# A Second Course in Linear Algebra

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## Contents

**Lecture 1: Review**

# 1 Vectors and Matrices

For the time being, everything indicated in this course is in $\mathbb{R}$.

> **Definition 1.** A **vector** will be defined as a column vector, e.g.
> $$u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

**Notation.** Sometimes, they will be written as a column vector lying down, e.g. $(x_1, x_2, x_3) \in \mathbb{R}^3$

> **Definition 2.** Let $a$ be a scalar. Then multiplication between vector and scalar is defined as
> $$au = \begin{bmatrix} a \cdot x_1 \\ a \cdot x_2 \\ a \cdot x_3 \end{bmatrix}.$$

> **Definition 3.** Let $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.
> Then addition between vectors is defined as
> $$u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

> **Definition 4.** If $u, v$ are vectors and $a, b$ are scalars, then any $au + bv$ is a **linear combination** of $u$ and $v$.

> **Remark.** A **vector space** $V$ is a set of objects $u, v$ such that $au + bv \in V$.

> **Example.** Polynomials of degree $\leq 2$ in one variable can form a vector space.

> **Proof.** Let $p(x) = a_0 + a_1 x + a_2 x^2$, and $q(x) = b_0 + b_1 x + b_2 x^2$. Multiplying by scalars and adding are defined. Note that $p(x) \to \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

> **Example.** Let $f(x) : [0, 1] \to \mathbb{R}$ be a continuous function. We can multiply such functions by scalars and add together such functions, so they form a vector space as well.

Suppose we have two vectors $u, v \in \mathbb{R}^3$. Looking at the set of all linear combinations of $u, v$,

- if both $u$ and $v$ are the zero vectoor, then $W = \{0\}$.

- if $u = \lambda v$, $v \neq 0$, then $W$ is the line of all multiples of $v$.

- if $u$ and $v$ are **linearly independent**, then $W$ is a plane in $\mathbb{R}^3$.

> **Definition 5.** Vectors $u_1, u_2, u_3$ are **linearly independent** if and only if
> $$a_1 u_1 + a_2 u_2 + a_3 u_3 = 0 \Rightarrow a_1 = a_2 = a_3 = 0.$$

> **Definition 6.** Let $V, W$ be a vector spaces such that $W \subseteq V$. Then, $W$ is called a **subspace** of $V$.

> **Example.** Let $W = \{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \}$.
> Then, $W$ is a subspace of $\mathbb{R}^3$.

> **Theorem 1.** If $u, v \in V$, then the set of linear combinations of $u$ and $v$ is a subspace.

**Proof.** Let $W = \text{span}\{u, v\}$. We must show that $w_1, w_2 \in W \Rightarrow c_1 w_1 + c_2 w_2 \in W$. By assumption, $w_1 = a_1 u + b_1 v$, and $w_2 = a_2 u + b_2 v$, such that $w = (c_1 a_1 + c_2 a_2)u + (c_1 b_1 + c_2 b_2)v$. Therefore, $w$ is a linear combination of $u, v$. $\square$

**Example.** Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $v = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. Then, $\text{span}\{u, v\}$ is a proper subspace of $\mathbb{R}^3$.

**Definition 7.** $u \cdot v = x_1 y_1 + x_2 y_2 + x_3 y_3$ is the dot product of the vectors $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

**Definition 8.** We say that $u \perp v$ if $u \cdot v = 0$.

**Definition 9.** The length or **norm** of a vector $u$ is $\sqrt{u \cdot u} = \|u\|$

**Theorem 2.** The **Cauchy–Schwarz inequality** states that $|u \cdot v| \le \|u\|\|v\|$.

**Proof.**

$$(u + \lambda v) \cdot (u + \lambda v) \ge 0$$
$$u \cdot u + \lambda^2 v \cdot v + 2\lambda u \cdot v \ge 0.$$

The minimum lambda is $\frac{-b}{2a} = \frac{-u \cdot v}{v \cdot v}$, which results in this inequality being true. Therefore, all greater values for lambda will result in this inequality being true. $\square$

**Theorem 3.** The **triangle inequality theorem** states that $\|u + v\| \le \|u\| + \|v\|$.

**Definition 10.** The **unit vector** of a vector $u$, $\hat{u}$ is given by $\frac{u}{\|u\|}$.

**Theorem 4.** If $u$ and $v$ are vectors such that $\|u\| = \|v\| = 1$, then $u \cdot v = \cos(\theta)$ where $\theta$ is the angle between $u$ and $v$.

**Corollary.** If $u$ and $v$ are vectors, then $u \cdot v = \|u\|\|v\|\cos(\theta)$. Note that $u \cdot v = 0$ when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

## Lecture 2: Matrices

**Example.**

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

is a matrix. We can also write $A = \{a_{ij}\}$ such that $i = 1 \ldots n$ and $j = 1 \ldots m$.

What does it mean to take a product between a matrix and a vector?

**Definition 11.** This product is defined as

$$\begin{pmatrix} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\ a_{21} x_1 + a_{22} x_2 + a_{13} x_3 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 \end{pmatrix}.$$

i.e. a collection of dot products between the rows and $x$.

We can also see the product as a linear combination.