A Second Course in Linear Algebra

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Lecture 1: Review

1 Vectors and Matrices

For the time being, everything indicated in this course is in $\mathbb{R}$.

Definition 1. A **vector** will be defined as a column vector, e.g.

$$u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

Notation. Sometimes, they will be written as a column vector lying down, e.g. $(x_1, x_2, x_3) \in \mathbb{R}^3$

Definition 2. Let a be a scalar. Then multiplication between vector and scalar is defined as

$$au = \begin{bmatrix} a \cdot x_1 \\ a \cdot x_2 \\ a \cdot x_3 \end{bmatrix}.$$

Definition 3. Let $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then addition between vectors is defined as

$$u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

Definition 4. If u, v are vectors and a, b are scalars, then any $au + bv$ is a **linear combination** of u and v.

Remark. A **vector space** V is a set of objects u, v such that $au + bv \in V$.

Example. Polynomials of degree ≤ 2 in one variable can form a vector space.

Proof. Let $p(x) = a_0 + a_1x + a_2x^2$, and $q(x) = b_0 + b_1x + b_2x^2$. Multiplying by scalars and adding are defined. Note that $p(x) \to \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

Example. Let $f(x) : [0, 1] \to \mathbb{R}$ be a continuous function. We can multiply such functions by scalars and add together such functions, so they form a vector space as well.

Suppose we have two vectors $u, v \in \mathbb{R}^3$. Looking at the set of all linear combinations of u, v,

- if both u and v are the zero vectoor, then $W = \{0\}$.
- if $u = \lambda v$, $v \neq 0$, then W is the line of all multiples of v.
- if u and v are **linearly independent**, then W is a plane in $\mathbb{R}^3$.

Definition 5. Vectors u_1, u_2, u_3 are **linearly independent** if and only if

$$a_1u_1 + a_2u_2 + a_3u_3 = 0 \Rightarrow a_1 = a_2 = a_3 = 0.$$

Definition 6. Let V, W be a vector spaces such that $W \subseteq V$. Then, W is called a **subspace** of V.

Example. Let $W = \{\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R}\}$. Then, W is a subspace of $\mathbb{R}^3$.

Theorem 1. If $u, v \in V$, then the set of linear combinations of u and v is a subspace.

Proof. Let $W = \text{span}\{u, v\}$. We must show that $w_1, w_2 \in W \Rightarrow c_1 w_1 + c_2 w_2 \in W$. By assumption, $w_1 = a_1 u + b_1 v$, and $w_2 = a_2 u + b_2 v$, such that $w = (c_1 a_1 + c_2 a_2)u + (c_1 b_1 + c_2 b_2)v$. Therefore, w is a linear combination of u, v. □

Example. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $v = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. Then, $\text{span}\{u, v\}$ is a proper subspace of $\mathbb{R}^3$.

Definition 7. $u \cdot v = x_1 y_1 + x_2 y_2 + x_3 y_3$ is the dot product of the vectors $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Definition 8. We say that $u \perp v$ if $u \cdot v = 0$.

Definition 9. The length or **norm** of a vector u is $\sqrt{u \cdot u} = \|u\|$

Theorem 2. The **Cauchy–Schwarz inequality** states that $|u \cdot v| \leq \|u\|\|v\|$.

Proof.

$$(u + \lambda v) \cdot (u + \lambda v) \geq 0$$
$$u \cdot u + \lambda^2 v \cdot v + 2\lambda u \cdot v \geq 0.$$

The minimum lambda is $\frac{-b}{2a} = \frac{-u \cdot v}{v \cdot v}$, which results in this inequality being true. Therefore, all greater values for lambda will result in this inequality being true. □

Theorem 3. The **triangle inequality theorem** states that $\|u + v\| \leq \|u\| + \|v\|$.

Definition 10. The **unit vector** of a vector u, $\hat{u}$ is given by $\frac{u}{\|u\|}$.

Theorem 4. If u and v are vectors such that $\|u\| = \|v\| = 1$, then $u \cdot v = \cos(\theta)$ where θ is the angle between u and v.

Corollary. If u and v are vectors, then $u \cdot v = \|u\|\|v\| \cos(\theta)$. Note that $u \cdot v = 0$ when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Lecture 2: Matrices

Example.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

is a matrix. We can also write $A = \{a_{ij}\}$ such that $i = 1 \ldots n$ and $j = 1 \ldots m$.

What does it mean to take a product between a matrix and a vector?

Definition 11. This product is defined as

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{13}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}.$$

i.e. a collection of dot products between the rows and x.

We can also see the product as a linear combination of the columns of the matrix A.

Definition 12. Let the columns of A be A_1, A_2, A_3. Then, $Ax = x_1 A_1 + x_2 A_2 + x_3 A_3$.

Notation. A's columns are denoted A_1, A_2, A_3, while A's rows are denoted A^1, A^2, A^3.

If we look at the linear equation $Ax = b$, we can say that b is a linear combination of the columns of A. Instead, looking at it like an equation, "can b be written as a linear combination of the columns of A"?

Looking at $A^1 x = b_1$, there are two free variables, such as this is a plane in $\mathbb{R}^3$. The only time this is not a plane is if a_{11}, a_{12}, a_{13} are all zero, and b_1 is nonzero.

If we have x, y, $A^1 x = 0$ and $A^1 y = 0$ implies $ax + by = z$, which solves $A^1 z = 0$. The set of solutions is a subspace.

Now, suppose we have all solutions of $A^1 x = 0$. Call this V. How do we then write the solutions to $A^1 x = b$? We find any such c such that $A^1 c = b_1$. Then, we claim that the set of solutions of $A^1 x = b_1$ is $V + c = \{x + c | x \in V\}$. Checking our solution, $A^1 \cdot (x + c) = \underbrace{A^1 \cdot x}_{0} + \underbrace{A^1 \cdot c}_{b_1} = b_1$.

Let $W = V + c$. We want to show if $x \in W \Rightarrow A^1 \cdot x = 0$. Assume $A^1 z = b_1$. If we set $x = z - c$, then $A^1 x = A^1 z - A^1 c = 0$. Therefore, $z = x + c \in W$.

All in all, solving all three equations $A^1 x = b_1, A^2 x = b_2, A^3 x = b_3$ is now just finding the intersection of three translated planes. **This is what solving $Ax = b$ means.**

Another viewpoint is this. Consider the equation $A_1 x_1 + A_2 x_2 + A_3 x_3 = b$. Consider the span of

A_1, A_2, A_3. Does this span contain b?

Example. Let's say that

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solving $Ax = b$, we have $x_3 = b_3$, $x_2 = b_2 + b_3$, and $x_1 = b_1 + b_2 + b_3$ such that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Let C denote this matrix. Then, $Ax = b \Leftrightarrow Cb = x$, such that $C = A^{-1}$. Then, C is the **inverse** of A.

Definition 13. We want to say that every $n \times n$ matrix can be written as the product as an upper triangular and lower triangular matrix, called **LU factorization**.

Definition 14. **Matrix multiplication** is defined as $(AB)_{ij} = \sum_k a_{ik} + b_{kl}$ where $A = \{a_{ij}\}$ and $B = \{b_{kl}\}$

The other way to see AB is if $B = \begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix}$, then $AB = \begin{pmatrix} AB_1 & AB_2 & \dots & AB_n \end{pmatrix}$. In other words, $(AB)_{ij} = A^i \cdot B_j$.

Lecture 3: Matrix Algebra

Example. Solve

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & 4 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{b}.$$

Proof.

$$x = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Let

$$E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have

$$E_{12} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}.$$

Note that this is also $E_{12}(Ax) = E_{12}b = (E_{12}A)x$

Definition 15. AB is such that

$$A(Bx) = (AB)x.$$

for every vector x. It is defined as

$$AB = [AB^1, AB^2, \dots, AB^n].$$

where B^i is the i-th column of B.

Theorem 5. $Ax = b \Rightarrow (CA)x = Cb$

Theorem 6. Let $\mathbb{R}^n$ be a vector space and $A, B : \mathbb{R}^n \to \mathbb{R}^n$ linear mappings. Then,

$$A \circ B : \mathbb{R}^n \to \mathbb{R}^n.$$

is also a linear transformation. Also

$$A \circ B(x) = ABx.$$

Theorem 7. If $\hat{A}$ is a linear map from $\mathbb{R}^n \to \mathbb{R}^n$ then $\hat{A}(x) = Ax$ for a matrix A.

Proof. For a linear map, we have $\hat{A}(x + y) = \hat{A}(x) + \hat{A}(y)$ and $\hat{A}(\alpha x) = \alpha\hat{A}(x)$. We want to show that any linear mapping is a matrix multiplication. Let

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{bmatrix}$$

where the 1 is in the ith place. Let $A^i = \hat{A}(e_i)$. Let $A = \begin{bmatrix} A^1 & A^2 & \dots & A^n \end{bmatrix}$. Then, by construction

$$\begin{aligned} \hat{A}(x) &= \hat{A}(x_1 e_1 + x_2 e_2) + \dots + x_n e_n) \\ &= x_1 \hat{A}(e_1) + x_2 \hat{A}(e_2) + \dots + x_n \hat{A} e_n) \\ &= x_1 A^1 + x_2 A^2 + \dots + x_n A^n \\ &= Ax. \end{aligned}$$

□

We can also calculate matrix multiplication as $(AB)_{i,j} = \sum_k A_{i,k} \cdot B_{k,j}$.

Theorem 8. Suppose we take a third matrix C.

Then,
$$A(BC) = (AB)C.$$
This is the **associative property**.

Proof. We saw that
$$A(Bx) = (AB)x.$$
Applying this, we have:
$$\begin{aligned}(AB)C &= \begin{bmatrix}(AB)C^1 & \dots & (AB)C^n\end{bmatrix} \\ &= \begin{bmatrix}A(BC^1) & \dots & A(BC^n)\end{bmatrix} \\ &= A\begin{bmatrix}BC^1 & \dots & BC^n\end{bmatrix} \\ &= A(BC).\end{aligned}$$
□