MATH 3338 Probability

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Lecture 3 - 3338

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Outline

Chapter 3: Combinatorics

Permutations

- **Counting Problems** When a task requires a few steps (r) to finish, the first step has n_1 ways to complete, the second step has n_2 ways to complete, ..., the r-th step has n_r ways to complete,the total number of ways to complete the task is $N = n_1 ... n_r$.
  Note that the Birthday problem in the textbook is a bit surprising, because it takes the prob of more (less) favorable, based on the prob 0.5, does not mean much if prob - 0.51 or so.
- **Permutations Definition 3.1** Let A be any finite set. A permutation of A is a one-to-one mapping of A onto itself.
  Explanation: A rearrangement or reshuffle of all elements in A, in different order, potentially, where the order is important and makes difference.
  Theorem 3.1 The total number of permutations of a set A of n elements is given by the factorial number $n! = n \cdot (n-1) \cdot ... \cdot 2 \cdot 1$.

Permutations

- **Counting Problems Theorem 3.3** (Stirling's formula) The sequence $n!$ is asymptotically equal to

$$n^n e^{-n}\sqrt{2\pi n}$$

- **Definition 3.2** Let A be an n-element set, and let k be an integer between 0 and n. Then a k-permutation of A is an ordered listing of a subset of A of size k.

  Theorem 3.2 The total number of k-permutations of a set A of n elements is given by $n \cdot (n-1) \cdot ... \cdot (n-k+1)$, which is denoted sometimes by P_k^n.

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- **Binomial Coefficients**
  Let U be a set with n elements. we count the number of distinct subsets of the set U that have exactly j elements. The empty set $\emptyset$ and the set U are considered to be subsets of U.
  The number of distinct subsets with j elements that can be chosen from a set with n elements is denoted by $\binom{n}{j}$, or sometimes denoted by C_j^n. It is called a binomial coefficient.

  Adding all binomial coefficients together we have

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3.$$

$$\binom{n}{0} = \binom{n}{n} = 1; \ \binom{n}{1} = \binom{n}{n-1} = n$$

  How to interpret the last two identities?

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- **Theorem 3.4**
  For integers n and j, with $0 < j < n$, the binomial coefficients satisfy:
  $$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$
- **Theorem 3.5**
  For integers n and j, with $0 < j < n$, the binomial coefficients satisfy:
  $$\binom{n}{j} = \frac{P_j^n}{j!} = \frac{n!}{j!(n-j)!};$$
  It is easy to see that
  $$\binom{n}{j} = \binom{n}{n-j}$$

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- **Pascal's Triangle**
  Consider the coefficients of the polynomial $(1+x)^n$. They can be written in the form of the combination number C_i^n, $i = 1, ..., n$

	$j=0$	1	2	3	4
$n=0$	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1
...					

- Let $x = 1$ in the polynomial below,

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i,$$

we have $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

- **Bernoulli Trials**
  Definition 3.5 A Bernoulli trials process is a sequence of n chance experiments such that
  1. Each experiment has two possible outcomes, *success* and *failure*.
  2. The probability p of success on each experiment is the same for all experiment, and this probability is not affected by any knowledge of previous outcomes. $q = 1 - p$ is the probability of failure.
- Example 1. Tossing a coin with prob $P(head) = p$, either a fair ($p = .5$) or unfair ($p \neq .5$) coin is okay.
  Example 2. Consider student's grade at the end of semester in a class. No matter how many different grades the student may get, consider the event of receiving one specific grade, such as "B", and not getting it, constitutes a Bernoulli trial among the students.

- **Binomial Probabilities**
  Consider the probability that in n Bernoulli trials, there are exactly j successes.
  First, within each trial, there are j successes, and $n - j$ failures. Hence the prob is $p^j(1-p)^{n-j}$.
  Furthermore, there are n trials, and only j successes, hence there are $\binom{n}{j}$ ways to have exactly j successes. Hence the above prob needs to be multiplied by the combination number. Hence the total prob of having j successes out of n trials is in the Theorem below.
- **Theorem 3.6** Given n Bernoulli trials with prob p of success on each experiment, the prob of exactly j successes is

$$b(n, p, j) = \binom{n}{j} p^j (1-p)^{n-j}$$

- **Binomial Probabilities**

  Example 3.8 A fair coin is tossed 6 times. Find the prob $Pr(3heads)$.

$$\binom{6}{3}.5^3.5^3 = 20/64 = .3125$$

  Example 3.9 Rolling a die 4 times. Find prob of having exactly one 6 ?

  $p = Pr(\text{one 6 in each toss}) = 1/6.$

$$Pr\left(\begin{array}{c}\text{exactly one 6}\\ \text{out of four tosses}\end{array}\right) = \binom{4}{1}p(1-p)^3 = 4/6(5/6)^3 = .386$$

- **Binomial Distributions**

  Definition 3.6 Let $n > 0$ an integer, and real number $0 \leq p \leq 1$. Random variable (r.v.) X counts the number of successes in a Bernoulli trial process with n trials and prob. p. Then the distribution $Bin(n, p, k)$ that X follows is called a Binomial probability distribution.

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Combinations

- **Galton Board experiment**
  Q: The same results of the bell-shape curve with either equal prob to bumped to the left or right or even unequal prob?

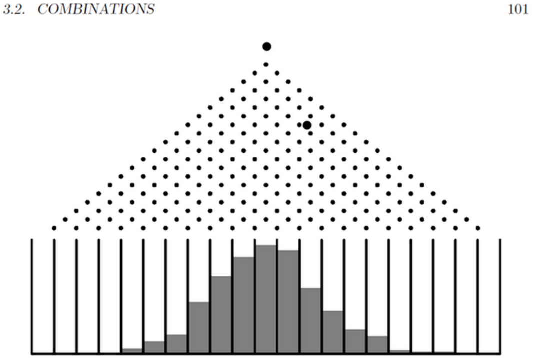

Figure 3.6: Simulation of the Galton board.

- **Theorem 3.7 Binomial Theorem**
  The quantity $(a+b)^n$ can be expressed in the form

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$$

Corollary 3.1 The sum of the elements in the nth row of Pascal's

triangle is 2^n. If the elements in the nth row are added with alternating signs, the sum is 0.
Proof In the above expansion, let $a=1, b=1$, we have the results in the first half. Let $a=-1, b=1$, we have the results in the second half.

Binomial Expansion

- **Inclusion-Exclusion Principle**
  Theorem 3.8 Let P be a prob distribution on a sample space Ω, and let $\{A_1, A_2, ..., A_n\}$ be a finite set of events. Then

$$P(A_1 \cup A_2 \cup ... \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j)$$

$$+ \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) - ...$$

  The proof in the textbook is very interesting. Read it at home.
- Card Shuffling
  Interesting stuff! Enjoy reading it at home.
  An interesting paper is cited in this section by Bayer and Diaconis (1992). The paper is posted in the canvas. The last reference cited in the paper is an article published in “The Magician Monthly”.

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