Introduction to Graph Theory

From Binary Relations to Feynman Diagrams

Ray D. Sameshima

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About

Graph theory has long served as a powerful tool in physics – from its early role in solving puzzles to its essential use in circuit analysis, modeling state transitions, and representing quantum interactions among fields and particles. This talk presents selected insights from the book Lectures on Graph Theory – Insights into Feynman Diagrams, highlighting how graph-theoretical methods, in combination with linear algebra, form a unified framework for understanding Feynman diagrams and their applications in quantum field theory. We will review the mathematical foundations and classical applications such as Kirchhoff's laws, and then discuss their modern implications in fundamental physics.

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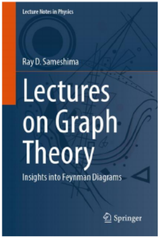
Lectures on Graph Theory

Insights into Feynman Diagrams

Provides a clear and accessible introduction to mastering Feynman integral computations

Offers a detailed exploration of essential techniques for practical calculations in particle physics

Features numerous examples and algorithms to help readers learn quickly and effectively



Brief History

- Euler – the father of graph theory (1736 Königsberg Bridge Problem)

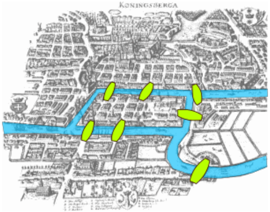


Figure: The Seven Bridges of Königsberg. (Wikipedia) Can you traverse all seven bridges exactly once?

- Kirchhoff – the theory of trees (1847)
- Cayley – chemical problem of trees (1874)
- Veblen – simplicial complex (1922)
- Feynman – Feynman diagram (1949)

Graphs are almost everywhere

- functions
- function “type” annotations
- stock-flow diagrams
  - Equation of Motion (Newton’s second law of motion)
  - Work-Energy Theorem
  - Thermodynamics
- programming languages and data types
- Markov chain
- Electric circuits
- Feynman diagrams

Binary Relations and Functions

Given two sets A, B, a binary relation over A and B is a subset of $A \times B$. If a binary relation $f \subset A \times B$ satisfies additional conditions:

- for each $a \in A$, there is some $b \in B$ with $(a, b) \in f$,
- if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$,

namely, for each $a \in A$ a unique $b \in B$ exists, we write $b = f(a)$ and call f a function from A to B:

$$f \colon A \to B.$$

- We can identify an ordered pair (a, b) of a relation a line from a to b.
- The type annotation itself is a line from A to B.

Binary Relations and Functions – Example

For a function $f\colon \mathbb{N} \to \mathbb{N}$:

$$f(x) = 2x + 1,$$

we can depict:

$$\begin{tikzcd}
0 & 1 & 2 & 3 & 4 & 5 \cdots & & \text{“codomain”} \\
0 \arrow[ur, mapsto] & 1 \arrow[urr, mapsto] & 2 \arrow[urrr, mapsto] & \cdots & & & \text{“domain”}
\end{tikzcd}$$

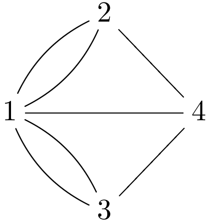
A function can be identified as a set of rooted trees of height 2.
As functions are everywhere in mathematics, so are graphs.

Graphs and Directed Graphs

A directed graph consists of

- a set of lines;
- a set of vertices;
- an incidence relation.

Pictorially, a graph is the vertices connected with directed lines among them; if the direction is not important, e.g.,



This is a pictorial representation of seven bridges in Königsberg. Euler's negative solution to the problem would be the first discovery of graph theory: **A graph is eulerian – there is a walk that traverses each line exactly once – iff every point has even degree.**

Simpler Examples

- EoM: The change in linear momentum of a system from $\vec{p}_i$ to $\vec{p}_f$ is the impulse $\vec{f}\Delta t$ on the system: $\vec{p}_i \overset{\Downarrow \vec{f}\Delta t}{\dashrightarrow} \vec{p}_f$.

  For a rotational motion: $\vec{L}_i \overset{\Downarrow \vec{\tau}\Delta t}{\dashrightarrow} \vec{L}_f$.
- WET and TMEC:
  The net work on a system is the change in kinetic energy: $K_i \overset{\Downarrow W}{\dashrightarrow} K_f$. If the work is provided by a "good source," namely a conservative force, there is an associated potential energy: $P_i \underset{\Downarrow W}{\dashrightarrow} P_f$, and $P_i + K_i = P_f + K_f$.
- Thermodynamics
  - The work done by an adiabatic process is given by the change in the internal energy: $U_i \underset{\Downarrow W_{\text{ad.}}}{\dashrightarrow} U_f$.
  - The heat flow Q into a thermodynamical system is defined via $U_i \underset{\Downarrow W}{\overset{\Downarrow Q}{\dashrightarrow}} U_f$

Slightly Involved Examples

A two-state Markov process:

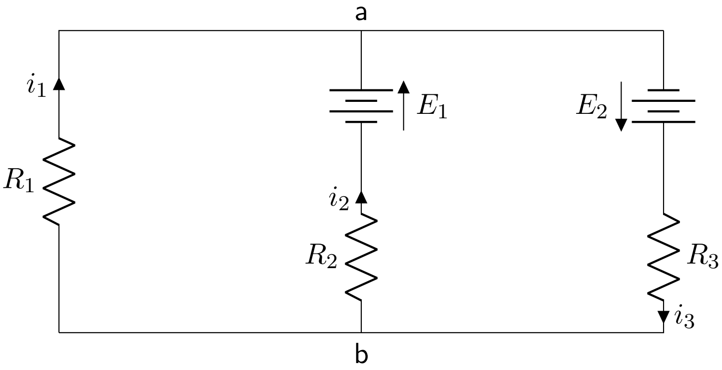
$$\begin{tikzcd} 1 \arrow[loop left, "1-p"] \arrow[r, bend left, "p"] & 2 \arrow[l, bend left, "p'"] \arrow[loop right, "1-p'"] \end{tikzcd}$$

where p represents the transition probability from 1 to 2.
The absolute temperature is defined via the following Carnot cycle:

$$\begin{tikzcd}[column sep=huge, row sep=large] (T_H; X_0') \arrow[r, "\text{quasistatic}", "\text{isothermal}"'] & (T_H; X_1') \arrow[d, "\text{quasistatic}"', "\text{adiabatic}"] \\ (T_L; X_0) \arrow[u, "\text{quasistatic}", "\text{adiabatic}"'] & (T_L; X_1) \arrow[l, "\text{quasistatic}"', "\text{isothermal}"] \end{tikzcd}$$

Electric Circuits

Given the following circuit configuration with EMFs, can we find the currents i_1, i_2, i_3?



A day of a graph-theorist

1. Find a mathematical object.
2. Determine the corresponding graph(s).
3. Apply graph-theoretical analyses.

Various Subgraphs

For a given graph, we can extract several important subgraphs such as:

- Circuits (also known as Loops)
- (Spanning) Trees
- Cut-sets

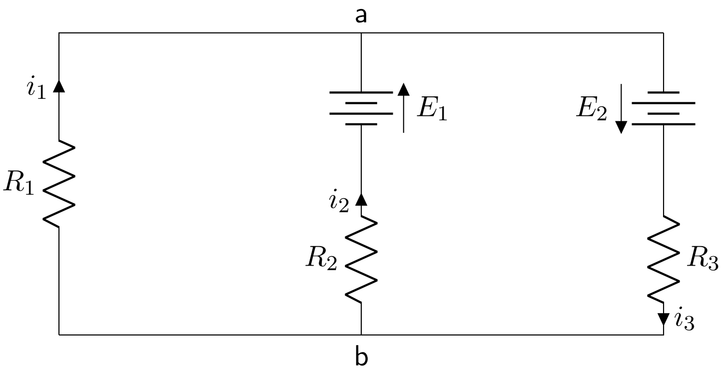
Circuits – Definition

A circuit of a given graph is a closed and non-self-intersecting walk.

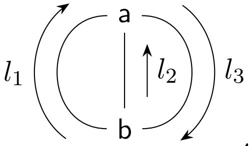
- Walk
  An alternating sequence of vertices and lines
- Closed walk
  A walk with the initial vertex = the final vertex
- non-self-intersecting
  No vertices appear more than once, except for both initial and final vertices

Numbers of Circuits

Can you count the number of circuits, i.e., loops, in an electric circuit:



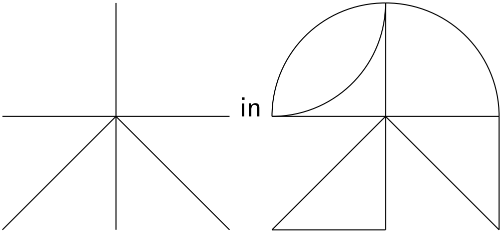
or that of the corresponding directed graph?



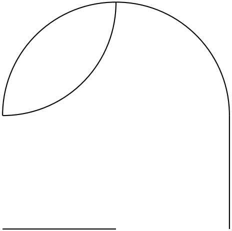
Trees

For simplicity, let us consider connected graphs.

Here is a tree of the underlying graph:



The complement is called the chord-set, or more abstractly, the cotree relative to the tree:



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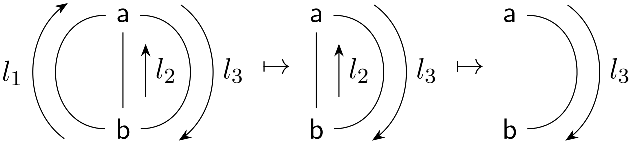
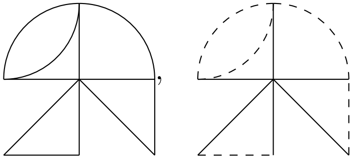
Removing lines one by one, we can extract a spanning tree.

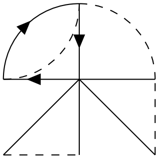


A fixed tree and associated circuits

For a graph and a spanning tree:



If we recover a deleted line into the spanning tree, we can build a unique loop associated with a spanning tree.



We call such a unique circuit for a line in the cotree the fundamental circuit.

Cut-sets

A cut-set of a given graph is ...

Cut-sets

A cut-set of a given graph is … it's involved!

Duality and Associated Matrices

Algebra is generous: she often gives more than is asked for. (Jean D' Alembert)

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Associated Matrix Representations are better for algebraic and computer processing than pictorial representations:

- Incidence matrix $\mathcal{A}$
  Vertices vs. Lines
  For a vertex a and a line l, $\mathcal{A}_{a,l}$ is $+1$ iff l is from a, -1 iff l is to a, 0 otherwise.
- Circuit matrix $\mathcal{C}$
  Circuits vs. Lines
- Cut-set matrix $\mathcal{S}$
  Cut-sets vs. Lines

Linear Algebra is generous

Let G be a connected graph with M vertices and N lines. Then the incidence matrix is $M \times N$. We call $r = M - 1$ the rank of G. With a fixed vertex a and a fixed spanning tree T in G, if we write

$$\mathcal{A}^a = \begin{pmatrix} \mathcal{A}^a_{T^*} & \mathcal{A}^a_T \end{pmatrix} \tag{4.79}$$

then the circuit matrix $\mathcal{C}_{T^*}$ and the cut-set matrix $\mathcal{S}_T$ can be written as

$$\mathcal{C}_{T^*} = \begin{pmatrix} 1_{\mathbb{b}_1} & \mathcal{B}_T \end{pmatrix}, \quad \mathcal{S}_T = \begin{pmatrix} -\mathcal{B}_T{}^\top & 1_r \end{pmatrix} \tag{4.75}$$

where $\mathbb{b}_1 = N - M + 1$ and

$$\mathcal{B}_T{}^\top = -\left(\mathcal{A}^a_T\right)^{-1} \mathcal{A}^a_{T^*}. \tag{4.81}$$

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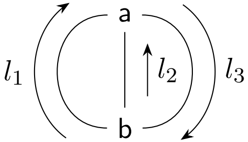
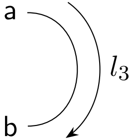
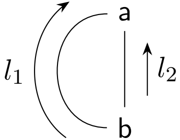
One of the bridging tools is the $rank\text{–}nullity$ theorem.

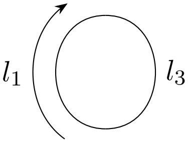
Electric Circuit Analysis

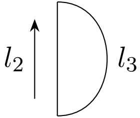
Let $G =$ 

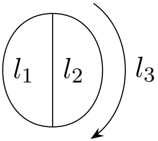
be an underlying directed graph. If we select a spanning tree $T =$ 

and the corresponding cotree $T^* =$ 

, we have the following "fundamental" circuits and a cut-set:

$C_1 =$ 

, $C_2 =$ 

, $S_1 =$ 

Electric Circuit Analysis (continued)

I.e., this graph and the electric circuit have two "loops." If we assign the loop currents j_1 and j_2 along the loops l_1 and l_2, respectively, Kirchhoff's junction rule is automatically guaranteed. Then the corresponding line currents are

$$i_1 = j_1, \quad i_2 = j_2, \quad i_3 = j_1 + j_2.$$

Kirchhoff's loop rule becomes the following 2×2 system:

$$\begin{aligned} 0 &= R_1 j_1 + (-E_2 + R_3 (j_1 + j_2)) \\ 0 &= (R_2 j_2 - E_1) + (-E_2 + R_3 (j_1 + j_2)) \end{aligned}$$

That is, the linear dimension of the problem is **the rank** of the graph.

Electric Circuit Analysis (continued)

Moreover, one can show the following identity – instantaneous energy conservation – known as Tellegen's theorem:

$$\mathcal{I}^\top \mathcal{V} = 0, \tag{4.99}$$

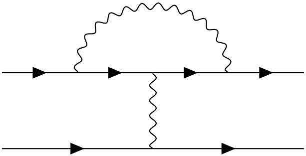
where $\mathcal{I}$ is the current vector across the circuit and $\mathcal{V}$ is the potential difference vector:

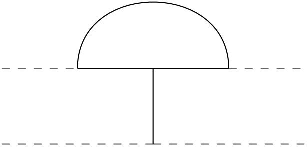
> *Among the theorems of network theory Tellegen's theorem is unusual in that it depends solely upon Kirchhoff's laws and the topology of the network. (Penfield, P. Spence, R. and Duinker, S. "Tellegen' s Theorem and Electrical Networks")*

Feynman Diagrams?

A graphical representation of the quantum interactions among particles and fields.

A Feynman diagram consists of a graph with external legs:





The external legs, the dashed lines, represent interactions with the outside world, like ports in an electric circuit.

Feynman Diagrams 101

They are graded with respect to the number of circuits in the graph parts; the number of circuits is the order of the asymptotic expansion of S-matrix:

$$\text{zero-loop diagrams} + \text{one-loop diagrams} + \text{two-loop diagrams} + \cdots$$

Each term in the expansion is an involved integral, represented as a Feynman diagram. To achieve higher precision, we require more diagrams, which makes the computation challenging in both memory space and processing speed.
Using the graph-theoretical tools, we have a systematic way to write the integral with polynomials associated with the underlying Feynman diagram. Not only their numerical values but also their analytical properties are of interest.

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Using the graph-theoretical tools, we have a systematic way to write the integral with **polynomials associated with the underlying Feynman diagram.** Not only their numerical values but also their analytical properties are of interest.

An Open Problem

Similar to electric currents across lines in an electric circuit, each line in a Feynman diagram has the corresponding momentum flow. **Question: Is there any voltage-like quantity associated with a Feynman diagram?** If such a quantity exists, thanks to Tellegen's theorem, we may obtain yet another quantum conservation law!

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