An Introduction to Hawkes Processes with Applications to Finance

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Simple point processes

Point process

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be some probability space. Let $(t_i)_{i \in \mathbb{N}^*}$ a sequence of non-negative random variables such that $\forall i \in \mathbb{N}^*, t_i < t_i + 1$. We call $(t_i)_{i \in \mathbb{N}^*}$ a (simple) *point process* on $\mathbb{R}_+$.

In particular, the variables t_i can represent the times of occurrence of transactions, or arrival of limit orders in an order book, etc. We start counting events with index 1. If needed, we will assume that $t_0 = 0$.

Counting process and durations

Counting process

Let $(t_i)_{i\in\mathbb{N}^*}$ be a point process. The right-continuous process

$$N(t) = \sum_{i\in\mathbb{N}^*} \mathbf{1}_{t_i \leq t} \tag{1}$$

is called the *counting process* associated with $(t_i)_{i\in\mathbb{N}^*}$.

Duration

The process $(\delta t_i)_{i\in\mathbb{N}^*}$ defined by

$$\forall i \in \mathbb{N}^*, \;\; \delta t_i = t_i - t_{i-1} \tag{2}$$

is called the *duration process* associated with $(t_i)_{i\in\mathbb{N}^*}$.

Representation of a simple point process

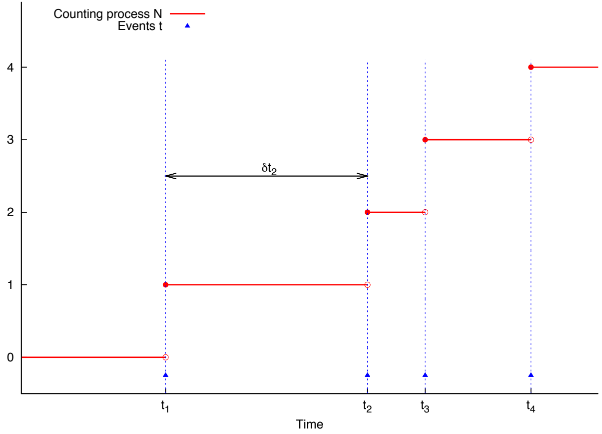

Figure: Illustration of a simple point process: events, counting process and duration process

Intensity process

Intensity

Let N be a point process adapted to a filtration $\mathcal{F}_t$. The left-continuous *intensity process* is defined as

$$\lambda(t|\mathcal{F}_t) = \lim_{h\downarrow 0} \mathrm{E}\left[\left. \frac{N(t+h) - N(t)}{h} \right| \mathcal{F}_t \right], \tag{3}$$

or equivalently

$$\lambda(t|\mathcal{F}_t) = \lim_{h\downarrow 0} \frac{1}{h} \mathbf{P}\left[N(t+h) - N(t) > 0 | \mathcal{F}_t \right]. \tag{4}$$

Intensity depends on the choice of filtration, but we will always assume that the filtration used is the natural one for the process N, denoted $\mathcal{F}_t^N$. We will therefore write $\lambda(t)$ instead of $\lambda(t|\mathcal{F}_t^N)$.

Example: the Poisson process

Homogeneous Poisson Process

Let $\lambda \in \mathbb{R}_+^*$. A Poisson process with constant rate λ is a point process defined by

$$\mathbf{P}\left[N(t+h) - N(t) = 1 | \mathcal{F}_t\right] = \lambda h + o(h), \quad (5)$$

$$\mathbf{P}\left[N(t+h) - N(t) > 1 | \mathcal{F}_t\right] = o(h). \quad (6)$$

- The intensity does not depend on the history of the process N, and the probability of occurrence of an event in $(t, t+h]$ is independent from $\mathcal{F}_t$.
- Durations $(\delta t_i)_{i \in \mathbb{N}^*}$ of an homogeneous Poisson process are independent and identically distributed (i.i.d.) according to an exponential distribution with parameter λ.

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Stochastic Time Change

Integrated intensity

The integrated intensity function Λ is defined as :

$$\forall i \in \mathbb{N}^*, \quad \Lambda(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} \lambda(s) ds. \tag{7}$$

Time change theorem

Let N be point process on $\mathbb{R}_+$ such that $\int_0^\infty \lambda(s) ds = \infty$. Let t_τ be the stopping time defined by

$$\int_0^{t_\tau} \lambda(s) ds = \tau. \tag{8}$$

Then the process $\tilde{N}(\tau) = N(t_\tau)$ is an homogeneous Poisson process with constant intensity $\lambda = 1$.

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Definition of a linear self-exciting process

Linear self-exciting process

A general definition for a linear self-exciting process N reads :

$$\begin{aligned} \lambda(t) &= \lambda_0(t) + \int_{-\infty}^{t} \nu(t-s)dN_s, \\ &= \lambda_0(t) + \sum_{t_i<t} \nu(t-t_i), \end{aligned} \tag{9}$$

where $\lambda_0 : \mathbb{R} \mapsto \mathbb{R}_+$ is a deterministic base intensity and $\nu : \mathbb{R}_+ \mapsto \mathbb{R}_+$ expresses the positive influence of the past events t_i on the current value of the intensity process.

Simple Hawkes process considered here

Hawkes process

Hawkes (1971) proposes an exponential kernel $\nu(t) = \sum_{j=1}^{P} \alpha_j e^{-\beta_j t} \mathbf{1}_{\mathbb{R}_+}$, so that the intensity of the model becomes :

$$\begin{aligned} \lambda(t) &= \lambda_0(t) + \int_0^t \sum_{j=1}^{P} \alpha_j e^{-\beta_j (t-s)} dN_s, \\ &= \lambda_0(t) + \sum_{t_i < t} \sum_{j=1}^{P} \alpha_j e^{-\beta_j (t-t_i)}, \end{aligned} \tag{10}$$

The simplest version with $P = 1$ and $\lambda_0(t)$ constant is defined as:

$$\lambda(t) = \lambda_0 + \int_0^t \alpha e^{-\beta(t-s)} dN_s = \lambda_0 + \sum_{t_i < t} \alpha e^{-\beta(t-t_i)}. \tag{11}$$

Sample path of a 1D-Hawkes process

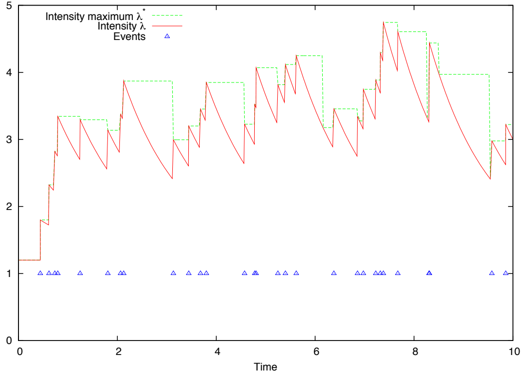

Figure: Simulation of a one-dimensional Hawkes process with parameters $P = 1, \lambda_0 = 1.2, \alpha_1 = 0.6, \beta_1 = 0.8$.

Stationarity (I)

Assuming stationarity gives $\mathrm{E}[\lambda(t)] = \mu$ constant. Thus,

$$
\begin{aligned}
\mu &= \mathrm{E}\left[\lambda(t)\right] = \mathrm{E}\left[\lambda_0 + \int_{-\infty}^{t} \nu(t-s) dN_s\right], \\
&= \lambda_0 + \mathrm{E}\left[\int_{-\infty}^{t} \nu(t-s)\lambda(s) ds\right], \\
&= \lambda_0 + \int_{-\infty}^{t} \nu(t-s)\mu ds, \\
&= \lambda_0 + \mu \int_{0}^{\infty} \nu(v) dv, \qquad (12)
\end{aligned}
$$

which gives :

$$
\mu = \frac{\lambda_0}{1 - \int_0^\infty \nu(v) dv}. \qquad (13)
$$

Stationarity (II)

Stationarity condition for a 1D-Hawkes process

$$\sum_{j=1}^{P} \frac{\alpha_j}{\beta_j} < 1. \tag{14}$$

Average intensity of a stationary process

Equation (13) immediately gives for the one-dimensional Hawkes process with $P = 1$ the unconditional expected value of the intensity process:

$$\mathrm{E}\left[\lambda(t)\right] = \frac{\lambda_0}{1 - \alpha/\beta}. \tag{15}$$

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Thinning procedure

Lewis & Shedler (1979) proposes a "thinning procedure" that allows the simulation of a point process with bounded intensity.

Basic thinning theorem

Consider a one-dimensional non-homogeneous Poisson process $\{N^*(t)\}_{t\geq 0}$ with rate function $\lambda^*(t)$, so that the number of points $N^*(T_0)$ in a fixed interval $(0, T_0]$ has a Poisson distribution with parameter $\mu_0^* = \int_0^{T_0} \lambda^*(s)ds$. Let $t_1^*, t_2^*, \dots, t_{N^*(T_0)}^*$ be the points of the process in the interval $(0, T_0]$. Suppose that for $0 \leq t \leq T_0, \lambda(t) \leq \lambda^*(t)$.

For $i = 1, 2, \dots, N^*(T_0)$*, delete the points* t_i^* *with probability* $1 - \frac{\lambda(t_i^*)}{\lambda^*(t_i^*)}$*. Then the remaining points form a non-homogeneous Poisson process* $\{N(t)\}_{t\geq 0}$ *with rate function* $\lambda(t)$ *in the interval* $(0, T_0]$*.*

Simulation algorithm (I)

Ogata (1981) proposes an algorithm for the simulation of Hawkes processes. Let us denote $\mathcal{U}_{[0,1]}$ the uniform distribution on the interval $[0,1]$ and $[0,T]$ the time interval on which the process is to be simulated. We'll assume here that $P = 1$.

Algorithm - Initialization

1. **Initialization** : Set $\lambda^* \leftarrow \lambda_0(0), \quad n \leftarrow 1$.
2. **First event** : Generate $U \rightsquigarrow \mathcal{U}_{[0,1]}$ and set $s \leftarrow -\frac{1}{\lambda^*} \ln U$.
   If $s \leq T$,
   Then $t_1 \leftarrow s$,
   Else go to last step.

Simulation algorithm (II)

Algorithm - General routine

3. **General routine** : Set $n \leftarrow n+1$.
   1. **Update maximum intensity**: Set $\lambda^* \leftarrow \lambda(t_{n-1}) + \alpha$.
      λ^* exhibits a jump of size α as an event has just occurred. λ being left-continuous, this jump is not counted in $\lambda(t_{n-1})$, hence the explicit addition.
   2. **New event** : Generate $U \rightsquigarrow \mathcal{U}_{[0,1]}$ and set $s \leftarrow s - \frac{1}{\lambda^*} \ln U$.
      If $s \geq T$,
      Then go to the last step.
   3. **Rejection test** : Generate $D \rightsquigarrow \mathcal{U}_{[0,1]}$.
      If $D \leq \frac{\lambda(s)}{\lambda^*}$,
      Then $t_n \leftarrow s$ and go through the general routine again,
      Else update $\lambda^* \leftarrow \lambda(s)$ and try a new date at step (b) of the general routine.
4. **Output**: Retrieve the simulated process $\{t_n\}$ on $[0, T]$.

Examples of simulations (I)

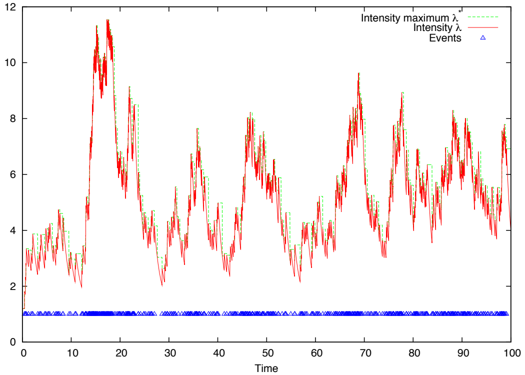

Figure: Simulation of a one-dimensional Hawkes process with parameters $P = 1, \lambda_0 = 1.2, \alpha_1 = 0.6, \beta_1 = 0.8$.

Examples of simulations (II)

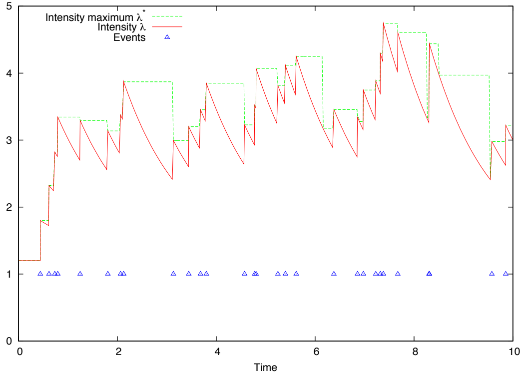

Figure: Simulation of a one-dimensional Hawkes process with parameters $P = 1, \lambda_0 = 1.2, \alpha_1 = 0.6, \beta_1 = 0.8$. (Zoom of the previous figure).

Testing the simulated process (I)

For any consecutive events t_{i-1} and t_i :

$$\begin{aligned}
\Lambda(t_{i-1}, t_i) &= \int_{t_{i-1}}^{t_i} \lambda(s) ds \qquad (16) \\
&= \int_{t_{i-1}}^{t_i} \lambda_0(s) ds + \int_{t_{i-1}}^{t_i} \sum_{t_k < s} \sum_{j=1}^{P} \alpha_j e^{-\beta_j (s - t_k)} ds \\
&= \int_{t_{i-1}}^{t_i} \lambda_0(s) ds + \int_{t_{i-1}}^{t_i} \sum_{t_k \leq t_{i-1}} \sum_{j=1}^{P} \alpha_j e^{-\beta_j (s - t_k)} ds \\
&= \int_{t_{i-1}}^{t_i} \lambda_0(s) ds + \sum_{t_k \leq t_{i-1}} \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j} \left[e^{-\beta_j (t_{i-1} - t_k)} - e^{-\beta_j (t_i - t_k)} \right].
\end{aligned}$$

Testing the simulated process (II)

This computation can be simplified with a recursive element. Let us denote

$$A_j(i-1) = \sum_{t_k \leq t_{i-1}} e^{-\beta_j(t_{i-1}-t_k)}. \tag{17}$$

We observe that

$$\begin{aligned} A_j(i-1) &= \sum_{t_k \leq t_{i-1}} e^{-\beta_j(t_{i-1}-t_k)} \\ &= 1 + e^{-\beta_j(t_{i-1}-t_{i-2})} \sum_{t_k \leq t_{i-2}} e^{-\beta_j(t_{i-2}-t_k)} \\ &= 1 + e^{-\beta_j(t_{i-1}-t_{i-2})} A_j(i-2). \end{aligned} \tag{18}$$

Testing the simulated process (III)

Finally, the integrated density can be written $\forall i \in \mathbb{N}^*$:

$$\Lambda(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} \lambda_0(s)ds + \sum_{j=1}^{P} \frac{\alpha_j}{\beta_j} \left(1 - e^{-\beta_j(t_i - t_{i-1})}\right) A_j(i-1), \quad (19)$$

where A is defined as in equation (17) with $\forall j = 1, \ldots, P, A_j(0) = 0$.

Time change property

Following theorem 2 and defining $\{\tau_i\}$ as

$$\tau_0 = \int_0^{t_0} \lambda(s)ds = \Lambda(0, t_0), \quad (20)$$

$$\tau_i = \tau_{i-1} + \Lambda(t_{i-1}, t_i), \quad (21)$$

the durations $\tau_i - \tau_{i-1} = \Lambda(t_{i-1}, t_i)$ are exponentially distributed.

Testing the simulated process (IV)

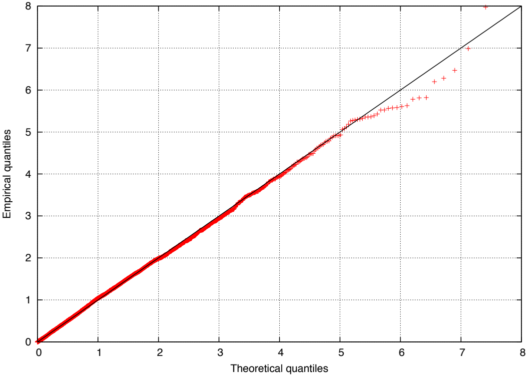

Figure: Quantile plot for one sample of simulated data of a one-dimensional Hawkes process with parameters $P = 1, \lambda_0 = 1.2, \alpha_1 = 0.6, \beta_1 = 0.8$, on an interval $[0, 10000]$.

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Computation of the log-likelihood function (I)

The log-likelihood of a simple point process N with intensity λ is written :

$$\ln \mathcal{L}((N_t)_{t\in[0,T]}) = \int_0^T (1-\lambda(s))\,ds + \int_0^T \ln \lambda(s) dN(s), \qquad (22)$$

which in the case of a Hawkes model can be explicitly computed as :

$$\begin{aligned} \ln \mathcal{L}(\{t_i\}_{i=1,\dots,n}) &= t_n - \Lambda(0,t_n) + \sum_{i=1}^{n} \ln \lambda(t_i) \\ &= t_n - \Lambda(0,t_n) \\ &+ \sum_{i=1}^{n} \ln \left[\lambda_0(t_i) + \sum_{j=1}^{P} \sum_{k=1}^{i-1} \alpha_j e^{-\beta_j (t_i - t_k)} \right]. \qquad (23) \end{aligned}$$

Computation of the log-likelihood function (II)

As noted by Ogata (1981), this log-likelihood function is easily computed with a recursive formula. We observe that:

$$\begin{aligned} R_j(i) &= \sum_{k=1}^{i-1} e^{-\beta_j(t_i - t_k)} \\ &= e^{-\beta_j(t_i - t_{i-1})} \sum_{k=1}^{i-1} e^{-\beta_j(t_{i-1} - t_k)} \\ &= e^{-\beta_j(t_i - t_{i-1})} \left(1 + \sum_{k=1}^{i-2} e^{-\beta_j(t_{i-1} - t_k)}\right) \\ &= e^{-\beta_j(t_i - t_{i-1})} \left(1 + R_j(i-1)\right). \end{aligned} \tag{24}$$

Computation of the log-likelihood function (III)

The log-likelihood can thus be recursively computed with :

$$\ln \mathcal{L}(\{t_i\}_{i=1,\dots,n}) = t_n - \Lambda(0,t_n) + \sum_{i=1}^{n} \ln\left[\lambda_0(t_i) + \sum_{j=1}^{P} \alpha_j R_j(i)\right], \quad (25)$$

where R is defined by equation (24) and $\forall j, R_j(1) = 0$.
Direct computation of $\Lambda(0,t_n)$ yields to :

Log-likelihood of a 1D-Hawkes process

$$\begin{aligned}\ln \mathcal{L}(\{t_i\}_{i=1,\dots,n}) &= t_n - \int_0^{t_n} \lambda_0(s)ds - \sum_{i=1}^{n}\sum_{j=1}^{P} \frac{\alpha_j}{\beta_j}\left(1 - e^{-\beta_j(t_n - t_i)}\right) \\ &+ \sum_{i=1}^{n} \ln\left[\lambda_0(t_i) + \sum_{j=1}^{P} \alpha_j R_j(i)\right], \end{aligned} \quad (26)$$

Properties of the maximum-likelihood estimator

Ogata (1978) shows that for a stationary one-dimensional Hawkes process with constant λ_0 and $P = 1$, the maximum-likelihood estimator $\hat{\theta}^T = (\hat{\lambda}_0, \hat{\alpha}_1, \hat{\beta}_1))$ is

- *consistent*, i.e. converges in probability to the true values $\theta = (\lambda_0, \alpha_1, \beta_1))$ as $T \to \infty$:

$$\forall \epsilon > 0, \quad \lim_{T\to\infty} P[|\hat{\theta}^T - \theta| > \epsilon] = 0. \tag{27}$$

- *asymptotically normal*, i.e.

$$\sqrt{T}\left(\hat{\theta}^T - \theta\right) \to \mathcal{N}(0, I^{-1}(\theta)) \tag{28}$$

  where $I^{-1}(\theta) = \left(\mathrm{E}\left[\frac{1}{\lambda}\frac{\partial\lambda}{\partial\theta_i}\frac{\partial\lambda}{\partial\theta_j}\right]\right)_{i,j}$.

- *asymptotically efficient*, i.e. asymptotically reaches the lower bound of the variance.

Numerical estimation of a simulated process

T	λ_0	α_1	β_1
100	1.210 (0.370)	0.588 (0.164)	0.833 (0.442)
1000	1.185 (0.133)	0.590 (0.044)	0.787 (0.068)
10000	1.204 (0.045)	0.602 (0.016)	0.804 (0.023)
100000	1.202 (0.014)	0.600 (0.004)	0.800 (0.007)
True values	1.200	0.600	0.800

Table: Maximum likelihood estimation of a one-dimensional Hawkes process on simulated data. Each estimation is the average result computed on 100 samples of length $[0, T]$. Standard deviations are given in parentheses. These results are obtained with a simple Nelder-Mead simplex algorithm.

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Multidimensional Hawkes processes

Let $M \in \mathbb{N}^*$. Let $\{(t_i^m)_i\}_{m=1,\ldots,M}$ be a M-dimensional point process. We will denote $\mathbf{N}_t = (N_t^1, \ldots, N_t^M)$ the associated counting process.

Definition

A multidimensional Hawkes process is defined with intensities $\lambda^m, m = 1, \ldots, M$ given by :

$$\lambda^m(t) = \lambda_0^m(t) + \sum_{n=1}^{M} \int_0^t \sum_{j=1}^{P} \alpha_j^{mn} e^{-\beta_j^{mn}(t-s)} dN_s^n, \tag{29}$$

i.e. in its simplest version with $P = 1$ and $\lambda_0^m(t)$ constant :

$$\lambda^m(t) = \lambda_0^m + \sum_{n=1}^{M} \int_0^t \alpha^{mn} e^{-\beta^{mn}(t-s)} dN_s^n = \lambda_0^m + \sum_{n=1}^{M} \sum_{t_i^n < t} \alpha^{mn} e^{-\beta^{mn}(t-t_i^n)}. \tag{30}$$

Stationarity condition (I)

We'll take here $P = 1$ to simplify the notations. Rewriting equation (30) using vectorial notation, we have :

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}_0 + \int_0^t \mathbf{G}(t-s) d\mathbf{N}_s, \tag{31}$$

where

$$\mathbf{G}(t) = \left(\alpha^{mn} e^{-\beta^{mn}(t-s)}\right)_{m,n=1,\ldots,M}. \tag{32}$$

Assuming stationarity gives $\mathrm{E}\left[\boldsymbol{\lambda}(t)\right] = \boldsymbol{\mu}$ constant vector, and thus stationary intensities must satisfy :

$$\boldsymbol{\mu} = \left(\mathbf{I} - \int_0^\infty \mathbf{G}(u) du\right)^{-1} \boldsymbol{\lambda}_0 \tag{33}$$

Stationarity condition (II)

Stationarity of a multivariate Hawkes process

A sufficient condition for a multivariate Hawkes process to be linear is that the spectral radius of the matrix

$$\mathbf{\Gamma} = \int_0^\infty \mathbf{G}(u)du = \left(\frac{\alpha^{mn}}{\beta^{mn}}\right)_{m,n=1,\dots,M} \tag{34}$$

be strictly smaller than 1.

We recall that the spectral radius of the matrix $\mathbf{G}$ is defined as :

$$\rho(\mathbf{G}) = \max_{a \in \mathcal{S}(\mathbf{G})} |a|, \tag{35}$$

where $\mathcal{S}(\mathbf{G})$ denotes the set of all eigenvalues of $\mathbf{G}$.

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Simulation of a multivariate Hawkes process (I)

We generalize the 1D-algorithm in a multidimensional setting. We recall that :

- $\mathcal{U}_{[0,1]}$ denotes the uniform distribution on the interval $[0, 1]$,
- $[0, T]$ is the time interval on which the process is to be simulated,

and we define

$$I^K(t) = \sum_{n=1}^{K} \lambda^n(t) \tag{36}$$

the sum of the intensities of the first K components of the multivariate process. $I^M(t) = \sum_{n=1}^{M} \lambda^n(t)$ is thus the total intensity of the multivariate process and we set $I^0 = 0$. The algorithm is then rewritten as follows.

Simulation of a multivariate Hawkes process (II)

Algorithm - Initialization

1. **Initialization** : Set $i \leftarrow 1$, $i^1 \leftarrow 1, \dots, i^M \leftarrow 1$ and $I^* \leftarrow I^M(0) = \sum_{n=i}^{M} \lambda_0^i(0)$.
2. **First event** : Generate $U \rightsquigarrow \mathcal{U}_{[0,1]}$ and set $s \leftarrow -\frac{1}{\lambda^*} \ln U$.
    1. **If** $s > T$ **Then** go to last step.
    2. **Attribution Test** : Generate $D \rightsquigarrow \mathcal{U}_{[0,1]}$ and set $t_1^{n_0} \leftarrow s$ where n_0 is such that $\frac{I^{n_0-1}(0)}{I^*} < D \leq \frac{I^{n_0}(0)}{I^*}$.
    3. Set $t_1 \leftarrow t_1^{n_0}$.

Simulation of a multivariate Hawkes process (III)

Algorithm - General routine

3. **General routine** : Set $i^{n_0} \leftarrow i^{n_0} + 1$ and $i \leftarrow i + 1$.
   1. **Update maximum intensity**: Set $I^* \leftarrow I^M(t_{i-1}) + \sum_{n=1}^{M} \sum_{j=1}^{P} \alpha_j^{nn_0}$.
   2. **New event** : Generate $U \rightsquigarrow \mathcal{U}_{[0,1]}$ and set $s \leftarrow s - \frac{1}{I^*} \ln U$.

      If $s > T$, **Then** go to the last step.
   3. **Attribution-Rejection test** : Generate $D \rightsquigarrow \mathcal{U}_{[0,1]}$.

      If $D \leq \frac{I^M(s)}{I^*}$,

      Then set $t_{i^{n_0}}^{n_0} \leftarrow s$ where n_0 is such that $\frac{I^{n_0-1}(s)}{I^*} < D \leq \frac{I^{n_0}(s)}{I^*}$, and $t_i \leftarrow t_{i^{n_0}}^{n_0}$ and go through the general routine again,
      Else update $I^* \leftarrow I^M(s)$ and try a new date at step (b) of the general routine.
4. **Output**: Retrieve the simulated process $(\{t_i^n\}_i)_{n=1,\dots,M}$ on $[0, T]$.

Sample paths of a bivariate Hawkes process (I)

We simulate a bivariate Hawkes process with $P = 1$ and the following parameters:

$$\begin{aligned} &\lambda_0^1 = 0.1, \alpha_1^{11} = 0.2, \beta_1^{11} = 1.0, \alpha_1^{12} = 0.1, \beta_1^{12} = 1.0, \\ &\lambda_0^2 = 0.5, \alpha_1^{21} = 0.5, \beta_1^{21} = 1.0, \alpha_1^{22} = 0.1, \beta_1^{22} = 1.0, \end{aligned} \tag{37}$$

Sample paths of a bivariate Hawkes process (II)

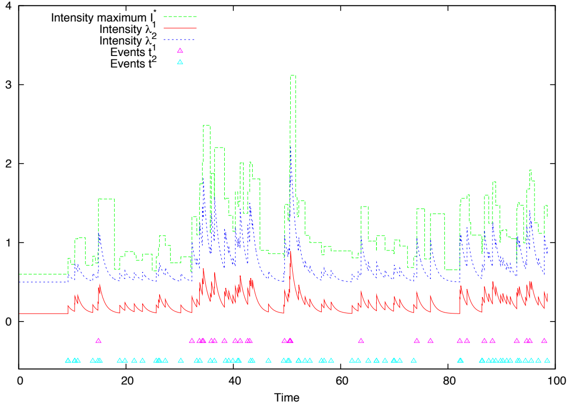

Figure: Simulation of a two-dimensional Hawkes process with $P = 1$ and parameters given in equation (37).

Sample paths of a bivariate Hawkes process (III)

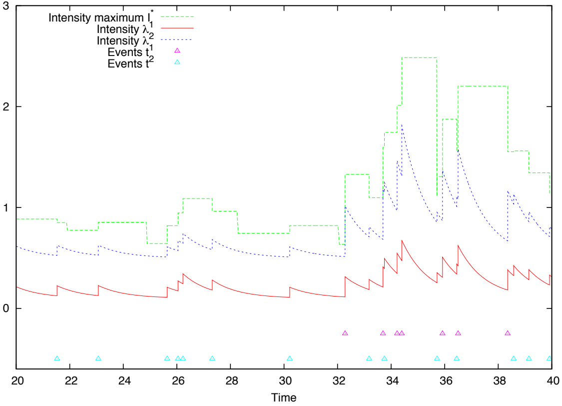

Figure: Simulation of a two-dimensional Hawkes process with $P = 1$ and parameters given in equation (37). (Zoom of the previous figure).

Testing the simulated data (I)

The integrated intensity of the m-th coordinate of a multidimensional Hawkes process between two consecutive events t_{i-1}^m and t_i^m of type m is computed as:

$$
\begin{aligned}
\Lambda^m(t_{i-1}^m, t_i^m) &= \int_{t_{i-1}^m}^{t_i^m} \lambda^m(s)ds \\
&= \int_{t_{i-1}^m}^{t_i^m} \lambda_0^m(s)ds + \int_{t_{i-1}^m}^{t_i^m} \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_k^n < s} \alpha_j^{mn} e^{-\beta_j^{mn}(s-t_k^n)} ds \\
&= \int_{t_{i-1}^m}^{t_i^m} \lambda_0^m(s)ds + \int_{t_{i-1}^m}^{t_i^m} \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_k^n < t_{i-1}^m} \alpha_j^{mn} e^{-\beta_j^{mn}(s-t_k^n)} ds \\
&\quad + \int_{t_{i-1}^m}^{t_i^m} \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_{i-1}^m \leq t_k^n < s} \alpha_j^{mn} e^{-\beta_j^{mn}(s-t_k^n)} ds
\end{aligned}
$$

Testing the simulated data (II)

$$\begin{aligned}\Lambda^m(t_{i-1}^m, t_i^m) &= \int_{t_{i-1}^m}^{t_i^m} \lambda_0^m(s)ds \\ &+ \sum_{n=1}^{M}\sum_{j=1}^{P}\sum_{t_k^n < t_{i-1}^m} \frac{\alpha_j^{mn}}{\beta_j^{mn}}\left[e^{-\beta_j^{mn}(t_{i-1}^m - t_k^n)} - e^{-\beta_j^{mn}(t_i^m - t_k^n)}\right] \\ &+ \sum_{n=1}^{M}\sum_{j=1}^{P}\sum_{t_{i-1}^m \leq t_k^n < t_i^m} \frac{\alpha_j^{mn}}{\beta_j^{mn}}\left[1 - e^{-\beta_j^{mn}(t_i^m - t_k^n)}\right]. \end{aligned} \tag{38}$$

This computation can be simplified with a recursive element. Let us denote

$$A_j^{mn}(i-1) = \sum_{t_k^n < t_{i-1}^m} e^{-\beta_j^{mn}(t_{i-1}^m - t_k^n)}. \tag{39}$$

Testing the simulated data (III)

We observe that

$$
\begin{aligned}
A_j^{mn}(i-1) &= \sum_{t_k^n < t_{i-1}^m} e^{-\beta_j^{mn}(t_{i-1}^m - t_k^n)} \\
&= e^{-\beta_j^{mn}(t_{i-1}^m - t_{i-2}^m)} \sum_{t_k^n < t_{i-2}^m} e^{-\beta_j^{mn}(t_{i-2}^m - t_k^n)} \\
&\quad + \sum_{t_{i-2}^m \leq t_k^n < t_{i-1}^m} e^{-\beta_j^{mn}(t_{i-1}^m - t_k^n)} \\
&= e^{-\beta_j^{mn}(t_{i-1}^m - t_{i-2}^m)} A_j^{mn}(i-2) \\
&\quad + \sum_{t_{i-2}^m \leq t_k^n < t_{i-1}^m} e^{-\beta_j^{mn}(t_{i-1}^m - t_k^n)}. \qquad (40)
\end{aligned}
$$

Testing the simulated data (IV)

The integrated density can thus be written $\forall i \in \mathbb{N}^*$:

$$\Lambda^m(t_{i-1}^m, t_i^m) = \int_{t_{i-1}^m}^{t_i^m} \lambda_0^m(s)ds + \sum_{n=1}^{M} \sum_{j=1}^{P} \frac{\alpha_j^{mn}}{\beta_j^{mn}} \Bigg[\left(1 - e^{-\beta_j^{mn}(t_i^m - t_{i-1}^m)}\right) \times A_j^{mn}(i-1) + \sum_{t_{i-1}^m \leq t_k^n < t_i^m} \left(1 - e^{-\beta_j^{mn}(t_i^m - t_k^n)}\right) \Bigg], \quad (41)$$

where A is defined as in equation (39) with $\forall j, A_j^{mn}(0) = 0$.

Time change property

As for the one-dimensional case, the durations $\tau_i^m - \tau_{i-1}^m = \Lambda^m(t_{i-1}^m, t_i^m)$ are exponentially distributed with parameter 1. See e.g. (Bowsher 2007).

Testing the simulated data (V)

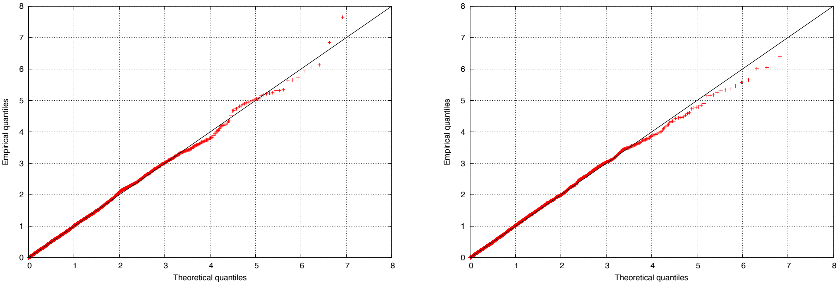

Figure: Quantile plots for one sample of simulated data of a two-dimensional Hawkes process with $P = 1$ and parameters given in equation (37). (Left) $m = 0$. (Right) $m = 1$.

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Computation of the log-likelihood function (I)

The log-likelihood of a multidimensional Hawkes process can be computed as the sum of the likelihood of each coordinate, i.e. is written:

$$\ln \mathcal{L}(\{t_i\}_{i=1,\dots,N}) = \sum_{m=1}^{M} \ln \mathcal{L}^m(\{t_i\}), \tag{42}$$

where each term is defined by:

$$\ln \mathcal{L}^m(\{t_i\}) = \int_0^T (1 - \lambda^m(s))\, ds + \int_0^T \ln \lambda^m(s) dN^m(s). \tag{43}$$

Computation of the log-likelihood function (II)

In the case of a multidimensional Hawkes process, denoting $\{t_i\}_{i=1,\dots,N}$ the ordered pool of all events $\{\{t_i^m\}_{m=1,\dots,M}\}$, this log-likelihood can be computed as:

$$\begin{aligned} \ln \mathcal{L}^m(\{t_i\}) &= T - \Lambda^m(0,T) \\ &+ \sum_{i=1}^{N} z_i^m \ln \left[\lambda_0^m(t_i) + \sum_{n=1}^{M} \sum_{j=1}^{P} \sum_{t_k^n < t_i} \alpha_j^{mn} e^{-\beta_j^{mn}(t_i - t_k^n)} \right], \end{aligned} \tag{44}$$

where z_i^m is equal to 1 if the event t_i is of type m, 0 otherwise.

Computation of the log-likelihood function (III)

As in the one dimensional case, this can be computed in a recursive way. We observe that

$$
\begin{aligned}
R_j^{mn}(l) &= \sum_{t_k^n < t_l^m} e^{-\beta_j^{mn}(t_l^m - t_k^n)} \\
= & \sum_{t_k^n < t_{l-1}^m} e^{-\beta_j^{mn}(t_l^m - t_k^n)} + \sum_{t_{l-1}^m \leq t_k^n < t_l^m} e^{-\beta_j^{mn}(t_l^m - t_k^n)} \\
= & \; e^{-\beta_j^{mn}(t_l^m - t_{l-1}^m)} \sum_{t_k^n < t_{l-1}^m} e^{-\beta_j^{mn}(t_{l-1}^m - t_k^n)} + \sum_{t_{l-1}^m \leq t_k^n < t_l^m} e^{-\beta_j^{mn}(t_l^m - t_k^n)} \\
= & \; e^{-\beta_j^{mn}(t_l^m - t_{l-1}^m)} R_j^{mn}(l-1) + \sum_{t_{l-1}^m \leq t_k^n < t_l^m} e^{-\beta_j^{mn}(t_l^m - t_k^n)} \\
= & \begin{cases} e^{-\beta_j^{mn}(t_l^m - t_{l-1}^m)} R_j^{mn}(l-1) + \sum_{t_{l-1}^m \leq t_k^n < t_l^m} e^{-\beta_j^{mn}(t_l^m - t_k^n)} & \text{if } m \neq n, \\ e^{-\beta_j^{mn}(t_l^m - t_{l-1}^m)} \left(1 + R_j^{mn}(l-1)\right) & \text{if } m = n. \end{cases}
\end{aligned}
\tag{45}
$$

Computation of the log-likelihood function (IV)

The final expression of the log-likelihood may be written:

Log-likelihood of a multivariate Hawkes process

$$\begin{aligned}\ln \mathcal{L}^m(\{t_i\}) &= T - \sum_{i=1}^{N}\sum_{n=1}^{M}\sum_{j=1}^{P} \frac{\alpha_j^{mn}}{\beta_j^{mn}}\left(1 - e^{-\beta_j^{mn}(T-t_i)}\right) \\ &+ \sum_{t_l^m} \ln\left[\lambda_0^m(t_l^m) + \sum_{n=1}^{M}\sum_{j=1}^{P}\alpha_j^{mn} R_j^{mn}(l)\right], \end{aligned} \tag{46}$$

where $R_j^{mn}(l)$ is defined with equation (45) and $R_j^{mn}(0) = 0$.

Numerical estimation of a simulated process

T	λ_0^1	α_1^{11}	β_1^{11}	α_1^{12}	β_1^{12}	λ_0^2	α_1^{21}	β_1^{21}	α_1^{22}	β_1^{22}
100	0.614	0.510	0.369	1.482	1.710	0.518	0.337	0.600	1.605	2.595
	(0.372)	(0.268)	(0.269)	(1.216)	(3.172)	(0.272)	(0.206)	(0.365)	(2.051)	(6.586)
500	0.516	0.505	0.268	1.043	0.865	0.518	0.264	0.507	0.814	1.048
	(0.112)	(0.085)	(0.080)	(0.214)	(0.479)	(0.120)	(0.080)	(0.084)	(0.278)	(0.221)
1000	0.507	0.492	0.254	1.018	0.761	0.513	0.255	0.488	0.794	1.003
	(0.079)	(0.054)	(0.052)	(0.122)	(0.203)	(0.092)	(0.052)	(0.061)	(0.387)	(0.152)
	0.500	0.500	0.250	1.000	0.750	0.500	0.250	0.500	0.750	1.000

Table: Maximum likelihood estimation of a two-dimensional Hawkes process on simulated data. Each estimation is the average result computed on 100 samples of length $[0, T]$. Standard deviations are given in parentheses.

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A model for buy and sell intensities (I)

Hewlett (2006) proposes to model the clustered arrivals of buy and sell trades using Hawkes processes. Using the exponent 'B' for buy variables and 'S' for sell variables, the model is written :

$$\lambda^B(t) = \lambda_0^B + \int_0^t \alpha^{BB} e^{-\beta^{BB}(t-u)} dN_u^B + \int_0^t \alpha^{BS} e^{-\beta^{BS}(t-u)} dN_u^S, \quad (47)$$

$$\lambda^S(t) = \lambda_0^S + \int_0^t \alpha^{SB} e^{-\beta^{SB}(t-u)} dN_u^B + \int_0^t \alpha^{SS} e^{-\beta^{SS}(t-u)} dN_u^S. \quad (48)$$

A model for buy and sell intensities (II)

Hewlett (2006) imposes some symmetry constraints, stating that mutual excitation and self-excitation should be the same for both processes, which is written :

$$\lambda_0^B = \lambda_0^S = \lambda_0 \quad (49)$$

$$\alpha^{SB} = \alpha^{BS} = \alpha^{cross} \quad (50)$$

$$\beta^{SB} = \beta^{BS} = \beta^{cross} \quad (51)$$

$$\alpha^{SS} = \alpha^{BB} = \alpha^{self} \quad (52)$$

$$\beta^{SS} = \beta^{BB} = \beta^{self} \quad (53)$$

Goodness of fit

Hewlett (2006) fits this model on two-month data of EUR/PLN transactions (no dates given): the Hawkes model is a much better fit of the empirical data than the Poisson model.

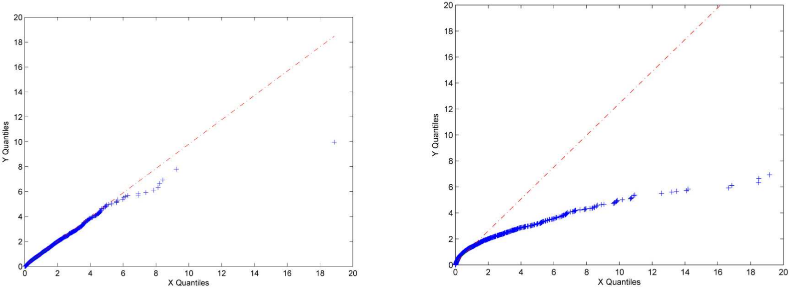

Figure: Quantile plots of integrated intensities for the Hawkes model (left) and a Poisson model (right) on EUR/PLN buy and sell data. Reproduced from (Hewlett 2006).

Numerical results

The numerical values obtained are :

$$\lambda_0 = 0.0033, \alpha^{cross} = 0, \alpha^{self} = 0.0169, \beta^{self} = 0.0286. \qquad (54)$$

In other words,

- the occurrence of a buy (resp. sell) order has an exciting effect on the stream of buy (resp. sell) orders, with a typical half-life of $\frac{\ln 2}{\beta^{self}} \approx 24$ seconds;
- the zero value of α^{cross} tends to indicate that there is no influence of buy orders on sell orders, and conversely.

Test on our own data (I)

We perform the fit of a bivariate Hawkes model on buy/sell market orders on the following data : BNPP.PA, Feb. 1st 2010 to Feb. 23rd, 2010 (14 trading days), 10am-12am without symmetry constraints. Numerical results are :

$$\lambda_0^B = 0.080, \quad \alpha^{BB} = 3.230, \beta^{BB} = 13.304, \quad \alpha^{BS} = 0.276, \beta^{BS} = 6.193$$
$$\lambda_0^B = 0.086, \quad \alpha^{SB} = 0.515, \beta^{SB} = 13.451, \quad \alpha^{SS} = 3.789, \beta^{SS} = 14.151$$

- Confirmation of the very limited cross-excitation effect.
- Change of magnitude of parameters β: difference in precision of data (second, millisecond)

Test on our own data (II)

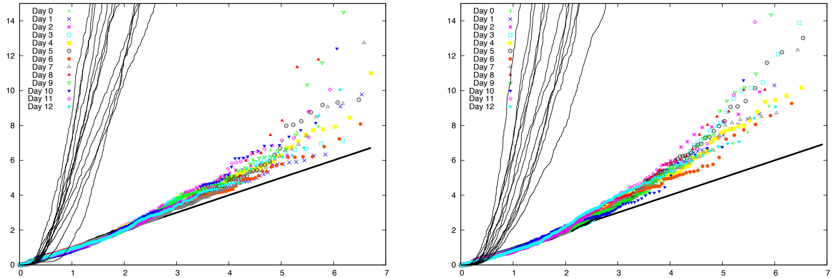

Figure: Quantile plots of integrated intensities for a bivariate Hawkes model on buy/sell market orders fitted on 13 trading days of the stock BNPP.PA (from Feb. 1st 2010 to Feb. 22nd, 2010), 10am-12am each day.

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A one-dimensional price model

Bacry, Delattre, Hoffmann & Muzy (2011) propose a one-dimensional case where the price p is written:

$$p(t) = N^1(t) - N^2(t), \tag{55}$$

where $N_i, i \in \{1,2\}$ is a Hawkes process with intensities $\lambda_i, i \in \{1,2\}$ such that

$$\lambda^1(t) = \lambda_0 + \int_{-\infty}^{t} \alpha e^{-\beta(t-s)} dN_s^2, \tag{56}$$

$$\lambda^2(t) = \lambda_0 + \int_{-\infty}^{t} \alpha e^{-\beta(t-s)} dN_s^1. \tag{57}$$

- No self-excitation of upward (resp. downward) jumps on following upward (resp. downward) jumps
- Only cross-excitation terms are kept, enforcing the mean-reversion empirically observed on the price p
- Cross-excitation is set to be symmetric

An analytical expression for the variance of the price

A volatility signature plot plots the realized variance as a function of the sampling period:

$$RV(\tau) = \frac{1}{\tau}\sum_{i=1}^{I}\left(\hat{p}(i\tau) - \hat{p}((i-1)\tau)\right)^2 \tag{58}$$

where I is the number of observations $\hat{p}$. Bacry et al. (2011) shows that the theoretical signature plot of the stationary model (55)-(57) can be theoretically computed as:

$$C(\tau) = \frac{1}{\tau}\mathrm{E}\left[p(\tau)^2\right] = \Lambda\left(\kappa^2 + (1-\kappa^2)\frac{1-e^{-\gamma\tau}}{\gamma\tau}\right), \tag{59}$$

where

$$\Lambda = \frac{2\lambda_0}{1-\alpha/\beta}, \quad \kappa = \frac{1}{1+\alpha/\beta}, \quad \text{and} \quad \gamma = \alpha + \beta.$$

(See (Bacry et al. 2011, Appendix 1).)

Fitting the volatility signature plot

Bacry et al. (2011) show that the signature plot of their model is in very good agreement with the one computed on 21 samples of two-hour Euro-Bund futures contracts in November and December 2009.

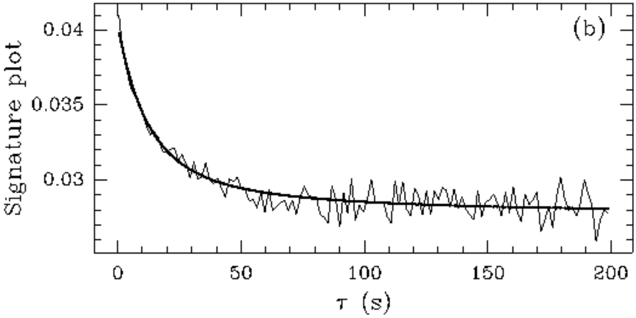

Figure: Empirical signature plot of the Euro-Bund prices (thin line) and theoretical Hawkes fit (thick line). Reproduced from (Bacry et al. 2011).

A two-dimensional model

Bacry et al. (2011) also propose a bivariate version of the model:

$$p_1(t) = N^1(t) - N^2(t), \quad (60)$$
$$p_2(t) = N^3(t) - N^4(t), \quad (61)$$

in which $\mathbf{N} = (N_i)_{i=1,\dots 4}$ is a Hawkes process with intensity:

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}_0 + \int_0^t \begin{pmatrix} 0 & \phi^{12} & \phi^{13} & 0 \\ \phi^{12} & 0 & 0 & \phi^{13} \\ \phi^{31} & 0 & 0 & \phi^{34} \\ 0 & \phi^{31} & \phi^{34} & 0 \end{pmatrix} (t-s) d\mathbf{N}_s, \quad (62)$$

where $\phi^{ij}(t-s) = \alpha^{ij} e^{-\beta^{ij}(t-s)}$.

- self-exciting terms are ruled out, $\phi^{ii} = 0 \forall i$;
- upward and downward effects are assumed to be symmetric *within* the processes p^1 and p^2 ($\phi^{12} = \phi^{21}, \phi^{23} = \phi^{43}$);
- prices p_1 and p_2 influence each other in a positive way, not a negative one ($\phi^{14} = \phi^{23} = \phi^{32} = \phi^{41} = 0$).

An explicit expression for the covariance matrix

For this model, Bacry et al. (2011) show that an explicit form of the correlation coefficient

$$\rho(\tau) = \mathrm{Corr}\left(p_1(t+\tau) - p_1(t), p_2(t+\tau) - p_2(t)\right) \tag{63}$$

can be explicitly computed, although the expected result is quite cumbersome (see (Bacry et al. 2011, Proposition 3.1)).

Addressing the Epps effect

The theoretical correlation is in good agreement with correlations mesured on empirical data, and that it is in accordance with the so-called Epps effect, stating that correlation measured on financial assets decreases when the sampling frequency increases.

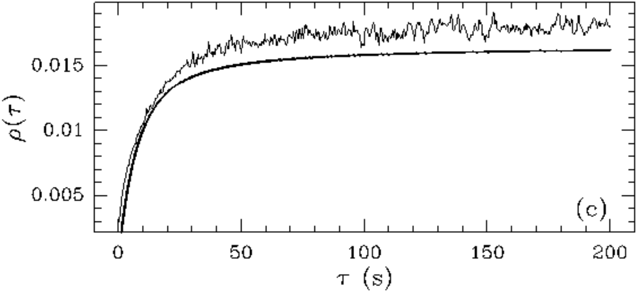

Figure: Empirical correlation measured between the Euro-Bund and Euro-Bobl prices (thin line) and theoretical Hawkes fit (thick line). Reproduced from (Bacry et al. 2011).

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Classifying orders according to their aggressiveness

Large (2007) models streams of orders by Hawkes processes, extending the model by Hewlett (2006) using a much finer description of orders.
Following classical typologies used in microstructure, events occurring in an order book are classified in ten categories :

Type	Description	Aggressiveness
1	Market order that moves the ask	Yes
2	Market order that moves the bid	Yes
3	Limit order that moves the ask	Yes
4	Limit order that moves the bid	Yes
5	Market order that doesn't move the ask	No
6	Market order that doesn't move the bid	No
7	Limit order that doesn't move the ask	No
8	Limit order that doesn't move the bid	No
9	Cancellation at ask	No
10	Cancellation at bid	No

A 10-variate Hawkes model for aggressive orders

Events of type 1 to 4 are Hawkes processes whose intensities depend on the 10 different sorts of events, i.e. can be written for $m = 1, \dots, 4$:

$$\lambda^m(t) = \lambda_0(t) + \sum_{n=1}^{10} \int_0^t \alpha^{mn} e^{-\beta_m n(t-u)} dN_u^n. \tag{64}$$

Hawkes parameters for aggressive limit orders

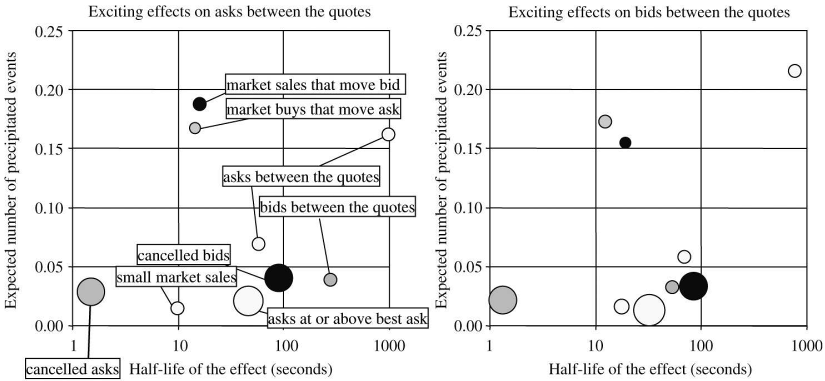

Figure: Representation of the influences on aggressive limit orders measured by the fitting of a Hawkes model on the Barclay's order book on January 2002. β^{mn} are in abscissas, α^{mn} are in ordinates, the size of the discs are proportional to the number of observed events. Reproduced from (Large 2007).

Hawkes parameters for aggressive market orders

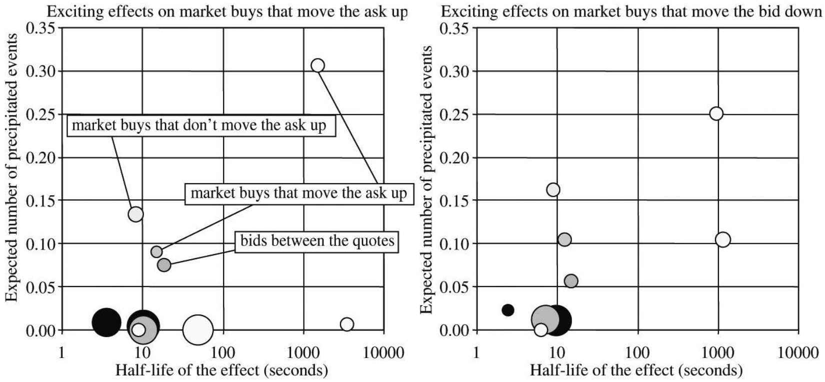

Figure: Representation of the influences on aggressive market orders measured by the fitting of a Hawkes model on the Barclay's order book on January 2002. β^{mn} are in abscissas, α^{mn} are in ordinates, the size of the discs are proportional to the number of observed events. Reproduced from (Large 2007).

Empirical conclusions in Large (2007)

Previous figures can be used to draw some conclusions on the way order book events influence each other. The main findings reported are the followings:

- aggressive limit orders are firstly influenced by aggressive market orders: this is an evidence of some "resiliency" in the order book ;
- aggressive limit orders are secondly influenced by aggressive limit orders ;
- aggressive market orders are firstly influenced by market orders (aggressive or not) ;
- aggressive market orders are secondly influenced by aggressive limit orders: this is an evidence of some "rush to liquidity".

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The basic Zero-Intelligence Poisson model ("HP")

Liquidity provider

1. arrival of new limit orders: homogeneous Poisson process $N^L(\lambda^L)$
2. arrival of cancelation of orders: homogeneous Poisson process $N^C(\lambda^C)$
3. new limit orders' placement: Student's distribution with parameters (ν_1^P, m_1^P, s_1^P) around the same side best quote
4. volume of new limit orders: exponential distribution $\mathcal{E}(1/m_1^V)$;
5. in case of a cancelation, orders are deleted with probability δ

Noise trader (liquidity taker)

1. arrival of market orders: homogeneous Poisson process $N^M(\mu)$
2. volume of market orders: exponential distribution with mean $\mathcal{E}(1/m_2^V)$.

Need for physical time in order book models

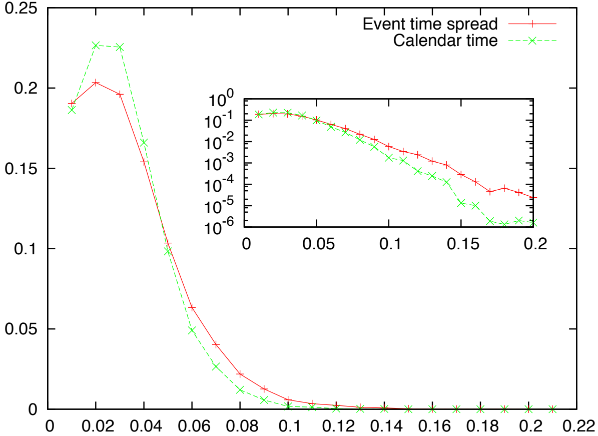
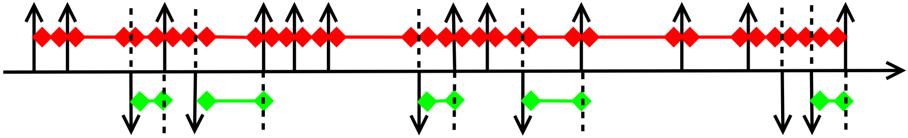


Figure: Empirical density function of the distribution of the bid-ask spread in event time and in physical time.

Measures of inter arrival times



- (red) Inter arrival times of the counting process of all orders (limit orders and market orders mixed), i.e. the time step between any order book event (other than cancelation)
- (green) Interval time between a market order and immediatly following limit order

Empirical evidence of “market making”

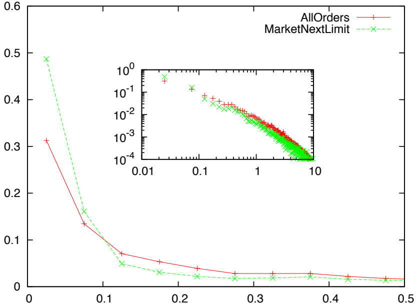


Figure: Empirical density function of the distribution of the timesteps between two consecutive orders (any type, market or limit) and empirical density function of the distribution of the time steps between a market order and the immediatly following limit order. X-axis is scaled in seconds. In insets, same data using a semi-log scale. Studied assets : BNPP.PA (left). Reproduced from (Muni Toke 2011).

Adding dependance between order flows (I)

Liquidity provider

1. arrival of new limit orders: Hawkes process $N^L(\lambda^L)$
2. arrival of cancelation of orders: homogeneous Poisson process $N^C(\lambda^C)$
3. new limit orders' placement: Student's distribution with parameters (ν_1^P, m_1^P, s_1^P) around the same side best quote
4. volume of new limit orders: exponential distribution $\mathcal{E}(1/m_1^V)$;
5. in case of a cancelation, orders are deleted with probability δ

Noise trader (liquidity taker)

1. arrival of market orders: Hawkes process $N^M(\mu)$
2. volume of market orders: exponential distribution with mean $\mathcal{E}(1/m_2^V)$.

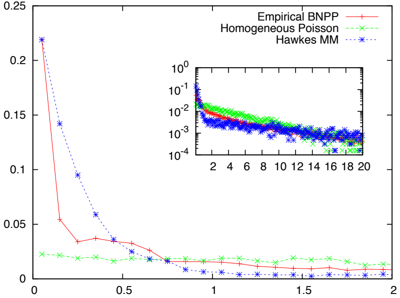
Adding dependance between order flows (II)

Hawkes processes N^L and N^M

$$\begin{cases} \mu(t) & = & \mu_0 & + \int_0^t \alpha_{MM} e^{-\beta_{MM}(t-s)} dN_s^M \\ \lambda^L(t) & = & \lambda_0^L & + \int_0^t \alpha_{LM} e^{-\beta_{LM}(t-s)} dN_s^M & + \int_0^t \alpha_{LL} e^{-\beta_{LL}(t-s)} dN_s^L \end{cases} \tag{65}$$

- MM and LL effect for clustering of orders
- LM effect as observed on data
- no ML effect

Impact on arrival times (I)



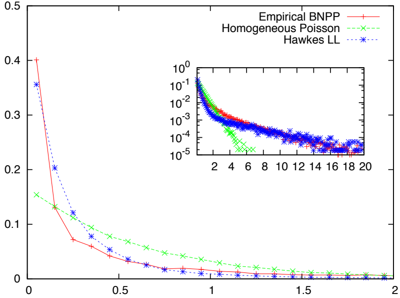


Figure: Empirical density function of the distribution of the interarrival times of market orders (left) and limit orders (right) for three simulations, namely HP, MM, LL, compared to empirical measures. In inset, same data using a semi-log scale. Reproduced from (Muni Toke 2011).

Impact on arrival times (II)

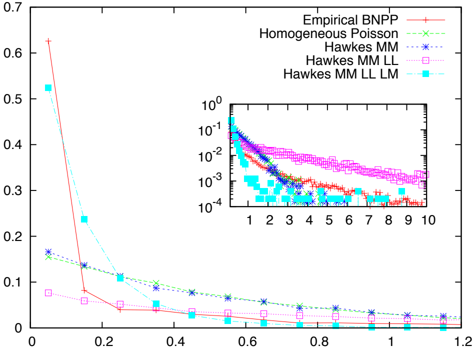


Figure: Empirical density function of the distribution of the interval times between a market order and the following limit order for three simulations, namely HP, MM+LL, MM+LL+LM, compared to empirical measures. In inset, same data using a semi-log scale. Reproduced from (Muni Toke 2011).

Impact on the bid-ask spread (I)

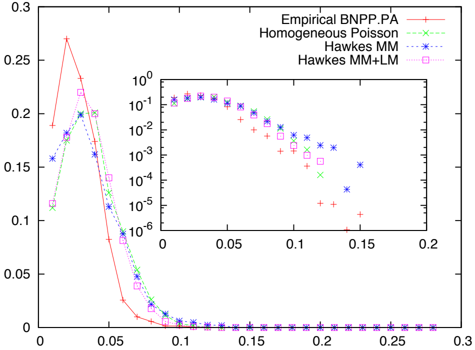


Figure: Empirical density function of the distribution of the bid-ask spread for three simulations, namely HP, MM, MM+LM, compared to empirical measures. In inset, same data using a semi-log scale. X-axis is scaled in euro (1 tick is 0.01 euro). Reproduced from (Muni Toke 2011).

Impact on the bid-ask spread (II)

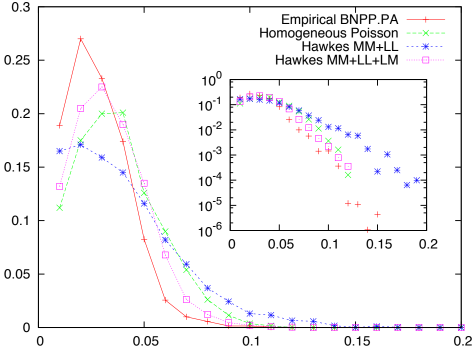


Figure: Empirical density function of the distribution of bid-ask spread for three simulations, namely HP, MM+LL, MM+LL+LM. In inset, same data using a semi-log scale. X-axis is scaled in euro (1 tick is 0.01 euro). Reproduced from (Muni Toke 2011).

Limitations of the model

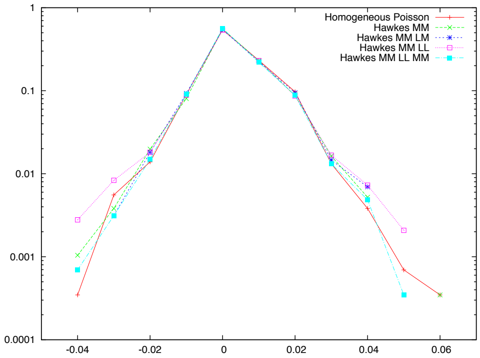


Figure: Empirical density function of the distribution of the variations of the mid-price sampled every 30 seconds for five simulations, namely HP, MM, MM+LM, MM+LL, MM+LL+LM, compared to empirical measures. X-axis is scaled in euro (1 tick is 0.01 euro). Reproduced from (Muni Toke 2011).

Summary

- Self- and mutual-exciting processes (epidemic, earthquakes,... finance!)
- Exponential kernel allows easy manipulation (simulation, estimation)
- Quite good fit on tested data (buy/sell, market/limit)
- See (Bowsher 2007) for a generalized econometric framework
- Lots of possible models/strategies to be imagined

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