Exercise Sheet 10

The field k is assumed to be algebraically closed.

Exercise 1 Let C be a smooth projective curve and D a divisor on C. Consider the linear system $|D|$, i.e. the set of effective divisors linearly equivalent to D.

1. Identify $|D|$ with $\mathbb{P}(L(D))$.
2. Let $Bs(|D|) := \bigcap_{E \in |D|} Supp(E)$ be the *base locus* of $|D|$. Show that $Bs(|D|) = \emptyset$ iff $\forall p \in C$
$$\dim |D - p| = \dim |D| - 1.$$
We say in this case that $|D|$ is basepoint free.
3. We say that D is *very ample* if $\forall p, q \in C$
$$\dim |D - p - q| = \dim |D| - 2.$$
Show that
   - if $\deg D \geqslant 2g$, then $|D|$ is basepoint free.
   - if $\deg D \geqslant 2g + 1$, then D is very ample.
4. Show that a basepoint free linear system $|D|$ induces a morphism
$$\phi_{|D|} \colon C \to \mathbb{P}(L(D)) \simeq \mathbb{P}^n(k)$$
$$p \mapsto [s_0(p) : \cdots s_n(p)],$$
where the $s_i \in L(D)$ form a basis of $L(D)$ and that $\phi_{|D|}$ is unique up to a projectivity of the target.
5. Show that $\phi_{|D|}$ is injective iff for any $p, q \in C$ *distinct* points,
$$\dim |D - p - q| = \dim |D| - 2.$$
6. Assume that $\phi_{|D|}$ is injective. Prove that $\phi_{|D|}$ is a closed immersion iff for any $p \in C$,
$$\dim |D - 2p| = \dim |D| - 2.$$
(hint: start proving that $\phi_{|D|}$ is a closed immersion iff for any $p \in C$ the set $\{s \in L(D) \mid s_p \in \mathfrak{m}_p\}$ generate $\mathfrak{m}_p/\mathfrak{m}_p^2$).
7. Conclude that D is very ample iff $\phi_{|D|}$ is a closed immersion.