Exercise Sheet 5

The field k is assumed to be algebraically closed.

Exercise 1 (Irreducible components of $Z(F)$ and $Z_+(G)$)

Let $F(T_1, \ldots, T_n) \in k[T_1, \ldots, T_n]$ be a non-constant polynomial.

1. Show that the minimal prime ideals over (F) are principal.
2. Show that the irreducible components of $Z(F) \subseteq \mathbb{A}^n(k)$ are exactly the $Z(P)$'s where P runs through the irreducible factors of F.

Let $G(T_0, \ldots, T_n) \in k[T_0, \ldots, T_n]$ be a non-zero homogeneous polynomial.

3. Using the decomposition into homogeneous components, show that the irreducible factors of G are homogeneous.
4. Show that the irreducible components of $Z_+(G) \subset \mathbb{P}^n(k)$ are exactly the $Z_+(P)$'s where P runs through the irreducible factors of G.

Exercise 2 (Dimension of affine subvarieties) Let X be an integral affine algebraic variety of dimension n and let $f_1, \ldots, f_s \in A(X)$. Show that every irreducible component of $Z(f_1, \ldots, f_s)$ has dimension $\geqslant n - s$

Exercise 3 (Parameters of affine subvarieties) Let X be an integral affine algebraic variety of dimension n and let Z be an integral affine algebraic subvariety of dimension $r < n$. Show that for every s such that $1 \leqslant s \leqslant n - r$, there exist $f_1, \ldots, f_s \in A(X)$ such that:

- $Z \subseteq Z(f_1, \ldots, f_s)$;
- All irreducible components of $Z(f_1, \ldots, f_s)$ have dimension $n - s$.

Deduce that Z is an irreducible component of $Z(f_1, \ldots, f_{n-r})$.

Exercise 4 (Dimension of fibres) Let $f\colon X \to Y$ be a dominant morphism of integral algebraic varieties and let y be a point of Y. Show that every irreducible component of the fibre $f^{-1}(y)$ has dimension at least $\dim X - \dim Y$. (Hint: you may want to use previous exercises)