Combinatorial Analysis

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Lecture 1: Syllabus and Review

1 Introduction

This course is basically just a second course in Combinatorics, and will cover a range of topics.

Definition 1. **Matroids** are the structures that capture whether or not the greedy algorithm works. They will be covered later in the course.

Now, for some examples and review:

Definition 2. We say points are in **convex position** if no point is inside a triangle made by 3 other points.

Example. Given a finite set of points on the plane, what is the maxmimum number of points such that no 3 are on a line, and no 4 are in convex position.

Proof. Informally, we know that the "outside" of our points has at most 3 points in the shape of a triangle. We can then place a point in the middle. However, if we try to add another point, then we find that 4 points are in convex position, which is a contradiction. Therefore, 4 points is the maximum size of such a set.

This example is actually part of a more general problem, shown below.

Theorem 1. (ES, 1935) The maxmimum number of points such that no 3 are on a line and no n are in convex position is $\leq 4^n$ and $\geq 2^{n-2}$.

Theorem 2. (Suk, 2017) This number is actually $\leq 2^{n+o(1)}$

Notation. Think of $o(1)$ as standing for a function $f(n)$ such that $\lim_{n\to\infty} f(n) = 0$. In other words, for every $\varepsilon > 0$, there exists n_0 such that $|f(n)| < \varepsilon$ for every $n \geq n_0$.

Example. How many distinct 5-letter words are there on the 26-letter english alphabet?

Proof. There are 26 options for each of the 5 slots, so there are 26^5 words.

Example. What if repetitions aren't allowed?

Proof. Each slot you lose an option, so there are $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = \frac{26!}{21!}$ words.

Example. How many ways are there to choose 5 students out of 35 to present?

Proof. There are $\binom{35}{5} = \frac{35!}{5! \cdot 30!}$ ways.

Lecture 2: Review of Proofs

We will now review the types of proofs covered in Math-3012, as well as guidelines for writing them in this class.

Notation. If F is a mapping from N to M, we write $F : N \to M$.

Notation. Sometimes, $N \backslash \{a\}$ will be instead written as $N - \{a\}$.

Proposition 1. Let N be an n-element set and M be an m-element set. Then, there are m^n mappings (or functions) from N to M.

Proof. (Inductive) We go by induction on n.

Base case. For the base case $n = 0$, we consider the empty set $\varnothing$ to be a mapping from the empty set to M. So $m^0 = 1$ and the base case holds.

Inductive step. Now, let $n \geq 1$ and assume that the proposition holds for $n-1$ by induction. So, let $a \in N$. There are m^{n-1} mappings $F' : N \backslash \{a\} \to M$. For each such F', we have m choices for where to send a. These mappings are all distinct, and every $F : N \to M$ can be obtained in this way. So, the number of mappings $F : N \to M$ is $m^{n-1} \cdot m = m^n$, as desired.

$\square$

Definition 3. A **bijection** is a function $f : X \to Y$ such that f is one-to-one and onto.

Corollary. An n-element set X has 2^n many subsets.

Proof. (Bijective) For each $A \subseteq X$, let $F_A : X \to \{0,1\}$ such that for each $x \in X$,

$$F_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}.$$

These mappings $F_A, F_{A'}$ are distinct for distinct subsets $A, A' \subseteq X$, and every mapping $F : X \to \{0,1\}$ is equal to F_A for some $A \subseteq X$. So by proposition 1, the corollary holds. $\square$

Lemma 1. For any non-negative integers n, k ($n, k \in Z_{\geq 0}$), we have $\binom{n}{k} = \binom{n}{n-k}$.

Proof. (Algebraic) We have

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(n-(n-k))!(n-k)!} \\ &= \binom{n}{n-k}, \end{aligned}$$

as desired. $\square$

Theorem 3. (Binomial Theorem) Let $n \in \mathbb{Z}_{\geq 0}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. Consider

$$\underbrace{(x+y)(x+y)\dots(x+y)}_{n \text{ times}}.$$

For each $(x+y)$ term, we select either the x or the y, and there are $\binom{n}{k}$ ways to select k x's and $n-k$ y's. The formula follows. $\square$

Corollary. For any $n \in \mathbb{Z}_{\geq 0}$, we have

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \text{ and } 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

Proof. Apply the binomial theorem with $x = y = 1$ to yield the first result, and with $x = -1, y = 1$ to yield the second. $\square$

1.1 Counting Review

Definition 4. A **permutation** is a bijection from a finite set to itself.

Example. One such bijection could be $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3$.

Lemma 2. The number of such bijections is $n!$.

Proof. Exercise to the student! $\square$

Lecture 3: Permutations and Cycles

Notation. $\tau : X \to X$ is a permutation **on** X. Can also be denoted by $\sigma : X \to X$.

We will show that all permutations τ can be "decomposed" into "cycles".

Example. From the example earlier, $(1,2)$ is a cycle, and $(3,4,5)$ is another cycle.

For the following, let $\tau : X \to X$.

Definition 5. A **cycle** of τ is a tuple (ordered set of elements) $(x_1, x_2, \dots, x_k)$ such that $x_1, x_2, \dots, x_k$ are distinct elements of X, and $\tau(x_1) = x_2, \tau(x_2) = x_3, \dots, \tau(x_{k-1}) = x_k, \tau(x_k) = x_1$. We call $x_1, x_2, \dots x_k$ the **elements** of the cycle.

Lemma 3. If $(x_1, x_2 \dots, x_k)$ and $(y_1, y_2, \dots y_r)$ have an element in common, then $\{x_1, x_2, \dots x_k\} = \{y_1, y_2, \dots y_r\}$.

Proof. Note that since $(x_1, x_2, \dots, x_k)$ is a cycle, $(x_2, x_3, \dots, x_k, x_1)$ is also a cycle. Because of this, we can assume that $x_1 = y_1$. So

$x_2 = \tau(x_1) = \tau(y_1) = y_2$. Then, we have that $x_2 = y_2$. We can repeat this process until $x_k = y_k$ (swap x, y if $k > r$). Then, we have $x_1 = \tau(x_k) = \tau(y_k) = y_1$, which means that $r = k$. Therefore, all cycles are pairwise disjoint. $\square$

Lemma 4. For every $x \in X$, there exists a cycle of τ which has x as an element.

Proof. Consider visiting each element $x, \tau(x), \tau(\tau(x)), \ldots,$ until the first time we re-visit any element. This will eventually happen, because X is finite. Then, let's suppoose that we have visited elements $x_1, x_2, \ldots x_k$ so far, such that $x_1, x_2, \ldots, x_k$ are distinct, and that $\tau(x_k) = x_i$ for some $i \in \{1, 2, \ldots, k\}$. We cannot have $i \geq 2$ because then both x_{i-1} and x_k would both map to x_i, which is a contradiction because a permutation is a bijection. Therefore, $i = 1$ and we have established our cycle. $\square$

Corollary. There exists cycles $C_1, C_2, \ldots, C_t$, so that every element of X is an element in exactly one such cycle.

Definition 6. The **cycle notation** for τ is written as

$$\tau = C_1 C_2 \ldots C_t.$$

Example. Find the cycle notation for the permutation τ of $\{1, 2, 3, 4, 5, 6\}$ where

$$\begin{aligned} \tau(1) &= 4 \\ \tau(2) &= 6 \\ \tau(3) &= 2 \\ \tau(4) &= 5 \\ \tau(5) &= 1 \\ \tau(6) &= 3. \end{aligned}$$

Proof. By inspection, we have a cycle $(1, 4, 5)$ and another cycle $(2, 3, 6)$. Therefore, $\tau = (1, 4, 5)(2, 3, 6)$.

Definition 7. A **transposition** is a cycle with exactly two elements.

Problem. How quickly does $n!$ grow as n gets large?