

Combinatorial Analysis

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Theorem 2. (Suk, 2017) This number is actually $\leq 2^{n+o(1)}$

Notation. Think of $o(1)$ as standing for a function $f(n)$ such that $\lim_{n \rightarrow \infty} f(n) = 0$. In other words, for every $\varepsilon > 0$, there exists n_0 such that $|f(n)| < \varepsilon$ for every $n \geq n_0$.

Lecture 1: Syllabus and Review

1 Introduction

This course is basically just a second course in Combinatorics, and will cover a range of topics.

Definition 1. Matroids are the structures that capture whether or not the greedy algorithm works. They will be covered later in the course.

Now, for some examples and review:

Definition 2. We say points are in **convex position** if no point is inside a triangle made by 3 other points.

Example. Given a finite set of points on the plane, what is the maximum number of points such that no 3 are on a line, and no 4 are in convex position.

Explanation. Informally, we know that the “outside” of our points has at most 3 points in the shape of a triangle. We can then place a point in the middle. However, if we try to add another point, then we find that 4 points are in convex position, which is a contradiction. Therefore, 4 points is the maximum size of such a set.

This example is actually part of a more general problem, shown below.

Theorem 1. (ES, 1935) The maximum number of points such that no 3 are on a line and no n are in convex position is $\leq 4^n$ and $\geq 2^{n-2}$.

Example. How many distinct 5-letter words are there on the 26-letter english alphabet?

Explanation. There are 26 options for each of the 5 slots, so there are 26^5 words.

Example. What if repetitions aren't allowed?

Explanation. Each slot you lose an option, so there are $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = \frac{26!}{21!}$ words.

Example. How many ways are there to choose 5 students out of 35 to present?

Explanation. There are $\binom{35}{5} = \frac{35!}{5! \cdot 30!}$ ways.

Lecture 2: Review of Proofs

We will now review the types of proofs covered in Math-3012, as well as guidelines for writing them in this class.

Notation. If F is a mapping from N to M , we write $F : N \rightarrow M$.

Notation. Sometimes, $N \setminus \{a\}$ will be instead written as $N - \{a\}$.

Proposition 1. Let N be an n -element set and M be an m -element set. Then, there are m^n mappings (or functions) from N to M .

Proof. (Inductive) We go by induction on n .

Base case.: For the base case $n = 0$, we con-

sider the empty set \emptyset to be a mapping from the empty set to M . So $m^0 = 1$ and the base case holds.

Inductive step.: Now, let $n \geq 1$ and assume that the proposition holds for $n - 1$ by induction. So, let $a \in N$. There are m^{n-1} mappings $F' : N \setminus \{a\} \rightarrow M$. For each such F' , we have m choices for where to send a . These mappings are all distinct, and every $F : N \rightarrow M$ can be obtained in this way. So, the number of mappings $F : N \rightarrow M$ is $m^{n-1} \cdot m = m^n$, as desired. ■

Definition 3. A **bijection** is a function $f : X \rightarrow Y$ such that f is one-to-one and onto.

Corollary. An n -element set X has 2^n many subsets.

Proof. (Bijective) For each $A \subseteq X$, let $F_A : X \rightarrow \{0, 1\}$ such that for each $x \in X$,

$$F_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}.$$

These mappings $F_A, F_{A'}$ are distinct for distinct subsets $A, A' \subseteq X$, and every mapping $F : X \rightarrow \{0, 1\}$ is equal to F_A for some $A \subseteq X$. So by proposition 1, the corollary holds. ■

Lemma 1. For any non-negative integers n, k ($n, k \in \mathbb{Z}_{\geq 0}$), we have $\binom{n}{k} = \binom{n}{n-k}$.

Proof. (Algebraic) We have

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(n-(n-k))!(n-k)!} \\ &= \binom{n}{n-k}, \end{aligned}$$

as desired. ■

Theorem 3. (Binomial Theorem) Let $n \in \mathbb{Z}_{\geq 0}$. Then

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Proof. Consider

$$\underbrace{(x + y)(x + y) \dots (x + y)}_{n \text{ times}}.$$

For each $(x + y)$ term, we select either the x or the y , and there are $\binom{n}{k}$ ways to select k x 's and $n - k$ y 's. The formula follows. ■

Corollary. For any $n \in \mathbb{Z}_{\geq 0}$, we have

$$2^n = \sum_{k=0}^n \binom{n}{k} \text{ and } 0 = \sum_{k=0}^n \binom{n}{k} (-1)^k.$$

Proof. Apply the binomial theorem with $x = y = 1$ to yield the first result, and with $x = -1, y = 1$ to yield the second. ■

1.1 Counting Review

Definition 4. A **permutation** is a bijection from a finite set to itself.

Example. One such bijection could be $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3$.

Lemma 2. The number of such bijections is $n!$.

Proof. Exercise to the student! ■

Lecture 3: Permutations and Cycles

Notation. $\tau : X \rightarrow X$ is a permutation on X . Can also be denoted by $\sigma : X \rightarrow X$.

We will show that all permutations τ can be “decomposed” into “cycles”.

Example. From the example earlier, $(1, 2)$ is a cycle, and $(3, 4, 5)$ is another cycle.

For the following, let $\tau : X \rightarrow X$.

Definition 5. A **cycle** of τ is a tuple (ordered set of elements) (x_1, x_2, \dots, x_k) such that x_1, x_2, \dots, x_k are distinct elements of X , and $\tau(x_1) = x_2, \tau(x_2) = x_3, \dots, \tau(x_{k-1}) = x_k, \tau(x_k) = x_1$. We call x_1, x_2, \dots, x_k the **elements** of the cycle.

Lemma 3. If (x_1, x_2, \dots, x_k) and (y_1, y_2, \dots, y_r) have an element in common, then $\{x_1, x_2, \dots, x_k\} = \{y_1, y_2, \dots, y_r\}$.

Proof. Note that since (x_1, x_2, \dots, x_k) is a cycle, $(x_2, x_3, \dots, x_k, x_1)$ is also a cycle. Because of this, we can assume that $x_1 = y_1$. So $x_2 = \tau(x_1) = \tau(y_1) = y_2$. Then, we have that $x_2 = y_2$. We can repeat this process until $x_k = y_k$ (swap x, y if $k > r$). Then, we have $x_1 = \tau(x_k) = \tau(y_k) = y_1$, which means that $r = k$. Therefore, all cycles are pairwise disjoint. ■

Lemma 4. For every $x \in X$, there exists a cycle of τ which has x as an element.

Proof. Consider visiting each element $x, \tau(x), \tau(\tau(x)), \dots$, until the first time we re-visit any element. This will eventually happen, because X is finite. Then, let's suppose that we have visited elements x_1, x_2, \dots, x_k so far, such that x_1, x_2, \dots, x_k are distinct, and that $\tau(x_k) = x_i$ for some $i \in \{1, 2, \dots, k\}$. We cannot have $i \geq 2$ because then both x_{i-1} and x_k would both map to x_i , which is a contradiction because a permutation is a bijection. Therefore, $i = 1$ and we have established our cycle. ■

Corollary. There exists cycles C_1, C_2, \dots, C_t , so that every element of X is an element in exactly one such cycle.

Definition 6. The **cycle notation** for τ is written as

$$\tau = C_1 C_2 \dots C_t.$$

Example. Find the cycle notation for the permutation τ of $\{1, 2, 3, 4, 5, 6\}$ where

$$\begin{aligned}\tau(1) &= 4 \\ \tau(2) &= 6 \\ \tau(3) &= 2 \\ \tau(4) &= 5 \\ \tau(5) &= 1 \\ \tau(6) &= 3.\end{aligned}$$

Explanation. By inspection, we have a cycle $(1, 4, 5)$ and another cycle $(2, 3, 6)$. Therefore, $\tau = (1, 4, 5)(2, 3, 6)$.

Definition 7. A **transposition** is a cycle with exactly two elements.

Problem. How quickly does $n!$ grow as n gets large?

Lecture 4: Estimates for $n!$

Lemma 5. (Simplest) For any positive integer $n \in \mathbb{Z}_{>0}$,

$$2^{n-1} \leq n! \leq n^{n-1}.$$

Proof. We have for the lower bound

$$n! = \prod_{i=2}^n i \geq \prod_{i=2}^n 2 = 2^{n-1}.$$

And for the upper bound,

$$n! = \prod_{i=2}^n i \leq \prod_{i=2}^n n = n^{n-1}.$$

■

Note that these bounds are very far off. Here is a motivating example.

Example. Suppose n students draw a card from a deck of n cards, replacing the card afterwards. What is the likelihood that all n cards drawn are distinct?

Explanation. The probability is the number of desirable outcomes over the total number of outcomes. This is just

$$\frac{n!}{n^n}.$$

Note that if we use the upper bound from this lemma, we would get that the probability is at most $\frac{1}{n}$. In reality however, the true probability is much, much smaller.

Lemma 6. A better set of bounds are the following:

$$\left(\frac{n}{2}\right)^{\frac{n}{2}} \leq n! \leq \frac{(n+1)^n}{2^{\frac{n}{2}}}.$$

Proof. Left as an exercise! ■

Lemma 7. For any two $a, b \geq 2$, we have $a \cdot b \geq a + b$.

Lemma 8. (Arithmetic-Geometric Mean Inequality) For any two $a, b \geq 0$, we have

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

With these last two lemmas, we can show the following:

Theorem 4. (Gauss) For any $n \in \mathbb{Z}_{>0}$,

$$n^{\frac{n}{2}} \leq n! \leq \frac{(n+1)^n}{2^n}.$$

Proof. We instead look at $(n!)^2$. Pairing 1 with n , 2 with $n-1$, etc, we have for the lower bound

$$\begin{aligned} n! &= \left(\prod_{i=1}^n i \right) \left(\prod_{i=1}^n i \right) \\ &= \prod_{i=1}^n i(n+1-i) \\ &= \prod_{i=1}^n \sqrt{i(n+1-i)} \\ &\geq \prod_{i=1}^n \sqrt{n} \quad (\text{Lemma 7}) \\ &\geq n^{\frac{n}{2}}. \end{aligned}$$

And for the upper bound, we have

$$\begin{aligned} n! &= \left(\prod_{i=1}^n i \right) \left(\prod_{i=1}^n i \right) \\ &= \prod_{i=1}^n i(n+1-i) \\ &= \prod_{i=1}^n \sqrt{i(n+1-i)} \\ &\leq \prod_{i=1}^n \frac{i+n+1-i}{2} \\ &= \frac{(n+1)^n}{2^n}. \end{aligned}$$

■

Theorem 5. (Even better bound) For any $n \in \mathbb{Z}_{>0}$, we have

$$e \left(\frac{n}{e} \right)^n \leq n! \leq en \left(\frac{n}{e} \right)^n.$$

Proof. The lower bound will be given as a homework problem. The upper bound is as follows. Note that $\ln(n!) = \sum_{i=1}^n \ln(i)$. Then, we can take the integral of $\ln(x)$, which is greater than this sum.

$$\begin{aligned} \sum_{i=1}^n \ln(i) &\leq \int_1^{n+1} \ln(x) dx \\ &= (n+1) \ln(n+1) - n. \end{aligned}$$

Thus

$$\begin{aligned} n! &\leq e^{(n+1) \ln(n+1) - n} \\ &= \frac{e^{(n+1) \ln(n+1)}}{e^n} \\ &= \frac{(e^{\ln(n+1)})^{n+1}}{e^n} \\ &= \frac{(n+1)^{n+1}}{e^n}. \end{aligned}$$

Applying this for $n(n-1)!$ gives the bound. ■

Lecture 5: Asymptotic Analysis

Theorem 6. Stirling's Formula says that

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n.$$

Definition 8. For two functions $f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$, we write $f \sim g$ and say f is asymptotic to g if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1.$$

Also note that $f \sim g \Leftrightarrow g \sim f$.

Example. $2n + \sqrt{n} \sim 2n$.

Explanation. This is because

$$\lim_{n \rightarrow \infty} \frac{2n + \sqrt{n}}{2n} = 1.$$

Example. (Informal) How many digits are in $100!$?

Explanation. Using Stirling's Formula, we have that

$$100! \sim \sqrt{2\pi 100} \left(\frac{100}{e} \right)^{100} = 9.324 \dots \times 10^{157}.$$

, whereas $100! = 9.332 \dots \times 10^{157}$ (very close approximation).

Definition 9. The n -th harmonic number

$$H_n = \sum_{i=1}^n \frac{1}{i}.$$

Theorem 7. (Euler-Mascheroni) $H_n \sim \ln(n)$.

Proof. Omitted. ■

Lemma 9. For any positive integer $n \in \mathbb{Z}_{>0}$, we have

$$\frac{\lfloor \log_2(n) \rfloor}{2} \leq H_n \leq \lfloor \log_2(n) \rfloor + 1.$$

Proof. We can break up the proof into parts of size 2, 4, 8, 16, ... Let $S_k = \{i \in \mathbb{Z}_{>0} : 2^{k-1} \leq i \leq 2^k - 1\}$ for any $k \in \mathbb{Z}_{>0}$. Note that $|S_k| = 2^{k-1}$. Also, for every $x \in S_k$, we have

$$\frac{1}{2^k} < \frac{1}{x} \leq \frac{1}{2^{k-1}}.$$

Therefore, we have

$$\begin{aligned} H_n &= \sum_{i=1}^n \frac{1}{i} = \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \sum_{x \in S_k} \frac{1}{x} \\ &\geq \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \sum_{x \in S_k} \frac{1}{2^k} \\ &= \sum_{k=1}^{\lfloor \log_2(n) \rfloor} 2^{k-1} / 2^k \\ &= \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \frac{1}{2} \\ &= \frac{\lfloor \log_2(n) \rfloor}{2}. \end{aligned}$$

In the other direction, we have

$$\begin{aligned} H_n &\leq \sum_{k=1}^{\lfloor \log_2(n) \rfloor + 1} \sum_{x \in S_k} \frac{1}{x} \\ &\leq \sum_{k=1}^{\lfloor \log_2(n) \rfloor + 1} \frac{|S_k|}{2^{k-1}} \\ &= \lfloor \log_2(n) \rfloor + 1. \end{aligned}$$

■

Definition 10. Let $f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$. We say $f = O(g)$ or f is big-O of g if there exists n_0, C , such that

$$|f(n)| \leq C \cdot g(n) \quad \forall n \geq n_0.$$

Note. If $f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$ and $f \sim g$, we have

$$f = O(g) \quad \text{and} \quad g = O(f).$$

If $\varepsilon = 1$ for all significantly large n , $\frac{f(n)}{g(n)} \leq 2$.

Example. $\sum_{i=1}^n \frac{1}{i} = O(\log n)$.

Lemma 10. Let $a, \alpha, \beta > 0$ be fixed. Then as $n \rightarrow \infty$,

- $n^\alpha = O(n^\beta)$ if $\alpha < \beta$.
- $n^\alpha = O(a^n)$ if $a > 1$.
- $(\ln(n))^\alpha = O(n^\beta)$.

Lecture 6: Binomial Coefficients and Counting Primes

Note. Note that we can also write, for functions f, g, h $f = g + O(h)$, which means that $|f - g| = O(h)$.

Example.

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} = \frac{n^2}{2} + O(n).$$

Definition 11. $f(n) = \Theta(g(n))$ if $f = O(g)$ and $g = O(f)$.

Definition 12. $f = o(g)$ if $\lim_{n \rightarrow \infty} \frac{f}{g} = 0$.

Example. What are all primes less than 20?

Explanation. 2, 3, 5, 7, 11, 13, 17, 19

Definition 13. Let $\pi(n)$ be the number of primes that are $\leq n$.

Theorem 8. The **prime number theorem** states that

$$\pi(n) \sim \frac{n}{\ln(n)}.$$

The **Riemann Hypothesis** states that

$$\pi(n) = \int_1^n \frac{1}{\ln(x)} dx + O(\sqrt{n} \ln(n)).$$

It's called a hypothesis because it is often used in other mathematical proofs, even if not proved yet. For example, determining whether a knot could be un-knotted is in NP if the RH is true.

Lemma 11. For any $k \geq 1$, we have that

$$\binom{2k+1}{k} \leq 4^k.$$

Proof. Later. ■

Lemma 12. For any $n \geq 2$, the product of all primes $\leq n$ is at most 16^n .

Proof. We wish to prove that

$$\prod_{i=1}^{\pi(n)} p_i \leq 16^n.$$

where p_i denotes the i -th prime. We proceed with induction on n .

Base case: $n = 2, 3$. Holds trivially.

Step case 1: n is even. Note that n cannot be prime, such that by induction,

$$\prod_{i=1}^{\pi(n)} p_i = \prod_{i=1}^{\pi(n-1)} p_i \leq 16^{n-1} \leq 16^n.$$

Step case 2: n is odd. We write $n = 2k + 1$ for some $k \geq 1$. Note that every prime p such that $k + 2 \leq p \leq 2k + 1$ divides $\binom{2k+1}{k}$. This is because

$$\binom{2k+1}{k} = \frac{(2k+1)!}{k!(k+1)!}.$$

such that p divides the numerator but not the denominator.

By induction the product of primes p such that $0 \leq p \leq k + 1$ is bound by

$$\prod_{i=1}^{\pi(k+1)} p_i \leq 16^{k+1}.$$

Combining our bounds and lemma, we have

$$\begin{aligned} \prod_{i=1}^{\pi(n)} p_i &= \left(\prod_{i=1}^{\pi(k+1)} p_i \right) \left(\prod_{i=\pi(k+1)+1}^{\pi(n)} p_i \right) \\ &\leq 16^{k+1} \binom{2k+1}{k} \\ &\leq 16^{k+1} \cdot 4^k \\ &\leq 16^{2k+1}. \end{aligned}$$

Note that bounding by 4^n may work here, but 16^n is presented due to a mistake in the lecture notes. ■

Theorem 9. The weak prime number theorem

states that

$$\pi(n) = \Theta\left(\frac{n}{\ln n}\right).$$

Proof. We will show the upper bound, i.e.

$$\pi(n) = O\left(\frac{n}{\ln n}\right).$$

Let p_1, p_2, \dots be the sequence of primes. Then,

$$\pi(n)! \leq \prod_{i=1}^{\pi(n)} p_i \leq 16^n.$$

because $p_1 \geq 1, p_2 \geq 2$, etc. We have also shown that

$$e \left(\frac{\pi(n)}{e} \right)^{\pi(n)} \leq \pi(n)!.$$

As such,

$$e \left(\frac{\pi(n)}{e} \right)^{\pi(n)} \leq 16^n.$$

Taking the \ln of both sides, we have

$$\begin{aligned} \ln \left(e \left(\frac{\pi(n)}{e} \right)^{\pi(n)} \right) &\leq \ln(16^n) \\ \pi(n) \cdot \ln \left(\frac{\pi(n)}{e} \right) &\leq n \ln(16) \end{aligned}$$

Assume towards a contradiction that

$$\pi(n) \geq \frac{100n}{\ln n}$$

. Then,

$$\begin{aligned} \frac{100n}{\ln n} \cdot \ln \left(\frac{100n}{e \ln n} \right) &\leq n \ln 16 \\ \frac{100}{\ln 16 \cdot \ln n} \ln \left(\frac{100n}{e \ln n} \right) &\leq 1 \\ \frac{100}{\ln 16 \cdot \ln n} (\ln(100n) - \ln(e \ln n)) &\leq 1 \end{aligned}$$

which is a contradiction (after many calculations). Therefore, there exists some $C < 100$ such that $\pi(n) = O\left(\frac{n}{\ln n}\right)$. ■

Lecture 7: More Bounds, Inclusion Exclusion

The proof from previous lecture will not be given, as it is on the homework.

Lemma 13. For any $n \geq k \geq 1$,

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Observation. If $n = 2k$, then

$$\binom{2k}{k} \leq \left(\frac{e \cdot 2k}{k}\right)^k \leq (2e)^k.$$

Lemma 14. (Bernoulli-type inequality) For any real number $x \geq 0$, we have

$$1 + x \leq e^x.$$

Proof. This is true for $x = 0$ because $1 + 0 \leq e^0 = 1$. Therefore, we just need to show that

$$f(x) = e^x - (1 + x).$$

is increasing. This holds because $f'(x) = e^x - 1 \geq 0$. ■

Lemma 15. For any $n \geq k \geq 1$, we have

$$\sum_{i=0}^k \binom{n}{i} \leq \left(\frac{en}{k}\right)^k.$$

Proof. Recall that for any x, y , we have

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}.$$

When $y = 1$,

$$(1 + x)^n = \sum_{i=0}^n \binom{n}{i} x^i.$$

Choosing $x = \frac{k}{n}$ with the intuition that $x < 1$ allows us to continue:

$$\sum_{i=0}^n \binom{n}{i} x^i \geq \sum_{i=0}^k \binom{n}{i} x^i.$$

Dividing everything by x^k ,

$$\frac{(1+x)^n}{x^k} = \sum_{i=0}^n \binom{n}{i} x^{i-k} \geq \sum_{i=0}^k \binom{n}{i} x^{i-k}.$$

Since $x \leq 1$,

$$\frac{(e^x)^n}{x^k} \geq \frac{(1+x)^n}{x^k} \geq \sum_{i=0}^k \binom{n}{i} x^{i-k} \geq \sum_{i=0}^k \binom{n}{i}.$$

Plugging in $x = \frac{k}{n}$ gets our result. ■

Lemma 16. For any $n \geq k \geq 1$, we have

$$\binom{n}{k} \geq \frac{n^k}{k^k}.$$

Proof.

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \\ &= \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \frac{n-2}{k-2} \cdot \dots \cdot \frac{n-k+1}{1} \\ &\geq \frac{n^k}{k^k}. \end{aligned}$$

■

1.2 Inclusion Exclusion

Example. Say there are 20 math majors, 15 CS majors, and 5 who are majoring in both in one class. How many people are in the class?

Explanation. $20 + 15 - 5 = 30$.

Lemma 17. If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof. Count. ■

Lemma 18. If A, B, C are finite sets, then

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |B \cap C| - |C \cap A| \\ &\quad + |A \cap B \cap C|. \end{aligned}$$

Definition 14. Given a set S , and a positive integer $k \leq |S|$, we write

$$\binom{S}{k}.$$

to denote the set of subsets of S whose size is exactly k .

Note.

$$\left| \binom{S}{k} \right| = \binom{|S|}{k}.$$

Theorem 10. (Inclusion Exclusion Principle) If A_1, A_2, \dots, A_n are finite sets, then

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{k=1}^n (-1)^{k-1} \sum_{I \in \binom{\{1,2,\dots,n\}}{k}} \left| \bigcap_{i \in I} A_i \right|.$$

Lecture 8: Derangements

Lecture 9: Extremal Combinatorics

We will start this unit by looking at graphs and hypergraphs.

Definition 15. A **hypergraph** is a pair $H = (V, E)$ such that V is a set, and E is a subset of the powerset of V ($E \subset 2^V$). Unless otherwise noted, V is a finite set.

Definition 16. Given a hypergraph $H = (V, E)$, then the elements $v \in V$ are called **vertices** and the elements $e \in E$ are called **edges**.

Definition 17. A hypergraph $H = (V, E)$ is **isomorphic** to another hypergraph $H' = (V', E')$ if there is a bijection $\phi : V \rightarrow V'$ between the vertex sets such that for any $S \subseteq V$, we have $S \in E$ iff $\phi(S) \in E'$.

Definition 18. A **graph** is a hypergraph in which every edge has size 2. In other words, $E \subseteq \binom{V}{2}$.

Definition 19. Given $H = (V, E)$, a vertex $v \in V$ is **incident** to an edge $e \in E$ if $v \in e$.

Definition 20. The **degree** of a vertex $v \in V$ is the number of edges it is incident to. We write this as

$$d_H(v) = |\{e \in E : v \in e\}|.$$

Definition 21. The **incidence matrix** M of a hypergraph $H = (V, E)$ is a $V \times E$ matrix (V rows and E columns) so that each entry

$$M_{v,e} : \begin{cases} 1, & \text{if } v \in e \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 11. (Hypergraph handshaking). If $H = (V, E)$ is a hypergraph, then

$$\sum_{v \in V} d_H(v) = \sum_{e \in E} |e|.$$

Proof. Let M be the incidence matrix of H . Then $\sum_{v \in V} d_H(v)$ counts the number of 1's of M by summing along rows, and $\sum_{e \in E} |e|$ counts the number of 1's by summing along columns. As the number of 1's in the matrix is the same, these two values are the same. ■

Corollary. For any graph, we have

$$\sum_{v \in V} d_G(v) = 2|E|.$$

Proof. This is because $|e| = 2$ for all $e \in E$ by definition of a graph. ■

Lecture 10: Extremal Combinatorics Continued

Lemma 19. For any graph $G = (V, E)$, there exists a vertex v of small degree such that

$$\deg(v) \leq \frac{2|E|}{|V|}.$$

Proof. By the handshaking lemma,

$$\frac{\sum_{v \in V} \deg(v)}{|V|} = \frac{\sum_{e \in E} |e|}{|V|} = \frac{2|E|}{|V|}.$$

is the average degree of a vertex in the graph. Then, there exists $v \in V$ whose degree is at most the average. ■

Definition 22. A **triangulation** is a sequence of triangles T, T_1, T_2, \dots, T_n such that

1. $T, T_1, T_2, \dots, T_n \subseteq \mathbb{R}^2$,
2. $T = \bigcup_{i=1}^n T_i$,
3. For any distinct $i, j \in \{1, \dots, n\}$, $T_i \cap T_j$ is either empty, consists of one vertex, or one edge.

Lemma 20. (Sperner) For any vertices such that

1. The 3 outer vertices get different colors,
2. Vertices on the edge of T are colored the

same as one of the edge's endpoints.

Then, there exists an inner triangle with 3 different colors. More formally, if T, T_1, T_2, \dots, T_n , and v_1, v_2, v_3 are the vertices of T , and the vertices are colored by $\{1, 2, 3\}$, such that

1. v_i is color i for $i = 1, 2, 3$, and
2. vertices on the edge of T between v_i and v_j are colored either i or j for $i \neq j \in \{1, 2, 3\}$,

then there exists a triangle T_k , $k \in \{1, 2, \dots, n\}$, such that the three vertices of T_k receive colors 1, 2, 3.

Proof. Define a graph $G = (V, E)$ such that $V = \{T, T_1, T_2, \dots, T_n\}$. Connect two vertices if and only if the edge the two triangles share have colors 1 and 2.

Observation. It is enough to prove that a triangle has odd degree in G if and only if three vertices of the triangle receive distinct colors. Then, as the outer triangle has odd degree, some other smaller triangle must also have odd degree. ■

Theorem 12. (Brower's Fixed Point) Every continuous map from the unit disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ to itself has a fixed point (a point p such that $p \mapsto p$).

Theorem 13. (Product Structure Theorem) Also uses Sperner's Lemma.

Lecture 11: More on Hypergraphs

Continuing with the proof of Sperner's Lemma:

Definition 23. Two vertices are **adjacent** if they share a common edge.

Proof. Consider the outer triangle T . All smaller triangles T_i adjacent to T must lie on the edge of T which has colors 1, 2. Note that there must be odd triangles adjacent to T , as there must be odd number of changes from 1 to 2 on the edge.

Now, consider a smaller triangle's degree $d_G(T_i)$, where $i \in \{1, 2, \dots, n\}$. If T_i has 3 different colors, then $d_G(T_i) = 1$. For the other

direction, assume that $d_G(T_i)$ is odd. Then, there must be at least one edge that receives colors 1 and 2. There are then 3 cases for the third vertex:

Case 1: It is of color 1. Then, there are two edges that have colors 1, 2.

Case 2: It is of color 2. Then, there are again two edges that have colors 1, 2.

Case 3: It is of color 3. Then, there is only one edge.

Therefore, the degree of T_i is odd if and only if it is colored vertices of different colors. There must be one such T_i in the graph, because the number of odd degree vertices in the graph is even. ■

Definition 24. A hypergraph $H = (V, E)$ is **laminar** if for all pairs of its edges $A, B \in E$, A, B is either disjoint, or one is a subset of the other. More formally, $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$.

Lemma 21. Every laminar hypergraph with n vertices has at most $2n - 1$ edges.

Proof. Suppose that this is not the case. Let $H = (V, E)$ be a counterexample such that $|V|$ is minimum, and $|E|$ is maximum. Because H is a counterexample, there must exist an edge $A \in E$ such that $A \neq V$. Choose A such that $|A|$ is maximum. By the choice of H , $\bar{A} = V \setminus A$ is an edge of H (because otherwise, we would have a counterexample with even more edges, but our counterexample is maximum). Consider

$$H_1 = (A, \{e \in E : e \subseteq A\})$$

$$H_2 = (\bar{A}, \{e \in E : e \subseteq \bar{A}\}).$$

Then, by the choice of H , neither H_1 nor H_2 is a counterexample. Therefore, the number of edges contained in A is at most $2|A| - 1$. Likewise, the number of edges contained in \bar{A} is $2|\bar{A}| - 1$. Then, the only edge that is not a subset of A or \bar{A} is V . Therefore,

$$|E| \leq (2|A| - 1) + (2|\bar{A}| - 1) - 1 = 2|V| - 1.$$

as desired. ■

Definition 25. A hypergraph is **Sperner** if there are no distinct edges A and B with $A \subset B$.

Corollary. For any integer $n \geq 1$, there exists a Sperner hypergraph on n vertices and

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

edges.

Theorem 14. This bound is tight.

Lecture 12: Sperner Graphs

Note. All normal graphs are Sperner, as all edges have the same size.

Lemma 22. (LMY-Inequality) If $H = (V, E)$ is a Sperner hypergraph, and $n = |V|$, then

$$\sum_{A \in E} \frac{1}{\binom{n}{|A|}} \leq 1.$$

Proof. Without loss of generality, let $V = \{1, 2, \dots, n\}$. We proceed with a double counting argument. We let $A \subseteq \{1, 2, \dots, n\}$ be **initial** for τ if $\tau(\{1, 2, \dots, |A|\}) = A$. In other words, the first $|A|$ elements map to the set A . Note that $\tau(\emptyset) \subseteq \tau(\{1\}) \subseteq \tau(\{1, 2\}) \subseteq \dots \subseteq \tau(\{1, 2, \dots, n\})$ are all initial sets.

Observation. For every permutation τ , there is at most 1 edge $A \in E$ such that A is initial for τ . This is because in a Sperner hypergraph, no edge is contained in the other.

Observation. Every $A \in E$ is initial for $|A|!(n - |A|)!$ permutations. This is because there are $|A|!$ many ways to map $\{1, 2, \dots, |A|\}$ to A . There are then $(n - |A|)!$ ways to map the rest.

Let

$$\chi_{A,\tau} = \begin{cases} 1, & \text{if } A \text{ initial for } \tau \\ 0, & \text{otherwise} \end{cases}.$$

Then,

$$\sum_{\tau} \sum_{A \in E} \chi_{A,\tau} = \sum_{A \in E} \sum_{\tau} \chi_{A,\tau} = \sum_{A \in E} |A|!(n - |A|)!.$$

Together, we have that

$$n! \geq \sum_{A \in E} |A|!(n - |A|)!.$$

such that

$$1 \geq \sum_{A \in E} \frac{|A|!(n - |A|)!}{n!} = \sum_{A \in E} \frac{1}{\binom{n}{|A|}}.$$

And now with our proof for Sperner's theorem: ■

Theorem 15. Sperner's

Proof. Recall that

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \binom{n}{k}.$$

for any $1 \leq k \leq n$. Therefore,

$$\frac{|E|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \sum_{A \in E} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \sum_{A \in E} \frac{1}{\binom{n}{|A|}}.$$

which must be at most 1 from LMY. Therefore, $|E| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. ■

Lecture 13: Extremal Graph Theory

Definition 26. If we have that G and G' are **isomorphic**, we write that

$$G \cong G'.$$

Definition 27. If $e \in E$, then the **ends** of e are the two vertices $v \in e$.

Definition 28. We say that a graph G' is a **sub-graph** of G if $V' \subseteq V$ and $E' \subseteq E$.

If we fix a graph G' , what is the maximum number of edges in an n -vertex graph such that G does not contain a subgraph $\cong G'$?

Example. If G' is the single edge, then the maximum number of edges in a graph that does not contain G' is 0.

Example. If G' is the line graph with 2 edges, then for every vertex v we have that $d_G(v) \leq 1$. By the handshaking lemma, we have that

$$2|E| \leq \sum_{v \in V} d_G(v) \leq |V|.$$

Therefore,

$$|E| \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Observation. Note that if $G'' \subseteq G'$, and $G' \subseteq G$,

then $G'' \subseteq G$.

Definition 29. For any integer $r \geq 1$, the **complete graph** on r vertices, denoted K_r , has r vertices and $\binom{r}{2}$ edges.

We now look at the case where $G' = K_3$ is the triangle graph. One such example is the complete bipartite graph with $\lfloor \frac{n}{2} \rfloor$ vertices on one side and $\lceil \frac{n}{2} \rceil$ vertices on the other. this graph has $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ edges, which is actually the unique graph to the extremal problem.

Theorem 16. (Turan) Every n -vertex graph with no subgraph isomorphic to K_3 has at most

$$\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil.$$

edges.

Proof. (Erdos) Let $G = (V, E)$ be an n -vertex graph with no subgraph $\cong K_3$. Let $v \in V$ be a vertex of maximum degree. Let $N_G(v) = \{x \in V : (v, x) \in E\}$. Let this set be called the **neighborhood** of v in G .

Let $\hat{G} = (\hat{V}, \hat{E})$ be the graph with vertex set $\hat{V} = V$ and edge set

$$\hat{E} = \{(x, y) : x \in N_G(v) \text{ and } y \notin N_G(v)\}.$$

This graph \hat{G} has no subgraph $\cong K_3$. We claim that

$$|\hat{E}| \geq |E|.$$

Continuing,

$$\begin{aligned} |E| &\leq \sum_{y \notin N_G(v)} d_G(y) \\ &\leq \sum_{y \notin N_G(v)} d_G(v) \\ &= |\hat{E}|. \end{aligned}$$

Remember to maximize ab such that $a + b = n$, we set $a = \lfloor \frac{n}{2} \rfloor$ and $b = \lceil \frac{n}{2} \rceil$. Therefore, the maximum number of edges in our graph is $|\hat{E}| = ab \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. ■

Lecture 14: Turan's Theorem

This graph is called the **Turan Graph** $T(n, 2)$.

Definition 30. $T(n, r)$ is the unique (up to isomorphism) n -vertex graph whose vertex set V can be partitioned into r disjoint parts V_1, V_2, \dots, V_r such that each part has roughly

equal size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$, and whose edge set is defined by

$$E = \{(u, v) : u \in V_i \text{ and } v \in V_j \text{ for some } i \neq j\}.$$

Theorem 17. For any $n, r \in \mathbb{Z}_{>0}$, any n -vertex graph $G = (V, E)$ with no subgraph isomorphic to K_{r+1} has at most as many edges as the Turan graph $T(n, r)$. Furthermore, if this graph has the same amount of edges as the Turan graph, $T(n, r)$, then it is isomorphic to the Turan graph.

Proof. On HW, but can use the following: ■

Definition 31. A **complete multipartite graph** is a graph $G = (V, E)$ such that V has a partition into sets of vertices V_1, \dots, V_n such that

$$E = \{(u, v) : u \in V_i \text{ and } v \in V_j \text{ for } i \neq j\}.$$

Denote each set V_1, \dots, V_n as a **part**.

Lemma 23. Let $n, r \in \mathbb{Z}_{>0}$, and let $G = (V, E)$ be an n -vertex complete multipartite graph with $\leq r$ parts. Then G has at most as many edges as $T(n, r)$, and is $T(n, r)$ if = holds.

Proof. Suppose G has ≥ 2 vertices in its largest part than in its smallest part. We will show that $|E| <$ the number of edges in $T(n, r)$.

While there are ≥ 2 more vertices in the largest part of G_i than in its smallest part, let G_{i+1} be the complete multipartite graph formed from moving one such vertex from the smallest part to the largest part of G_i . ■

Lemma 24. This process will end, and when it does, that graph is isomorphic to $T(n, r)$.

Proof. HW. ■

Lemma 25. The number of edges in G_i is strictly less than the number of edges in G_{i+1} .

Proof. Let v be the vertex of G_i which was moved. Then the number of edges of G_i , which are not incident to v , is the number of edges in G_{i+1} that are not incident to v . Also, we have that

$$d_{G_i}(v) = n - \text{vertices in part with } v.$$

and

$$d_{G_{i+1}}(v) = n - \text{vertices in new part with } v.$$

Therefore, the number of edges in G_{i+1} is strictly greater than that of G_i . ■

Lecture 15: Kovari-Sos-Turan

Definition 32. A graph G is **bipartite** if $G \subseteq T(n, 2)$ for some n . Equivalently, G is **bipartite** if its vertex set can be partitioned into two sets A and B such that every edge of G has one end in A and one end in B . The two sets A and B are called a **bipartition** of the graph.

Theorem 18. (Kovari-Sos-Turan) Let $G' = (V', E')$ be a bipartite graph. Then there exists $\varepsilon > 0$ (which depends on G') such that every n -vertex graph with no subgraph isomorphic $\cong G'$ has $O(n^{2-\varepsilon})$ edges. In fact, we can take

$$\varepsilon = \frac{1}{\text{size of larger part of bipartition}}.$$

Theorem 19. (Special case of KST) If G' is the complete bipartite graph with bipartition (A, B) such that $|A| = 2$ and $|B| = 2$, then every n -vertex graph with no subgraph $\cong G'$ has

$$O(n^{\frac{3}{2}}).$$

Note that G' is also the 4-cycle C_4 .

Note. The same for $K_{3,3}$ is an open problem, known as the **Zarankiewicz Problem**.

We will now look at a construction for the lower bound.

Definition 33. In its essence, a **projective plane** is a system of $q^2 + q + 1$ lines and points such that every pair of points determines a line, every pair of lines determines a unique point, and every line contains $q + 1$ points.

Explanation. Consider the (bipartite) incidence graph of a projective plane such that each point is connected to the lines that it is in. Formally,

$$V = \{\text{points}\} \cup \{\text{lines}\}$$

and

$$E = \{\{x, y\} : x \text{ is a point in line } y\}.$$

Observe that G has no subgraph isomorphic to C_4 . Also note that the number of edges is the number of lines times the number of points in each line, which is

$$(q^2 + q + 1)(q + 1) \geq q^3.$$

Also note that the number of vertices is $2(q^2 + q + 1) = O(q^2)$. Therefore,

$$\text{number edges} = \Omega(\text{number vertices}^{1.5}).$$

Lecture 16: Graphs with no 4-Cycles

Theorem 20. (Cauchy-Schwarz) Given $x_1, x_2, \dots, x_n \in \mathbb{R}$ and $y_1, y_2, \dots, y_n \in \mathbb{R}$,

$$\sum_{i=1}^n x_i y_i \leq \sqrt{\sum_{i=1}^n x_i^2} \cdot \sqrt{\sum_{i=1}^n y_i^2}.$$

We will now prove the special case of the Kovari-Sos-Turan theorem.

Proof. We count the number of subgraphs isomorphic to cherries such that v is red and the two green star vertices are $\in N_G(v)$. One way to count such cherries is choose every two vertices in the neighborhood of G , which is $\sum_{v \in V} \binom{d_G(v)}{2}$.

The second way is to first choose the two green star vertices, which is just $\binom{n}{2}$. Then, there is at most one choice for the red vertex as there is no subgraph isomorphic to C_4 . This is therefore at most $\binom{n}{2}$.

Putting this all together, we have

$$\sum_{v \in V} \binom{d_G(v)}{2} \leq \binom{n}{2} \leq \frac{n^2}{2}.$$

Let the vertex set be $\{1, 2, \dots, n\}$. Apply the Cauchy-Schwarz inequality with all $y_i = 1$, and

$x_i = d_G(i) - 1$. Then,

$$\begin{aligned} \sum_{i=1}^n x_i &\leq \sqrt{\sum_{i=1}^n x_i^2} \sqrt{\sum_{i=1}^n 1} \\ &= \sqrt{\sum_{i=1}^n (d_G(i) - 1)^2} \cdot \sqrt{n} \\ &\leq \sqrt{2 \sum_{v \in V} \binom{d_G(v)}{2}} \cdot \sqrt{n}. \end{aligned}$$

All in all,

$$\begin{aligned} 2|E| - n &\leq \sqrt{2 \sum_{v \in V} \binom{d_G(v)}{2}} \cdot \sqrt{n} \\ \sum_{v \in V} \binom{d_G(v)}{2} &\leq \frac{n^2}{2} \\ 2|E| - n &\leq \sqrt{n^2} \cdot \sqrt{n} \\ &= n^{\frac{3}{2}}. \end{aligned}$$

Such that

$$|E| \leq \frac{n^{\frac{3}{2}} + n}{2} = O(n^{\frac{3}{2}}).$$

■

Theorem 21. (Szemerédi-Trotter) For incidences for points on lines in the plane in \mathbb{R}^2 , the number of incidences is $O(n^{\frac{4}{3}})$.

Lecture 17: Blanche Descartes

Definition 34. The **chromatic number** of a graph is the minimum integer k such that $G \subseteq$ a complete multipartite graph with $\leq k$ parts.

Theorem 22. (Tutte) For every $k \in \mathbb{Z}_{>0}$, there exists a graph $G = (V, E)$ with no subgraph $\cong K_3$ such that G is not a subgraph of any complete multipartite graph with $\leq k$ parts.

Lemma 26. The minimum **chromatic number** of G is the minimum integer k such that there exists $\phi: V \rightarrow \{1, 2, \dots, k\}$ such that the ends of every edge have different colors.

Definition 35. Another way to define the **chromatic number** is $\chi(G)$. Any function $\phi: V \rightarrow \{1, \dots, k\}$ such that for all $(u, v) \in E$, $\phi(u) \neq \phi(v)$ is called a **proper k-coloring**.

We now show the proof of the theorem:

Proof. We proceed with induction on k . Formally, set $G_{k+1} = (V_{k+1}, E_{k+1})$ where

$$V_{k+1} = V \cup \underbrace{\{\hat{a} : a \in V_k\}}_{\hat{V}_k} \cup \{v\}$$

and

$$\begin{aligned} E_{k+1} &= E_k \cup \{\{a, \hat{b}\} : \{a, b\} \in E_k\} \\ &\quad \cup \{\{\hat{a}, v\} : a \in V_k\}. \end{aligned}$$

We now show that there is no $k+1$ coloring of a graph. Suppose there is for the sake of contradiction. WLOG assume $\phi(v) = k+1$. Define $\phi' : G_k \rightarrow \{1, \dots, k\}$ by setting, for each $a \in V_k$,

$$\phi'(a) = \begin{cases} \phi(a) & \text{if } \phi(a) \in \{1, \dots, k\} \\ \phi(\hat{a}) & \text{if } \phi(a) = k+1 \end{cases}.$$

Note that $\phi'(a) \in \{1, \dots, k\}$. This is a proper k coloring, a contradiction. ■

Lecture 18: Triangle-Free Graphs

Definition 36. The n -sphere S_n is $\{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$.

Note that S_2 is a ball, like a soccer ball.

Definition 37. A set $X \subseteq S_n$ is **open** if for every $x \in X$, there exists $\varepsilon > 0$ such that all points $y \in S_n$ with $\text{dist}(x, y) < \varepsilon$ are in X .

Definition 38. A set $X \subseteq S_n$ is **closed** if $S_n \setminus X$ is open.

Lemma 27. Any union of open sets is open.

Proof. Omitted. ■

Theorem 23. (Borsuk-Ulam) Let $X_1, X_2, \dots, X_{n+1} \subseteq S_n$ such that

$$\bigcup_{i=1}^{n+1} X_i = S_n.$$

and each of X_i is either open or closed. Then there exists $i \in \{1, 2, \dots, n+1\}$ and $x \in S_n$ such that X_i contains both x and $-x$. These

points are called **antipodal**.

Definition 39. The **Kneser** graph $G_{n,k} = (V_{n,k}, E_{n,k})$ has a vertex set

$$V_{n,k} = \{I \subseteq \{1, \dots, n\} : |I| = k\}.$$

and

$$E_{n,k} = \{\{A, B\} : A \cap B = \emptyset\}.$$

Lemma 28. The graph $G_{3k-1,k}$ has no subgraph $\cong K_3$.

Proof. There are no 3 disjoint sets $A, B, C \in \{1, 2, \dots, 3k-1\}$ of size k . ■

Theorem 24. (Lovasz-Kneser) The chromatic number of $G_{3k-1,k}$ is $> k$.

Proof. Suppose not. Then ϕ is a proper k -coloring of $G_{3k-1,k}$. Let $P \subseteq S_k$ be a set of $3k-1$ points such that no points in P lie on a hyperplane through the origin. (For any vector $x \in \mathbb{R}^{k+1}$, the set of $\{y \in \mathbb{R}^{k+1} : x \cdot y = 0\}$ is a hyperplane through the origin). ■