

# Combinatorial Analysis

Raymond Bian

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**Theorem 2.** (Suk, 2017) This number is actually  $\leq 2^{n+o(1)}$

**Notation.** Think of  $o(1)$  as standing for a function  $f(n)$  such that  $\lim_{n \rightarrow \infty} f(n) = 0$ . In other words, for every  $\varepsilon > 0$ , there exists  $n_0$  such that  $|f(n)| < \varepsilon$  for every  $n \geq n_0$ .

## Lecture 1: Syllabus and Review

### 1 Introduction

This course is basically just a second course in Combinatorics, and will cover a range of topics.

**Definition 1. Matroids** are the structures that capture whether or not the greedy algorithm works. They will be covered later in the course.

Now, for some examples and review:

**Definition 2.** We say points are in **convex position** if no point is inside a triangle made by 3 other points.

**Example.** Given a finite set of points on the plane, what is the maximum number of points such that no 3 are on a line, and no 4 are in convex position.

**Explanation.** Informally, we know that the “outside” of our points has at most 3 points in the shape of a triangle. We can then place a point in the middle. However, if we try to add another point, then we find that 4 points are in convex position, which is a contradiction. Therefore, 4 points is the maximum size of such a set.

This example is actually part of a more general problem, shown below.

**Theorem 1.** (ES, 1935) The maximum number of points such that no 3 are on a line and no  $n$  are in convex position is  $\leq 4^n$  and  $\geq 2^{n-2}$ .

**Example.** How many distinct 5-letter words are there on the 26-letter english alphabet?

**Explanation.** There are 26 options for each of the 5 slots, so there are  $26^5$  words.

**Example.** What if repetitions aren't allowed?

**Explanation.** Each slot you lose an option, so there are  $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = \frac{26!}{21!}$  words.

**Example.** How many ways are there to choose 5 students out of 35 to present?

**Explanation.** There are  $\binom{35}{5} = \frac{35!}{5! \cdot 30!}$  ways.

## Lecture 2: Review of Proofs

We will now review the types of proofs covered in Math-3012, as well as guidelines for writing them in this class.

**Notation.** If  $F$  is a mapping from  $N$  to  $M$ , we write  $F : N \rightarrow M$ .

**Notation.** Sometimes,  $N \setminus \{a\}$  will be instead written as  $N - \{a\}$ .

**Proposition 1.** Let  $N$  be an  $n$ -element set and  $M$  be an  $m$ -element set. Then, there are  $m^n$  mappings (or functions) from  $N$  to  $M$ .

**Proof.** (Inductive) We go by induction on  $n$ .

**Base case.:** For the base case  $n = 0$ , we con-

sider the empty set  $\emptyset$  to be a mapping from the empty set to  $M$ . So  $m^0 = 1$  and the base case holds.

**Inductive step.:** Now, let  $n \geq 1$  and assume that the proposition holds for  $n - 1$  by induction. So, let  $a \in N$ . There are  $m^{n-1}$  mappings  $F' : N \setminus \{a\} \rightarrow M$ . For each such  $F'$ , we have  $m$  choices for where to send  $a$ . These mappings are all distinct, and every  $F : N \rightarrow M$  can be obtained in this way. So, the number of mappings  $F : N \rightarrow M$  is  $m^{n-1} \cdot m = m^n$ , as desired. ■

**Definition 3.** A **bijection** is a function  $f : X \rightarrow Y$  such that  $f$  is one-to-one and onto.

**Corollary.** An  $n$ -element set  $X$  has  $2^n$  many subsets.

**Proof.** (Bijective) For each  $A \subseteq X$ , let  $F_A : X \rightarrow \{0, 1\}$  such that for each  $x \in X$ ,

$$F_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}.$$

These mappings  $F_A, F_{A'}$  are distinct for distinct subsets  $A, A' \subseteq X$ , and every mapping  $F : X \rightarrow \{0, 1\}$  is equal to  $F_A$  for some  $A \subseteq X$ . So by proposition 1, the corollary holds. ■

**Lemma 1.** For any non-negative integers  $n, k$  ( $n, k \in \mathbb{Z}_{\geq 0}$ ), we have  $\binom{n}{k} = \binom{n}{n-k}$ .

**Proof.** (Algebraic) We have

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(n-(n-k))!(n-k)!} \\ &= \binom{n}{n-k}, \end{aligned}$$

as desired. ■

**Theorem 3.** (Binomial Theorem) Let  $n \in \mathbb{Z}_{\geq 0}$ . Then

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

**Proof.** Consider

$$\underbrace{(x + y)(x + y) \dots (x + y)}_{n \text{ times}}.$$

For each  $(x + y)$  term, we select either the  $x$  or the  $y$ , and there are  $\binom{n}{k}$  ways to select  $k$   $x$ 's and  $n - k$   $y$ 's. The formula follows. ■

**Corollary.** For any  $n \in \mathbb{Z}_{\geq 0}$ , we have

$$2^n = \sum_{k=0}^n \binom{n}{k} \text{ and } 0 = \sum_{k=0}^n \binom{n}{k} (-1)^k.$$

**Proof.** Apply the binomial theorem with  $x = y = 1$  to yield the first result, and with  $x = -1, y = 1$  to yield the second. ■

## 1.1 Counting Review

**Definition 4.** A **permutation** is a bijection from a finite set to itself.

**Example.** One such bijection could be  $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3$ .

**Lemma 2.** The number of such bijections is  $n!$ .

**Proof.** Exercise to the student! ■

## Lecture 3: Permutations and Cycles

**Notation.**  $\tau : X \rightarrow X$  is a permutation on  $X$ . Can also be denoted by  $\sigma : X \rightarrow X$ .

We will show that all permutations  $\tau$  can be “decomposed” into “cycles”.

**Example.** From the example earlier,  $(1, 2)$  is a cycle, and  $(3, 4, 5)$  is another cycle.

For the following, let  $\tau : X \rightarrow X$ .

**Definition 5.** A **cycle** of  $\tau$  is a tuple (ordered set of elements)  $(x_1, x_2, \dots, x_k)$  such that  $x_1, x_2, \dots, x_k$  are distinct elements of  $X$ , and  $\tau(x_1) = x_2, \tau(x_2) = x_3, \dots, \tau(x_{k-1}) = x_k, \tau(x_k) = x_1$ . We call  $x_1, x_2, \dots, x_k$  the **elements** of the cycle.

**Lemma 3.** If  $(x_1, x_2, \dots, x_k)$  and  $(y_1, y_2, \dots, y_r)$  have an element in common, then  $\{x_1, x_2, \dots, x_k\} = \{y_1, y_2, \dots, y_r\}$ .

**Proof.** Note that since  $(x_1, x_2, \dots, x_k)$  is a cycle,  $(x_2, x_3, \dots, x_k, x_1)$  is also a cycle. Because of this, we can assume that  $x_1 = y_1$ . So  $x_2 = \tau(x_1) = \tau(y_1) = y_2$ . Then, we have that  $x_2 = y_2$ . We can repeat this process until  $x_k = y_k$  (swap  $x, y$  if  $k > r$ ). Then, we have  $x_1 = \tau(x_k) = \tau(y_k) = y_1$ , which means that  $r = k$ . Therefore, all cycles are pairwise disjoint. ■

**Lemma 4.** For every  $x \in X$ , there exists a cycle of  $\tau$  which has  $x$  as an element.

**Proof.** Consider visiting each element  $x, \tau(x), \tau(\tau(x)), \dots$ , until the first time we re-visit any element. This will eventually happen, because  $X$  is finite. Then, let's suppose that we have visited elements  $x_1, x_2, \dots, x_k$  so far, such that  $x_1, x_2, \dots, x_k$  are distinct, and that  $\tau(x_k) = x_i$  for some  $i \in \{1, 2, \dots, k\}$ . We cannot have  $i \geq 2$  because then both  $x_{i-1}$  and  $x_k$  would both map to  $x_i$ , which is a contradiction because a permutation is a bijection. Therefore,  $i = 1$  and we have established our cycle. ■

**Corollary.** There exists cycles  $C_1, C_2, \dots, C_t$ , so that every element of  $X$  is an element in exactly one such cycle.

**Definition 6.** The **cycle notation** for  $\tau$  is written as

$$\tau = C_1 C_2 \dots C_t.$$

**Example.** Find the cycle notation for the permutation  $\tau$  of  $\{1, 2, 3, 4, 5, 6\}$  where

$$\begin{aligned}\tau(1) &= 4 \\ \tau(2) &= 6 \\ \tau(3) &= 2 \\ \tau(4) &= 5 \\ \tau(5) &= 1 \\ \tau(6) &= 3.\end{aligned}$$

**Explanation.** By inspection, we have a cycle  $(1, 4, 5)$  and another cycle  $(2, 3, 6)$ . Therefore,  $\tau = (1, 4, 5)(2, 3, 6)$ .

**Definition 7.** A **transposition** is a cycle with exactly two elements.

**Problem.** How quickly does  $n!$  grow as  $n$  gets large?

## Lecture 4: Estimates for $n!$

**Lemma 5.** (Simplest) For any positive integer  $n \in \mathbb{Z}_{>0}$ ,

$$2^{n-1} \leq n! \leq n^{n-1}.$$

**Proof.** We have for the lower bound

$$n! = \prod_{i=2}^n i \geq \prod_{i=2}^n 2 = 2^{n-1}.$$

And for the upper bound,

$$n! = \prod_{i=2}^n i \leq \prod_{i=2}^n n = n^{n-1}.$$

■

Note that these bounds are very far off. Here is a motivating example.

**Example.** Suppose  $n$  students draw a card from a deck of  $n$  cards, replacing the card afterwards. What is the likelihood that all  $n$  cards drawn are distinct?

**Explanation.** The probability is the number of desirable outcomes over the total number of outcomes. This is just

$$\frac{n!}{n^n}.$$

Note that if we use the upper bound from this lemma, we would get that the probability is at most  $\frac{1}{n}$ . In reality however, the true probability is much, much smaller.

**Lemma 6.** A better set of bounds are the following:

$$\left(\frac{n}{2}\right)^{\frac{n}{2}} \leq n! \leq \frac{(n+1)^n}{2^{\frac{n}{2}}}.$$

**Proof.** Left as an exercise! ■

**Lemma 7.** For any two  $a, b \geq 2$ , we have  $a \cdot b \geq a + b$ .

**Lemma 8.** (Arithmetic-Geometric Mean Inequality) For any two  $a, b \geq 0$ , we have

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

With these last two lemmas, we can show the following:

**Theorem 4.** (Gauss) For any  $n \in \mathbb{Z}_{>0}$ ,

$$n^{\frac{n}{2}} \leq n! \leq \frac{(n+1)^n}{2^n}.$$

**Proof.** We instead look at  $(n!)^2$ . Pairing 1 with  $n$ , 2 with  $n-1$ , etc, we have for the lower bound

$$\begin{aligned} n! &= \left( \prod_{i=1}^n i \right) \left( \prod_{i=1}^n i \right) \\ &= \prod_{i=1}^n i(n+1-i) \\ &= \prod_{i=1}^n \sqrt{i(n+1-i)} \\ &\geq \prod_{i=1}^n \sqrt{n} \quad (\text{Lemma 7}) \\ &\geq n^{\frac{n}{2}}. \end{aligned}$$

And for the upper bound, we have

$$\begin{aligned} n! &= \left( \prod_{i=1}^n i \right) \left( \prod_{i=1}^n i \right) \\ &= \prod_{i=1}^n i(n+1-i) \\ &= \prod_{i=1}^n \sqrt{i(n+1-i)} \\ &\leq \prod_{i=1}^n \frac{i+n+1-i}{2} \\ &= \frac{(n+1)^n}{2^n}. \end{aligned}$$

■

**Theorem 5.** (Even better bound) For any  $n \in \mathbb{Z}_{>0}$ , we have

$$e \left( \frac{n}{e} \right)^n \leq n! \leq en \left( \frac{n}{e} \right)^n.$$

**Proof.** The lower bound will be given as a homework problem. The upper bound is as follows. Note that  $\ln(n!) = \sum_{i=1}^n \ln(i)$ . Then, we can take the integral of  $\ln(x)$ , which is greater than this sum.

$$\begin{aligned} \sum_{i=1}^n \ln(i) &\leq \int_1^{n+1} \ln(x) dx \\ &= (n+1) \ln(n+1) - n. \end{aligned}$$

Thus

$$\begin{aligned} n! &\leq e^{(n+1) \ln(n+1) - n} \\ &= \frac{e^{(n+1) \ln(n+1)}}{e^n} \\ &= \frac{(e^{\ln(n+1)})^{n+1}}{e^n} \\ &= \frac{(n+1)^{n+1}}{e^n}. \end{aligned}$$

Applying this for  $n(n-1)!$  gives the bound. ■

## Lecture 5: Asymptotic Analysis

**Theorem 6.** Stirling's Formula says that

$$n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$

**Definition 8.** For two functions  $f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$ , we write  $f \sim g$  and say  $f$  is asymptotic to  $g$  if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1.$$

Also note that  $f \sim g \Leftrightarrow g \sim f$ .

**Example.**  $2n + \sqrt{n} \sim 2n$ .

**Explanation.** This is because

$$\lim_{n \rightarrow \infty} \frac{2n + \sqrt{n}}{2n} = 1.$$

**Example.** (Informal) How many digits are in  $100!$ ?

**Explanation.** Using Stirling's Formula, we have that

$$100! \sim \sqrt{2\pi 100} \left( \frac{100}{e} \right)^{100} = 9.324 \dots \times 10^{157}.$$

, whereas  $100! = 9.332 \dots \times 10^{157}$  (very close approximation).

**Definition 9.** The  $n$ -th harmonic number

$$H_n = \sum_{i=1}^n \frac{1}{i}.$$

**Theorem 7.** (Euler-Mascheroni)  $H_n \sim \ln(n)$ .

**Proof.** Omitted. ■

**Lemma 9.** For any positive integer  $n \in \mathbb{Z}_{>0}$ , we have

$$\frac{\lfloor \log_2(n) \rfloor}{2} \leq H_n \leq \lfloor \log_2(n) \rfloor + 1.$$

**Proof.** We can break up the proof into parts of size 2, 4, 8, 16, ... Let  $S_k = \{i \in \mathbb{Z}_{>0} : 2^{k-1} \leq i \leq 2^k - 1\}$  for any  $k \in \mathbb{Z}_{>0}$ . Note that  $|S_k| = 2^{k-1}$ . Also, for every  $x \in S_k$ , we have

$$\frac{1}{2^k} < \frac{1}{x} \leq \frac{1}{2^{k-1}}.$$

Therefore, we have

$$\begin{aligned} H_n &= \sum_{i=1}^n \frac{1}{i} = \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \sum_{x \in S_k} \frac{1}{x} \\ &\geq \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \sum_{x \in S_k} \frac{1}{2^k} \\ &= \sum_{k=1}^{\lfloor \log_2(n) \rfloor} 2^{k-1} / 2^k \\ &= \sum_{k=1}^{\lfloor \log_2(n) \rfloor} \frac{1}{2} \\ &= \frac{\lfloor \log_2(n) \rfloor}{2}. \end{aligned}$$

In the other direction, we have

$$\begin{aligned} H_n &\leq \sum_{k=1}^{\lfloor \log_2(n) \rfloor + 1} \sum_{x \in S_k} \frac{1}{x} \\ &\leq \sum_{k=1}^{\lfloor \log_2(n) \rfloor + 1} \frac{|S_k|}{2^{k-1}} \\ &= \lfloor \log_2(n) \rfloor + 1. \end{aligned}$$

■

**Definition 10.** Let  $f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$ . We say  $f = O(g)$  or  $f$  is big-O of  $g$  if there exists  $n_0, C$ , such that

$$|f(n)| \leq C \cdot g(n) \quad \forall n \geq n_0.$$

**Note.** If  $f, g : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$  and  $f \sim g$ , we have

$$f = O(g) \quad \text{and} \quad g = O(f).$$

If  $\varepsilon = 1$  for all significantly large  $n$ ,  $\frac{f(n)}{g(n)} \leq 2$ .

**Example.**  $\sum_{i=1}^n \frac{1}{i} = O(\log n)$ .

**Lemma 10.** Let  $a, \alpha, \beta > 0$  be fixed. Then as  $n \rightarrow \infty$ ,

- $n^\alpha = O(n^\beta)$  if  $\alpha < \beta$ .
- $n^\alpha = O(a^n)$  if  $a > 1$ .
- $(\ln(n))^\alpha = O(n^\beta)$ .

## Lecture 6: Binomial Coefficients and Counting Primes

**Note.** Note that we can also write, for functions  $f, g, h$   $f = g + O(h)$ , which means that  $|f - g| = O(h)$ .

**Example.**

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} = \frac{n^2}{2} + O(n).$$

**Definition 11.**  $f(n) = \Theta(g(n))$  if  $f = O(g)$  and  $g = O(f)$ .

**Definition 12.**  $f = o(g)$  if  $\lim_{n \rightarrow \infty} \frac{f}{g} = 0$ .

**Example.** What are all primes less than 20?

**Explanation.** 2, 3, 5, 7, 11, 13, 17, 19

**Definition 13.** Let  $\pi(n)$  be the number of primes that are  $\leq n$ .

**Theorem 8.** The **prime number theorem** states that

$$\pi(n) \sim \frac{n}{\ln(n)}.$$

The **Riemann Hypothesis** states that

$$\pi(n) = \int_1^n \frac{1}{\ln(x)} dx + O(\sqrt{n} \ln(n)).$$

It's called a hypothesis because it is often used in other mathematical proofs, even if not proved yet. For example, determining whether a knot could be un-knotted is in NP if the RH is true.

**Lemma 11.** For any  $k \geq 1$ , we have that

$$\binom{2k+1}{k} \leq 4^k.$$

**Proof.** Later. ■

**Lemma 12.** For any  $n \geq 2$ , the product of all primes  $\leq n$  is at most  $16^n$ .

**Proof.** We wish to prove that

$$\prod_{i=1}^{\pi(n)} p_i \leq 16^n.$$

where  $p_i$  denotes the  $i$ -th prime. We proceed with induction on  $n$ .

**Base case:**  $n = 2, 3$ . Holds trivially.

**Step case 1:**  $n$  is even. Note that  $n$  cannot be prime, such that by induction,

$$\prod_{i=1}^{\pi(n)} p_i = \prod_{i=1}^{\pi(n-1)} p_i \leq 16^{n-1} \leq 16^n.$$

**Step case 2:**  $n$  is odd. We write  $n = 2k + 1$  for some  $k \geq 1$ . Note that every prime  $p$  such that  $k + 2 \leq p \leq 2k + 1$  divides  $\binom{2k+1}{k}$ . This is because

$$\binom{2k+1}{k} = \frac{(2k+1)!}{k!(k+1)!}.$$

such that  $p$  divides the numerator but not the denominator.

By induction the product of primes  $p$  such that  $0 \leq p \leq k + 1$  is bound by

$$\prod_{i=1}^{\pi(k+1)} p_i \leq 16^{k+1}.$$

Combining our bounds and lemma, we have

$$\begin{aligned} \prod_{i=1}^{\pi(n)} p_i &= \left( \prod_{i=1}^{\pi(k+1)} p_i \right) \left( \prod_{i=\pi(k+1)+1}^{\pi(n)} p_i \right) \\ &\leq 16^{k+1} \binom{2k+1}{k} \\ &\leq 16^{k+1} \cdot 4^k \\ &\leq 16^{2k+1}. \end{aligned}$$

Note that bounding by  $4^n$  may work here, but  $16^n$  is presented due to a mistake in the lecture notes. ■

**Theorem 9.** The weak prime number theorem

states that

$$\pi(n) = \Theta\left(\frac{n}{\ln n}\right).$$

**Proof.** We will show the upper bound, i.e.

$$\pi(n) = O\left(\frac{n}{\ln n}\right).$$

Let  $p_1, p_2, \dots$  be the sequence of primes. Then,

$$\pi(n)! \leq \prod_{i=1}^{\pi(n)} p_i \leq 16^n.$$

because  $p_1 \geq 1, p_2 \geq 2$ , etc. We have also shown that

$$e \left( \frac{\pi(n)}{e} \right)^{\pi(n)} \leq \pi(n)!.$$

As such,

$$e \left( \frac{\pi(n)}{e} \right)^{\pi(n)} \leq 16^n.$$

Taking the  $\ln$  of both sides, we have

$$\begin{aligned} \ln \left( e \left( \frac{\pi(n)}{e} \right)^{\pi(n)} \right) &\leq \ln(16^n) \\ \pi(n) \cdot \ln \left( \frac{\pi(n)}{e} \right) &\leq n \ln(16) \end{aligned}$$

Assume towards a contradiction that

$$\pi(n) \geq \frac{100n}{\ln n}$$

. Then,

$$\begin{aligned} \frac{100n}{\ln n} \cdot \ln \left( \frac{100n}{e \ln n} \right) &\leq n \ln 16 \\ \frac{100}{\ln 16 \cdot \ln n} \ln \left( \frac{100n}{e \ln n} \right) &\leq 1 \\ \frac{100}{\ln 16 \cdot \ln n} (\ln(100n) - \ln(e \ln n)) &\leq 1 \end{aligned}$$

which is a contradiction (after many calculations). Therefore, there exists some  $C < 100$  such that  $\pi(n) = O\left(\frac{n}{\ln n}\right)$ . ■

## Lecture 7: More Bounds, Inclusion Exclusion

The proof from previous lecture will not be given, as it is on the homework.

**Lemma 13.** For any  $n \geq k \geq 1$ ,

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

**Observation.** If  $n = 2k$ , then

$$\binom{2k}{k} \leq \left(\frac{e \cdot 2k}{k}\right)^k \leq (2e)^k.$$

**Lemma 14.** (Bernoulli-type inequality) For any real number  $x \geq 0$ , we have

$$1 + x \leq e^x.$$

**Proof.** This is true for  $x = 0$  because  $1 + 0 \leq e^0 = 1$ . Therefore, we just need to show that

$$f(x) = e^x - (1 + x).$$

is increasing. This holds because  $f'(x) = e^x - 1 \geq 0$ . ■

**Lemma 15.** For any  $n \geq k \geq 1$ , we have

$$\sum_{i=0}^k \binom{n}{i} \leq \left(\frac{en}{k}\right)^k.$$

**Proof.** Recall that for any  $x, y$ , we have

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}.$$

When  $y = 1$ ,

$$(1 + x)^n = \sum_{i=0}^n \binom{n}{i} x^i.$$

Choosing  $x = \frac{k}{n}$  with the intuition that  $x < 1$  allows us to continue:

$$\sum_{i=0}^n \binom{n}{i} x^i \geq \sum_{i=0}^k \binom{n}{i} x^i.$$

Dividing everything by  $x^k$ ,

$$\frac{(1+x)^n}{x^k} = \sum_{i=0}^n \binom{n}{i} x^{i-k} \geq \sum_{i=0}^k \binom{n}{i} x^{i-k}.$$

Since  $x \leq 1$ ,

$$\frac{(e^x)^n}{x^k} \geq \frac{(1+x)^n}{x^k} \geq \sum_{i=0}^k \binom{n}{i} x^{i-k} \geq \sum_{i=0}^k \binom{n}{i}.$$

Plugging in  $x = \frac{k}{n}$  gets our result. ■

**Lemma 16.** For any  $n \geq k \geq 1$ , we have

$$\binom{n}{k} \geq \frac{n^k}{k^k}.$$

**Proof.**

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \\ &= \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \frac{n-2}{k-2} \cdot \dots \cdot \frac{n-k+1}{1} \\ &\geq \frac{n^k}{k^k}. \end{aligned}$$

■

## 1.2 Inclusion Exclusion

**Example.** Say there are 20 math majors, 15 CS majors, and 5 who are majoring in both in one class. How many people are in the class?

**Explanation.**  $20 + 15 - 5 = 30$ .

**Lemma 17.** If  $A$  and  $B$  are finite sets, then  $|A \cup B| = |A| + |B| - |A \cap B|$ .

**Proof.** Count. ■

**Lemma 18.** If  $A, B, C$  are finite sets, then

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |B \cap C| - |C \cap A| \\ &\quad + |A \cap B \cap C|. \end{aligned}$$

**Definition 14.** Given a set  $S$ , and a positive integer  $k \leq |S|$ , we write

$$\binom{S}{k}.$$

to denote the set of subsets of  $S$  whose size is exactly  $k$ .

**Note.**

$$\left| \binom{S}{k} \right| = \binom{|S|}{k}.$$

**Theorem 10.** (Inclusion Exclusion Principle) If  $A_1, A_2, \dots, A_n$  are finite sets, then

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{k=1}^n (-1)^{k-1} \sum_{I \in \binom{\{1,2,\dots,n\}}{k}} \left| \bigcap_{i \in I} A_i \right|.$$

## Lecture 8: Derangements

## Lecture 9: Extremal Combinatorics

We will start this unit by looking at graphs and hypergraphs.

**Definition 15.** A **hypergraph** is a pair  $H = (V, E)$  such that  $V$  is a set, and  $E$  is a subset of the powerset of  $V$  ( $E \subset 2^V$ ). Unless otherwise noted,  $V$  is a finite set.

**Definition 16.** Given a hypergraph  $H = (V, E)$ , then the elements  $v \in V$  are called **vertices** and the elements  $e \in E$  are called **edges**.

**Definition 17.** A hypergraph  $H = (V, E)$  is **isomorphic** to another hypergraph  $H' = (V', E')$  if there is a bijection  $\phi : V \rightarrow V'$  between the vertex sets such that for any  $S \subseteq V$ , we have  $S \in E$  iff  $\phi(S) \in E'$ .

**Definition 18.** A **graph** is a hypergraph in which every edge has size 2. In other words,  $E \subseteq \binom{V}{2}$ .

**Definition 19.** Given  $H = (V, E)$ , a vertex  $v \in V$  is **incident** to an edge  $e \in E$  if  $v \in e$ .

**Definition 20.** The **degree** of a vertex  $v \in V$  is the number of edges it is incident to. We write this as

$$d_H(v) = |\{e \in E : v \in e\}|.$$

**Definition 21.** The **incidence matrix**  $M$  of a hypergraph  $H = (V, E)$  is a  $V \times E$  matrix ( $V$  rows and  $E$  columns) so that each entry

$$M_{v,e} : \begin{cases} 1, & \text{if } v \in e \\ 0 & \text{otherwise} \end{cases}.$$

**Theorem 11.** (Hypergraph handshaking). If  $H = (V, E)$  is a hypergraph, then

$$\sum_{v \in V} d_H(v) = \sum_{e \in E} |e|.$$

**Proof.** Let  $M$  be the incidence matrix of  $H$ . Then  $\sum_{v \in V} d_H(v)$  counts the number of 1's of  $M$  by summing along rows, and  $\sum_{e \in E} |e|$  counts the number of 1's by summing along columns. As the number of 1's in the matrix is the same, these two values are the same. ■

**Corollary.** For any graph, we have

$$\sum_{v \in V} d_G(v) = 2|E|.$$

**Proof.** This is because  $|e| = 2$  for all  $e \in E$  by definition of a graph. ■

## Lecture 10: Extremal Combinatorics Continued

**Lemma 19.** For any graph  $G = (V, E)$ , there exists a vertex  $v$  of small degree such that

$$\deg(v) \leq \frac{2|E|}{|V|}.$$

**Proof.** By the handshaking lemma,

$$\frac{\sum_{v \in V} \deg(v)}{|V|} = \frac{\sum_{e \in E} |e|}{|V|} = \frac{2|E|}{|V|}.$$

is the average degree of a vertex in the graph. Then, there exists  $v \in V$  whose degree is at most the average. ■

**Definition 22.** A **triangulation** is a sequence of triangles  $T, T_1, T_2, \dots, T_n$  such that

1.  $T, T_1, T_2, \dots, T_n \subseteq \mathbb{R}^2$ ,
2.  $T = \bigcup_{i=1}^n T_i$ ,
3. For any distinct  $i, j \in \{1, \dots, n\}$ ,  $T_i \cap T_j$  is either empty, consists of one vertex, or one edge.

**Lemma 20.** (Sperner) For any vertices such that

1. The 3 outer vertices get different colors,
2. Vertices on the edge of  $T$  are colored the



same as one of the edge's endpoints.

Then, there exists an inner triangle with 3 different colors. More formally, if  $T, T_1, T_2, \dots, T_n$ , and  $v_1, v_2, v_3$  are the vertices of  $T$ , and the vertices are colored by  $\{1, 2, 3\}$ , such that

1.  $v_i$  is color  $i$  for  $i = 1, 2, 3$ , and
2. vertices on the edge of  $T$  between  $v_i$  and  $v_j$  are colored either  $i$  or  $j$  for  $i \neq j \in \{1, 2, 3\}$ ,

then there exists a triangle  $T_k$ ,  $k \in \{1, 2, \dots, n\}$ , such that the three vertices of  $T_k$  receive colors 1, 2, 3.

**Proof.** Define a graph  $G = (V, E)$  such that  $V = \{T, T_1, T_2, \dots, T_n\}$ . Connect two vertices if and only if the edge the two triangles share have colors 1 and 2.

**Observation.** It is enough to prove that a triangle has odd degree in  $G$  if and only if three vertices of the triangle receive distinct colors. Then, as the outer triangle has odd degree, some other smaller triangle must also have odd degree. ■

**Theorem 12.** (Brouwer's Fixed Point) Every continuous map from the unit disc  $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$  to itself has a fixed point (a point  $p$  such that  $p \mapsto p$ ).

**Theorem 13.** (Product Structure Theorem) Also uses Sperner's Lemma.

## Lecture 11: More on Hypergraphs

Continuing with the proof of Sperner's Lemma:

**Definition 23.** Two vertices are **adjacent** if they share a common edge.

**Proof.** Consider the outer triangle  $T$ . All smaller triangles  $T_i$  adjacent to  $T$  must lie on the edge of  $T$  which has colors 1, 2. Note that there must be odd triangles adjacent to  $T$ , as there must be odd number of changes from 1 to 2 on the edge.

Now, consider a smaller triangle's degree  $d_G(T_i)$ , where  $i \in \{1, 2, \dots, n\}$ . If  $T_i$  has 3 different colors, then  $d_G(T_i) = 1$ . For the other

direction, assume that  $d_G(T_i)$  is odd. Then, there must be at least one edge that receives colors 1 and 2. There are then 3 cases for the third vertex:

**Case 1:** It is of color 1. Then, there are two edges that have colors 1, 2.

**Case 2:** It is of color 2. Then, there are again two edges that have colors 1, 2.

**Case 3:** It is of color 3. Then, there is only one edge.

Therefore, the degree of  $T_i$  is odd if and only if it is colored vertices of different colors. There must be one such  $T_i$  in the graph, because the number of odd degree vertices in the graph is even. ■

**Definition 24.** A hypergraph  $H = (V, E)$  is **laminar** if for all pairs of its edges  $A, B \in E$ ,  $A, B$  is either disjoint, or one is a subset of the other. More formally,  $A \cap B = \emptyset$ ,  $A \subseteq B$ , or  $B \subseteq A$ .

**Lemma 21.** Every laminar hypergraph with  $n$  vertices has at most  $2n - 1$  edges.

**Proof.** Suppose that this is not the case. Let  $H = (V, E)$  be a counterexample such that  $|V|$  is minimum, and  $|E|$  is maximum. Because  $H$  is a counterexample, there must exist an edge  $A \in E$  such that  $A \neq V$ . Choose  $A$  such that  $|A|$  is maximum. By the choice of  $H$ ,  $\bar{A} = V \setminus A$  is an edge of  $H$  (because otherwise, we would have a counterexample with even more edges, but our counterexample is maximum). Consider

$$H_1 = (A, \{e \in E : e \subseteq A\})$$

$$H_2 = (\bar{A}, \{e \in E : e \subseteq \bar{A}\}).$$

Then, by the choice of  $H$ , neither  $H_1$  nor  $H_2$  is a counterexample. Therefore, the number of edges contained in  $A$  is at most  $2|A| - 1$ . Likewise, the number of edges contained in  $\bar{A}$  is  $2|\bar{A}| - 1$ . Then, the only edge that is not a subset of  $A$  or  $\bar{A}$  is  $V$ . Therefore,

$$|E| \leq (2|A| - 1) + (2|\bar{A}| - 1) - 1 = 2|V| - 1.$$

as desired. ■

**Definition 25.** A hypergraph is **Sperner** if there are no distinct edges  $A$  and  $B$  with  $A \subset B$ .

**Corollary.** For any integer  $n \geq 1$ , there exists a Sperner hypergraph on  $n$  vertices and

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

edges.

**Theorem 14.** This bound is tight.

## Lecture 12: Sperner Graphs

**Note.** All normal graphs are Sperner, as all edges have the same size.

**Lemma 22.** (LMY-Inequality) If  $H = (V, E)$  is a Sperner hypergraph, and  $n = |V|$ , then

$$\sum_{A \in E} \frac{1}{\binom{n}{|A|}} \leq 1.$$

**Proof.** Without loss of generality, let  $V = \{1, 2, \dots, n\}$ . We proceed with a double counting argument. We let  $A \subseteq \{1, 2, \dots, n\}$  be **initial** for  $\tau$  if  $\tau(\{1, 2, \dots, |A|\}) = A$ . In other words, the first  $|A|$  elements map to the set  $A$ . Note that  $\tau(\emptyset) \subseteq \tau(\{1\}) \subseteq \tau(\{1, 2\}) \subseteq \dots \subseteq \tau(\{1, 2, \dots, n\})$  are all initial sets.

**Observation.** For every permutation  $\tau$ , there is at most 1 edge  $A \in E$  such that  $A$  is initial for  $\tau$ . This is because in a Sperner hypergraph, no edge is contained in the other.

**Observation.** Every  $A \in E$  is initial for  $|A|!(n - |A|)!$  permutations. This is because there are  $|A|!$  many ways to map  $\{1, 2, \dots, |A|\}$  to  $A$ . There are then  $(n - |A|)!$  ways to map the rest.

Let

$$\chi_{A,\tau} = \begin{cases} 1, & \text{if } A \text{ initial for } \tau \\ 0, & \text{otherwise} \end{cases}.$$

Then,

$$\sum_{\tau} \sum_{A \in E} \chi_{A,\tau} = \sum_{A \in E} \sum_{\tau} \chi_{A,\tau}.$$

$\leq \sum_{\tau} n! \quad \quad \quad \sum_{A \in E} |A|!(n - |A|)!$

Together, we have that

$$n! \geq \sum_{A \in E} |A|!(n - |A|)!.$$

such that

$$1 \geq \sum_{A \in E} \frac{|A|!(n - |A|)!}{n!} = \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}.$$

And now with our proof for Sperner's theorem: ■

**Theorem 15.** Sperner's

**Proof.** Recall that

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \binom{n}{k}.$$

for any  $1 \leq k \leq n$ . Therefore,

$$\frac{|E|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \sum_{A \in E} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \sum_{A \in E} \frac{1}{\binom{n}{|A|}}.$$

which must be at most 1 from LMY. Therefore,  $|E| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ . ■

## Lecture 13: Extremal Graph Theory

**Definition 26.** If we have that  $G$  and  $G'$  are **isomorphic**, we write that

$$G \cong G'.$$

**Definition 27.** If  $e \in E$ , then the **ends** of  $e$  are the two vertices  $v \in e$ .

**Definition 28.** We say that a graph  $G'$  is a **sub-graph** of  $G$  if  $V' \subseteq V$  and  $E' \subseteq E$ .

If we fix a graph  $G'$ , what is the maximum number of edges in an  $n$ -vertex graph such that  $G$  does not contain a subgraph  $\cong G'$ ?

**Example.** If  $G'$  is the single edge, then the maximum number of edges in a graph that does not contain  $G'$  is 0.

**Example.** If  $G'$  is the line graph with 2 edges, then for every vertex  $v$  we have that  $d_G(v) \leq 1$ . By the handshaking lemma, we have that

$$2|E| \leq \sum_{v \in V} d_G(v) \leq |V|.$$

Therefore,

$$|E| \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

**Observation.** Note that if  $G'' \subseteq G'$ , and  $G' \subseteq G$ ,

then  $G'' \subseteq G$ .

**Definition 29.** For any integer  $r \geq 1$ , the **complete graph** on  $r$  vertices, denoted  $K_r$ , has  $r$  vertices and  $\binom{r}{2}$  edges.

We now look at the case where  $G' = K_3$  is the triangle graph. One such example is the complete bipartite graph with  $\lfloor \frac{n}{2} \rfloor$  vertices on one side and  $\lceil \frac{n}{2} \rceil$  vertices on the other. this graph has  $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$  edges, which is actually the unique graph to the extremal problem.

**Theorem 16.** (Turan) Every  $n$ -vertex graph with no subgraph isomorphic to  $K_3$  has at most

$$\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil.$$

edges.

**Proof.** (Erdos) Let  $G = (V, E)$  be an  $n$ -vertex graph with no subgraph  $\cong K_3$ . Let  $v \in V$  be a vertex of maximum degree. Let  $N_G(v) = \{x \in V : (v, x) \in E\}$ . Let this set be called the **neighborhood** of  $v$  in  $G$ .

Let  $\hat{G} = (\hat{V}, \hat{E})$  be the graph with vertex set  $\hat{V} = V$  and edge set

$$\hat{E} = \{(x, y) : x \in N_G(v) \text{ and } y \notin N_G(v)\}.$$

This graph  $\hat{G}$  has no subgraph  $\cong K_3$ . We claim that

$$|\hat{E}| \geq |E|.$$

Continuing,

$$\begin{aligned} |E| &\leq \sum_{y \notin N_G(v)} d_G(y) \\ &\leq \sum_{y \notin N_G(v)} d_G(v) \\ &= |\hat{E}|. \end{aligned}$$

Remember to maximize  $ab$  such that  $a + b = n$ , we set  $a = \lfloor \frac{n}{2} \rfloor$  and  $b = \lceil \frac{n}{2} \rceil$ . Therefore, the maximum number of edges in our graph is  $|\hat{E}| = ab \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ . ■

equal size  $\lfloor \frac{n}{r} \rfloor$  or  $\lceil \frac{n}{r} \rceil$ , and whose edge set is defined by

$$E = \{(u, v) : u \in V_i \text{ and } v \in V_j \text{ for some } i \neq j\}.$$

**Theorem 17.** For any  $n, r \in \mathbb{Z}_{>0}$ , any  $n$ -vertex graph  $G = (V, E)$  with no subgraph isomorphic to  $K_{r+1}$  has at most as many edges as the Turan graph  $T(n, r)$ . Furthermore, if this graph has the same amount of edges as the Turan graph,  $T(n, r)$ , then it is isomorphic to the Turan graph.

**Proof.** On HW, but can use the following: ■

**Definition 31.** A **complete multipartite graph** is a graph  $G = (V, E)$  such that  $V$  has a partition into sets of vertices  $V_1, \dots, V_n$  such that

$$E = \{(u, v) : u \in V_i \text{ and } v \in V_j \text{ for } i \neq j\}.$$

Denote each set  $V_1, \dots, V_n$  as a **part**.

**Lemma 23.** Let  $n, r \in \mathbb{Z}_{>0}$ , and let  $G = (V, E)$  be an  $n$ -vertex complete multipartite graph with  $\leq r$  parts. Then  $G$  has at most as many edges as  $T(n, r)$ , and is  $T(n, r)$  if = holds.

**Proof.** Suppose  $G$  has  $\geq 2$  vertices in its largest part than in its smallest part. We wil show that  $|E| <$  the number of edges in  $T(n, r)$ . ■

## Lecture 14

This graph is called the **Turan Graph**  $T(n, 2)$ .

**Definition 30.**  $T(n, r)$  is the unique (up to isomorphism)  $n$ -vertex graph whose vertex set  $V$  can be partitioned into  $r$  disjoint parts  $V_1, V_2, \dots, V_r$  such that each part has roughly