MATH 3338 Probability

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Test 1 Review

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Date of Test and Covered Chapters

- Dates: Feb 22, 2024
- Chapters: 1, 2, 3 and 4 and part of Ch 5 on discrete distributions

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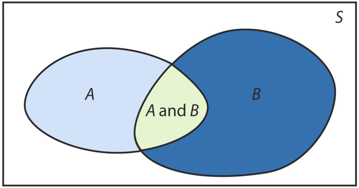
- Probability function, sample space, set, etc.
  Given $A \subset \Omega$, $P(A) = \sum_{\omega \in A} m(\omega)$.

- Operation of sets.
  A, B are sets in sample space Ω.
  $A \subset B$ $\quad$ $A \cup B$
  $A \cap B$ $\quad$ $\overline{A}$

- Venn diagram



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Chapter 2

- Properties of probability
  1. $P(E) \geq 0$ for every $E \subset \Omega$.
  2. $P(\Omega) = 1$.
  3. If $E \subset F \subset \Omega$, then $P(E) \leq P(F)$.
  4. If A and B are disjoint subsets of Ω, then $P(A \cup B) = P(A) + P(B)$.
  5. $P(\bar{A}) = 1 - P(A)$ for every $A \subset \Omega$.

- **Theorem 1.2** If $A_1, ..., A_n$ are pairwise disjoint subsets of Ω (i.e. no two of the A_i's have an element in common), Then

$$P(A_1 \cup ... \cup A_n) = \sum_{i=1}^{n} P(A_i).$$

  Theorem 1.3 Let $A_1, ..., A_n$ be disjoint events with $\Omega = A_1 \cup ... \cup A_n$, and let E be any event. Then

$$P(E) = \sum_{i=1}^{n} P(E \cap A_i)$$

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- **Corollary 1.1** For any two events A and B,

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

Theorem 1.4 If A and B are two subsets of Ω, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Chapter 2

- Continuous density function

- Prob density function of continuous RVs.
  A density function for X is a real-valued function f which satisfies
  $$P(a \leq X \leq b) = \int_a^b f(x)dx$$
  for all $a, b \in \mathbb{R}$. The Probability is equal to the area under the density function.

- Uniform Density
  Other forms of density function, e.g. $f(x) = 2x$ on $[0, 1]$.

- The cumulative distribution function for a random variable X is defined as $F(x) = \int_{-\infty}^{x} f(x)dx$ for any given $x \in \mathbb{R}$
  The CDF $F(x)$ is an increasing and right continuous function.
  $$0 \leq F(x) \leq 1, \text{ and } \frac{d}{dx}F(x) = f(x).$$

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- **Counting Problems** When a task requires a few steps (r) to finish, the first step has n_1 ways to complete, the second step has n_2 ways to complete, ..., the r-th step has n_r ways to complete,the total number of ways to complete the task is $N = n_1 ... n_r$.
- Permutation number, when the order of selected matters $P_r^n = n(n-1)...(n-r+1)$
- Combination number, when the order of selected does not matter.

$$C_r^n = \binom{n}{r} = n(n-1)...(n-r+1)/r!$$

- Properties.
  1) $P_r^n = C_r^n r! > C_r^n$
  2) Adding all binomial coefficients together we have

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3.$$

$$\binom{n}{0} = \binom{n}{n} = 1; \ \binom{n}{1} = \binom{n}{n-1} = n$$

- **Theorem 3.4**
  For integers n and j, with $0 < j < n$, the binomial coefficients satisfy:
  $$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$
- **Theorem 3.5**
  For integers n and j, with $0 < j < n$, the binomial coefficients satisfy:
  $$\binom{n}{j} = \frac{P_j^n}{j!} = \frac{n!}{j!(n-j)!};$$
  It is easy to see that
  $$\binom{n}{j} = \binom{n}{n-j}$$

- Bernoulli Trials. Binary outcome, success (1) or failure (0) with Prob $P(X=1)=p$, and $P(X=0)=1-p=q$.
- Binomial prob of having exact k successes out of n trials is

$$P(X=k)=b(n,p,k)=\binom{n}{k}p^k(1-p)^{n-k}, \quad k=0,1,...,n$$

A sum of n indep Bernoulli trials.

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- **Conditional Probability** $P(F|E)$

$$P(F|E) = \frac{P(F \cap E)}{P(E)}$$

- Independent events E, F.

$$P(F|E) = P(F); \quad P(E|F) = P(E); \quad P(F \cap E) = P(F)P(E)$$

- Mutually independent events $A_1, A_2, ..., A_n$ if for any subsets $\{A_i, A_j, ..., A_m\}$ of these events,

$$P(A_i \cap A_j \cap ... \cap A_m) = P(A_i)P(A_j)...P(A_m),$$

or equivalently, their complement events $\overline{A_i}$, ..., satisfy the above equation.

Joint distributions

1. Joint discrete RVs are also provided with a table or tables that display the prob of each joint coordinates (x, y). How to use the table to explain or demonstrate the independence ?
   Mutually independent The random variables $X_1, X_2, ..., X_n$ are mutually independent if

   $$P(X_1 = r_1, X_2 = r_2, ..., X_n = r_n)$$

   $$= P(X_1 = r_1)P(X_2 = r_2)...P(X_n = r_n)$$

   for any choice of $r_1, r_2, ..., r_n$. Thus if $X_1, X_2, ..., X_n$ are mutually independent, then the joint distribution function of the random variable $X = (X_1, , X_2, ..., X_n)$ is just the product of individual distribution functions.
2. Marginal distributions. Take the prob of event on only 1 RV, and ignore the others.

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Joint distribution

- **Bayes Probabilities** Suppose a set of events $H_1, H_2, ..., H_m$ that are pairwise disjoint and exhaustive (mutually exclusive and exhaustive):

$$\Omega = H_1 \cup H_2 \cup ... \cup H_m; \quad H_i \cap H_j = \emptyset \; \forall i \neq j$$

Bayes' Formula

$$P(H_i|E) = \frac{P(H_i)P(E|H_i)}{\sum_{j=1}^{m} P(H_j)P(E|H_j)}$$

- **Conditional Density Function**

  Normalize the density function by dividing it with prob of condition, $f(x)/P(E)$.

$$f(x|E) = \begin{cases} f(x)/P(E), & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

  For joint distribution of RVs $X_1, ..., X_n$, the CDF is

$$F(x_1, x_2, .., x_n) = P(X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n).$$

  The joint density function of X satisfies the following equation

$$F(x_1, ..., x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(t_1, t_2, ..., t_n) dt_n dt_{n-1} ... dt_1.$$

  It can be shown that

$$f(x_1, x_2, .., x_n) = \frac{\partial^n F(x_1, ..., x_n)}{\partial x_1 \partial x_2 ... \partial x_n}.$$

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- **Definition 4.7** Let $X_1, ..., X_n$ be continuous rvs with CDFs $F_1(x_1), ..., F_n(x_n)$. Then these RVs are mutually indep if

$$F(x_1, x_2, .., x_n) = F_1(x_1)F_2(x_2)...F_n(x_n)$$

for any choice of $x_1, ..., x_n$.

- If RVs are mutually indep, then the CDF of $X = (X_1, ..., X_n)$ is the product of individual CDFs of $X_1, ..., X_n$.

- **Theorem 4.2** Let $X_1, ..., X_n$ be continuous rvs with density functions $f_1(x_1), .., f_n(x_n)$. Then the RVs are mutually indep iff

$$f(x_1, ..., x_n) = f_1(x_1)...f_n(x_n)$$

for all choice of $x_1, ., x_n$.

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- **Theorem 4.3** Let $X_1, ..., X_n$ be mutually indep continuous rvs and let $\phi_1(x), ..., \phi_n(x)$ be continuous functions. Then $\phi_1(X_1), ..., \phi_n(X_n)$ are mutually indep.

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