A Second Course in Linear Algebra

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Lecture 1: Review

1 Vectors and Matrices

For the time being, everything indicated in this source is in $\mathbb{R}$.

Definition 1. A **vector** will be defined as a column vector, e.g.

$$u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

Notation. Sometimes, they will be written as a column vector lying down, e.g. $(x_1, x_2, x_3) \in \mathbb{R}^3$

Definition 2. Let a be a scalar. Then multiplication by a scalar is defined as

$$au = \begin{bmatrix} a \cdot x_1 \\ a \cdot x_2 \\ a \cdot x_3 \end{bmatrix}.$$

Definition 3. Let $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then addition between vectors is defined as

$$u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

Definition 4. If u, v are vectors and a, b are scalars, then any $au + bv$ is a **linear combination** of u and v.

Remark. A **vector space** V is a set of objects $u.v$ such that $au + bv \in V$.

Example. Polynomials of degree ≤ 2 in one variable can form a vector space.

Proof. Let $p(x) = a_0 + a_1x + a_2x^2$, and $q(x) = b_0 + b_1x + b_2x^2$. Multiplying by scalars and adding are defined. Note that $p(x) \to \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

Example. Let $f(x) : [0, 1] \to \mathbb{R}$ be a continuous function. We can multiply such functions by scalars and add together such functions, so they form a vector space as well.

Suppose we have two vectors $u, v \in \mathbb{R}^3$. Looking at the set of all linear combinations of u, v,

- if both u and v are the zero vectoor, then $W = \{0\}$.
- if $u = \lambda v$, $v \neq 0$, then W is the line of all multiples of v.
- if u and v are **linearly independent**, then W is a plane in $\mathbb{R}^3$.

Definition 5. Vectors u_1, u_2, u_3 are **linearly independent** if and only if

$$a_1u_1 + a_2u_2 + a_3u_3 = 0 \Rightarrow a_1 = a_2 = a_3 = 0.$$

Definition 6. Let V, W be a vector spaces such that $W \subseteq V$. Then, W is called a **subspace** of V.

Example. Let $W = \{\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R}\}$. Then, W is a subspace of $\mathbb{R}^3$.

Theorem 1. If $u, v \in V$, then the set of linear combinations of u and v is a subspace.

Proof. Let $W = \text{span}\{u, v\}$. We must show that $w_1, w_2 \in W \Rightarrow c_1 w_1 + c_2 w_2 \in W$. By assumption, $w_1 = a_1 u + b_1 v$, and $w_2 = a_2 u + b_2 v$, such that $w = (c_1 a_1 + c_2 a_2) u + (c_1 b_1 + c_2 b_2) v$. Therefore, w is a linear combination of u, v. □

Example. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $v = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. Then, $\text{span}\{u, v\}$ is a proper subspace of $\mathbb{R}^3$.

Definition 7. $u \cdot v = x_1 y_1 + x_2 y_2 + x_3 y_3$ is the dot product of the vectors $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Definition 8. We say that $u \perp v$ if $u \cdot v = 0$.

Definition 9. The length or **norm** of a vector u is $\sqrt{u \cdot u} = \|u\|$

Theorem 2. The **Cauchy–Schwarz inequality** states that $|u \cdot v| \leq \|u\| \|v\|$.

Proof.

$$\begin{aligned} (u + \lambda v) \cdot (u + \lambda v) &\geq 0 \\ u \cdot u + \lambda^2 v \cdot v + 2\lambda u \cdot v &\geq 0. \end{aligned}$$

The minimum lambda is $\frac{-b}{2a} = \frac{-u \cdot v}{v \cdot v}$, which results in this inequality being true. Therefore, all greater values for lambda will result in this inequality being true. □

Theorem 3. The **triangle inequality theorem** states that $\|u + v\| \leq \|u\| + \|v\|$.

Definition 10. The **unit vector** of a vector u, $\hat{u}$ is given by $\frac{u}{\|u\|}$.

Theorem 4. If u and v are vectors such that $\|u\| = \|v\| = 1$, then $u \cdot v = \cos(\theta)$ where θ is the angle between u and v.

Corollary. If u and v are vectors, then $u \cdot v = \|u\| \|v\| \cos(\theta)$.