MATH 3339
Statistics for the Sciences
Sec 4.4-4.7

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Lecture 7 - 3339

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Outline

1. Bernoulli Distribution
2. Cumulative Distributions
3. Hypergeometric Distribution

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Special cases for discrete distribution

Special cases for Discrete distribution:

- Bernoulli distribution
- Binomial distribution
- Hypergeometric distribution
- Poisson distribution
- Jointly distributed Variables

The Bernoulli Probability Distribution

Definition: A **Bernoulli** random varilable is a random experiment (a Bernoulli Trial) with the following characteristics:

1. The outcome can be classified as either **success** or **failure** (where these are mutually exclusive and exhaustive).

2. The probability of **success** is **p**, so the probability of **failure** is **q=1-p** . e.g. a coin is flipped (heads or tails), someone is pulled over for speeding (ticket or warning), etc.

Suppose that a coin is flipped. Let X be the random variable that indicates that heads was flipped. Here heads represents “success" and tails represents “failure" so that X is a Bernoulli random variable.

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$E(X) = \sum x P(X=x)$

$= 0 * (1-p)$
$+ 1 * p$
$= p$

(1)

X	0 (tail)	1 (Head)
$P(x)$	$1-p$	p

(2)

$$f(x) = P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1-p, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0, 1 \end{cases}$$

A compact way of writing this is:

(3)

$$f(x) = P(X = x) = p^x(1-p)^{1-x}$$

$f(0) = P(X=0) = p^0(1-p)^{1-0} = 1-p$

$f(1) = P(X=1) = p^1(1-p)^{(1-1)} = p$

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Mean and Variance of a Bernoulli Distribution

If X has the Bernoulli distribution with probability of success p, the **mean** and **variance** of X are

$$\mu_X = E[X] = p$$

$$\sigma_X^2 = Var[X] = p(1-p) \quad \text{proof}$$

Then the standard deviation is the square root of the variance.

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= 1^2 * p + \cancel{0^2(1-p)} - p^2$$
$$= p - p^2 = p(1-p)$$

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The Binomial Probability Distribution

Suppose that we flip a coin 10 times. This is a sequence of Bernoulli trials. We are interested in calculating the probability of obtaining a certain number of heads. Let X_i indicate heads on the i-th flip.

$X_i = \begin{cases} 1 & (\text{head}) \\ 0 & (\text{tail}) \end{cases}$

Define $Y = X_1 + X_2 + ... + X_{10}$. What does Y represent?

total # of "head" in this 10 trials.

What is the probability that $Y = 0$? Y can be 0, 1, 2, ..., 10

$0.5 * 0.5 \cdots * 0.5$

What is the probability that $Y = 1$? $P(Y=1) = C_1^{10}\, 0.5^1 * (1-0.5)^{(10-1)}$

HTTTT....T ⟶ $0.5 * (1-0.5) * (1-0.5) * \cdots$

THTTT....T ⟶ $(1-0.5) * 0.5 * (1-0.5) \cdots$

TTHTT....T ⟶ $(1-0.5) * (1-0.5) * 0.5 * (1-0.5) \cdots$

TTTTTTTTTH

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The Binomial Probability Distribution

Suppose that we flip a coin 10 times. This is a sequence of Bernoulli trials. We are interested in calculating the probability of obtaining a certain number of heads. Let X_i indicate heads on the i-th flip.

Define $Y = X_1 + X_2 + ... + X_{10}$. What does Y represent?

What is the probability that $Y = 2$?

What is the probability that $Y = n$?

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The Binomial Probability Distribution

X	0	$\cdots$	n
$P(X=x)$			

Here Y is the sum of 10 independent Bernoulli trials. We call this type of random variable a **Binomial random variable.**

A random variable X is a Binomial random variable if the following conditions are satisfied:

1. X represents the number of successes on n Bernoulli trials.
2. The probability of success for each trial is p.
3. The trials are mutually independent.

If X is a binomial random variable with probability p of success on each of n trials, we write $X \sim Binomial(n, p)$

X can be 0, 1, ..., n

If $X \sim Binomial(n, p)$, then $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$ where $x = 0, 1, 2, ..., n$

$P(X=x) = C_x^n \, p^x (1-p)^{(n-x)}$

Mean and Variance of a Binomial Distribution

If a count X has the Binomial distribution with number of observations n and probability of success p, the **mean** and **variance** of X are

$\mu_x = \sum x P(X = x)$

$$\begin{aligned} \mu_X &= E[X] = np \\ \sigma_X^2 &= Var[X] = np(1-p) \end{aligned}$$

Then the standard deviation is the square root of the variance.

The Binomial Probability Distribution

R commands:

$$X \sim \text{Binomial}(n, p)$$

$$P(X = x) = \text{dbinom}(x, n, p)$$

$$P(X \le x) = \text{pbinom}(x, n, p)$$

the largest possible value for $X \le x$

$$P(X > x) = 1 - P(X \le x)$$
$$= 1 - \text{pbinom}(x, n, p)$$

The Binomial Probability Distribution

$n = 8$

Example: Suppose that at a 4-way stop in a certain subdivision, only 12% of drivers come to a complete stop. What is the probability that among 8 drivers, at least 6 of them will run the stop sign?

"Success" = "run the stop sign"

$P(\text{success}) = 1 - 0.12 = 0.88$ / X = # of drivers run of stop sign

$P(\text{at least } 6) = P(X \geq 6) = 1 - P(X < 6)$

$= 1 - P(X = 0, 1, 2, 3, 4, 5)$

$= 1 - P(X \leq 5)$

$= 1 - \text{pbinom}(5, 8, 0.88)$

What is the expected number of drivers who will run the stop sign?

$\mu = n \cdot p$

$= 8 * 0.88$

$= \boxed{7.04}$

On ave, we expect 7.04 drivers run the stop sign among 8.

```
> 1-pbinom(5,8,0.88)
[1] 0.9392108
```

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The Binomial Probability Distribution

$n=12$

Suppose $X \sim Binomial(12, 0.3)$, find the following:

$p=0.3$

$P(2 \leq X < 5) =$ P(X can be 2, 3, 4) $= P(X=2) + P(X=3) + P(X=4)$

$P(X=3) = \binom{12}{3} 0.3^3 (1-0.3)^{(12-3)}$

$=$ dbinom(3, 12, 0.3)

~~$P(X$ [illegible] $5) =$~~

$P(2 \le X < 5)$ $\quad X \sim \text{Binom}(12, 0.3)$

$= P(X \text{ can be } 2, 3, 4)$

$= P(X = 2) + P(X = 3) + P(X = 4)$

$= \text{dbinom}(2, 12, 0.3) + \text{dbinom}(3, 12, 0.3) + \text{dbinom}(4, 12, 0.3)$

$P(2 \le X < 5) = P(X < 5) - P(X < 2)$

$= \text{pbinom}(4, 12, 0.3) - \text{pbinom}(1, 12, 0.3)$

$$\underline{P(X>5)} = 1 - P(X \le 5)$$

$$\doteq 1 - \text{pbinom}(5, 12, 0.3)$$

Example

$n = 5$ $p = 0.80$

Suppose it is known that 80% of the people exposed to the flu virus will contract the flu. Out of a family of five exposed to the virus, what is the probability that:

X = # of ppl in this family contract the flu

$X: 0, 1, 2, 3, 4, 5$

1. No one will contract the flu?

$P(X = 0) = \text{dbinom}(0, 5, 0.8)$

2. All will contract the flu?

$P(X=5) = \text{dbinom}(5, 5, 0.8)$.

3. Exactly two will get the flu?

$P(X=2) = \text{dbinom}(2, 5, 0.8)$

4. At least two will get the flu?

$P(X \geq 2) = P(X \text{ can be } 2, 3, 4, 5)$
$= 1 - P(X \text{ can be } 0, 1) = 1 - \text{pbinom}(1, 5, 0.8)$

5. at most 3 will get flu?

$P(X \text{ at most } 3) = P(X \le 3)$

$= P(X \text{ can be } 0, 1, 2, \underline{\underline{3}})$

$= \text{pbinom}(3, 5, 0.8)$

6. fewer than 4 will get flu?

$P(X < 4) = P(X \text{ can be } 0, 1, 2, \underline{\underline{3}})$

$= \text{pbinom}(3, 5, 0.8)$

Cumulative Distribution Function

Recall that a quantitative random variable X has a **cumulative distribution function** given by

$$F_X(x) = P(X \leq x)$$

for all $x \in \mathbb{R}$.

When we have a discrete random variable X, the cdf is related to the pmf in the following way:

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i)$$

where $x_1, x_2, \ldots$ are the values of X.

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Cumulative Distribution Function Properties

Any cdf F has the following properties:

1. F is a non-decreasing function defined on $\mathbb{R}$
2. F is right-continuous, meaning for each a, $F(a) = F(a+) = \lim_{x \to a^+} F(x)$
3. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
4. $P(a < X \leq b) = F(b) - F(a)$ for all real a and b, where $a < b$.
5. $P(X > a) = 1 - F(a)$
6. $P(X < b) = F(b-) = \lim_{x \to b^-} F(x)$.
7. $P(a < X < b) = F(b-) - F(a)$.
8. $P(X = b) = F(b) - F(b-)$.

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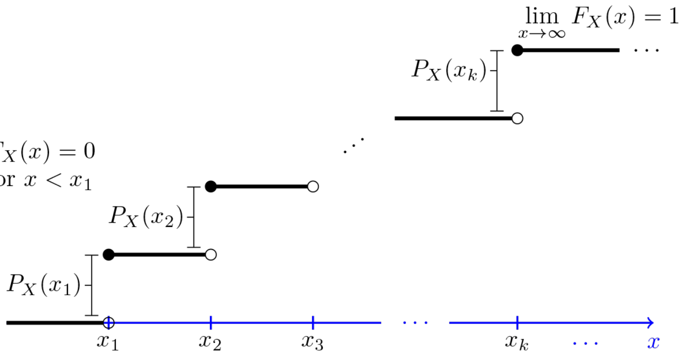
Graph of CDF



CDF of a Binomial R.V.

If $X \sim Binomial(n, p)$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \sum_{k=0}^{x^*} \binom{n}{k} p^k (1-p)^{n-k}, & \text{if } 0 \le x \le n \\ 1, & \text{if } n \le x \end{cases}$$

where $x^* = x$, the first integer less than or equal to x.

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Hypergeometric Distribution

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Each of 12 refrigerators of a certain type has been returned to a distributor because of an audible, high-pitched, oscillating noise when the refrigerator is running. Suppose that 7 of these refrigerators have a defective compressor and the other 5 have less serious problems. The technition looks at 6 refrigerators, what is the probability that exactly 5 have a defective compressor?

$m = 7$ def. $\leftarrow$ 5

$n = 5$ good $\leftarrow$ 1

$N = 12$ 6

X = # of def. compressor. among the 6.

$X \sim$

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Conditions for a Hypergeometric Distribution

1. The population or set to be sampled consists of N individuals, objects or elements (a *finite* population).
2. Each individual can be characterized as a "success" or "failure." There are m successes in the population, and n failures in the population. Notice: $m + n = N$. $m=7$, $n=5$
3. A sample size of k individuals is selected without replacement in such a way that each subset of size k is equally likely to be chosen. $k=6$ $P(X=5)=?$

The **parameters** of a hypergeometric distribution is m, n, k. We write $X \sim Hyper(m, n, k)$. The probability mass function for a hypergeometric is:

$$f_X(x) = P(X = x) = \frac{\binom{m}{x}\binom{n}{k-x}}{\binom{m+n}{k}}$$

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Conditions for a Hypergeometric Distribution

binomial : $Y \sim binom(n, p)$

1. The population or set to be sampled consists of N individuals, objects or elements (a *finite* population).
2. Each individual can be characterized as a "success" or "failure." There are m successes in the population, and n failures in the population. Notice: $m + n = N$.
3. A sample size of k individuals is selected without replacement in such a way that each subset of size k is equally likely to be chosen.

$m+n=N$

The **parameters** of a hypergeometric distribution is m, n, k. We write $X \sim Hyper(m, n, k)$. The probability mass function for a hypergeometric is:

$$f_X(x) = P(X = x) = \frac{\binom{m}{x}\binom{n}{k-x}}{\binom{m+n}{k}}$$

dbinom

- R commands: $P(X = x) =$ dhyper(x,m,n,k) and $P(X \leq x) =$ phyper(x,m,n,k)

pbinom

- Going back to the refrigerator example, $m = 7$, $n = 5$, $k = 6$. $P(X = 5)$

```
> dhyper(5,7,5,6)
[1] 0.1136364
```

- $P(X \leq 4)$

P(X can be 0, 1, 2, 3, 4)

```
> phyper(4,7,5,6)
[1] 0.8787879
```

$P(X<4) = P(X \text{ can be } 0,1,2,3)$

$= \text{phyper}(3,7,5,6)$

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Mean and Variance of a Hypergeometric Distribution

Let Y have a hypergeometric distribution with parameter, m, n, and k.

- The mean of Y is:

$$\mu_Y = E(Y) = k\left(\frac{m}{m+n}\right) = kp.$$

- The variance of Y is:

$$\sigma_Y^2 = var(Y) = kp(1-p)\left(1 - \frac{k-1}{m+n-1}\right).$$

- $1 - \frac{k-1}{m+n-1}$ is called the **finite population correction factor.** As, the population increases, this factor will get closer to 1.

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Digital Cameras

3-mega 6
4-mega 9
15

A certain type of digital camera comes in either a 3-megapixel version or a 4-megapixel version. A camera store has received a shipment of 15 of these cameras, of which 6 have 3-megapixel resolution. Suppose that 5 of these cameras are randomly selected to be stored behind the counter; the other 10 are placed in a storeroom. Let X = the number of 3-megapixel cameras among the 5 selected for behind the counter storage.

$X \sim hyper(6, 9, 5)$

1. What is the probability that exactly 2 of the 3-megapixel cameras are stored behind the counter?

$P(X=2) = dhyper(2, 6, 9, 5)$

```
> dhyper(2,6,9,5)
[1] 0.4195804
```

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Digital Cameras

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2. What is the probability that at least one of the 3-megapixel cameras are stored behind the counter?

$$P(X \text{ at least } 1) = P(X \geq 1)$$
$$= 1 - P(X = 0)$$
$$= 1 - \text{dhyper}(0, 6, 9, 5)$$

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Digital Cameras

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3. Calculate the mean and standard deviation of X.

$m = 6 \quad n = 9 \quad k = 5$