Digital Image Processing

COSC 6380/4393

Lecture – 17

Oct. 17th, 2023

Pranav Mantini

Slides from Dr. Shishir K Shah and S. Narasimhan

2D Discrete Fourier Transform

- Let $\tilde{I}$ be the DFT of the I

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} I(i,j)\, e^{-\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

- Also,

$$I(i,j) = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1} \tilde{I}(u,v)\, e^{\sqrt{-1}\frac{2\pi}{N}(ui+vj)}$$

Example

$$I = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$$

$$\tilde{I}(u,v) = \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} I(i,j)\, e^{-\sqrt{-1}\left(\frac{2\pi}{N}ui + \frac{2\pi}{M}vj\right)}$$

$$\tilde{I}(0,0) = \sum_{i=0}^{2-1} \sum_{j=0}^{3-1} I(i,j)\, e^{-\sqrt{-1}\left(\frac{2\pi}{N}0*i + \frac{2\pi}{M}0*j\right)}$$

$$= \sum_{i=0}^{1} \sum_{j=0}^{2} I(i,j) = 21$$

$\tilde{I}(0,1) = -3 + 1.732051\mathrm{j}$

$\tilde{I}(0,2) = -3 - 1.732051\mathrm{j}$

$\tilde{I}(1,0) = -9$

$\tilde{I}(1,1) = 0 + 0\mathrm{j}$

$\tilde{I}(1,2) = 0 + 0\mathrm{j}$

$\tilde{I} =$

$21 + 0\sqrt{-1}$	$-3 + 1.73\sqrt{-1}$	$-3 - 1.73\sqrt{-1}$
$-9 + 0\sqrt{-1}$	$0 + 0\sqrt{-1}$	$0 + 0\sqrt{-1}$

Complex Image

Properties of DFT Matrix

- We can understand the DFT matrix better by studying some of its properties.
- Any image **I** of interest to us is composed of **real integers.**
- However, the DFT of **I** is generally **complex**.
- It can be written in the form

$$\tilde{\mathbf{I}} = \tilde{\mathbf{I}}_{real} + \sqrt{-1}\ \tilde{\mathbf{I}}_{imag}$$

where $\tilde{\mathbf{I}}_{real}$ and $\tilde{\mathbf{I}}_{imag}$ have components

$$\tilde{I}_{real}(u, v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \cos\left[\frac{2\pi}{N}(ui + vj)\right]$$

$$\tilde{I}_{imag}(u, v) = -\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i, j) \sin\left[\frac{2\pi}{N}(ui + vj)\right]$$

i.e.,

$$\tilde{I}(u, v) = \tilde{I}_{real}(u, v) + \sqrt{-1}\ \tilde{I}_{imag}(u, v) \text{ for } 0 \leq u, v \leq N-1$$

(These are taken directly from the original DFT equation).

Therefore $\tilde{I}$ has a **magnitude** and a **phase**.

$21 + 0\sqrt{-1}$	$-3 + 1.73\sqrt{-1}$	$-3 - 1.73\sqrt{-1}$
$-9 + 0\sqrt{-1}$	$0 + 0\sqrt{-1}$	$0 + 0\sqrt{-1}$

21	−3	−3
−9	0	0

0	1.73	− 1.73
0	0	0

Magnitude and Phase of DFT

- The **magnitude** of the DFT is the matrix

$$|\tilde{\mathbf{I}}| = \left[|\tilde{I}(u, v)| \ ; 0 \leq u, v \leq N\text{-}1 \right]$$

with elements

$$|\tilde{I}(u, v)| = \sqrt{\tilde{I}^2_{real}(u,v) + \tilde{I}^2_{imag}(u,v)}$$

which are just the magnitudes of the complex components of $\tilde{\mathbf{I}}$

21	3.46	3.46
9	0	0

- The **phase** of the DFT is the matrix

$$\angle\tilde{\mathbf{I}} = \left[\angle\tilde{I}(u, v) \ ; 0 \leq u, v \leq N\text{-}1 \right]$$

with elements

$$\angle\tilde{I}(u, v) = \tan^{-1}\left[\tilde{I}_{imag}(u, v) \, / \, \tilde{I}_{real}(u, v) \right]$$

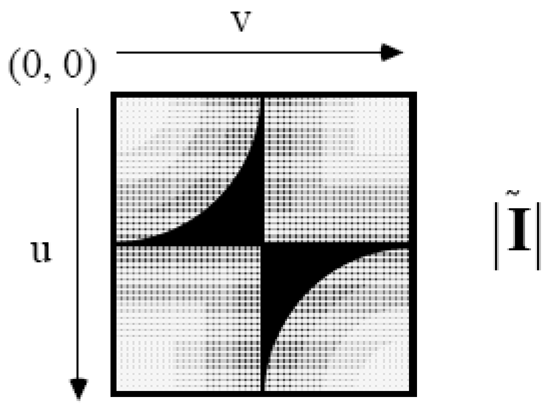
which are just the phases of the complex components of $\tilde{\mathbf{I}}$.

- Therefore

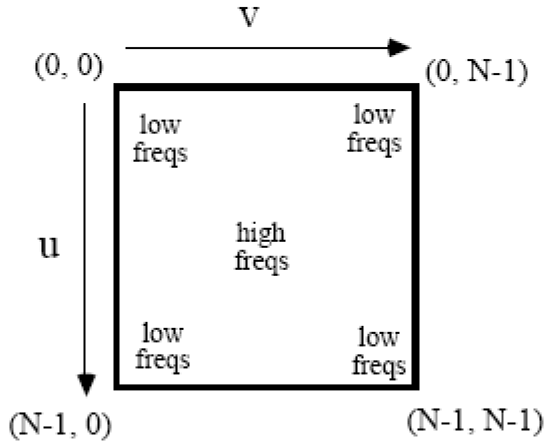
$$\tilde{I}(u, v) = |\tilde{I}(u, v)| \exp\left\{ \sqrt{-1} \angle\tilde{I}(u, v) \right\}$$

Symmetry of DFT

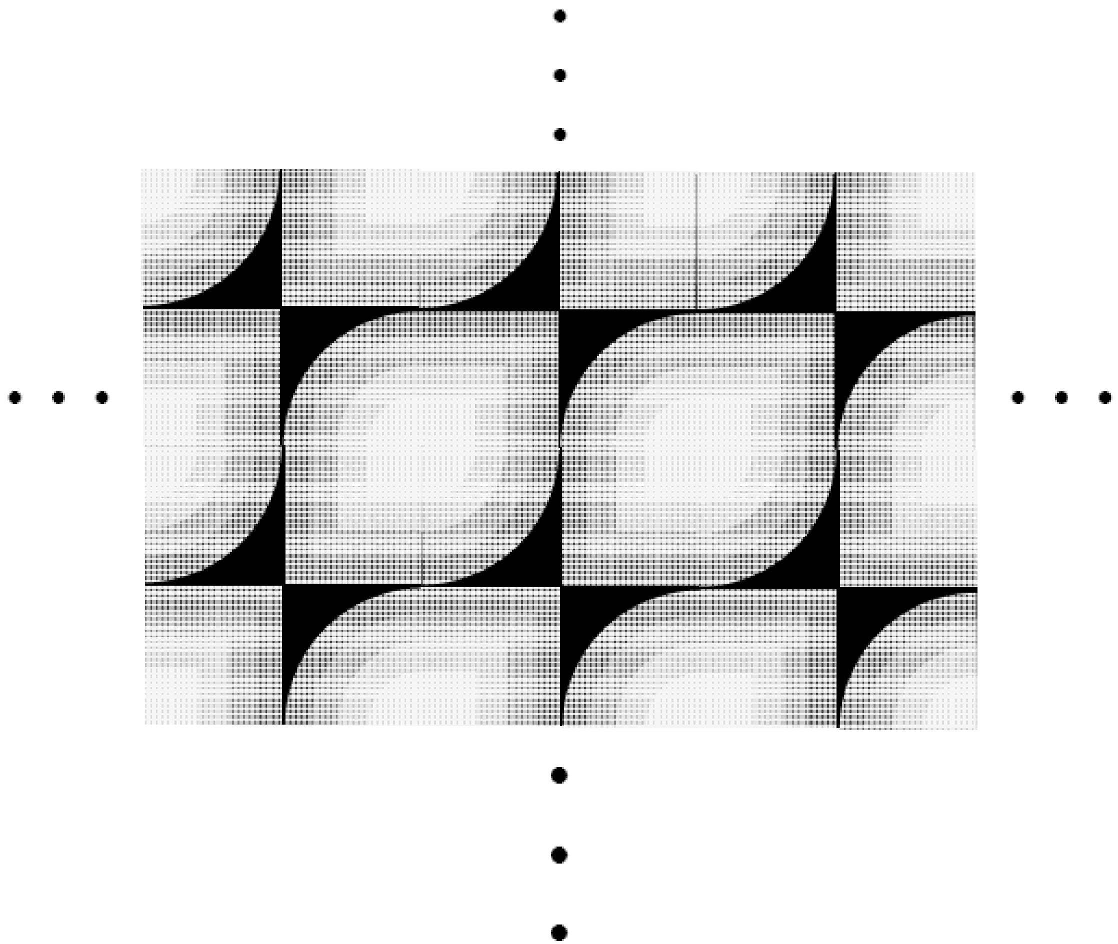
- Depiction of the symmetry of the DFT (magnitude).



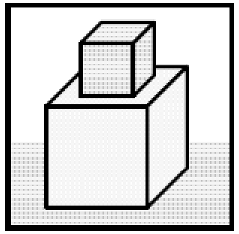
- The highest frequencies are represented near (u, v) = (N/2, N/2).



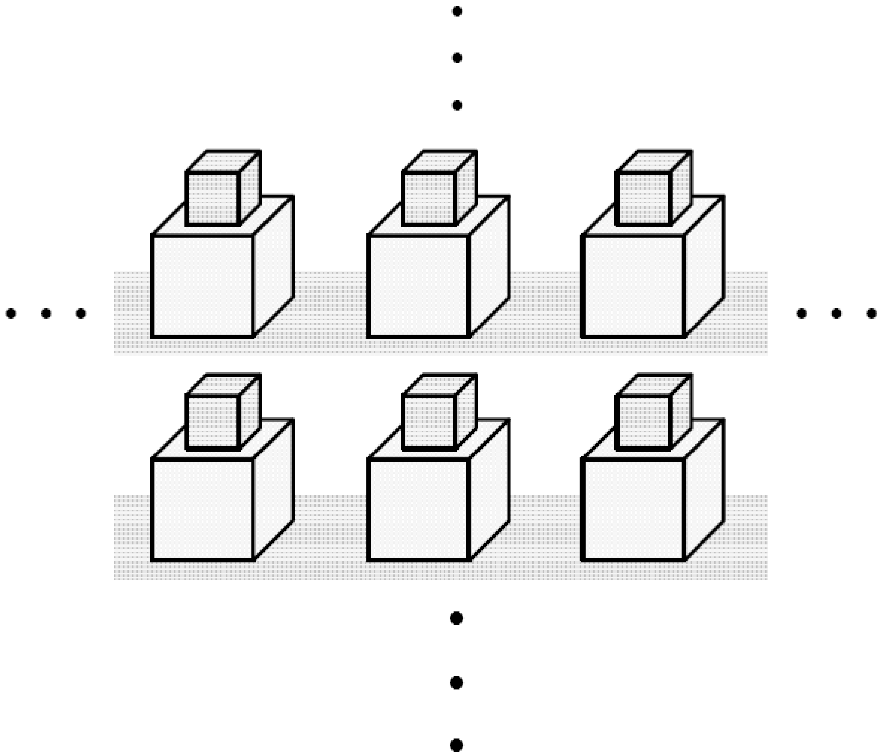
Periodic Extension of DFT



Periodic Extension of Image

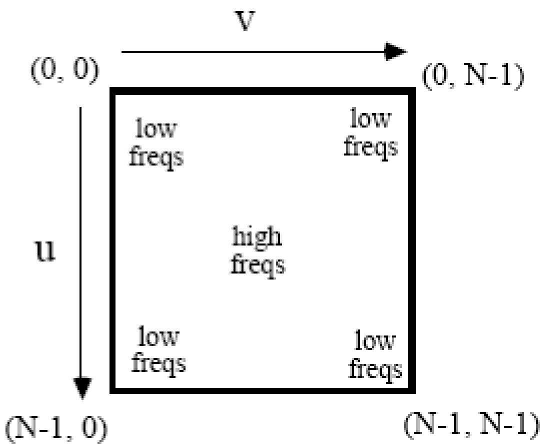


Image



Frequencies DFT

- The highest frequencies are represented near (u, v) = (N/2, N/2).



Displaying the DFT

- Usually, the DFT is displayed with its center coordinate (u, v) = (0, 0) at the center of the image.
- This way, the lower frequency information (which usually dominates an image) is clustered together near the origin at the center of the display.
- This can be accomplished in practice by taking the DFT of the alternating image (for display purposes only!)

$$[(-1)^{i+j} I(i,j) \; ; \; 0 \leq i, j \leq N-1]$$

- Observe that

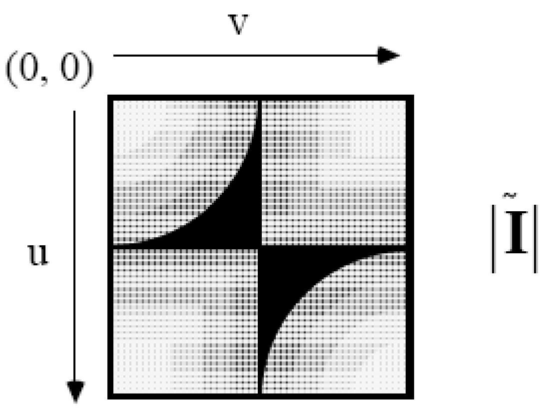
$$(-1)^{i+j} = e^{\sqrt{-1}\pi (i+j)} = e^{\sqrt{-1} \frac{2\pi}{N} N(i+j)/2} = W_N^{-N(i+j)/2}$$

so

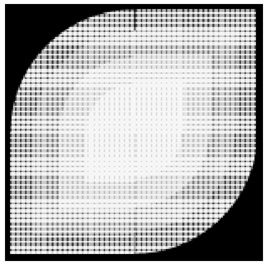
$$\begin{aligned} DFT\left[(-1)^{i+j} I(i,j)\right] &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\,(-1)^{i+j}\, W_N^{(ui+vj)} \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, W_N^{(ui+vj)}\, W_N^{-N(i+j)/2} \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} I(i,j)\, W_N^{[(u-N/2)i+(v-N/2)j]} \\ &= \tilde{I}(u - \tfrac{N}{2}, v - \tfrac{N}{2}) \end{aligned}$$

- A simple shift of the DF

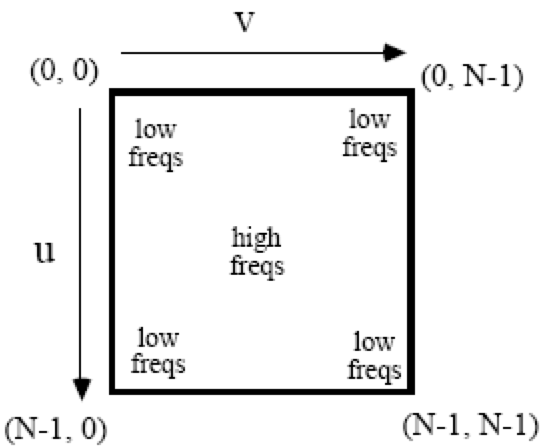
Centered DFT

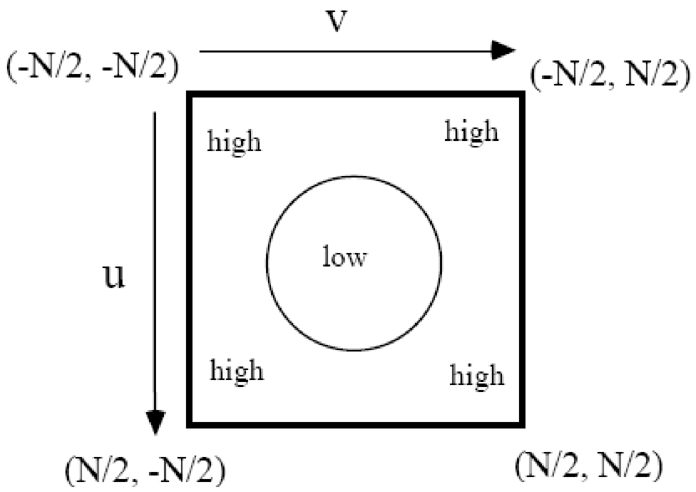


Original DFT



Centered DFT





Displaying the DFT

- Since the DFT is **complex** one can display only either the magnitude or phase as an image at a time.
- Usually the phase is very difficult to interpret visually.
- To display the magnitude, usually it's best to **logarithmically compress** it:

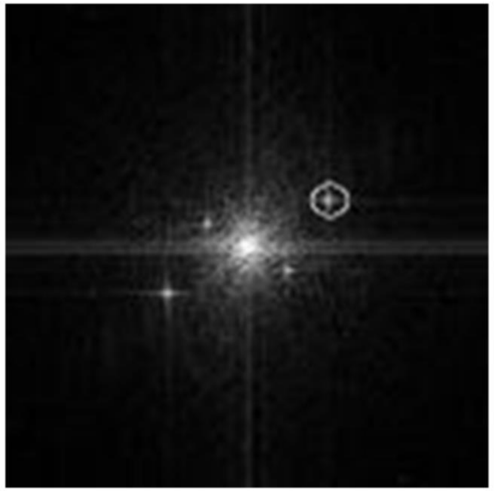
$$\log \left[1 + \left| \tilde{I}(u, v) \right| \right]$$

prior to display, since (visually) the low-amplitude frequencies will be hard to see.

- Following the logarithm, it is necessary to use a linear point operation to stretch the contrast, since the log values will be very small.

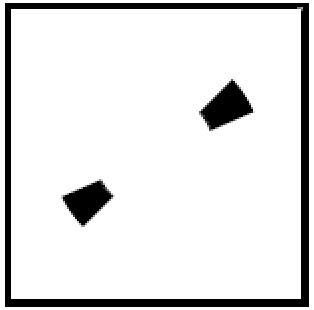
Periodic Noise removal

Periodic Noise removal



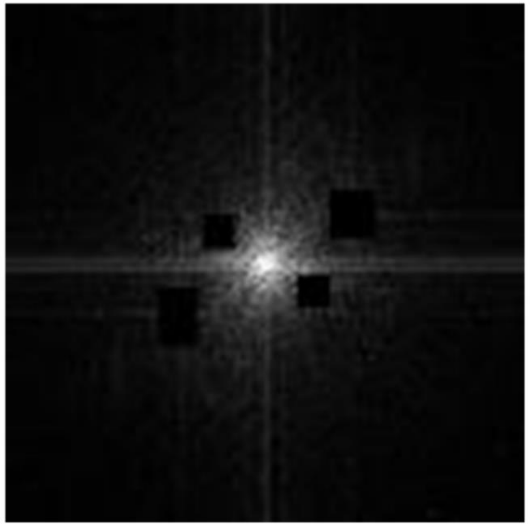
Narrowband Image

- It is also possible to produce an images that are highly granular **and** highly oriented:

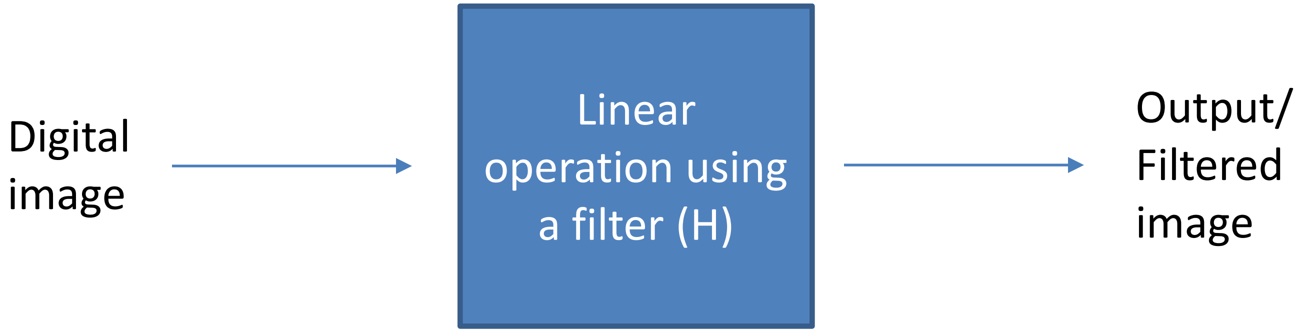


- This mask was created by (pointwise) multiplying the mid-frequency mask with one of the oriented masks.

Filtered Image



Linear Image Filtering



Linear Image Filtering

- Correlation and Convolution are basic operations that we will perform to extract information from images
- Two operations
  - Correlation
    - Used as a tool to measure the similarity between two signals
  - Convolution
    - Used to modify one signal using another signal.
- The two operations in essence are the same with a minor difference.

Spatial Correlation Operator

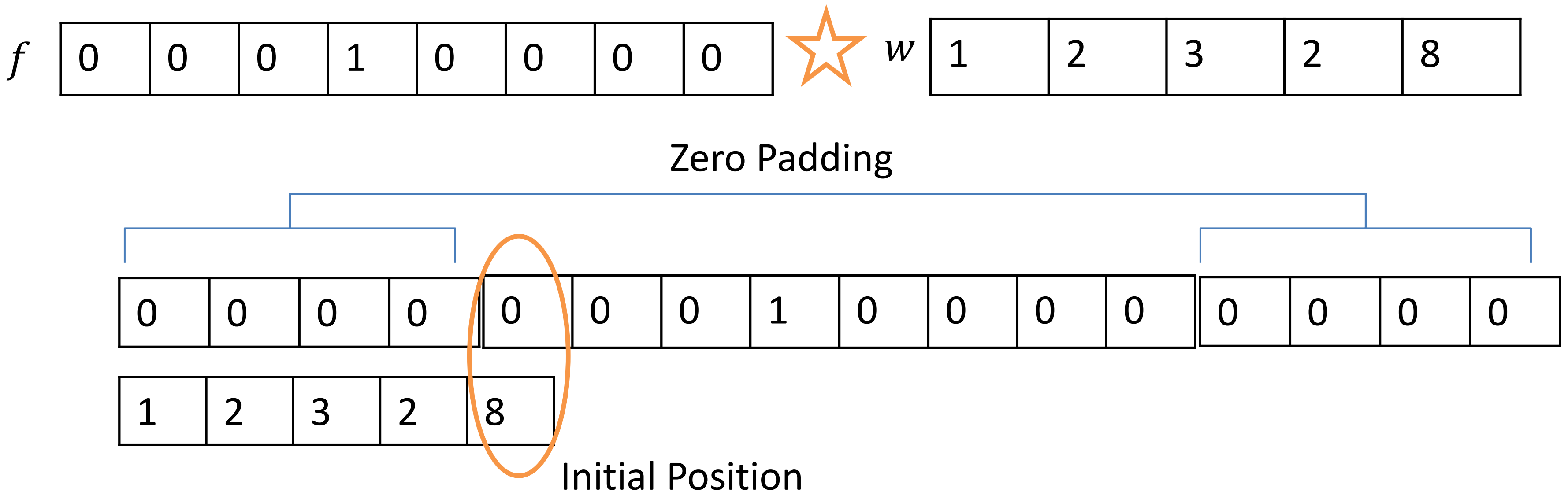
f	0	0	0	1	0	0	0	0

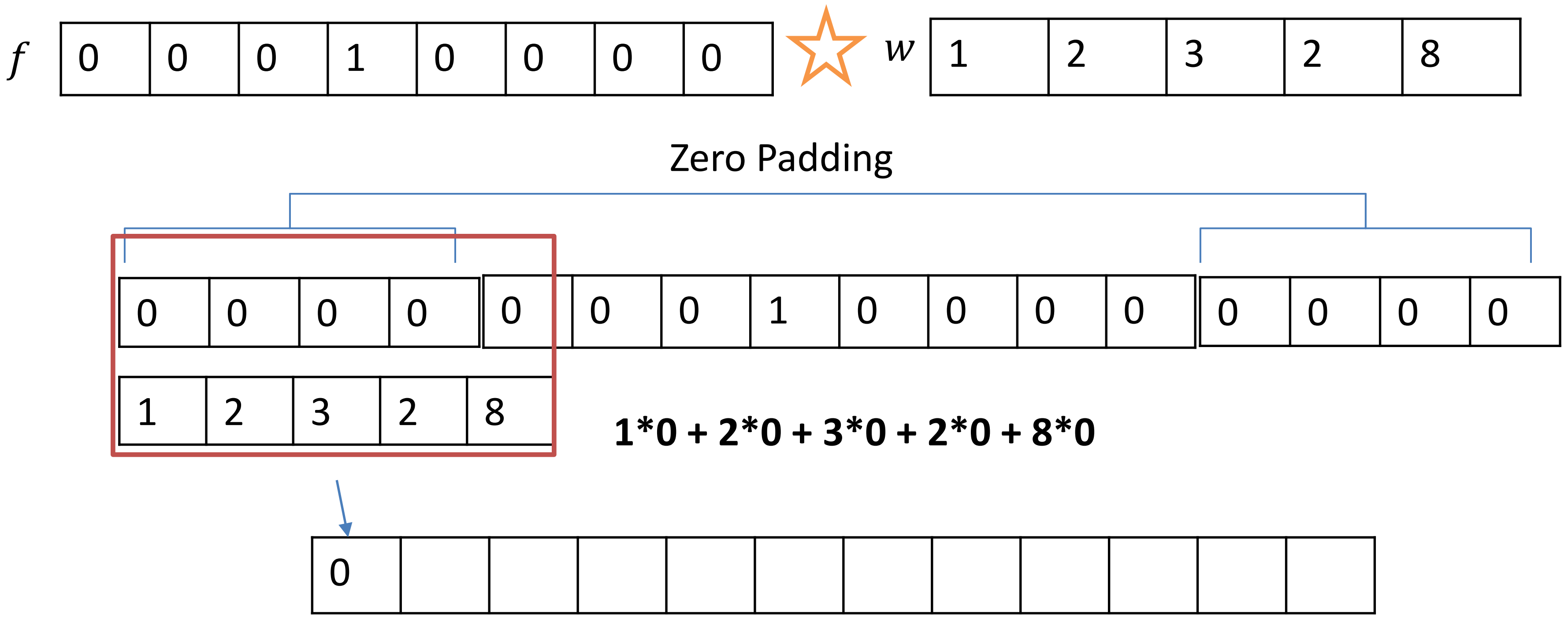
☆

w	1	2	3	2	8

f	0	0	0	1	0	0	0	0	$\star$	w	1	2	3	2	8

				0	0	0	1	0	0	0	0
1	2	3	2	8							


f
0 0 0 1 0 0 0 0
w
1 2 3 2 8
Zero Padding
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
1 2 3 2 8
Initial Position



f
0 0 0 1 0 0 0 0
w
1 2 3 2 8
Zero Padding
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
1 2 3 2 8
1*0 + 2*0 + 3*0 + 2*0 + 8*0
0

$f \star w$

f							
0	0	0	1	0	0	0	0

w				
1	2	3	2	8

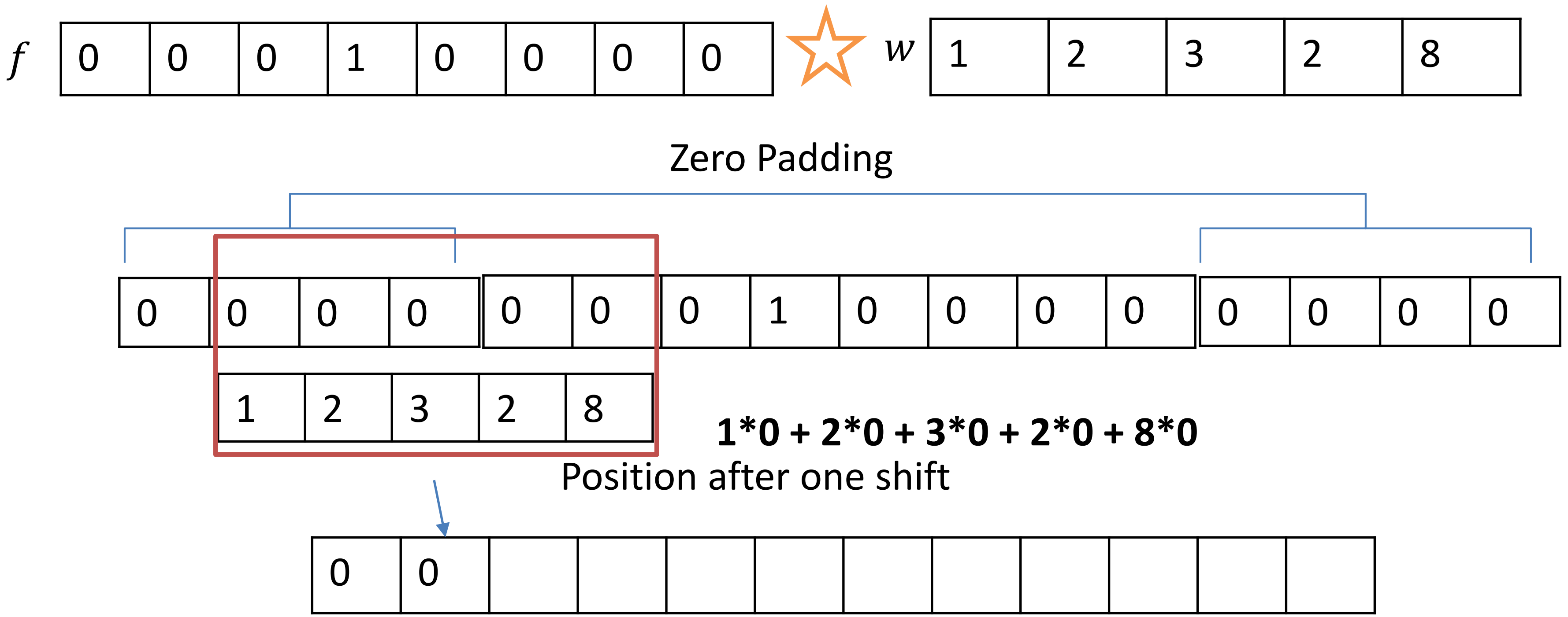
Zero Padding

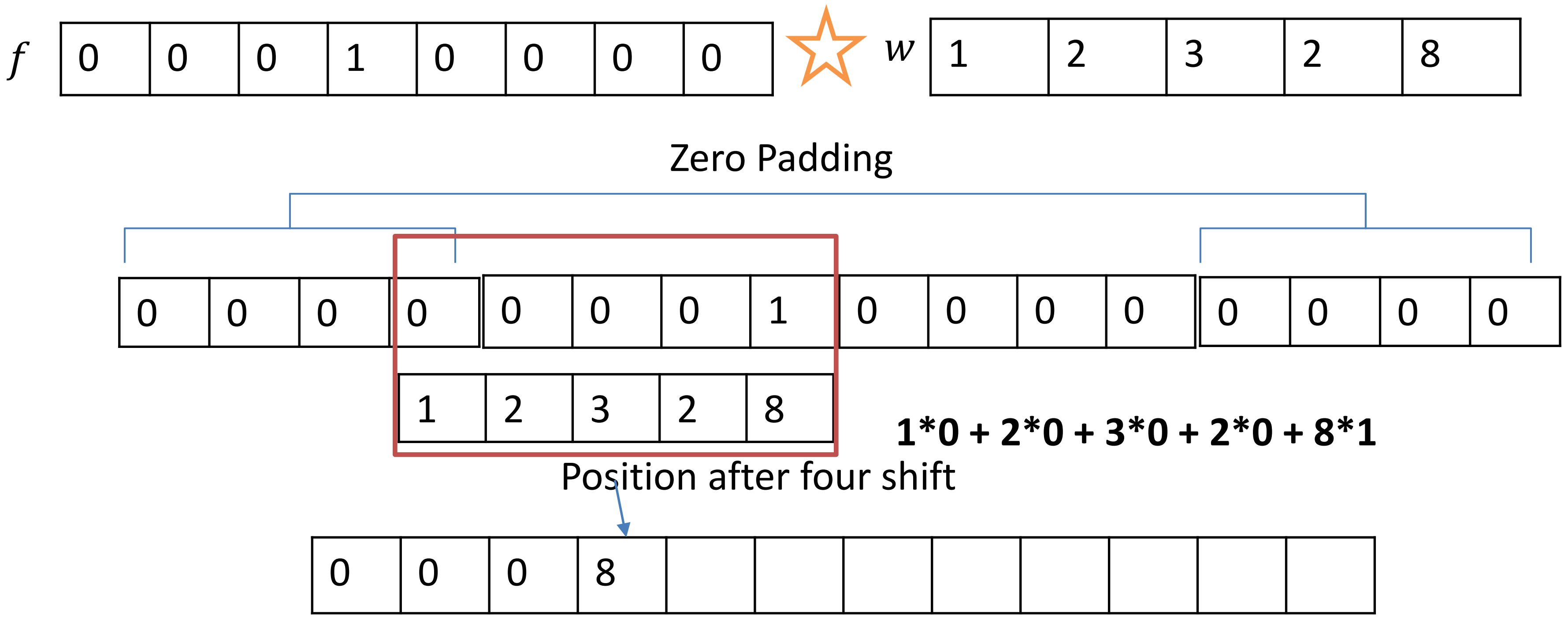
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

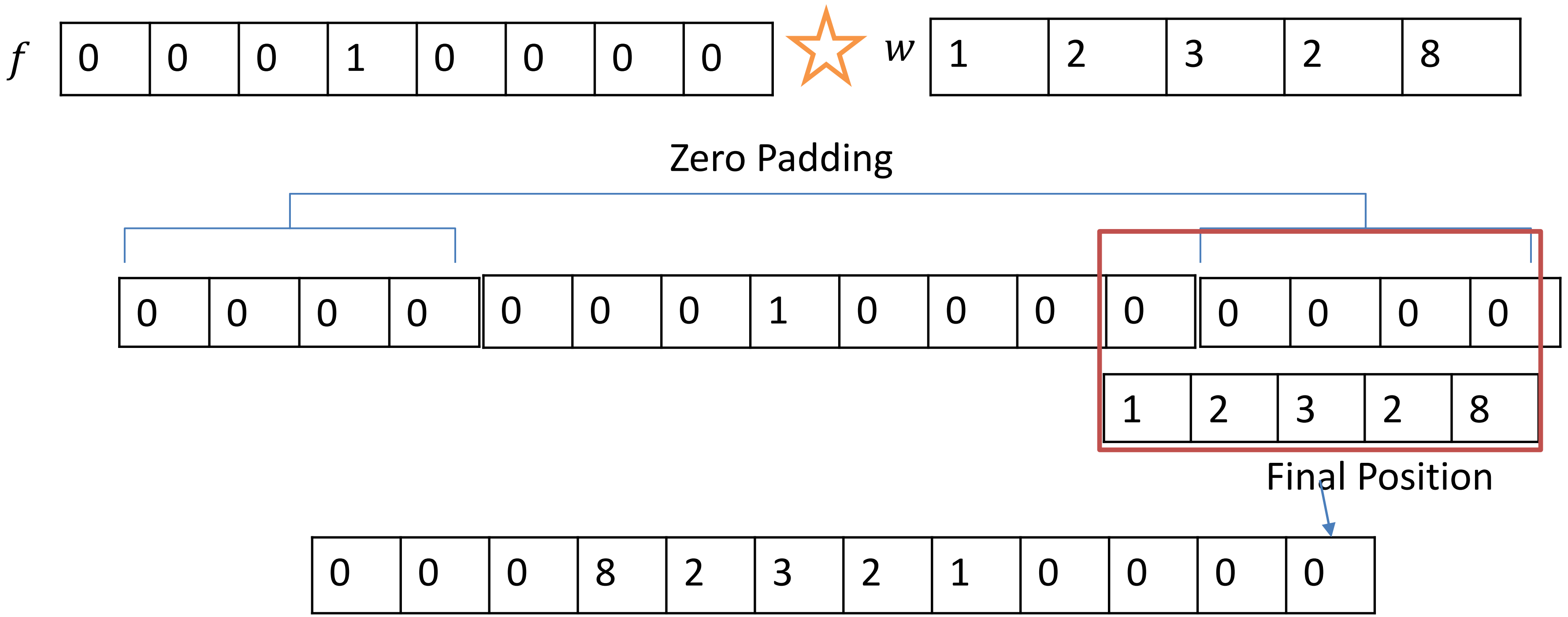
1	2	3	2	8

Position after one shift

0	0										


f
0 0 0 1 0 0 0 0
w
1 2 3 2 8
Zero Padding
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
1 2 3 2 8
1*0 + 2*0 + 3*0 + 2*0 + 8*0
Position after one shift
0 0


f
0 0 0 1 0 0 0 0
w
1 2 3 2 8
Zero Padding
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
1 2 3 2 8
1*0 + 2*0 + 3*0 + 2*0 + 8*1
Position after four shift
0 0 0 8



f
0 0 0 1 0 0 0 0
w
1 2 3 2 8
Zero Padding
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
1 2 3 2 8
Final Position
0 0 0 8 2 3 2 1 0 0 0 0

1*0 + 2*0 + 3*0 + 2*0 + 8*0

f

0	0	0	1	0	0	0	0

☆ w

1	2	3	2	8

Full Correlation result

0	0	0	8	2	3	2	1	0	0	0	0

f

0	0	0	1	0	0	0	0

☆ w

1	2	3	2	8

Cropped Correlation result

0	8	2	3	2	1	0	0

Spatial Correlation Operator

The correlation of a filter $w(x)$ of size m
with an signal $f(x)$, denoted as $w(x) \star f(x)$

$$w(x) \star f(x) = \sum_{s=-a}^{a} w(s) f(x+s)$$

Spatial Convolution Operator

f	0	0	0	1	0	0	0	0

$\otimes$

w	1	2	3	2	8

f

0	0	0	1	0	0	0	0

$\otimes$

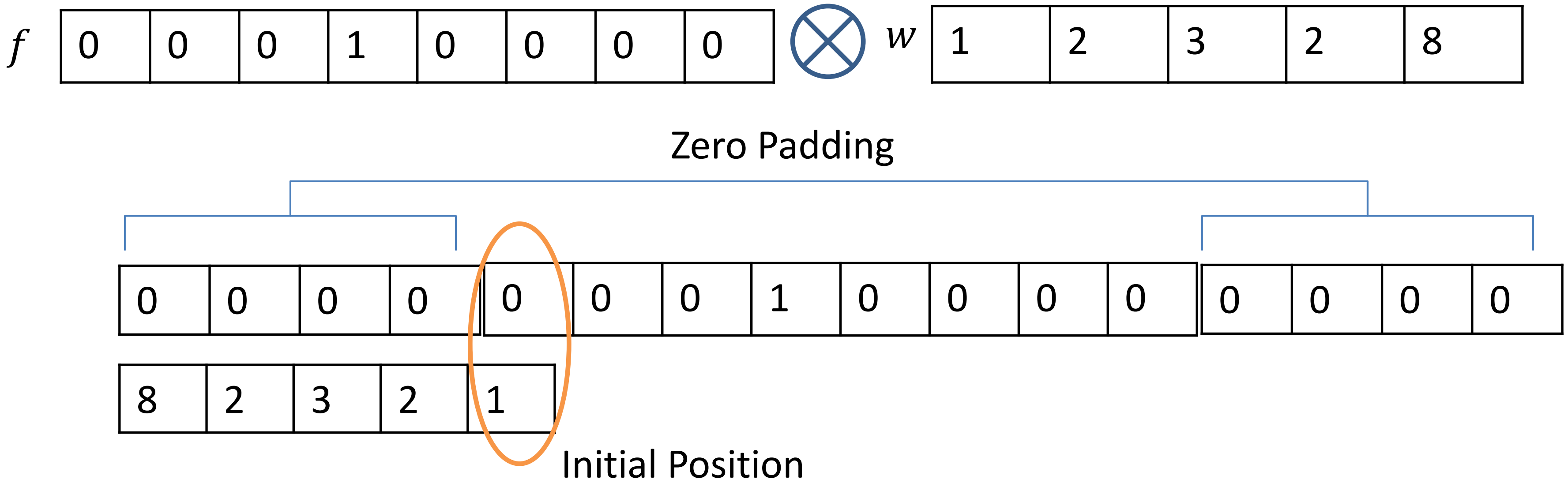
w

1	2	3	2	8

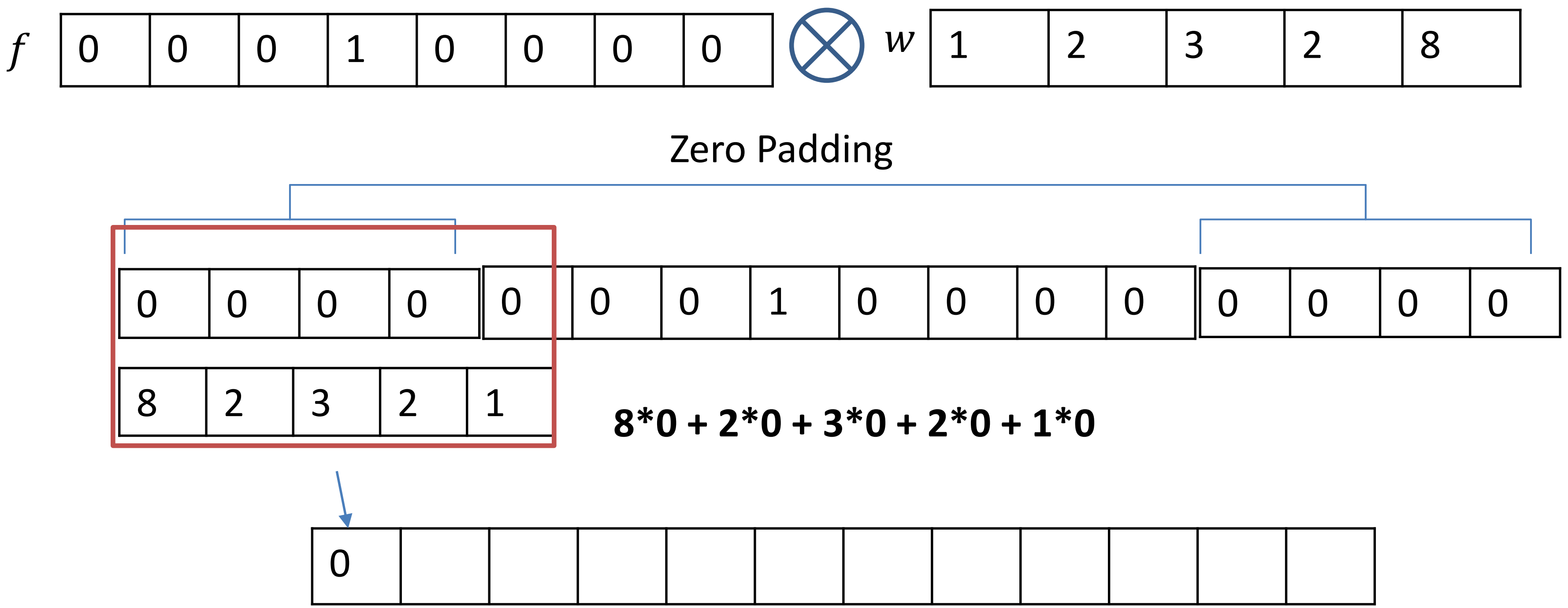
0	0	0	1	0	0	0	0

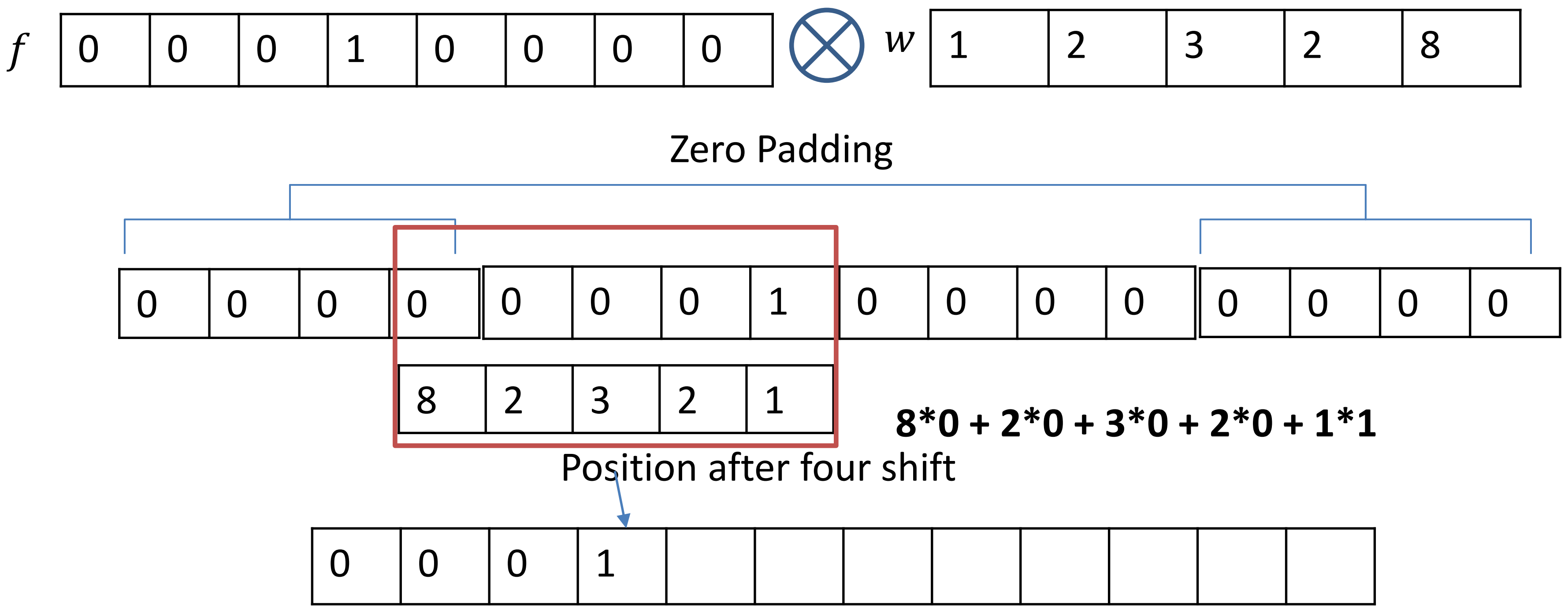
8	2	3	2	1

$w \text{ rotated by } 180^0$



f
0 0 0 1 0 0 0 0
$\otimes$
w
1 2 3 2 8
Zero Padding
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
8 2 3 2 1
Initial Position


f
0 0 0 1 0 0 0 0
w
1 2 3 2 8
Zero Padding
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
8 2 3 2 1
8*0 + 2*0 + 3*0 + 2*0 + 1*0
0



f
0 0 0 1 0 0 0 0
w
1 2 3 2 8
Zero Padding
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
8 2 3 2 1
8*0 + 2*0 + 3*0 + 2*0 + 1*1
Position after four shift
0 0 0 1

f

0	0	0	1	0	0	0	0

$\otimes$ w

1	2	3	2	8

Full Convolution result

0	0	0	1	2	3	2	8	0	0	0	0

f

0	0	0	1	0	0	0	0

$\otimes$ w

1	2	3	2	8

Cropped Convolution result

0	1	2	3	2	8	0	0

Spatial Correlation Operator

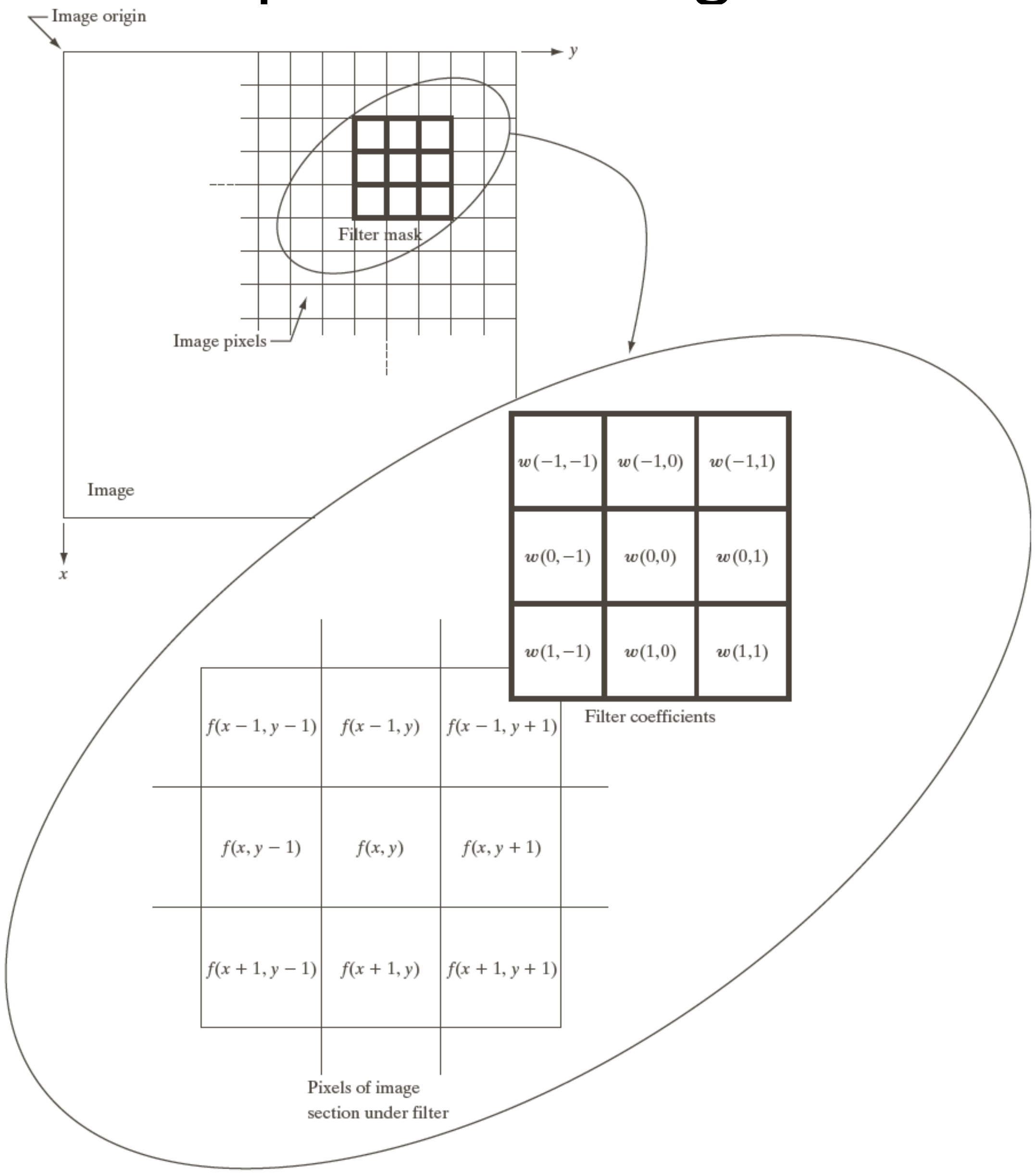
The correlation of a filter $w(x)$ of size m with an signal $f(x)$, denoted as $w(x) \otimes f(x)$

$$w(x) \otimes f(x) = \sum_{s=-a}^{a} w(s) f(x-s)$$

Spatial Filtering

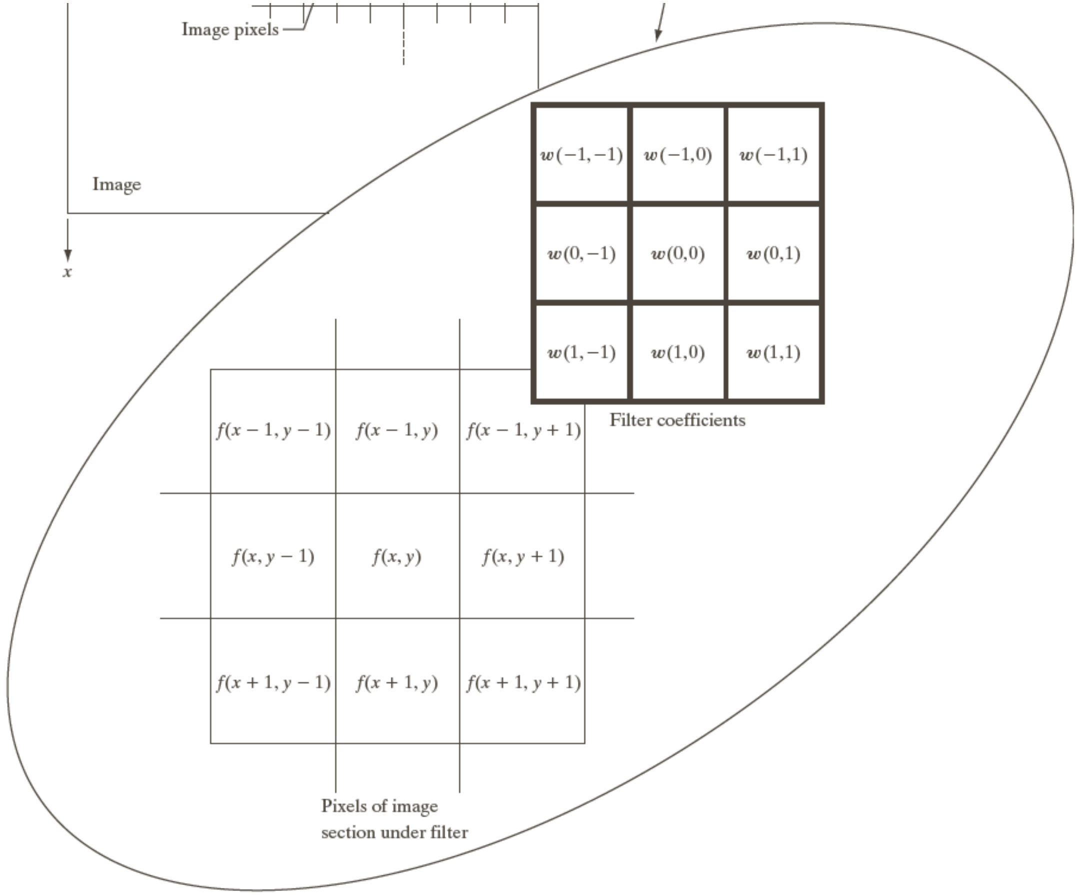
Linear spatial filtering of an image of size MXN with a filter of size mXn is given by

Spatial Filtering



Spatial Correlation Operator

$$w(-1,-1)f(x-1,y-1) + w(-1,0)f(x-1,y) + \cdots$$
$$+ w(0,0)f(x,y) + \cdots + w(1,0)f(x+1,y) + w(1,1)f(x+1,y+1),$$



Spatial Correlation Operator

The correlation of a filter $w(x, y)$ of size $m \times n$ with an image $f(x, y)$, denoted as $w(x, y) \star f(x, y)$

$$w(x, y) \star f(x, y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s, t) f(x+s, y+t)$$

Spatial Convolution Operator

The convolution of a filter $w(x, y)$ of size $m \times n$

with an image $f(x, y)$, denoted as $w(x, y) \otimes f(x, y)$

$$w(x, y) \otimes f(x, y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x-s, y-t)$$

Origin $f(x, y)$

0	0	0	0	0
0	0	0	0	0
0	0	1	0	0
0	0	0	0	0
0	0	0	0	0

$w(x, y)$

1	2	3
4	5	6
7	8	9

(a)

FIGURE 3.30 Correlation (middle row) and convolution (last row) of a 2-D filter with a 2-D discrete, unit impulse. The 0s are shown in gray to simplify visual analysis.