MATH 3339
Statistics for the Sciences
Sec 8.1-8.5

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Lecture 15 - 3339

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Outline

1. Components of a significance test
2. Decision of the test
3. Errors
4. Assumptions
5. Hypothesis Tests for proportions

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What is the actual mean body temperature?

- We believe the mean body temperature to be 98.6°F. But is the true population mean body temperature really 98.6°F?
- University of Maryland researchers obtained temperatures from 100 healthy adults.
- The following is the characteristics of the sample:
  - The distribution is approximately bell shaped.
  - The sample mean is 98.2°F.
  - The standard deviation is σ = 0.62°F.
  - The sample size is n = 100.

What is the actual mean body temperature?

- It is commonly believed that the mean body temperature is 98.6°F, but the researchers in Maryland suggest that it might be less.
- Could the mean body temperature actually be less than 98.6°F?
- Could the sample mean of $\bar{x} = 98.2^\circ$F be a result of a chance sample fluctuation?
- What is the sampling distribution of $\bar{X}$?

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Hypothesis Test

- To assess the evidence provided by data about some claim concerning a population.
- The reasoning is based on what would happen if we repeated the sample or experiment many times.
- The test of significance answers the question: ”Is the observed effect due to chance?”

Components of a significance test

- Null and alternative hypothesis
- Rejection region
- Test Statistic
- P-value
- Decision of test
- Conclusion in context of the test

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Null Hypothesis of significant tests

- The null hypothesis is the statement that is assumed to be true. We assume "no effect" or "no difference" for the parameter tested.
- Abbreviate the null hypothesis by H_0.
- From mean body temperature example, $H_0 : \mu = 98.6°$F.
- For a significant test of the mean, the null hypothesis is always equal to some value of what we assume the mean to be.
- The null hypothesis is always $H_0 : \mu = \mu_0$, where μ_0 is some value that is assumed to be the true mean.

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Alternative Hypothesis of Significance Tests

- The alternative hypothesis is the statement we hope or suspect is true instead of the null hypothesis.
- Abbreviate the alternative hypothesis by H_a.
- From the mean body temperature example, $H_a : \mu < 98.6^\circ$F.
- The test of significance is designed to assess the strength of the evidence **against** the null hypothesis.

Possible values for the Alternative Hypothesis

There are three possible ways that we would want to test against the null hypothesis.

1. Test to prove that the mean is really lower than what is assumed. This is called a **left-tailed test**. $H_a : \mu < \mu_0$

2. Test to prove that the mean is greater than what is assumed. This is called a **right-tailed test**. $H_a : \mu > \mu_0$

3. Test to prove that the mean is not equal (either higher or lower) than what is assumed. This is called a **two-tailed test.** $H_a : \mu \neq \mu_0$

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You try

1. Last year, a company's service technicians took an average of 2.6 hours to respond to trouble calls from business customers who had purchased service contracts. Do this year's data show a different average response time?

   a) $H_0 : \mu = 2.6$ and $H_a : \mu \neq 2.6$
   b) $H_0 : \mu = 2.6$ and $H_a : \mu > 2.6$
   c) $H_0 : \mu = 2.6$ and $H_a : \mu < 2.6$
   d) $H_0 : \mu = 2.6$ and $H_a : \mu = 0$

2. The manager of an automobile dealership is considering a new bonus plan designed to increase sales volume. Currently, the mean sales volume is 14 automobiles per month. The manager wants to conduct a research study to see whether the new bonus plan increases sales volume. To collect data on the plan, a sample of sales personnel will be allowed to sell under the new bonus plan for a one-month period.

   a) $H_0 : \mu = 14$ and $H_a : \mu \neq 14$
   b) $H_0 : \mu = 14$ and $H_a : \mu > 14$
   c) $H_0 : \mu = 14$ and $H_a : \mu < 14$
   d) $H_0 : \mu = 14$ and $H_a : \mu = 0$

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Decision

- Since there are only two hypotheses, there are only two possible decisions.
- **Reject** the null hypothesis in favor of the alternative hypothesis. (RH0)
- **Fail to** reject the null hypothesis. (FTRH0)
- We will **never** say that we accept the null hypothesis.

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Analogy of the decision

- Suppose a person is on trial for murder. In the U.S. court of law what is the assumption?
- Assumption - Innocent (Null hypothesis)
- Need to prove - Guilty (Alternative hypothesis)
- Jury's choices for a decision - Guilty or not guilty (Conclusion)

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Analogy of the decision

Can the jury make a wrong decision?

Jury's Decision	Correct Condition	
	Person is Innocent	Person is Guilty
Guilty	Error	Correct
Not guilty	Correct	Error

Analogy of the decision

According to the person on trial which error would be worse to get?

Jury's Decision	Correct Condition	
	Person is Innocent	Person is Guilty
Guilty	Error	Correct
Not guilty	Correct	Error

Analogy of the decision

- According to the person on trial which error would be worse to get?
- This error is called the type I error: Rejecting the null hypothesis when in fact it is true.
- Since this is the worst conclusion we try to control for this error.
- P(Type I error) = α the level of significance.
- By predetermining α usually 0.05, we are saying that we make this type I error only 5% of the time.

Errors in our decision

In the same way we can make an error in our decisions.

Our Decision	Correct Condition	
	H_0 is true	H_0 is false
Reject H_0	Type I Error	Correct
Fail to reject H_0	Correct	Type II Error

Thus by determining α, the level of significance, we try to control for the Type I error.

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Type I and Type II Errors

Type I error only occurs when we reject the null hypothesis.

- If we reject H_0 when it is true this is a **Type I error**.
- If we do not reject H_0 when it is false, this is a **Type II error.**
- In the example of the mean body temperature:
  - Type I error: We would conclude that the mean body temperature is less than 98.6 degrees, when in fact it truly is 98.6.
  - Type II error: We would conclude that the mean body temperature is not less than 98.6 degrees, when in fact the mean body temperature is less than 98.6.

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Example of Type I and Type II Errors

You are considering whether or not to play the lottery with your favorite numbers. What situations denote a type I and II errors?

	H_0 is true (Won't win)	H_a is true (Would win)
Reject H_0 (buy a ticket)	type I error	correct
Fail to reject H_0 (don't buy)	correct	type II error

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You try

A can of Pepsi is supposed to have a mean volume of 12 ounces. Both overfilling and under-filling are undesirable. If either occurs, the machine that fills the cans has to be readjusted.

3. The bottling company wants to set up a hypothesis test so that the machine has to be readjusted if the null hypothesis is rejected. Set up the null and alternative hypothesis for this test.

   a) $H_0 : \mu = 12$ and $H_a : \mu \neq 12$
   b) $H_0 : \mu = 12$ and $H_a : \mu < 12$
   c) $H_0 : \bar{x} = 12$ and $H_a : \bar{x} \neq 12$
   d) $H_0 : \mu = 12$ and $H_a : \mu > 12$

4. Once completing the test, they readjusted the machine but they did not have to. This is an example of:

   a) Type I error
   b) Type II error
   c) Type III error
   d) correct decision

Test Statistic

- A value calculated based on sample data and the type of distribution.
- Test Statistic is used to measure the difference between the data and what is expected on the null hypothesis.
- For a hypothesis about the population mean if σ is known.

$$z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

- For a hypothesis about the population mean if σ is not known.

$$t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}}$$

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How to Make the Decision

How to Make the Decision for a hypothesis test

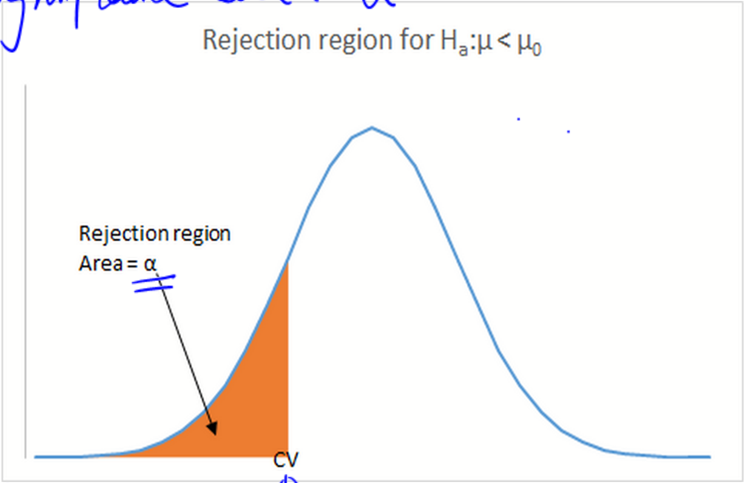
Rejection Region

Rejection region vs. Non-rejection region.
(R.R)

- A **rejection region** is the set of value for which the test statistic least to a rejection of the null hypothesis.
- The critical value is the boundary of the rejection region, based on the alternative hypothesis and the level of significance, α.

Rejection Region for $H_a : \mu < \mu_0$ left tailed test

significance level: α



if z, $\alpha = 1\%$. $CV =$ qnorm$(0.01, 0, 1)$

if t, $\alpha = 1\%$, n, $CV =$ qt$(0.01, n-1)$

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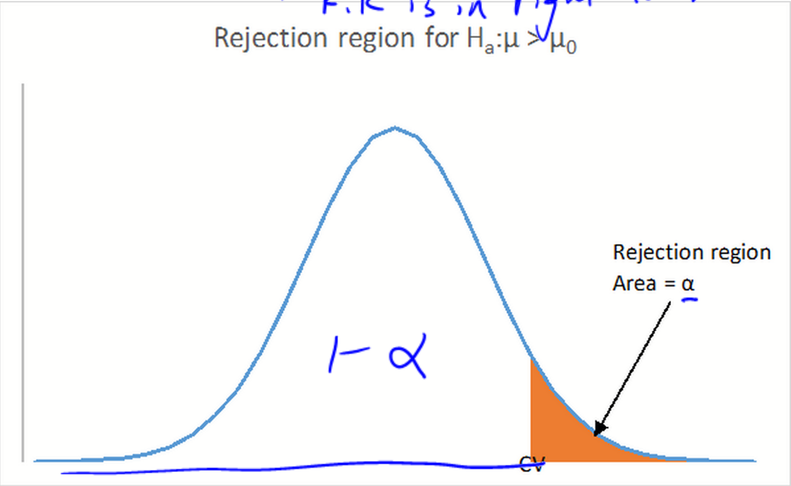
Rejection Region for $H_a : \mu > \mu_0$ right-tailed test

$\Rightarrow$ R.R is in right tail.

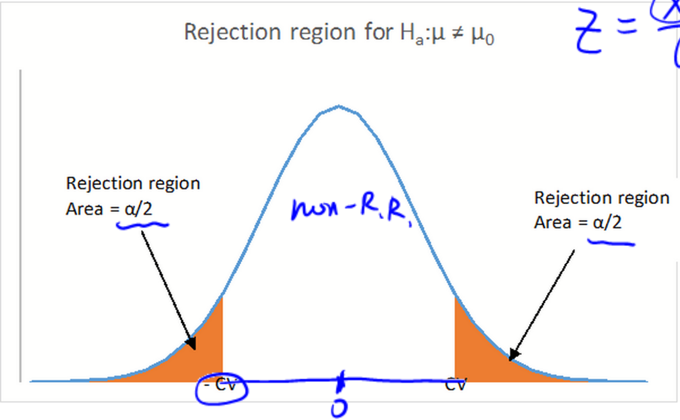


if z, α, CV= qnorm(1-α, 0, 1)

if t, n, α, CV= qt(1-α, n-1)

Rejection Region for $H_a : \mu \neq \mu_0$ two tailed test

$$z = \frac{\bar{x} - \mu}{(\sigma/\sqrt{n})}$$



if z, α, $CV(s) = \pm\, qnorm(\frac{\alpha}{2}, 0, 1)$

if t, n, α, $CV(s) = \pm\, qt(\frac{\alpha}{2}, n-1)$

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Decision of the test using P-values

- One approach is to announce how much evidence **against** H_0 we will require to reject H_0. We compare the P-value with a level that says "this evidence is strong enough." This decisive level is called the **significance level** denoted by α. This significance level is given. If not we will assume $\alpha = 0.05$. When we compare the P-value to α, we have to choose from these two decisions.
- **Reject** H_0 if the P-value is as small or smaller than α. Thus we say that the data are statistically significant at level α.
- **Do not reject** H_0 if the P-value is larger than α.

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P-value

- The probability (assuming that H_0 is true) that the test statistic would take a value as extreme or more extreme (in the way of H_a) than that actually observed.
- The smaller the P-value, the stronger the evidence **against** H_0 provided by the data.
- To calculate the P-value for the mean we will use the sampling distribution of the means which by the central limit theorem is the Normal distribution.
- Hence, this probability is the same as the area under a normal curve depending on the alternative hypothesis.

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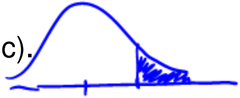
Determining P-values

$z = \frac{\bar{x}-\mu}{(\sigma/\sqrt{n})}$

- If $H_a : \mu < \mu_0$, then P-value = P(Z $<$ test statistic).
- If $H_a : \mu > \mu_0$, then P-value = P(Z $>$ test statistic).



$z = \frac{\bar{x}-\mu}{(\sigma/\sqrt{n})}$

- If $H_a : \mu \neq \mu_0$, then

$$\begin{aligned}\text{P-value} &= P(Z < -\text{test statistic or} Z > +\text{test statistic}) \\ &= P(Z < -\text{test statistic}) + P(Z > +\text{test statistic}) \\ &= 2P(Z > |\text{test statistic}|).\end{aligned}$$

$= 2 * P(Z < -|\text{test-stat}|)$

- If we use the t as the test statistic replace Z with T.

if σ is NOT given, t-dist.

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Assumptions of the Tests

For a *z*-test:

1. An SRS of size *n* from the population.
2. Known population standard deviation, σ.
3. Either a Normal population or a large sample ($n \geq 30$).

For a *t*-test:

1. An SRS of size *n* form the population.
2. Unknown population standard deviation. S
3. Either a Normal population or a large sample ($n \geq 30$).

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Steps of a Significance Test

When performing a significance test, we follow these steps:

1. Check assumptions. z or t?
2. State the null and alternative hypothesis. H_0 "=" H_a: $<$ $>$ $\neq$
3. Graph the rejection region, labeling the critical values.
4. Calculate the test statistic. using sample data
5. Find the *p*-value. If this answer is less than the significance level, α, we can reject the null hypothesis in favor of the alternative hypothesis.
6. Give your conclusion using the context of the problem. When stating the conclusion give results with a confidence of $(1-\alpha)(100)\%$.

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What if we are not given α?

If the P-value for testing H_0 is less than:

- 0.1 we have **some evidence** that H_0 is false.
- 0.05 we have **strong evidence** that H_0 is false.
- 0.01 we have **very strong evidence** that H_0 is false.
- 0.001 we have **extremely strong evidence** that H_0 is false.

If the P-value is greater than 0.1, we **do not have any evidence** that H_0 is false.

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Example of Hypothesis test

We believe the mean body temperature to be 98.6°F. But is the true population mean body temperature really less than 98.6°F? The University of Maryland researchers obtained temperatures from 100 healthy adults. From the sample the mean body temperature was $\bar{x} = 98.2$°F. We assume a population standard deviation of $\sigma = 0.62$°F. Test at the 1% significance level.

① assumption σ is given $\rightarrow z$

② $H_0: \mu = 98.6$

$H_a: \mu < 98.6$

③ R.R ↓ left-tailed test

$\alpha = 0.01$

CV = -2.33 = qnorm(0.01, 0, 1) = -2.33

④ test statistics

$$z = \frac{(\bar{x} - \mu_0)}{(\sigma/\sqrt{n})} = \frac{(98.2 - 98.6)}{\left(\frac{0.62}{\sqrt{100}}\right)} = -6.45$$

⑤ Decision

-6.45 is in R.R

$\Rightarrow$ Reject H_0

p-value = P(z < test statistic)

= P(z < -6.45)

= pnorm(-6.45, 0, 1) ≈ 0

$P\text{-value} < 0.00001$

$\Rightarrow$ extremely strong evidence to reject H_0.

⑥ Conclusion.

We have ex. strong evidence that $\underline{\text{xxxxx}}$.

$\downarrow$

H_a. is <u>true</u>.

Example

σ is given

Mr. Murphy is an avid golfer. Suppose he has been using the same golf clubs for quite some time. Based on this experience, he knows that his average distance when hitting a ball with his current driver (the longest-hitting club) under ideal conditions is 200 yards with a standard deviation of 9. After some preliminary swings with a new driver, he obtained the following sample of driving distances:

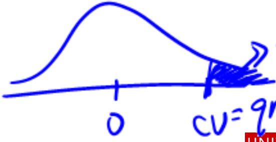
205	198	220	210	194	201	213	191	211	203

He feels that the new club does a better job. Do you agree?

z or t? $H_0: \mu = 200$ $H_a: \mu > 200$.

R.R: $\alpha = 5\%$ $\bar{X} = \text{mean}(\quad)$

$$z = \frac{\bar{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$$



$\alpha = 0.05$

CV = qnorm(0.95,0,1)

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Mean Textbook Costs

```
> qt(.005,74)
[1] -2.643913
```

An association of college bookstores reported that the average amount of money spent by students on textbooks for the Fall 2010 semester was $325.16. A random sample of 75 students at the local campus of the state university indicated an average bill for textbooks for the semester in question to be $312.34 with a standard deviation of $76.42. Do these data provide significant evidence that the actual average bill is different from the $325.16 reported? Test at the 1% significance level.

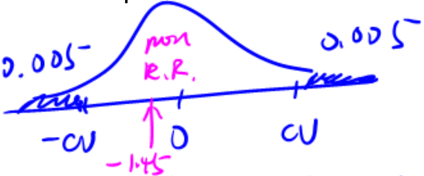
Z or t ?

$H_0: \mu = 325.16$

$H_a: \mu \neq 325.16$

$\alpha = 1\%$

$\frac{\alpha}{2} = 0.005$



$\pm CV = \pm qt(0.005, 75-1)$

Reject region: reject H_0 if $t > 2.64$ or $t < -2.64$

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$$t = \frac{\bar{x} - \mu}{(s/\sqrt{n})} = \frac{(312.34 - 325.16)}{(76.42/\sqrt{75})} = -1.45$$

since -1.45 is not in R.R.,

we fail to reject H_0

The data does not provide sufficient evidence to reject the H_0.

$$\text{P-value} = 2 * P(t < -|\text{test stat}|)$$
$$= 2 * P(t < -1.45)$$
$$= 2 * pt(-1.45, 74) = 0.1513 > \alpha.$$

P-value is not smaller than α, so we cannot reject H_0,

Mean Amount of Coffee Dispensed

A coffee machine dispenses coffee into paper cups. Here are the amounts measured in a random sample of 20 cups.

9.9,	9.7,	10.0,	10.1,	9.9,	9.6,	9.8,	9.8,	10.0,	9.5,
9.7,	10.1,	9.9,	9.6,	10.2,	9.8,	10.0,	9.9,	9.5,	9.9

The machine is supposed to dispense a mean of 10 ounces. Is there significant evidence to conclude that the mean is 10 ounces?

$H_0: \mu = 10$

$H_a: \mu \neq 10$,

z or t?

σ is NOT given

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Hypothesis Tests for proportions

Inference for a Population Proportion

- For these inferences, p_0 represents the given population proportion and the hypothesis will be
  - $H_0 : p = p_0$
  - $H_a : p \neq p_0$ or $p < p_0$ or $p > p_0$

- Conditions:
  1. The sample must be an SRS from the population of interest.
  2. The population must be at least 10 times the size of the sample.
  3. The number of successes and the number of failures must each be at least 10 (both $n\hat{p} \geq 10$ and $n(1 - \hat{p}) \geq 10$).

- Recall, the statistic used for proportions is: $\hat{p} = \frac{\text{\# of successes}}{\text{\# of observations}} = \frac{x}{n}$.

- For tests involving proportions that meet the above conditions, we will use the z-test statistic:

sample prop.

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

hypothesized value H_0

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Are 10% of M&Ms Blue?

A bag of M&Ms was randomly selected from the grocery store shelf, and the color counts were:

Brown	14	Red	14	Yellow	5
Orange	7	Green	6	Blue	10

Test if the proportion of M&Ms that are blue are 10%. Use $\alpha = 0.01$.

$H_0: p = 0.10$

$H_a: p \neq 0.10$

$$\hat{p}_{blue} = \frac{10}{56}$$

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{\frac{10}{56} - 0.10}{\sqrt{\frac{0.10 * (1-0.10)}{56}}} = 1.9599$$

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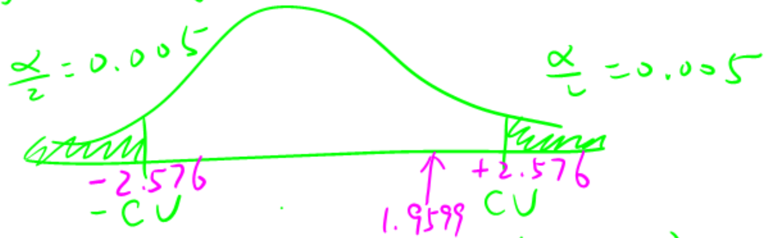
Rejection region $\alpha = 1\% = 0.01$



$$\pm CU = \pm \, qnorm\left(\frac{0.01}{2}, 0, 1\right)$$

$$= \pm 2.576$$

since 1.9599 is in non-rejection region,

we <u>fail to reject</u> H_0.

Conclusion: The data does not provide enough evidence that the prop. of blue M&M is not 10%

p-value = 2*(1-pnorm(test-stat))

= 2*(1-pnorm(1.9599))

= 0.05 $> \alpha$.

$\Rightarrow$ Fail to reject H_0.

```
> 2*(1-pnorm(1.9599))
[1] 0.05000748
```

(if $P \le \alpha$, we can reject H_0)

H_0: $p_0 = 10\%$ $\qquad$ H_a: $p > 10\%$

$\hat{p} = 0.178 \ldots > 10\%$

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = 1.9599 \qquad \alpha = 0.01$$

RR:

$CV = \text{qnorm}(1-0.01) = 2.33$

1.9599

1.9599 is in non-rejection region $\Rightarrow$ F.T.R. H_0.

Conclusion:

The data does <u>not</u> provide enough evidence that the prop. of blue M&M is greater than 10%.

$$\text{Pvalue} = \text{Prob}(Z > 1.9599)$$
$$= 1 - \text{pnorm}(1.9599, 0, 1)$$

Example

A new shampoo is being test-marketed. A large number of 16-ounce bottles were mailed out at random to potential customers in the hope that the customers will return an enclosed questionnaire. Out of the 1,000 returned questionnaires, 575 indicated that they like the shampoo and will consider buying it when it becomes available on the market. Perform a hypothesis test to determine if the proportion of potential customers is more than 50%.

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Example

A new brand of chocolate bar is being market tested. Five hundred of the new chocolate bars were given to consumers to try. The consumers were asked whether they liked or disliked the chocolate bar. The company that produces the new brand of chocolate bars said they will put the chocolate bar on the shelf if more than half of consumers (50%) like this chocolate bar.

1. Give the null and alternative hypothesis.

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Example

A new brand of chocolate bar is being market tested. Five hundred of the new chocolate bars were given to consumers to try. The consumers were asked whether they liked or disliked the chocolate bar. The company that produces the new brand of chocolate bars said they will put the chocolate bar on the shelf if more than half of consumers (50%) like this chocolate bar.

1. From the sample of 500, 265 people liked the candy bar. Test the claim that more than half of consumers will like this candy bar at the level $\alpha = 0.01$.

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Summary of Hypothesis Tests

The following table gives you a step by step approach for the significance tests:

population mean

→ population proportion

Parameter	μ given σ	μ **not** given σ	p proportions
1. Null hypothesis	$H_0 : \mu = \mu_0$		$H_0 : p = p_0$
2. Alternative	Choose either $<$, $>$, or $\neq$ in place of $=$ in H_0.		
3. Rejection Region Depending on H_a.	$z_{\alpha/2}$	$t_{\alpha/2}$ with df = n - 1	$z_{\alpha/2}$
4. Test statistic	$z = \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}$	$t = \frac{\bar{x}-\mu_0}{\frac{s}{\sqrt{n}}}$	$z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
5. P-value	pnorm(z)	pt(t,n-1)	pnorm(z)
This is the area under the density curve shaded according to H_a.			
6. Decision	**Reject** H_0 if P-value $\leq \alpha$ **Fail** to reject H_0 if P-value $> \alpha$		

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