Exercise Sheet 6

The field k is assumed to be algebraically closed.

Exercise 1 (Connected rings) A ring R is said to be *connected* if every idempotent in R is trivial (i.e., if every element $e \in R$ such that $e^2 = e$ is equal to 0 or 1).

1. Prove that every integral domain is connected.
2. If R is the product of two non-trivial rings, prove that R is not connected.
3. Conversely, if R possesses a non-trivial idempotent e, prove that $R \simeq R/(e) \times R/(1-e)$.
4. Let X be an affine algebraic variety over k. Prove that X is connected (in the Zariski topology) if and only if $A(X)$ is connected.

Exercise 2 (Separated varieties)

1. Show that affine varieties are separated.
2. Show that open subvarieties and closed subvarieties of a separated variety are separated.
3. Let X be a separated algebraic variety. Show that then the diagonal map $\Delta\colon X \to X \times X$ is a closed immersion.
4. Show that an algebraic variety X is separated if and only if there exists an affine covering $\{X_i\}_i$ of X such that for all i, j, the intersection $X_i \cap X_j$ is affine, and the canonical homomorphism $\mathcal{O}_X(X_i) \otimes_k \mathcal{O}_X(X_j) \to \mathcal{O}_X(X_i \cap X_j)$ defined as $f_i \otimes f_j \to f_i|_{X_i \cap X_j} f_j|_{X_i \cap X_j}$ is surjective.
5. Show that products of separated varieties are separated.
6. Show that projective varieties are separated.

Exercise 3 (Finite morphisms) Let X and Y be affine integral varieties and let $f\colon X \to Y$ a dominant morphism. Then the induced homomorphism $A(Y) \to A(X)$ is injective (by Partial Exam). We say that f is a finite morphism if $A(X)$ is integral over $A(Y)$.

1. Prove that a finite morphism is surjective.
2. Deduce that a finite morphism is closed.
3. Let $g\colon X \to Y$ be a morphism of affine integral varieties. Assume that every point $y \in Y$ has an affine neighbourhood $U \ni y$ such that $V = g^{-1}(U)$ is affine and $f\colon V \to U$ is finite. Prove that g is finite.

Exercise 4 (Dimension of intersections) Let X and Y be two integral algebraic subvarieties of respective dimensions r and s in $\mathbb{A}^n(k)$. Our aim is to prove that any irreducible component of $X \cap Y$ is of dimension $\geqslant r + s - n$.

1. Prove that this result is true if X is a hypersurface.
2. Let $\Delta = \Delta(\mathbb{A}^n(k))$ be the diagonal in $\mathbb{A}^n(k) \times \mathbb{A}^n(k)$. Prove that the variety $X \cap Y$ is isomorphic to $(X \times Y) \cap \Delta$.
3. Conclude using explicit equations for Δ.