

Exercise 4

Let $(K, |\;|)$ be an ultrametric valued field, L be a finite extension of K and $|\;|_L$ be an absolute value on L extending $|\;|$.

a. Show that if $\alpha \in L$ is integral over $\mathcal{O}_K$, then $\alpha \in \mathcal{O}_L$ (in particular, $\mathcal{O}_K$ is integrally closed).

b. Prove that the converse holds if K is complete.

c. Show with an example that it does not hold in general.

Exercise 5

Let $(K, |\ |)$ be a complete ultrametric valued field, and L be a finite extension of K. Prove that if $\|\ \|$ is any norm on the K-vector space L, the map $x \mapsto \lim_{n \to +\infty} \|x^n\|^{1/n}$ coincides with the unique absolute value extending $|\ |$ on L.

Exercise 6

Let p be a prime number such that $p \equiv 3 \mod 4$. Let $\overline{\mathbb{Q}_p}$ be an algebraic closure of $\mathbb{Q}_p$.

a. Verify that $X^2 + 1$ is irreducible in $\mathbb{Q}_p[X]$.

Let $i \in \overline{\mathbb{Q}_p}$ be a root of X^2+1, $K = \mathbb{Q}_p[i]$ and $|\ |_K$ be the extension of $|\ |_p$ to K.

b. If $(a, b) \in \mathbb{Q}_p^2$ and $x = a + bi$, show that $|x|_K = \max(|a|_p, |b|_p)$.

c. Determine the value group and the residue field of $(K, |\ |_K)$.

Exercise 7

Let $(K, |\ |)$ be a complete ultrametric valued field and $P \in K[X] \setminus K$.

a. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements in K such that $\lim_{n \to +\infty} |P(x_n)| = 0$. Prove that there is a subsequence of $(x_n)_{n \in \mathbb{N}}$ that converges to a root of P in K.

b. Deduce that if $F \subset K$ is closed, then its image $P(F)$ is closed.

c. Show that if $C \subset K$ is compact, then its inverse image $P^{-1}(C)$ is compact.

Année universitaire 2024-2025
Master 2 de mathématiques
Théorie des nombres - Feuille 5

Exercise 1

Let $(K, |\ |)$ be an ultrametric valued field. Denote by $|\ |_G$ the Gauss absolute value on $K(X)$. What is the residue field of $(K(X), |\ |_G)$?

Exercise 2

Let $(K, |\ |)$ be a complete valued field and L be a quadratic extension of K. The goal of this exercise is to prove that the map $x \mapsto |\mathrm{N}_{L/K}(x)|^{1/2}$ is an absolute value on L that extends $|\ |$.

a. Let $x \in L \smallsetminus K$; put $\alpha = \mathrm{Tr}_{L/K}(x)$ and $\beta = \mathrm{N}_{L/K}(x)$. Assume that $4|\beta| < |\alpha|^2$. Consider the map $f : K^* \to K$ defined by $f(t) = \alpha - \dfrac{\beta}{t}$ for all $t \in K^*$ and the set $C = \{t \in K \mid 2|t| \geqslant |\alpha|\}$. Show that $f(C) \subset C$ and that $f_{|C}$ is contractive, and use the fixed point theorem to contradict the hypothesis.

b. Conclude.

Exercise 3

Prove that a complete archimedean valued field is isomorphic to $(\mathbb{R}, |\ |_\infty^s)$ or $(\mathbb{C}, |\ |_\infty^s)$ for some $s \in]0, 1]$. *Hint :* use exercise 8 of sheet 3.

Exercise 4

Let A be a semi-local Dedekind ring. Show that A is a principal ideal domain. *Hint :* use the Chinese remainder theorem.

Exercise 5

Let $\overline{\mathbb{Q}_p}$ be an algebraic closure of $\mathbb{Q}_p$. Let $P = X^3 - 17$ and $j \in \overline{\mathbb{Q}_3}$ be a primitive cubic root of unity.

a. Verify that $j \notin \mathbb{Q}_3$.

b. What are the degrees of the irreducible factors of P in $\mathbb{Q}_3[X]$? *Hint :* compute $P(5)$.

c. How many extensions to $\mathbb{Q}(\sqrt[3]{17})$ does the 3-adic absolute value have ?

Exercise 6

Let p be a prime number. Let $x \in \mathbb{Q}_p^*$ and $x = \sum_{n=v_p(x)}^{+\infty} a_n p^n$ (with $a_n \in [[0, p-1]]$ for all n) be its p-adic development. What is the p-adic development of $-x$?

Exercise 7

The p-adic absolute value on $\mathbb{Q}_p$ extends uniquely to an absolute value $|\ |_p$ on $\overline{\mathbb{Q}_p}$. For $n \in \mathbb{N}^*$, put $H_n = \{x \in \overline{\mathbb{Q}_p} \mid [\mathbb{Q}_p(x) : \mathbb{Q}_p] \leqslant n\}$.

a. Prove that H_n is closed. *Hint :* use the fact that $\mathbb{Z}_p$ is compact.

b. Show that $H_n \neq \overline{\mathbb{Q}_p}$ for all $n \in \mathbb{N}^*$.

c. Verify that $H_n + H_m \subset H_{nm}$ for all $(n, m) \in \mathbb{N}^{*2}$.

d. Using Baire's theorem, deduce that $\overline{\mathbb{Q}_p}$ isn't complete for $|\ |_p$.

$(\mathbb{Q}(j\sqrt[3]{2}), |.|_2)$ $\qquad$ $(\mathbb{Q}(\sqrt[3]{2}), |.|)$

où $|j\sqrt[3]{2}| = |j^2\sqrt{2}|_2$

$(\mathbb{Q}, |.|_2)$

$\hookrightarrow X^3 - \frac{1}{2} \in \mathbb{Q}[X]$
$\notin \mathcal{O}_{\mathbb{Q}}[X]$