Exercise Sheet 8

The field k is assumed to be algebraically closed.

Exercise 1 (Open subsets of curves) Let X be an integral smooth proper curve and $D > 0$ a divisor on X. Let $U = X \setminus Supp(D)$.

1. Prove that $\mathcal{O}_X(U) = \bigcup_{n \geqslant 0} L(nD)$ as subsets of $k(X)$.
2. Suppose that $L(D) \neq k$ and prove that $Frac(\mathcal{O}_X(U)) = k(X)$.

Let V be any strict open subset of X.

3. Show that V is affine.

Exercise 2 (Easy consequences of Riemann-Roch) Let X be an integral smooth proper curve of genus g.

1. Show that for any divisor D of degree $degD > 2g - 2$, we have $l(D) = degD + 1 - g$.
2. Show that $g = 0$ iff $X \simeq \mathbb{P}^1(k)$.

Exercise 3 (Genus-one curves) Let X be an integral smooth proper curve of genus $g = 1$. Let $x_0 \in X$ be a fixed point.

1. Show that $L([x_0]) = k$ and there exist $x \in L(2[x_0]) \setminus k$ and $y \in L(3[x_0]) \setminus L(3[x_0])$.
2. Let $I = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i \geqslant 0, 0 \leqslant j \leqslant 1, 2i + 3j \leqslant n\}$. Prove that
$$L(n[x_0]) = \bigoplus_{(i,j) \in I} kx^i y^j.$$
3. Show that X is isomorphic to a cubic in $\mathbb{P}^2(k)$ defined by the equation
$$X_1^2 X_3 + (a_1 X_0 X_3 + a_3 X_3^2) X_1 = X_0^3 + a_2 X_0^2 X_3 + a_4 X_0 X_3^2 + a_6 X_3^3$$
for some $a_1, a_3, a_2, a_4, a_6 \in k$. (Hint: use Exercise 1).
4. Show that the map $\theta \colon X \to Pic^0(X)$, defined by $x \mapsto [x] - [x_0]$ is bijective.
In particular this induces a commutative group structure on X.

Exercise 4 (Genus-two curves) Let X be an integral smooth proper curve of genus $g = 2$.
Show that there exists a finite separable morphism $X \to \mathbb{P}^1(k)$ of degree 2.
(Hint: use the same strategy via Riemann-Roch as in Exercise 3).