Exercise Sheet 4

The field k is assumed to be algebraically closed.

Exercise 1 (Integral varieties) Let X be an algebraic variety over k. Then the following properties are equivalent:

1. X is integral;
2. There exists an open affine covering $X = \cup_i U_i$ such that U_i is irreducible for all i and $U_i \cap U_j \neq \emptyset$ for all i, j.
3. Any non-empty affine open subset of X is integral.

Exercise 2 (Zero-dimensional varieties) Let X be an algebraic variety. Show that $\dim X = 0$ if and only if X is a finite set.

Exercise 3 (Homogeneous ideals) Let $R = k[T_0, \ldots, T_n]$ and denote by R_d the vector space of homogeneous polynomials of degree d.

1. Show that an ideal I of R is homogeneous if and only if $I = \oplus_{d \geqslant 0} I \cap R_d$.
2. If I is homogeneous, show that $\sqrt{I}$ is homogeneous.
3. If I, J are homogeneous, show that $I + J, IJ$ and $I \cap J$ are homogeneous.
4. Let I be an ideal, let $I^h = \oplus_{d \geqslant 0} I \cap R_d$. Show that I^h is the homogeneous ideal generated by the homogeneous elements of I. Show that if I is prime then so is I^h.
5. If I is homogeneous, show that any prime ideal over I (prime ideal minimal among those containing I) is homogeneous and that $\sqrt{I}$ is a finite intersection of homogeneous prime ideals.

Exercise 4 (Induced projective morphism) Let $\phi : k[T_0, \ldots, T_n] \to k[S_0, \ldots, S_m]/J$ be a homogeneous homomorphism (there exists $r \geqslant 1$ such that the image of any homogeneous element of degree d has degree rd). Show that ϕ induces a morphism of algebraic varieties

$$Z_+(J) \setminus Z_+(\phi(T_0, \ldots, T_n)) \to \mathbb{P}^n.$$

Exercise 5 (Dominant and birational morphisms) Let $f : X \to Y$ be a morphism of integral varieties over k. We say that f is *dominant* if $f(X)$ is dense in Y.
Show that f then induces an injective homomorphism $k(Y) \to k(X)$, and $\dim Y \leqslant \dim X$.

We say that f is *birational* if f is dominant and if $k(Y) \to k(X)$ is an isomorphism.
Show that this is equivalent to say that there are dense open subsets U of X and V of Y such that $f|_U : U \to V$ is an isomorphism.