Combinatorial Analysis

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Contents

Lecture 1: Syllabus and Review

1 Introduction

This course is basically just a second course in Combinatorics, and will cover a range of topics.

Definition 1. Matroids are the structures that capture whether or not the greedy algorithm works. They will be covered later in the course.

Now, for some examples and review:

Definition 2. We say points are in **convex position** if no point is inside a triangle made by 3 other points.

Example. Given a finite set of points on the plane, what is the maxmimum number of points such that no 3 are on a line, and no 4 are in convex position.

Explanation. Informally, we know that the "outside" of our points has at most 3 points in the shape of a triangle. We can then place a point in the middle. However, if we try to add another point, then we find that 4 points are in convex position, which is a contradiction. Therefore, 4 points is the maximum size of such a set.

This example is actually part of a more general problem, shown below.

Theorem 1. (ES, 1935) The maxmimum number of points such that no 3 are on a line and no n are in convex position is $\leq 4^n$ and $\geq 2^{n-2}$.

Theorem 2. (Suk, 2017) This number is actually $\leq 2^{n+o(1)}$

Notation. Think of $o(1)$ as standing for a function $f(n)$ such that $\lim_{n \to \infty} f(n) = 0$. In other words, for every $\varepsilon > 0$, there exists n_0 such that $|f(n)| < \varepsilon$ for every $n \geq n_0$.

Example. How many distinct 5-letter words are there on the 26-letter english alphabet?

Explanation. There are 26 options for each of the 5 slots, so there are 26^5 words.

Example. What if repetitions aren't allowed?

Explanation. Each slot you lose an option, so there are $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = \frac{26!}{21!}$ words.

Example. How many ways are there to choose 5 students out of 35 to present?

Explanation. There are $\binom{35}{5} = \frac{35!}{5! \cdot 30!}$ ways.

Lecture 2: Review of Proofs

We will now review the types of proofs covered in Math-3012, as well as guidelines for writing them in this class.

Notation. If F is a mapping from N to M, we write $F : N \to M$.

Notation. Sometimes, $N \backslash \{a\}$ will be instead written as $N - \{a\}$.

Proposition 1. Let N be an n-element set and M be an m-element set. Then, there are m^n mappings (or functions) from N to M.

Proof. (Inductive) We go by induction on n.

Base case.: For the base case $n = 0$, we con-

sider the empty set $\varnothing$ to be a mapping from the empty set to M. So $m^0 = 1$ and the base case holds.

Inductive step.: Now, let $n \geq 1$ and assume that the proposition holds for $n-1$ by induction. So, let $a \in N$. There are m^{n-1} mappings $F' : N\backslash\{a\} \to M$. For each such F', we have m choices for where to send a. These mappings are all distinct, and every $F : N \to M$ can be obtained in this way. So, the number of mappings $F : N \to M$ is $m^{n-1} \cdot m = m^n$, as desired.

∎

Definition 3. A **bijection** is a function $f : X \to Y$ such that f is one-to-one and onto.

Corollary. An n-element set X has 2^n many subsets.

Proof. (Bijective) For each $A \subseteq X$, let $F_A : X \to \{0, 1\}$ such that for each $x \in X$,

$$F_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}.$$

These mappings $F_A, F_{A'}$ are distinct for distinct subsets $A, A' \subseteq X$, and every mapping $F : X \to \{0, 1\}$ is equal to F_A for some $A \subseteq X$. So by proposition 1, the corollary holds. ∎

Lemma 1. For any non-negative integers n, k ($n, k \in Z_{\geq 0}$), we have $\binom{n}{k} = \binom{n}{n-k}$.

Proof. (Algebraic) We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
$$= \frac{n!}{(n-(n-k))!(n-k)!}$$
$$= \binom{n}{n-k},$$

as desired. ∎

Theorem 3. (Binomial Theorem) Let $n \in \mathbb{Z}_{\geq 0}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. Consider

$$\underbrace{(x+y)(x+y)\ldots(x+y)}_{n \text{ times}}.$$

For each $(x+y)$ term, we select either the x or the y, and there are $\binom{n}{k}$ ways to select k x's and $n-k$ y's. The formula follows. ∎

Corollary. For any $n \in \mathbb{Z}_{\geq 0}$, we have

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \text{ and } 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

Proof. Apply the binomial theorem with $x = y = 1$ to yield the first result, and with $x = -1, y = 1$ to yield the second. ∎

1.1 Counting Review

Definition 4. A **permutation** is a bijection from a finite set to itself.

Example. One such bijection could be $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3$.

Lemma 2. The number of such bijections is $n!$.

Proof. Exercise to the student! ∎

Lecture 3: Permutations and Cycles

Notation. $\tau : X \to X$ is a permutation **on** X. Can also be denoted by $\sigma : X \to X$.

We will show that all permutations τ can be "decomposed" into "cycles".

Example. From the example earlier, $(1, 2)$ is a cycle, and $(3, 4, 5)$ is another cycle.

For the following, let $\tau : X \to X$.

Definition 5. A **cycle** of τ is a tuple (ordered set of elements) $(x_1, x_2, \ldots, x_k)$ such that $x_1, x_2, \ldots, x_k$ are distinct elements of X, and $\tau(x_1) = x_2, \tau(x_2) = x_3, \ldots, \tau(x_{k-1}) = x_k, \tau(x_k) = x_1$. We call $x_1, x_2, \ldots x_k$ the **elements** of the cycle.

Lemma 3. If $(x_1, x_2 \ldots, x_k)$ and $(y_1, y_2, \ldots y_r)$ have an element in common, then $\{x_1, x_2, \ldots x_k\} = \{y_1, y_2, \ldots y_r\}$.

Proof. Note that since $(x_1, x_2, \ldots, x_k)$ is a cycle, $(x_2, x_3, \ldots, x_k, x_1)$ is also a cycle. Because of this, we can assume that $x_1 = y_1$. So $x_2 = \tau(x_1) = \tau(y_1) = y_2$. Then, we have that $x_2 = y_2$. We can repeat this process until $x_k = y_k$ (swap x, y if $k > r$). Then, we have $x_1 = \tau(x_k) = \tau(y_k) = y_1$, which means that $r = k$. Therefore, all cycles are pairwise disjoint. ∎

Lemma 4. For every $x \in X$, there exists a cycle of τ which has x as an element.

Proof. Consider visiting each element $x, \tau(x), \tau(\tau(x)), \ldots,$ until the first time we re-visit any element. This will eventually happen, because X is finite. Then, let's suppoose that we have visited elements $x_1, x_2, \ldots x_k$ so far, such that $x_1, x_2, \ldots, x_k$ are distinct, and that $\tau(x_k) = x_i$ for some $i \in \{1, 2, \ldots, k\}$. We cannot have $i \geq 2$ because then both x_{i-1} and x_k would both map to x_i, which is a contradiction because a permutation is a bijection. Therefore, $i = 1$ and we have established our cycle. ∎

Corollary. There exists cycles $C_1, C_2, \ldots, C_t$, so that every element of X is an element in exactly one such cycle.

Definition 6. The **cycle notation** for τ is written as
$$\tau = C_1 C_2 \ldots C_t.$$

Example. Find the cycle notation for the permutation τ of $\{1, 2, 3, 4, 5, 6\}$ where
$$\tau(1) = 4$$
$$\tau(2) = 6$$
$$\tau(3) = 2$$
$$\tau(4) = 5$$
$$\tau(5) = 1$$
$$\tau(6) = 3.$$

Explanation. By inspection, we have a cycle $(1, 4, 5)$ and another cycle $(2, 3, 6)$. Therefore, $\tau = (1, 4, 5)(2, 3, 6)$.

Definition 7. A **transposition** is a cycle with exactly two elements.

Problem. How quickly does $n!$ grow as n gets large?

Lecture 4: Estimates for $n!$

Lemma 5. (Simplest) For any positive integer $n \in \mathbb{Z}_{>0}$,
$$2^{n-1} \leq n! \leq n^{n-1}.$$

Proof. We have for the lower bound
$$n! = \prod_{i=2}^{n} i \geq \prod_{i=2}^{n} 2 = 2^{n-1}.$$

And for the upper bound,
$$n! = \prod_{i=2}^{n} i \leq \prod_{i=2}^{n} n = n^{n-1}.$$
∎

Note that these bounds are very far off. Here is a motivating example.

Example. Suppose n students draw a card from a deck of n cards, replacing the card afterwards. What is the likelihood that all n cards drawn are distinct?

Explanation. The probability is the number of desireable outcomes over the total number of outcomes. This is just
$$\frac{n!}{n^n}.$$

Note that if we use the upper bound from this lemma, we would get that the probability is at most $\frac{1}{n}$. In reality however, the true probability is much, much smaller.

Lemma 6. A better set of bounds are the following:
$$\left(\frac{n}{2}\right)^{\frac{n}{2}} \leq n! \leq \frac{(n+1)^n}{2^{\frac{n}{2}}}.$$

Proof. Left as an exercise! ∎

Lemma 7. For any two $a, b \geq 2$, we have $a \cdot b \geq a + b$.

Lemma 8. (Arithmetic-Geometric Mean Inequality) For any two $a, b \geq 0$, we have
$$\sqrt{ab} \leq \frac{a + b}{2}.$$

With these last two lemmas, we can show the following:

Theorem 4. (Gauss) For any $n \in \mathbb{Z}_{>0}$,

$$n^{\frac{n}{2}} \le n! \le \frac{(n+1)^n}{2^n}.$$

Proof. We instead look at $(n!)^2$. Pairing 1 with n, 2 with $n-1$, etc, we have for the lower bound

$$n! = \left(\prod_{i=1}^{n} i\right)\left(\prod_{i=1}^{n} i\right)$$
$$= \prod_{i=1}^{n} i(n+1-i)$$
$$= \prod_{i=1}^{n} \sqrt{i(n+1-i)}$$
$$\ge \prod_{i=1}^{n} \sqrt{n} \qquad \text{(Lemma 7)}$$
$$\ge n^{\frac{n}{2}}.$$

And for the upper bound, we have

$$n! = \left(\prod_{i=1}^{n} i\right)\left(\prod_{i=1}^{n} i\right)$$
$$= \prod_{i=1}^{n} i(n+1-i)$$
$$= \prod_{i=1}^{n} \sqrt{i(n+1-i)}$$
$$\le \prod_{i=1}^{n} \frac{i+n+1-i}{2}$$
$$= \frac{(n+1)^n}{2^n}.$$

$\blacksquare$

Theorem 5. (Even better bound) For any $n \in \mathbb{Z}_{>0}$, we have

$$e\left(\frac{n}{e}\right)^n \le n! \le en\left(\frac{n}{e}\right)^n.$$

Proof. The lower bound will be given as a homework problem. The upper bound is as follows. Note that $\ln(n!) = \sum_{i=1}^{n} \ln(i)$. Then, we can take the integral of $\ln(x)$, which is greater than this sum.

$$\sum_{i=1}^{n} \ln(i) \le \int_{1}^{n+1} \ln(x)\,dx$$
$$= (n+1)\ln(n+1) - n.$$

Thus

$$n! \le e^{(n+1)\ln(n+1)-n}$$
$$= \frac{e^{(n+1)\ln(n+1)}}{e^n}$$
$$= \frac{\left(e^{\ln(n+1)}\right)^{n+1}}{e^n}$$
$$= \frac{(n+1)^{n+1}}{e^n}.$$

Applying this for $n(n-1)!$ gives the bound. $\blacksquare$

Lecture 5: Asymptotic Analysis

Theorem 6. Stirling's Formula says that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Definition 8. For two functions $f, g : \mathbb{Z}_{>0} \to \mathbb{R}$, we write $f \sim g$ and say f is asymptotic to g if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Also note that $f \sim g \Leftrightarrow g \sim f$.

Example. $2n + \sqrt{n} \sim 2n$.

Explanation. This is because

$$\lim_{n\to\infty} \frac{2n + \sqrt{n}}{2n} = 1.$$

Example. (Informal) How many digits are in 100!?

Explanation. Using Stirling's Formula, we have that

$$100! \sim \sqrt{2\pi 100}\left(\frac{100}{e}\right)^{100} = 9.324\ldots \times 10^{157}.$$

, whereas $100! = 9.332\ldots \times 10^{157}$ (very close approximation).

Definition 9. The n-th harmonic number

$$H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

Theorem 7. (Euler-Mascheroni) $H_n \sim \ln(n)$.

Proof. Omitted. $\blacksquare$

Lemma 9. For any positive integer $n \in \mathbb{Z}_{>0}$, we have

$$\frac{\lfloor \log_2(n) \rfloor}{2} \leq H_n \leq \lfloor \log_2(n) \rfloor + 1.$$

Proof. We can break up the proof into parts of size $2, 4, 8, 16 \ldots$. Let $S_k = \{i \in \mathbb{Z}_{>0} : 2^{k-1} \leq i \leq 2^k - 1\}$ for any $k \in \mathbb{Z}_{>0}$. Note that $|S_k| = 2^{k-1}$. Also, for every $x \in S_k$, we have

$$\frac{1}{2^k} < \frac{1}{x} \leq \frac{1}{2^{k-1}}.$$

Therefore, we have

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \sum_{x \in S_k} \frac{1}{x}$$

$$\geq \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \sum_{x \in S_k} \frac{1}{2^k}$$

$$= \sum_{k=1}^{\lfloor \log_2(n) \rfloor} 2^{k-1}/2^k$$

$$= \sum_{k=1}^{\lfloor \log_2 n \rfloor} \frac{1}{2}$$

$$= \frac{\lfloor \log_2(n) \rfloor}{2}.$$

In the other direction, we have

$$H_n \leq \sum_{k=1}^{\lfloor \log_2(n) \rfloor + 1} \sum_{x \in S_k} \frac{1}{x}$$

$$\leq \sum_{k=1}^{\lfloor \log_2(n) \rfloor + 1} \frac{|S_k|}{2^{k-1}}$$

$$= \lfloor \log_2(n) \rfloor + 1.$$

∎

Definition 10. Let $f, g : \mathbb{Z}_{>0} \to \mathbb{R}$. We say $f = O(g)$ or f is big-O of g if there exists n_0, C, such that

$$|f(n)| \leq C \cdot g(n) \quad \forall n \geq n_0.$$

Note. If $f, g : \mathbb{Z}_{>0} \to \mathbb{R}$ and $f \sim g$, we have

$$f = O(g) \quad \text{and} \quad g = O(f).$$

If $\varepsilon = 1$ for all significantly large n, $\frac{f(n)}{g(n)} \leq 2$.

Example. $\sum_{i=1}^{n} \frac{1}{i} = O(\log n)$.

Lemma 10. Let $a, \alpha, \beta > 0$ be fixed. Then as $n \to \infty$,

- $n^\alpha = O(n^\beta)$ if $\alpha < \beta$.
- $n^\alpha = O(a^n)$ if $a > 1$.
- $(\ln(n))^\alpha = O(n^\beta)$.

Lecture 6: Binomial Coefficients and Counting Primes

Note. Note that we can also write, for functions f, g, h $f = g + O(h)$, which means that $|f - g| + O(h)$.

Example.

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} = \frac{n^2}{2} + O(n).$$

Definition 11. $f(n) = \Theta(g(n))$ if $f = O(g)$ and $g = O(f)$.

Definition 12. $f = o(g)$ if $\lim_{n \to \infty} \frac{f}{g} = 0$.

Example. What are all primes less than 20?

Explanation. 2, 3, 5, 7, 11, 13, 17, 19

Definition 13. Let $\pi(n)$ be the number of primes that are $\leq n$.

Theorem 8. The **prime number theorem** states that

$$\pi(n) \sim \frac{n}{\ln(n)}.$$

The **Riemann Hypothesis** states that

$$\pi(n) = \int_1^n \frac{1}{\ln(x)} dx + O(\sqrt{n} \ln(n)).$$

It's called a hypothesis because it is often used in other mathematical proofs, even if not proved yet. For example, determining whether a knot could be un-knotted is in NP if the RH is true.

Lemma 11. For any $k \geq 1$, we have that

$$\binom{2k+1}{k} \leq 4^k.$$

Proof. Later. ∎

Lemma 12. For any $n \geq 2$, the product of all primes $\leq n$ is at most 16^n.

Proof. We wish to prove that

$$\prod_{i=1}^{\pi(n)} p_i \leq 16^n.$$

where p_i denotes the i-th prime. We proceed with induction on n.

Base case: $n = 2, 3$. Holds trivially.

Step case 1: n is even. Note that n cannot be prime, such that by induction,

$$\prod_{i=1}^{\pi(n)} p_i = \prod_{i=1}^{\pi(n-1)} p_i \leq 16^{n-1} \leq 16^n.$$

Step case 2: n is odd. We write $n = 2k+1$ for some $k \geq 1$. Note that every prime p such that $k + 2 \leq p \leq 2k + 1$ divides $\binom{2k+1}{k}$. This is because

$$\binom{2k+1}{k} = \frac{(2k+1)!}{k!(k+1)!}.$$

such that p divides the numberator but not the denominator.

By induction the product of primes p such that $0 \leq p \leq k + 1$ is bound by

$$\prod_{i=1}^{\pi(k+1)} p_i \leq 16^{k+1}.$$

Combining our bounds and lemma, we have

$$\prod_{i=1}^{\pi(n)} p_i = \left(\prod_{i=1}^{\pi(k+1)} p_i \right) \left(\prod_{i=\pi(k+1)+1}^{\pi(n)} p_i \right)$$
$$\leq 16^{k+1} \binom{2k+1}{k}$$
$$\leq 16^{k+1} \cdot 4^k$$
$$\leq 16^{2k+1}.$$

Note that bounding by 4^n may work here, but 16^n is presented due to a mistake in the lecture notes.

∎

Theorem 9. The weak prime number theorem

states that

$$\pi(n) = \Theta\left(\frac{n}{\ln n} \right).$$

Proof. We will show the upper bound, i.e.

$$\pi(n) = O\left(\frac{n}{\ln n} \right).$$

Let $p_1, p_2, \ldots$ be the sequence of primes. Then,

$$\pi(n)! \leq \prod_{i=1}^{\pi(n)} p_i \leq 16^n.$$

because $p_1 \geq 1, p_2 \geq 2$, etc. We have also shown that

$$e \left(\frac{\pi(n)}{e} \right)^{\pi(n)} \leq \pi(n)!.$$

As such,

$$e \left(\frac{\pi(n)}{e} \right)^{\pi(n)} \leq 16^n.$$

Taking the ln of both sides, we have

$$\ln \left(\textcolor{red}{e \to 1} \left(\frac{\pi(n)}{e} \right)^{\pi(n)} \right) \leq \ln (16^n)$$
$$\pi(n) \cdot \ln \left(\frac{\pi(n)}{e} \right) \leq n \ln(16)$$

.

Assume towards a contradiction that

$$\pi(n) \geq \frac{100n}{\ln n}$$

. Then,

$$\frac{100n}{\ln n} \cdot \ln \left(\frac{100n}{e \ln n} \right) \leq n \ln 16$$
$$\frac{100}{\ln 16 \cdot \ln n} \ln \left(\frac{100n}{e \ln n} \right) \leq 1$$
$$\frac{100}{\ln 16 \cdot \ln n} (\ln(100n) - \ln(e \ln n)) \leq 1$$

.

which is a contradiction (after many calculations). Therefore, there exists some $C < 100$ such that $\pi(n) = O(\frac{n}{\ln n})$. ∎

Lecture 7: More Bounds, Inclusion Exclusion

The proof from previous lecture will not be given, as it is on the homework.

Lemma 13. For any $n \geq k \geq 1$,

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Observation. If $n = 2k$, then

$$\binom{2k}{k} \leq \left(\frac{e \cdot 2k}{k}\right)^k \leq (2e)^k.$$

Lemma 14. (Bernoulli-type inequality) For any real number $x \geq 0$, we have

$$1 + x \leq e^x.$$

Proof. This is true for $x = 0$ because $1 + 0 \leq e^0 = 1$. Therefore, we just need to show that

$$f(x) = e^x - (1 + x).$$

is increasing. This holds because $f'(x) = e^x - 1 \geq 0$. ∎

Lemma 15. For any $n \geq k \geq 1$, we have

$$\sum_{i=0}^{k} \binom{n}{i} \leq \left(\frac{en}{k}\right)^k.$$

Proof. Recall that for any x, y, we have

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

When $y = 1$,

$$(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Choosing $x = \frac{k}{n}$ with the intuition that $x < 1$ allows us to continue:

$$\sum_{i=0}^{n} \binom{n}{i} x^i \geq \sum_{i=0}^{k} \binom{n}{i} x^i.$$

Dividing everything by x^k,

$$\frac{(1 + x)^n}{x^k} = \sum_{i=0}^{n} \binom{n}{i} x^{i-k} \geq \sum_{i=0}^{k} \binom{n}{i} x^{i-k}.$$

Since $x \leq 1$,

$$\frac{(e^x)^n}{x^k} \geq \frac{(1 + x)^n}{x^k} \geq \sum_{i=0}^{k} \binom{n}{i} x^{i-k} \geq \sum_{i=0}^{k} \binom{n}{i}.$$

Plugging in $x = \frac{k}{n}$ gets our result. ∎

Lemma 16. For any $n \geq k \geq 1$, we have

$$\binom{n}{k} \geq \frac{n^k}{k^k}.$$

Proof.

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} \\
&= \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \frac{n-2}{k-2} \cdot \ldots \cdot \frac{n-k+1}{1} \\
&\geq \frac{n^k}{k^k}.
\end{aligned}$$

∎

1.2 Inclusion Exclusion

Example. Say there are 20 math majors, 15 CS majors, and 5 who are majoring in both in one class. How many people are in the class?

Explanation. $20 + 15 - 5 = 30$.

Lemma 17. If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof. Count. ∎

Lemma 18. If A, B, C are finite sets, then

$$\begin{aligned}
|A \cup B \cup C| = {}& |A| + |B| + |C| \\
& - |A \cap B| - |B \cap C| - |C \cap A| \\
& + |A \cap B \cap C|.
\end{aligned}$$

Definition 14. Given a set S, and a positive integer $k \leq |S|$, we write

$$\binom{S}{k}.$$

to denote the set of subsets of S whose size is exactly k.

Note.

$$\left|\binom{S}{k}\right| = \binom{|S|}{k}.$$

Theorem 10. (Inclusion Exclusion Principle) If $A_1, A_2, \ldots, A_n$ are finite sets, then

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \binom{\{1,2,\ldots,n\}}{k}} \left| \bigcap_{i \in I} A_i \right|.$$

Lecture 8: Derangements

Lecture 9: Extremal Combinatorics

We will start this unit by looking at graphs and hypergraphs.

Definition 15. A **hypergraph** is a pair $H = (V, E)$ such that V is a set, and E is a subset of the powerset of V ($E \subset 2^V$). Unless otherwise noted, V is a finite set.

Definition 16. Given a hypergraph $H = (V, E)$, then the elements $v \in V$ are called **vertices** and the elements $e \in E$ are called **edges**.

Definition 17. A hypergraph $H = (V, E)$ is **isomorphic** to another hypergraph $H' = (V', E')$ if there is a bijection $\phi : V \to V'$ between the vertex sets such that for any $S \subseteq V$, we have $S \in E$ iff $\phi(S) \in E'$.

Definition 18. A **graph** is a hypergraph in which every edge has size 2. In other words, $E \subseteq \binom{V}{2}$.

Definition 19. Given $H = (V, E)$, a vertex $v \in V$ is **incident** to an edge $e \in E$ if $v \in e$.

Definition 20. The **degree** of a vertex $v \in V$ is the number of edges it is incident to. We write this as

$$d_H(v) = |\{e \in E : v \in e\}|.$$

Definition 21. The **incidence mamtrix** M of a hypergraph $H = (V, E)$ is a $V \times E$ matrix (V rows and E columns) so that each entry

$$M_{v,e} : \begin{cases} 1, & \text{if } v \in e \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 11. (Hypergraph handshaking). If $H = (V, E)$ is a hypergraph, then

$$\sum_{v \in V} d_H(V) = \sum_{e \in E} |e|.$$

Proof. Let M be the incidence matrix of H. Then $\sum_{v \in V} d_H(v)$ counts the number of 1's of M by summing along rows, and $\sum_{e \in E} |e|$ counts the number of 1's by summing along columns. As the number of 1's in the matrix is the same, these two values are the same. ∎

Corollary. For any graph, we have

$$\sum_{v \in V} d_G(v) = 2|E|.$$

Proof. This is because $|e| = 2$ for all $e \in E$ by definition of a graph. ∎