Université de Bordeaux
M2 AGTN/ALGANT

2024-2025
Algebraic Geometry

Exercise Sheet 7

The field k is assumed to be algebraically closed.

Exercise 1 (Projective Jacobian criterion) Let $X \subseteq \mathbb{P}^n(k)$ be an integral projective projective variety and let $x = [X_0, \ldots, X_n] \in X$. Assume that $I(X) = (F_1, \ldots, F_r)$, with F_i's homogeneous. Let M be the matrix defined by $M_{ij} = \partial F_i / \partial X_j(x)$, with $i = 1, \ldots, r$ and $j = 0, \ldots, n$.
Show that X is non-singular in x if and only if $rk(M) = n - \dim X$.

Exercise 2 (Stalks) Let X be an integral variety.

1. For any $x \in X$, show that we have a canonical map $\mathcal{O}_{X,x} \to K(X)$.
2. Consider all the $\mathcal{O}_{X,x}$ as subrings of $K(X)$, show for any open affine $U \subseteq X$ the equality
$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}$$
in $K(X)$.

Exercise 3 (Characterise normality) Let X be an affine integral variety.

1. Assume that X is normal. Show that for any irreducible closed subvariety $Y \subset X$, the local ring $A(X)_{I(Y)}$ is integrally closed.
2. Prove that if for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is integrally closed, then X is normal.

Exercise 4 (Some normal varieties) Suppose that $char(k) \neq 2$.

1. Show that every conic in $\mathbb{A}^2(k)$ is normal.
2. Show that the quadric surfaces
   - $Q_1 = Z(xy - zw) \subset \mathbb{P}^3(k)$;
   - $Q_2 = Z(xy - z^2) \subset \mathbb{P}^3(k)$

   are normal.
3. Are the previous surfaces smooth?