Université de Bordeaux
M2 AGTN/ALGANT

2024-2025
Algebraic Geometry

Exercise Sheet 6

The field k is assumed to be algebraically closed.

Exercise 1 (Connected rings) A ring R is said to be *connected* if every idempotent in R is trivial (i.e., if every element $e \in R$ such that $e^2 = e$ is equal to 0 or 1).

1. Prove that every integral domain is connected.
2. If R is the product of two non-trivial rings, prove that R is not connected.
3. Conversely, if R possesses a non-trivial idempotent e, prove that $R \simeq R/(e) \times R/(1-e)$.
4. Let X be an affine algebraic variety over k. Prove that X is connected (in the Zariski topology) if and only if $A(X)$ is connected.

Exercise 2 (Separated varieties)

1. Show that affine varieties are separated.
2. Show that open subvarieties and closed subvarieties of a separated variety are separated.
3. Let X be a separated algebraic variety. Show that then the diagonal map $\Delta \colon X \to X \times X$ is a closed immersion.
4. Show that an algebraic variety X is separated if and only if there exists an affine covering $\{X_i\}_i$ of X such that for all i, j, the intersection $X_i \cap X_j$ is affine, and the canonical homomorphism $\mathcal{O}_X(X_i) \otimes_k \mathcal{O}_X(X_j) \to \mathcal{O}_X(X_i \cap X_j)$ defined as $f_i \otimes f_j \to f_i|_{X_i \cap X_j} f_j|_{X_i \cap X_j}$ is surjective.
5. Show that products of separated varieties are separated.
6. Show that projective varieties are separated.

Exercise 3 (Finite morphisms) Let X and Y be affine integral varieties and let $f \colon X \to Y$ a dominant morphism. Then the induced homomorphism $A(Y) \to A(X)$ is injective (by Partial Exam). We say that f is a finite morphism if $A(X)$ is integral over $A(Y)$.

1. Prove that a finite morphism is surjective.
2. Deduce that a finite morphism is closed.
3. Let $g \colon X \to Y$ be a morphism of affine integral varieties. Assume that every point $y \in Y$ has an affine neighbourhood $U \ni y$ such that $V = g^{-1}(U)$ is affine and $f \colon V \to U$ is finite. Prove that g is finite.

Exercise 4 (Dimension of intersections) Let X and Y be two integral algebraic subvarieties of respective dimensions r and s in $\mathbb{A}^n(k)$. Our aim is to prove that any irreducible component of $X \cap Y$ is of dimension $\geqslant r + s - n$.

1. Prove that this result is true if X is a hypersurface.
2. Treat the general case considering the product $X \times Y \subseteq \mathbb{A}^{2n}(k)$ and looking at the restriction of the diagonal and using Exercise 2, Sheet 5.

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Exercise Sheet 8

The field k is assumed to be algebraically closed.

Exercise 1 (Open subsets of curves) Let X be an integral smooth proper curve and $D > 0$ a divisor on X. Let $U = X \setminus Supp(D)$.

1. Prove that $\mathcal{O}_X(U) = \bigcup_{n \geqslant 0} L(nD)$ as subsets of $k(X)$.
2. Suppose that $L(D) \neq k$ and prove that $Frac(\mathcal{O}_X(U)) = k(X)$.

Let V be any strict open subset of X.

3. Show that V is affine.

Exercise 2 (Easy consequences of Riemann-Roch) Let X be an integral smooth proper curve of genus g.

1. Show that for any divisor D of degree $degD > 2g-2$, we have $l(D) = degD + 1 - g$.
2. Show that $g = 0$ iff $X \simeq \mathbb{P}^1(k)$.

Exercise 3 (Genus-one curves) Let X be an integral smooth proper curve of genus $g = 1$. Let $x_0 \in X$ be a fixed point.

1. Show that $L([x_0]) = k$ and there exist $x \in L(2[x_0]) \setminus k$ and $y \in L(3[x_0]) \setminus L(2[x_0])$.
2. Let $I = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i \geqslant 0, 0 \leqslant j \leqslant 1, 2i + 3j \leqslant n\}$. Prove that
$$L(n[x_0]) = \bigoplus_{(i,j) \in I} kx^i y^j.$$
3. Show that X is isomorphic to a cubic in $\mathbb{P}^2(k)$ defined by the equation
$$X_1^2 X_3 + (a_1 X_0 X_3 + a_3 X_3^2) X_1 = X_0^3 + a_2 X_0^2 X_3 + a_4 X_0 X_3^2 + a_6 X_3^3$$
for some $a_1, a_3, a_2, a_4, a_6 \in k$. (Hint: use Exercise 1).
4. Show that the map $\theta \colon X \to Pic^0(X)$, defined by $x \mapsto [x] - [x_0]$ is bijective.
In particular this induces a commutative group structure on X.

Exercise 4 (Genus-two curves) Let X be an integral smooth proper curve of genus $g = 2$. Show that there exists a finite separable morphism $X \to \mathbb{P}^1(k)$ of degree 2.
(Hint: use the same strategy via Riemann-Roch as in Exercise 3).