Problem Introduction

Let $A = \{a_1, a_2, ..., a_n\}$ be the set of points in the k-center problem. We restrict to the L_2 metric on the 2D plane. We seek to choose a subset $C = \{c_1, c_2, ..., c_k\}$ that minimizes

$$max_{a_i \in A} min_{c_j \in C} d(a_i, c_j)$$

In the LP, for a given distance R, we define choice variables $0 \leq y_i \leq 1$ corresponding to point a_i such that the following are satisfied:

$$\Sigma_{i=1}^{n} y_i \leq k$$

$$\forall a_i \in A,\ \Sigma_{dist(y_i, a) \leq R} y_i \geq 1$$

The goal is for a given k to compare the minimum R_{OPT} that satisfies the integral program, and the minimum R_{LP} that satisfies the LP.

Integrality Gap

For any $k \geq 1$, consider a regular $(3k+1) - gon$ with side length 1. Then for a radius 1, it requires at least $k+1$ circles opened to cover all points. This is because each circle contains exactly 3 points.

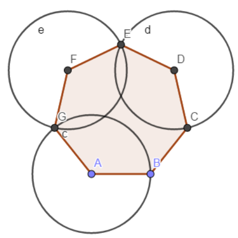

Figure 1. A regular $7 - gon$ requires at least 3 circles to cover all points

Now in the LP, we can let each choice variable $y_i = \frac{1}{3}$. Since all points are contained in exactly 3 circles, they are all filled. So we only require $k + \frac{1}{3}$ centers in the LP.

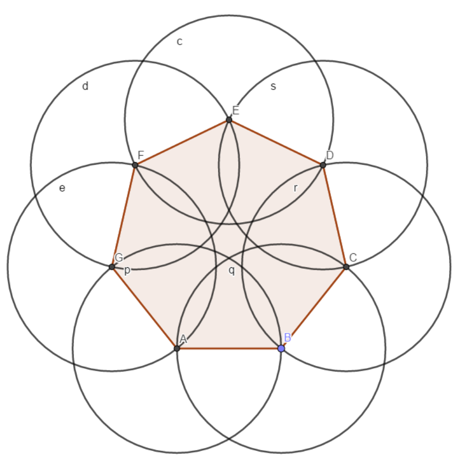

Figure 2. Every point is contained in exactly 3 circles

Now let's take three instances of the $(3k+1)-gon$, we see that for $R = 1$, the LP only requires $3k+1$ centers, while OPT requires $3k+3$. In order for a circle centered at a point of the polygon to contain more than 3 points, it must contain a point 2-adjacent away.

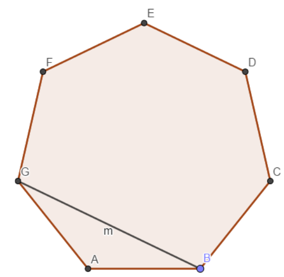

Figure 3. R_{OPT} is the distance between these 2-adjacent points

We then note that for a fixed k, R_{OPT} is the distance between 2-adjacent points. And the integrality gap, as $k \to \infty$ goes to 2.