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Jets and Kinematics

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Abstract

In $p\overline{p}$ physics, it is advantageous to transform particles and jets from the coordinates $(E, \vec{\mathbf{p}})$ to (p_t, m, η, ϕ). In this paper we review the motivations for this transformation and derive some useful formulas. Using these coordinates, we then outline how one goes from clusters in the calorimeter to "jets" and include a short discussion on the meaning of "jet" quantities such as transverse energy and mass.

1 Lorentz Invariant Phase Space

1.1 Rapidity

The 4-momentum of a particle ($p_\mu \equiv (E, \vec{\mathbf{p}})$) can be described in a more clearly relativistically invariant way using the variables p_t (transverse momentum), ϕ (azimuth angle), y (rapidity), and m (mass) where rapidity is defined by

$$y \;\equiv\; \frac{1}{2}\ln\frac{E+p_z}{E-p_z} \;=\; \frac{1}{2}\ln\frac{1+\beta\cos\theta}{1-\beta\cos\theta} \tag{1}$$

It is then straight-forward to see that a boost along the z direction with velocity β_{boost} will change the rapidity y by

$$y \rightarrow y + y_{boost}$$

where

$$y_{boost} = \ln\gamma_{boost}(1+\beta_{boost})$$

or equivalently

$$\beta_{boost} = \tanh y_{boost}$$

The advantage of using the coordinates p_t, y, ϕ, m (as opposed to $E, \vec{\mathbf{p}}$) are that as seen in the above equations, the effect of a boost along the z-axis changes only y, and then only by an additive constant.

We can now construct some useful kinematic formulas from the 4 variables p_t, y, ϕ, and m. One can define the "transverse energy" of a particle as the energy of the particle in the rest frame where $p_z = 0$ (which is the frame with a boost given by the rapidity) as

$$E_t^2 \equiv E^2 - p_z^2 = p_t^2 + m^2 \tag{2}$$

or equivalently

$$E \;=\; E_t \cosh y. \tag{3}$$

Note that the p_t is always defined in terms of the 3-momentum $\vec{p}$ as

$$p_t \equiv p\sin\theta. \tag{4}$$

Using these definitions, one can write the invariant mass of two particles with 4-momenta p_1^μ and p_2^μ (defined by $M_{12}^2 \equiv (p_1^\mu + p_2^\mu)^2$) as

$$M_{12}^2 = m_1^2 + m_2^2 + 2E_{t1}E_{t2}(\cosh\delta y - \beta_{t1}\beta_{t2}\cos\delta\phi).$$

where $\beta_t \equiv p_t/E_t = \beta \sin\theta \cosh y$, E_t is as defined in equation 2, and $\delta y \equiv y_1 - y_2$.

1.2 Pseudo-Rapidity

In a detector where one measures calorimeter pulse height, it is often difficult to determine the true rapidity of a particle. We can then define the "pseudo-rapidity" η as the rapidity of a particle with zero mass, or $\eta \equiv y|_{m=0}$. Using equation 1 for y and setting $\beta = 1$ one obtains

$$\eta \equiv \frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta} = -\ln\tan\frac{\theta}{2}.$$

and the inverse equation, which is often very useful

$$\cos\theta = \tanh\eta.$$

Since β is less than 1, the pseudo-rapidity η for a particle with a given p_t will have a *larger* pseudo-rapidity than rapidity y, or $|\eta| > |y|$. We would therefore expect a single particle spectrum which is flat in y to be depleted in the central region when plotting $dn/d\eta$, but the effect is only important for particles at low p_t (or $\beta \ll 1$). The following tables give one a feeling for how the rapidity y differs from the pseudo-rapidity η at various locations in the detector specified by η. Each entry is the difference between η and y ($\eta - y$). The first table is for pions, the second for protons. All momenta are in GeV. Note that rapidity and pseudo-rapidity are equal to within 0.05 (of order size of smallest pad) for pion momenta of 0.4 GeV and proton momenta of 3.0 GeV at all rapidities. This means that one can use η in place of y in making kinematic calculations for pions and protons in this range.

PION	$\eta = 0$	$\eta = 1$	$\eta = 2$	$\eta = 3$
$p = 0.2$	0.20	0.12	0.11	0.10
$p = 0.4$	0.06	0.03	0.03	0.03
$p_t = 0.2$	0.20	0.05	0.01	< 0.01
$p_t = 0.4$	0.06	0.01	< 0.01	< 0.01

PROTON	$\eta = 0$	$\eta = 1$	$\eta = 2$	$\eta = 3$
$p = 0.5$	0.75	0.49	0.45	0.45
$p = 1.5$	0.17	0.10	0.09	0.09
$p = 3.0$	0.05	0.03	0.02	0.02
$p_t = 0.5$	0.75	0.28	0.06	< 0.01
$p_t = 1.5$	0.17	0.04	< 0.01	< 0.01
$p_t = 3.0$	0.05	0.01	< 0.01	< 0.01

In the limit $m \to 0$, $\eta = y$, $\beta = \beta_t = 1$, and so we can write the invariant mass of two massless particles as:

$$M_{12}^2 = 2E_{t1}E_{t2}(\cosh\delta\eta - \cos\delta\phi). \tag{5}$$

In $p\overline{p}$ physics where most of the (longitudinal) momentum from the collision is lost down the beam pipe, energy conservation is restricted by measurement to the transverse plane. Therefore, one has to settle with associating the missing *transverse* energy in the detector with the *transverse* energy of neutrinos or other supersymmetric and non-interacting particles. In this case it is only possible to directly "measure" the invariant "transverse " mass (M_T) of pairs of particles using energy conservation. We can thus define the transverse mass as

$$M_T^2 \equiv (p_{t1}^\mu + p_{t2}^\mu)^2$$

where $p_t^\mu = (E_t, \vec{\mathbf{p}}_t)$. The "transverse mass" can also be defined as the invariant mass that the two particles would have if they were at the same rapidity ($\delta\eta = 0$). For the case where $m_1 = m_2 = 0$ (*e.g.* $W \to e\nu$), we can use equation 5 and set $\delta\eta = 0$ to see that the transverse mass is given by

$$M_T^2 = 2E_{t1}E_{t2}(1 - \cos\delta\phi) = 4E_{t1}E_{t2}\sin^2\delta\phi/2.$$

It is important to remember that in order to use these formulas for M_{12} one must *always* define E_t using equation 2 and <u>not</u> $E\sin\theta$. They are equivalent only in the limit $m/p_t \to 0$, but equation 4 holds always.

1.3 Phase Space Volume

The relativistically invariant phase space volume $d\tau$, defined as

$$d\tau \equiv \frac{d^3p}{E} = \frac{dp_x dp_y dp_z}{E}$$

can be transformed from the coordinates $E, \vec{\mathbf{p}}$ to the coordinates p_t, y, ϕ, m using the fact that

$$dy = \frac{dp_z}{E}$$

as

$$d\tau = \frac{1}{2} dp_t^2 dy d\phi.$$

One sees that particles produced by processes with matrix elements varying slowly (if at all) with rapidity y should be distributed uniformly in rapidity. This is the origin of the famous "rapidity plateau" for single particles in minimum bias events.

2 From Clusters to Jets

2.1 Definition of a Jet

In high energy physics, there are an infinite number of ways to define a "jet". (Most of the LEP, PEP, and PETRA e^+e^- experiments define jets using just the tracking chamber information [1].) However, there is an unambiguous way to define jets, motivated by what we understand to be the physics of fragmentation, namely that a jet is the result of the hadronization of a parent parton, and such hadronization occurs thru the pairing of $q\bar{q}$ pairs from virtual gluon emission. Define ψ to be the angle of a gluon with respect to some reference in the plane perpendicular to the momentum direction of the parent parton. Then these gluons are emitted uniformly in ψ. The momentum (k_t) distributions of the gluons in this plane has been phenomenologically found to be exponential in $-k_t^2$ with $\langle k_t \rangle \simeq 300$ MeV at SPEAR [2] and DORIS [3] energies ($\sqrt{s} \simeq 10GeV$) increasing to $\simeq 1GeV$ at

[1] S. Bethke, Proceedings of the XXIII International Conference on High Energy Physics, Volume II, p. 1086-1087

[2] Hanson, G.G., 13th Renc.de Moriond, 1978, Vol. II, p. 15.

[3] PLUTO Collaboration, BERGER, CH. *et al.*, Phys.Lett. **78B** (1980) 176.

$Sp\overline{p}S$ energies [4]. If one represents the energy E_i in each cell i as a particle with zero mass, then $\vec{\mathbf{p}}_i = E_i\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit vector pointing to the cell. The "jet" is then defined in the calorimeter as the "cluster" of cells whose coordinates in real space is given by the equation

$$\sum \vec{\mathbf{k}}_{\psi i} = 0 \tag{6}$$

and whose 4-vector is given by

$$p^{\mu}_{jet} \ = \ (\sum E_i, \sum \vec{\mathbf{p}}_i) \ = (\sum E_i, \sum E_i\hat{\mathbf{n}}) \tag{7}$$

How does this translate clusters into jets? First, define a D0 cartesian coordinate system where z is along the beam direction pointing counter-clockwise (when looking at the ring from above), x is in the plane of the ring pointing outwards, and y points up. For simplicity, picture a jet which is along the x axis. Then the angle ψ is in the yz plane and equation 6 can be rewritten as

$$\sum E_{yi} = 0 \ , \quad \sum E_{zi} = 0.$$

Using equation 3 above (and remembering the approximation $m_i = 0$) we have

$$\sum E_{ti} \sin\phi_i = 0, \qquad \sum E_{ti} \sinh\eta_i = 0.$$

Since this jet has $\eta_{jet} = \phi_{jet} = 0$, we can generalize the formula by defining

$$\delta\eta_i \equiv \eta_i - \eta_{jet} \tag{8}$$

and

$$\delta\phi_i \equiv \phi_i - \phi_{jet} \tag{9}$$

to rewrite the defining equation 6 as

$$\sum E_{ti} \sin\delta\phi_i = 0, \quad \sum E_{ti} \sinh\delta\eta_i = 0 \tag{10}$$

We can expand equation 10 and solve for the true jet coordinations:

[4] G. Thompson, Proceedings of the XXIII International Conference on High Energy Physics, Volume II, p.1153.

$$\tan\phi_{jet} = \frac{\sum E_{yi}}{\sum E_{xi}}$$

and

$$\cos\theta_{jet} = \frac{\sum E_i \cos\theta_i}{\sum E_i}$$

or since $\cos\theta = \tanh\eta$ we can write this as

$$\tanh\eta_{jet} = \frac{\sum E_i \tanh\eta_i}{\sum E_i}.$$

This is in fact equivalent to equation 7 above.

Note that the last missing ingredient guides the decision as to which cells to include in a cluster. One such scheme, usually referred to as a "fixed-cone" algorithm and used by UA1 [5], and CDF [6] involves an iteration over the choices of cells using equation 6 employing a soft cutoff in the distance in $\eta\phi$ space between the jet centroid and the cell. This distance is often referred to as "r" and is defined as

$$r \equiv \sqrt{\delta\eta^2 + \delta\phi^2}.$$

(Note UA2 uses a so-called "nearest-neighbor" algorithm [7].) Figure 1 shows how well a jet which is defined using equation 10 is approximated by a circle. For the usual cone size of $r = .7$, the approximation is quite good. Note that the approximation is even better than what is in figure 1, since for a true jet, there will be much more energy nearer the jet center than at the outer radii where one would look for such deviations. Also, CDF could not look for circular jets due to the magnetic field distorting the jet shape in the ϕ direction.

2.2 The E_t of a Jet

In the QCD literature for jets at the hadron colliders, one will often refer to the "E_t" of a jet (*e.g* the inclusive E_t spectrum will deviate from QCD

[5] Arnison, G. *et al.*, Phys.Lett.**132B**, 214 (1983) and **172B**, 461 (1986).
[6] F. Abe *et al.*, PRL. **62**, 613 (1989).
[7] M. Barner *et al.*Phys.Lett. **118B**, 203 (1982).

at very high E_t if there is compositness at a reasonable scale). The quantity "E_t" is always calculated using the relation

$$E_t \equiv E \sin\theta \tag{11}$$

where θ is the polar angle. As long as one is consistent in comparing theory to experiment, it is completely arbitrary how one defines E_t. However, from the previous chapter, it is clear that equation 11 is incorrect *if* one is going to use E_t in invariant mass formulas such as equation 5. In that case, one must use E_t as defined in equation 2. The difference, again, is only significant in the limit $\beta_{jet} \ll 1$. To see this clearly, we use the relation $E^2 = p^2 + m^2$ and write

$$(E\sin\theta)^2 \;=\; (p^2 + m^2)\sin^2\theta \;=\; p_t^2 + m^2\sin^2\theta.$$

We see that this definition gives a *systematically* lower value for "E_t" than the relativistically correct definition from equation 2. The implications of this can be seen by noting that the the inclusive jet cross-section $d\sigma/dE_t$ falls extremely fast with E_t (*e.g.* over 3 decades between 50 and 150 GeV [8]) for just about all classes of QCD events (jet physics, heavy quark physics, W/Z physics, *etc.*). For heavy quark (top) physics where one is looking in the e+jets channel, a logical initial requirement is that there be at least 2 jets in the event with E_t above some threshold (and of course and electron candidate). Since one will want to make this jet E_t cut as low as possible, one will be cutting in the region where $E_t - E\sin\theta$ is largest and where the acceptance is changing the fastest. Therefore, one will want to be sure how one is defining the jet transverse energy in order to know one's acceptance when making cuts in such a steeply falling spectra.

It is important to note that E_{tjet} is not made up of a sum of the E_t's of the individual cells. To be consistent with the definition of the jet from equation 7 we have

$$p_{tjet} = \sqrt{p_{xjet}^2 + p_{yjet}^2}$$

or equivalently

$$p_{tjet} = \frac{\sum E_{ti}\cos\phi_i}{\cos\phi_{jet}} \;=\; \frac{\sum E_{ti}\sin\phi_i}{\sin\phi_{jet}}$$

[8]F. Abe *et al.*, PRL. **62**, 613 (1989).

and similarly for E_{tjet} using equation 2 or equation 11 as desired depending on the context. If we make use of equations 8 and 9 we have the following formulas for E_{tjet} and p_{tjet} in terms of the angles $\delta\phi_i$ and $\delta\eta_i$:

$$E_{tjet} \;=\; \sum E_{ti}\cosh\delta\eta_i, \qquad p_{tjet} \;=\; \sum E_{ti}\cos\delta\phi_i$$

2.3 The Small Angle Limit

We expect that jets with a large amount of energy will be relatively "thin" ($\delta\phi_i \to 0$ and $\delta\eta_i \to 0$) or in other words will trace out a very small circle in $\eta\phi$ space. In this "small angle" limit, there are some very interesting results from the above equations.

Using equation 10, we can write k_t in this limit as

$$k_{ti}^2 \simeq E_{ti}^2(\delta\phi_i^2 + \delta\eta_i^2)$$

or equivalently

$$k_{ti} \simeq E_{ti}r_i$$

where r_i is the distance between the jet and the cell in $\eta\phi$ space defined by

$$r_i^2 \equiv \; \delta\eta_i^2 \; + \; \delta\phi_i^2. \tag{12}$$

Thus, the momentum of each "particle" in the plane perpendicular to the jet axis (the $\eta\phi$ plane) is given by the product of the E_t of the particle (in the lab frame) and the distance in $\eta\phi$ space from the axis.

Another interesting result is that all particles with a given E_{ti} should be circularly distributed in $\eta\phi$ space, or in other words a jet to some approximation traces out a circle in $\eta\phi$ space. This is a good approximation if one is only including particles near the jet.

In the small angle limit, we can expand E_{tjet} and p_{tjet} to show

$$E_{tjet} \;\to\; \sum E_{ti}(1 + \frac{\delta\eta_i^2}{2}), \qquad p_{tjet} \;\to\; \sum E_{ti}(1 - \frac{\delta\phi_i^2}{2}).$$

We can see that the difference between the true E_{tjet} and p_{tjet} and the quantity $\sum E_{ti}$ is in the opening angle between the "cell" and the jet centroid in η and ϕ. This angle is intuitively associated with the "mass" of the "cell" as if the "cell" were a single particle.

2.4 The Obesity of a Jet

The difference between the quantities E_{tjet} and p_{tjet} is sometimes referred to as the "obesity" of the jet:

$$\Omega \;\equiv\; E_{tjet} - p_{tjet} \tag{13}$$

and in the small angle limit

$$\Omega \;\rightarrow\; \frac{1}{2}\sum E_{ti}(\delta\eta_i^2 + \delta\phi_i^2)$$

or using equation 12

$$\Omega \;\simeq\; \frac{1}{2}\sum k_{ti} r_i. \tag{14}$$

It is believed that the average number of particles in a jet and the average k_t grow (approximately) with the logarithm of the jet energy (from longitudinal phase space considerations). At TEVATRON energies, the average number of charged particles in a jet is estimated to be in the range of 5-10, with an average k_t usually taken to be about $.5 - 1GeV$ (see previous references). If we include neutral particles, we can estimate the obesity as being in the range of $\simeq 1 - 10GeV$ for most jets.

2.5 The Mass of a Jet

It is interesting to use equation 7 and calculate the invariant mass of a jet. The fastest way to do this is to write

$$M^2 \equiv E^2 - p^2 \;=\; E_t^2 - p_t^2 \;=\; (E_t + p_t)(E_t - p_t)$$

and using equation 13 we have

$$M^2 \;=\; 2\Omega E_t - \Omega^2. \tag{15}$$

In the small angle limit where the jets are thin and thus have large E_{tjet}, we can drop terms in Ω/E_{tjet} to get

$$M^2 \rightarrow E_t \sum E_{ti}(\delta\eta_i^2 + \delta\phi_i^2).$$

To get a feeling for this formula, look again at equation 10 and take the small angle limit. The result is that

$$\sum E_{ti}\delta\phi_i \ = \ 0, \quad \sum E_{ti}\delta\eta_i \ = \ 0.$$

This says that the cell E_t weighted 1st moment (in the difference coordinates) of the jet vanishes, which is nothing more than the definition of $\delta\phi$ and $\delta\eta$. To extend this, we can define the 2nd moment of the jet in $\eta\phi$ space as

$$\sigma_\eta^2 \equiv \frac{\sum E_{ti}\delta\eta_i^2}{\sum E_{ti}}, \ \sigma_\phi^2 \equiv \frac{\sum E_{ti}\delta\phi_i^2}{\sum E_{ti}}$$

which being quadratic in $\delta\eta$ and $\delta\phi$ will not vanish. Now, define

$$\sigma_r^2 \ \equiv \ \sigma_\eta^2 \ + \ \sigma_\phi^2$$

and we have

$$M^2 \ = \ (E_t)^2\sigma_r^2.$$

If we define

$$\Delta R \ \equiv \sqrt{\sigma_\phi^2 \ + \ \sigma_\eta^2}$$

we can write the invariant mass of the jet as

$$M_{jet} \ = \simeq \Delta R \ E_{tjet}.$$

This equation is interesting because it confirms what we already know about the invariant mass of a multiparticle system, namely that the mass of the system is equivalent to (or generated by) the angles between the particles.