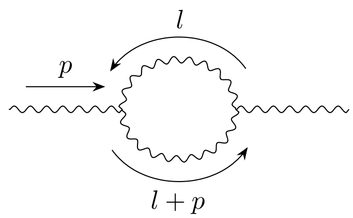
An example: one loop bubble diagram and its IBP relations

Let us consider the following integral



where we consider general powers on denominators:

$$
\begin{aligned}
I(n_1, n_2) &= \int d^d l_1 d^d l_2 \frac{1}{D_1^{n_1} D_2^{n_2}} \\
D_1 &:= l^2 \\
D_2 &:= (l+p)^2.
\end{aligned}
$$

Consider an IBP relation generated by

$$
\begin{aligned}
\frac{\partial}{\partial l} \cdot p \frac{1}{D_1^{n_1} D_2^{n_2}} &= \frac{-n_1}{D_1^{n_1+1}} p \cdot \frac{\partial D_1}{\partial l} \frac{1}{D_2^{n_2}} + \frac{1}{D_1^{n_1}} \frac{-n_2}{D_2^{n_2+1}} p \cdot \frac{\partial D_2}{\partial l} \\
&= \frac{-n_1 2p \cdot l}{D_1^{n_1+1} D_2^{n_2}} + \frac{-n_2 p \cdot (2p + 2l)}{D_1^{n_1} D_2^{n_2+1}}
\end{aligned}
$$

Since p^2 is an "external" parameter of this event, and

$$
D_2 - D_1 - p^2 = 2p \cdot l
$$

implies that

$$
\begin{aligned}
rhs &= \frac{-n_1(D_2 - D_1 - p^2)}{D_1^{n_1+1} D_2^{n_2}} + \frac{-n_2 p \cdot (2p^2 + D_2 - D_1 - p^2)}{D_1^{n_1} D_2^{n_2+1}} \\
&= \frac{n_1 - n_2}{D_1^{n_1} D_2^{n_2}} + \frac{n_1 p^2}{D_1^{n_1+1} D_2^{n_2}} + \frac{-n_2 p^2}{D_1^{n_1} D_2^{n_2+1}} + \frac{n_2}{D_1^{n_1-1} D_2^{n_2+1}} + \frac{-n_1}{D_1^{n_1+1} D_2^{n_2-1}}
\end{aligned}
$$

and we get a linear equation

$$
\begin{aligned}
0 = (n_1 - n_2) I(n_1, n_2) &+ n_1 p^2 I(n_1 + 1, n_2) + (-n_2 p^2) I(n_1, n_2 + 1) \\
&+ n_2 I(n_1 - 1, n_2 + 1) + (-n_1) I(n_1 + 1, n + 2 - 1)
\end{aligned}
$$

Similarly, we can take

$$
\frac{\partial}{\partial l} \cdot \frac{l}{D_1^{n_1} D_2^{n_2}} = \frac{\partial l}{\partial l} \frac{1}{D_1^{n_1} D_2^{n_2}} + l \cdot \frac{\partial}{\partial l} \frac{1}{D_1^{n_1} D_2^{n_2}} = d \frac{1}{D_1^{n_1} D_2^{n_2}} + l \cdot \frac{\partial}{\partial l} \frac{1}{D_1^{n_1} D_2^{n_2}}
$$

and get another linear relation

$$
0 = (d - n_1 - 2n_2) I(n_1, n_2) + (-n_2) I((n_1 - 1, n_2 + 1) + n_1 p^2 I(n_1, n_2 + 1)
$$

If we restrict $0 < n_1, n_2$ then with these equations, we can reduce any indices into $(1,1)$. Thus, any integral is written as a product of some rational function of d, n_1, n_2 and $I(1,1)$, where $I(1,1)$ is called a Master Integral of this system.