Exercise Sheet 9

The field k is assumed to be algebraically closed.

Exercise 1 (Consequences of Hurwitz formula)
Let $f\colon X \to Y$ a separable morphism of smooth proper curves.

1. Prove that $g(X) \geqslant g(Y)$.

Let $f\colon X \to Y$ be a separable unramified cover of smooth proper curves.

2. Prove that if $g(Y) = 0$, then f is an isomorphism.

Exercise 2 (Finite morphism of curves, $char(k) = 0$)
Let $C_0 = Z(x^4 + y^5 - x) \subset \mathbb{A}^2(k)$ and let C be the projective closure of C_0 in $\mathbb{P}^2(k)$.

1. Show that C is smooth.

Let $\pi_x\colon C_0 \to \mathbb{A}^1(k)$ the projection on the x-coordinate and $f\colon C \to \mathbb{P}^1(k)$ be the extension of π_x.

2. Determine zeros and poles of f.
3. Determine the degree and the ramification points of f.
4. Use Hurwitz formula to determine the genus of C.

Exercise 3 (Rational functions on curves, $char(k) = 0$)
Let $C = Z(X_0^4 + X_0X_1^3 + X_2^4) \subset \mathbb{P}^2(k)$ and let f be the rational function on C defined by X_0/X_2.

1. Determine zeros and poles of f.
2. Determine the degree and the ramification points of f.
3. Use Hurwitz formula to determine the genus of C.
4. Find a basis of regular differentials on C. (Try to produce local regular differentials on an affine chart which extend to regular differentials on C)

Exercise 4 (Genus formula, $char(k) = 0$) Let $C = Z(F(X_0, X_1, X_2)) \subset \mathbb{P}^2(k)$ be an integral smooth plane curve of degree d and genus $g = g(C)$. Let $U_i := \{X_i \neq 0\}$ with $i = 0, 1, 2$ the affine cover of $\mathbb{P}^2(k)$ and $(x_0 = X_1/X_0, y_0 = X_2/X_0)$ the affine coordinates on U_0.

1. Show that
$$\omega_0 = \frac{dx_0}{\partial F(1, x_0, y_0)/\partial y_0}$$
defines a regular differential on $C \cap U_0$.
2. Extend ω_0 to a rational differential ω on C compute a canonical divisor (ω).
3. Deduce the genus formula
$$g = \frac{(d-1)(d-2)}{2}.$$