

Probability Theory

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Lecture 1: Intro to Probability

1 Basics of Probability

What data do you need to specify probability? You need the **set of all outcomes**, a list of everything that could possibly occur as a consequence, and the likelihood of each event.

Example. For a roll of a dice, the set of all outcomes would be $\{1, 2, 3, 4, 5, 6\}$. The list could include things like “the result is 3”, or “the result is ≥ 4 ”, and the likelihood would be $\frac{1}{6}$ for each of the results.

1.1 Basics of Set Theory

Definition 1. A **set** is an unordered collection of elements. **Elements** are objects within sets.

Definition 2. A set A is a **subset** of a set B if $a \in A \Rightarrow a \in B$

Definition 3. The **union** of two sets A and B is the collection of elements that are in A or B .

Definition 4. The **intersection** of two sets A and B is the collection of elements that are in

both A and B .

Definition 5. The **complement** of a set A is everything not in A .

Definition 6. A **finite set** is a set with finite number of elements.

Definition 7. The **cartesian** product of two sets A and B denoted $A \times B$ is

$$\{(a, b) : a \in A \wedge b \in B\}.$$

Then, $|A \times B| = |A| \cdot |B|$.

1.2 Back to Probability

Definition 8. A **sample space** is the set of all possible outcomes in an experiment.

Example. The sample space Ω for a coin flip is $\{H, T\}$.

Note that **events** are just subsets of the sample space, and **elementary events** are just elements of the sample space.

Example. For a dice roll: $\Omega = \{1, 2, 3, 4, 5, 6\}$, some events could be $\{1, 2\}$, $\{3, 6\}$, $\{3\}$. There are a total of 2^6 events.

Definition 9. If Ω is a finite set, a probability P on Ω is a function: $P: 2^\Omega \rightarrow [0, 1]$ such that $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$.

Lemma 1. If $A_1, \dots, A_\alpha \subset \Omega$ are disjoint, $\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)$.

Proposition 1. Let $A = \{a_1, a_2, \dots, a_l\}$ such

that a_i are elementary events. Then,

$$\mathbb{P}(A) = \sum_{i=1}^I \mathbb{P}(\{a_i\}).$$

Example. For the dice roll, if $A = \{1, 3, 5\}$, then $\mathbb{P}(A) = 3 \cdot \frac{1}{6} = \frac{1}{2}$.

Definition 10. Equiprobable outcomes: Let's say we have the set $\Omega = \{\omega_1, \dots, \omega_N\}$ and $\mathbb{P}(\omega_i) = \mathbb{P}(\omega_j)$ for all i and j . Then, $\mathbb{P}(\omega) = \frac{1}{N}$ for all $\omega \in \Omega$ and $\mathbb{P}(A) = \frac{|A|}{N}$. In other words, when outcomes are probable,

$$\mathbb{P}(\text{event}) = \frac{\text{number of outcomes for that event}}{\text{number of possible outcomes}}.$$

1.3 Counting

Suppose 2 experiments are being performed. Let's say that experiment 1 has m possible outcomes, and experiment 2 has n possible outcomes. Then together, there are total of $n \cdot m$ total outcomes.

Example. Rolling a dice and then flipping a coin, how many possible outcomes are there?

Proof. You have $6 \cdot 2 = 12$ outcomes.

Example. Let's say you have a college planning committee that consists of 3 freshman, 4 sophomores, 5 juniors, and 2 seniors. How many ways are there to select a subcommittee of 4 with one person from each grade?

Proof. There are 4 events with 3, 4, 5, and 2 possible outcomes for each. Therefore, there are $3 \cdot 4 \cdot 5 \cdot 2 = 120$ total subcommittees.

Example. How many 7-place license plates are there if the first 3 are letters and the last 4 are numbers?

Proof. There are $26^3 \cdot 10^4$ license plates.

Definition 11. A **permutation** is an ordering of elements in a set. The number of ways to order n elements is given by $n!$.

Example. Alex has a bunny ranch with 10 bunnies. They are going to run an obstacle course

and ranked 1-10 based on completion time. How many possible rankings are there (no ties)?

Proof. There are $10!$ possible rankings.

Example. Assume 6 bunnies have straight ears and 4 have floppy ears. We rank the bunnies separately. How many possible rankings are there?

Proof. There are $6! \cdot 4!$ possible outcomes.

Definition 12. A **combination** denotes the number of ways to choose k elements from n total elements (counting subsets).

Example. How many ways are there to pick a 2 person team from a set of 5 people?

Proof. There are $C(5, 2) = \binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$ ways.

Example. How many committees consisting of 2 women and 3 men can be formed from a group of 5 women and 7 men?

Proof. We have $C(5, 2) \cdot C(7, 3)$ possible committees.

Example. What if two of the men do not want to serve on the committee together?

Proof. The number of ways to choose the women stays the same. However, for the men we must subtract the number of committees that have both men. Therefore, we have $C(5, 2) \cdot (C(7, 3) - C(5, 1))$ possible committees.

Example. How many ways can we divide a 10 person class into 3 groups, sizes 3, 3, and 4?

Proof. We just have 3 events, multiplying: $C(10, 3) \cdot C(7, 3) \cdot C(4, 4)$.

Definition 13. This is known as a **multinomial**, and is given by

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_r!}.$$

It counts the number of ways to partition a set of size n into sets of sizes n_1, n_2, \dots, n_r .

1.4 Back to Probability Again

Example. Flip 10 fair coins. What is the likelihood of flipping 3 heads?

Proof. Number of events of 3 heads is $C(10, 3)$. Total number of events is 2^{10} . Therefore,

$$\mathbb{P}(10 \text{ heads}) = \frac{C(10, 3)}{2^{10}}.$$

In general, we have $\sum_{k=0}^n \mathbb{P}(k \text{ heads}) = 1$. In other words,

$$\frac{1}{2^{10}} \cdot \sum_{k=0}^{10} \binom{10}{k} = 1.$$

such that

$$\sum_{k=0}^{10} \binom{10}{k} = 2^{10}.$$

More generally,

Definition 14. The **binomial theorem** states that for all $x, y \in \mathbb{R}$, $n \geq 1$, $n \in \mathbb{N}$,

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Example. Rolling 10 dice, what is the likelihood of exactly 2 outcomes each of 1,2,3,4, 1 outcome of 6, and 1 outcome of 5.

Proof. There are total 6^{10} outcomes, and there are $\binom{10}{2,2,2,2,1,1}$ desired outcomes. Therefore, the probability of this event is $\frac{\binom{10}{2,2,2,2,1,1}}{6^{10}}$.

Definition 15. The **multinomial theorem** states that $(x_1 + \dots + x_r)^n =$

$$\sum_{n_1 + \dots + n_r = n} \binom{n}{n_1, \dots, n_r} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}.$$

1.5 Measure Theory

This is just a generalization of what we have seen before.

Definition 16. Let $\mathcal{F} \subset 2^\Omega$ be an “event space”. A mapping $P : \mathcal{F} \rightarrow \mathbb{R}$ is a **probability measure** on (Ω, \mathcal{F}) if

- $\mathbb{P}(A) \geq 0 \quad \forall A \in \mathcal{F}$
- $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$

- If A_1, A_2, \dots are disjoint,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Lecture 2: More Probability

1.6 Properties of Event Spaces

Definition 17. A collection \mathcal{F} of subsets of the sample space Ω is called an **event space** if

- \mathcal{F} is non-empty.
- if $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$.
- if $A_1, A_2, \dots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Theorem 1. If $A \in \mathcal{F}$, then $\mathbb{P}(A) + \mathbb{P}(\Omega \setminus A) = 1$

Proof. Notice that A and $\Omega \setminus A$ are disjoint. And, that $A \cup (\Omega \setminus A) = \Omega$. Then,

$$\mathbb{P}(A \cup (\Omega \setminus A)) = \mathbb{P}(\Omega) = 1.$$

□

Theorem 2. If $A, B \in \mathcal{F}$ then $\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B)$.

Proof. Note that $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$. This is a union of disjoint sets, such that $\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)$. Then, we have $\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$, of which the RHS simplifies to $\mathbb{P}(A) + \mathbb{P}(B)$. □

Theorem 3. If $A, B \in \mathcal{F}$, and $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Proof. We wish to show $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$. Then, $B = (B \setminus A) \cup (B \cap A) = (B \setminus A) \cup A$, such that $\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A) \geq \mathbb{P}(A)$ because $\mathbb{P}(B \setminus A) \geq 0$. □

1.7 Examples

Example. What is the probability that one is dealt a full house?

Proof. This is the number of ways one can get a full house, divided by the total number of poker hands (5 card). The total number of poker

hands is $\binom{52}{5}$. The number of full houses is $\frac{52 \cdot \binom{3}{2} \cdot 48 \cdot 3}{2!3!}$. Another way we can count the number of full houses is $\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{1} \cdot \binom{4}{2}$. The result of the division is our answer.

Example. A box contains 3 marbles, 1 red 1 green and 1 blue. Consider an experiment that consists of us taking 1 marble, replacing it, and drawing another marble. What is the sample space?

Proof.

$$\Omega = \{(r, r), (r, b), (r, g), (b, r), (b, g), (b, b), (g, r), (g, g), (g, b)\}.$$

Example. What about if we don't replace the first marble?

Proof. Everything without $(r, r), (b, b), (g, g)$.

Example. What is the probability of being dealt a flush?

Proof. This is just number of flushes divided by number of poker hands. The number of flushes is $\binom{4}{1} \cdot \binom{13}{5}$.

Example. What is the probability of being dealt a straight?

Proof. We can do the probability of any straight, minus probability of straight flush. The number of straights is 10 number-wise. Therefore, the number of straights is $10 \cdot (4^5 - 4)$. The probability can be then calculated.

Example. An urn contains n balls. If k balls are withdrawn one at a time, what is the probability that a special ball is chosen?

Proof. $\mathbb{P}(\text{special}) = 1 - \mathbb{P}(\text{special}^c)$. If the special ball is not chosen, it would be $\frac{(n-1)!}{(n-k-1)!}$. The total number of withdrawals is $\frac{n!}{k!}$. Then, the total probability is $1 - \frac{n-k}{n}$.

Example. If n people are present in a room, what is the prob that no two celebrate their birthday on the same date? How large must n be such that this probability is $< \frac{1}{2}$.

Proof. $\mathbb{P}(\text{no people with same birthday})$ is the number of no same birthday situations divided by the number of possibilities. Total possibilities is 365^n . No same birthday situations is $\mathbb{P}(365, n) = \frac{365!}{(365-n)!}$. For the second question, $n = 23$.

1.8 Conditional Probability

Definition 18. If $A, B \in \mathcal{F}$ and $\mathbb{P}(B) > 0$ then the **conditional probability** if A given B is denoted by $\mathbb{P}(A | B)$ and defined by

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Theorem 4. If $B \in \mathcal{F}$ and $\mathbb{P}(B) > 0$ then $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space where $\mathbb{Q} : \mathcal{F} \rightarrow \mathbb{R}$ is defined by $\mathbb{Q}(A) = \mathbb{P}(A | B)$

Example. Let's say a coin is flipped twice. What is the conditional probability that both flips land on heads, given that the first flip lands on heads?

Proof. $\frac{\mathbb{P}(\text{two heads} \cap \text{first heads})}{\mathbb{P}(\text{first heads})} = \frac{\mathbb{P}(\text{two heads})}{\mathbb{P}(\text{first heads})}$. This is just $\frac{1}{2}$.

Example. What if given at least one lands on heads?

Proof. $\frac{\mathbb{P}(\text{two heads} \cap \text{at least one head})}{\mathbb{P}(\text{at least one head})} = \frac{\mathbb{P}(\text{two heads})}{\mathbb{P}(\text{at least one head})} = \frac{2}{3}$.

Example. In the card game bridge, the 52 cards are dealt equally. If North and South have a total of 8 spades among them, what is the probability that East has 3 of the 5 remaining spades?

Proof. No rule: $\mathbb{P}(\text{E has 3 spades}) = \frac{\binom{5}{3} \cdot \binom{21}{10}}{\binom{26}{13}}$.

Theorem 5. Probability of intersection of three sets (insert from canvas).

Definition 19. We call two events A, B **independent** if the occurrence of one does not affect the other. Formally,

$$\mathbb{P}(A | B) = \mathbb{P}(A) \text{ and } \mathbb{P}(B | A) = \mathbb{P}(B).$$

We can also check that $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

Example. Flip three fair coins. What is likelihood that all three come up heads?

Proof. With the sample space approach: $\Omega = \{H, T\}^3$. Of 8 total elementary events, 1 has three heads, so the probability is $\frac{1}{8}$.

With independence: we know that each event is independent, and all events are $\frac{1}{2}$, so the probability is $(\frac{1}{2})^3 = \frac{1}{8}$.

Definition 20. Independence can be expanded to more than just two events (insert from canvas). However, note that events can be pairwise independent, but may not be all together independent.

Lemma 2.

$$\mathbb{P}(B | A) = \mathbb{P}(A | B) \frac{\mathbb{P}(B)}{\mathbb{P}(A)}.$$

Proof. The RHS is the same as $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \cdot \frac{\mathbb{P}(B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \mathbb{P}(B | A)$. \square

Example. There are n balls that are sequentially chosen without replacement from r red balls and b blue balls. Given that k of the n balls are blue, what is the conditional probability that the first chosen is blue?

Proof.

$$\begin{aligned} & \mathbb{P}(\text{first is blue} | k \text{ are blue}) \\ &= \mathbb{P}(k \text{ are blue} | \text{first is blue}) \\ & \cdot \frac{\mathbb{P}(\text{first is blue})}{\mathbb{P}(k \text{ are blue})} \cdots \end{aligned}$$

Lecture 3

Continuing on with conditional probability from last time,

Example. A total of n balls are sequentially and randomly chosen without replacement from an urn containing r red balls and b blue balls ($n \leq r + b$). Given that k of the n balls are blue, what is the conditional probability that the first ball chosen is blue?

Proof. We can use Lemma 2. Then, we have

$$\mathbb{P}(\text{first blue}) = \frac{b}{r + b}$$

$$\mathbb{P}(\text{first } k \text{ are blue}) = \frac{\binom{n}{k} P(b, k) P(r, n - k)}{P(r + b, n)}.$$

$\mathbb{P}(k - 1 \text{ of remaining } n - 1 \text{ slots are blue.}) =$

$$\frac{\binom{n-1}{k-1} P(b - 1, k - 1) \cdot P(r, n - k)}{P(r + b - 1, n - 1)}.$$

$$\mathbb{P}(k - 1 \text{ of rest } n - 1 \text{ are blue}) \cdot \frac{\mathbb{P}(\text{first blue})}{\mathbb{P}(\text{first } k \text{ are blue})}.$$

will then be our answer.

1.9 Bayes Theorem

Definition 21. A **partition** of Ω is a collection $\{B_i : i \in I\}$ of disjoint events with union $\bigcup_i B_i = \Omega$.

Theorem 6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $\{B_1, B_2, \dots\}$ is such a partition with $\mathbb{P}(B_i) > 0$, then

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A | B_i) \mathbb{P}(B_i) \quad \text{for } A \in \mathcal{F}.$$

Example. Flip a fair coin. If heads, roll a 6-sided fair die. If tails, roll two 4-sided dice and sum the total. What is the overall likelihood of an outcome of 3?

Proof. Look at the event tree, and count the probabilities. The heads case is $\frac{1}{2} \cdot \frac{1}{6}$ and the tails case is $\frac{1}{2} \cdot \frac{1}{8}$. This is an informal Bayes Theorem.

Theorem 7. We can also rearrange Bayes' Theorem to yield

$$\mathbb{P}(B_j | A) = \frac{\mathbb{P}(A | B_j) \mathbb{P}(B_j)}{\sum_i \mathbb{P}(A | B_i) \mathbb{P}(B_i)}.$$

2 Random Variables

Definition 22. A **random variable** on (Ω, \mathbb{P}) , $|\Omega| < \infty$ is a function $X : \Omega \rightarrow \mathbb{R}$.

Notation. $\{X = a\} = \{\omega \in \Omega : X(\omega) = a\} = \dots = X^{-1}(a)$.

Example. 3 balls are to be selected without replacement from an urn containing 20 balls numbered 1 to 20. What is the probability that at least one of the balls that are drawn has a number as large or larger than 17?

Proof. $\Omega = \{1, 2, 3, \dots, 20\}$. $|\Omega| = \binom{20}{3}$. Let our random variable $X : \Omega \rightarrow \mathbb{R}$, $X =$ largest of the three values. Let $E = \{X \geq 17\}$. Then, $\mathbb{P}(E) = 1 - \mathbb{P}(E^c) = \mathbb{P}(\text{all} < 17)$.

$$\mathbb{P}(\text{all} < 17) = \frac{|E^c|}{|\Omega|} = \frac{\binom{16}{3}}{\binom{20}{3}}$$

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c) = 1 - \frac{\binom{16}{3}}{\binom{20}{3}}.$$

Definition 23. X is called **discrete** if \exists a countable set $S \subset \mathbb{R}$ such that $\mathbb{P}(X \in S) = 1$.

Definition 24. The **probability mass function** $p(a) = \mathbb{P}(X = a)$ is positive for most a countable number of values of a .

Example. The pmf of a random variable X is given by $p_X(i) = \frac{c\lambda^i}{i!}$, $i = 0, 1, 2, \dots$ where λ is some positive value. What is $\mathbb{P}(X = 0)$ and $\mathbb{P}(X > 2)$?

Proof.

$$\sum_{i=0}^{\infty} p_X(i) = \sum_{i=0}^{\infty} \frac{c\lambda^i}{i!} = 1$$

$$\Rightarrow X = \frac{1}{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}} = e^{-\lambda}.$$

Then, $\mathbb{P}(X = 0) = p_X(0) = \frac{c\lambda^0}{0!} = c = e^{-\lambda}$. Also, $\mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leq 2)$.

$$\mathbb{P}(X \leq 2) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2)$$

$$= e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2} \right).$$

Definition 25. If X is a discrete random variable, the **expectation** of X is denoted by $\mathbb{E}(X)$ and is defined by

$$\mathbb{E}(X) = \sum_{x \in \text{Im} X} x \mathbb{P}(X = x).$$

Example. We say that I is an indicator variable

for the event A if

$$I = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}.$$

Find $\mathbb{E}(I)$.

Proof.

$$\mathbb{E}(I_A) = 0 \cdot \mathbb{P}(I_A = 0) + 1 \cdot \mathbb{P}(I_A = 1)$$

$$= \mathbb{P}(I_A = 1)$$

$$= \mathbb{P}(\{\omega \in \Omega : I_A(\omega) = 1\})$$

$$= \mathbb{P}(\{\omega \in A\})$$

$$= \mathbb{P}(A).$$

Example. A class of 120 students is driven in 3 buses to a performance, with 36, 40, and 44 students in the busees. Let X denote the number of students on the bus of a randomly chosen student. Find $\mathbb{E}(X)$.

Proof. Note that $\mathbb{P}(B_1) = \frac{36}{120}$, $\mathbb{P}(B_2) = \frac{40}{120}$ and $\mathbb{P}(B_3) = \frac{44}{120}$. Then,

$$\mathbb{E}(X) = 36 \cdot \frac{36}{120} + 40 \cdot \frac{40}{120} + 44 \cdot \frac{44}{120}$$

$$= \frac{36^2 + 40^2 + 44^2}{120}$$