Digital Image Processing

COSC 6380/4393

Lecture – 7

Sept 12th, 2023

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Slides from Dr. Shishir K Shah and Frank (Qingzhong) Liu

Review: DIGITAL IMAGE REPRESENTATION

- Once an image is **digitized** (A/D) and stored it is an array of **voltage or magnetic potentials**
- Not easy to work with from an algorithmic point of view
- The **representation** that is easiest to work with from an **algorithmic perspective** is that of a **matrix of integers**

Matrix Image Representation

- Denote a (square) image matrix **I** = [I(i, j); 0 < i, j < N-1] where
- (i, j) = (row, column)
- I(i, j) = image value at coordinate or pixel (i, j)

Review: **DIGITAL IMAGE REPRESENTATION (contd.)**

- **Example** - Matrix notation

$$\mathbf{I} = \begin{bmatrix} I(0,0) & I(0,1) & \cdots & I(0,N-1) \\ I(1,0) & I(1,1) & \cdots & I(1,N-1) \\ \vdots & \vdots & & \vdots \\ I(N-1,0) & I(N-1,1) & \cdots & I(N-1,N-1) \end{bmatrix}$$

- **Example** - Pixel notation - an N x N image

What's the minimum number of bits/pixel allocated?

columns

rows	0	1	2	3	4	5	6	7	8	. . .	N-1
0	193	191	189	194	196	200	225	227	224	• • •	57
1	189	185	187	190	193	198	223	229	222	• • •	62
2	186	188	185	192	194	193	219	228	223	• • •	59
3	180	176	179	178	193	193	199	231	221	• • •	54
	•	•	•	•	•	•	•	•	•		• •
	•	•	•	•	•	•	•	•	•		•
N-1	0	0	1	11	13	11	12	10	15	• • •	189

Image **I**

Review: **DIGITAL IMAGE REPRESENTATION (contd.)**

- **Example** - Binary Image (2-valued, usually BLACK and WHITE)

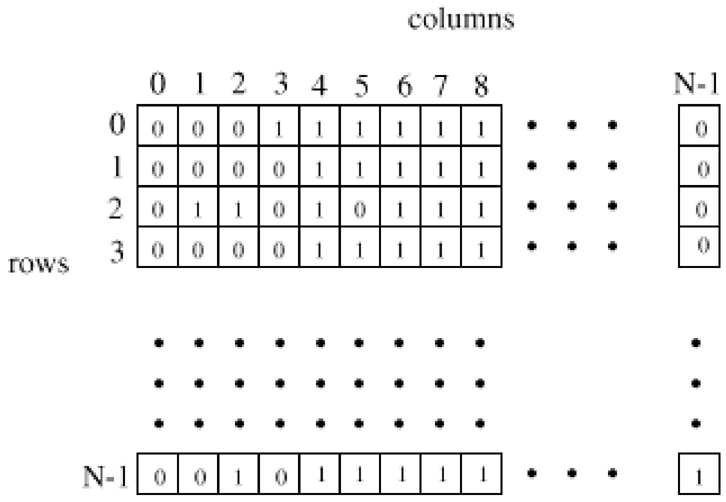


Image **I**

- Another way of depicting the image:

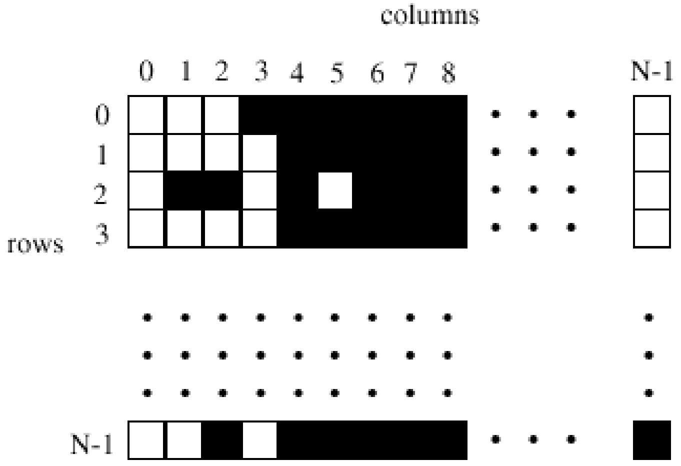


Image **I**

BINARY IMAGES

- Since **binary = bi-valued,** the (logical) values ’0’ or ’1’ usually indicate the **absence** or **presence** of an **image** property in an associated gray-level image:
  - Points of high or low intensity (brightness)
  - Points where an object is present or absent
  - More abstract properties, such as smooth vs. nonsmooth, etc.
- **Convention** - We will make the associations
- ’1’ = BLACK
- ’0’ = WHITE

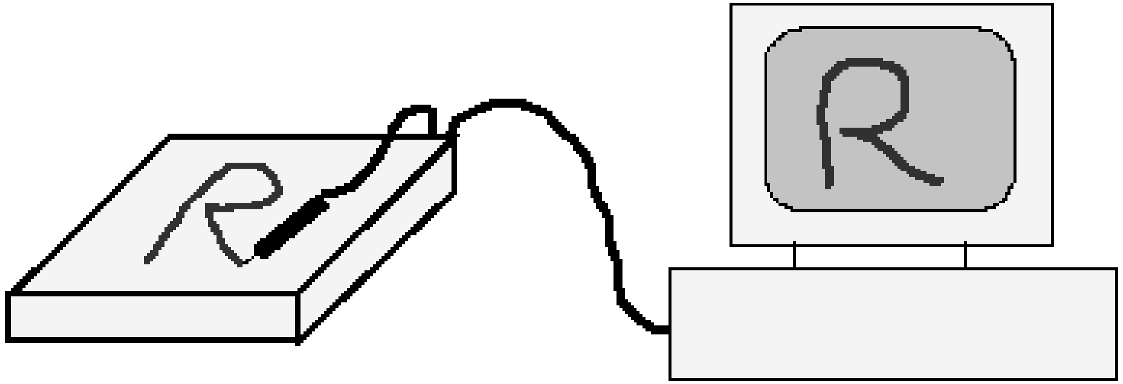
BINARY IMAGE

- Usually a binary image is obtained from a gray-level image
- Advantages:
  - B-fold reduction in required storage
  - Simple abstraction of information
  - Fast processing - **logical operators**
  - Can be further compressed

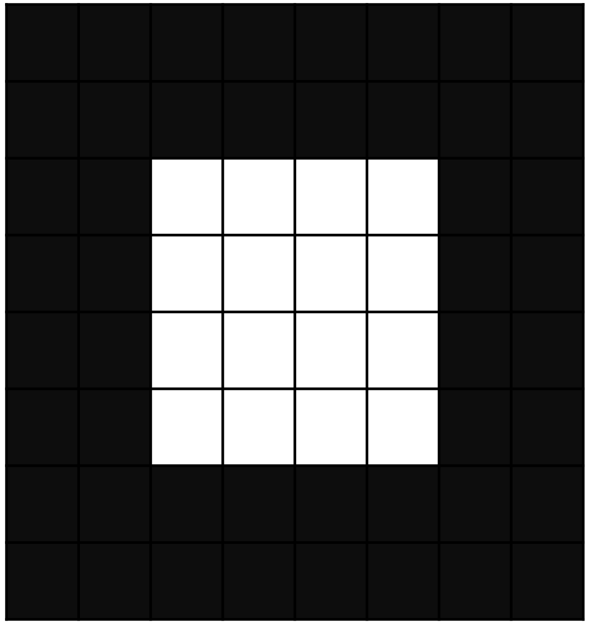
BINARY IMAGE GENERATION

- **Tablet-Based Input:**
- Binary images can derive from **simple sensors** with binary output
- Simplest example: **tablet, resistive pad,** or **light pen**
- All pixels **initially** assigned value ’0’:

  I = [I(i, j)], I(i, j) = ’0’ for all (i, j) = (row column)
- When pressure or light is applied at (i_0 , j_0), the image is assigned the value ’1’: I(i_0 , j_0) = ’1’
- This continues until the user completes the drawing



Grey Level → Binary Image



8X8 image → Black box on white background

Grey Level → Binary Image

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	255	255	255	255	0	0
0	0	255	255	255	255	0	0
0	0	255	255	255	255	0	0
0	0	255	255	255	255	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Grey scale Pixels values

→

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Binary image

GRAY-LEVEL THRESHOLDING

Simple Thresholding

- The simplest of image processing operations
- An extreme form of **gray-level quantization**
- Define an integer **threshold** T (in the gray-scale range)
- Compare each pixel intensity to T

THRESHOLDING

- Suppose **gray-level** image **I** has K gray-levels: 0, 1, 2, ..., K-1
- Select threshold T $\in$ { 0, 1, 2, ..., K-1}
- Compare **every** gray-level in **I** to T
- Define a new **binary image J** as follows:
- J(i, j) = ' 0' if I(i, j) <= T
- J(i, j) = '1' if I(i, j) > T
- A new binary image **J** is created from a gray-level image **I**

Grey Level → Binary Image

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	255	255	255	255	0	0
0	0	255	255	255	255	0	0
0	0	255	255	255	255	0	0
0	0	255	255	255	255	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Grey scale Pixels values

$\xrightarrow{Threshold(\mathrm{T})}$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Binary image

Grey Level → Binary Image

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	255	255	255	255	0	0
0	0	255	255	255	255	0	0
0	0	255	255	255	255	0	0
0	0	255	255	255	255	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Grey scale Pixels values

$\mathit{Threshold}$(T)

→

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Binary image

What is good value of T?

Grey Level → Binary Image

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

8X8 image → grey box on black background

What is good value of T?

Grey Level → Binary Image

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

8X8 image → grey box on black background

What is good value of T?

240	240	240	240	240	240	240	240
240	240	240	240	240	240	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	240	240	240	240	240	240
240	240	240	240	240	240	240	240

8X8 image → white box on dark white background

What is good value of T?

THRESHOLD SELECTION

- The quality of the **binary image J** obtained by thresholding **I** depends very heavily on the **threshold T**
- Indeed it is instructive to observe the result of thresholding an image at many different levels in sequence
- Different thresholds can produce different valuable abstractions of the image
- Some images do not produce any interesting results when thresholded by any T
- So: How does one decide if thresholding is possible ?
- How does one decide on a threshold T ?

Grey Level → Binary Image

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

8X8 image → black box on grey background

240	240	240	240	240	240	240	240
240	240	240	240	240	240	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	240	240	240	240	240	240
240	240	240	240	240	240	240	240

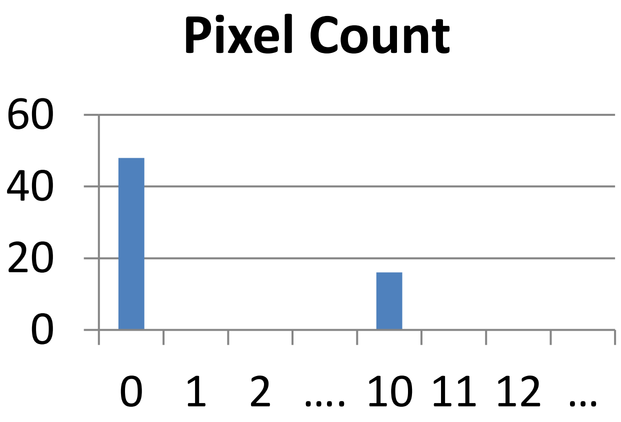
8X8 image → light white box on white background

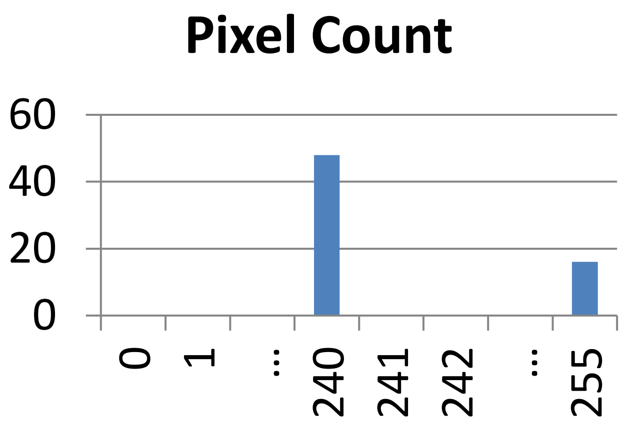
How do we determine T?

Determine modes

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

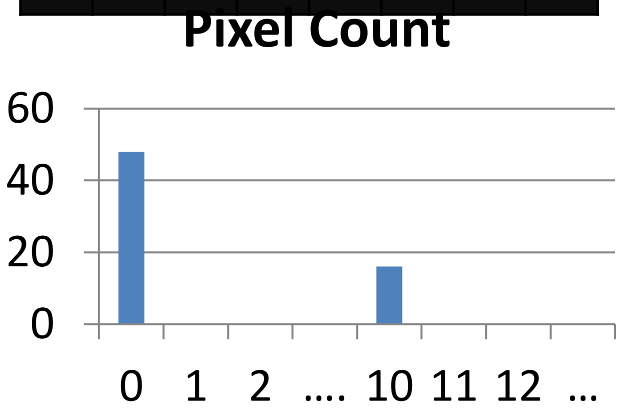
240	240	240	240	240	240	240	240
240	240	240	240	240	240	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	240	240	240	240	240	240
240	240	240	240	240	240	240	240





Determine modes

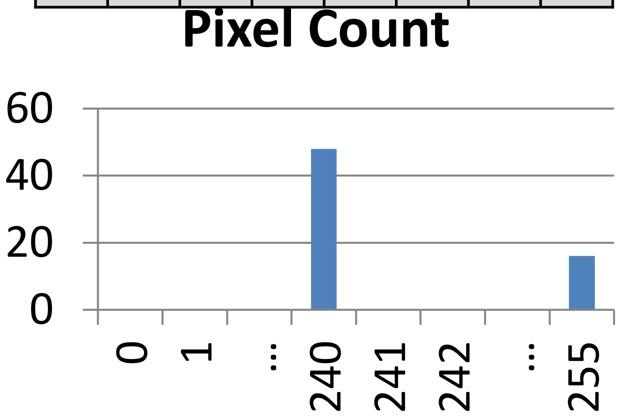
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	10	10	10	10	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0



$mode_1 = 0; mode_2 = 10$

$T = avg(mode) = \mathbf{5}$

240	240	240	240	240	240	240	240
240	240	240	240	240	240	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	255	255	255	255	240	240
240	240	240	240	240	240	240	240
240	240	240	240	240	240	240	240



$mode_1 = 240; mode_2 = 255$

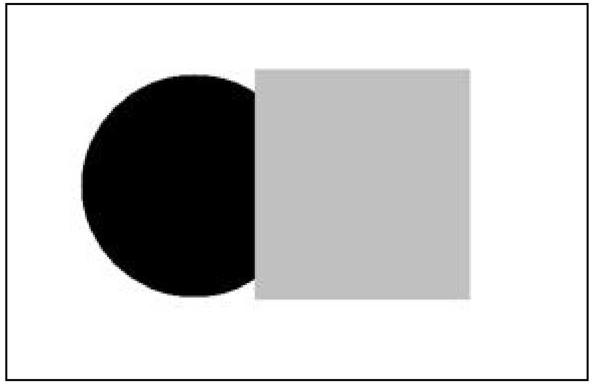
$T = avg(mode) = \mathbf{247.5}$

GRAY-LEVEL IMAGE HISTOGRAM

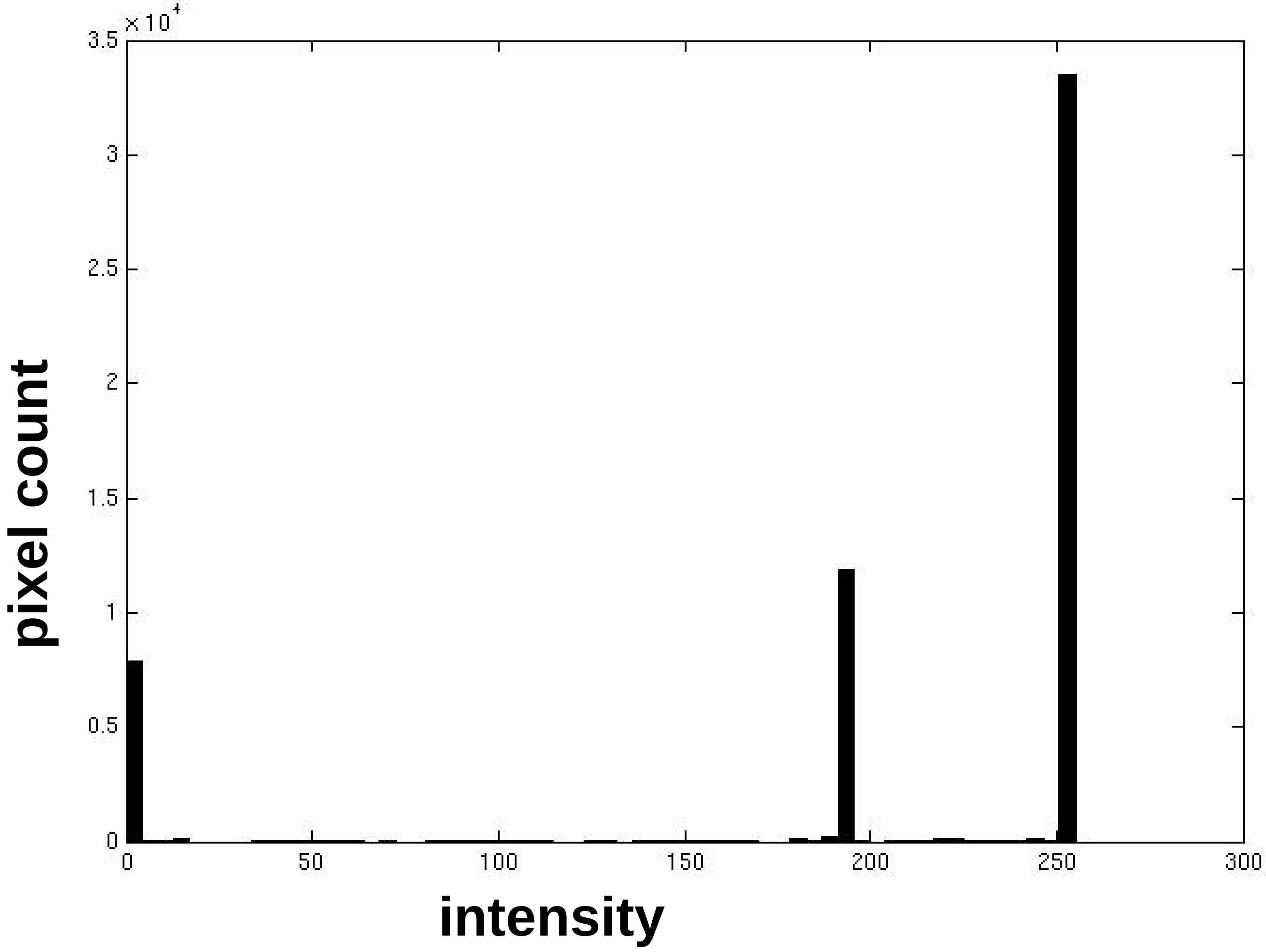
- The **histogram** $\mathbf{H_I}$ of image **I** is a **plot** or **graph** of the **frequency of occurrence** of each gray level in **I**
- $\mathbf{H_I}$ is a one-dimensional function with domain 0, ... , K-1
- $\mathbf{H_I}(x) = n$ if **I** contains **exactly** n occurrences of gray level x, for each x = 0, ... K-1

Histogram Example

Black = 0
Gray = 190
White = 254

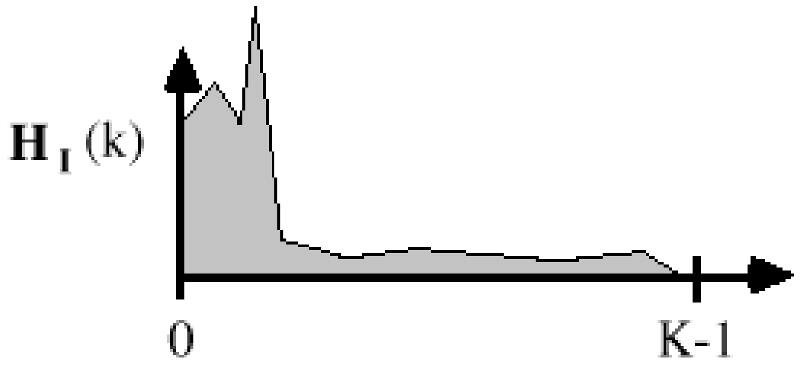


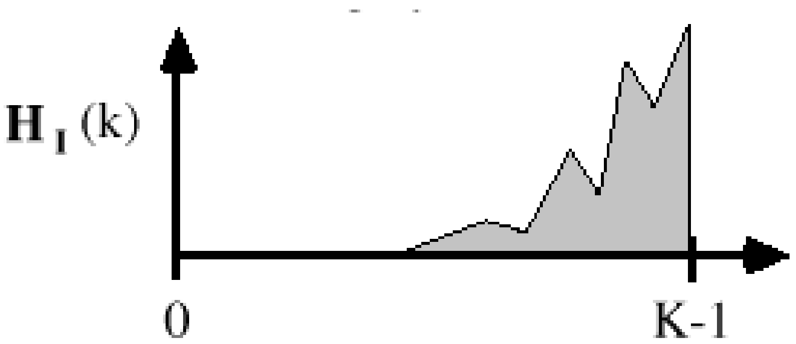
input image



HISTOGRAM APPEARANCE

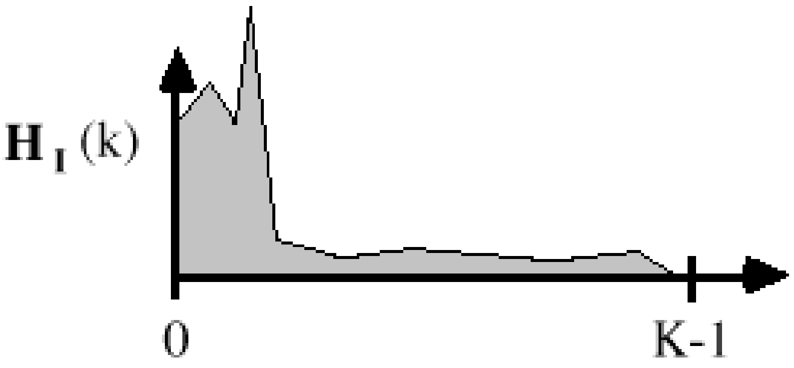
- The **appearance** of a histogram suggests much about the image



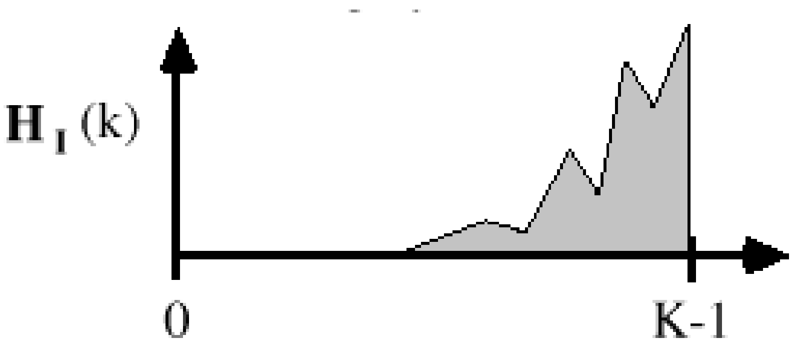


HISTOGRAM APPEARANCE

- The **appearance** of a histogram suggests much about the image



predominantly
dark image

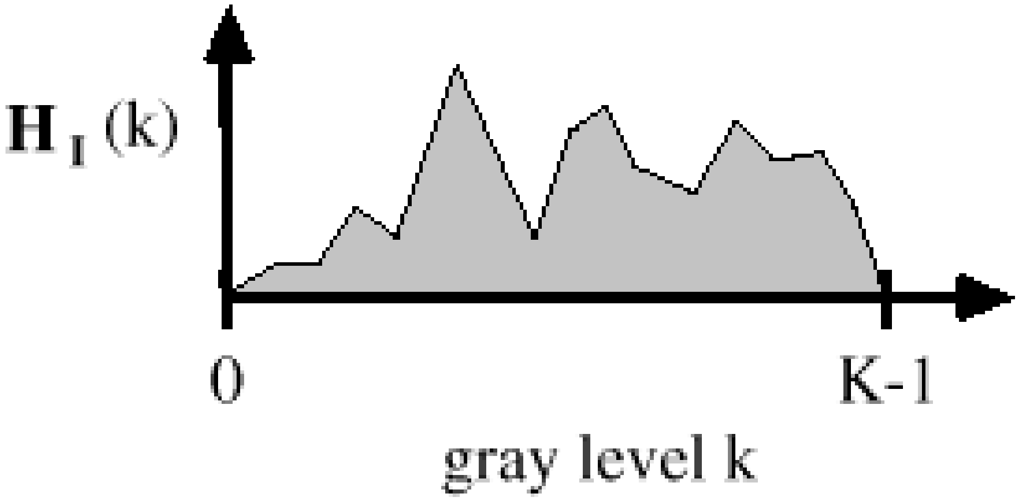


predominantly
light image

- These could be histograms of **underexposed** and **overexposed** images, respectively

HISTOGRAM APPEARANCE

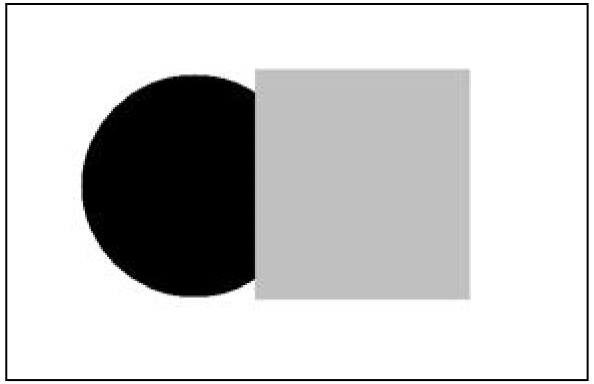
- This histogram may show better use of the gray-scale range



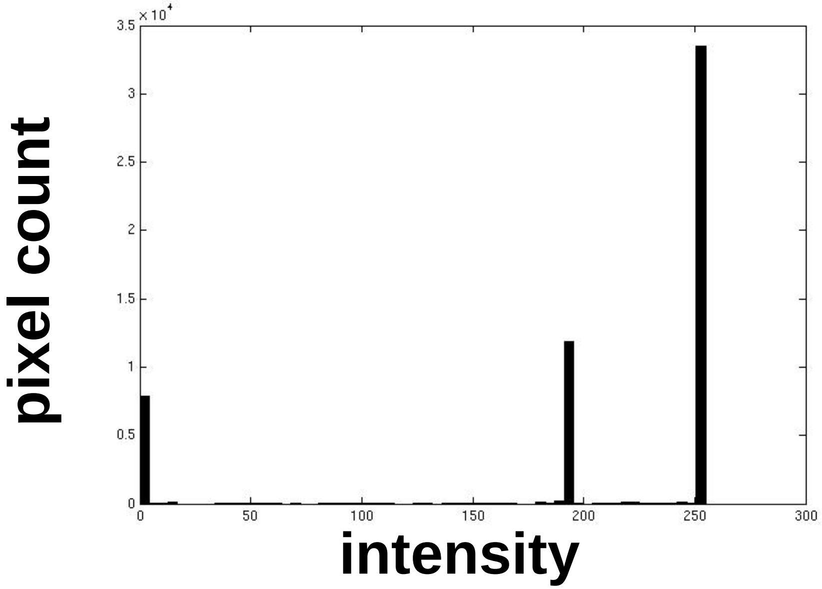
- Well-distributed histogram

Histogram Example

Black = 0
Gray = 190
White = 254

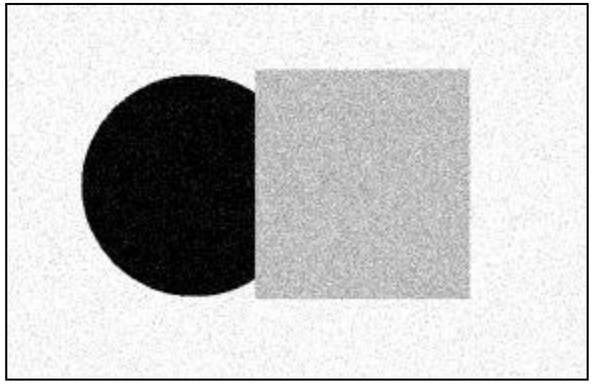


input image



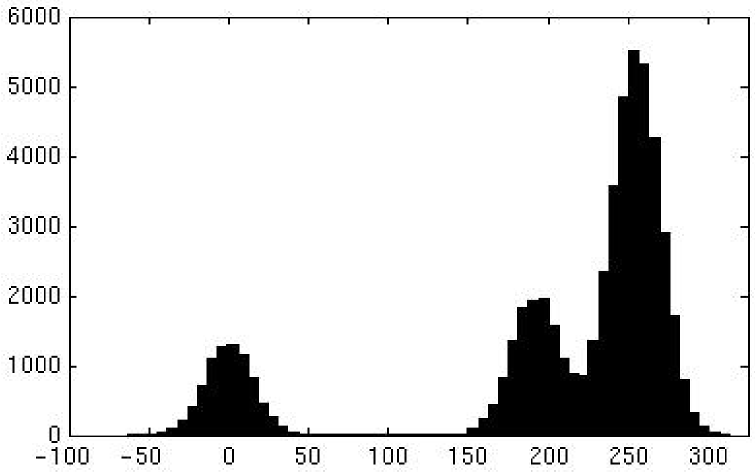
Histogram Example

- **Reality**



input image

pixel count



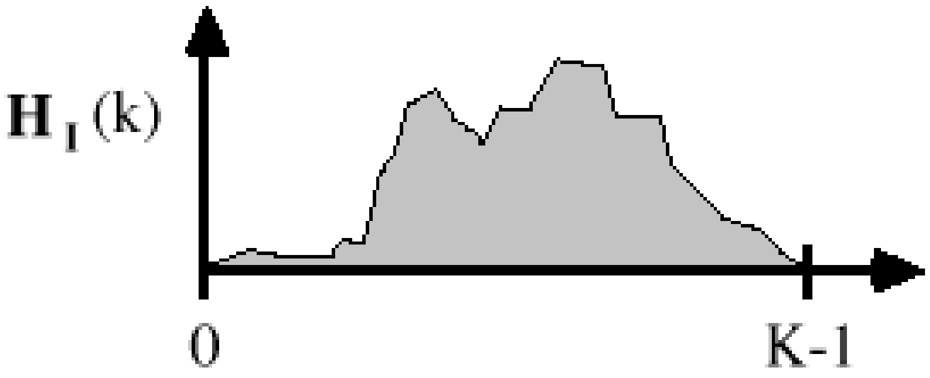
intensity

BIMODAL HISTOGRAM

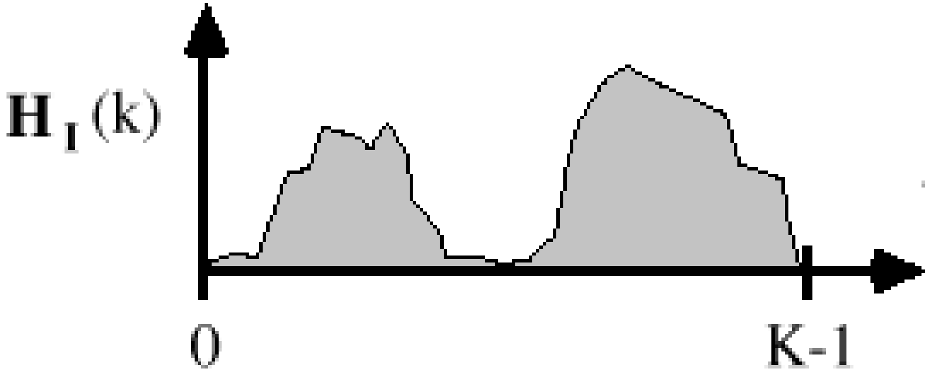
- Thresholding usually works best when there are **dark objects** on a **light background**
- Or when there are **light objects** on a **dark background**
- Images of this type tend to have histograms with **multiple distinct peaks or modes in them**

BIMODAL HISTOGRAM

- If the peaks are well-separated, threshold selection can be easy



bimodal histogram
poorly separated

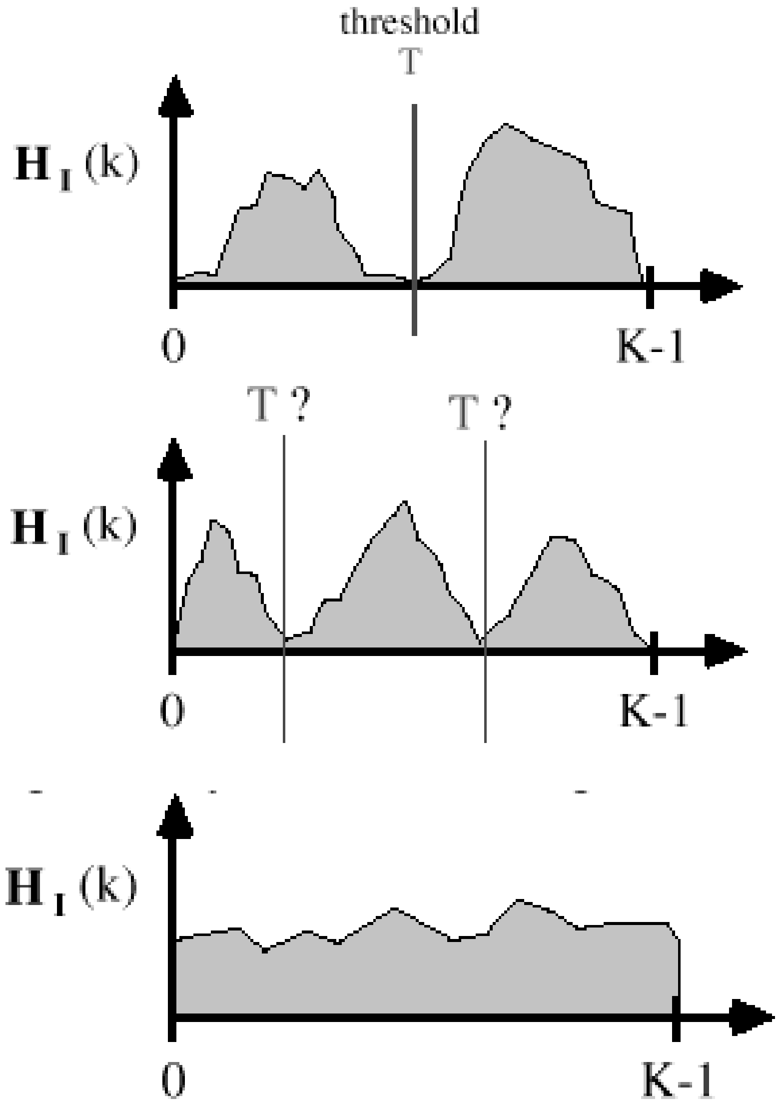


bimodal histogram
well separated peaks

- Set the threshold T somewhere between the peaks
- It may be an interactive trial-and-error process

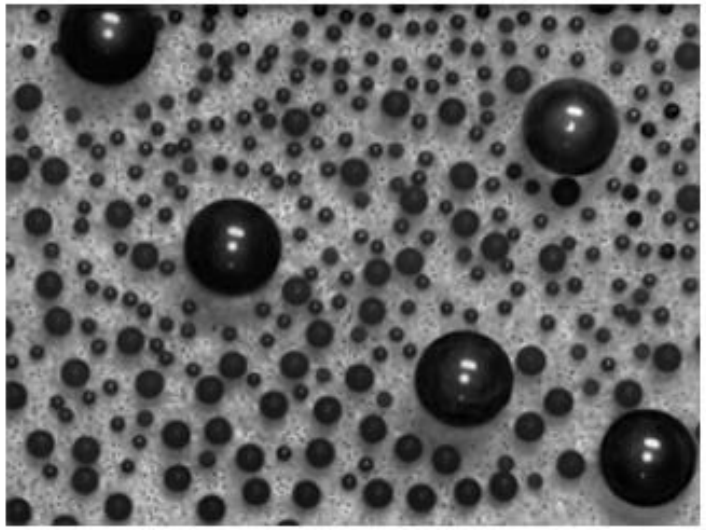
THRESHOLD SELECTION FROM HISTOGRAM

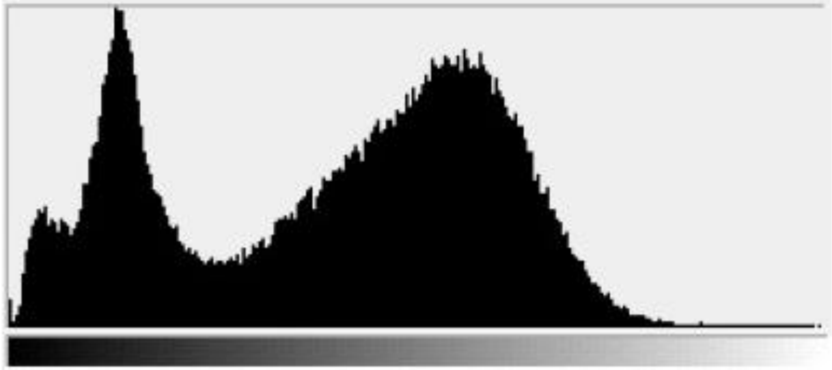
- Placing threshold T between modes may yield acceptable results
- Exactly **where in between** can be difficult to determine
- An image histogram may contain multiple modes. Placing the threshold in different places will produce very different results
- Histogram may be "flat," making threshold selection **difficult**



Example: How to find T

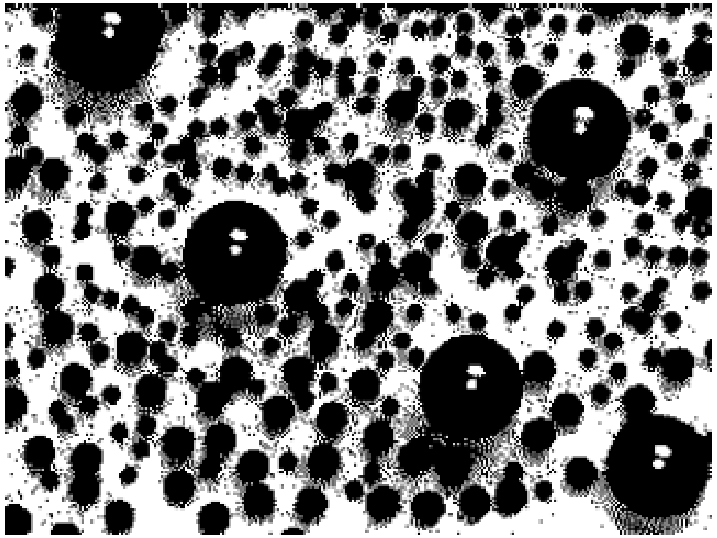
- Microscopic image
- Grey level → binary image
- Binary: 1-cell present, 0-cell absent

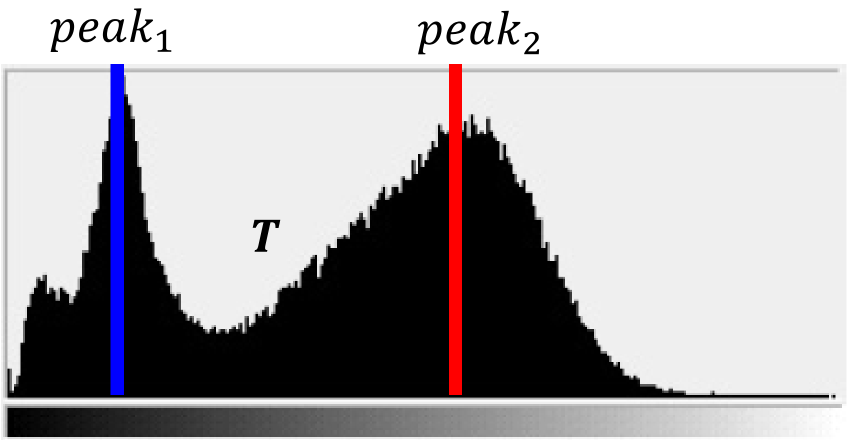




Example: How to find T

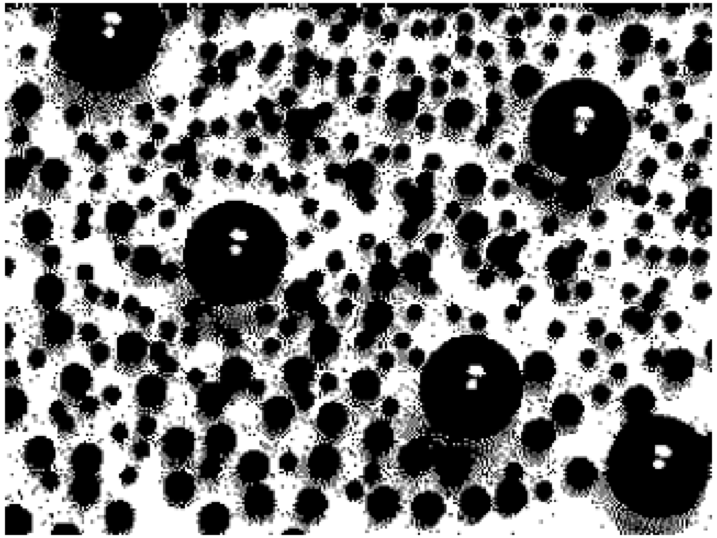
- If the peaks are known
- We can choose T between peaks (say average)

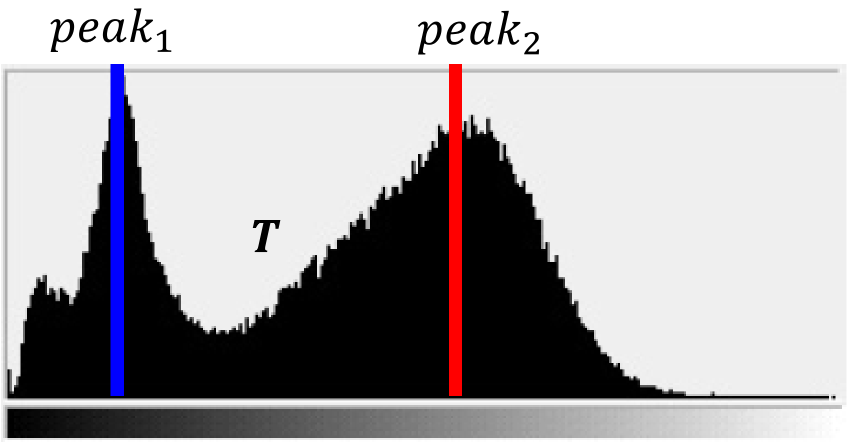




Example: How to find T

- If the peaks are known
- We can choose T between peaks (say average)

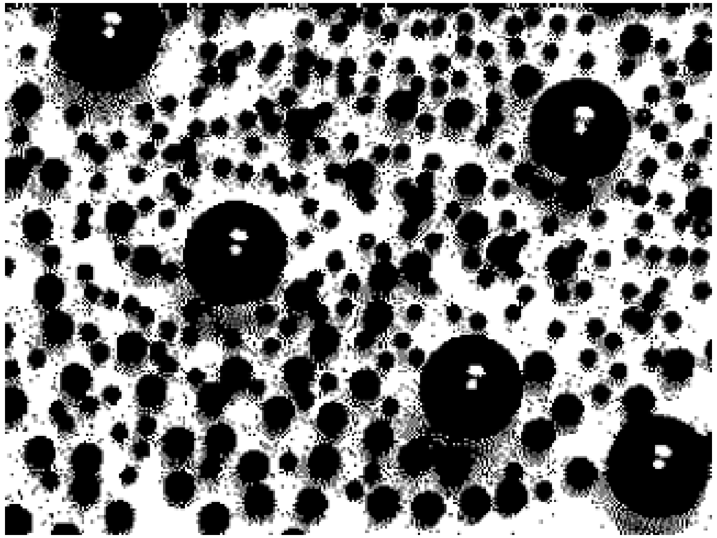


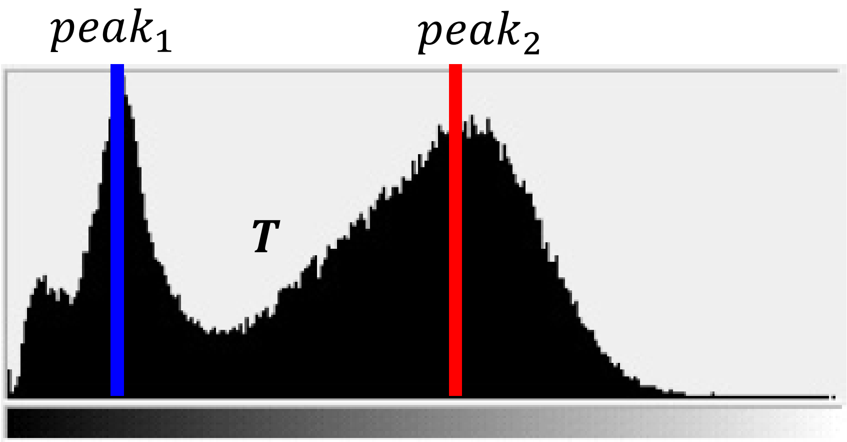


We do not know the peaks!!

Example: How to find T

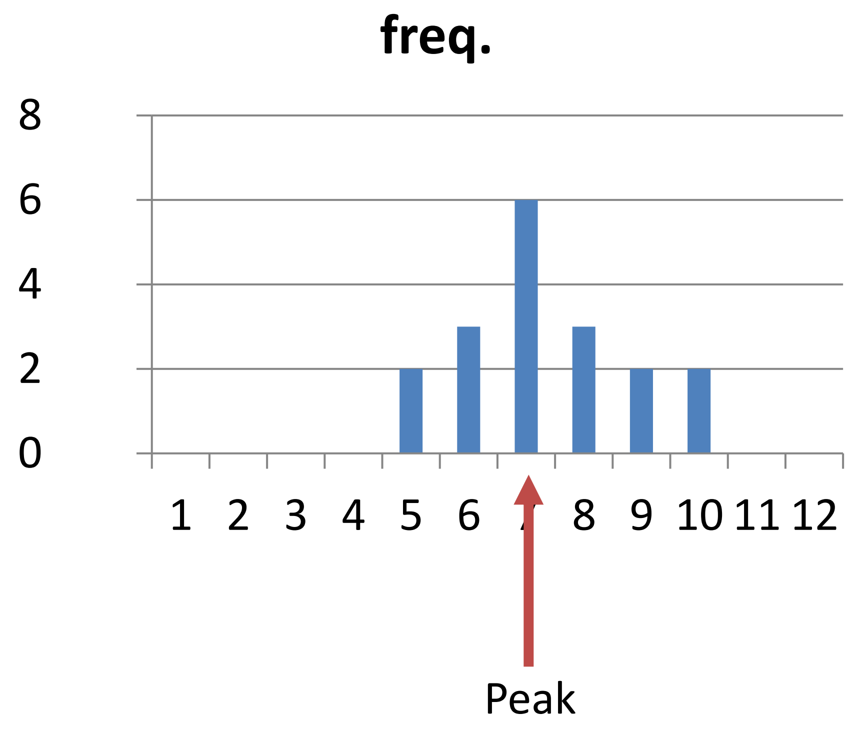
- If the Threshold (T) is known
- Can we determine the peaks?





Recap: Probability

- Data: {5,5,6,6,6,7,7,7,7,7,7,8,8,8,9,9,10,10}
- X: random variable



Recap: Probability

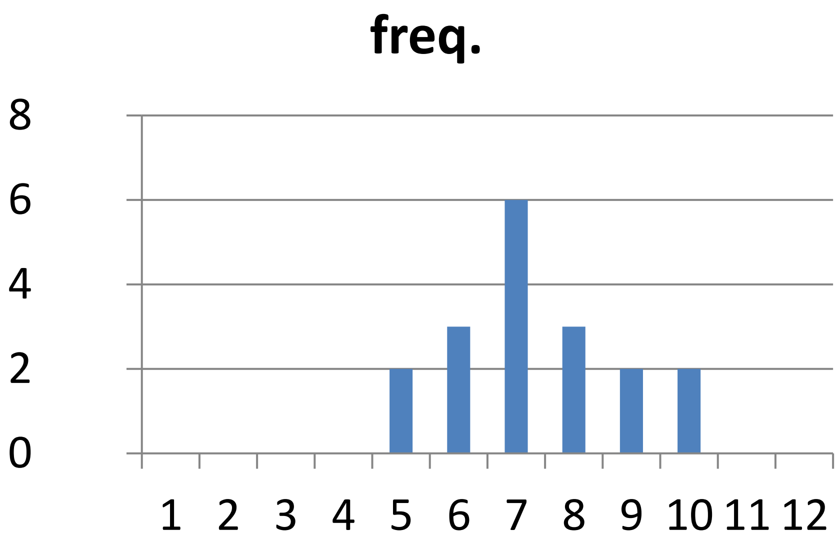
- Data: {5,5,6,6,6,7,7,7,7,7,7,8,8,8,9,9,10,10}
- X: random variable
- P: X → [0,1] probability function
- $P(X = 7) = ?$

Recap: Probability

- Data: {5,5,6,6,6,7,7,7,7,7,7,8,8,8,9,9,10,10}
- X: random variable
- P: probability function
- $P(X = 7) = 0.33$

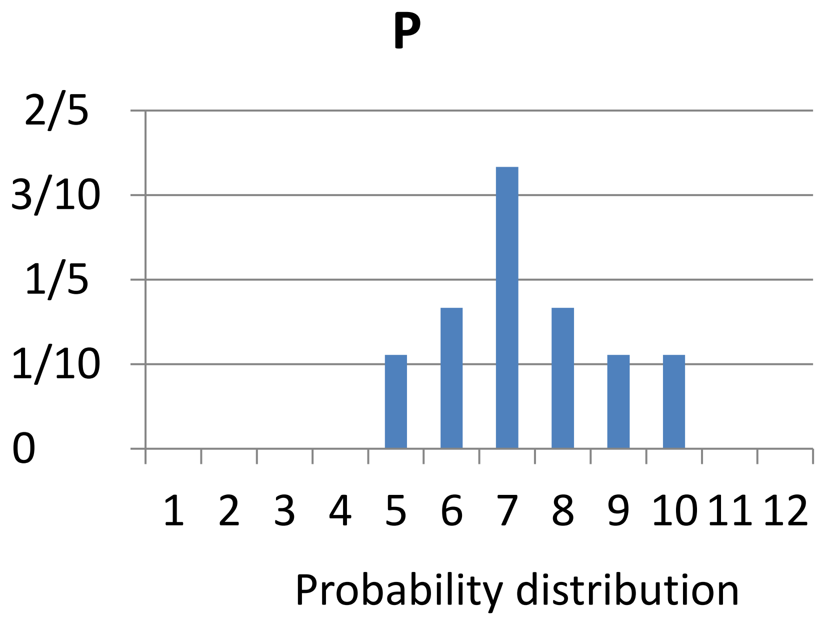
Recap: Histogram

- Data: {5,5,6,6,6,7,7,7,7,7,7,8,8,8,9,9,10,10}
- X: random variable
- p: probability function
- $p(X = 7) = 0.33$
- Histogram



Recap: Probability Distribution

- Data: {5,5,6,6,6,7,7,7,7,7,7,8,8,8,9,9,10,10}
- X: random variable
- p: probability function
- $\mathrm{p}(X = 7) = 0.33$
- p→Normalize(Histogram)



- Where is the peak for this case?

Expectation

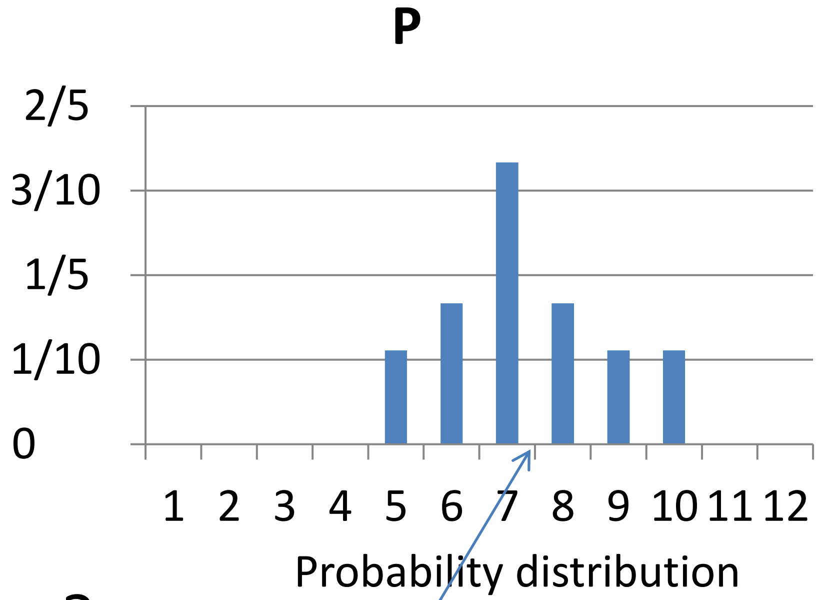
- $E(X)$ → Expected value of random variable X
- ~Average of all the expected values of random variable X
- $E(X) = ?$

Expectation

- $E(X) \rightarrow$ Expected value of random variable X
- ~Average of all the values
- $E(X) = \sum X\, p(X)$

Recap: Probability Distribution

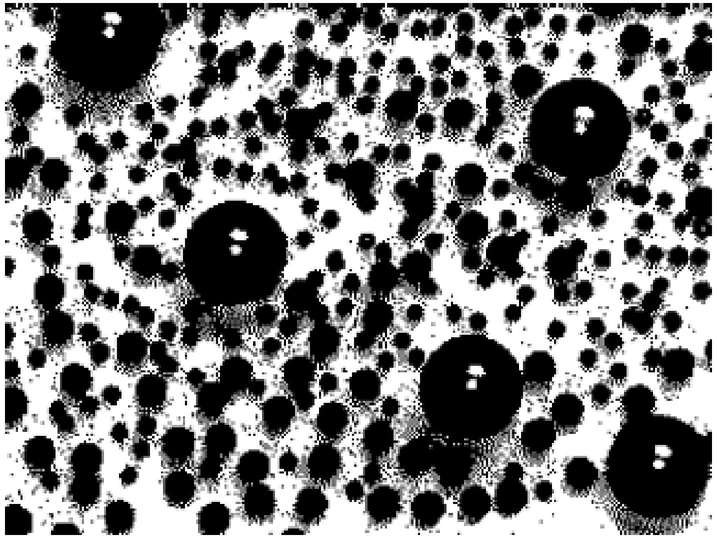
- Data: {5,5,6,6,6,7,7,7,7,7,7,8,8,8,9,9,10,10}
- X: random variable
- p: probability function
- $p(X = 7) = 0.33$
- p→Normalize(Histogram)

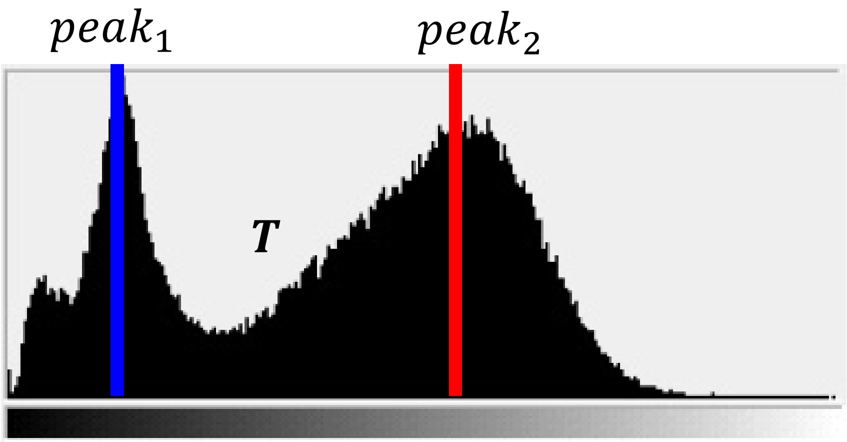


- Where is the peak for this case?
- E(X) = 5 * 0.11 + 6*0.17 + 7*0.33 +… = **7.33**

Example: How to find T

- If the Threshold (T) is known
- Can we determine the peaks?
- Yes, Compute Expectation on either side of the threshold.

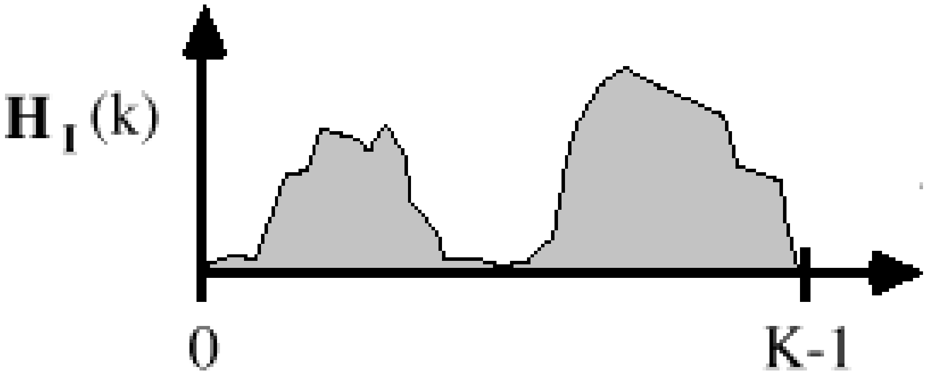




Algorithm

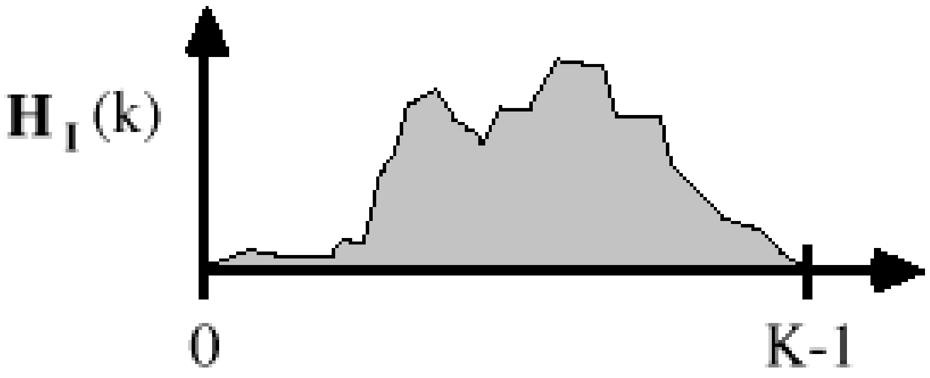
$$Initialize\ T = K/2$$
$$Do$$
$$\quad Compute\ \mu_1 = E(X)\forall X < T$$
$$\quad Compute\ \mu_2 = E(X)\forall X \geq T$$
$$\quad Set\ T = \frac{\mu_1 + \mu_2}{2}$$
$$While\ \Delta\mu_1 ! = 0\ \&\Delta\mu_2 ! = 0$$

AKA: Expectation Maximization (simple version)

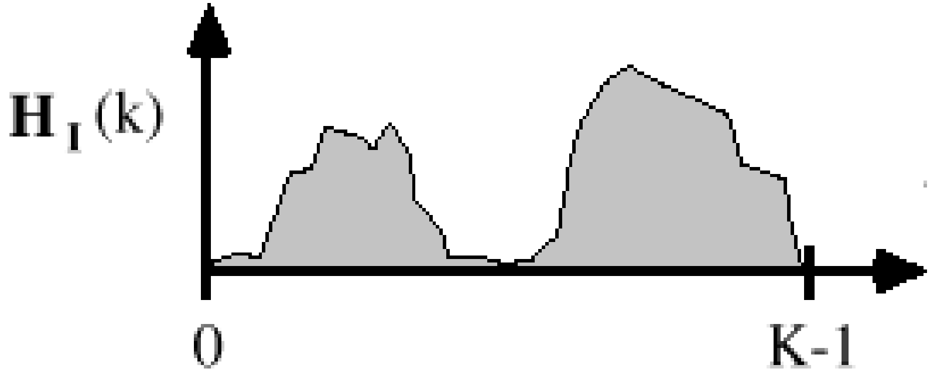


bimodal histogram
well separated peaks

BIMODAL HISTOGRAM



bimodal histogram
poorly separated

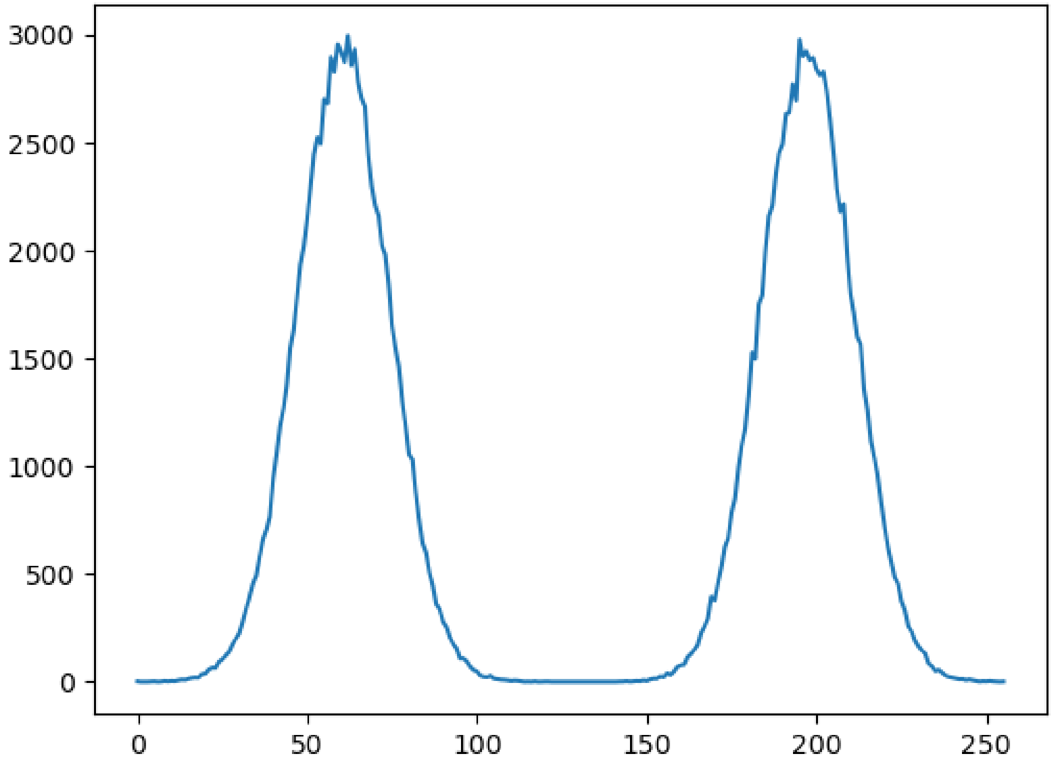


bimodal histogram
well separated peaks

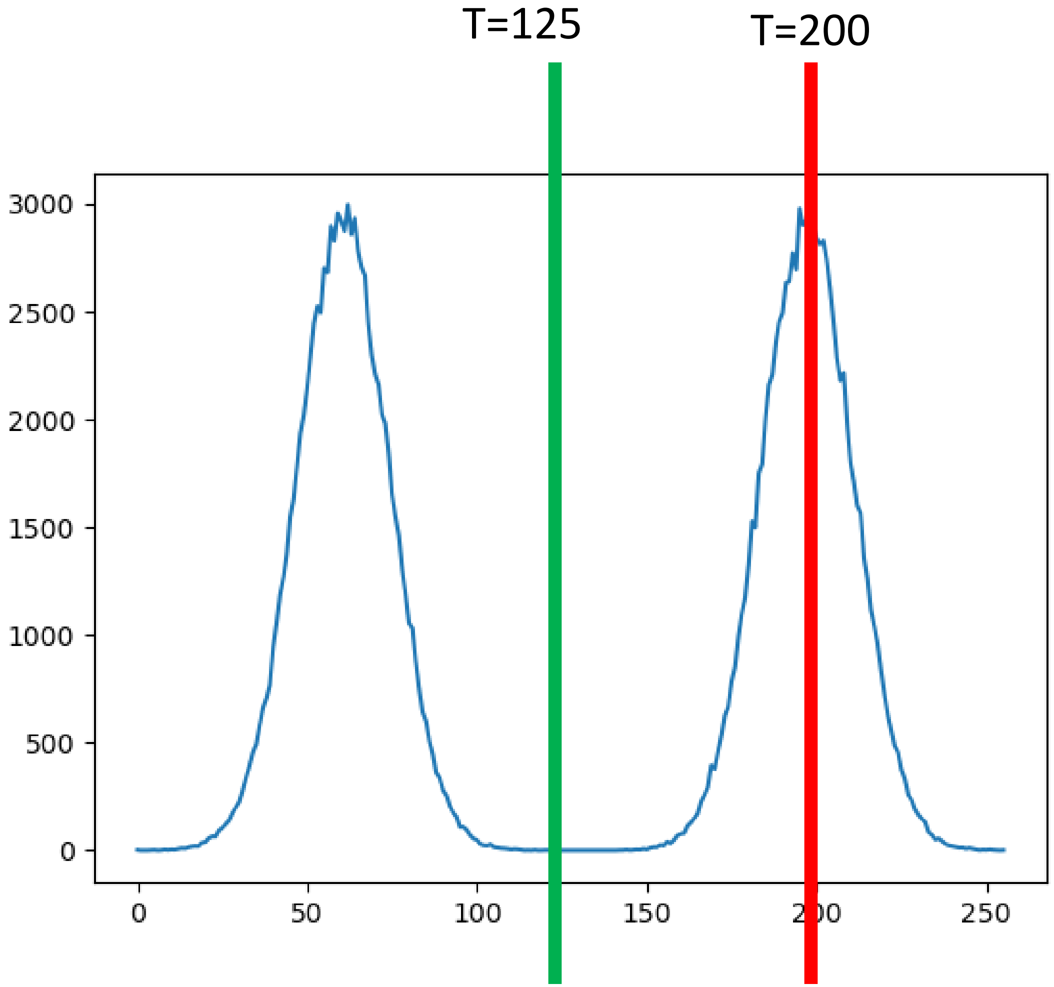
Otsu's Binarization

- Popular thresholding method
- It works on the histogram of the image
- It assumes that the histogram is bimodal

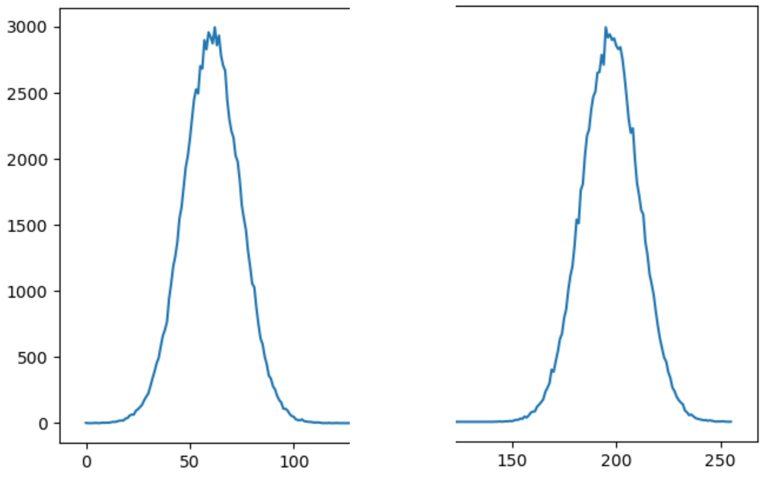
Bimodal Histogram



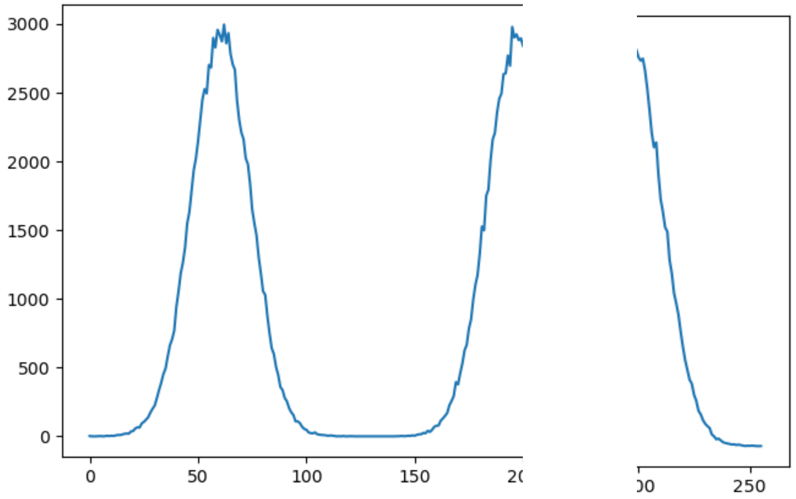
Bimodal Histogram



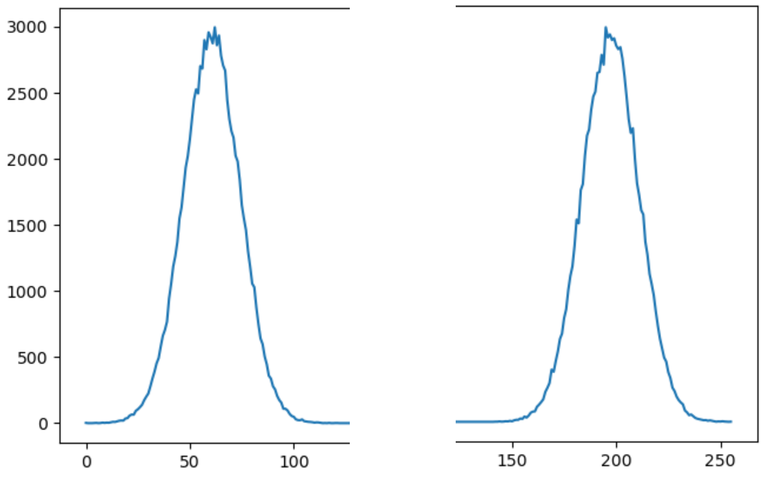
T=125


3000
2500
2000
1500
1000
500
0
0
50
100
150
200
250

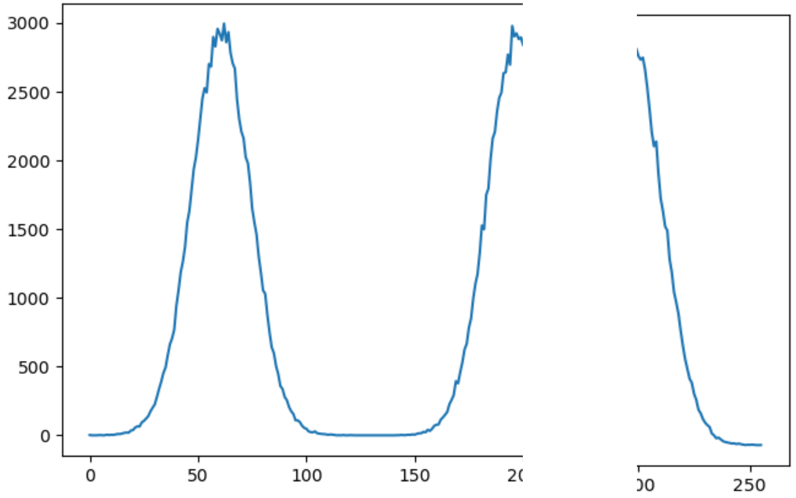
T=200


3000
2500
2000
1500
1000
500
0
0
50
100
150
250

T=125



T=200



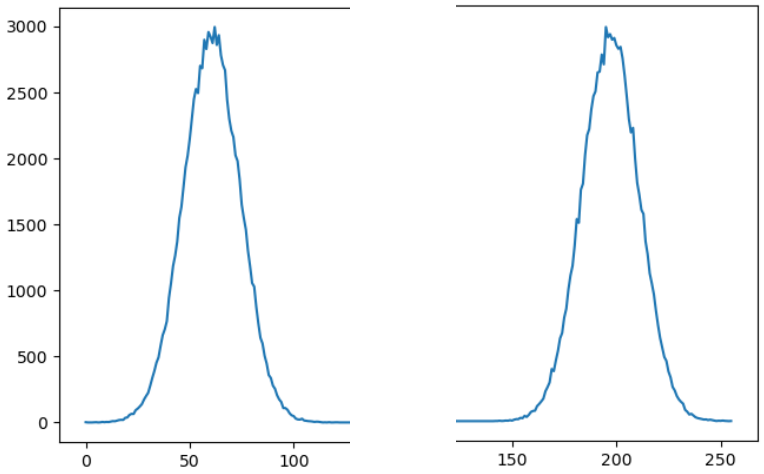
Compute a statistical value

Compute a statistical value

Compare

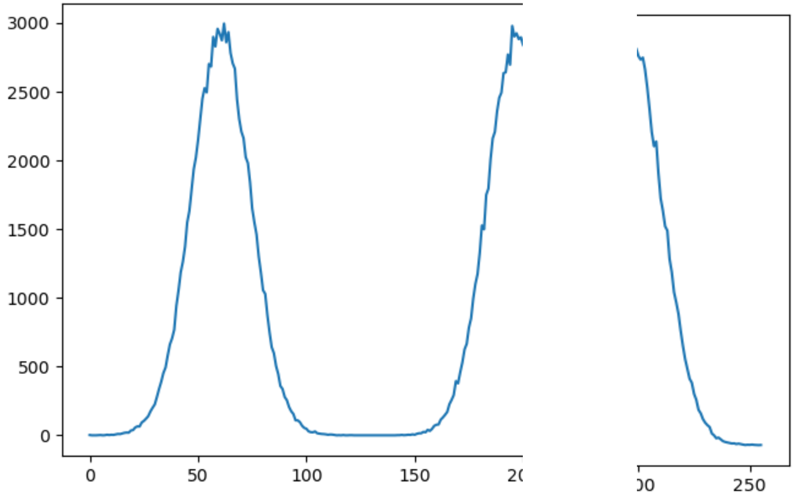
Pick Threshold

T=125



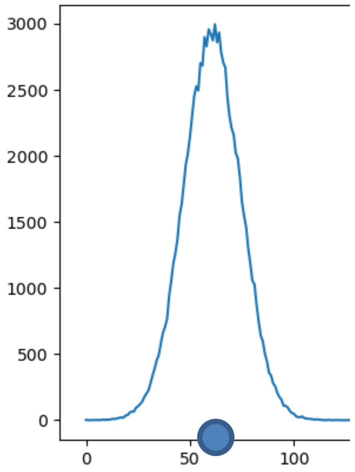
What metric to compute?

T=200

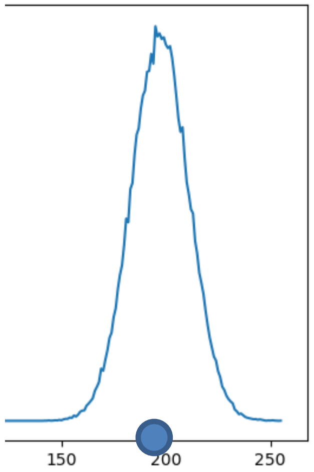


What metric to compute?

T=125

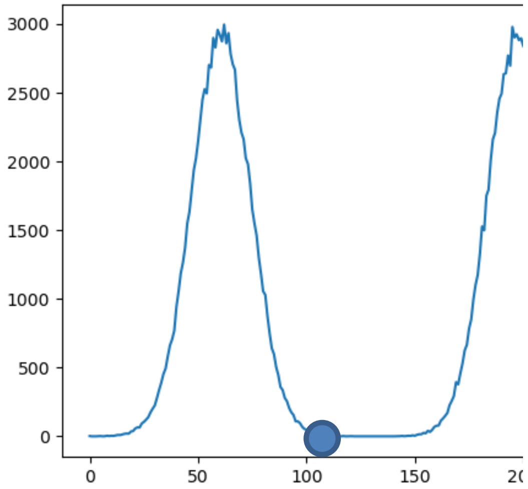


Mean = 62.03

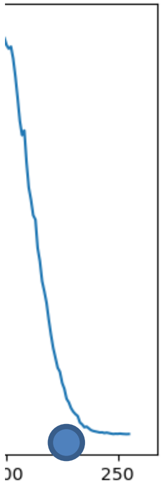


Mean = 195.8

T=200

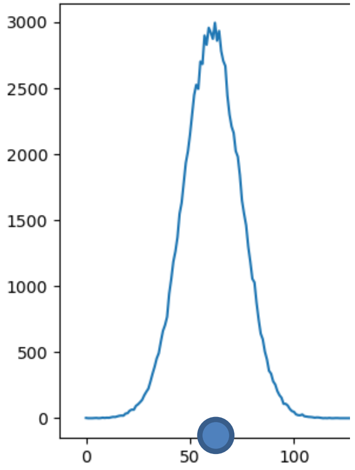


Mean = 107.54

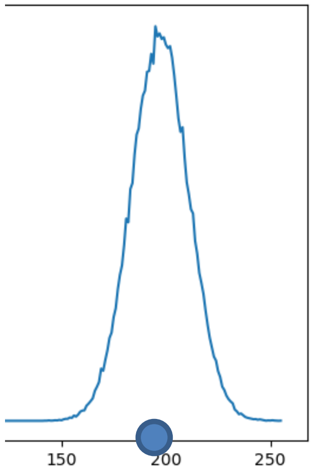


Mean = 225.4

T=125

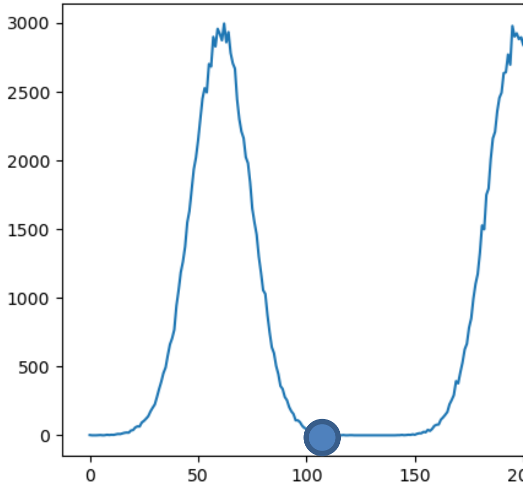


Mean = 62.03
Var = 189.3

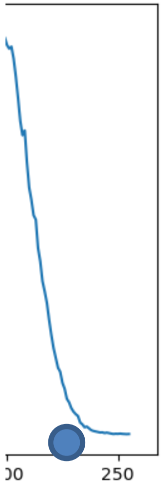


Mean = 195.8
Var = 190.0

T=200

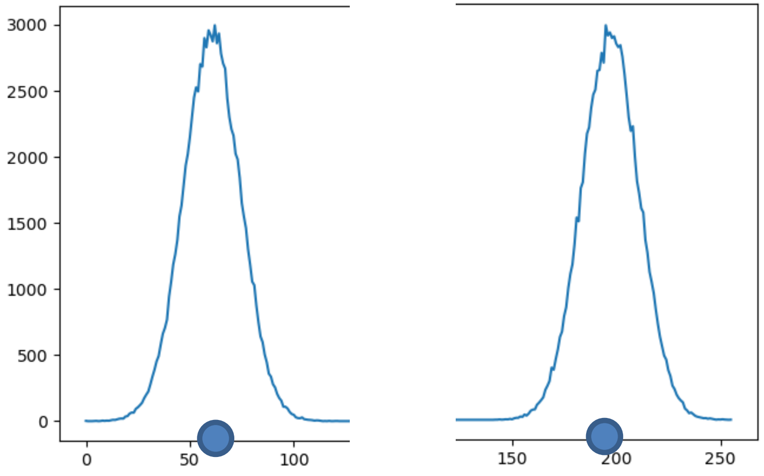


Mean = 107.54
Var = 3709.54



Mean = 225.4
Var = 259.1

T=125



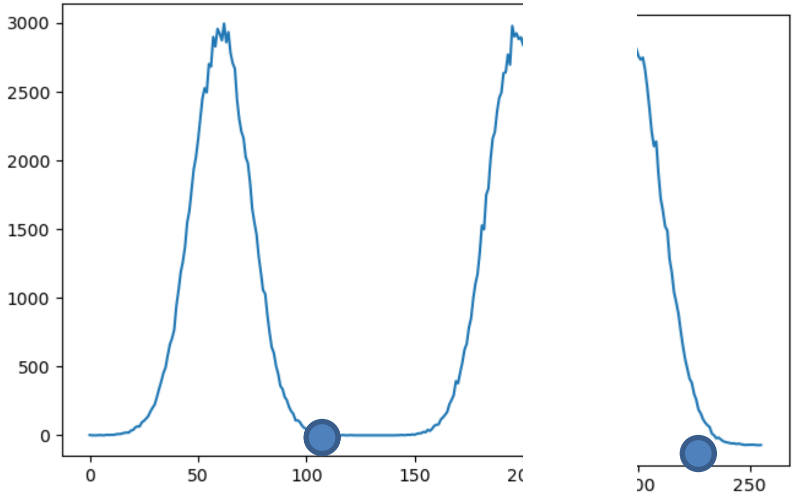
Mean = 62.03

Var = 189.3

Mean = 195.8

Var = 190.0

T=200



Mean = 107.54

Var = 3709.54

Mean = 225.4

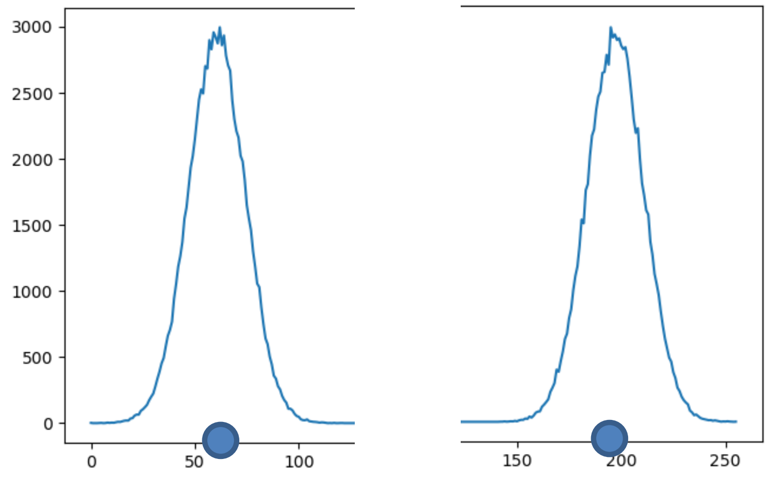
Var = 259.1

Intra-class Variance

Compare intra-class var

Pick Threshold that minimizes this value

T=125



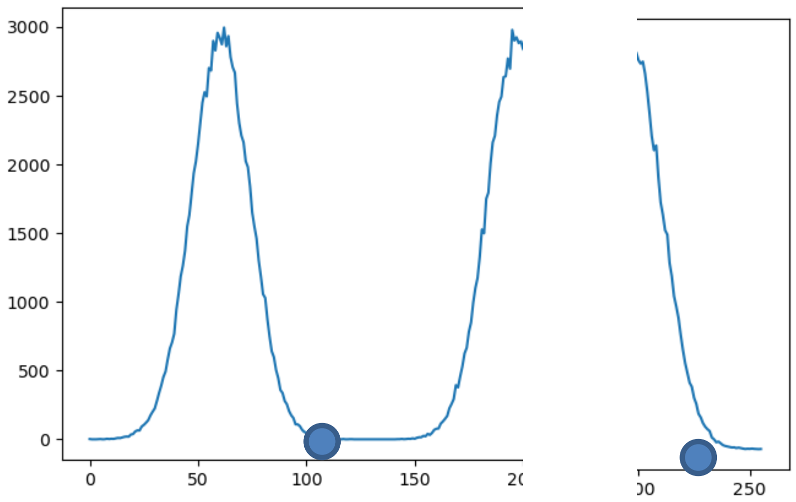
Mean = 62.03

Var = 189.3 +

Mean = 195.8

Var = 190.0

T=200



Mean = 107.54

Var = 3709.54 +

Mean = 225.4

Var = 259.1

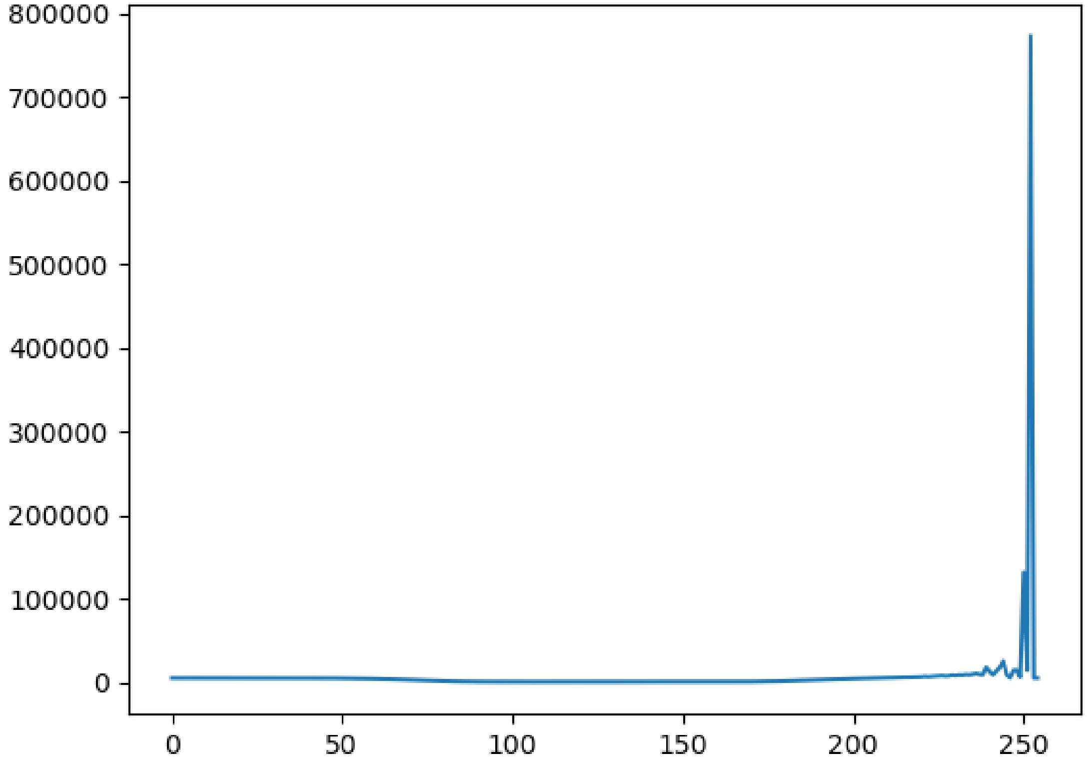
364 $\underbrace{\hspace{10em}}_{}$ Intra-class Variance 4370

Compare intra-class var

Pick Threshold that minimizes this value

Fn = Var1 + var 2

fn



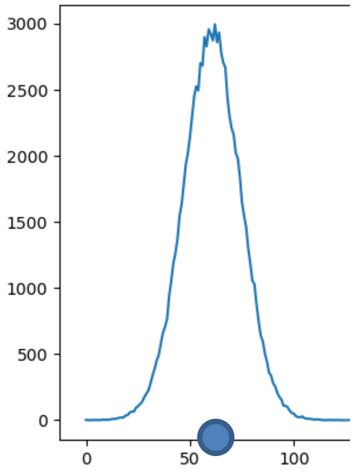
Threshold

Extreme values

Low values on either side of the valley

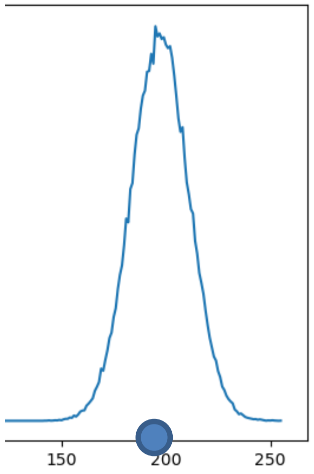
Weighted Sum

T=125



Mean = 62.03

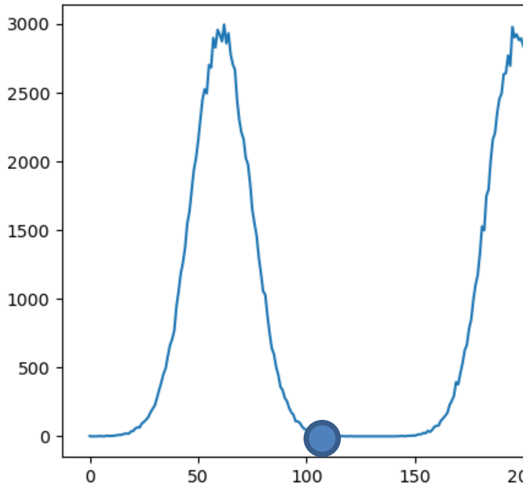
Var = 189.3



Mean = 195.8

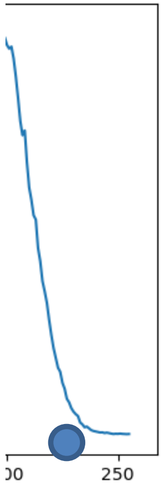
Var = 190.0

T=200



Mean = 107.54

Var = 3709.54



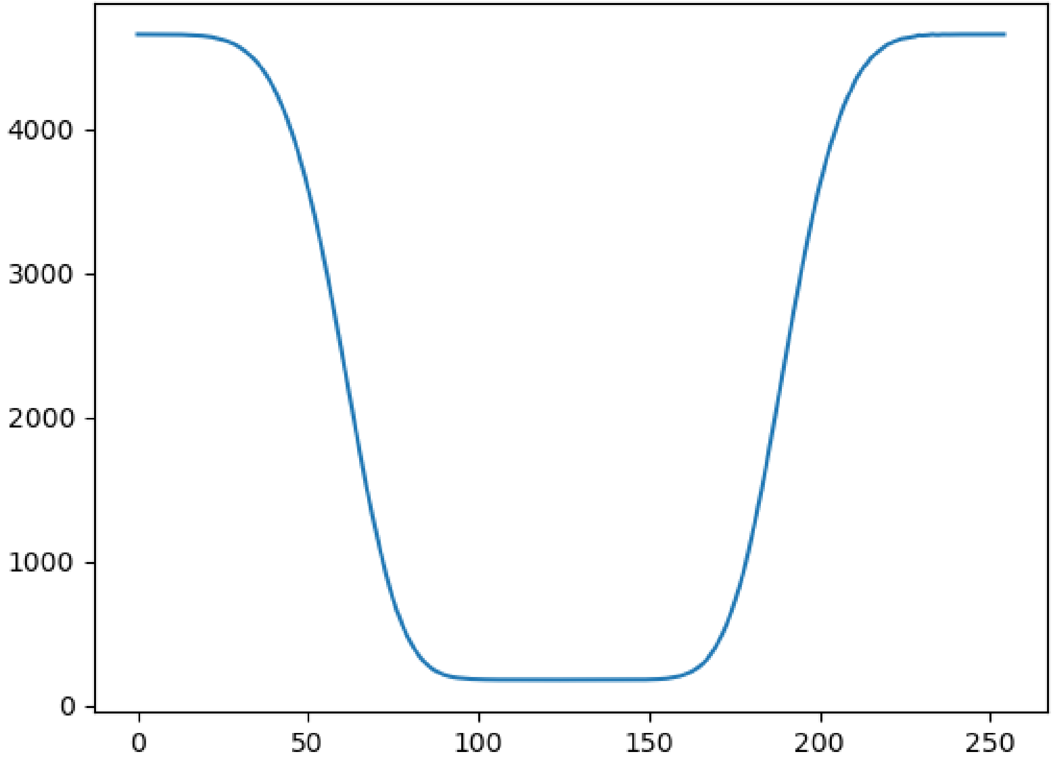
Mean = 225.4

Var = 259.1

$$fn = w_1\ var1 + \ w_2 var2$$

$$w_1 = \sum_{i=0}^{t} p(i),\ w_2 = \sum_{i=t+1}^{255} p(i),$$

$$fn = w_1\, var1 + w_2 var2$$



fn

Threshold

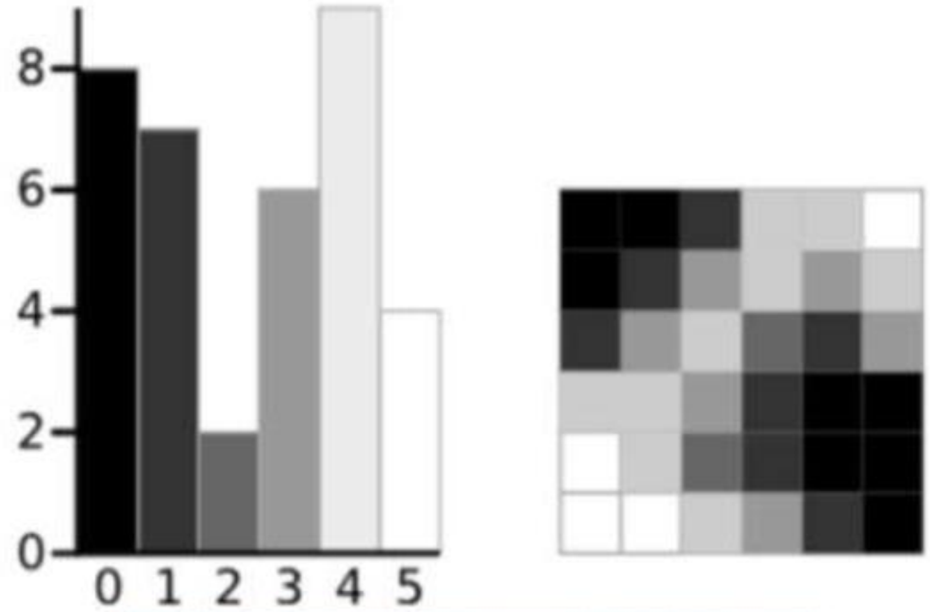
Extreme values

Low values on either side of the valley

Algorithm

1. Compute Histogram
2. Compute probabilities
3. Iterate through all possible threshold values (t=0 to t= 255)
   1. Compute weights (q_1, q_2)
   2. Compute mean (μ_1, μ_2)
   3. Compute intra-class variance $(\boldsymbol{\sigma_1^2}, \boldsymbol{\sigma_2^2})$
   4. Compute weighted sum of intra-class variance $(\boldsymbol{q_1\sigma_1^2} + \boldsymbol{q_2\sigma_2^2})$
4. Pick threshold that minimizes the weighted sum of intraclass variance

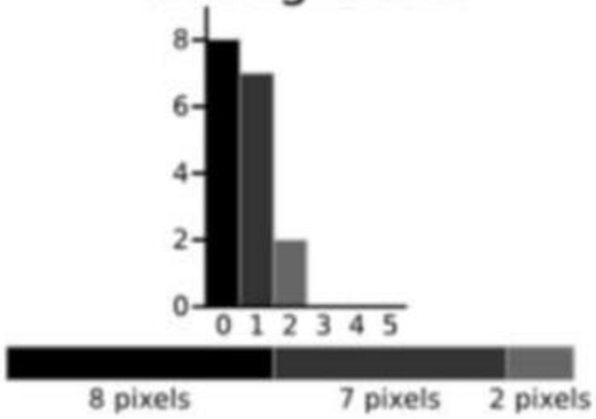
Example



A 6-level greyscale image and its histogram

- The calculations for finding the foreground and background variances (the measure of spread) for a single threshold are now shown in next slide.

Background

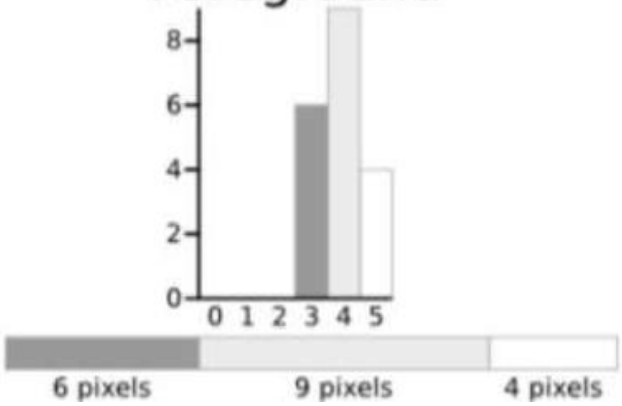


$$\text{Weight} \quad W_b = \frac{8+7+2}{36} = 0.4722$$

$$\text{Mean} \quad \mu_b = \frac{(0 \times 8) + (1 \times 7) + (2 \times 2)}{17} = 0.6471$$

$$\begin{aligned}\text{Variance} \quad \sigma_b^2 &= \frac{((0-0.6471)^2 \times 8) + ((1-0.6471)^2 \times 7) + ((2-0.6471)^2 \times 2)}{17} \\ &= \frac{(0.4187 \times 8) + (0.1246 \times 7) + (1.8304 \times 2)}{17} \\ &= 0.4637\end{aligned}$$

Foreground



$$\text{Weight} \quad W_f = \frac{6+9+4}{36} = 0.5278$$

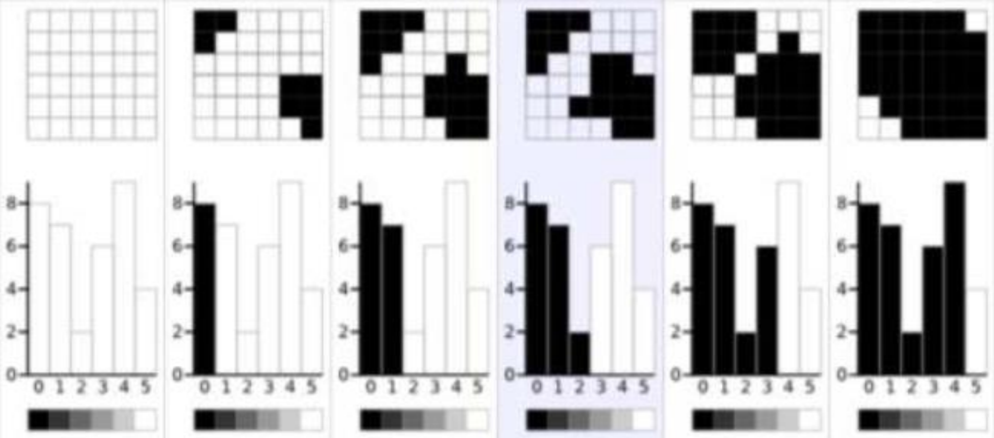
$$\text{Mean} \quad \mu_f = \frac{(3 \times 6) + (4 \times 9) + (5 \times 4)}{19} = 3.8947$$

$$\begin{aligned}\text{Variance} \quad \sigma_f^2 &= \frac{((3-3.8947)^2 \times 6) + ((4-3.8947)^2 \times 9) + ((5-3.8947)^2 \times 4)}{19} \\ &= \frac{(4.8033 \times 6) + (0.0997 \times 9) + (4.8864 \times 4)}{19} \\ &= 0.5152\end{aligned}$$

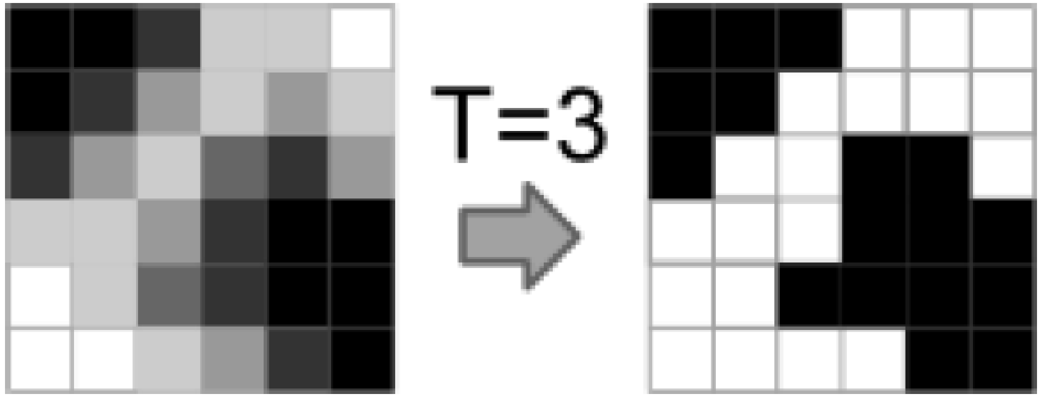
- The next step is to calculate the 'Within-Class Variance'. This is simply the sum of the two variances multiplied by their associated weights

$$\text{Within Class Variance} \quad \sigma_W^2 = W_b\,\sigma_b^2 + W_f\,\sigma_f^2 = 0.4722 * 0.4637 + 0.5278 * 0.5152$$
$$= 0.4909$$

- This final value is the 'sum of weighted variances' for the threshold value 3. This same calculation needs to be performed for all the possible threshold values 0 to 5. The table below shows the results for these calculations. The highlighted column shows the values for the threshold calculated above

Threshold	T=0	T=1	T=2	T=3	T=4	T=5
						
Weight, Background	$W_b = 0$	$W_b = 0.222$	$W_b = 0.4167$	$W_b = 0.4722$	$W_b = 0.6389$	$W_b = 0.8889$
Mean, Background	$M_b = 0$	$M_b = 0$	$M_b = 0.4667$	$M_b = 0.6471$	$M_b = 1.2609$	$M_b = 2.0313$
Variance, Background	$\sigma^2_b = 0$	$\sigma^2_b = 0$	$\sigma^2_b = 0.2489$	$\sigma^2_b = 0.4637$	$\sigma^2_b = 1.4102$	$\sigma^2_b = 2.5303$
Weight, Foreground	$W_f = 1$	$W_f = 0.7778$	$W_f = 0.5833$	$W_f = 0.5278$	$W_f = 0.3611$	$W_f = 0.1111$
Mean, Foreground	$M_f = 2.3611$	$M_f = 3.0357$	$M_f = 3.7143$	$M_f = 3.8947$	$M_f = 4.3077$	$M_f = 5.000$
Variance, Foreground	$\sigma^2_f = 3.1196$	$\sigma^2_f = 1.9639$	$\sigma^2_f = 0.7755$	$\sigma^2_f = 0.5152$	$\sigma^2_f = 0.2130$	$\sigma^2_f = 0$
Within Class Variance	$\sigma^2_W = 3.1196$	$\sigma^2_W = 1.5268$	$\sigma^2_W = 0.5561$	$\sigma^2_W = 0.4909$	$\sigma^2_W = 0.9779$	$\sigma^2_W = 2.2491$

- It can be seen that for the threshold equal to 3, as well as being used for the example, also has the lowest sum of weighted variances. Therefore, this is the final selected threshold. All pixels with a level less than 3 are background, all those with a level equal to or greater than 3 are foreground. As the images in the table show, this threshold works well.



Algorithm

1. Compute Histogram $\left(H(i)\right)$, with N different intensity values.
2. Compute probabilities $\boldsymbol{P(i)} = \frac{\boldsymbol{H(i)}}{\sum_i \boldsymbol{H(i)}}$

3. Iterate through all possible threshold values (t=0 to t= 255)

   3.1 Calculate Weights

$$\boldsymbol{q_1(t) = \sum_{i=0}^{t} P(i)} \quad \boldsymbol{q_2(t) = \sum_{i=t+1}^{N} P(i)}$$

   3.2 Compute mean

$$\mu_1(t) = \sum_{i=0}^{i=t} \frac{iP(i)}{q_1(t)}, \mu_2(t) = \sum_{i=t+1}^{i=N} \frac{iP(i)}{q_2(t)}$$

3.3 Compute intra-class variance

$$\boldsymbol{\sigma_1^2(t) = \sum_{i=0}^{t} \frac{\left(i - \mu_1(t)\right)^2 P(i)}{q_1(t)}} \qquad \boldsymbol{\sigma_2^2(t) = \sum_{i=t+1}^{N} \frac{\left(i - \mu_2(t)\right)^2 P(i)}{q_2(t)}}$$

3.4 Compute weighted sum of intra-class variance

$$\sigma_w{}^2(t) = q_1(t) \quad \sigma_1{}^2(t) + q_2(t) \quad \sigma_2{}^2(t)$$

4. Pick threshold that minimizes the weighted sum of intra-class variance