MATH 3338 Probability

Wenjiang Fu

Lecture 8 - MATH 3338
Ch 8 Law of Large Numbers

UNIVERSITY of HOUSTON
DEPARTMENT OF MATHEMATICS

Outline

LLN for Discrete RVs

Law of Large Numbers (LLN)

UNIVERSITY of HOUSTON
DEPARTMENT OF MATHEMATICS

LLN for Discrete RVs

- **Chebyshev Inequality**

  Theorem 8.1 (Chebyshev Inequality) Let X be a discrete RV with expected value $\mu = E(X)$, and let $\varepsilon > 0$ be any positive real number. Then

  $$P(|X-\mu| > \varepsilon) \leq \frac{Var(X)}{\varepsilon^2}.$$

  Proof. Let $m(x)$ denote the distr function of X. Then the prob that X differs from μ by at least ε is given by

  $$P(|X-\mu| \geq \varepsilon) = \sum_{|X-\mu|\geq\varepsilon} m(x)$$

  Consider the variance

  $$Var(X) = \sum_x (x-\mu)^2 m(x) \geq \sum_{|x-\mu|\geq\varepsilon} (x-\mu)^2 m(x) \geq \varepsilon^2 \sum_{|x-\mu|\geq\varepsilon} m(x)$$

  Thus

  $$P(|x-\mu| \geq \varepsilon) \leq \frac{Var(X)}{\varepsilon^2}$$

LLN for Discrete RVs

- **Chebyshev Inequality**
  Example 8.1 Let X be any RV with $\mu = E(X)$, and $Var(X) = \sigma^2$. Then, if $\varepsilon = k\sigma$, Chebyshev's Inequality states that

$$P(|X - \mu| > k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}.$$

  The prob of a deviation from the mean of more than k standard deviations is $\leq 1/k^2$. For $k = 4, 5$, $1/k^2 = 1/16, 1/25$, etc. This means that the tail probability is well controlled by the variance and the distance from the mean.

LLN for Discrete RVs

- **Chebyshev Inequality**
- **Law of Large Numbers**

  Theorem 8.2 (Law of Large Numbers) Let $X_1, X_2, ..., X_n$ be an indep trials process, with finite expected value $\mu = E(X_i)$ and finite variance $\sigma^2 = Var(X_i)$. Let $S_n = X_1 + X_2 + ... + X_n$. Then for any $\varepsilon > 0$,

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \to 0$$

  as $n \to \infty$. Equivalently,

$$P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) \to 1$$

  as $n \to \infty$. This is the weak Law of Large Numbers (WLLN). Since the average sequence S_n/n converges in probability to the mean μ.

UNIVERSITY of HOUSTON
DEPARTMENT OF MATHEMATICS

LLN for Discrete RVs

- **Chebyshev Inequality**
- **Law of Large Numbers**

  Proof Since $X_1, X_2, ..., X_n$ are indep trials process, with finite $\mu = E(X_i)$ and finite variance $\sigma^2 = Var(X_i)$, $E(S_n/n) = \mu$ and $Var(S_n/n) = \sigma^2/n$. Following Chebyshev inequality,

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{Var(S_n/n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0$$

  as $n \to \infty$.

  Note No matter how small $\varepsilon > 0$ is, the average distance S_n/n from its mean μ will make the probability outside the interval $(\mu - \varepsilon, \mu + \varepsilon)$ converges to 0. So, eventually, all probability of the average will focus on the mean only.

LLN for Discrete RVs

- **LLN examples**
- **Coin Tossing**
  Consider tossing a coin n times with S_n be the number of heads up. Then S_n/n represents the fraction of times heads up, between 0 and 1. The LLN predicts the fraction converges to the probability of head up.
- **Die Rolling** Consider rolling a die n times. Let X_i be the outcome of the i-th time. Then $S_n = X_1, X_2, ..., X_n$ is the sum of the n outcomes. This is an indep trials process with mean $E(X_i) = 3.5$. By the LLN,

$$P\left(\left|\frac{S_n}{n} - 3.5\right| < \varepsilon\right) \to 1$$

LLN for Continuous RVs

- **Chebyshev Inequality**
- **Theorem 8.3** (Chebyshev inequality)
  Let X be a continuous RV with density function $f(x)$. Suppose X has a finite expected value $\mu = E(X)$. and finite variance $\sigma^2 = Var(X)$. Then for any positive number $\varepsilon > 0$, we have

  $$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.$$

  Proof It is analogous to the discrete case.
- **Example 8.4** Let X be any continuous RV wih $E(X) = \mu$ and $Var(X) = \sigma^2$. If $\varepsilon = k\sigma$, i.e. k standard deviations for some integer k, then

  $$P(|X - \mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}.$$

  The tail prob converges to 0 quickly.

LLN for Continuous RVs

- **Law of Large Numbers**
- **Theorem 8.4** (LLN)
  Let $X_1, ..., X_n$ be an indep trials process with a continuous density function f, finite expected value $\mu = E(X)$ and finite variance $\sigma^2 = Var(X)$. Let $S_n = X_1 + ... + X_n$ be the sum of the X_i. Then for any positive number $\varepsilon > 0$, we have

$$\lim_{n\to\infty} P(|\frac{S_n}{n} - \mu| \geq \varepsilon) = 0,$$

or equivalently

$$\lim_{n\to\infty} P(|\frac{S_n}{n} - \mu| < \varepsilon) = 1.$$

Note 1. This theorem may not be true if $Var(X) = \infty$.
2. The proof of the LLN is analogous to the discrete case.

LLN for Continuous RVs

- **Law of Large Numbers**
- **Examples** (LLN)

  1. Uniform distribution. Let $X_1, ..., X_n$ be an indep trials process following Unif[0,1]. It is known that $E(X_i) = 1/2$, and $Var(X_i) = 1/12$. Hence $E(S_n/n) = 1/2$, and $Var(S_n/n) = 1/(12n)$. For any $\varepsilon > 0$,

  $$P\left(\left|\frac{S_n}{n} - .5\right| \geq \varepsilon\right) \leq \frac{1}{12n\varepsilon^2}$$

  2. Normal distribution. Assume $X_1, .., X_n \sim N(0,1)$. Then $E(S_n/n) = 0$, $Var(S_n/n) = 1/n$.

  $$P\left(|\frac{S_n}{n}| \geq \varepsilon\right) \leq \frac{1}{n\varepsilon^2} \to 0$$

LLN for Continuous RVs

- **Law of Large Numbers**
- **Example 8.7** (Monte Carlo Method)
  Suppose we need to find an integral $\int_0^1 g(x)dx$ for a continuous function $g(x)$ on $[0, 1]$. We can use the following Monte Carlo method.
  1. Take a large number of random variates $X_n \sim Unif[0, 1]$, and calculate $Y_n = g(X_n)$.
  2. Take the mean $\overline{Y_n}$.

$$\overline{Y_n} \to E(Y) = \int_0^1 g(x)dx$$

by the LLN. Variance $\sigma^2 = E((Y_n - \mu)^2) = \int_0^1 (g(x) - \mu)^2 dx < B$, an upper bound. By Chebyshev inequality, for any small $\varepsilon > 0$, such as 0.001, .00001, etc. as long as ε is fixed,

$$P(|\overline{Y_n} - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0$$

LLN for Continuous RVs

- **Law of Large Numbers Example** (Monte Carlo Method)
  Take the integral of function $g(x) = e^{-x^2/2}$ over the interval [0,3] using the Monte Carlo method.
  1. Find the density $f(x)$ of a uniform distribution over interval [0,3].
  2. Generate a large number ($n \geq 1000$) random variates $x_1, ..., x_n$.
  3. Calculate the mean $\overline{y_n}$ of $y_1 = g(x_1), .., y_n = g(x_n)$.
  4. Make sure the mean $\overline{y_n}$ converges to the integral $\int_0^3 g(x)dx$ using the LLN.

$$\overline{Y_n} \to E(Y) = \int_0^3 g(x)dx$$

Solution 1-3. Density of unif[0,3] is $f(x) = 1/3$.

```
x=runif(5000,0,3); y=exp(-x**2/2);
ymean=mean(y)
```

4. $\overline{y_n} \to E(Y) = \int_0^3 g(x)f(x)dx = (1/3)\int_0^3 g(x)dx$.

$\Rightarrow 3\overline{y_n} \to \int_0^3 g(x)dx$. $3\overline{y_n} = 1.252$ vs. $.997\sqrt{\pi/2} = 1.250$

UNIVERSITY of HOUSTON
DEPARTMENT OF MATHEMATICS