Probability Theory

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Lecture 1: Intro to Probability

1 Basics of Probability

What data do you need to specify probability? You need the **set of all outcomes**, a list of everything that could possibly occur as a consequence, and the likelihood of each event.

Example. For a roll of a dice, the set of all outcomes would be $\{1, 2, 3, 4, 5, 6\}$. The list could include things like "the result is 3", or "the result is ≥ 4", and the likelihood would be $\frac{1}{6}$ for each of the results.

1.1 Basics of Set Theory

Definition 1. A **set** is an unordered collection of elements. **Elements** are objects within sets.

Definition 2. A set A is a **subset** of a set B if $a \in A \Rightarrow a \in B$

Definition 3. The **union** of two sets A and B is the collection of elements that are in A or B.

Definition 4. The **intersection** of two sets A and B is the collection of elements that are in both A and B.

Definition 5. The **complement** of a set A is everything not in A.

Definition 6. A **finite set** is a set with finite number of elements.

Definition 7. The **cartesian** product of two sets A and B denoted $A \times B$ is

$$\{(a, b) : a \in A \wedge b \in B\}.$$

Then, $|A \times B| = |A| \cdot |B|$.

1.2 Back to Probability

Definition 8. A **sample space** is the set of al possible outcomes in an experiment.

Example. The sample space Ω for a coin flip is $\{H, T\}$.

Note that **events** are just subsets of the sample space, and **elementary events** are just elements of the sample space.

Example. For a dice roll: $\Omega = \{1, 2, 3, 4, 5, 6\}$, some events could be $\{1, 2\}$, $\{3, 6\}$, $\{3\}$. There are a total of 2^6 events.

Definition 9. If Ω is a finite set, a probability P on Ω is a function: $P : 2^{\Omega} \to [0, 1]$ such that $P(\varnothing) = 0$ and $P(\Omega) = 1$.

Lemma 1. If $A_1, \ldots, A_\alpha \subset \Omega$ are disjoint, $P(\bigcup_i A_i) = \sum_i P(A_i)$.

Proposition 1. Let $A = \{a_1, a_2, \ldots a_l\}$ such that a_i are elementary events. Then,

$$P(A) = \sum_{i=1}^{l} P(\{a_i\}).$$

Example. For the dice roll, if $A = \{1, 3, 5\}$, then $P(A) = 3 \cdot \frac{1}{6} = \frac{1}{2}$.

Definition 10. **Equiprobable outcomes**: Let's say we have the set $\Omega = \{\omega_1, \ldots, \omega_N\}$ and $P(\omega_i) = P(\omega_j)$ for all i and j. Then, $P(\omega) = \frac{1}{N}$ for all $\omega \in \Omega$ and $P(A) = \frac{|A|}{N}$. In other words, when outcomes are probable,

$$P(\text{event}) = \frac{\text{number of outcomes for that event}}{\text{number of possible outcomes}}.$$

1.3 Counting

Suppose 2 experiments are being performed. Let's say that experiment 1 has m possible outcommes, and experiment 2 has n possible outcomes. Then together, there are total of $n \cdot m$ total outcomes.

Example. Rolling a dice and then flipping a coin, how many possible outcomes are there?

Proof. You have $6 \cdot 2 = 12$ outcomes.

Example. Let's say you have a college planning committee that consists of 3 freshman, 4 sophomores, 5 juniors, and 2 seniors. How many ways are there to select a subcommittee of 4 with one person from each grade?

Proof. There are 4 events with 3, 4, 5, and 2 possible outcomes for each. Therefore, there are $3 \cdot 4 \cdot 5 \cdot 2 = 120$ total subcommittees.

Example. How many 7-place license plates are there if the first 3 are letters and the last 4 are numbers?

Proof. There are $26^3 \cdot 10^4$ license plates.