Contents

Chapter 1

Algebraic varieties

Algebraic geometry studies sets of solutions of polynomial systems, e.g. for a fix $n \geq 1$, sets of $(x, y, z) \in \mathbb{Z}^3$ such that $x^n + y^n = z^n$; or $(x, y) \in \mathbb{Q}^2$ such that $y^2 = x^3 + 1$; or $(x, y, z, u) \in \mathbb{C}^4$ s.t. $x^2 + y^2 + z^2 + u^2 = xy - zu = 0$. Algebraic geometry has a long history (see J. Dieudonné, Cours de géomètrie algébrique, vol. 1, Presse Universitaire de France). Tremendous developments were made in 1950's by Serre (Faisceaux algébriques cohérents...), and then by Grothendieck and his school (theory of schemes, EGA, SGA...) and al. The present course is an introduction to this "modern" algebraic geometry.

Convention. All rings considered are commutative with a unit.

1.1 Algebraic sets, regular functions and morphisms

We try to give a first definition of affine algebraic varieties.

1.1.1 Affine algebraic sets

We start with the very basic aspect of varieties: algebraic sets. In this (set-theoretical) point of view, we deal with equations. In all this section, k is a fixed **field**. For any positive integer n, $k[T_1, ..., T_n]$ denotes the polynomial ring in n variables and with coefficients in k (it is k if $n = 0$).

Definition 1.1.1. *The* affine n-space over k*:* $\mathbb{A}^n(k)$ *(or loosely k^n) is just the Cartesian product k^n. The affine line is $\mathbb{A}^1(k)$, the affine plane is $\mathbb{A}^2(k)$.*

Definition 1.1.2. *Let k be a field. Let S be a subset of $k[T_1, \dots, T_n]$. Let*

$$Z(S) := \{x \in \mathbb{A}^n(k) : P(x) = 0, \textit{ for all } P \in S\}.$$

We call any set of this form algebraic set.

Remark 1.1.3. *Let S be a subset of $k[T_1, ..., T_n]$ and let I be the ideal of $k[T_1, ..., T_n]$ generated by S. Then $Z(S) = Z(I)$. So every algebraic set is defined by an ideal. In the following we always suppose that S is an ideal.*

Now let's play with algebraic sets in $\mathbb{A}^n(k)$.

Proposition 1.1.4. *Let $\{Z_\alpha = Z(I_\alpha)\}_\alpha$ be a collection of algebraic sets in $\mathbb{A}^n(k)$.*

(1) *If $J \supseteq I$, $Z(J) \subseteq Z(I)$ (note that we change the inclusion directions).*

(2) *$Z(0) = \mathbb{A}^n(k)$, $Z(1) = \emptyset$.*

(3) *$\cap_\alpha Z_\alpha = Z(\sum_\alpha I_\alpha)$, so it is algebraic.*

(4) *Any* finite *union $\cup_{i=1}^n Z(I_i)$ is algebraic and equal to $Z(\cap_{i=1}^n I_i) = Z(I_1 \cdots I_n)$.*

So there exists a unique topology on $\mathbb{A}^n(k)$ whose closed sets are the sets of the form $Z(I)$, with I ideals of $k[T_1, \ldots, T_n]$.

Proof. 1. It is clear.

2. It is clear.

3. The inclusion $\supseteq$ follows from 1.. For the other inclusion we remark that if $z \in \cap_\alpha Z_\alpha$ then $f(z) = 0$ for any $f \in I_\alpha$, for any α. So $f(z) =$ for any $f \in \sum_\alpha I_\alpha)$.

4. By 1. we get that $\cup_{i=1}^n Z(I_i) \subseteq Z(\cap_{i=1}^n I_i) \subseteq Z(I_1 \cdots I_n)$. Now we prove the inclusion $Z(I_1 \cdots I_n)$. Let us suppose, without loss of generality, that $z \in Z(I_1 \cdots I_n)$ but $z \notin Z(I_i)$ for $i = 1, \ldots, n-1$ and we will porve that $z \in Z(I_n)$. This means that, for $i = 1, \ldots, n-1$, there exist $f_i \in I_i$ such that $f_i(z) \neq 0$. Now let $f \in I_n$ then from one hand $(f_1 \cdots f_{n-1} f)(z) = 0$ since $z \in Z(I_1 \cdots I_n)$, on the other hand $(f_1 \cdots f_{n-1} f)(z) = (f_1 \cdots f_{n-1})(z) f(z)$, with $(f_1 \cdots f_{n-1})(z) \neq 0$, therefore $f(z) = 0$ and then $z \in Z(I_n)$.

□

Remark 1.1.5. *We observe that $Z(I) = Z(\sqrt{I})$, since $P(x) = 0$ if and only if $P^n(x) = P(x)^n = 0$ for some n. Similarly we have that $I(Z)$ is a radical ideal.*

Definition 1.1.6. *If Z is a subset of $\mathbb{A}^n(k)$ we set*

$$I(Z) := \{P \in k[X_1, \ldots, X_n] : P(x) = 0, \text{ for all } x \in Z\}.$$

It is an ideal of $k[T_1, \ldots, T_n]$

Lemma 1.1.7. *1. For any pair of affine algebraic subsets Z, W, $Z \subseteq W$ if and only if $I(W) \subseteq I(Z)$.*

2. For any family Z_λ of subsets of k^n we have $\cap_\lambda I(Z_\lambda) = I(\cup_\lambda Z_\lambda)$.

Proof. Both are clear.

□

Remark 1.1.8. *Let $Z = Z(I)$. It is clear that $\sqrt{I} \subseteq I(Z(I))$ and $Z \subseteq Z(I(Z))$.*

But in general $\sqrt{I} \neq I(Z(I))$. Over a non algebraically closed field k, Z could by empty with $I \neq k[T_0, \dots, T_n]$ However, $Z(I(Z)) = Z$ since $\sqrt{I} \subseteq I(Z)$ implies, by the previous Proposition, that $Z(I(Z)) \subseteq Z = Z(\sqrt{I})$.

For any ring A, we denote by $\operatorname{Spm} A$ the set of the maximal ideals of A, call it the *maximal spectrum* of A. Next we will describe the maximal spectrum of the polynomial rings.

Let A be a ring. We denote by $\operatorname{Spm} A$ the set of all maximal ideals of A. We call it the *maximal spectrum* of A. By convention the unit ideal is not prime. And $\operatorname{Spm}\{0\} = \emptyset$.

We will now endow $\operatorname{Spm} A$ with a topological structure. For any ideal I of A we set $V(I) = \{\mathfrak{p} \in \operatorname{Spm} A : I \subseteq \mathfrak{p}\}$. These sets will be the closed sets in the Zariski topology of $\operatorname{Spm} A$. And, if $f \in A$, we sets $D(f) = \operatorname{Spm} A \setminus V(f)$.

We prove now that they form a family of closed sets.

Lemma 1.1.9. *Let A be a ring.*

1. *For any pair I, J of ideals of A if $I \subseteq J$ we have $V(J) \subseteq V(I)$.*
2. *$V(\{0\}) = \operatorname{Spm} A$ and $V(A) = \emptyset$.*
3. *For any family I_λ of ideals of A we have $\cap_\lambda V(I_\lambda) = V(\sum_\lambda I_\lambda)$.*
4. *For any finite collections of ideals of A we have $\cup_{i=1}^n V(I_i) = V(\cap_{i=1}^n I_i) = V(I_1 \cdots I_n)$.*

So there exists a unique topology on $\operatorname{Spm} A$ whose closed sets are the sets of the form $V(I)$, with I ideals of A. And the open sets $D(f)$ constitutes a basis of open sets for this topology.

Proof.

1. This is clear.
2. This is clear.
3. The inclusion $\supseteq$ is clear. We now prove the opposite inclusion. Let $\mathfrak{p}$ be a prime ideal which contains I_λ for any λ. Then it contains $\sum_\lambda I_\lambda$ since this one is the smallest ideal which contains all I_λ.
4. It is enough to prove that $\cup_{i=1}^n V(I_i) = V(I_1 \dots I_n)$ since

$$\cup_{i=1}^n V(I_i) \subseteq V(\cap_{i=1}^n I_i) \subseteq V(I_1 \cdots I_n),$$

given that $I_1 \cdots I_n \subseteq \cap_{i=1}^n$ and $\cap_{i=1}^n$ is contained in I_i for any $i = 1, \dots, n$. Now prove the inclusion $V(I_1 \cdots I_n) \subseteq \cup_{i=1}^n V(I_i)$. We suppose that a prime ideal $\mathfrak{p}$ contains $I_1 \cdots I_n$. With no loss of generality we can suppose that it does not contain I_i, for $i = 1, \dots, n-1$. This means that there exists $a_i \in I_i$, for $i = 1, \dots, n-1$ such that $a_i \notin \mathfrak{p}$. Let $a \in J$. Then $a_1 \cdots a_{n-1} a \in I_1 \cdots I_n \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal and $a_i \notin \mathfrak{p}$ for $i = 1, \dots, n-1$, then $a \in \mathfrak{p}$. Since this is true for any $a \in J$ then we have that $J \subseteq \mathfrak{p}$.

Now it is clear, by definition, the existence of the topology demanded. For the last sentence we remark that if $U = \operatorname{Spm} A \setminus V(I)$, with I an ideal of A generated by a family of elements $\{f_\lambda\}$ then, by (3), we have that $V(I) = \cap_\lambda V(f_\lambda)$, so $U = \cup_\lambda D(f_\lambda)$. This proves the statement. □

Lemma 1.1.10. *Let k be a field and let $x = (x_1, \dots, x_n) \in k^n$. Then $\mathfrak{m}_x := (X_1 - x_1, \dots, X_n - x_n)$ is maximal and we have an injection $\psi : k^n \mapsto \mathbb{A}^n_k$ given by $x \mapsto \mathfrak{m}_x$.*

Moreover if I is an ideal of $k[X_1, \dots, X_n]$ then $x \in Z(I)$ if and only if $\mathfrak{m}_x \in V(I)$. So the algebraic sets are the closed sets for the topology over k^n induced by the Zariski topology.

Proof. We consider the evaluation map

$$\varphi_x : k[X_1, \dots, X_n] \to k$$

given by $X_i \mapsto x_i$. We now prove that its kernel is $\mathfrak{m}_x$. Let $Q(Y_1, \dots, Y_n) = P(Y_1 + x_1, \dots, P_n + x_n) \in k[Y_1, \dots, Y_n]$. Then

$$Q = Q(0, \dots, 0) + \tilde{Q}$$

with $\tilde{Q} \in (Y_1, \dots, Y_n)$. Since $P(X_1, \dots, X_n) = Q(X_1 - x_1, \dots, X_n - x_n)$, this implies that

$$P = P(x_1, \dots, x_n) + \tilde{P}$$

with $\tilde{P} \in \mathfrak{m}_x$. Then $P \in \ker \varphi$ if and only if $P \in (X_1 - x_1, \dots, X_n - x_n)$. So this ideal is maximal. Clearly ψ is injective.

Finally $x \in Z(I)$ means $I \subseteq \ker \varphi_x = \mathfrak{m}$. So $x \in Z(I)$ if and only if $\mathfrak{m}_x$. Therefore for any ideal I of $k[X_1, \dots, X_n]$ we have $\psi(Z(I)) = V(I) \cap \psi(k^n)$ which means that the induced topology on k^n by Zariski topology is given by algebraic sets. □

If by the previous lemma we identify $k^n \subseteq \mathbb{A}^n(k)$ we have simply that $Z(I) = V(I) \cap k^n$.

Theorem 1.1.11 (Nullstellensatz)**.** *Let k be an algebraically closed closed field. Let I be an ideal of $k[T_1, ..., T_n]$. Then $I(Z(I)) = \sqrt{I}$. In other word, if $F \in k[T_1, ..., T_n]$ vanishes at the commun zeros of I, then some positive power of F belongs to I.*

Corollary 1.1.12. *If k is algebraically closed and $P \in k[T_1, \dots, T_n]$ then $Z(P) = \emptyset$ if and only if $P \in k^*$.*

Corollary 1.1.13. *If k is algebraically close d we have $Z(I) \subseteq Z(J)$ if and only if $J \subseteq \sqrt{I}$.*

1.1.2 Proof of Nullstellensatz

We will use the Noether normalization lemma which has its own interest.

Definition 1.1.14. *Let A be a (commuatative unitary) ring. An A-*algebra *B is a ring endowed with a ring homomorphism $A \to B$. A* morphism of A-algebras *is a ring homomorphism compatible with the structure of A-algebra.*

We say that B is an A algebra of finite type over A *(or finitely generated) if there exists a surjective A-algebras morphism $A[T_1, \dots, T_n] \to B$.*

Here we are going to consider the case of a finitely generated algebra A over a field k, i.e. $A = k[X_1, \dots, X_n]/I$ for some n and some ideal I of $k[X_1, \dots, X_n]$. If $I = 0$ then we call $\mathrm{Spm}(k[X_1, \dots, X_n])$ the space of dimension n, $\mathbb{A}^n_k$. We first begin with some definitions.

Definition 1.1.15. *We say that a morphism of rings $\phi : A \to B$ is integral if for any element $b \in B$ there exists a monic polynomial $\sum_i a_i T^i$ such that $\sum_i \phi(a_i) b^i = 0$. And we say that the ϕ is finite if B is finite as A-module.*

We remark that ϕ is finite if and only if it is integral and B is a finitely generated A-algebra.

Proposition 1.1.16 (Emmy Noether's normalization)**.** *Let A be a finitely generated algebra over a field a k. Then there exists an integer d and an injection $k[Y_1, \dots, Y_d] \to A$ of k-algebras such that A is finite as $k[Y_1, \dots, Y_d]$-module.*

Proof. We proceed by induction on n. There is nothing to show if $n = 0$. Suppose $n \geq 1$ and that the lemma holds for $n-1$ variables. We can suppose $I \neq 0$. Let $P \in I$ be non-zero. We will first change variables to make P unitary in T_n.

Let $m_1, \dots, m_{n-1} \geq 1$ be positive integers and consider the automorphism

$$\sigma : k[T_1, \dots, T_n] \to k[T_1, \dots, T_n]$$

of k-algebras defined by $\sigma(T_i) = T_i + T_n^{m_i}$ for $i \leq n-1$ and $\sigma(T_n) = T_n$. It is enough to prove the lemma for the ideal $\sigma(I)$. We have

$$\sigma(P) = P(T_1 + T_n^{m_1}, \dots, T_{n-1} + T_n^{m_{n-1}}, T_n).$$

Claim: we can find the m_i in such a way that

$$\sigma(P) = cT_n^r + e_{r-1}(T_1, \dots, T_{n-1})T_n^{r-1} + \cdots, \quad c \in k^*, r \geq 1.$$

Thus we can suppose I contains a monic polynomial $P \in k[T_1, \dots, T_{n-1}][T_n]$. Consider the homomorphism of k-algebras

$$k[T_1, \dots, T_{n-1}]/(I \cap k[T_1, \dots, T_{n-1}]) \to k[T_1, \dots, T_n]/I.$$

It is injective, finite type and integral, thus finite. We conclude by induction on n

So, now, we prove the claim. Let $P = \sum_{\mathbf{l}\in\mathbb{N}^n} a_{\mathbf{l}}\mathbf{X}^{\mathbf{l}}$ be a non-zero element of I. We recall that if $\mathbf{l} = (l_1, \ldots, l_n)$ then $\mathbf{X}^{\mathbf{l}}$ means $X_1^{l_1} \cdots X_n{}^{l_n}$. Let $\mathbf{m} = (m_1, \ldots, m_{n-1}, 1) \in (\mathbb{N} \setminus \{0\})^n$. So we have that

$$\sigma(P) = \sum_{\mathbf{l}\in\mathbb{N}^n} a_{\mathbf{l}} X_n^{\mathbf{l}\cdot\mathbf{m}} + g(T_1, \ldots, T_{n-1}, T_n)$$

where g is a polynomial which does not contain monomials purely in T_n and such that $\deg_{T_n} g < \max\{\mathbf{l}\cdot\mathbf{m}|a_{\mathbf{l}} \neq 0\}$

Let $\tilde{l}$ be the maximum $\mathbf{l}$, for the lexicographic order, such that $a_{\mathbf{l}} \neq 0$. Now we take $\mathbf{m} = (m_1, \ldots, m_{n-1}, 1)$ such that the scalar product $\mathbf{m}\cdot\tilde{\mathbf{l}}$ is larger than $\mathbf{m}\cdot\mathbf{l}$ for any $\mathbf{l}$ such that $a_{\mathbf{l}} \neq 0$. To find $\mathbf{m}$ we can, for instance, proceed as follows. We set $\mathbf{m} = (d^{n-i})_{i=1,\ldots,n}$, with $d \in \mathbb{N}$. Then for any $\mathbf{l}$ we have that

$$\mathbf{m}\cdot\mathbf{l} = \sum_{i=1}^{n} d^{n-i} l_i.$$

So, since $\sum_{i=1}^n T^{n-i}\tilde{l}_i$ grows to infinity faster then $\sum_{i=1}^n T^{n-i}l_i$, with $a_{\mathbf{l}} \neq 0$, it is clear that if we take $d >> 0$ we have that $e = \mathbf{m}\cdot\tilde{\mathbf{l}}$ is the larger one. So we have that

$$\sigma(P) = a_{\tilde{\mathbf{l}}} T_n^e + \sum_{i=0}^{e-1} Q_i(T_1, \ldots, T_{n-1}) T_n^i.$$

as wanted.

□

Remark 1.1.17. *We will see later how to interpret this geometrically.*

Corollary 1.1.18. *Let A be a non-zero finitely generated algebra over a field k. Let $\mathfrak{m}$ be a maximal ideal. Then the field $A/\mathfrak{m}$ is a finite algebraic extension of k. In particular if k is algebraically closed then $A/\mathfrak{m} = k$.*

Proof. Since $A/\mathfrak{m}$ is a finitely generated algebra over k then by the previous Proposition there exists an positive integer r and a finite injective k-algebras morphism $\phi : k[Y_1, \ldots, Y_r] \to A/\mathfrak{m}$. Now since $A/\mathfrak{m}$ is a field then $k[Y_1, \ldots, Y_r]$ is a field by the following Lemma, which implies that $r = 0$.

□

Lemma 1.1.19. *Let $\phi : A \to B$ be an integral injection of integral domains. Then A is a field if and only if B is a field.* [1]

Proof. If part. Since B is a field then for any non-zero $a \in A$ we have $\phi(a)$ is invertible. Now ϕ is integral so there exist $a_i \in A$ for $i = 1, \ldots, n$, with $a_n = 1$, such that $\sum_{i=0}^n \phi(a_i)\frac{1}{\phi(a)^i} = 0$. If we multiply by $\phi(a)^n$ we obtain

$$1 = \phi(-\sum_{i=0}^{n-1} a_i a^{n-i})$$

[1] If B is a field then automatically A is an integral domain.

which says that a is invertible in B since ϕ is injective.

Only if part. Let $b \in B \setminus 0$. Then there exist $a_i \in A$ for $i = 1, \dots, n$, with $a_n = 1$, such that $\sum_{i=0}^{n} \phi(a_i) b^i = 0$. We chose n minimal. Since B is an integral domain and n minimal we have that $\phi(a_0) \neq 0$. but a_0 is invertible, since different from zero, then we have that

$$1 = -\frac{\sum_{i=1}^{n} \phi(a_i) b^i}{\phi(a_0)},$$

which implies that b is invertible.

□

Exercise 1.1.1. *Let k be a finite field. Then k^n with Zariski topology is a finite topological space where any point is closed. So the topology is discrete. This shows that in general it is not enough to work with algebraic sets since we lose informations. For instance we have that $(\mathbb{F}_p)^n$ is homeomorphic to $\mathbb{F}_{p^n}$ for any n.*

Lemma 1.1.20. *Let A be a non-zero finitely generated algebra over a field k and let I be an ideal of A. Then*

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{m} \in V(I), \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m}.$$

In particular $I \mapsto V(I)$ gives a bijection, inclusion reversing, between radical ideals and closed subsets of $\operatorname{Spm} A$.

Proof. We can suppose that $I = 0$ up to pass to quotient by I. The inclusion $\subseteq$ is clear. Let f be an element of $\bigcap_{\mathfrak{m} \in \operatorname{Spm} A} \mathfrak{m}$. we prove it is nilpotent. We suppose it isn't. Then we have that A_f is a non-zero ring. Then it has a maximal ideal $\mathfrak{m}$, by Zorn's lemma. Let $\varrho : A \to A_f$ the localization map. Let us consider $\varrho^{-1}(\mathfrak{m})$. Now we remark that $A_f = A[T]/(fT-1)$ is a finitely generated algebra over k. So, by 1.1.18, we have that $A_f/\mathfrak{m}$ is finite algebraic extension of k. Since we have the following injections

$$k \to A/\varrho^{-1}(\mathfrak{m}) \to A_f/\mathfrak{m}$$

and the composition is a finite algebraic extension of fields then in particular $A/\varrho^{-1}(\mathfrak{m}) \to A_f/\mathfrak{m}$ is an integral extension. Then by Lemma 1.1.19 we have that $A/\varrho^{-1}(\mathfrak{m})$ is a field and so $\varrho^{-1}(\mathfrak{m})$ is maximal. But it does not contain f, which is absurd.

The last part is immediate since we can reconstruct I by the maximal ideal which contain it.

□

Example 1.1.21. *This is false in general. For instance take R is a discrete valuation ring. Then $\sqrt{(0)} = (0)$ and there is only one maximal ideal, which is not zero.*

Corollary 1.1.22. *Let k be an algebraically closed field Then the application $\psi : k^n \to \mathbb{A}^n_k$ given by $x = (x_1, \dots, x_n) \mapsto \mathfrak{m}_x := (X_1 - x_1, \dots, X_n - x_n)$ is an homeomorphism.*

Proof. It is enough to prove that it is surjective. Let $\mathfrak{m}$ be a maximal ideal of $k[X_1, \dots, X_n]$. Then $k[X_1, \dots, X_n]/\mathfrak{m} \simeq k$ by Corollary 1.1.18 and the fact that k is algebraically closed. The quotient map is necessarily the evaluation map $P \mapsto P(x_1, \dots, x_n)$ in a point $(x_1, \dots, x_n)$. So $\mathfrak{m} = \mathfrak{m}_x = \psi(x)$. □

Remark 1.1.23. *The above result is false if k is not algebraically closed. For instance if $k = \mathbb{R}$ and $I = (T^2 + 1) \subseteq \mathbb{R}[T]$ then $Z(I) = \emptyset$ but $V(I)$ is a closed point of $\operatorname{Spm} \mathbb{R}[T]$.*

Proposition 1.1.24 (Nullstellensatz)**.** *Let k be an algebraically closed field. For any ideal I of $k[X_1, \dots, X_n]$ we have that*

$$I(Z(I)) = \sqrt{I}.$$

Proof. If $x = (x_1, \dots, x_n) \in k^n$ then $I(\{x\}) = \{f \in k[X_1, \dots, X_n] | f(x) = 0\} = \mathfrak{m}_x$. So

$$I(Z(I)) = I(\bigcup_{x \in Z(I)} \{x\}) = \bigcap_{x \in Z(I)} I(\{x\}) = \bigcap_{x \in Z(I)} \mathfrak{m}_x \overset{*}{=} \bigcap_{\substack{\mathfrak{m} \in V(I) \\ \mathfrak{m} \text{ maximal}}} \mathfrak{m} = \sqrt{I}.$$

where the equality $*$ follows by the fact that any maximal ideals is of the form $\mathfrak{m}_x$, and the last equality follows by the previous Lemma. □

Remark 1.1.25. *From the Nullstellensatz it follows for instance that, over an algebraicaly closed field, if $I \neq A$ then $Z(I) \neq \emptyset$. Indeed $Z(I) = \emptyset$ implies $\sqrt{I} = I(Z(I)) = I(\emptyset) = A$.*

Corollary 1.1.26. *Let k be an algebraically closed field then the application $Z \mapsto I(Z)$ gives a bijection, inclusion reversing, between the algebraic sets of k^n and radical ideals of $k[X_1, \dots, X_n]$ whose inverse is $I \mapsto Z(I)$.*

Proof. By Nullstellensatz we have that if I is radical then

$$I(Z(I)) = \sqrt{I} = I.$$

Moreover if Z is an algebraic set then $Z(I(Z)) = Z$. Indeed $Z = Z(I)$ for some ideal I. Then

$$Z(I(Z(I)) = Z(\sqrt{I}) = Z(I).$$

□

Let us introduce some notation. Let $Z = Z(I) \subseteq \mathbb{A}^n(k)$ with I radicial. Let

$$A(Z) = k[T_1, \dots, T_n]/I.$$

For any $f \in A(Z)$, let $\tilde{f} \in k[T_1, \ldots, T_n]$ be an arbitrary lifting of f. For any $z \in Z$, denote by

$$f(z) = \tilde{f}(z) \in k.$$

This is independent on the choice of the lifting $\tilde{f}$. Denote by

$$D(f) = D(\tilde{f}) \cap Z = \{z \in Z \mid f(z) \neq 0\}.$$

This open subset of Z is independent on the choice of $\tilde{f}$ and is called a *principal open subset of Z*.

If J is an ideal of $A(Z)$, denote

$$Z(J) = Z(\tilde{J}) \cap Z = \{z \in Z \mid f(z) = 0 \ \forall z \in J\}$$

where the ideal $\tilde{J}$ is the preimage of J in $k[T_1, \ldots, T_n]$. Corollary 1.1.13 can be easily generalized to the following :

Corollary 1.1.27. *In Z, we have*

1. $Z(J_1) \subseteq Z(J_2)$ *if and only if* $J_2 \subseteq \sqrt{J_1}$;

2. $D(f) \subseteq D(g)$ *if and only if there exists* $N \geq 1$ *and* $h \in A$ *such that* $f^N = gh$.

Lemma 1.1.28. *Let Z be an algebraic set.*

(1) *The familly of principal open subsets $\{D(f) \cap Z\}_{f \in A}$ is a basis of topology for Z.*

(2) *Let U be an open subset of Z. Then the topological space U is quasi-compact.*

Proof. (1) Any open subset U of Z is the complement of some $Z(J) \subseteq Z$ for some ideal $J \subseteq A := A(Z)$. We have $Z(J) = \cap_{f \in J} Z(f)$, so $U = \cup_{f \in J} D(f)$. Note that as A is noetherian, J is generated by finitely many $f_1, \ldots, f_m$, and then U is covered by finitely many principal open subsets $D(f_1), \ldots, D(f_m)$.

(2) Let $U = \cup_j U_j$ be an open covering, by (1) we can refine U_j into a covering by principal open subsets and restrict ourselves to the case when each U_j is a principal open subset $D(g_j)$. Thus

$$U = \cup_j D(g_j).$$

This means that $X \setminus U = \cap_j Z(g_j) = Z(\sum_j g_j A)$. As A is noetherian, $\sum_j g_j A$ is generated by finitely many g_j's : $g_{j_1}, \ldots, g_{j_r}$. This implies that

$$X \setminus U = Z(\sum_{q \leq r} g_{j_q} A) = \cap_{q \leq r} Z(g_{j_q})$$

and $U = \cup_{q \leq r} D(g_{j_q})$. This proves the quasi-compactness. $\square$

1.1.3 Topology and irreducible components

Remark 1.1.29. *The Zariski topology on $\mathbb{A}^n(k)$ is very different from the usual (say metric) topologies. It has rather few closed subsets and is far from being (T_1-)separated. When k is a topological field ($\mathbb{C}$, $\mathbb{C}_p$ etc...), we can use the stronger product topology on $\mathbb{A}^n(k)$. This being said, in general, the Zariski topology contains enough information for everyday's life. To go further Grothendieck introduced étale topology, flat topology etc...*

Example 1.1.30. *Closed subsets of $\mathbb{A}^1(k)$: all finite subsets and $\mathbb{A}^1(k)$ itself.*

Definition 1.1.31. *We want to decompose a topological space into small pieces.* Irreducible topological spaces*: X is* irreducible *if it is non-empty and can not be the union of two strict closed subsets. A closed subset of X is said to be irreducible if it is irreducible for the induced topology.* Irreducible components *$\{X_i\}_i$ of X : these are irreducible closed subsets, not strictly contained in another irreducible closed subset, and such that $X = \cup_i X_i$.*

Remark 1.1.32. *Again, usual topological spaces are not irreducible.*

Proposition 1.1.33. *An affine algebraic set Z is irreducible if and only if $I(Z)$ is a prime ideal. If k is algebraically closed then $Z(I)$ is irreducible if and only if $\sqrt{I}$ is prime.*

Exercise 1.1.2.

1. *Prove that a topological space X is irreducible if and only if any non-empty open subset is dense. Or equivalently any two non-empty open subsets have non-empty intersection.*

2. *Prove that if $Z \subseteq X$ is irreducible then $\overline{Z}$ is also irreducible.*

Using Zorn Lemma one sees that any $x \in X$ belongs to a maximal connected. and moreover by previous exercise we have that irreducible components are closed.

Proof. Let us suppose that Z is irreducible and let $f, g \in k[T_1, \ldots, T_n]$ such that $fg \in I(Z)$. Since $Z(I(Z)) = Z$ then $Z \subseteq Z((fg)) = Z((f)) \cup Z((g))$. But Z is irreducible so we can suppose that $Z \subseteq Z((f))$ then $f \in I(Z)$.

Converesely let us suppose that $I(Z)$ is prime and $Z = Z_1 \cup Z_2$ union of two nonempty affine algebraic sets such that $Z_i \neq Z$ for $i = 1, 2$. Then $I(Z) = I(Z_1) \cap I(Z_2)$. Let $f_i \in I(Z_i) \setminus I(Z)$, for $i = 1, 2$. This is possibile since $Z_i \subsetneq Z$ implies $I(Z) \subsetneq I(Z_i)$.

If k is algebraically closed then $\sqrt{I} = I(Z(I))$ so the result follows by the first part. □

Remark 1.1.34. *The second part is not true in both direction if k is not algebraically closed. Take for instance $Z = Z((0)) \in \mathbb{A}^1(\mathbb{F}_p)$ or $Z((X(X^2+1)))$ in $\mathbb{A}^1(\mathbb{R})$.*

Corollary 1.1.35. *If k is algebraically closed $\mathbb{A}^n(k)$ is irreducible.*

Lemma 1.1.36. *Let A be a noetherian ring. Then any radical ideal is a finite interesection of prime ideals.*

Proof. Consider the set of radical ideals which are not finite intersections of prime ideals and choose a maximal element I (use noetherianity). Then I is not prime. Choose $a, b \in A \setminus I$ such that $ab \in I$. Consider $\sqrt{I + aA}$ and $\sqrt{I + bA}$. They are both finite intersections of prime ideals and their intersection is I. Therefore I is also a finite intersection of prime ideals, contradiction ! □

Proposition 1.1.37. *Let k be algebraically closed. Let Z be an algebraic set in $\mathbb{A}^n(k)$. Then there are unique irreducible algebraic sets $Z_1, ..., Z_m$ such that $Z = Z_1 \cup ... \cup Z_m$, and that there is no inclusion relation between the Z_i's. The Z_i's are the irreducible components of Z.*

Proof. The uniqueness is easy. Applying Lemma 1.1.36 to $A = k[T_1, \ldots, T_n]$ and $I(Z)$, we see (*need Nullstellensatz*) that Z is a finite union of irreducible closed subsets. Now remove those which are not minimal and we are done. □

Exercise 1.1.3. *Some properties of irreducible topological spaces:*

1. *$X \neq \emptyset$ is irreducible if and only if any non-empty open subset is dense in X.*
2. *If X is irreducible, then any non-empty open subset of X is irreducible.*
3. *If $Y \subset X$, then Y is irreducible and only if its closure $\overline{Y}$ in X is irreducible.*
4. *Let $X = \cup_i U_i$ be an open covering. Then Y (supposed to be non-empty) is irreducible if and only if for all i, $Y \cap U_i$ is either empty or irreducible.*

Exercise 1.1.4.

1. *Show that the intersection of an algebraic set in $\mathbb{A}^n(k)$ with a principal open subset can be naturally identified with an algebraic set in $\mathbb{A}^{n+1}(k)$.*
2. *Give a natural structure of algebraic sets to $Sl_n(k)$ and $Gl_n(k)$.*
3. *Finite subsets of $\mathbb{A}^n(k)$ are algebraic.*
4. *Let Z be an algebraic set in $\mathbb{A}^2(k)$ and let L be a line in $\mathbb{A}^2(k)$. Show that either we have either $L \subseteq Z$ or $L \cap Z$ is finite. Show the similar statement in the projective case.*
5. *Is $\{(x, y) \in \mathbb{C}^2 | \sin x = y^2\}$ an algebraic subset in the affine plane ?*
6. *Show that $\{(a^2, ab, b^2) \in \mathbb{P}^2(k) \mid (a, b) \in \mathbb{P}^1(k)\}$ is a projective algebraic set.*
7. *Describe the image of the map*
$$\mathbb{A}^2(k) \to \mathbb{A}^2(k), \quad (x, y) \mapsto (x, xy)$$
and show that the image is not open nor closed in $\mathbb{A}^2(k)$.
8. *Let k be endowed with the Zariski topology. Show that the Zariski topology is strictly finer (has more open subsets) on k^2 than the product topology.*

1.1.4 Projective algebraic sets

In algebraic geometry, there is a class of varieties who share a lot of properties with compact complex manifolds, they are called proper algebraic varieties. A special kind of proper varieties are projective varieties.

Definition 1.1.38. *A polynomial $f \in k[T_0, ..., T_n]$ is homogeneous of degree d if $f = \sum a_{i_0 \dots i_n} T_0^{i_0} \cdots T_n^{i_n}$ with $a_{i_0 \dots i_n} \neq 0$ only if $i_0 + \cdots + i_n = d$. Any $f \in k[X_0, \dots, X_n]$ has a unique expression $f = f_0 + f_1 + \cdots + f_N$ in which f_d is homogeneous of degree d for each $d = 0, 1, \dots, N$.*

Proposition 1.1.39. *If f is homogeneous of degree d then $f(\lambda T_0, \dots, \lambda T_n) = \lambda^d f(T_0, \dots, T_n)$ for all $\lambda \in k$; if k is an infinite field then the converse also holds.*

Definition 1.1.40. *An ideal $I \subseteq k[X_0, \dots, X_n]$ is homogeneous if for all $f \in I$, the homogeneous decomposition $f = f_0 + f_1 + \cdots + f_N$ of f satisfies $f_i \in I$ for all i.*

This is equivalent to ask that I is generated by homogenous polynomials.

Let $\mathbb{P}^n(k)$ be the n-dimensional projective space over a field k Then $f \in k[T_0, \dots, T_n]$ is not a function on $\mathbb{P}^n(k)$: by definition, $\mathbb{P}^n(k) = kn+1 \setminus \{0\}/\simeq$, where $\simeq$ is the equivalence relation given by $(t_0, \dots, t_n) \simeq (\lambda t_0, \dots, \lambda t_n)$ for $\lambda \in k \setminus \{0\}$; f is a function on k^{n+1}. We denote $\pi : k^{n+1} \setminus \{0\} \to \mathbb{P}^n(k)$. Nevertheless, for $p \in \mathbb{P}^n(k)$, the condition $f(p) = 0$ is well defined provided that f is homogeneous: suppose $p = (t_0 : \cdots : t_n)$, so that $(t_0, \dots, t_n)$ is a representative in $k^{n+1} \setminus \{0\}$ of the equivalence class of p. Then since $f(\lambda t_0, \dots, \lambda t_n) = \lambda^d f(t_0, \dots, t_n) = 0$, the condition $f(p) = 0$ is independent of the choice of representative.

Definition 1.1.41. *Let I be an homogenous ideal of $k[T_0, \dots, T_n]$. We define $Z_+(I) = \{p \in \mathbb{P}^n(k) : f(p) = 0 \text{ for all } f \in I\}$. If k is infinite and $Z \subseteq \mathbb{P}^n(k)$, we define $I_+(Z) = I(\pi^{-1}Z)$. It is an homogenous ideal.*

We have that Z_+ and I_+ satisfies the same conditions of Z and I. We have Zariski topology in which closed subsets are algebraic closed subset of $\mathbb{P}^n(k)$.

We have a projective version of Nullstellensatz.

Theorem 1.1.42. *Assume that k is an algebraically closed field. Let J be an homogenous ideal. Then*

1. *$Z_+(J) = \emptyset$ if and only if $\sqrt{J} \supseteq (T_0, \dots, T_n)$ (the so called* irrelevant ideal*);*
2. *If $Z_+(J) \neq \emptyset$ then $I_+(Z_+(J)) = \sqrt{J}$.*

Proof. For a homogeneous ideal $J \subseteq k[T_0, ..., T_n]$, we consider the affine algebraic set $Z(J) \subseteq \mathbb{A}^{n+1}(k)$. Then, since J is homogeneous, $Z(J)$ is empty or $Z(J) = \pi^{-1} Z_+(J) \cup \{0\}$ Hence $Z_+(J) = \emptyset$ if and only if $Z(J) \subseteq \{0\}$ if and only if $\sqrt{J} \supseteq (T_0, \dots, T_n)$, where the last implication uses the affine Nullstellensatz. Also, if $Z_+(J) \neq \emptyset$ then $f \in I_+(Z_+(J))$ if and only if $f \in I(Z(J)) = \sqrt{J}$. □

Proposition 1.1.43. *Let k be an algebraically closed field. Let I, J be proper homogeneous ideals of $k[T_0, \dots, T_n]$. Then $Z_+(I) \subseteq Z_+(J)$ if and only if $J \subseteq \sqrt{I}$.*

Corollary 1.1.44. *$Z_+(I)$ is irreducible if and only if $\sqrt{I}$ is prime and different from the irrelevant ideal $\mathfrak{m}_0 = (T_0, \dots, T_n) \subset k[T_0, \dots, T_n]$.*

Definition 1.1.45. *Let f be an homogenous polynomial we define $D_+(f) = \{p \in \mathbb{P}^n(k) : f(p) \neq 0\} = \mathbb{P}^n(k) \setminus Z_+((f))$.*

We have $\mathbb{P}^n(k) = \cup_i D_+(T_i)$. Moreover, as in the affine case, the $D_+(F)$ form a basis for the Zariski topology.

Proposition 1.1.46. *For any $0 \leq i \leq n$, $D_+(T_i)$ is homeomorphic to $\mathbb{A}^n(k)$.*

Proof. Let us consider the function

$$\phi : \mathbb{A}^n(k) \to D_+(T_i)$$

given by

$$(x_1, \dots, x_n) \mapsto [x_1 : \dots : 1 : \dots : x_n]$$

where the 1 is at the i-th place. It is bijective and, for any homogenous polynomial $F(T_0, \dots, T_n)$, $\phi^{-1}(D_+(F)) = D(f)$ where $f(S_1, \dots, S_n) = F(S_1, \dots, 1, \dots, S_n)$ and, for any polynomial $f(S_1, \dots, S_n)$, $\phi(D(f)) = D_+(F)$ where $F(T_0, \dots, T_n) = T_i^{\deg f}(f(T_0/T_i, \dots, T_n/T_i))$. □

1.1.5 Regular functions

Let k be an algebraically closed field and let $Z = Z(I) \subseteq \mathbb{A}^n_k$ be some algebraic set with I radicial. We want to define the set of the regular functions $\mathcal{O}_Z(U)$ on any open subset U of Z. This is a particular subset of k^U. By *functions* we means maps with values in k.

There are some reasonable requirements.

(1) The restriction to Z of any polynomial function $\in k[T_1, \dots, T_n]$ should be regular on Z. So we should have a map $k[T_1, \dots, T_n] \to \mathcal{O}_Z(Z)$ given by restriction. This map factors naturally into an injective map $k[T_1, \dots, T_n]/I \to \mathcal{O}_Z(Z)$.

(2) Let $z \in Z$. Let $F \in k[T_1, \dots, T_n]$ such that $F(z) \neq 0$. Then $F(z') \neq 0$ for any z' in the open neighborhood $D(F) \cap Z$ of z in Z. So $1/F$ should be regarded as a regular function. Combining with (1), we see that $G/F \in k(T_1, \dots, T_n)$ should be considered as regular in an open neighborhood of z. Again, this makes sense modulo I, so the classe of G/F in $k[T_1, \dots, T_n, 1/F]/I)$ should be regular at z.

(3) Note that $D(F) \cap Z$ is naturally an algebraic subset in k^{n+1} with variables $k[T_1, \dots, T_n, S]$ and the relation $FS - 1 = 0$. So (1) implies that the class of S should be a regular function on $D(F) \cap Z$ and this coincides with (2).

Definition 1.1.47. *Let $Z = Z(I)$ be an affine algebraic set in $\mathbb{A}^n(k)$ with I radicial. Let U be an open subset of Z. A* regular function *on U is a map $f : U \to k$ such that for any $z \in U$, there exist $F, G \in k[T_1, \ldots, T_n]$ such that $F(z) \neq 0$ and $f(z') = G(z')/F(z')$ for all z' in some open neighborhood of z contained in $D(F) \cap U$.*

We will denote by $\mathcal{O}_Z(U)$ the k-algebra of regular functions on U. We observe that $\mathcal{O}_Z(U)$ is naturally a k-algebra and if V is an open subset in U, then the restriction of maps takes elements of $\mathcal{O}_Z(U)$ to elements of $\mathcal{O}_Z(V)$. Note that in general $\mathcal{O}_Z(U)$ is not of finite type over k.

We have a natural map $k[T_1, \ldots, T_n] \to \mathcal{O}_Z(Z)$ whose kernel is exactly I (radicial). So we get a natural injective homomorphism of k-algebras

$$A(Z) := k[T_1, \ldots, T_n]/I \to \mathcal{O}_Z(Z).$$

Proposition 1.1.48. *Let $Z = Z(I) \subseteq \mathbb{A}^n(k)$ be an algebraic set defined by a radicial ideal I.*

(1) *Let $U \subseteq Z$ be an open subset. Then any regular function $U \to k$ is continuous for the Zariski topology.*

(2) *For any $f \in A(Z)$, the natural map $A(Z) \to \mathcal{O}_Z(D(f))$ induces an isomorphism $A(Z)_f \to \mathcal{O}_Z(D(f))$, where $A(Z)_f = A(Z)[S]/(fS-1)$ and the kernel is $\{g \in A(Z) | fg = 0\}$*

(3) *The natural map $A(Z) \to \mathcal{O}_Z(Z)$ above is an isomorphism.*

Proof. (1) Let h be a regular function on U. Let $z \in U$. Then there exists $z \in V \subseteq U$ and $F, G \in k[T_1, \ldots, T_n]$ such that $V \subseteq D(F)$ and $h = G/F$ on V. It is enough to show that the function $h_0 = G/F$ is continuous on $D(f)$ (where f is the image of F is $A(Z)$). The proper closed subsets of k are finite, so it is enough to show that $h_0^{-1}(\lambda)$ is closed in $D(f)$ for any $\lambda \in k$. But this set is nothing but $\{z \in D(f) \mid (g - \lambda f)(z) = 0\}$. So it is closed in $D(f)$.

(2) The statement about the kernel is clear. The above injection allows us to view f as a regular function on Z. Its restriction on $D(f)$ never vanishes. If we look at the definition of regular functions, we see that $1/f : D(f) \to k$, $z \mapsto 1/f(z)$ is a regular function on $D(f)$. Consider the homomorphism of k-algebras $A(Z)[S] \to \mathcal{O}_Z(D(f))$ which takes S to $1/f$. Then it factors through the quotient $A(Z)[S]/(fS-1) \to \mathcal{O}_Z(D(f))$. Now let $g \in \mathcal{O}_Z(D(f))$ and let us take $D(f) = \cup_{i=1}^n D(f_i)$ such that $g_{|D(f_i)} = g_i/f_i$. The last condition implies that $gf_i^2 = g_i f_i$ in $\mathcal{O}_Z(Z)$. Now $D(f) = \cup_{i=1}^n D(f_i^2)$ and therefore there exists r such that $f^r = \sum_i^n a_i f_i^2$ therefore $gf^r = \sum_i^n a_i g_i f_i$ which means that $g \in A[Z]_f$.

(3) This follows from 2 with $f = 1$.

□

Example 1.1.49. *1. If $Z = \mathbb{A}_k^n$ and $g \in k[T_1, \ldots, T_n]$, then $\mathcal{O}_Z(D(g)) = k[T_1, \ldots, T_n, 1/g] = k[T_1, \ldots, T_n, S]/(gS-1)$. This also corresponds to the fact $D(g)$ is an algebraic set in k^{n+1}.*

2. If $Z = \mathbb{A}_k^2$ and $U = Z \setminus \{(0,0)\}$. Then the restriction $\mathcal{O}_Z(Z) = k[T_1, T_2] \to \mathcal{O}_Z(U)$ is an isomorphism.

1.1.6 Morphisms of affine algebraic sets

The topological spaces $\mathbb{A}^1(k)$ and $Z(y^2-x^3) \subset \mathbb{A}^2(k)$ are homeomorphic by the map $t \mapsto (t^2, t^3)$. But visually they are different. The second has a singular point (a cusp). What make them different k and $Z(y^2-x^3) \subset k^2$? The answer is that they are not isomorphic as algebraic sets. To give a sense to this claim, we have to define morphisms of algebraic sets. There are some natural requirements:

(1) $\mathrm{Mor}(Z, \mathbb{A}^1(k)) = \mathcal{O}_Z(Z)$;

(2) $\mathrm{Mor}(Z, \mathbb{A}^m(k)) = \mathcal{O}_Z(Z)^n = \mathrm{Hom}_{k-\mathrm{alg.}}(k[S_1, \ldots, S_m], \mathcal{O}_Z(Z))$; Note that for $f : Z \to \mathbb{A}^m(k)$, the corresponding homomorphism $\varphi : k[S_1, \ldots, S_m] \to \mathcal{O}_Z(Z)$ takes S_i to f_i the i-th coordinate of f.

(3) If $Z' = Z(J) \subseteq \mathbb{A}^m(k)$, then $f(Z) \subseteq Z'$ if and only if $J \subseteq \ker \varphi$. So $\mathrm{Mor}(Z, Z') = \mathrm{Hom}_{k-\mathrm{alg.}}(\mathcal{O}_{Z'}(Z'), \mathcal{O}_Z(Z))$;

Definition 1.1.50. *A morphism of affine algebraic sets*

$$f : Z = Z(I) \to Y = Z(J), \quad I \subseteq k[T_1, \ldots, T_n],\ J \subseteq k[S_1, \ldots, S_m]$$

(with I, J radicial) is given by

$$f(z) = (F_1(z), \ldots, F_m(z))$$

for some $F_1, \ldots, F_m \in k[T_1, \ldots, T_n]$ such that $f(z) \in Y$, for any $z \in Z$. So f is just the restriction to $Z(I)$ of a polynomial function $k^n \to k^m$.

It is clear that the composition of morphisms of algebraic sets (when applicable) is again a morphism of algebraic set and the identity on Z is a morphism. So we have the notion of isomorphisms of algebraic sets.

An bijective morphism of algebraic sets, even when it is a homeomorphism, is not necessarily an isomorphism.

Let f be a morphism as above. We are going to associate a homomorphism of k-algebras $\varphi : A(Y) \to A(Z)$ as follows. Let $\Phi : k[S_1, \ldots, S_m] \to k[T_1, \ldots, T_n]$ be the unique homomorphism of k-algebras defined by $\Phi(S_j) = F_j(T_1, \ldots, T_n)$ for all $j \leq m$. Let $z \in \mathbb{A}^n(k)$ and let $P(S_1, \ldots, S_m) \in k[S_1, \ldots, S_m]$. Then

$$P(f(z)) = P(F_1(z), \ldots, F_m(z)) = P(F_1, \ldots, F_m)(z) = \Phi(P)(z) \tag{1.1}$$

(one can write $P = \sum \alpha_v S_1^{\nu_1} \ldots S_m^{\nu_m}$ and check directly the above equality). Thus for all $z \in Z$ and for all $P \in J$, we have $\Phi(P)(z) = 0$. Therefore $\Phi(J) \subseteq I$ and Φ induces a unique commutative diagram

$$\begin{tikzcd} k[S_1, \ldots, S_m] \arrow[r, "\Phi"] \arrow[d] & k[T_1, \ldots, T_n] \arrow[d] \\ A(Y) \arrow[r, "\varphi"] & A(Z) \end{tikzcd}$$

with a homomorphism of k-algebras φ. Moreover for any $h \in A(Y)$ we have

$$h(f(z)) = \varphi(h)(z), \quad \forall z \in Z \tag{1.2}$$

In other words, $h \circ f = \varphi(h)$ in $A(Z)$.

Proposition 1.1.51. *The functor from the category of affine algebraic sets to the category of reduced k-algebra of finite type given by $Z \to A(Z)$, is anti-equivalence of category.*

Proof. Let $\varphi : k[S_1, \ldots, S_m]/J \to k[T_1, \ldots, T_n]/I$ be a k-algebra homomorphism. Let $F_j \in k[T_1, \ldots, T_n]$ be a lifting of $\varphi(\bar{S}_j)$. Define $f : Z \to \mathbb{A}_k^m$ by $f(z) = (F_1(z), \ldots, F_m(z))$. Let us prove $f(Z) \subseteq Y$ (this will imply that f is a morphism from Z to Y). The map φ fits in a commutative diagram

$$\begin{tikzcd} k[S_1, \ldots, S_m] \arrow[r, "\Phi"] \arrow[d] & k[T_1, \ldots, T_n] \arrow[d] \\ A(Y) \arrow[r, "\varphi"] & A(Z) \end{tikzcd}$$

For any $P \in J$, we have

$$P(f(z)) = P(F_1(z), \ldots, F_m(z)) = P(F_1, \ldots, F_m)(z) = \Phi(P)(z) = 0.$$

So $f(z) \in Y$ and $f \in \mathrm{Mor}(Z, Y)$. It is straightforward to check that $\varphi \mapsto f$ is the inverse construction of the construction $f \mapsto \varphi$. $\square$

Corollary 1.1.52. *Two algebraic sets Y, Z are isomorphic if and only if the k-algebras $A(Z)$ and $A(Y)$ are isomorphic.*

Example 1.1.53. *The map $t \mapsto (t^2, t^3)$ induces a homeomorphism*

$$\mathbb{A}^1(k) \to Z := Z(y^2 - x^3) \subset \mathbb{A}^2(k).$$

But the two algebraic sets are not isomorphic because the $A(Z)$ is not a principal ideal domain (the ideal generated by x, y in $A(Z)$ is not principal).

Any algebraic set Z induces a reduced k-algebra of finite type $A(Z)$. Conversely, any reduced k-algebra of finite type A is isomorphic to $A(Z)$ for some (not unique) Z. Moreover, morphisms of algebraic sets correspond to homomorphisms of k-algebras. So, up to isomorphisms, algebraic sets are equivalent to reduced k-algebras of finite type.

1.1.7 Exercises

Exercise 1.1.5. *Let $A = k[T_0, \ldots, T_n]$ and denote by A_d the vector space of homogeneous polynomials of degree d.*

1. *Show that an ideal I of A is homogeneous if and only if $I = \oplus_{d \geq 0} I \cap A_d$.*

2. *If I is homogeneous, show that $\sqrt{I}$ is homogeneous.*

3. *If I, J are homogeneous, show that $I + J, IJ$ and $I \cap J$ are homogeneous.*

4. *Let I be an ideal, let $I^h = \oplus_{d \geq 0} I \cap A_d$. Show that I^h is the homogeneous ideal generated by the homogeneous elements of I. Show that if I is prime then so is I^h.*

5. *If I is homogeneous, show that any prime ideal over I (prime ideal minimal among those containing I) is homogeneous and that I is a finite intersection of homogeneous prime ideals.*

Exercise 1.1.6. *Show that*

$$\mathbb{P}^1(k) \to \mathbb{P}^2(k), \quad (u; v) \mapsto (u^2; uv; v^2)$$

is well-defined an is a homeomorphism onto $Z_+(xz - y^2)$.

Exercise 1.1.7. *Find the irreducible components of $Z(xy - z^2, xz - x) \subset \mathbb{A}^3(k)$ and that of $Z_+(xy - z^2, xz - xt) \subset \mathbb{P}^3(k)$.*

Exercise 1.1.8. *Let $\phi : A \to B$ be a finite homomorphism (of k-algebras of finite type, but this does not matter). Let $\mathfrak{m}$ be a maximal ideal of A. Then the set of maximal ideals of B containing $\phi(\mathfrak{m})$ is finite.*

Exercise 1.1.9. *Let $f : X \to Y$ be a continuous map of topological spaces. Suppose X is non-empty.*

1. *X irreducible implies $f(X)$ is irreducible and X is connected.*

2. *X is irreducible iff any non-empty open subset of X is dense in X. Then any non-empty open subset of X is irreducible.*

3. *If Z is a subspace of X. Then Z is irreducible iff $\overline{Z}$ is irreducible.*

4. *if X, Y are irreducible, then the product topological space $X \times Y$ is irreducible.*

Exercise 1.1.10. *Let $P, Q \in k[x]$ be separable of different degrees. Show that $k[x, 1/P]$ is not isomorphic to $k[x, 1/Q]$.*

Exercise 1.1.11. *Show that any finite subset of $\mathbb{A}^2(k)$ is defined by two equations.*

Exercise 1.1.12. *Points of $\mathbb{P}^n(k)$. Consider the set $\operatorname{Proj} k[T_0, \dots, T_n]$ of homogeneous prime ideals, different from $(T_0, \dots, T_n)$, and maximal for this property.*

1. *Let $(a_0, \dots, a_n) \in \mathbb{P}^n(k)$. Show that $(a_i T_j - a_j T_i)_{i,j} \in \operatorname{Proj} k[T]$ (consider the quotient ring).*

2. *Let $f \in k[T_0, \dots, T_n]$ be homogeneous. Show that $f(a_0, \dots, a_n) = 0$ if and only if $f \in (a_i T_j - a_j T_i)_{0 \leq i,j \leq n}$.*

3. *Show the above processus defines an injective map $\mathbb{P}^n(k) \to \operatorname{Proj} k[T]$.*

4. *Show the above map is surjective (localize at some T_i).*

1.2 Abstract algebraic varieties

1.2.1 Ringed topological spaces

Definition 1.2.1. *Let k be field. A ringed topological space X over k consists of a topological space $|X|$ endowed with a sheaf $\mathcal{O}_X$ of k-algebras. We denote it $(|X|, \mathcal{O}_X)$ and $\mathcal{O}_X$ is called the structure sheaf.*

A ringed topological space is called locally ringed topological space if for any $x \in |X|$ the stalk $\mathcal{O}_{X,x}$ is a local k-algebra.

Definition 1.2.2. *A morphism $(|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$ of ringed topological spaces is a pair $(|f|, f^{\#})$ of a continuous function $|f| : |X| \to |Y|$ and of a morphism of sheaves $f^{\#} : \mathcal{O}_Y \to |f|_* \mathcal{O}_X$.*

A morphism of locally ringed spaces is a morphism $f = (|f|, f^{\#})$ of ringed spaces such that for any $x \in X$, if $y = |f|(x)$, the natural induced morphism

$$f_y^{\#} : \mathcal{O}_{Y,y} \to (|f|_* \mathcal{O}_X)_y \to \mathcal{O}_{X,x}$$

is a morphism of local rings, i.e. $f_y^{\#}(\mathfrak{m}_y) \subseteq \mathfrak{m}_x$, where $\mathfrak{m}_y$ and $\mathfrak{m}_x$ are, respectively, the maximal ideals of $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$.

So we have the category of locally ringed topological spaces which we note by **LocRingSp**. Of course the same is true for ringed spaces. The composition is defined as follows. Let $f = (|f|, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g = (|g|, g^{\#}) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ two morphisms of locally ringed spaces. Then we define the composition as

$$(|g| \circ |f|, g_* f^{\#} \circ g^{\#}) : (|X|, \mathcal{O}_X) \to (|Z|, \mathcal{O}_Z).$$

If $X = (|X|, \mathcal{O}_X)$ is a ringed topological space over k, then any open subset U of X inherits a structure of ringed topological space, the sheaf $\mathcal{O}_U := \mathcal{O}_X|_U$ being defined by $\mathcal{O}_U(V) = \mathcal{O}_X(V)$ for all $V \subseteq U$. An open subset U is always endowed with this sheaf of functions unless the contrary is stated. A morphism $f : X \to Y$ such that $f(X) = V$ is open and $f : X \to V$ is an isomorphism is called an *open immersion.*

Example 1.2.3. *Let X be an affine algebraic set and let $\mathcal{O}_X(U)$ be the set of the regular functions on U. Then $(X, \mathcal{O}_X)$ is a ringed topological space over k.*

Proposition 1.2.4. *Let k be algebraically closed. The natural functor, given by $Z \mapsto (Z, \mathcal{O}_Z)$, from the category affine algebraic sets to the category of locally ringed space is an equivalence of categories.*

Proof. Let $f : X \to Y$ be a morphism of algebraic sets. Let V be an open subset of $Y \subseteq \mathbb{A}^m(k)$ and let $s \in \mathcal{O}_Y(V)$. We have to show that $s \circ f \in \mathcal{O}_X(f^{-1}(V))$. Let $\varphi : A(Y) \to A(X)$ be the homomorphism of k-algebras corresponding to f, induced by some $\Phi : k[S_1, \dots, S_m] \to k[T_1, \dots, T_n]$. Let $x_0 \in f^{-1}(V)$. By definition, there exists an open neighborhood $f(x_0) \in V_0 \subseteq V$ and $G, H \in$

$k[S_1, \ldots, S_m]$ such that $H(y) \neq 0$ for all $y \in V_0$ and that $s(y) = G(y)/H(y)$. For all $x \in f^{-1}(V_0)$, we have $\Phi(H)(x) = H(f(x)) \neq 0$ and

$$s(f(x)) = G(f(x))/H(f(x)) = \Phi(G)(x)/\Phi(H)(x)$$

by the relation (1.1). So $s \circ f$ is defined by a rational function without pole in a neighborhood of x_0 and $s \circ f \in \mathcal{O}_X(f^{-1}(V))$. Therefore f is a morphism of locally ringed topological spaces.

Let $(|f|, f^\sharp) : X \to Y$ be a morphism of locally ringed topological spaces. The composition with the inclusion $Y \subseteq \mathbb{A}^m(k)$ is a morphism of locally ringed topological spaces and it is enough to show that $f : X \to \mathbb{A}^m(k)$ is a defined by a polynomial map. Write $X = Z(I) \subseteq \mathbb{A}^n(k)$ and $S_1, \ldots, S_m$ be the coordinate functions on $\mathbb{A}^m(k)$. First of all, since it is a morphism of locally ringed space then $f^\sharp(g) = g \circ f$ for any $g \in k[T_1, \ldots, T_n]$.

Then

$$f_j := f^\sharp(S_j) = S_j \circ f \in A(X) = k[T_1, \ldots, T_n]/I$$

for all $j = 1, \ldots, m$. Let $F_j \in k[T_1, \ldots, T_n]$ be a lifting of f_j. For any $x \in X \subseteq \mathbb{A}^n(k) = k^n$ and for any $j \leq m$ we have

$$F_j(x) = f_j(x) = (S_j \circ f)(x) = S_j(f(x))$$

is the j-th coordinate of $f(x) \in \mathbb{A}^m(k)$. So the polynomial map $(F_1, \ldots, F_m) : \mathbb{A}^n(k) \to \mathbb{A}^m(k)$ coincides with f on X and the proposition is proved. $\square$

Definition 1.2.5. *An* affine algebraic variety over k *is a ringed topological space over k isomorphic to $(X, \mathcal{O}_X)$ for some affine algebraic set X over k.*

Note that in Hartshorne, an affine variety is required to be irreducible. Here we do not impose this condition.

Lemma 1.2.6. *Let X be an algebraic set. Let $f \in A(X)$. Then $(D(f), \mathcal{O}_X|_{D(f)})$ is an affine variety.*

Proof. Write $X = Z(I) \subseteq \mathbb{A}^n(k)$, $F \in k[T_1, \ldots, T_n]$ a lifting of f and let $Z = Z(I, FS - 1) \subseteq \mathbb{A}^{n+1}(k)$. The projection $\mathbb{A}^{n+1}(k) \to \mathbb{A}^n(k)$ to the n-th first coordinates induces a morphisms of algebraic sets $p : Z \to \mathbb{A}^n(k)$. For points $(t, s) \in \mathbb{A}^{n+1}(k)$ with $t \in \mathbb{A}^n(k)$, we have $(t, s) \in Z$ if and only if $t \in X$ and $sf(t) = 1$. This implies easily that $p(Z) = D(f)$. We are going to prove that p is an isomorphism of ringed topological spaces $Z \to D(f)$.

First it is clear that $p : Z \to D(f)$ is bijective. Let us show it is a homeomorphism.It is enough to show that p is open, or that it maps a principal open subset onto a principal open subset. Let $G(\underline{T}, S) \in k[T_1, \ldots, T_n, S]$. Using Euclidean division by $FS - 1 \in K[\underline{T}][S]$, we get a relation

$$F^N G(\underline{T}, S) = (FS - 1)H(\underline{T}, S) + R(\underline{T})$$

for some $N \geq 1$. So $(t, s) \in D(G) \cap Z$ if and only if $t \in X$ and $R(t) \neq 0$, thus $p(D(G) \cap Z) = D(R) \cap X$. This implies that p is an open map, hence a homeomorphism.

To show that p is an isomorphism, we need to show that for any open subset V of X contained in $D(f)$, the composition with p induces an isomorphism $\mathcal{O}_X(V) \to \mathcal{O}_Z(p^{-1}(V))$ of k-algebras. Again one can restrict to the case when $V = D(r)$ for some $r \in A(X)$. Let $R \in k[\underline{T}]$ be a lifting of r. Then $p^{-1}(D(r)) = D(R) \cap Z = D(r')$ where $r' \in A(Z)$ is the image of r by $\varphi : A(X) \to A(Z)$. The composition with p induces a commutative diagram of homomorphisms of k-algebras

$$\begin{tikzcd} A(X) \arrow[r, "\varphi"] \arrow[d] & A(Z) \arrow[d] \\ \mathcal{O}_X(D(r)) \arrow[r, "\rho"] & \mathcal{O}_Z(D(r')) \end{tikzcd}$$

where the vertical arrows are the restriction maps (use Proposition 1.2.4). It remains to show that ρ is an isomorphism. The above commutative diagram can be explicitely described as

$$\begin{tikzcd} A(X) \arrow[r, "\varphi"] \arrow[d] & A(X)[S]/(fS-1) = A(Z) \arrow[d] \\ A(X)[W]/(rW-1) \arrow[r, "\rho"] & A(Z)[U]/(rU-1) \end{tikzcd}$$

where φ and the vertical arrows are the canonical ones. Let w, u be the respective images of W and U in the quotients. We have $rw = 1$, so $r\rho(w) = 1$ and $\rho(w) = u$. Then ρ can be interpreted as the natural map

$$A(X)[W]/(rW-1) \to (A(X)[W]/(rW-1))[S]/(fS-1).$$

The condition $D(r) \subseteq D(f)$ in X implies that $r^N = fg$ for some $N \geq 1$ and some $g \in A(X)$ (see Corollary 1.1.27(2)). This implies that f is invertible in $A(X)[W]/(rW-1)$. We can now apply the next lemma to conclude. □

Lemma 1.2.7. *Let A be a ring and let $f \in A$. If f is invertible, then the canonical homomorphism $A \to A[S]/(fS-1)$ is an isomorphism.*

Proof. Indeed, $(fS-1) = (S-f^{-1})$ as ideals and $A \to A[S]/(S-f^{-1})$ is an isomorphism. □

Definition 1.2.8. *An* algebraic variety over k *is a locally ringed topological space $(X, \mathcal{O}_X)$ over k such that there is a finite open covering $X = \cup_i X_i$ where $(X_i, \mathcal{O}_X|_{X_i})$ is an affine algebraic variety for all i.*

Remark 1.2.9. *Equivalently, an algebraic variety over $k$is a locally ringed topological space $(X, \mathcal{O}_X)$ over k such that $|X|$ is quasi compact and there is an open covering $X = \cup_i X_i$ where $(X_i, \mathcal{O}_X|_{X_i})$ is an affine algebraic variety for all i.*

Proposition 1.2.10. *If X is affine, then any open subset $U \subseteq X$ is an algebraic variety.*

Proof. It is enough to show that for any principal open subset $D(f) \subseteq X$, the space $(D(f), \mathcal{O}_X|_{D(f)})$ is affine. But this is just Lemma 1.2.6. $\square$

Remark 1.2.11. *In general, U is not an affine variety.*

Corollary 1.2.12. *Let X be an algebraic variety, let U be an open subset. Then U is an algebraic variety.*

Proposition 1.2.13. *Let k be algebraically closed. For any algebraic variety X and an affine algebraic variety Y we have a natural bijection*

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

In particular $\mathcal{O}_X(X) = \mathrm{Hom}(X, \mathbb{A}^1(k))$.

1.2.2 Constructing algebraic varieties by glueing

This is the method of charts and altas.

Let $X_1, \ldots, X_n$ be algebraic varieties over k. Suppose we are given for any pair $i, j \leq n$, an open subset X_{ij} of X_i and an isomorphism (of algebraic varieties) $f_{ij} : X_{ij} \to X_{ji}$ such that the diagram

$$\begin{tikzcd}
X_{ij} \cap X_{iq} \arrow[rr, "f_{ij}|_{X_{ij} \cap X_{iq}}"] \arrow[dr, "f_{iq}"'] & & X_{ji} \cap X_{jq} \\
& X_{qi} \cap X_{qj} \arrow[ur, "f_{qj}"'] &
\end{tikzcd}$$

is commutative for all $i, j, q \leq n$. Then there exist an algebraic variety X and open immersion $f_i : X_i \to X$ such that $X = \cup_i f_i(X_i)$ and

$$f_i|_{X_{ij}} = f_j|_{X_{ji}} \circ f_{ij}, \quad i, j \leq n.$$

Moreover, if $X', f_1', \ldots, f_n'$ are another tuple satisfying the same conditions, then there exists a unique isomorphism $f : X \to X'$ such that $f_i = f \circ f_i'$ for all $i \leq n$.

Example 1.2.14. *Let $X_1 = \mathbb{A}^1(k)$ with coordinate function t and $X_2 = \mathbb{A}^1(k)$ with coordinate function s, $X_{12} = D(t)$ and $X_{21} = D(s)$. Then there are two different isomorphisms*

$$f_{12} : X_{12} \to X_{21}, \quad s \mapsto t;$$

$$g_{12} : X_{12} \to X_{21}, \quad s \mapsto 1/t.$$

The glueing with the first isomorphism gives rise to the "line with a doubled point" and the second gives $\mathbb{P}^1(k)$.

1.2.3 Subvarieties

Definition 1.2.15. *Let $(X, \mathcal{O}_X)$ be an algebraic variety. An* open subvariety *if an open subset U of X endowed with the sheaf of regular functions $\mathcal{O}_X|_U$.*

An open subset U of X is called affine open subset *if the corresponding open subvariety is affine.*

An affine open covering *of X is an open covering of X by affine open subsets.*

Let Z be a closed subset of X. Define a presheaf $\mathcal{O}_Z$ on Z as follows:

Let $V \subseteq Z$ be an open subset of Z. A function $f : V \to k$ belongs to $\mathcal{O}_Z(V)$ if for any $z \in V$, there exists an open neighborhood U of z in X and $g \in \mathcal{O}_X(U)$ such that $U \cap Z \subseteq V$ and $f(z') = g(z')$ for all $z' \in U \cap Z$.

Lemma 1.2.16. *Let Z be a closed subset of X. Then $\mathcal{O}_Z$ is a sheaf and $(Z, \mathcal{O}_Z)$ is an algebraic variety. If X is affine, then so is Z.*

Proof. It is clearly a sheaf. Now suppose X is affine $Z(I) \subseteq \mathbb{A}^n(k)$ and Z is given by $Z = Z(J) \cap X \subseteq \mathbb{A}^n(k)$. Then it is easy to check that $\mathcal{O}_Z$ is the sheaf of regular functions on the algebraic set $Z(I, J) \subseteq \mathbb{A}^n(k)$. □

Definition 1.2.17. *A* closed subvariety of X *is a closed subset Z of X endowed with the sheaf $\mathcal{O}_Z$ defined as above.*

We now define projective varieties as closed subvarieties of $\mathbb{P}^n(k)$. Let X be a projective algebraic set in $\mathbb{P}^n(k)$ defined by a radicial homogeneous ideal $I \subseteq k[T_0, \ldots, T_n]$. Let us define a sheaf $\mathcal{O}_X$ on X. Let U be an open subset of X. Let $f : U \to k$ be a function on U. We will say that f is a regular function if for any $x \in U$, there exist an open neighborhood $x \in V \subseteq U$ and homogeneous polynomials $F(T_0, \ldots, T_n), G(T_0, \ldots, T_n)$ of the same degree such that $G(t) \neq 0$ and $f(t) = F(t)/G(t)$ for all $t \in V$. This defines a sheaf of k-valued regular functions $\mathcal{O}_X$ on X and $(X, \mathcal{O}_X)$ is a locally ringed space.

Let $B = k[T_0, \ldots, T_n]/I$. Let $P \in k[T_0, \ldots, T_n]$ be homogeneous of degree $d \geq 0$ and let $p \in B$ be the image of P in B (such a p is called a *homogeneous element of B of degree p*). We denote by

$$k[T_0, \ldots, T_n]_{(P)} = \{\frac{Q}{P^r} \in k(T_0, \ldots, T_n) \ \mid \ r \geq 0,\ Q \in k_{rd}[T_0, \ldots, T_n]\},$$

(this is a k-algebra of finite type, generated by M/P for monomials M of degree d),

$$I_{(P)} = \{\frac{Q}{P^r} \in k[T_0, \ldots, T_n]_{(P)} \ \mid \ Q \in I \cap k_{rd}[T_0, \ldots, T_n]\}.$$

This is an ideal of $k[T_0, \ldots, T_n]_{(P)}$, and

$$B_{(p)} = k[T_0, \ldots, T_n]_{(P)}/I_{(P)}.$$

Remark 1.2.18. *The above $B_{(p)}$ is independent of the lifting $P \in k_d[T_0, \ldots, T_n]$ of p as well as the open subset*

$$D_+(p) := D_+(P) \cap X,$$

called a principal open subset of X.

Similarly to Proposition 1.1.48, we have

Proposition 1.2.19. *Let X be a closed subset of $\mathbb{P}^n(k)$ defined by a radicial homogeneous ideal I.*

(1) *Let U be an open subset of X. Then any $f \in \mathcal{O}_Z(U)$ is continuous (when $k = \mathbb{A}^1(k)$ is endowed with the Zariski topology).*

(2) *If $P \in k[T_0, \ldots, T_n]$ is homogeneous of degree $d \geq 0$, p is the image of P in B and $U = X \cap D_+(P)$, then*

$$\mathcal{O}_X(U) = B_{(p)}.$$

(3) *If $p \in B$ is homogeneous of degree ≥ 1, then $D_+(p)$ is an affine open subvariety with $A(D_+(p)) = B_{(p)}$.*

Proof. 1. This follows by point 3) and by the affine case, since X can be covered by affine open subvarieties.

2. We have a natural injective map

$$B_{(p)} \to \mathcal{O}_X(U).$$

The statement about the kernel is clear. The above injection allows us to view f as a regular function on Z.

We have to prove that it is surjective. Let $g \in \mathcal{O}_X(U)$ and let us take $D_+(p) = \cup_{i=1}^n D(f_i)$ such that $g_{|D_+(f_i)} = G_i/F_i$, with F_i and G_i homogeneous of the same degree. The last condition implies that $gF_i^2 = G_iF_i$ in $\mathcal{O}_X(X)$. Now $D_+(p) = \cup_{i=1}^n D_+(f_i^2)$ and therefore there exists r such that $p^r = \sum_i^n a_i f_i^2$. We can suppose that $\deg a_i f_i^2 = r \deg p$ therefore $gp^r = \sum_i^n a_i g_i f_i$ which means that $g \in B_{(p)}$.

3. Let $d = \deg P$. Let us consider the morphism $\mathbb{P}_k^n \to \mathbb{P}_k^N$ given by

$$\mu : [x_0, \ldots, x_n] \mapsto [M_0, \ldots, M_N]$$

where the M_i are the monomials of degree $\deg P$ ($N = \binom{n+d}{n} - 1$) and $\mu^{\#}$ is defined by composition (verify that it is well defined). Then μ is an isomorphism between $\mathbb{P}_k^n$ and $Z_+(I)$ where I is the kernel of the map $k[T_0, \ldots, T_M] \to k[S_0, \ldots, S_n]$ given by $T_i \mapsto M_i$ (Exercise!).

Then $\mu(D_+(P)) = D_+(H) \cap \mu(\mathbb{P}_k^n))$ where H is an homogenous polynomial of degree 1 given by the coefficients of P). It's enough to prove that $D_+(H)$ is affine since $\mu(\mathbb{P}_k^n))$ is closed. Let us suppose that the polynomial H 'contains' T_0 then we can consider the isomorphism

$$\mathbb{P}_k^n \to \mathbb{P}_k^n$$

given by $[t_0, \ldots, t_n] \times [H, t_1, \ldots, t_n]$ whixhe sends $D_+(H)$ on $D_+(T_0)$. So it is enough to prove that $D_+(T_i)$ is an isomorphic to $\mathbb{A}_k^n$.

We proved that the application $|\phi_i| : D_+(T_i) \to \mathbb{A}^n(k)$ given by $[x_0 : x_1 : \cdots : x_n] \mapsto (x_0/x_i, \ldots, \widehat{x_i/x_i}, x_n/x_i)$ is an homeomorphism. Moreover for any open subset U of $\mathbb{A}^n(k)$, the morphism $\phi_i^{\#}(U) : \mathcal{O}_{\mathbb{A}^n_k}(U) \to \mathcal{O}_{D_+(T_i)}(\phi_i^{-1}(U))$ given by $f \mapsto f \circ |\phi_i|$ is well defined, as it is easy to verify using the definition of regular functions. The inverse of $(|\phi_i|, \phi_i^{\#})$ is $(|\psi_i|, \psi_i^{\#})$ where $|\psi_i| : \mathbb{A}^n(k) \to D_+(T_i)$ is given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, 1, \ldots, x_n)$ and, for any V open subset of $D_+(T_i)$, $\psi_i^{\#} : \mathcal{O}_{D_+(T_i)}(V) \to \mathcal{O}_{\mathbb{A}^n(k)}(\psi_i^{-1}(V))$ is given by $g \mapsto g \circ |\psi_i|$. It is easy to check that it is well defined, homogeneizing locally. This also proves that if X is a closed subset of $\mathbb{P}^n_k$ then $X \cap D_+(T_i)$ is isomorphic to a an affine closed subset of X and $\mathcal{O}_X(X \cap D_+(T_i)) = B_{(t_i)}$.

□

Notation 1.2.20. *The projective space will also be denoted by $\mathbb{P}^n_k$ when we want to insist in the fact that this is a ringed topological space. The same for $\mathbb{A}^n_k$.*

For any arbitrary radicial homogeneous ideal I, we find

Corollary 1.2.21. *$X = Z_+(I)$ endowed with the sheaf $\mathcal{O}_X$ is a closed*

Definition 1.2.22. *A* projective variety *is a ringed topological space isomorphic to a projective algebraic set X endowed with the sheaf of regular functions $\mathcal{O}_X$ as above.*

Remark 1.2.23. *Projective varieties X share lot of properties with compact complex manifolds. For example, the algebra of regular functions on X are locally constant.*

Definition 1.2.24. *A* quasi-projective variety *is a ringed topological space isomorphic to an open subvariety of a projective variety.*

Proposition 1.2.25. *Affine algebraic varieties are quasi-projective.*

Remark 1.2.26. *Upon to 1950's (when abstract algebraic varieties are defined), algebraic geometers were only concerned with quasi-projective varieties. Of course, there are non-quasiprojective varieties, e.g. the line with a doubled point (Example 1.2.14) and also more sophisticate ones.*

1.3 Dimension of varieties

We want to define the first numerical invariant of an algebraic variety X, its dimension. Let X be any topological space. The *Krull dimension* of X is to the supremum of integers $n \leq +\infty$ such that there exists a strictly increasing chain

$$Z_0 \subset Z_1 \subset \ldots \subset Z_n$$

of irreducible closed subsets (Definition 1.1.31) of X. By convention $\dim \emptyset = -\infty$.

Example 1.3.1. *If X is a discrete (i.e. all subsets are open) and non-empty topological space, then* $\dim X = 0$ *because the only irreducible subsets are singletons.*

Example 1.3.2. *If X is a Hausdorff topological space, then* $\dim X \leq 0$*. Indeed, if Z is an irreducible subset of X, then Z is a singleton (if $z_1 \neq z_2$ are points of Z, there exists open subsets $V_1 \ni z_1$, $V_2 \ni z_2$ in Z with $V_1 \cap V_2 = \emptyset$, hence V_1 is a non-empty open subset, but is not dense in Z). So* $\dim X = 0$ *if X is non-empty.*

Recall that an irreducible component of a topological space X is an irreducible subset Z of X, maximal for the inclusion. As $\overline{Z}$ is also irreducible (Exercise 1.1.9(3)), we have necessarily $Z = \overline{Z}$, so Z is closed. Using Zorn's lemma, we see easily that any irreducible subset of X is contained in an irreducible component.

If $X \neq \emptyset$, as $\{x\}$ is irreducible for any $x \in X$, there is at least one chain of length ≥ 0, so then $\dim X \geq 0$.

Proposition 1.3.3. *Let X be a topological space.*

(1) *If Y is a subset of X with the induced topology, then* $\dim Y \leq \dim X$.

(2) *If X is irreducible of finite dimension and $Y \subseteq X$, then $Y = X$ if and only if Y is closed of dimension* $\dim X$.

(3) *Let $\{X_i\}_i$ be the irreducible components of X. Then* $\dim X = \sup_i \dim X_i$.

(4) *If $\{U_i\}_i$ is a covering of X by open subsets, then* $\dim X = \sup_i \dim U_i$.

Proof. (1) If $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ is a chain of irreducible closed subsets of Y, then their closures $\overline{Z}_i$ in X is an increasing sequence of irreducible closed subsets of X. As $\overline{Z}_i \cap Y = Z_i$, the sequence is strictly increasing. Thus $\dim Y \leq \dim X$.

(2) - (3) are straightforward from the definition.

(4) If $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ is a chain of closed irreducible subsets of X, Z_0 meets one of the X_i's. Then the intersection of the Z_j's with this X_i is a chain of irreducible closed subsets in X_i. This implies the statement □

The *dimension* of an algebraic variety is the Krull dimension of the underlying topological space.

Example 1.3.4. *As expected,* $\dim \mathbb{A}^1_k = \dim \mathbb{P}^1_k = 1$*. Actually, using Proposition 1.1.33, we see that irreducible closed subsets of $\mathbb{A}^1_k$ are $\mathbb{A}^1_k$ and singletons $\{x\}$, $x \in \mathbb{A}^1_k$. So the maximal length of chains of irreducible closed subsets in $\mathbb{A}^1_k$ is 1. Hence* $\dim \mathbb{A}^1_k = 1$*. As $\mathbb{P}^1_k$ is union of two open subsets $D_+(T_0)$, $D_+(T_1)$ of dimension 1, it is also of dimension 1 by Prop. 1.3.3. By an exercise we did before (irreducible closed subsets of $\mathbb{A}^2(k)$ are points, principal closed subsets defined by irreducible polynomials and $\mathbb{A}^2(k)$).*

Definition 1.3.5. *Let R be a (commutative unitary) ring. The* Krull dimension *of R is the supremum of the lengths of chains of prime ideals of R:*

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n.$$

Example 1.3.6. *If $R = \mathbb{C}[T_1, T_2, \ldots,]$ (infinitely many variables), then for all $n \geq 1$, the ideal $\mathfrak{p}_n := (T_1, \ldots, T_n)$ is prime because $R/\mathfrak{p}_n$ is a domain. As $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots$ is an infinite chain of prime ideals of R, we have* $\dim R = +\infty$.

Remark 1.3.7.

1. *Let I be an ideal of R and let $\pi : R \to R/I$ be the canonical surjection. Then there is a canonical bijection between the set of the prime ideals $\mathfrak{p}$ of R containing I and that of R/I, which is given by $\mathfrak{p} \mapsto \overline{\mathfrak{p}} = \mathfrak{p}/I$. The inverse correspondence is given by $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$.*
2. *As a consequence, and because all prime ideals contain the nilradical $\sqrt{0}$ of R, we have* $\dim R = \dim(R/\sqrt{0})$.
3. *If X is an affine algebraic set. Then by Proposition 1.1.33*
$$\dim X = \dim A(X).$$

If $A \to B$ is finite, and $b \in B$, how to find a monic polynomial in $A[T]$ vanishing at b ? Fix a system of generators $e_1, \ldots, e_n$ of B over A. Then we can write a matrix of the A-linear map $[b] : B \to B$, $y \mapsto by$ in the given system of generators. The characteristic polynomial of this matrix then vanishes at b (in B) by Cayley-Hamilton theorem.

Proposition 1.3.8. *Let $\varphi : A \to B$ be a finite ring homomorphism.*

(1) *We have* $\dim B \leq \dim A$.

(2) *The preimage of a prime ideal is maximal if and only if the ideal is maximal.*

(3) *If φ is moreover injective, then for any prime ideal $\mathfrak{p}$ of A, there exists a prime ideal $\mathfrak{q}$ of B such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.*

(4) *Under the hypothesis of (2), we have* $\dim B = \dim A$.

Proof. (1) Let $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1$ be two prime ideals of B. Let us show that $\varphi^{-1}(\mathfrak{q}_0) \subsetneq \varphi^{-1}(\mathfrak{q}_1)$. This will imply the inequality on the dimensions. The homomorphism φ induces by factorisation an injective and finite homomorphism $A/\varphi^{-1}(\mathfrak{q}_0) \to B/\mathfrak{q}_0$. Replacing A by $A/\varphi^{-1}(\mathfrak{q}_0)$ and B by $B/\mathfrak{q}_0$, we may assume that B is an integral domain and that $\mathfrak{q}_0 = 0$. We must then show that $\varphi^{-1}(\mathfrak{q}_1) \neq 0$. Let $b \in \mathfrak{q}_1$ be non-zero. Let $b^n + \varphi(a_{n-1})b^{n-1} + \cdots + \varphi(a_0) = 0$ be an integral equation for b over A, of minimal degree n. Then $\varphi(a_0) = b(-\varphi(a_1) - \varphi(a_2)b - \cdots - b^{n-1}) \in \mathfrak{q}_1 \setminus \{0\}$ and $a_0 \in \varphi^{-1}(\mathfrak{q}_1) \setminus \{0\}$.

(2)Let $\mathfrak{p}$ a prime ideal of B. Then we have an injective inetgral morphism of integral domains $A/varphi^{-1}(\mathfrak{p}) \to B/\mathfrak{p}$. Therefor $\mathfrak{p}$ is maximal if and only if $\varphi^{-1}(\mathfrak{p})$ is maximal.

(3) Now suppose that φ is injective.

$A_\mathfrak{p} \to B \otimes_A A_\mathfrak{p}$ is integral. and since ϕ is injective then $T = (\phi(A \setminus \mathfrak{p}))$ and $B \otimes_A A_\mathfrak{p} = T^{-1}B \neq 0$.

(4) Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1$ be prime ideals of A. Let $\mathfrak{q}_0 \subset B$ be a prime ideal such that $\varphi^{-1}(\mathfrak{q}_0) = \mathfrak{p}_0$. By considering the injective integral homomorphism $A/\mathfrak{p}_0 \hookrightarrow$

$B/\mathfrak{q}_0$, we obtain a prime ideal $\mathfrak{q}_1$ of B such that $\varphi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$ and $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1$. By repeatedly applying this result to any finite chain of prime ideals of A, we get $\dim A \leq \dim B$. □

Remark 1.3.9. *Any quotient homomorphism* $A := k[T_1, \ldots, T_n] \to B := A/I$ *is finite, so* $\dim B = \dim Z(I) < \dim \mathbb{A}^n(k) = \dim A$ *if* $I \neq 0$. *Therefore the equality of dimensions does not holds in general if we do not assume injectivity of* φ.

Corollary 1.3.10. *Let* K *be any field. Let* $n \geq 0$ *and let* $F \in K[T_1, \ldots, T_n] \backslash K$.

(1) $\dim K[T_1, \ldots, T_n] = n$.

(2) $\dim K[T_1, \ldots, T_n]/(F) = \dim K[T_1, \ldots, T_{n-1}]$.

Proof. (2) In the course of the proof of Noether's normalization lemma (**??**), we showed that up to a K-automorphism of $K[T_1, \ldots, T_{n+1}]$, we can suppose that F is monic and of positive degree in T_{n+1}. Then the K-homomorphism

$$K[T_1, \ldots, T_n] \to K[T_1, \ldots, T_{n+1}]/(F), \quad T_i \mapsto \overline{T}_i$$

is finite and injective. We then apply the above proposition.

(1) Note that

$$\{0\} \subset (T_1) \subset (T_1, T_2) \subset \cdots \subset (T_1, \ldots, T_n)$$

is a chain of prime ideals of length n. So $\dim K[T_1, \ldots, T_n] \geq n$.

We proceed by induction on n to prove the converse inequality. The property is true for $n \leq 1$ (Example 1.3.4). Suppose it is true for $n - 1$. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots. \subset \mathfrak{p}_m$ be a chain of prime ideals of $K[T_1, ..., T_n]$. We want to show that $m \leq n$. Let $P \in \mathfrak{p}_1$ be non-zero. The images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ is a chain of prime ideals in $K[T_1, \ldots, T_n]/(P)$, so $m - 1 \leq \dim K[T_1, \ldots, T_n]/(P) = \dim K[T_1, \ldots, T_{n-1}] = n - 1$ By (2) we just proved and the induction hypothesis. Therefore $m \leq n$ and the corollary is proved. □

Corollary 1.3.11. *Algebraic varieties have finite dimensions.*

Proof. Any algebraic variety is a finite union of affine open subvariety, and any affine variety has finite dimension by the above corollary and Proposition 1.3.3. □

Corollary 1.3.12. *Let* $P \in k[T_1, \ldots, T_n] \backslash k$. *Then* $\dim \mathbb{A}^n(k) = n$ *and* $\dim Z(P) = n - 1$.

1.3.1 Function fields

Definition 1.3.13. *Let* k *be a field. A* field of finite type over k *or* a function field over k *is a field extension* K/k *such that there exists an sub-k-algebra* A *of* K, *of finite type over* k *with* $K = \mathrm{Frac}(A)$.

If $A = k[t_1, \ldots, t_n]$ *(this means that* A *is a quotient of* $k[T_1, \ldots, T_n]$ *and we denote by* t_i *the class of* T_i *in the quotient), then* $k(t_1, \ldots, t_n)$ *is by definition* $\mathrm{Frac}(A)$.

Applying Noether's normalization lemma to A, we see that K is then a finite extension of a field of rational functions $k(T_1, \dots, T_d)$ (the T_i's are variables).

Definition 1.3.14. *Let K/k be a field extension. A familly $\{t_1, \dots, t_n\}$ of elements of K are said* algebraically independent *if for any non-zero $F(T_1, \dots, T_n) \in k[T_1, \dots, T_n]$, we have $F(t_1, \dots, t_n) \neq 0$. This is equivalent to saying that the k-homomorphism*

$$k[T_1, \dots, T_n] \to K, \quad T_i \mapsto t_i$$

is injective. Such a k-homomorphism induces a homomorphism of k-extensions $k(T_1, \dots, T_n) \to K$.

If moreover K is algebraic over $k(t_1, \dots, t_n)$, then $\{t_1, \dots, t_n\}$ is called a transcendence base *of K over k.*

Lemma 1.3.15. *A field extension K/k is of finite type if and only if there are algebraically independent elements $t_1, \dots, t_n \in K$ such that K is a finite extension of $k(t_1, \dots, t_n) := \mathrm{Frac}(k[t_1, \dots, t_n]) \simeq k(T_1, \dots, T_n)$.*

Proposition 1.3.16. *Let K/k be an extension of finite type. If K is algebraic over $k(t_1, ..., t_n)$ and over $k(s_1, ..., s_m)$, then $n = m$.*

Proof. Suppose $m > n$. Write a polynomial dependence of s_1 over $k(t_1, \dots, t_n)$. At least one variable, say t_1, appears in the relation. Then t_1 is algebraic over $s_1, t_2, \dots, t_n$. Hence K is algebraic over $k(s_1, t_2, \dots, t_n)$. In particular we have an algebraic relation of s_2 over this field. In this relation, at least one of the t_i's, $i \geq 2$, appears because s_1, s_2 are algebraically independent. We can suppose that t_2 appears. Then t_2 (hence K) is algebraic over $k(s_1, s_2, t_3, \dots, t_n)$. Repeating the same argument we find that K (hence s_{n+1}) is algebraic over $k(s_1, s_2, \dots, s_n)$, contradiction. □

Definition 1.3.17. *The cardinality n of a transcendence base in K is called the* transcendence degree *of K over k. We denote it by $\mathrm{degtr}_k K$. We say that K is a* function field in n variables. *A finite extension has $\mathrm{degtr}_k K = 0$.*

Definition 1.3.18. *An algebraic variety X is* integral *if its underlying topological space is irreducible.*

Lemma 1.3.19. *X is an integral variety if and only if for any non-empty open subset U of X, the ring $\mathcal{O}_X(U)$ is an integral domain.*

Proof. Let us suppose that X is integral. Then U is integral for any open subset U of X. So we can suppose $U = X$ and we prove that $\mathcal{O}_X(X)$ is integral. Let $f, g \in \mathcal{O}_X(X)$ such that $fg = 0$. Let W be an open subset of X and let $Z(f) := \{x \in X | f(x) = 0\}$ and $Z(g) := \{x \in W | g(x) = 0\}$, whic are closed since f and g are continuous. Then $X = Z(f) \cup Z(g)$. Since X is irreducible then $f = 0$ or $g = 0$.

Conversely, let us suppose that there exists U, V open subsets of X such that $U \cap V = \emptyset$. Since $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$, then it is not integral. □

Lemma 1.3.20. *Let X be an integral algebraic variety. Let $V \subseteq U$ open subsets of X with $V \neq \emptyset$. Then the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is injective. If moreover U, V are affine, then* $\mathrm{Frac}(\mathcal{O}_X(U)) \to \mathrm{Frac}(\mathcal{O}_X(V))$ *is an isomorphism.*

Proof. Let $f \in \mathcal{O}_X(U)$ such that $f_{|V} = 0$. Then $V \subseteq Z(f) \subseteq U$. Since X is irreducible then V is dense in X so $Z(f) = 0$ and then $f = 0$. If U and V are affine we take a principal oben subset W of U contained in V. So

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V) \to \mathcal{O}_X(W)$$

are two injections. So it is enough to prove that the morphism $\mathrm{Frac}(\mathcal{O}_X(U)) \to \mathrm{Frac}(\mathcal{O}_X(W))$ is an isomorphism. So we reduce to the case where U is affine and V is a principal open subset $D(f)$ with $f \in \mathcal{O}_X(U)$. In this case we have an inclusion

$$\mathcal{O}_X(U) \to \mathcal{O}_X(U)_f$$

and it is clear that we have an isomorphism passing to the fraction field.[2]

□

Definition 1.3.21. *Let X be an integral algebraic variety. The* field of rational functions of X *is the inductive limit*

$$k(X) = \varinjlim_{U} \mathcal{O}_X(U)$$

where U runs through the non-empty open subsets of X. So the elements of $k(X)$ are regular functions on some non-empty open subsets of X.

Proposition 1.3.22. *Let X be an integral algebraic variety. Let U_0 be a non-empty affine open subset of X. Then $k(X)$ is canonically isomorphic to the field of fractions* $\mathrm{Frac}(\mathcal{O}_X(U_0))$. *If U is a non-empty open subset, then $k(X) \simeq k(U)$.*

Proof. Let U be any non-empty open subset of X. Let $V = D(f)$ be a non-empty principal open subset of U_0 contained in $U \cap U_0$. Then we have canonical maps

$$\mathcal{O}_X(U) \to \mathcal{O}_X(U \cap U_0) \to \mathcal{O}_X(V) = \mathcal{O}_X(U_0)_f \subseteq \mathrm{Frac}(\mathcal{O}_X(U_0)).$$

This gives a canonical injective map $k(X) \to \mathrm{Frac}(\mathcal{O}_X(U_0))$. This map is surjective because any element $f/g \in \mathrm{Frac}(\mathcal{O}_X(U_0))$ is regular over the non-empty principal open subset $D(g)$ of U_0.

For a given non-empty open subset U, there exists a non-empty affine open subset $U_0 \subseteq U$. So $k(X) \simeq k(U_0) = \mathrm{Frac}(\mathcal{O}_X(U_0)) \simeq k(U)$. □

Next we will relate the dimension of an integral variety X to a numerical invariant of its field of rational functions, the transcendental degree. We need some terminology in field extensions.

Proposition 1.3.23. *Let X be an integral algebraic variety. Then*

$$\dim X = \mathrm{degtr}_k k(X).$$

If U is a non-empty open subset of X, then $\dim U = \dim X$.

Proof. First suppose that X is affine. By Noether's normalization lemma, there is an injective finite homomorphism $k[T_1, ..., T_n] \to A(X)$. So on the one hand, $\dim X = n$ by Proposition 1.3.8, and on the other hand, $k(X) = \mathrm{Frac}(A)$ is finite over $k(T_1, ..., T_n)$, thus $\mathrm{degtr}_k k(X) = n = \dim X$. In general, let $X = \cup_i U_i$ be an affine covering of X. Then

$$\dim X = \max_i \dim U_i = \max_i \mathrm{degtr}_k k(U_i) = \max_i \mathrm{degtr}_k k(X) = \mathrm{degtr}_k k(X).$$

Finally, $k(U) = k(X)$, so $\dim U = \dim X$. □

Proposition 1.3.24. *If X is an integral algebraic variety and $f \in \mathcal{O}_X(X)$ is non-zero and non-invertible, then $\dim Z(f) = \dim X - 1$.*

Proof. As X is irreducible and $Z(f) \neq X$ (because $f \neq 0$), we have $\dim Z(f) \leq \dim X - 1$. There exists an affine open subset U of X such that $f|_U$ is non-invertible. It is enough to show that $\dim Z(f|_U) \geq \dim U - 1$ because $\dim Z(f) \geq \dim Z(f|_U)$ and $\dim U = \dim X$. So we can suppose X is affine and integral.

Let $k[T_1, \ldots, T_n] \to A = A(X)$ be an injective finite homomorphism. Then $\dim X = n$. We have a finite injective k-homomorphism

$$k[T_1, \ldots, T_n]/(fA \cap k[T_1, \ldots, T_n]) \to A/fA.$$

It is enough to show that the left-hand side term has dimension $n - 1$ because $\dim Z(f) = \dim A(Z(f)) = \dim(A/\sqrt{fA}) = \dim(A/fA)$.

Let F be the norm of f in the extension $\mathrm{Frac}(A)/k(T_1, \ldots, T_n)$. This is some power of the constant term of the monic minimal polynomial of f over $k(T_1, \ldots, T_n)$. As f is integral over $k[T_1, \ldots, T_n]$ and the latter is a UFD, it is known that the minimal polynomial of f has coefficients in $k[T_1, \ldots, T_n]$. In particular $F \in k[T_1, \ldots, T_n]$. We then have

$$fA \cap k[T_1, \ldots, T_n] \subseteq \sqrt{(F)} \subseteq \sqrt{fA \cap k[T_1, \ldots, T_n]}.$$

The first inclusion comes frome the fact that if $x = fa \in fA \cap k[T_1, \ldots, T_n]$ then $N(x) = x^r = N(f)N(a) = FN(a)$ where r is the degree of the extension. So it is enough to show that $\dim k[T_1, \ldots, T_n]/(F) = n - 1$. But this is just Corollary 1.3.12. □

Exercise 1.3.1. *Under the hypothesis of Proposition 1.3.24, show that each irreducible component of $Z(f)$ has dimension $n - 1$.*

Corollary 1.3.25. *Let $n \geq 1$. Then $\dim \mathbb{P}^n(k) = n$. Any hypersurface $Z_+(F)$ of $\mathbb{P}^n(k)$ with F non-constant homogenous polynomial has dimension $n - 1$.*

Proof. Use Proposition 1.3.3 and Corollary 1.3.10. □