Rk: since $\mathcal{C}^*(\mathcal{F})$ is a $\Gamma_{\mathcal{U}}$-acyclic resolution, it computes the right cohomology: $\forall n \geq 0$, $H^n(\mathcal{U}, \mathcal{F}) = H^n(\mathcal{C}^*(\mathcal{F}|_{\mathcal{U}}))$

4.3) Hypercohomology

We defined coh of a top space with coef in $\mathcal{F} \in Sh(X)$.

Want to define coh with coef in a complex $\mathcal{F}^* \in Ch^+(Sh(X))$.

Let $\mathcal{C}$ be an abelian category with enough injectives.

Definition:

A complex $(A^*, d^*) \in Ch(\mathcal{C})$ is bounded below if $A^n = 0_{\mathcal{C}}$ $\forall n \ll 0$.

We denote by $Ch^+(\mathcal{C}) \subseteq Ch(\mathcal{C})$ the full subcategory of bounded below complexes.

Notation: let $A^*, B^* \in Ch(\mathcal{C})$

- $Hom^{n.h.}_{Ch(\mathcal{C})}(A^*, B^*) \subseteq Hom_{Ch(\mathcal{C})}(A^*, B^*)$ be the subgroup of null-homotopic morphisms of complexes.

- $$[A^*, B^*]_{\mathcal{C}} = \frac{Hom_{Ch(\mathcal{C})}(A^*, B^*)}{Hom^{n.h.}_{Ch(\mathcal{C})}(A^*, B^*)}$$ the group of morphism of complexes $A^* \to B^*$ modulo homotopy.

Remark: $f^*, g^* : A^* \to B^*$.

If f^*, g^* are null-homotopic then so is $f^* + g^*$ (exercise)

If $f^* \sim g^*$ then $\overline{f^*} = \overline{g^*}$ in $[A^*, B^*]$

Theorem:

1) Any bounded below complex in $\mathcal{C}$ has an injective resolution.

i.e. $\forall A^* \in Ch^+(\mathcal{C})$, $\exists I^* \in Ch^+(\mathcal{C})$ a complex of injective objects in $\mathcal{C}$ and a quasi-isomorphism $A^* \xrightarrow{q\text{-is}} I^*$ (i.e. $H^n \xrightarrow{\sim} H^n(I^*)$ $\forall n \in \mathbb{Z}$)

Moreover, if $A^n = 0$ $\forall n < 0$, then we can chose I^* such that $I^n = 0$ $\forall n < 0$.

2) let $I^* \in Ch^+(\mathcal{C})$ be a cplex of injectives.

If $A^* \to B^*$ is a quasi-iso in $Ch^+(\mathcal{C})$ then $[B^*, I^*]_{\mathcal{C}} \xrightarrow{\sim} [A^*, I^*]_{\mathcal{C}}$ is an iso in Ab.

Remark: let $A^*, B^* \in Ch^+(\mathcal{C})$ and let $A^* \to I^*$, $B^* \to J^*$ be injective resolutions in the sense of the theorem.

If $f : A^* \to B^*$ is a map in $Ch^+(\mathcal{C})$, $\exists \tilde{f} : I^* \to J^*$

$$\begin{tikzcd} A^* \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\sim"] & B^* \arrow[d, "\beta"] \\ I^* \arrow[r, "\tilde{f}"'] & J^* \end{tikzcd}$$

Indeed, since α is a quasi iso, we have

$$[I^*, J^*] \xrightarrow{\sim} [A^*, J^*]$$
$$\exists!\ \tilde{f} \longmapsto \beta \circ f$$

$\exists!\ \tilde{f} \in [I^*, J^*]$ st $\tilde{f} \circ \alpha = \beta \circ f$ in $[A^*, B^*]$

Remark: We already knew that for A^*, B^* satisfying $A^n = B^n = 0 \ \forall n \neq 0$

Note that an injective resolution $A^* \xrightarrow{q\text{-iso}} I^*$ is an injective resolution of A^0

$$0 \to A^0 \to I^0 \to I^1 \to \dots$$

Definition:

The homotopy category of bounded below complexes $K^+(\mathcal{C})$:

- $\mathrm{Obj}(K^+(\mathcal{C})) = \mathrm{Obj}(Ch^+(\mathcal{C}))$
- $\forall A^*, B^* \in \mathrm{Ob}(K^+(\mathcal{C}))$, $\mathrm{Hom}_{K^+(\mathcal{C})}(A^*, B^*) = [A^*, B^*]_{\mathcal{C}}$
- We denote by $\mathbb{I}_{\mathcal{C}} \subseteq \mathcal{C}$ the full sub-category of injective objects and by $Ch^+(\mathbb{I}_{\mathcal{C}}) \subseteq Ch^+(\mathcal{C})$ the full subcategory of $Ch^+(\mathcal{C})$ consisting of b. below complexe of injectives.

The derived category of b. below complexes $D^+(\mathcal{C})$:

$\mathrm{Obj}(D^+(\mathcal{C})) := Ch^+(\mathbb{I}_{\mathcal{C}})$

$\forall A^*, B^* \in \mathrm{Obj}(D^+(\mathcal{C}))$, $\mathrm{Hom}_{D^+(\mathcal{C})}(A^*, B^*) := [A^*, B^*]_{\mathcal{C}}$

Remark: We have an obvious fully faithful functor

$$i : D^+(\mathcal{C}) \longrightarrow K^+(\mathcal{C})$$

We have a functor $\rho : K^+(\mathcal{C}) \longrightarrow D^+(\mathcal{C})$ where $A^* \xrightarrow{q\text{-iso}} I^*$ is an injective resolution.
$A^* \longmapsto I^*$

Note that $\rho(A) := I^* \in D^+(\mathcal{C})$ is well defined in $D^+(\mathcal{C})$ up to a unique isomorphism

Consider
$$\begin{tikzcd} A^* \arrow[r, "q\text{-iso}"] \arrow[d, "\mathrm{Id}"', "\wr"] & I^* \\ A^* \arrow[r, "q\text{-iso}"'] & J^* \end{tikzcd}$$

$\exists$ maps $I^* \to J^*$ and $J^* \to I^*$ unique up to homotopy whose compositions are homotopic to Id_{I^*} and Id_{J^*}.

We obtain an adjunction $\rho : K^+(\mathcal{C}) \rightleftarrows D^+(\mathcal{C}) : i$

Indeed, for $A^* \in K^+(\mathcal{C})$, $B^* \in D^+(\mathcal{C})$

$$\begin{aligned} \mathrm{Hom}_{D^+(\mathcal{C})}(\rho A^*, B^*) &= [\rho A^*, B^*]_{\mathcal{C}} \\ &\xrightarrow{\sim} [A^*, B^*] \\ &= \mathrm{Hom}_{K^+(\mathcal{C})}(A^*, i(B^*)) \end{aligned}$$

We get an iso since $A^* \to \rho A^*$ is a quasi-iso and B^* is a complexes of injectives.

<u>Construction</u>: (Godement resolution)

of b. below complexes.

Let X be a top space and let $(\mathcal{F}^*, \delta^*) = [0 \to \mathcal{F}^0 \xrightarrow{\delta^0} \mathcal{F}^1 \xrightarrow{\delta^1} \mathcal{F}^2 \to \cdots]$
$\in Ch(Sh(X))$

Recall that we have a functor
$$\begin{aligned} C^*(-) : Sh(X) &\longrightarrow Ch^+(Sh(X)) \\ \mathcal{F} &\longmapsto (C^*(\mathcal{F}), d^*) \end{aligned}$$

Apply this functor to the complex $(\mathcal{F}^*, \delta^*)$ and get $C^*(\mathcal{F}^*) \in Ch^+(Ch^+(Sh(X)))$

$$
\begin{tikzcd}
\uparrow & & & & \\
C^0(\mathcal{F}^2) \arrow[r] \arrow[u] & C^1(\mathcal{F}^2) \arrow[r] & C^2(\mathcal{F}^2) \arrow[r] & C^3(\mathcal{F}^2) \arrow[r] & \cdots \\
C^0(\mathcal{F}^1) \arrow[r, "d^{0,1}"] \arrow[u] & C^1(\mathcal{F}^1) \arrow[r, "d^{1,1}"] \arrow[u] & C^2(\mathcal{F}^1) \arrow[r, "d^{2,1}"] \arrow[u] & C^3(\mathcal{F}^1) \arrow[r] \arrow[u] & \cdots \\
C^0(\mathcal{F}^0) \arrow[r, "d^{0,0}"] \arrow[u, "C^0(\delta^0)"] & C^1(\mathcal{F}^0) \arrow[r, "d^{1,0}"] \arrow[u] & C^2(\mathcal{F}^0) \arrow[r, "d^{2,0}"] \arrow[u] & C^3(\mathcal{F}^0) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}
$$

is a commutative diagram !

$\delta^0 : \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \quad \rightsquigarrow \quad C^0(\delta^0) : C^0(\mathcal{F}^0) \longrightarrow C^0(\mathcal{F}^1)$

We (redefine ?) $\delta^{i,j} := (-1)^i C^i(\delta^j)$ and get

$$
\begin{tikzcd}
\uparrow & \uparrow & \uparrow & \\
C^0(\mathcal{F}^1) \arrow[r, "d^{0,1}"] \arrow[u] & C^1(\mathcal{F}^1) \arrow[r, "d^{1,1}"] \arrow[u] & C^2(\mathcal{F}^1) \arrow[u] & \\
C^0(\mathcal{F}^0) \arrow[r, "d^{0,0}"] \arrow[u, "\delta^{0,0}"] & C^1(\mathcal{F}^0) \arrow[r] \arrow[u, "\delta^{1,0}"'] & C^2(\mathcal{F}^0) \arrow[r] \arrow[u] & {}
\end{tikzcd}
$$

where the squares are now anti-commutative :

$$\delta^{1,0} \circ d^{0,0} + d^{0,1} \circ \delta^{0,0} = 0$$

We define the associated total cplex :

$$\mathrm{Tot}^n(C^*(\mathcal{F}^*)) := \bigoplus_{i+j=n} C^i(\mathcal{F}^j) \text{ with differential}$$

$$\bigoplus_{i+j=n} \delta^{i,j} + d^{i,j} : \mathrm{Tot}^n(C^*(\mathcal{F}^*)) \longrightarrow \mathrm{Tot}^{n+1}(C^*(\mathcal{F}^*))$$

ex: $\mathrm{Tot}^1(C^*(\mathcal{F}^*)) = C^0(\mathcal{F}^1) \oplus C^1(\mathcal{F}^0)$

This is indeed a complex because the squares are all anti-commutative !

Moreover, $\mathrm{Tot}^n(C^*(\mathcal{F}^*))$ is a finite direct sum of flasque sheaves

$\forall j \geq 0$, we have a map $\mathcal{F}^j \to C^0(\mathcal{F}^j) \to \mathrm{Tot}^j(C^*(\mathcal{F}^*))$

we obtain a morphism of complexes $\mathcal{F}^* \to \mathrm{Tot}^*(C^*(\mathcal{F}^*))$ which is a quasi-isomorphism (not easy, need spectral theory).

Hence, any bounded below complex of sheaves has a canonical flasque resolution!

Definition:

Let X be a topological space and let $\mathcal{F}^* \in \mathrm{Ch}^+(\mathrm{Sh}(X))$ be a b. below cplx of ab. sheaves. We define the hypercohomology with coef in $\mathcal{F}^*$ as

$\mathbb{H}^n(X, \mathcal{F}^*) := H^n(I^*(X)) \ \forall n \in \mathbb{Z}$ where $\mathcal{F}^* \xrightarrow{\text{q.iso}} I^*$ is an <u>injective</u> resolution or a <u>flasque resolution</u>.

4.4) Continuous maps and cohomology

$f: X \to Y$ continuous map of top. spaces.

Then he did recall about f^* and f_* ...

Example: Suppose that $Y = \{*\}$, $f: X \to Y = \{*\}$ the canonical map

$$\begin{aligned} \mathrm{Sh}(\{*\}) &\xrightarrow{\sim} \underline{Ab} \quad \text{is an equivalence.} \\ \mathcal{G} &\longmapsto \mathcal{G}(\{*\}) \end{aligned}$$

$$\Gamma(X,-) : Sh(X) \xrightarrow{f_*} Sh(\{*\}) \simeq \underline{Ab}$$

$$\mathcal{F} \longrightarrow f_*\mathcal{F} \longmapsto (f_*\mathcal{F})(\{*\}) = \mathcal{F}(f^{-1}(\{*\})) = \mathcal{F}(X)$$

Let $A \in \underline{Ab} \simeq Sh(\{*\})$

$\forall \mathcal{U} \subseteq X$ open, $(f^{p}A)(\mathcal{U}) = \lim\limits_{\{*\} \supseteq V \supseteq f(\mathcal{U})} A(V) = \begin{cases} A & \text{if } \mathcal{U} \neq \emptyset \\ 0 & \text{if } \mathcal{U} = \emptyset \end{cases}$

So f^*A is the constant sheaf A_X associated to A.

$f^* : \underline{Ab} \longrightarrow Sh(X)$ is the constant sheaf functor which is left adjoint to the global sections functor.

$$f_* = \Gamma(X,-) : Sh(X) \longrightarrow \underline{Ab}$$

Notation: If X is a topological space we set $K^+(X) := K^+(Sh(X))$

$$D^+(X) := D^+(Sh(X))$$

Proposition: Let $f: X \to Y$ be a continuous map. Then there is an adjunction

$$Lf^* : D^+(Y) \rightleftarrows D^+(X) : Rf_*$$

Proof: We have adjunctions $\rho_X : K^+(X) \rightleftarrows D^+(X) : i_X$

$$\rho_Y : K^+(Y) \rightleftarrows D^+(Y) : i_Y$$

Let $I^* \in D^+(Y)$. Then I^* is a b. below cplx of injective ab. sheaves

Then $f^* I^* = [\dots \to f^* I^n \longrightarrow f^* I^{n+1} \to \dots] \in K^+(X)$ is a b. below

cplx of sheaves on X.

Define $Lf^*(I^*) := \rho_X(f^* I^*) \in D^+(X)$

More formally, define
$$\begin{tikzcd} D^+(Y) \arrow[r, "i_Y"] \arrow[drr, "Lf^*"'] & K^+(Y) \arrow[r, "f^*"] & K^+(X) \arrow[d, "\rho_X"] \\ & & D^+(X) \end{tikzcd}$$

Let $J^* \in D^+(X)$. Since f_* preserves injectives

$$f_* J^* = [\dots \longrightarrow f_* J^n \longrightarrow f_* J^{n+1} \longrightarrow \dots] \in D^+(Y)$$

Define $Rf_*(J^*) := f_* J^* \in D^+(Y)$

We have a commutative square
$$\begin{tikzcd} D^+(X) \arrow[r, "Rf_*"] \arrow[d, hook, "i_X"'] & D^+(Y) \arrow[d, hook] \\ K^+(X) \arrow[r, "f^*"] & K^+(Y) \end{tikzcd}$$

Let $I^* \in D^+(Y)$ and $J^* \in D^+(X)$

$$
\begin{aligned}
\mathrm{Hom}_{D^+(X)}(Lf^* I^*, J^*) &= \mathrm{Hom}_{D^+(X)}(\rho_X(f^* I), J^*) \\
&= \mathrm{Hom}_{K^+(X)}(f^* I^*, i_X(J^*)) \\
&= \mathrm{Hom}_{K^+(X)}(f^* I^*, J^*) \\
&= \mathrm{Hom}_{K^+(Y)}(I^*, f_* J^*) \\
&= \mathrm{Hom}_{D^+(Y)}(I^*, Rf_* J^*)
\end{aligned}
$$

by adjunction $\rho_X \rightleftarrows i_X$

by adj $f^*: K^+(Y) \rightleftarrows K^+(X): f_*$

since $D^+(Y) \subseteq K^+(Y)$ full sub-category.