

# Combinatorial Analysis

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## Contents

<b>1 Introduction</b>	<b>1</b>
1.1 Counting Review . . . . .	2

**Notation.** Think of  $o(1)$  as standing for a function  $f(n)$  such that  $\lim_{n \rightarrow \infty} f(n) = 0$ . In other words, for every  $\varepsilon > 0$ , there exists  $n_0$  such that  $|f(n)| < \varepsilon$  for every  $n \geq n_0$ .

## Lecture 1: Syllabus and Review

### 1 Introduction

This course is basically just a second course in Combinatorics, and will cover a range of topics.

**Definition 1. Matroids** are the structures that capture whether or not the greedy algorithm works. They will be covered later in the course.

Now, for some examples and review:

**Definition 2.** We say points are in **convex position** if no point is inside a triangle made by 3 other points.

**Example.** Given a finite set of points on the plane, what is the maximum number of points such that no 3 are on a line, and no 4 are in convex position.

**Explanation.** Informally, we know that the “outside” of our points has at most 3 points in the shape of a triangle. We can then place a point in the middle. However, if we try to add another point, then we find that 4 points are in convex position, which is a contradiction. Therefore, 4 points is the maximum size of such a set.

This example is actually part of a more general problem, shown below.

**Theorem 1.** (ES, 1935) The maximum number of points such that no 3 are on a line and no  $n$  are in convex position is  $\leq 4^n$  and  $\geq 2^{n-2}$ .

**Theorem 2.** (Suk, 2017) This number is actually  $\leq 2^{n+o(1)}$

**Example.** How many distinct 5-letter words are there on the 26-letter english alphabet?

**Explanation.** There are 26 options for each of the 5 slots, so there are  $26^5$  words.

**Example.** What if repetitions aren’t allowed?

**Explanation.** Each slot you lose an option, so there are  $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = \frac{26!}{21!}$  words.

**Example.** How many ways are there to choose 5 students out of 35 to present?

**Explanation.** There are  $\binom{35}{5} = \frac{35!}{5! \cdot 30!}$  ways.

## Lecture 2: Review of Proofs

We will now review the types of proofs covered in Math-3012, as well as guidelines for writing them in this class.

**Notation.** If  $F$  is a mapping from  $N$  to  $M$ , we write  $F : N \rightarrow M$ .

**Notation.** Sometimes,  $N \setminus \{a\}$  will be instead written as  $N - \{a\}$ .

**Proposition 1.** Let  $N$  be an  $n$ -element set and  $M$  be an  $m$ -element set. Then, there are  $m^n$  mappings (or functions) from  $N$  to  $M$ .

**Proof.** (Inductive) We go by induction on  $n$ .

**Base case.:** For the base case  $n = 0$ , we consider the empty set  $\emptyset$  to be a mapping from the empty set to  $M$ . So  $m^0 = 1$  and the base case holds.

**Inductive step.:** Now, let  $n \geq 1$  and assume that the proposition holds for  $n - 1$  by induction. So, let  $a \in N$ . There are  $m^{n-1}$  mappings  $F' : N \setminus \{a\} \rightarrow M$ . For each such  $F'$ , we have  $m$  choices for where to send  $a$ . These mappings are all distinct, and every  $F : N \rightarrow M$  can be obtained in this way. So, the number of mappings  $F : N \rightarrow M$  is  $m^{n-1} \cdot m = m^n$ , as desired.  $\square$

**Definition 3.** A **bijection** is a function  $f : X \rightarrow Y$  such that  $f$  is one-to-one and onto.

**Corollary.** An  $n$ -element set  $X$  has  $2^n$  many subsets.

**Proof.** (Bijective) For each  $A \subseteq X$ , let  $F_A : X \rightarrow \{0, 1\}$  such that for each  $x \in X$ ,

$$F_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

These mappings  $F_A, F_{A'}$  are distinct for distinct subsets  $A, A' \subseteq X$ , and every mapping  $F : X \rightarrow \{0, 1\}$  is equal to  $F_A$  for some  $A \subseteq X$ . So by proposition 1, the corollary holds.  $\square$

**Lemma 1.** For any non-negative integers  $n, k$  ( $n, k \in \mathbb{Z}_{\geq 0}$ ), we have  $\binom{n}{k} = \binom{n}{n-k}$ .

**Proof.** (Algebraic) We have

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(n-(n-k))!(n-k)!} \\ &= \binom{n}{n-k}, \end{aligned}$$

as desired.  $\square$

**Theorem 3.** (Binomial Theorem) Let  $n \in \mathbb{Z}_{\geq 0}$ . Then

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

**Proof.** Consider

$$\underbrace{(x + y)(x + y) \dots (x + y)}_{n \text{ times}}.$$

For each  $(x + y)$  term, we select either the  $x$  or the  $y$ , and there are  $\binom{n}{k}$  ways to select  $k$   $x$ 's and  $n - k$   $y$ 's. The formula follows.  $\square$

**Corollary.** For any  $n \in \mathbb{Z}_{\geq 0}$ , we have

$$2^n = \sum_{k=0}^n \binom{n}{k} \text{ and } 0 = \sum_{k=0}^n \binom{n}{k} (-1)^k.$$

**Proof.** Apply the binomial theorem with  $x = y = 1$  to yield the first result, and with  $x = -1, y = 1$  to yield the second.  $\square$

## 1.1 Counting Review

**Definition 4.** A **permutation** is a bijection from a finite set to itself.

**Example.** One such bijection could be  $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3$ .

**Lemma 2.** The number of such bijections is  $n!$ .

**Proof.** Exercise to the student!  $\square$

## Lecture 3: Permutations and Cycles

**Notation.**  $\tau : X \rightarrow X$  is a permutation on  $X$ . Can also be denoted by  $\sigma : X \rightarrow X$ .

We will show that all permutations  $\tau$  can be “decomposed” into “cycles”.

**Example.** From the example earlier,  $(1, 2)$  is a cycle, and  $(3, 4, 5)$  is another cycle.

For the following, let  $\tau : X \rightarrow X$ .

**Definition 5.** A **cycle** of  $\tau$  is a tuple (ordered set of elements)  $(x_1, x_2, \dots, x_k)$  such that  $x_1, x_2, \dots, x_k$  are distinct elements of  $X$ , and  $\tau(x_1) = x_2, \tau(x_2) = x_3, \dots, \tau(x_{k-1}) = x_k, \tau(x_k) = x_1$ . We call  $x_1, x_2, \dots, x_k$  the **elements** of the cycle.

**Lemma 3.** If  $(x_1, x_2, \dots, x_k)$  and  $(y_1, y_2, \dots, y_r)$  have an element in common, then  $\{x_1, x_2, \dots, x_k\} = \{y_1, y_2, \dots, y_r\}$ .

**Proof.** Note that since  $(x_1, x_2, \dots, x_k)$  is a cycle,  $(x_2, x_3, \dots, x_k, x_1)$  is also a cycle. Because of this, we can assume that  $x_1 = y_1$ . So

$x_2 = \tau(x_1) = \tau(y_1) = y_2$ . Then, we have that  $x_2 = y_2$ . We can repeat this process until  $x_k = y_k$  (swap  $x, y$  if  $k > r$ ). Then, we have  $x_1 = \tau(x_k) = \tau(y_k) = y_1$ , which means that  $r = k$ . Therefore, all cycles are pairwise disjoint.  $\square$

**Lemma 4.** For every  $x \in X$ , there exists a cycle of  $\tau$  which has  $x$  as an element.

**Proof.** Consider visiting each element  $x, \tau(x), \tau(\tau(x)), \dots$ , until the first time we re-visit any element. This will eventually happen, because  $X$  is finite. Then, let's suppose that we have visited elements  $x_1, x_2, \dots, x_k$  so far, such that  $x_1, x_2, \dots, x_k$  are distinct, and that  $\tau(x_k) = x_i$  for some  $i \in \{1, 2, \dots, k\}$ . We cannot have  $i \geq 2$  because then both  $x_{i-1}$  and  $x_k$  would both map to  $x_i$ , which is a contradiction because a permutation is a bijection. Therefore,  $i = 1$  and we have established our cycle.  $\square$

**Corollary.** There exists cycles  $C_1, C_2, \dots, C_t$ , so that every element of  $X$  is an element in exactly one such cycle.

**Definition 6.** The **cycle notation** for  $\tau$  is written as

$$\tau = C_1 C_2 \dots C_t.$$

**Example.** Find the cycle notation for the permutation  $\tau$  of  $\{1, 2, 3, 4, 5, 6\}$  where

$$\begin{aligned}\tau(1) &= 4 \\ \tau(2) &= 6 \\ \tau(3) &= 2 \\ \tau(4) &= 5 \\ \tau(5) &= 1 \\ \tau(6) &= 3.\end{aligned}$$

**Explanation.** By inspection, we have a cycle  $(1, 4, 5)$  and another cycle  $(2, 3, 6)$ . Therefore,  $\tau = (1, 4, 5)(2, 3, 6)$ .

**Definition 7.** A **transposition** is a cycle with exactly two elements.

**Problem.** How quickly does  $n!$  grow as  $n$  gets large?

## Lecture 4: Estimates for $n!$

**Lemma 5.** (Simplest) For any positive integer  $n \in \mathbb{Z}_{>0}$ ,

$$2^{n-1} \leq n! \leq n^{n-1}.$$

**Proof.** We have for the lower bound

$$n! = \prod_{i=2}^n i \geq \prod_{i=2}^n 2 = 2^{n-1}.$$

And for the upper bound,

$$n! = \prod_{i=2}^n i \leq \prod_{i=2}^n n = n^{n-1}.$$

$\square$

Note that these bounds are very far off. Here is a motivating example.

**Example.** Suppose  $n$  students draw a card from a deck of  $n$  cards, replacing the card afterwards. What is the likelihood that all  $n$  cards drawn are distinct?

**Explanation.** The probability is the number of desirable outcomes over the total number of outcomes. This is just

$$\frac{n!}{n^n}.$$

Note that if we use the upper bound from this lemma, we would get that the probability is at most  $\frac{1}{n}$ . In reality however, the true probability is much, much smaller.

**Lemma 6.** A better set of bounds are the following:

$$\left(\frac{n}{2}\right)^{\frac{n}{2}} \leq n! \leq \frac{(n+1)^n}{2^{\frac{n}{2}}}.$$

**Proof.** Left as an exercise!  $\square$

**Lemma 7.** For any two  $a, b \geq 2$ , we have  $a \cdot b \geq a + b$ .

**Lemma 8.** (Arithmetic-Geometric Mean Inequality) For any two  $a, b \geq 0$ , we have

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

With these last two lemmas, we can show the following:

**Theorem 4.** (Gauss) For any  $n \in \mathbb{Z}_{>0}$ ,

$$n^{\frac{n}{2}} \leq n! \leq \frac{(n+1)^n}{2^n}.$$

**Proof.** We instead look at  $(n!)^2$ . Pairing 1 with  $n$ , 2 with  $n-1$ , etc, we have for the lower bound

$$\begin{aligned} n! &= \left( \prod_{i=1}^n i \right) \left( \prod_{i=1}^n i \right) \\ &= \prod_{i=1}^n i(n+1-i) \\ &= \prod_{i=1}^n \sqrt{i(n+1-i)} \\ &\geq \prod_{i=1}^n \sqrt{n} \quad (\text{Lemma 7}) \\ &\geq n^{\frac{n}{2}}. \end{aligned}$$

And for the upper bound, we have

$$\begin{aligned} n! &= \left( \prod_{i=1}^n i \right) \left( \prod_{i=1}^n i \right) \\ &= \prod_{i=1}^n i(n+1-i) \\ &= \prod_{i=1}^n \sqrt{i(n+1-i)} \\ &\leq \prod_{i=1}^n \frac{i+n+1-i}{2} \\ &= \frac{(n+1)^n}{2^n}. \end{aligned}$$

□

**Theorem 5.** (Even better bound) For any  $n \in \mathbb{Z}_{>0}$ , we have

$$e \left( \frac{n}{e} \right)^n \leq n! \leq en \left( \frac{n}{e} \right)^n.$$

**Proof.** The lower bound will be given as a homework problem. The upper bound is as follows. Note that  $\ln(n!) = \sum_{i=1}^n \ln(i)$ . Then, we can take the integral of  $\ln(x)$ , which is greater than this sum.

$$\begin{aligned} \sum_{i=1}^n \ln(i) &\leq \int_1^{n+1} \ln(x) dx \\ &= (n+1) \ln(n+1) - n. \end{aligned}$$

Thus

$$\begin{aligned} n! &\leq e^{(n+1) \ln(n+1) - n} \\ &= \frac{e^{(n+1) \ln(n+1)}}{e^n} \\ &= \frac{(e^{\ln(n+1)})^{n+1}}{e^n} \\ &= \frac{(n+1)^{n+1}}{e^n}. \end{aligned}$$

Applying this for  $n(n-1)!$  gives the bound. □

## Lecture 5: Asymptotic Analysis

**Theorem 6.** Stirling's Formula says that

$$n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$

**Definition 8.** For two functions  $f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$ , we write  $f \sim g$  and say  $f$  is asymptotic to  $g$  if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1.$$

Also note that  $f \sim g \Leftrightarrow g \sim f$ .

**Example.**  $2n + \sqrt{n} \sim 2n$ .

**Explanation.** This is because

$$\lim_{n \rightarrow \infty} \frac{2n + \sqrt{n}}{2n} = 1.$$

**Example.** (Informal) How many digits are in  $100!$ ?

**Explanation.** Using Stirling's Formula, we have that

$$100! \sim \sqrt{2\pi 100} \left( \frac{100}{e} \right)^{100} = 9.324 \dots \times 10^{157}.$$

, whereas  $100! = 9.332 \dots \times 10^{157}$  (very close approximation).

**Definition 9.** The  $n$ -th harmonic number

$$H_n = \sum_{i=1}^n \frac{1}{i}.$$

**Theorem 7.** (Euler-Mascheroni)  $H_n \sim \ln(n)$ .

**Proof.** Omitted. □

**Lemma 9.** For any positive integer  $n \in \mathbb{Z}_{>0}$ , we have

$$\frac{\lfloor \log_2(n) \rfloor}{2} \leq H_n \leq \lfloor \log_2(n) \rfloor + 1.$$

**Proof.** We can break up the proof into parts of size 2, 4, 8, 16, ... Let  $S_k = \{i \in \mathbb{Z}_{>0} : 2^{k-1} \leq i \leq 2^k - 1\}$  for any  $k \in \mathbb{Z}_{>0}$ . Note that  $|S_k| = 2^{k-1}$ . Also, for every  $x \in S_k$ , we have

$$\frac{1}{2^k} < \frac{1}{x} \leq \frac{1}{2^{k-1}}.$$

Therefore, we have

$$\begin{aligned} H_n &= \sum_{i=1}^n \frac{1}{i} = \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \sum_{x \in S_k} \frac{1}{x} \\ &\geq \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \sum_{x \in S_k} \frac{1}{2^k} \\ &= \sum_{k=1}^{\lfloor \log_2(n) \rfloor} 2^{k-1} / 2^k \\ &= \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \frac{1}{2} \\ &= \frac{\lfloor \log_2(n) \rfloor}{2}. \end{aligned}$$

In the other direction, we have

$$\begin{aligned} H_n &\leq \sum_{k=1}^{\lfloor \log_2(n) \rfloor + 1} \sum_{x \in S_k} \frac{1}{x} \\ &\leq \sum_{k=1}^{\lfloor \log_2(n) \rfloor + 1} \frac{|S_k|}{2^{k-1}} \\ &= \lfloor \log_2(n) \rfloor + 1. \end{aligned}$$

□

**Definition 10.** Let  $f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$ . We say  $f = O(g)$  or  $f$  is big-O of  $g$  if there exists  $n_0, C$ , such that

$$|f(n)| \leq C \cdot g(n) \quad \forall n \geq n_0.$$

**Note.** If  $f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$  and  $f \sim g$ , we have

$$f = O(g) \quad \text{and} \quad g = O(f).$$

If  $\varepsilon = 1$  for all significantly large  $n$ ,  $\frac{f(n)}{g(n)} \leq 2$ .

**Example.**  $\sum_{i=1}^n \frac{1}{i} = O(\log n)$ .

**Lemma 10.** Let  $a, \alpha, \beta > 0$  be fixed. Then as  $n \rightarrow \infty$ ,

- $n^\alpha = O(n^\beta)$  if  $\alpha < \beta$ .
- $n^\alpha = O(a^n)$  if  $a > 1$ .
- $(\ln(n))^\alpha = O(n^\beta)$ .

## Lecture 6: Binomial Coefficients and Counting Primes

**Note.** Note that we can also write, for functions  $f, g, h$   $f = g + O(h)$ , which means that  $|f - g| = O(h)$ .

**Example.**

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} = \frac{n^2}{2} + O(n).$$

**Definition 11.**  $f(n) = \Theta(g(n))$  if  $f = O(g)$  and  $g = O(f)$ .

**Definition 12.**  $f = o(g)$  if  $\lim_{n \rightarrow \infty} \frac{f}{g} = 0$ .

**Example.** What are all primes less than 20?

**Explanation.** 2, 3, 5, 7, 11, 13, 17, 19

**Definition 13.** Let  $\pi(n)$  be the number of primes that are  $\leq n$ .

**Theorem 8.** The **prime number theorem** states that

$$\pi(n) \sim \frac{n}{\ln(n)}.$$

The **Riemann Hypothesis** states that

$$\pi(n) = \int_1^n \frac{1}{\ln(x)} dx + O(\sqrt{n} \ln(n)).$$

It's called a hypothesis because it is often used in other mathematical proofs, even if not proved yet. For example, determining whether a knot could be un-knotted is in NP if the RH is true.

**Lemma 11.** For any  $k \geq 1$ , we have that

$$\binom{2k+1}{k} \leq 4^k.$$

**Proof.** Later.  $\square$

**Lemma 12.** For any  $n \geq 2$ , the product of all primes  $\leq n$  is at most  $16^n$ .

**Proof.** We wish to prove that

$$\prod_{i=1}^{\pi(n)} p_i \leq 16^n.$$

where  $p_i$  denotes the  $i$ -th prime. We proceed with induction on  $n$ .

**Base case:**  $n = 2, 3$ . Holds trivially.

**Step case 1:**  $n$  is even. Note that  $n$  cannot be prime, such that by induction,

$$\prod_{i=1}^{\pi(n)} p_i = \prod_{i=1}^{\pi(n-1)} p_i \leq 16^{n-1} \leq 16^n.$$

**Step case 2:**  $n$  is odd. We write  $n = 2k + 1$  for some  $k \geq 1$ . Note that every prime  $p$  such that  $k + 2 \leq p \leq 2k + 1$  divides  $\binom{2k+1}{k}$ . This is because

$$\binom{2k+1}{k} = \frac{(2k+1)!}{k!(k+1)!}.$$

such that  $p$  divides the numerator but not the denominator.

By induction the product of primes  $p$  such that  $0 \leq p \leq k + 1$  is bound by

$$\prod_{i=1}^{\pi(k+1)} p_i \leq 16^{k+1}.$$

Combining our bounds and lemma, we have

$$\begin{aligned} \prod_{i=1}^{\pi(n)} p_i &= \left( \prod_{i=1}^{\pi(k+1)} p_i \right) \left( \prod_{i=\pi(k+1)+1}^{\pi(n)} p_i \right) \\ &\leq 16^{k+1} \binom{2k+1}{k} \\ &\leq 16^{k+1} \cdot 4^k \\ &\leq 16^{2k+1}. \end{aligned}$$

Note that bounding by  $4^n$  may work here, but  $16^n$  is presented due to a mistake in the lecture notes.  $\square$

**Theorem 9.** The weak prime number theorem

states that

$$\pi(n) = \Theta\left(\frac{n}{\ln n}\right).$$

**Proof.** We will show the upper bound, i.e.

$$\pi(n) = O\left(\frac{n}{\ln n}\right).$$

Let  $p_1, p_2, \dots$  be the sequence of primes. Then,

$$\pi(n)! \leq \prod_{i=1}^{\pi(n)} p_i \leq 16^n.$$

because  $p_1 \geq 1, p_2 \geq 2$ , etc. We have also shown that

$$e \left( \frac{\pi(n)}{e} \right)^{\pi(n)} \leq \pi(n)!.$$

As such,

$$e \left( \frac{\pi(n)}{e} \right)^{\pi(n)} \leq 16^n.$$

Taking the  $\ln$  of both sides, we have

$$\begin{aligned} \ln \left( e \left( \frac{\pi(n)}{e} \right)^{\pi(n)} \right) &\leq \ln(16^n) \\ \pi(n) \cdot \ln \left( \frac{\pi(n)}{e} \right) &\leq n \ln(16) \end{aligned}$$

Assume towards a contradiction that

$$\pi(n) \geq \frac{100n}{\ln n}$$

. Then,

$$\begin{aligned} \frac{100n}{\ln n} \cdot \ln \left( \frac{100n}{e \ln n} \right) &\leq n \ln 16 \\ \frac{100}{\ln 16 \cdot \ln n} \ln \left( \frac{100n}{e \ln n} \right) &\leq 1 \\ \frac{100}{\ln 16 \cdot \ln n} (\ln(100n) - \ln(e \ln n)) &\leq 1 \end{aligned}$$

which is a contradiction (after many calculations). Therefore, there exists some  $C < 100$  such that  $\pi(n) = O\left(\frac{n}{\ln n}\right)$ .  $\square$