Digital Image Processing

COSC 6380/4393

Lecture – 13

Oct. 3rd, 2023

Pranav Mantini

Slides from Dr. Shishir K Shah and Frank (Qingzhong) Liu, S. Narasimhan

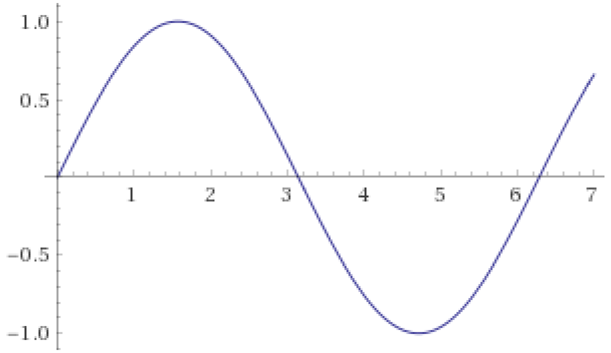
Discrete Fourier Transform (DFT)

Frequency domain analysis and Fourier Transform

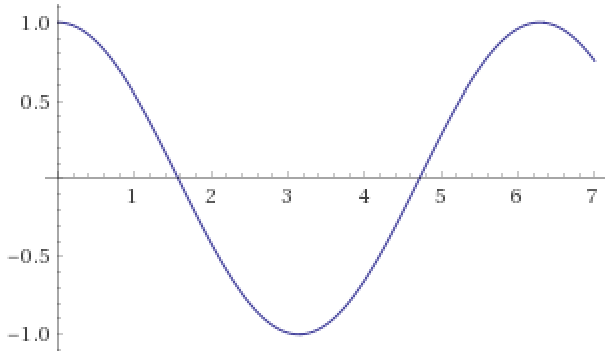
- Periodic Function

Recap: Sin and Cos

sin (t)

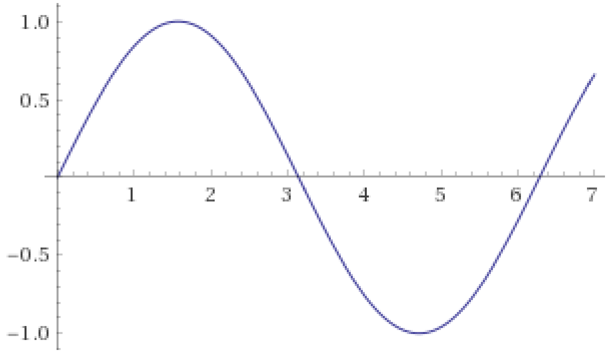


cos (t)



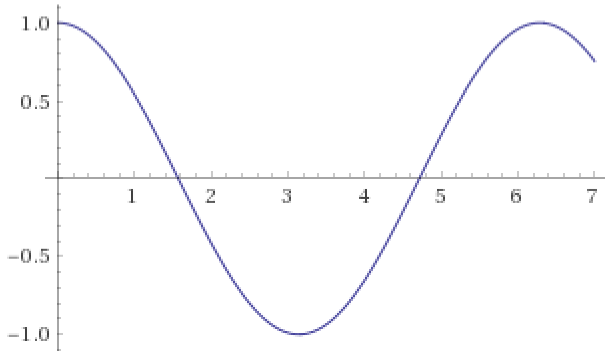
Recap: Sin and Cos

sin (t)



Period = 2π

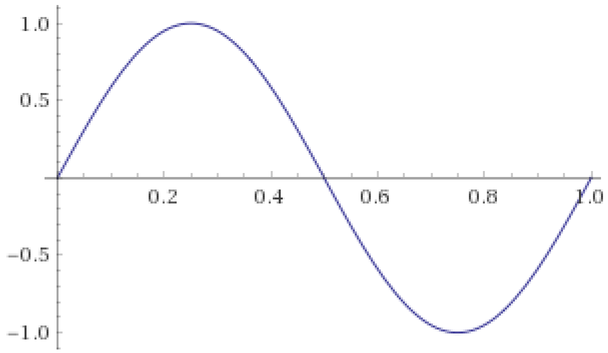
cos (t)



Period = 2π

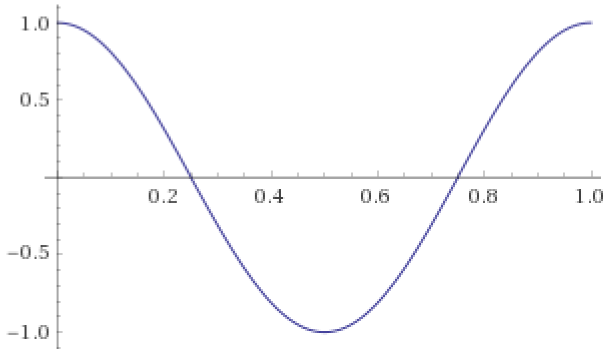
Recap: Sin and Cos

sin (?)



Period = 1

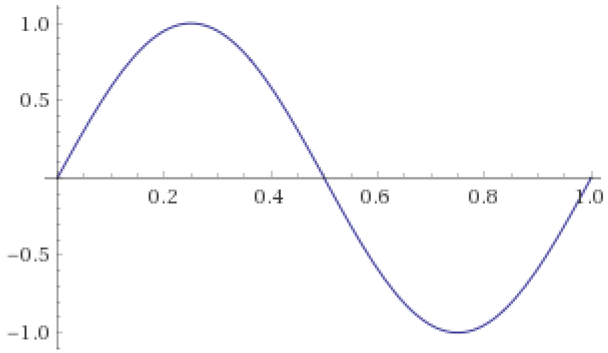
cos (?)



Period = 1

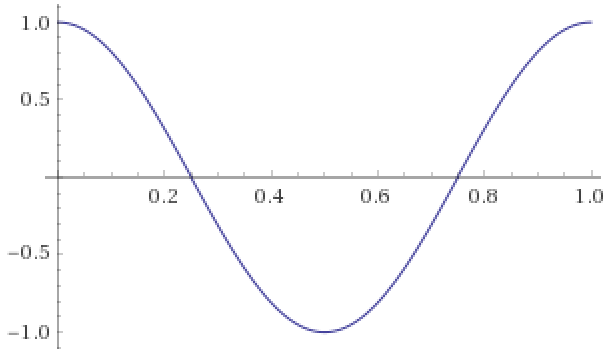
Recap: Sin and Cos

$\sin\,(\mathbf{2}\boldsymbol{\pi}\; t)$



Period = 1

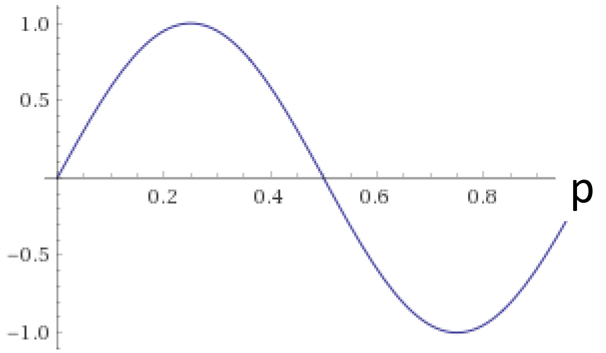
$\cos\,(\mathbf{2}\boldsymbol{\pi}\; t)$



Period = 1

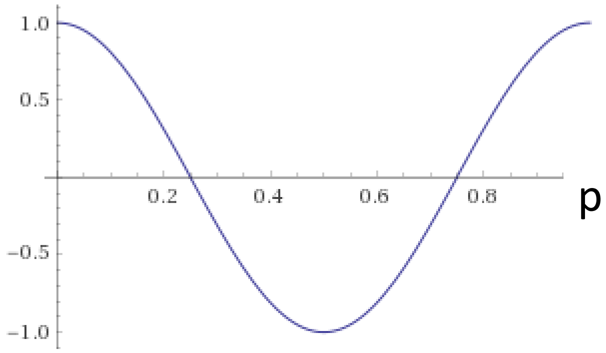
Recap: Sin and Cos

sin (?)



Period = p

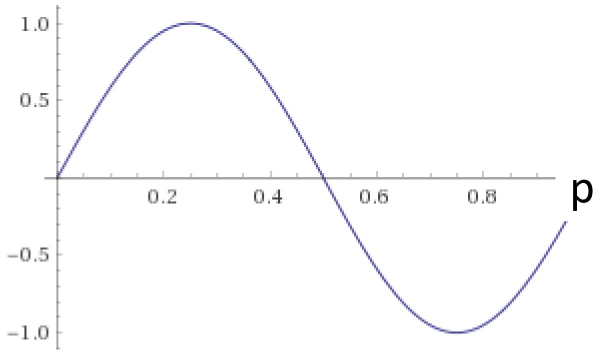
cos (?)



Period = p

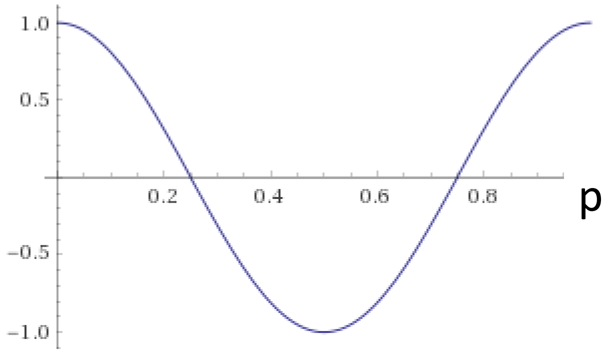
Recap: Sin and Cos

$\sin\left(2\pi \left(\frac{1}{p}\right) t\right)$



Period = p

$\cos\left(2\pi \left(\frac{1}{p}\right) t\right)$



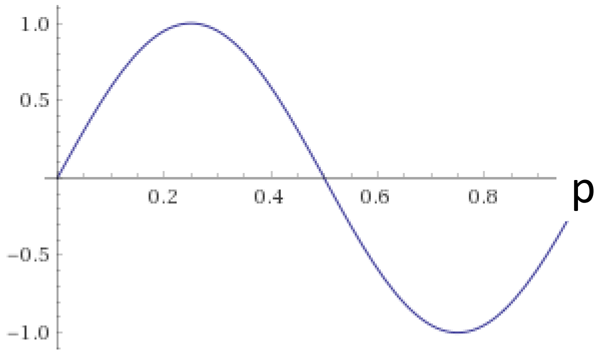
Period = p

Recap: Sin and Cos

Frequency (f) = Number of times it repeats in unit time.

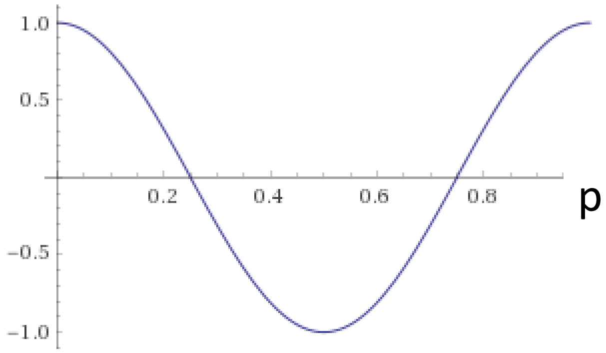
→ fp = 1

$\sin\left(2\pi\left(\frac{1}{p}\right)t\right)$



Period = p

$\cos\left(2\pi\left(\frac{1}{p}\right)t\right)$



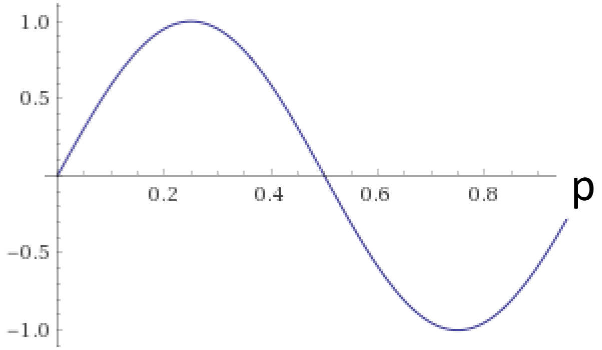
Period = p

Recap: Sin and Cos

Frequency (f) = Number of times it repeats in unit time.

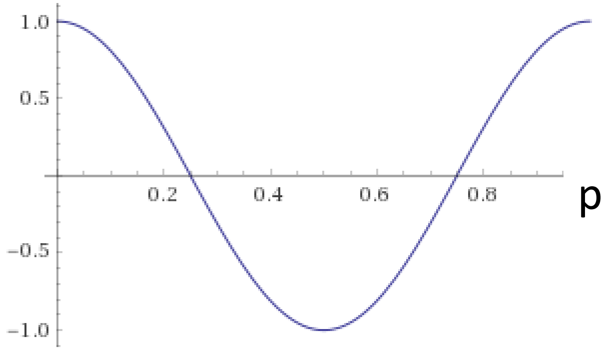
→ f = 1/p

$\sin(2\pi \boldsymbol{f}\, t)$



Period = p

$\cos(2\pi \boldsymbol{f} t)$

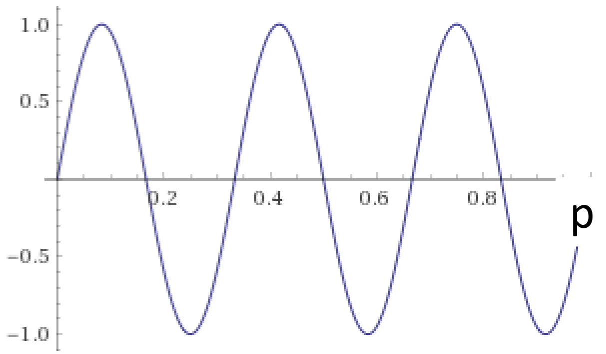


Period = p

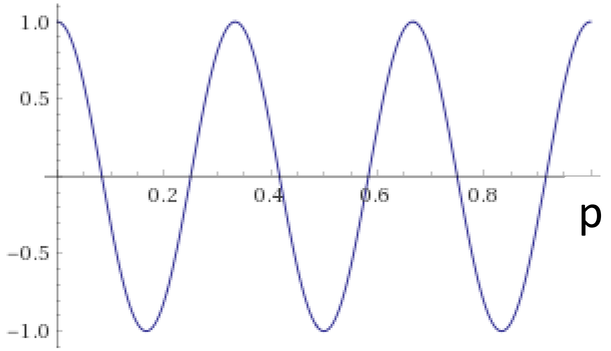
Recap: Sin and Cos

f = 1/p

sin (?)



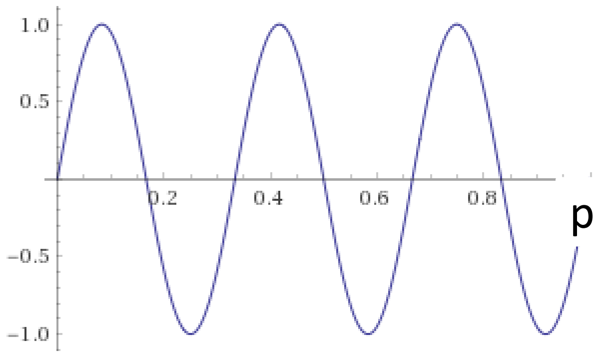
cos (?)



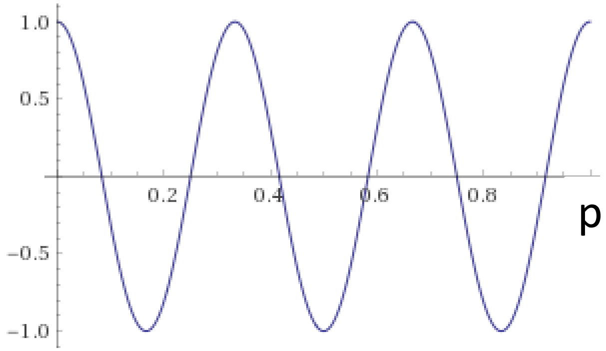
Recap: Sin and Cos

f = 1/p

$\sin(2\pi\, \mathbf{3} f\ t)$

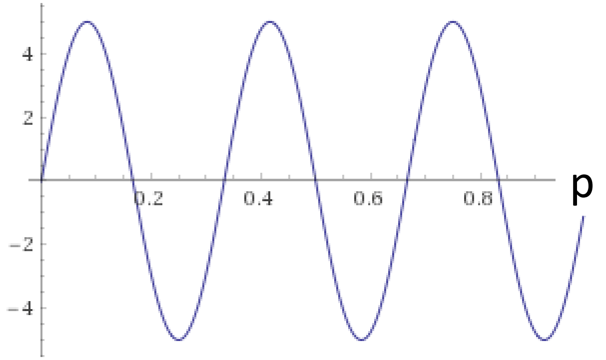


$\cos(2\pi\, \mathbf{3} f\ t)$

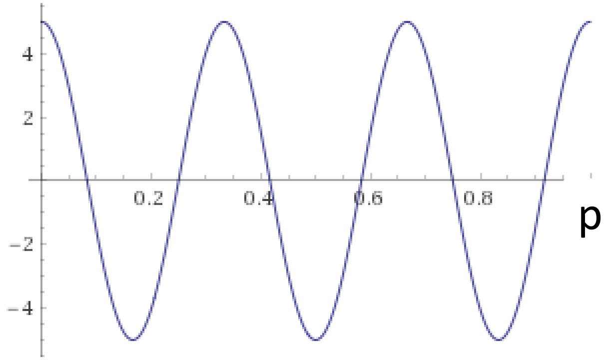


Recap: Sin and Cos

sin (?)



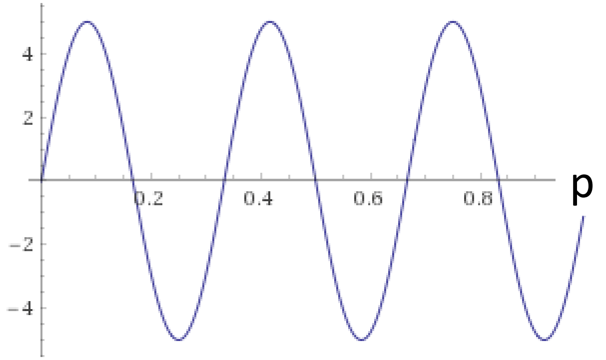
cos (?)



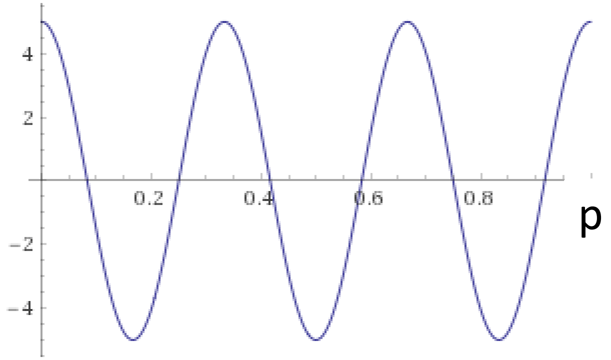
Recap: Sin and Cos

Amplitude = 5

$\mathbf{5}\sin(2\pi\, 3f\, t)$



$\mathbf{5}\cos(2\pi\, 3f\, t)$



Recap: Sin and Cos

Amplitude

$a\,\sin\,(2\pi\ k\ t)$ $\qquad$ $\mathrm{b}\cos\,(2\pi\ l\ t)$

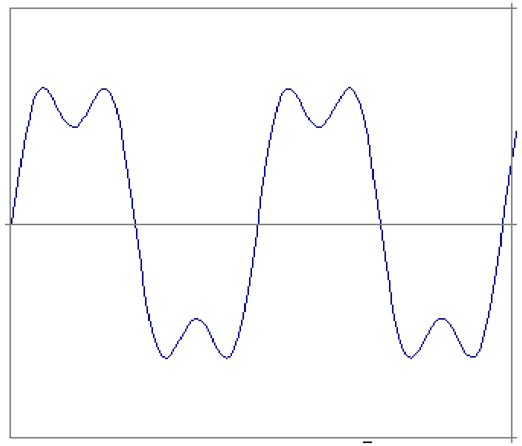
Frequency

Jean Baptiste Joseph Fourier (1768-1830)

- Had crazy idea (1807): **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

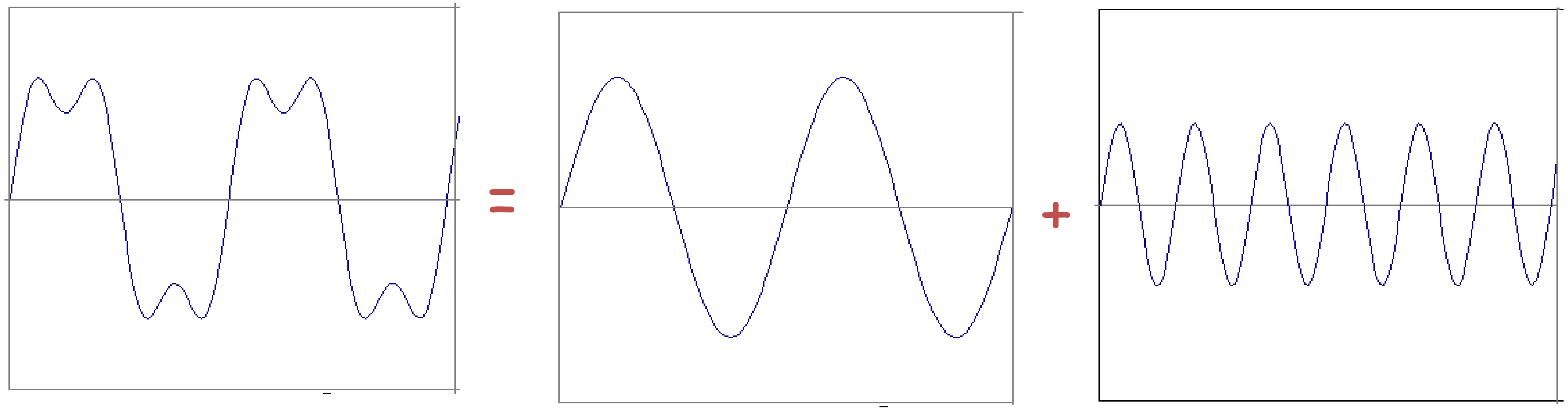
Time and Frequency

- example : $g(t)$



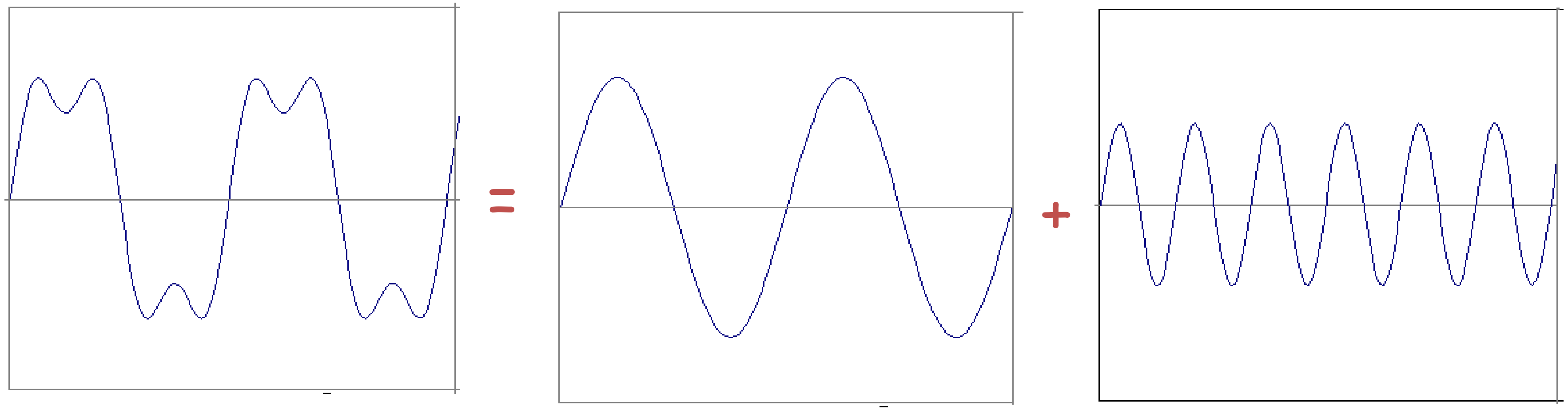
Time and Frequency

- example : $g(t)$



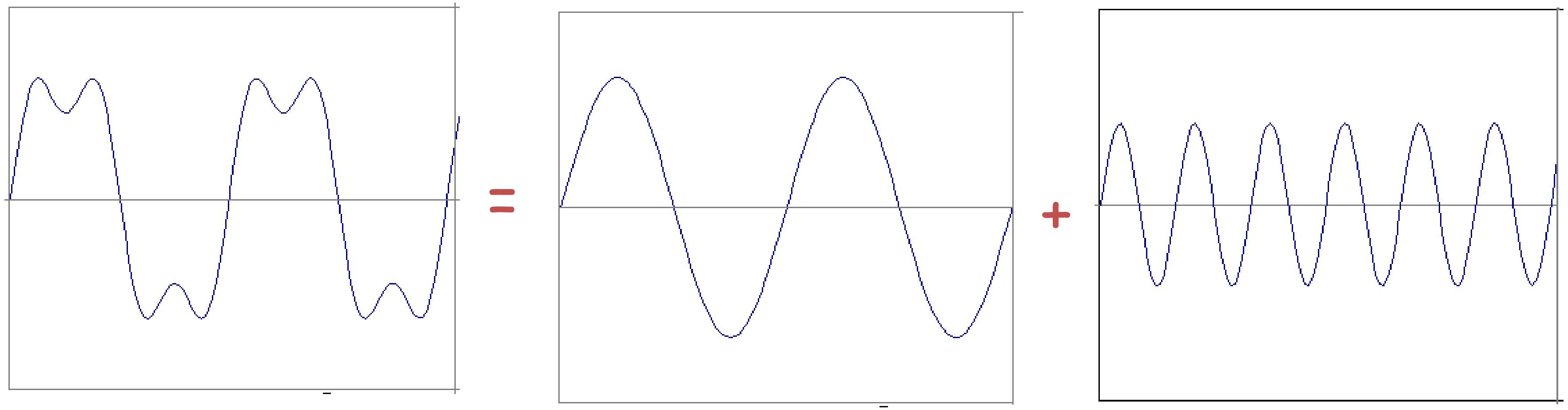
Time and Frequency

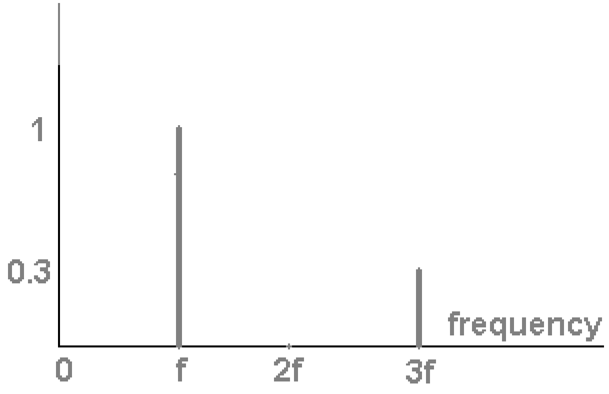
- example : $g(t) = \sin(2pi\ f\ t) + (1/3)\sin(2pi\ (3f)\ t)$



Frequency Spectra

- example : $g(t) = \sin(2pi\, f\, t) + (1/3)\sin(2pi\, (3f)\, t)$





Periodic Function

Sum of sine and cosine waves:

$f(t)$

Periodic Function

Sum of sine and cosine waves:

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
$$b_1 \sin(t) + b_2 \sin(2t) + \cdots$$

Recap

$\int_0^{2\pi} \sin(mt)\, dt = ?$

$\int_0^{2\pi} cos(mt)\, dt = ?$

$$\int_0^{2\pi} \sin(mt)\, cos(nt)\, dt = ?$$

- $\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = ?$ $(\forall m\, != n)$
- $\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = ?$ $(m = n)$
- $\int_0^{2\pi} \cos(mt)\, cos(nt)\, dt = ?$ $(\forall m\, != n)$
- $\int_0^{2\pi} \cos(mt)\, cos(nt)\, dt = ?$ $(m = n)$

Recap

$\int_0^{2\pi} \sin(mt)\, dt = 0$

$\int_0^{2\pi} cos(mt)\, dt = 0$

$$\int_0^{2\pi} \sin(mt)\, cos(nt)\, dt = 0$$

- $\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = ?$ $(\forall m\, != n)$
- $\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = ?$ $(m = n)$
- $\int_0^{2\pi} \cos(mt)\, cos(nt)\, dt = ?$ $(\forall m\, != n)$
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Recap

$\int_0^{2\pi} \sin(mt)\, dt = 0$

$\int_0^{2\pi} cos(mt)\, dt = 0$

$$\int_0^{2\pi} \sin(mt)\, cos(nt)\, dt = 0$$

Product Identities

$$\sin(x)\cos(y) = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$\cos(x)\sin(y) = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$$

$$\cos(x)\cos(y) = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$

$$\sin(x)\sin(y) = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

- $\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = ?$
  $(\forall m\, != n)$
- $\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = ?\ (m = n)$
- $\int_0^{2\pi} \cos(mt)\, cos(nt)\, dt = ?$
  $(\forall m\, != n)$
- $\int_0^{2\pi} \cos(mt)\, cos(nt)\, dt = ?\ (m = n)$

Recap

$$\int_0^{2\pi} \sin(mt)\, dt = 0$$

$$\int_0^{2\pi} cos(mt)\, dt = 0$$

$$\int_0^{2\pi} \sin(mt)\, cos(nt)\, dt = 0$$

$$\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = 0$$

$$(\forall m\, != n)$$

$$\int_0^{2\pi} \sin(mt)\, sin(nt)\, dt = \pi\ (m = n)$$

$$\int_0^{2\pi} \cos(mt)\, cos(nt)\, dt = 0$$

$$(\forall m\, != n)$$

$$\int_0^{2\pi} \cos(mt)\, cos(nt)\, dt = \pi(m = n)$$

Periodic Function

Sum of sine and cosine waves:

$$f(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots$$
$$b_1 \sin(t) + b_2 \sin(2t) + \cdots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)dt = \int_0^{2\pi} a_0 dt \quad + \cancel{\int_0^{2\pi} a_1 \cos(t)\, dt} + \cancel{\int_0^{2\pi} a_2 \cos(2t)\, dt} + \cdots$$

$$\cancel{\int_0^{2\pi} b_1 \sin(t) dt} \quad + \cancel{\int_0^{2\pi} b_2 \sin(2t) dt} \quad + \cdots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)dt = \int_0^{2\pi} a_0 dt = a_0(2\pi)$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

Periodic Function

Sum of sine and cosine waves:

$$f(t)\cos(nt)$$
$$= a_0\cos(nt) + a_1\cos(t)\cos(nt) + a_2\cos(2t)\cos(nt) + \cdots$$
$$b_1\sin(t)\cos(nt) + b_2\sin(2t)\cos(nt) + \cdots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$= \int_0^{2\pi} a_0\cos(nt)\,dt + \int_0^{2\pi} a_1\cos(t)\cos(nt)\,dt + \int_0^{2\pi} a_2\cos(2t)\cos(nt)\,dt + \cdots$$

$$\int_0^{2\pi} b_1\sin(t)\cos(nt)\,dt + \int_0^{2\pi} b_2\sin(2t)\cos(nt)\,dt + \cdots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$= \int_0^{2\pi} a_0\cos(nt)\,dt + \int_0^{2\pi} a_1\cos(t)\cos(nt)\,dt$$

$$+ \int_0^{2\pi} a_2\cos(2t)\cos(nt)\,dt + \cdots \int_0^{2\pi} a_n\cos(nt)\cos(nt)\,dt + \cdots$$

$$\int_0^{2\pi} b_1\sin(t)\cos(nt)\,dt + \int_0^{2\pi} b_2\sin(2t)\cos(nt)\,dt + \cdots$$

Periodic Function

Sum of sine and cosine waves:

$$\int_0^{2\pi} f(t)\cos(nt)\,dt = \int_0^{2\pi} a_n \cos(nt)\cos(nt)\,dt$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$Similarly, b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$

Periodic Function

Sum of sine and cosine waves:

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$