Integrand Reduction Reloaded; Integration By Parts technique

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Abstract

Scattering amplitudes in quantum field theories allow us to compare the phenomenological prediction of particle theories with the measurements at collider experiments. The study of scattering amplitudes, in terms of their symmetries and analytic properties, provides a framework to develop techniques and efficient algorithms to evaluate cross sections.

In this poster, we describe an interesting technique based on the generation of Integration By Parts identities for the reduction of a generic E+1 legs L loops integral toward the sum of a minimal basis of Master Integrals. We also introduce Baikov parametrization which translates IBP identities into polynomial forms from integrals.

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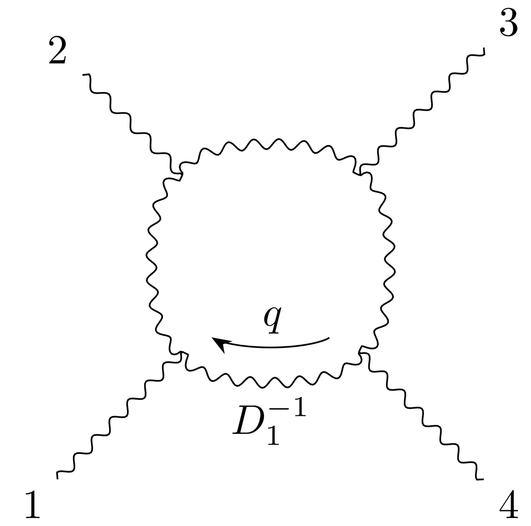
Scattering amplitudes

A scattering amplitude is given in a diagrammatic form → Feynman diagrams:



This diagram represents a 1 loop, $(2 \to 2)$ scattering process. We translate this kind of diagram into momentum integral form → a scattering amplitude.

$$\int d^d q \frac{N(q)}{D_1 D_2 D_3 D_4}$$

where $N(q)$ is determined by the theoretical model, e.g., the Standard Model.

Integrand Reduction and IBP

Integrand reduction is a kind of partial fractioning; for given Feynman integral kernel

$$\frac{N(q)}{D_0 \cdots, D_n}$$

we decompose the numerator via denominators

$$N(q) = R(q) + \sum_i Q_i(q) D_i$$

Once we substitute this alternative expression for $N(q)$, the second terms reduce a denominator. Algebraic geometry provide a systematic way of this reduction as a multivariate polynomial division.

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Integral By Parts identities

Integral by parts identities are a set of the following $i, j, n_1, \cdots, n_K$ indexed relations

$$0 = \int d^d \mathbf{q} \frac{\partial}{\partial q_i} \frac{p_j}{D_1^{n_1} \cdots D_N^{n_N}}$$

where p_j is either external or loop momentum, $\mathbf{q}$ represents L-fold integration momenta. Taking differentiation, we can rewrite this IBP identity as

$$0 = \sum_i c_i * I(\mathbf{n_i})$$

where

$$I(\mathbf{n}) := \int d^d \mathbf{q} \frac{1}{D_1^{n_1} \cdots D_N^{n_N}}$$

of $\mathbf{n}$ indexed scalar integral, c_i's are polynomials of $\mathbf{n}$, d, and external kinematical parameters.

Baikov parametrization

Baikov parametrization is essentially a coordinate change ($\mathbf{q} \mapsto x := D$), which allows us to write a scalar Feynman integral in the following form:

$$\int d^d \mathbf{q} \frac{1}{D_1^{n_1} \cdots D_N^{n_N}} = C \int \frac{dx_1 \cdots dx_N}{x_1^{n_1} \cdots x_N^{n_N}} P^{\frac{d-M-1}{2}}$$

where $M = L + E$ and P is the Jacobi determinant of the coordinate change. We call this P Baikov polynomial, which is a polynomial of x's and external kinematics.

IBP identities in Baikov parametrization

Under Baikov parametrization, IBP identities $0 = \int d^d \mathbf{q} \frac{\partial}{\partial q_i} \frac{p_j}{D_1^{n_1} \cdots D_N^{n_N}}$ become the following polynomial form, without integration:

$$0 = \hat{O}\left(x, \frac{\partial}{\partial x}\right) \left[P^{\frac{d-M-1}{2}}\right]$$

where d is regulated dimension, $M = L + E$, and P is Baikov polynomial.

Laporta algorithm

For a fixed N, IBP identities among the family of integral $I(\mathbf{n})$ gives us infinite set of linear equations, but the linear space spanned by these integrals is proved to be finite dimensional. So, by creating new identities of different $\mathbf{n}$, we can eventually create enough number of linear equations to solve. A solution is called Master Integrals; any integral can be represented as a finite linear combination of MI's.

A well-known public code is Reduze. More recent implementation is called Kira, which eliminates linear dependent equations before it solves system. Since linear dependent equations carry no new information toward the system, eliminating such dependent subset can reduce the computational space and time.

Sketch of a variant Laporta algorithm

We represent an IBP identity as a descending list of terms:

$$0 = \sum_i c_i I_i \Leftrightarrow \texttt{[c1*I1,c2*I2, ..]}$$

where I_1 is the most complicated integral. Among the system of IBP identities, we group the equations with the same highest integral, and place like the following form:

$$\left(\begin{array}{cccccccc}
\bullet & \bullet & \bullet & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
 & \bullet & \bullet & & & & & \\
 & \bullet & \bullet & \bullet & & & & \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & \bullet & \bullet & & & \\
 & & \bullet & \bullet & \bullet & \bullet & \bullet & \\
 & & & \vdots & & & &
\end{array}\right)$$

I.e., in each subset of equations, they share the highest integral (the left most $\bullet$) and they line up from shorter to longer.

Within the subset of equations, substituting the top element into the successors, we eventually have the right triangular form:

$$\left(\begin{array}{cccccccc}
\bullet & \bullet & \bullet & & & & & \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & & \\
 & & \bullet & & & & & \\
 & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & & \\
 & & & & & \vdots & &
\end{array}\right)$$

Once we reach this right triangular form, use back substitution; for the first two lists, using second equation, we substitute it in the first and represent the highest integral via lower integrals:

$$\left(\begin{array}{cccccc}
\bullet & \bullet & \bullet & & & \\
 & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}\right) \mapsto \left(\begin{array}{cccccc}
\bullet & 0 & \bullet & \bullet & \bullet & \bullet \\
 & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}\right)$$

Finally we have the following form of equations:

$$\left(\begin{array}{cccccccc}
\bullet & 0 & 0 & 0 & 0 & \star & \star & \star \\
 & \bullet & 0 & 0 & 0 & \star & \star & \star \\
 & & \bullet & 0 & 0 & \star & \star & \star \\
 & & & \bullet & 0 & \star & \star & \star \\
 & & & & \bullet & \star & \star & \star \\
 & & & & & \star & \star & \star
\end{array}\right)$$

The $\star$'s are the Master Integrals of this system.

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