Exercise Sheet 3

The field k is assumed to be algebraically closed.

Exercise 1 (More on irreducible spaces) Let X be a topological space and U a non-empty open of X.

1. Show that the irreducible components of U are the $\{X_i \cap U\}$, where the X_i sre the irreducible components of X such that $X_i \cap U \neq \emptyset$.
2. Let Z be an irreducible component of U, show that the closure $\overline{Z}$ is an irreducible component of X.

Exercise 2 (Affine varieties & isomorphisms)

1. Show that any conic in $\mathbb{A}^2(k)$ is isomorphic either to $\mathbb{A}^1(k)$ or $\mathbb{A}^1(k) \setminus \{0\}$.
2. Show that $\mathbb{A}^1(k)$ is not isomorphic to any proper open subset of itself.
3. Show that $\mathbb{A}^2(k) \setminus \{(0,0)\}$ is not an affine variety.

Exercise 3 (Blow-up construction) Let $U_1 \cong \mathbb{A}^2(k)$ and $U_2 \cong \mathbb{A}^2(k)$ be two affine varieties isomorphic to the plane, with coordinates (x_1, y_1) and (x_2, y_2) respectively. Consider the two morphisms $f_i \colon U_1 \to \mathbb{A}^2(k)$ of affine schemes induced by

$$f_1^* \colon k[x,y] \to k[x_1,y_1] \quad x \mapsto x_1, \quad y \mapsto x_1 y_1$$

$$f_2^* \colon k[x,y] \to k[x_2,y_2] \quad x \mapsto x_2 y_2, \quad y \mapsto y_2$$

1. Show that f_1 and f_2 induce isomorphisms $D(x_1) \xrightarrow{\sim} D(x)$ and $D(y_2) \xrightarrow{\sim} D(y)$;
2. Let X be the algebraic variety obtained by glueing U_1 and U_2 via the isomorphism
$$f_{12} \colon U_{12} = D(x_1) \cap D(y_2) \xrightarrow{\sim} D(x) \cap D(y) \xleftarrow{\sim} D(x_2) \cap D(y_2) = U_{21}.$$
Write down the homomorphism of k-algebras induced by f_{12} and construct an induced glued morphism $f \colon X \to \mathbb{A}^2(k)$.
3. Show that f induces an isomorphism on $f|_{X \setminus f^{-1}(0,0)} \colon X \setminus f^{-1}(0,0) \xrightarrow{\sim} \mathbb{A}^2(k) \setminus \{(0,0)\}$. Can you describe $f^{-1}(0,0)$?

Exercise 4 (Points of $\mathbb{P}^n(k)$) Consider the set $\operatorname{Proj} k[T_0, \dots, T_n]$ of homogeneous prime ideals, different from $(T_0, \dots, T_n)$, and maximal for this property.

1. Let $[a_0 : \dots : a_n] \in \mathbb{P}^n(k)$. Show that $(a_i T_j - a_j T_i)_{i,j} \in \operatorname{Proj} k[T_0, \dots, T_n]$ (consider the quotient ring).
2. Let $f \in k[T_0, \dots, T_n]$ be homogeneous.
Show that $f(a_0, \dots, a_n) = 0$ if and only if $f \in (a_i T_j - a_j T_i)_{0 \leqslant i,j \leqslant n}$.
3. Show the above processus defines an injective map $\mathbb{P}^n(k) \to \operatorname{Proj} k[T_0, \dots, T_n]$.
4. Show the above map is surjective (localise at some T_i).

Exercise 5 (Finite actions on affine varieties) Assume $\operatorname{char}(k) = 0$.
We define the following action of $\mathrm{GL}_n(k)$ on $k[x_1, \dots, x_n]$:

$$\forall g \in G, \quad g \cdot f(x) := f(g^{-1} \cdot x),$$

where the action of the RHS is the left action of $\mathrm{GL}_n(k)$ on k^n.
We fix a finite subgroup $G < \mathrm{GL}_n(k)$. The algebra of G-invariant polynomials is

$$k[x_1, \dots, x_n]^G := \{f \in k[x_1, \dots, x_n] \mid g \cdot f = f, \quad \forall g \in G\}.$$

1. Define the operator

$$\begin{aligned} \phi\colon k[x_1,\dots,x_n] &\to k[x_1,\dots,x_n]^G \\ f &\mapsto \frac{1}{|G|}\sum_{g\in G} g\cdot f \end{aligned}$$

and prove that

$$\phi(fh) = f\phi(h), \quad \forall f \in k[x_1,\dots,x_n]^G \text{ and } h \in k[x_1,\dots,x_n]$$

2. Let $I \subseteq k[x_1,\dots,x_n]$ be the ideal generated by homogeneous polynomials of positive degree in $k[x_1,\dots,x_n]^G$. Prove that there exist $f_1,\dots,f_k \in I$ homogeneous and G-invariant such that the k-algebra $k[x_1,\dots,x_n]^G$ is generated by $f_1,\dots,f_k$.

3. Identify the quotient $\mathbb{A}^n(k)/G$ with an affine variety.

4. Describe $k[x_1,\dots,x_n]^G$ and $\mathbb{A}^n(k)/G$ in the following cases

   a. $n=2$, $G = \left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1\end{pmatrix}\right\}$.

   b. $n=2$, $G = \left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\right\}$.

   c. $n \geqslant 2$, $G = \mathcal{S}_n$.