Exercise Sheet 2

The field k is assumed to be algebraically closed.

Exercise 1 (Coordinate ring) Let $X \subseteq \mathbb{A}^n(k)$ be an algebraic set. One defines the *affine coordinate ring* of X:

$$k[X] := k[x_1, \ldots, x_n]/I(X).$$

1. Show that $k[X]$ is an integral domain if and only if X is an irreducible affine set.
2. Prove that a k-algebra B is isomorphic to the affine coordinate ring of some affine set if and only if B is finitely generated with no nilpotent elements.

Exercise 2 (Conics)

1. Let $P := V(x^2 - y) \subseteq \mathbb{A}^2(k)$ be the parabola and prove that $k[P] \cong k[x]$.
2. Let $H := V(xy - 1) \subseteq \mathbb{A}^2(k)$ be the hyperbola and prove that $k[H] \ncong k[x]$.

Exercise 4 (Twisted cubic curve) Let $X \subset \mathbb{A}^2_k$ be the subset defined as $X := \{(t, t^2, t^3) \mid t \in k\}$. Show that X is an irreducible algebraic set, finding the generators of $I(X)$.

Exercise 5 (Projective closure) Let $X \subseteq \mathbb{A}^n(k)$ be an affine variety. We identify $\mathbb{A}^n(k)$ with the open set $U_0 := \{[X_0 : \ldots : X_n] \in \mathbb{P}^n(k) \mid X_0 \neq 0\} \subset \mathbb{P}^n(k)$ via

$$\phi \colon U_0 \to \mathbb{A}^n(k)$$
$$[X_0 : \ldots : X_n] \mapsto \left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right) = (x_1, \ldots, x_n)$$

and consider the closure $\overline{X}$ of X in $\mathbb{P}^n(k)$ (*projective closure* of X).

1. Define
$$\eta \colon k[x_1, \ldots, x_n] \to k[X_0, \ldots, X_n]$$
$$f \mapsto X_0^{\deg f} f\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right)$$
and prove that $I(\overline{X})$ is the ideal generated by $\eta(I(X))$.
2. Let $X \subset \mathbb{A}^2(k)$ be the twisted cubic curve (Exercise 4). Find generators of $I(\overline{X})$ and deduce that, in general,
$$f_1, \ldots, f_r \text{ generate } I(X) \;\not\Rightarrow\; \eta(f_1), \ldots, \eta(f_r) \text{ generate } I(\overline{X}).$$

Exercise 6 (More conics) Let $q(x, y) \in k[x, y]$ be an irreducible quadratic polynomial, and let $C := V(q)$ be the corresponding conic in $\mathbb{A}^2(k)$. Keep the notation of Exercise 2 and show that $k[C]$ is isomorphic either to $k[P]$ or $k[H]$. Which one is it when?