

A Second Course in Linear Algebra

Raymond Bian

January 16, 2024

Contents

1 Vectors and Matrices

1

Lecture 1: Review

1 Vectors and Matrices

For the time being, everything indicated in this course is in \mathbb{R} .

Definition 1. A **vector** will be defined as a column vector, e.g.

$$u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

Notation. Sometimes, they will be written as a column vector lying down, e.g. $(x_1, x_2, x_3) \in \mathbb{R}^3$

Definition 2. Let a be a scalar. Then multiplication between vector and scalar is defined as

$$au = \begin{bmatrix} a \cdot x_1 \\ a \cdot x_2 \\ a \cdot x_3 \end{bmatrix}.$$

Definition 3. Let $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Then addition between vectors is defined as

$$u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

Definition 4. If u, v are vectors and a, b are scalars, then any $au + bv$ is a **linear combination** of u and v .

Remark. A **vector space** V is a set of objects u, v such that $au + bv \in V$.

Example. Polynomials of degree ≤ 2 in one variable can form a vector space.

Proof. Let $p(x) = a_0 + a_1x + a_2x^2$, and $q(x) = b_0 + b_1x + b_2x^2$. Multiplying by scalars and adding are defined. Note that $p(x) \rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

Example. Let $f(x) : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. We can multiply such functions by scalars and add together such functions, so they form a vector space as well.

Suppose we have two vectors $u, v \in \mathbb{R}^3$. Looking at the set of all linear combinations of u, v ,

- if both u and v are the zero vector, then $W = \{0\}$.
- if $u = \lambda v$, $v \neq 0$, then W is the line of all multiples of v .
- if u and v are **linearly independent**, then W is a plane in \mathbb{R}^3 .

Definition 5. Vectors u_1, u_2, u_3 are **linearly independent** if and only if

$$a_1u_1 + a_2u_2 + a_3u_3 = 0 \Rightarrow a_1 = a_2 = a_3 = 0.$$

Definition 6. Let V, W be a vector spaces such that $W \subseteq V$. Then, W is called a **subspace** of V .

Example. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$. Then, W is a subspace of \mathbb{R}^3 .

Theorem 1. If $u, v \in V$, then the set of linear combinations of u and v is a subspace.

Proof. Let $W = \text{span}\{u, v\}$. We must show that $w_1, w_2 \in W \Rightarrow c_1 w_1 + c_2 w_2 \in W$. By assumption, $w_1 = a_1 u + b_1 v$, and $w_2 = a_2 u + b_2 v$, such that $w = (c_1 a_1 + c_2 a_2)u + (c_1 b_1 + c_2 b_2)v$. Therefore, w is a linear combination of u, v . \square

Example. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $v = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. Then, $\text{span}\{u, v\}$ is a proper subspace of \mathbb{R}^3 .

Definition 7. $u \cdot v = x_1 y_1 + x_2 y_2 + x_3 y_3$ is the dot product of the vectors $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Definition 8. We say that $u \perp v$ if $u \cdot v = 0$.

Definition 9. The length or **norm** of a vector u is $\sqrt{u \cdot u} = \|u\|$

Theorem 2. The **Cauchy-Schwarz inequality** states that $|u \cdot v| \leq \|u\| \|v\|$.

Proof.

$$(u + \lambda v) \cdot (u + \lambda v) \geq 0$$

$$u \cdot u + \lambda^2 v \cdot v + 2\lambda u \cdot v \geq 0.$$

The minimum lambda is $\frac{-b}{2a} = \frac{-u \cdot v}{v \cdot v}$, which results in this inequality being true. Therefore, all greater values for lambda will result in this inequality being true. \square

Theorem 3. The **triangle inequality theorem** states that $\|u + v\| \leq \|u\| + \|v\|$.

Definition 10. The **unit vector** of a vector u , \hat{u} is given by $\frac{u}{\|u\|}$.

Theorem 4. If u and v are vectors such that $\|u\| = \|v\| = 1$, then $u \cdot v = \cos(\theta)$ where θ is the angle between u and v .

Corollary. If u and v are vectors, then $u \cdot v = \|u\| \|v\| \cos(\theta)$. Note that $u \cdot v = 0$ when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Lecture 2: Matrices

Example.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

is a matrix. We can also write $A = \{a_{ij}\}$ such that $i = 1 \dots n$ and $j = 1 \dots m$.

What does it mean to take a product between a matrix and a vector?

Definition 11. This product is defined as

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}.$$

i.e. a collection of dot products between the rows and x .

We can also see the product as a linear combination of the columns of the matrix A .

Definition 12. Let the columns of A be A_1, A_2, A_3 . Then, $Ax = x_1 A_1 + x_2 A_2 + x_3 A_3$.

Notation. A 's columns are denoted A_1, A_2, A_3 , while A 's rows are denoted A^1, A^2, A^3 .

If we look at the linear equation $Ax = b$, we can say that b is a linear combination of the columns of A . Instead, looking at it like an equation, "can b be written as a linear combination of the columns of A ?"

Looking at $A^1 x = b_1$, there are two free variables, such as this is a plane in \mathbb{R}^3 . The only time this is not a plane is if a_{11}, a_{12}, a_{13} are all zero, and b_1 is nonzero.

If we have x, y , $A^1 x = 0$ and $A^1 y = 0$ implies $ax + by = z$, which solves $A^1 z = 0$. The set of solutions is a subspace.

Now, suppose we have all solutions of $A^1 x = 0$. Call this V . How do we then write the solutions to $A^1 x = b$? We find any such c such that $A^1 c = b_1$. Then, we claim that the set of solutions of $A^1 x = b_1$ is $V + c = \{x + c | x \in V\}$. Checking our solution, $A^1 \cdot (x + c) = \underbrace{A^1 \cdot x}_0 + \underbrace{A^1 \cdot c}_{b_1} = b_1$.

Let $W = V + c$. We want to show if $x \in W \Rightarrow A^1 \cdot x = 0$. Assume $A^1 z = b_1$. If we set $x = z - c$, then $A^1 x = A^1 z - A^1 c = 0$. Therefore, $z = x + c \in W$.

All in all, solving all three equations $A^1 x = b_1, A^2 x = b_2, A^3 x = b_3$ is now just finding the intersection of three translated planes. **This is what solving $Ax = b$ means.**

Another viewpoint is this. Consider the equation $A_1 x_1 + A_2 x_2 + A_3 x_3 = b$. Consider the span of

A_1, A_2, A_3 . Does this span contain b ?

Example. Let's say that

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solving $Ax = b$, we have $x_3 = b_3$, $x_2 = b_2 + b_3$, and $x_1 = b_1 + b_2 + b_3$ such that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Let C denote this matrix. Then, $Ax = b \Leftrightarrow Cb = x$, such that $C = A^{-1}$. Then, C is the **inverse** of A .

Note that now, we also have $E_{12}(Ax) = E_{12}b = (E_{12}A)x$

Definition 15. AB is such that

$$A(Bx) = (AB)x.$$

for every vector x .

Definition 13. We want to say that every $n \times n$ matrix can be written as the product as an upper triangular and lower triangular matrix, called **LU factorization**.

Definition 14. Matrix multiplication is defined as $(AB)_{ij} = \sum_k a_{ik} + b_{kj}$ where $A = \{a_{ij}\}$ and $B = \{b_{kl}\}$

The other way to see AB is if $B = (B_1 \ B_2 \ \dots \ B_n)$, then $AB = (AB_1 \ AB_2 \ \dots \ AB_n)$. In other words, $(AB)_{ij} = A^i \cdot B_j$.

Lecture 3: Matrix Algebra

Example. Solve

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & 4 \\ -2 & -3 & 7 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_b.$$

Proof.

$$x = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Let

$$E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have

$$E_{12} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}.$$