

Exercise Sheet 9

The field k is assumed to be algebraically closed.

Exercise 1 (Consequences of Hurwitz formula)

Let $f: X \rightarrow Y$ a separable morphism of smooth proper curves.

1. Prove that $g(X) \geq g(Y)$.

Let $f: X \rightarrow Y$ be a separable unramified cover of smooth proper curves.

2. Prove that if $g(Y) = 0$, then f is an isomorphism.

Exercise 2 (Finite morphism of curves, $\text{char}(k) = 0$)

Let $C_0 = Z(x^4 + y^5 - x) \subset \mathbb{A}^2(k)$ and let C be the projective closure of C_0 in $\mathbb{P}^2(k)$.

1. Show that C is smooth.

Let $\pi_x: C_0 \rightarrow \mathbb{A}^1(k)$ the projection on the x -coordinate and $f: C \rightarrow \mathbb{P}^1(k)$ be the extension of π_x .

2. Determine zeros and poles of f .
3. Determine the degree and the ramification points of f .
4. Use Hurwitz formula to determine the genus of C .

Exercise 3 (Rational functions on curves, $\text{char}(k) = 0$)

Let $C = Z(X_0^4 + X_0X_1^3 + X_2^4) \subset \mathbb{P}^2(k)$ and let f be the rational function on C defined by X_0/X_2 .

1. Determine zeros and poles of f .
2. Determine the degree and the ramification points of f .
3. Use Hurwitz formula to determine the genus of C .
4. Find a basis of regular differentials on C . (Try to produce local regular differentials on an affine chart which extend to regular differentials on C)

Exercise 4 (Genus formula, $\text{char}(k) = 0$) Let $C = Z(F(X_0, X_1, X_2)) \subset \mathbb{P}^2(k)$ be an integral smooth plane curve of degree d and genus $g = g(C)$. Let $U_i := \{X_i \neq 0\}$ with $i = 0, 1, 2$ the affine cover of $\mathbb{P}^2(k)$ and $(x_0 = X_1/X_0, y_0 = X_2/X_0)$ the affine coordinates on U_0 .

1. Show that

$$\omega_0 = \frac{dx_0}{\partial F(1, x_0, y_0)/\partial y_0}$$

defines a regular differential on $C \cap U_0$.

2. Extend ω_0 to a rational differential ω on C compute a canonical divisor (ω) .
3. Deduce the genus formula

$$g = \frac{(d-1)(d-2)}{2}.$$