

Exercise 1

Let K be a finite field. Prove that the only absolute value on K is the trivial one.

Exercise 2

a. Let $(K, |\cdot|)$ be a valued field. Show that $|\cdot|^s$ is an absolute value on K for all $s \in]0, 1]$.

b. What are the $s \in \mathbb{R}_+^*$ such that $|\cdot|_\infty^s$ is an absolute value on \mathbb{Q} ?

Exercise 3 : incompleteness of \mathbb{Q}

Let $(K, |\cdot|)$ be a complete valued field such that $|\cdot|$ is non-trivial. Using Baire's theorem, prove that K is uncountable. Deduce that \mathbb{Q} is non-complete for any of its non-trivial absolute values.

Exercise 4

Let $(K, |\cdot|)$ be a valued field.

a. Show that if $|\cdot|$ is ultrametric on a subfield of K , then it is ultrametric on K .

b. Deduce that if $\text{char}(K) \neq 0$, then every absolute value on K is ultrametric.

Exercise 5

Let K be a field and v be a non-trivial valuation on K . Prove that the following are equivalent :

- (a) \mathcal{O}_v is a principal ideal domain ;
- (b) \mathcal{O}_v is noetherian ;
- (c) the ideal \mathcal{M}_v is principal ;
- (d) $v(K^*)$ is a discrete subgroup of \mathbb{R} .

Exercise 6

Let $(K, |\cdot|)$ be an ultrametric valued field.

- a. Let $r \in]0, 1]$. Show that the open ball $B(1, r)$ is a subgroup of K^* .
- b. Prove that each open ball in K is closed.
- c. Deduce that the connected components of K are its points.

Exercise 7

Let $(K, | \cdot |)$ be a valued field. Let $(\widehat{K}, | \cdot |)$ be a complete valued field and $\iota : K \rightarrow \widehat{K}$ be a morphism of valued fields such that $\iota(K)$ is dense in \widehat{K} . Show that if $(L, | \cdot |')$ is a complete valued field and $f : K \rightarrow L$ a morphism of valued fields, then there exists a unique morphism of valued fields $\widehat{f} : \widehat{K} \rightarrow L$ such that $f = \widehat{f} \circ \iota$.

Exercise 8

Let $(K, | \cdot |)$ be an ultrametric locally compact valued field such that $| \cdot |$ isn't trivial. Prove that its residue field is finite and that $|K^*|$ is a discrete subgroup of \mathbb{R}_+^* .

✓ Exercise 1

Let $(K, | \cdot |)$ be an ultrametric valued field. Prove that $|K^*| = |\hat{K}^*|$ and that the residue fields of K and of \hat{K} are isomorphic.

✗ Exercise 2

Let p be a prime number. Show that the groups $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ and $\mathbb{Q}_p/\mathbb{Z}_p$ are isomorphic.

✗ Exercise 3

Find examples of two complete ultrametric valued fields whose respective residue fields and value groups are isomorphic, but which aren't isomorphic as fields.

Exercise 4

Let $(K, | \cdot |)$ be a valued field such that $| \cdot |$ is non-trivial. Let V be a K -vector space. Prove that two norms $\| \cdot \|_1, \| \cdot \|_2$ on V are equivalent if and only if there exist constants c, c' in \mathbb{R}_+^* such that $\forall v \in V \ c\|v\|_1 \leq \|v\|_2 \leq c'\|v\|_1$.

✗ Exercise 5

Let K be a field endowed with the trivial absolute value $| \cdot |_0$.

a. Let $\rho \in]0, 1[$; put $\|f\|_\rho = \rho^{v_X(f)}$ for all $f \in K[[X]]$. Check that $\| \cdot \|_\rho$ is a norm on the K -vector space $K[[X]]$.

b. Let $(r, s) \in \mathbb{R}^2$ be such that $0 < r < s < 1$. Show that $\| \cdot \|_r$ and $\| \cdot \|_s$ are equivalent, but that there doesn't exist $c \in \mathbb{R}_+^*$ such that $\| \cdot \|_s \leq c \| \cdot \|_r$.

Exercise 6

Let $(V, \| \cdot \|_1)$ and $(W, \| \cdot \|_2)$ be normed vector spaces over a complete valued field $(K, | \cdot |)$. Assume that V is finite dimensional.

✗ a. Prove that any K -linear map $f : V \rightarrow W$ is continuous.

b. Show that any surjective K -linear map $f : V \rightarrow W$ is open.

Exercise 7

2

Let $(K, | \cdot |)$ be a complete ultrametric valued field and $P \in \mathcal{O}_K[X]$ be a monic polynomial. Prove that if P is irreducible in $\mathcal{O}_K[X]$, its image in $\kappa_K[X]$ is the power of an irreducible polynomial.

Exercise 8

2

Consider $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$ as a \mathbb{Q} -vector space, endowed with the restriction $\| \cdot \|$ of the usual absolute value $| \cdot |_\infty$. Put $\|x + y\sqrt{2}\|' = \max(|x|_\infty, |y|_\infty)$ for all $(x, y) \in \mathbb{Q}^2$. Show that the norms $\| \cdot \|$ and $\| \cdot \|'$ aren't equivalent on $\mathbb{Q}(\sqrt{2})$.

Exercise 9

Let A be a local noetherian ring such that its maximal ideal \mathcal{M} is generated by a non-nilpotent element π . The goal of this exercise is to prove that A is a principal ideal domain.

a. Verify that $I = \{x \in A \mid \exists m \in \mathbb{N} \pi^m x = 0\}$ is an ideal of A and that there exists $N \in \mathbb{N}$ such that $\pi^N I = \{0\}$.

b. Show that $\bigcap_{n \in \mathbb{N}} \mathcal{M}^n = \{0\}$. *Hint* : if $y = \pi^n x_n$ for every $n \in \mathbb{N}$, then the sequence $(I + x_n A)_{n \in \mathbb{N}}$ is ascending.

c. Prove that any element $y \in A \setminus \{0\}$ can be written in a unique way $y = \pi^{v(y)} u$ with $v(y) \in \mathbb{N}$ and $u \in A^\times$, and that A is an integral domain.

d. Conclude.

Exercise 1

Let L be a finite extension of a field K , $x \in L$, and M be a finite extension of L . Show that $N_{M/K}(x) = N_{L/K}(x)^{[M:L]}$.

Exercise 2

Let p be a prime number.

- a. Let $n \in \mathbb{N}$ and $n = \sum_{i=0}^r a_i p^i$ (with $a_i \in [[0, p-1]]$ for all i) be its writing in base p . Prove that $v_p(n!) = \frac{n - s_n}{p-1}$, where $s_n = \sum_{i=0}^r a_i$ (sum of the digits).
- b. Let $x \in \mathbb{C}_p$. Deduce that the series $\sum_{n \geq 0} \frac{x^n}{n!}$ converges if and only if $v_p(x) > \frac{1}{p-1}$.

Exercise 3

Let p be a prime number.

- a. Let $u \in \mathbb{Q}_p^*$. Show that the following are equivalent :
- (a) $u \in \mathbb{Z}_p^\times$;
 - (b) u^{p^n-1} is a n -th power in \mathbb{Q}_p for infinitely many $n \in \mathbb{N}^*$.

b. Prove that the only field endomorphism of \mathbb{Q}_p is $\text{Id}_{\mathbb{Q}_p}$.

Exercise 4

Let p be a prime number. Show that $\mathbb{Q}_p^{*2} = \{x^2 ; x \in \mathbb{Q}_p^*\}$ is open in \mathbb{Q}_p .

Exercise 5

Let p be a prime number. Prove that $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid \exists y \in \mathbb{Q}_p \ y^2 = 1 + p^3 x^4\}$.

Exercise 6

Let $(K, |\cdot|)$ be an ultrametric valued field, and $\rho \in \mathbb{R}_+^*$. If $P = a_0 + a_1X + \dots + a_nX^n \in K[X]$, put $|P|_\rho = \max_{i \in \llbracket 0, n \rrbracket} |a_i| \rho^i$. Check that $|\cdot|_\rho$ extends into an absolute value on $K(X)$. When are two such absolute values equivalent?

Exercise 7

↳ jamais car si on a $|f| > 0$... et on considère $P = X \Rightarrow \rho = 1$

Show that $|\cdot|_\infty$ is the unique absolute value on \mathbb{C} that extends the usual absolute value of \mathbb{R} .

Exercise 8 : Ostrowski

Let $(K, |\cdot|)$ be a complete valued field such that $|2| = 2$. Assume that there exists $i \in K$ such that $i^2 = -1$.

a. Construct a morphism $(\mathbb{C}, |\cdot|_\infty) \rightarrow (K, |\cdot|)$ of valued fields. We identify \mathbb{C} with its image in K .

Let $a \in K$ and consider the map $f : \begin{matrix} \mathbb{C} & \rightarrow & \mathbb{R}_+ \\ z & \mapsto & |z - a| \end{matrix}$. Let $r = \inf_{\mathbb{C}} f$.

b. Verify that $f^{-1}(r)$ is closed, bounded and non-empty.

c. Prove that if $r > 0$ and $\gamma_0 \in f^{-1}(r)$, then $B(\gamma_0, r) \subset f^{-1}(r)$. *Hint : if $|\gamma - \gamma_0|_\infty < r$, consider $(\gamma_0 - a)^n - (\gamma_0 - \gamma)^n$.*

d. Conclude that $K = \mathbb{C}$.

Exercise 1

Let A be an integral domain and $\alpha \in A \setminus \{0\}$. Assume that $A[\alpha^{-1}]$ is integrally closed and that $A/\alpha A$ is reduced. Prove that A is integrally closed.

Exercise 2

Let B be a commutative ring and A be a subring of B such that B is integral over A .

a. Assume that B is an integral domain. Show that A is a field if and only if B is a field.

b. Let $\mathcal{Q} \subset B$ be a prime ideal. Deduce that $\mathcal{Q} \cap A$ is a maximal ideal in A if and only if \mathcal{Q} is maximal in B .

c. Let $\mathcal{Q}_1 \subset \mathcal{Q}_2$ be prime ideals in B such that $\mathcal{Q}_1 \cap A = \mathcal{Q}_2 \cap A$. Prove that $\mathcal{Q}_1 = \mathcal{Q}_2$.

Exercise 3

Let $(K, |\cdot|)$ be a complete ultrametric valued field. Let $P = a_n X^n + a_{n-1} X^{n-1} + \dots + a_1 X + a_0 \in K[X]$ be irreducible. Show that $|a_i|^n \leq |a_n|^i |a_0|^{n-i}$ for every $i \in [0, n]$ (*hint* : consider the splitting field of P over K). Deduce that $|P|_G = \max(|a_0|, |a_n|)$.

Exercise 4

Let $(K, |\cdot|)$ be an ultrametric valued field, L be a finite extension of K and $|\cdot|_L$ be an absolute value on L extending $|\cdot|$.

a. Show that if $\alpha \in L$ is integral over \mathcal{O}_K , then $\alpha \in \mathcal{O}_L$ (in particular, \mathcal{O}_K is integrally closed).

b. Prove that the converse holds if K is complete.

c. Show with an example that it does not hold in general.

Exercise 5

Let $(K, |\cdot|)$ be a complete ultrametric valued field, and L be a finite extension of K . Prove that if $\|\cdot\|$ is any norm on the K -vector space L , the map $x \mapsto \lim_{n \rightarrow +\infty} \|x^n\|^{1/n}$ coincides with the unique absolute value extending $|\cdot|$ on L .

Exercise 6

Let p be a prime number such that $p \equiv 3 \pmod{4}$. Let $\overline{\mathbb{Q}_p}$ be an algebraic closure of \mathbb{Q}_p .

a. Verify that $X^2 + 1$ is irreducible in $\mathbb{Q}_p[X]$.

Let $i \in \overline{\mathbb{Q}_p}$ be a root of $X^2 + 1$, $K = \mathbb{Q}_p[i]$ and $|\cdot|_K$ be the extension of $|\cdot|_p$ to K .

b. If $(a, b) \in \mathbb{Q}_p^2$ and $x = a + bi$, show that $|x|_K = \max(|a|_p, |b|_p)$.

c. Determine the value group and the residue field of $(K, |\cdot|_K)$.

Exercise 7

Let $(K, |\cdot|)$ be a complete ultrametric valued field and $P \in K[X] \setminus K$.

a. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements in K such that $\lim_{n \rightarrow +\infty} |P(x_n)| = 0$. Prove that there is a subsequence of $(x_n)_{n \in \mathbb{N}}$ that converges to a root of P in K .

b. Deduce that if $F \subset K$ is closed, then its image $P(F)$ is closed.

c. Show that if $C \subset K$ is compact, then its inverse image $P^{-1}(C)$ is compact.

Exercise 1

Let $(K, |\cdot|)$ be an ultrametric valued field. Denote by $|\cdot|_G$ the Gauss absolute value on $K(X)$. What is the residue field of $(K(X), |\cdot|_G)$?

Exercise 2

Let $(K, |\cdot|)$ be a complete valued field and L be a quadratic extension of K . The goal of this exercise is to prove that the map $x \mapsto |N_{L/K}(x)|^{1/2}$ is an absolute value on L that extends $|\cdot|$.

a. Let $x \in L \setminus K$; put $\alpha = \text{Tr}_{L/K}(x)$ and $\beta = N_{L/K}(x)$. Assume that $4|\beta| < |\alpha|^2$. Consider the map $f : K^* \rightarrow K$ defined by $f(t) = \alpha - \frac{\beta}{t}$ for all $t \in K^*$ and the set $C = \{t \in K \mid 2|t| \geq |\alpha|\}$. Show that $f(C) \subset C$ and that $f|_C$ is contractive, and use the fixed point theorem to contradict the hypothesis.

b. Conclude.

Exercise 3

Prove that a complete archimedean valued field is isomorphic to $(\mathbb{R}, |\cdot|_\infty)$ or $(\mathbb{C}, |\cdot|_\infty)$ for some $s \in]0, 1]$. *Hint* : use exercise 8 of sheet 3.

Exercise 4

Let A be a semi-local Dedekind ring. Show that A is a principal ideal domain.
Hint : use the Chinese remainder theorem.

Exercise 5

Let $\overline{\mathbb{Q}_p}$ be an algebraic closure of \mathbb{Q}_p . Let $P = X^3 - 17$ and $j \in \overline{\mathbb{Q}_3}$ be a primitive cubic root of unity.

a. Verify that $j \notin \mathbb{Q}_3$.

b. What are the degrees of the irreducible factors of P in $\mathbb{Q}_3[X]$? *Hint* : compute $P(5)$.

c. How many extensions to $\mathbb{Q}(\sqrt[3]{17})$ does the 3-adic absolute value have?

Exercise 6

Let p be a prime number. Let $x \in \mathbb{Q}_p^*$ and $x = \sum_{n=v_p(x)}^{+\infty} a_n p^n$ (with $a_n \in [0, p-1]$ for all n) be its p -adic development. What is the p -adic development of $-x$?

Exercise 7

The p -adic absolute value on \mathbb{Q}_p extends uniquely to an absolute value $|\cdot|_p$ on $\overline{\mathbb{Q}_p}$. For $n \in \mathbb{N}^*$, put $H_n = \{x \in \overline{\mathbb{Q}_p} \mid [\mathbb{Q}_p(x) : \mathbb{Q}_p] \leq n\}$.

- Prove that H_n is closed. *Hint* : use the fact that \mathbb{Z}_p is compact.
- Show that $H_n \neq \overline{\mathbb{Q}_p}$ for all $n \in \mathbb{N}^*$.
- Verify that $H_n + H_m \subset H_{nm}$ for all $(n, m) \in \mathbb{N}^{*2}$.
- Using Baire's theorem, deduce that $\overline{\mathbb{Q}_p}$ isn't complete for $|\cdot|_p$.

$$(\mathbb{Q}(\sqrt[3]{2}), 1.1_2)$$

$$(\mathbb{Q}(\sqrt[3]{2}), 1.1)$$

$$(\mathbb{Q}, 1.1_2)$$

$$\text{or } |\sqrt[3]{2}| = |j^2 \sqrt{2}|_2$$

$$\hookrightarrow x^3 - \frac{1}{2} \in \mathbb{Q}[x] \\ \notin \mathbb{Q}_q[x]$$