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Polynomial representation of IBP relations

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Abstract

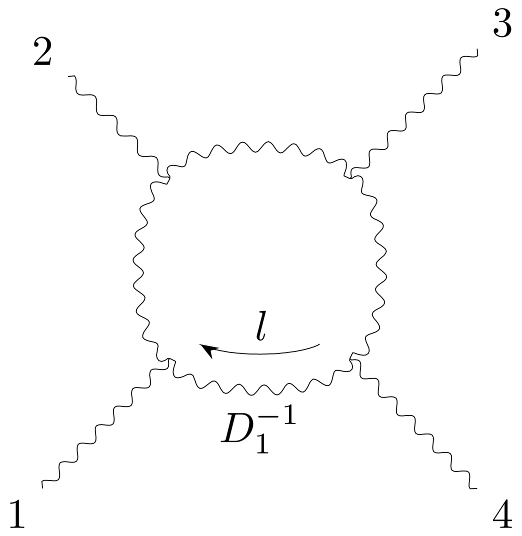
Scattering amplitudes in quantum field theories allow us to compare the phenomenological prediction of theoretical models with the measurement data at collider experiments. The study of scattering amplitudes, in terms of their symmetries and analytic properties, can provide a framework to develop techniques and efficient algorithms to evaluate cross sections. To evaluate higher-order amplitudes, a computational technique called Integration-By-Parts Reduction is at present an unavoidable step. In this process, a large set of linear relations between integrals is generated; by solving them, we can get a set Master integrals, which are necessary ingredients for our calculations. In our poster, we describe some interesting representations for this formalism and their applications to generate Integration-By-Parts relations.

Within these representations, integrals are fully characterized by a single polynomial, and the Integration-By-Parts relations becomes a polynomial identity.

This work was done under the supervision of Prof. A. Ferroglia and Prof. G. Ossola.

Diagrams and amplitudes

A scattering amplitude is given in a diagrammatic form → Feynman diagrams:



This diagram represents a 1 loop, $(2 \to 2)$ scattering process. We translate this kind of diagram into momentum integral form → a scattering amplitude.

$$\int d^d q \frac{N(l)}{D_1 D_2 D_3 D_4}$$

where $N(q)$ is determined by the theoretical model, e.g., the Standard Model. For simplicity, we consider scalar integrals which has $N(l) = 1$.

Integration By Parts identities

One can show that under the coordinate change

$$l_i \mapsto M_{ik} q_k = A_{i,k} l_k + B_{i,k} e_k, (\det A > 0)$$

the d dimensional regulated integral

$$\int d^d l_1 \cdots d^d l_L \frac{1}{D_1^{n_1} \cdots D_N^{n_N}}$$

is invariant. Therefore as its generator form, we have

$$\int d^d l_1 \cdots d^d l_L \frac{\partial}{\partial q_i} \cdot \frac{q_j}{D_1^{n_1} \cdots D_N^{n_N}} = 0,$$

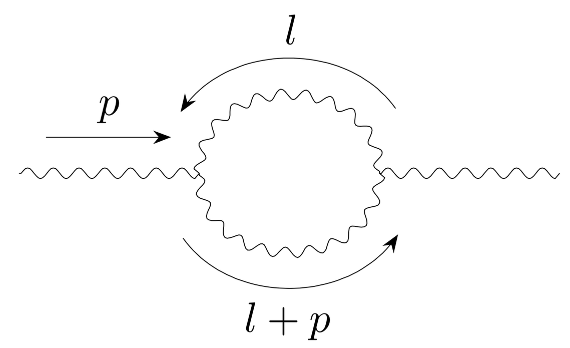
where

$$q_i \in \{l_1, \cdots, l_L\}$$
$$q_j \in \{l_1, \cdots, l_L, e_1, \cdots, e_E\}$$

This set of identities generated by $\frac{\partial}{\partial q_i} \cdot q_j$ is called Integration by Parts identities.

An example: one loop bubble diagram and its IBP relations

Let us consider the following integral



where we consider general powers on denominators:

$$I(n_1, n_2) = \int d^d l_1 d^d l_2 \frac{1}{D_1^{n_1} D_2^{n_2}}$$
$$D_1 := l^2$$
$$D_2 := (l+p)^2.$$

Consider an IBP relation generated by

$$\begin{aligned}\frac{\partial}{\partial l} \cdot p \frac{1}{D_1^{n_1} D_2^{n_2}} &= \frac{-n_1}{D_1^{n_1+1}} p \cdot \frac{\partial D_1}{\partial l} \frac{1}{D_2^{n_2}} + \frac{1}{D_1^{n_1}} \frac{-n_2}{D_2^{n_2+1}} p \cdot \frac{\partial D_2}{\partial l} \\ &= \frac{-n_1 2p \cdot l}{D_1^{n_1+1} D_2^{n_2}} + \frac{-n_2 p \cdot (2p + 2l)}{D_1^{n_1} D_2^{n_2+1}}\end{aligned}$$

Since p^2 is an "external" parameter of this event, and

$$D_2 - D_1 - p^2 = 2p \cdot l$$

implies that

$$\begin{aligned} rhs &= \frac{-n_1 (D_2 - D_1 - p^2)}{D_1^{n_1+1} D_2^{n_2}} + \frac{-n_2 p \cdot (2p^2 + D_2 - D_1 - p^2)}{D_1^{n_1} D_2^{n_2+1}} \\ &= \frac{n_1 - n_2}{D_1^{n_1} D_2^{n_2}} + \frac{n_1 p^2}{D_1^{n_1+1} D_2^{n_2}} + \frac{-n_2 p^2}{D_1^{n_1} D_2^{n_2+1}} + \frac{n_2}{D_1^{n_1-1} D_2^{n_2+1}} + \frac{-n_1}{D_1^{n_1+1} D_2^{n_2-1}}\end{aligned}$$

and we get a linear equation

$$\begin{aligned} 0 &= (n_1 - n_2) I(n_1, n_2) + n_1 p^2 I(n_1 + 1, n_2) + (-n_2 p^2) I(n_1, n_2 + 1) \\ &\quad + n_2 I(n_1 - 1, n_2 + 1) + (-n_1) I(n_1 + 1, n + 2 - 1)\end{aligned}$$

Similarly, we can take

$$\frac{\partial}{\partial l} \cdot \frac{l}{D_1^{n_1} D_2^{n_2}} = \frac{\partial l}{\partial l} \frac{1}{D_1^{n_1} D_2^{n_2}} + l \cdot \frac{\partial}{\partial l} \frac{1}{D_1^{n_1} D_2^{n_2}} = d \frac{1}{D_1^{n_1} D_2^{n_2}} + l \cdot \frac{\partial}{\partial l} \frac{1}{D_1^{n_1} D_2^{n_2}}$$

and get another linear relation

$$0 = (d - n_1 - 2n_2) I(n_1, n_2) + (-n_2) I((n_1 - 1, n_2 + 1) + n_1 p^2 I(n_1, n_2 + 1)$$

If we restrict $0 < n_1, n_2$ then with these equations, we can reduce any indices into $(1,1)$. Thus, any integral is written as a product of some rational function of d, n_1, n_2 and $I(1,1)$, where $I(1,1)$ is called a Master Integral of this system.

Analytical(geometrical) and topological data can be extracted from these associated polynomials.

Schwinger-Feynman parametrization can be used in evaluation phase of integrals. (One can compute $I(n_1, n_2)$ of the bubble diagram using Schwinger-Feynman parametrization relatively easily, as a textbook exercise.) T. Bitoun et. al. recently showed that the number of Master Integrals can be computed as the Euler characteristic of a super surface determined by $\mathcal{G}$.

Baikov found a practical criterion for the irreducibility of a given integral using Baikov parametrization. Lee and Pomeransky extended Baikov's idea and showed the number of Master Integrals is the number of proper critical points of P.

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Baikov parametrization

Under the integration variable change

$$(l_1, \cdots, l_L) \mapsto (D_1, \cdots, D_N),$$

we have

$$I(\vec{n}) \sim \int \frac{dD_1 \cdots dD_N}{D_1^{n_1} \cdots D_N^{n_N}} P^{\frac{d-L-E-1}{2}}$$

where P is the Jacobi determinant of this variable change

$$P = \det [q_i \cdot q_j] (D_1, \cdots, D_N)$$

that is, the determinant of scalar products expressed by denominators, and this P is called the Baikov polynomial. The integration domain is determined by the zeros of P.

P and the integration domain do not depend on $n_1, \cdots, n_N$, so the family of integrals are characterized by a polynomial P.

Schwinger-Feynman parametrization

Using Mellin transformation, one can show

$$I(\vec{n}) \sim \prod_{a=1}^{N} \int_0^\infty \frac{dx_a x_a^{n_a - 1}}{\Gamma(n_a)} \mathcal{G}^{-d/2}$$

where

$$\mathcal{G} := \mathcal{U} + \mathcal{F}$$
$$\mathcal{U} := \det \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$
$$\mathcal{F} := \det A$$

and the associated matrices are given in the following quadratic form:

$$\sum_a^N x_a D_a = l^t A l + 2 B^t l + C.$$

This polynomial $\mathcal{G}$ does not depend on $n_1, \cdots, n_N$ and characterizes the family of integrals.

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