

Group Theory 101

for Homology Theory

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0 Definitions and Basic Properties

This note explores basic group theory, following [1, 2].

0.1 Groups – Multiplicative Groups

Definition 0.1 (Groups). A group is given by

$$(G, \circ, 1, (_)^{-1}), \quad (1)$$

where

- G is a set and \circ is a binary product

$$\circ: G \times G \rightarrow G; (g_2, g_1) \mapsto g_2 \circ g_1. \quad (2)$$

- $(G, \circ, 1)$ forms a monoid. Namely, G is a non-empty set, \circ is an associative binary product, and $1 \in G$ is a multiplicative identity such that

$$g \circ 1 = g = g \circ 1, \quad (3)$$

for each $g \in G$. Note that the associativity is expressed by

$$g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1 \quad (4)$$

for any $g_1, g_2, g_3 \in G$.

- The unary operation $(\cdot)^{-1}: G \rightarrow G$ returns the multiplicative inverse g^{-1} of a given $g \in G$ such that

$$g^{-1} \circ g = 1 = g \circ g^{-1}. \quad (5)$$

Lemma 0.1. *Let $(G, \circ, 1, (\cdot)^{-1})$ be a group.*

- If $e \in G$ satisfies

$$g \circ e = g = e \circ g \quad (6)$$

for each $g \in G$, then $e = 1$. That is, the identity element is unique.

- For each $g \in G$, the inverse g^{-1} is unique.
- For each $g \in G$, $(g^{-1})^{-1} = g$.

Proof. Since $1 \in G$ is an multiplicative identity, $1 \circ e = e = e \circ 1$; $e \in G$ satisfies the same properties, we conclude $e = 1 \circ e = 1$. Let $g \in G$. Suppose $g' \in G$ satisfies $g' \circ g = g = g \circ g'$. Applying g^{-1} , we obtain $g' = g^{-1}$. We, then, have $g^{-1} \circ g = 1 = g^{-1} \circ g$, showing $(g^{-1})^{-1} = g$. \square

Definition 0.2 (Subgroups). A subset $H \subset G$ of a group $(G, \circ, 1, (\cdot)^{-1})$ is called a subgroup of G iff H forms a group and H is closed under group operations:

- For any $h_1, h_2 \in H$, $h_2 \circ h_1 \in H$.
- The identity $1 \in G$ is in H , $1 \in H$.
- For any $h \in H$, the inverse is in H , $h^{-1} \in H$.

A trivial subgroup is the singleton $\{1\} \subset G$; another example is G itself. We denote a subgroup H of G by $H < G$.

Lemma 0.2 (Subgroup-Test). *Let $(G, \circ, 1, (\cdot)^{-1})$ be a group. A non-empty subset $\emptyset \neq H \subset G$ is a subgroup iff $h_2 \circ h_1^{-1} \in H$ for any $h_1, h_2 \in H$.*

Proof. (\Rightarrow) If $H < G$ and $h_1, h_2 \in H$, then $h_1^{-1} \in H$ and hence $h_2 \circ h_1^{-1} \in H$.
(\Leftarrow) There is at least one element in H . Select $h \in H$. Then $h \circ h^{-1} = 1 \in H$. For any $k \in H$, $k^{-1} = 1 \circ k^{-1} \in H$. Suppose $h_1, h_2 \in H$. Then $h_2^{-1} \in H$, and hence $h_1 h_2 = h_1 (h_2^{-1})^{-1} \in H$. Since the underlying \circ is an associative binary product of G , $(H, \circ, 1, (\cdot)^{-1})$ forms an group. \square

From this point, the group operation can be written without the symbol \circ ; namely, for $g_1, g_2 \in G$, we write g_2g_1 instead of $g_2 \circ g_1$.

Definition 0.3 (Group Homomorphisms). Let G and H be groups. A map $\theta: G \rightarrow H$ is called a group homomorphism iff $\theta(g_2g_1) = (\theta g_2)(\theta g_1)$.

- If a group homomorphism $\theta: G \rightarrow H$ is injective:

$$\theta g_1 = \theta g_2 \Rightarrow g_1 = g_2 \quad (7)$$

for $g_1, g_2 \in G$, then θ is called a mono.

- If a group homomorphism $\theta: G \rightarrow H$ is subjective:

$$\theta G = H, \quad (8)$$

then θ is called an epi.

A group homomorphism is called an isomorphism iff it is both monic and epic.

Lemma 0.3. *Any group homomorphism preserves the identity and inverses.*

Proof. Let $\theta: G \rightarrow H$ be a group homomorphism. Since $1_G \circ 1_G = 1_G$, we have $(\theta 1_G) \circ (\theta 1_G) = \theta 1_G$. Applying the inverse $(\theta 1_G)^{-1}$ of $\theta 1_G \in H$, we obtain $\theta 1_G = 1_H$. Let $g \in G$ and consider the inverse $g^{-1} \in G$. Since $g^{-1} \circ g = 1_G = g \circ g^{-1}$, if we apply θ , we have $(\theta g^{-1}) \circ (\theta g) = 1_H = (\theta g) \circ (\theta g^{-1})$. Since the inverse is unique by Lemma 0.1, $\theta g^{-1} = (\theta g)^{-1}$. \square

Lemma 0.4. *A group homomorphism $\theta: G \rightarrow H$ is monic iff the kernel is singleton, $\theta^{\leftarrow} 1_H = \{1_G\}$.*

Proof. (\Rightarrow) Recall the very definition:

$$\theta^{\leftarrow} 1_H := \{g \in G \mid \theta g = 1_H\}. \quad (9)$$

The kernel is non-empty, $1_G \in \theta^{\leftarrow} 1_H$, since $\theta 1_G = 1_H$. Let $g \in \theta^{\leftarrow} 1_H$. Then $\theta g = 1_H = \theta 1_G$. Since θ is injective, $g = 1_G$, and hence $\theta^{\leftarrow} 1_H = \{1_G\}$.

(\Leftarrow) Conversely, suppose $\theta^{\leftarrow} 1_H = \{1_G\}$. Let $g_1, g_2 \in G$ and assume $\theta g_1 = \theta g_2$ for $g_1, g_2 \in G$. Then $1_H = (\theta g_2)(\theta g_1)^{-1} = (\theta g_2)(\theta g_1^{-1}) = \theta(g_2 g_1^{-1})$. Hence, $g_2 g_1^{-1} \in \theta^{\leftarrow} 1_H = \{1_G\}$, and we have $g_1 = g_2$ since $g_2 g_1^{-1} = 1_G$. \square

Theorem 0.5. *Both image and kernel of a group homomorphism are subgroups.*

Proof. Let $\theta: G \rightarrow H$ be a group homomorphism. Let $h_1, h_2 \in \theta G$. There are $g_1, g_2 \in G$ such that $h_1 = \theta g_1$ and $h_2 = \theta g_2$. Since $g_2 g_1^{-1} \in G$,

$$h_2 h_1^{-1} = \theta g_2 (\theta g_1)^{-1} = \theta(g_2 g_1^{-1}) \in \theta G. \quad (10)$$

By Lemma 0.2, $\theta G < H$.

Let $g_1, g_2 \in \theta^{\leftarrow} 1$. Then $\theta g_1 = \theta g_2$ and, hence

$$\theta(g_2 g_1^{-1}) = (\theta g_2)(\theta g_1)^{-1} = 1. \quad (11)$$

By Lemma 0.2, $\theta^{\leftarrow} 1 < G$. \square

0.2 Abelian Groups – Additive Groups

Definition 0.4 (Abelian Groups). Let $(G, \circ, 1, (_)^{-1})$ be a group. Two elements $a, b \in G$ are commutative iff $a \circ b = b \circ a$; G is called an abelian group iff any pair is commutative. We use $+$ sign for the commutative binary operator, and 0 for the additive identity, and $-g$ for the additive inverse of $g \in G$.

Definition 0.5 (Free Group). An abelian group $(A, +, 0, -(_))$ is called free iff there is $B \subset A$ such that every element $a \in A$ has a unique representation

$$a = \sum_{b \in B} n_b b, \quad (12)$$

where $\{n_b \in \mathbb{Z} \mid b \in B\}$ are all equal to zero but finitely many. In other words, the “sequence” is eventually zero. Such a subset $B \subset A$, which generates the given abelian group A , is called a basis for A .

Remark (Construction). Let S be a set. We will construct a free abelian group with basis S .

Let F_S be a set of functions from S to integers \mathbb{Z} such that $f \in F_S$ iff

$$fs = 0 \text{ for all but finitely many } s \in S. \quad (13)$$

Then the zero map is in F_S , $0 \in F_S$. For any $f, g \in F_S$, $f + g$ is defined componentwisely, $(f + g)s = fs + gs$ for each $s \in S$. The additive inverse of f is also defined componentwisely, $(-f)s = -(fs)$. By construction, the set of characteristic functions forms a basis for F_S :

$$B := \{\chi_{\{t\}} \mid t \in S\} \cong S, \quad (14)$$

where

$$\chi_{\{t\}}s := \begin{cases} 0 & s \neq t \\ 1 & s = t. \end{cases} \quad (15)$$

For each $f \in F_S$, let $I_f \subset S$ be a finite subset such that $fs \neq 0$ iff $s \in I_f$. Then any $f \in F_S$ is written as $f = \sum_{t \in I_f} (ft)\chi_{\{t\}}$.

We denote O for the abelian group with empty basis; since the empty sum is the additive identity, the unique element in O is 0:

$$O = (\{0\}, +, 0, -(_)). \quad (16)$$

Remark (Extension “by Linearity”). If an abelian group $(A, +, 0, -(_))$ is free with basis B and C is an abelian group, then any map $\theta: B \rightarrow C$ can be uniquely extended to an abelian group homomorphism $\theta: A \rightarrow C$ by linearity, namely for each $a \in A$ with its unique representation $\sum_{b \in B} n_b b$, $\theta a := \sum_{b \in B} n_b (\theta b)$.

1 Normal Subgroups and Quotient Groups

1.1 Normal Subgroups

Definition 1.1 (Normal Subgroups). Let G be a group. A subgroup $N < G$ is called a normal subgroup iff $gNg^{-1} \subset N$ for each $g \in G$, where

$$gNg^{-1} := \{gng^{-1} \mid n \in N\}. \quad (17)$$

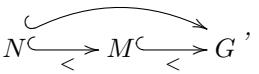
We denote $N \triangleleft G$ for a normal subgroup H of G .

Lemma 1.1. *The kernel of a group homomorphism is a normal subgroup.*

Proof. Let $\theta: G \rightarrow H$ be a group homomorphism. By Theorem 0.5, $\theta^{-1} < G$. Let $g \in G$ and $k \in \theta^{-1}$. Then $gkg^{-1} \in \theta^{-1}$ since

$$\theta(gkg^{-1}) = (\theta g) \theta(k) \theta(g)^{-1} = 1. \quad (18)$$

Therefore, $g(\theta^{-1})g^{-1} \subset \theta^{-1}$. \square

Lemma 1.2. *Let G be a group. If $N \triangleleft G$ and $N < M < G$,* *, then $N \triangleleft M$ holds.*

Proof. Since $N < M$, it suffices to show its normality. Let $m \in M$ and $n \in N$. Since $N \triangleleft G$ and $M < G$, $m \in G$ and hence, $mnm^{-1} \in N$. That is, $mNm^{-1} \subset N$. \square

Theorem 1.3. *Let $N \triangleleft G$ be a normal subgroup. If we define \sim by*

$$g_1 \sim g_2 \Leftrightarrow g_2g_1^{-1} \in N \quad (19)$$

then \sim is an equivalence relation on G relative to N .

Remark. For each $g \in G$, let

$$[g] := \{g' \in G \mid g' \sim g\} \quad (20)$$

be the set of equivalent elements represented by g . Let $G/\sim = \{[g] \mid g \in G\}$ denote the set of equivalent classes relative to $N \triangleleft G$.

Proof. Let $g \in G$. Since $gg^{-1} = 1 \in N$, $g \sim g$. Hence, \sim is reflexive.

Suppose $g_1 \sim g_2$ for $g_1, g_2 \in G$. Then $n := g_2g_1^{-1} \in N$. Since N is a subgroup of G , $n^{-1} \in N$, namely $n^{-1} = g_1g_2^{-1} \in N$. Hence, $g_2 \sim g_1$, and \sim is symmetric.

Finally, suppose $g_1 \sim g_2$ and $g_2 \sim g_3$. Then $g_3g_1^{-1} \in N$, since $g_2g_1^{-1} \in N$, $g_3g_2^{-1} \in N$ and $g_3g_1^{-1} = g_3g_2^{-1}g_3g_2^{-1}$. Therefore, \sim is transitive. \square

Corollary 1.3.1. *G/\sim with respect to $G \triangleleft G$ is a singleton.*

Proof. All elements are equivalent to $1 \in G$ since $g = g1^{-1} \in G$ for each $g \in G$. Hence $[1] = \{g \in G \mid g \sim 1\} = G$ is the unique element. \square

1.2 Quotient Groups

Lemma 1.4. *Let G be a group and $N \triangleleft G$ be a normal subgroup.*

- *For each $g \in G$, $gN = Ng = [g]$.*
- *For $g_1, g_2 \in G$, $[g_1][g_2] = [g_1g_2]$.*

Remark. Let G/N denote the set $G/\sim = \{[g] \mid g \in G\}$ of equivalent classes with respect to $N \triangleleft G$ with the binary product $[g_1][g_2] = [g_1g_2]$ for $[g_1], [g_2] \in G/N$.

Proof. Let $g \in G$. For each $n \in N$, since $N \triangleleft G$, $k := gng^{-1} \in N$. That is, $gn = kg \in Ng$, and hence $gN \subset Ng$. Conversely, for each $n \in N$, as $g^{-1} \in G$, $h := g^{-1}ng \in N$. It follows $Ng \subset gN$. Therefore, $gN = Ng$ holds.

Let $g \in G$ and consider $[g] \in G/\sim$. For each $g' \in [g]$, there exists $n \in N$ with $n = g'g^{-1}$. Then, $[g] \subset Ng$ since for each $g' \in [g]$, $g' = ng \in Ng$. Conversely, consider $gk \in gN$. As shown above, $gN = Ng$, there must be some $h \in N$ such that $gk = hg$. Therefore, $h = (gk)g^{-1} \in N$, showing $gk \sim g$, and $gk \in [g]$. Hence, we conclude $[g] = Ng$.

Finally, consider the binary operation. Let $g_1, g_2 \in G$.

$$\begin{aligned} [g_2][g_1] &= \{h_2h_1 \mid h_2 \in Ng_2 \wedge h_1 \in Ng_1\} \\ &= \{n_2g_2n_1g_1 \mid n_1, n_2 \in N\} \\ &= \{n_2(g_2n_1g_2^{-1})g_2g_1 \mid n_1, n_2 \in N\} \end{aligned} \tag{21}$$

Since $N \triangleleft G$, $(g_2n_1g_2^{-1}) \in N$ for each $g_2 \in G$ and $n_1 \in N$. Hence, we obtain

$$[g_2][g_1] = \{ng_2g_1 \mid n \in N\} = N(g_2g_1) = [g_2g_1]. \tag{22}$$

□

Theorem 1.5 (Quotient Groups). *Let G be a group and $N \triangleleft G$ be a normal subgroup. Then G/N forms a group:*

$$(G/N, \circ, N, [(\cdot)^{-1}]), \tag{23}$$

where the binary product \circ is defined by $[g_1] \circ [g_2] = [g_1g_2]$ for $[g_1], [g_2] \in G/N$.

Proof. Since the binary product is essentially the binary product of G , it suffices to show the identity and inverses.

- $N = 1_{G/N}$

Consider $[g]$ for $g \in G$. By Lemma 1.4, $[g] = gN$, and hence

$$[g]N = gNN = gN = [g]. \tag{24}$$

Similarly, one can show that $N[g] = [g]$.

- $[g]^{-1} = [g^{-1}]$

Consider $[g]$ for $g \in G$. By Lemma 1.4,

$$[g][g^{-1}] = [gg^{-1}] = [1] = N = [g^{-1}][g]. \quad (25)$$

Since the inverse is unique, by Theorem 0.1, $[g^{-1}]$ is the desired inverse of $[g]$.

We call the group G/N for $N \triangleleft G$ the quotient group of G modulo N . \square

1.3 Group Homomorphisms and Quotient Groups

Theorem 1.6. *Let G and H be groups. For a group homomorphism $\theta: G \rightarrow H$, as shown in Lemma 1.1, the kernel forms a normal subgroup, $\theta^{-1} \triangleleft G$. Conversely, if $N \triangleleft G$, then the canonical projection*

$$\pi: G \rightarrow G/N; g \mapsto [g] \quad (26)$$

is an epi with $\pi^{-1} = N$.

Proof. Suppose $N \triangleleft G$. For any $[g] \in G/N$, if we choose $g' \in [g]$, $\pi g' = [g'] = [g]$ since $g' \sim g$. Hence, the canonical projection π is surjective. For $g_1, g_2 \in G$, it follows

$$\pi(g_2 g_1) = [g_2 g_1] = [g_2][g_1] = (\pi g_2)(\pi g_1) \quad (27)$$

Hence, π is a group homomorphism. Finally, we have $\pi^{-1} = N$ since

$$\{g \in G \mid [g] = N = N1\} = \{g \in G \mid g \sim 1\} = \{g \in G \mid g = g1^{-1} \in N\}. \quad (28)$$

\square

Theorem 1.7. *Let G and H be groups, and $\theta: G \rightarrow H$ be a group homomorphism. Suppose $N \triangleleft G$ and $N \subset \theta^{-1}$. Then, there exists a unique mediator $\bar{\theta}: G/N \rightarrow H$ such that $\bar{\theta}[g] = \theta g$ for each $g \in G$. That is, the following diagram is commutative:*

$$\begin{array}{ccc} G & \xrightarrow{\theta} & H \\ \pi \downarrow & \nearrow \exists! \bar{\theta} & \\ G/N & & \end{array} \quad \bar{\theta} \circ \pi = \theta. \quad (29)$$

We also have $\bar{\theta}(G/N) = \theta G$ and $\bar{\theta}^{-1} = (\theta^{-1})/N$.

Proof. If $g \sim g'$ relative to $N \triangleleft G$, there is some $n \in N$ such that $n := g'g^{-1}$ and

$$\theta g' = \theta(ng) = (\theta n)(\theta g) = 1\theta g = \theta g, \quad (30)$$

since $n \in N \subset \theta^{-1}$. Thus $\bar{\theta}: G/N \rightarrow H; [g] \mapsto \theta g$ is a well-defined map among the corresponding sets. Since $\bar{\theta}$ is fully determined in terms of the given group homomorphism θ , it is unique.

The induced map $\bar{\theta}: G/N \rightarrow H$ is a group homomorphism, since θ is a group homomorphism:

$$\bar{\theta}([g_2][g_1]) = \bar{\theta}[g_2g_1] = \theta(g_2g_1) = (\theta g_2)(\theta g_1) = (\bar{\theta}[g_2])(\bar{\theta}[g_1]) \quad (31)$$

for $[g_1], [g_2] \in G/N$. The images coincide:

$$\theta G = \{h \in H \mid \exists g \in G : h = \theta g = \bar{\theta}[g]\} = \bar{\theta}(G/N). \quad (32)$$

Let $K := \theta^{\leftarrow}1 \triangleleft G$. Since $N < K \triangleleft G$ and $N \triangleleft G$, we may apply Lemma 1.2, $N \triangleleft K$. Hence, $K/N = (\theta^{\leftarrow}1)/N$ is a quotient group. For any $g \in G$, $[g] = gN \in K/N$ iff $\bar{\theta}[g] = fg = 1$ iff $[g] \in \bar{\theta}^{\leftarrow}1$. That is, $(\theta^{\leftarrow}1)/N = K/N = \bar{\theta}^{\leftarrow}1$. \square

Corollary 1.7.1. *The induced group homomorphism $\bar{\theta}: G/N \rightarrow H$ is an isomorphism iff θ is an epi and $N = \theta^{\leftarrow}1$.*

Proof. (\Rightarrow) Suppose $\bar{\theta}$ is an isomorphism. Since $\bar{\theta}$ is epic, so is θ since

$$\theta G = \bar{\theta}(G/N) = H. \quad (33)$$

By hypothesis, $N \subset \theta^{\leftarrow}1$; to show the other inclusion, let $k \in \theta^{\leftarrow}1$:

$$\bar{\theta}[k] = \theta k = 1 \quad (34)$$

Since $\bar{\theta}$ is monic, by Lemma 0.4, $\bar{\theta}^{\leftarrow}1 = \{1_{G/N}\}$, where $1_{G/N} = N$ by Theorem 1.5. Hence, $[k] = N$, i.e., $k \in N$.

(\Leftarrow) Suppose θ is epic and $N = \theta^{\leftarrow}1$. Then $\bar{\theta}$ is epic since $\bar{\theta}(G/N) = \theta G = H$. Recalling $(\theta^{\leftarrow}1)/N = \bar{\theta}^{\leftarrow}1$, and by hypothesis $N = \theta^{\leftarrow}1$, the kernel is a singleton by Corollary 1.3.1, $\bar{\theta}^{\leftarrow}1 = \{N\} = \{1_{G/N}\}$. \square

Theorem 1.8 (First Isomorphic Theorem). *A group homomorphism $\theta: G \rightarrow H$ induces an isomorphism between $G/\theta^{\leftarrow}1$ and θG .*

Proof. The corestriction $\theta: G \rightarrow \theta G$ is an epi. Thus, the induced group homomorphism via Corollary 1.7.1:

$$\bar{\theta}: G/N \rightarrow \theta G \quad (35)$$

is an isomorphism, where $N := \theta^{\leftarrow}1 \triangleleft G$. \square

1.4 Exact Sequences

Definition 1.2 (Exact Sequences). A pair of adjacent group homomorphisms $F \xrightarrow{\theta} G \xrightarrow{\varphi} H$ is called exact iff $\theta F = \varphi^{\leftarrow}1$. A sequence of group homomorphisms is called exact iff every adjacent pair of homomorphisms is exact. An exact sequence of the following form is called short exact:

$$O \longrightarrow F \xrightarrow{\theta} G \xrightarrow{\varphi} H \longrightarrow O \quad (36)$$

where $O \rightarrow F$ is essentially an inclusion map $\{1_F\} \subset F$ and $H \rightarrow O$ is a constant map toward a singleton group.

Lemma 1.9. *A group homomorphism $\theta: F \rightarrow G$ is an isomorphism iff $O \rightarrow F \xrightarrow{\theta} G \rightarrow O$ is exact.*

Proof. Since $O \rightarrow F$ is essentially an inclusion map $\{1_F\} \subset F$, the image is $\{1_F\}$. Hence, θ is monic, by Lemma 0.4 $\theta^{-1} = \{1_F\}$, iff $O \rightarrow F \xrightarrow{\theta} G$ is exact. Since $G \rightarrow O$ is a constant map toward the identity, its kernel is the domain G . Hence, θ is epic, $\theta F = G$, iff $F \xrightarrow{\theta} G \rightarrow O$ is exact. \square

Theorem 1.10. *For a short exact sequence $O \rightarrow F \xrightarrow{\theta} G \xrightarrow{\varphi} H \rightarrow O$, θ is monic and φ is epic.*

Proof. Since $O \rightarrow F \xrightarrow{\theta} G$ is exact, $\{1_F\} = \theta^{-1}$. By Lemma 0.4, θ is monic. Since $G \xrightarrow{\varphi} H \rightarrow O$ is exact, $\varphi G = H$. Hence, φ is epic. \square

Remark. The corestriction $\theta: F \rightarrow \theta F$ is isomorphic. Thus, we may identify $F \overset{\iota}{\subset} G$ through $F \cong \theta F \subset G$. Since $\varphi: G \rightarrow H$ is epic, $\varphi G = H$. With the kernel $\varphi^{-1} = \theta F$, Theorem 1.8 implies $\varphi G \cong G / (\varphi^{-1})$. Hence, we obtain $F \cong \theta F \triangleleft G$ and

$$G/F \cong G/(\theta F) = G/(\varphi^{-1}) \cong \varphi G = H. \quad (37)$$

Hence, the given short exact sequence is equivalent to

$$O \longrightarrow F \xrightarrow{\iota} G \xrightarrow{\pi} G/F \longrightarrow O \quad (38)$$

2 Direct Products

2.1 Direct Products

Definition 2.1 (Direct Products and Componentwise Binary Product). Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there is given a group G_λ . On their Cartesian product

$$\prod_{\lambda \in \Lambda} G_\lambda := \left\{ f: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda \mid \forall \lambda \in \Lambda : f_\lambda \in G_\lambda \right\}, \quad (39)$$

we define a binary operation componentwisely, namely for $f, g \in \prod_{\lambda \in \Lambda}$,

$$gf: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda; \lambda \mapsto g_\lambda f_\lambda. \quad (40)$$

We call $\prod_{\lambda \in \Lambda} G_\lambda$, with the associated binary operation, the direct product of $\{G_\lambda \mid \lambda \in \Lambda\}$.

Theorem 2.1. *For a set of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, their direct product $\prod_{\lambda \in \Lambda} G_\lambda$ forms a group.*

Proof. The corresponding binary product is essentially that of each component, hence it is associative. Let $1: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda; 1 \mapsto 1_{G_\lambda}$. Then 1 is the multiplicative identity. For each $f \in \prod_{\lambda \in \Lambda} G_\lambda$, let

$$f^{-1}: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda; \lambda \mapsto f_\lambda^{-1}. \quad (41)$$

It follows $f^{-1}f = 1 = ff^{-1}$, and $(_)^{-1}: \bigcup_{\lambda \in \Lambda} G_\lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda$ is the desired inverse unary operation. \square

Theorem 2.2. *For the direct product $\prod_{\lambda \in \Lambda} G_\lambda$ of a set of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, the canonical projection*

$$\pi_k: \prod_{\lambda \in \Lambda} G_\lambda \rightarrow G_k; f \mapsto f_k \quad (42)$$

is an epi for each $k \in \Lambda$.

Proof. Let $f, g \in \prod_{\lambda \in \Lambda} G_\lambda$. For each $k \in \Lambda$, by Definition 2.1,

$$\pi_k(gf) = (gf)_k = g_k f_k = (\pi_k g)(\pi_k f). \quad (43)$$

Hence, π_k is a group homomorphism for each $k \in \Lambda$.

Let $g_k \in G_k$ for a given $k \in \Lambda$. Define $h: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda$ by

$$h_\lambda := \begin{cases} 1_{G_\lambda} & \lambda \neq k \\ g_k & \lambda = k \end{cases} \quad (44)$$

Then $h \in \prod_{\lambda \in \Lambda} G_\lambda$ such that, $\pi_k h = g_k$. Hence, π_k is surjective. \square

Theorem 2.3 (Universal Property of Direct Products). *Let H be a group and Λ be a set, and suppose that for each $\lambda \in \Lambda$, there is given a group homomorphism $\theta_\lambda: H \rightarrow G_\lambda$. Then, there exists a unique mediator homomorphism $\theta: H \rightarrow \prod_{\lambda \in \Lambda} G_\lambda$ such that the following diagram for each $k \in \Lambda$ is commutative:*

$$\begin{array}{ccc} H & \xrightarrow{\theta} & \prod_{\lambda \in \Lambda} G_\lambda \\ \downarrow \exists! \theta & \searrow \theta_k & \xrightarrow{\pi_k} \\ & & G_k \end{array} \quad \pi_k \circ \theta = \theta_k. \quad (45)$$

Proof. Define

$$\theta_{_}: H \rightarrow \left(\Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda \right); h \mapsto \theta_{_} h, \quad (46)$$

by, for each $h \in H$,

$$\theta_{_} h: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda; k \mapsto \theta_k h. \quad (47)$$

For each $k \in \Lambda$, $\theta_k h \in G_k$ holds, hence $\theta_{\underline{\lambda}} h \in \prod_{\lambda \in \Lambda} G_\lambda$. Therefore, $\theta_{\underline{\lambda}}: H \rightarrow \prod_{\lambda \in \Lambda} G_\lambda$. By construction, $\theta_{\underline{\lambda}}$ is entirely given by $\{\theta_\lambda \mid \lambda \in \Lambda\}$, so it is unique.

For $h_1, h_2 \in H$, $\theta_{\underline{\lambda}}(h_2 h_1) = (\theta_{\underline{\lambda}} h_2)(\theta_{\underline{\lambda}} h_1)$ holds since θ_k is a group homomorphism and, hence,

$$\theta_k(h_2 h_1) = (\theta_k h_2)(\theta_k h_1) \quad (48)$$

for each $k \in \Lambda$.

Let $k \in \Lambda$. Then, $\pi_k \circ \theta_{\underline{\lambda}} = \theta_k$ holds, since for each $h \in H$,

$$(\pi_k \circ \theta_{\underline{\lambda}}) h = \pi_k(\theta_{\underline{\lambda}} h) = (\theta_{\underline{\lambda}} h)_k = \theta_k h. \quad (49)$$

□

2.2 Weak Direct Products

Definition 2.2 (Weak Direct Products). Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there is a given group G_λ . The weak direct product $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ is a subset of $\prod_{\lambda \in \Lambda} G_\lambda$ of all f such that $f_\lambda = 1_{G_\lambda}$ for all but finitely many $\lambda \in \Lambda$.

Remark (Direct Sums). For additive, i.e., abelian groups $\{A_\lambda \mid \lambda \in \Lambda\}$, the corresponding weak direct product is called the direct sum, denoted by $\sum_{\lambda \in \Lambda} A_\lambda$.

Theorem 2.4. Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there is given a group G_λ . Their weak direct product is a normal subgroup of the direct product:

$$\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda. \quad (50)$$

Proof. Let $f, g \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$. There are finite subsets $I, J \subset \Lambda$ such that $j \in J$ iff $f_j \neq 1$, and $k \in K$ iff $g_k \neq 1$. Consider gf^{-1} . Recalling $(gf^{-1})_i = g_i f_i^{-1}$ for each $i \in \Lambda$,

$$\{i \in \Lambda \mid g_i f_i^{-1} \neq 1\} \subset J \cup K. \quad (51)$$

Since $J \cup K \subset \Lambda$ is finite, $gf^{-1} \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$. By Lemma 0.2, $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda < \prod_{\lambda \in \Lambda} G_\lambda$.

Suppose $f \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ and $g \in \prod_{\lambda \in \Lambda} G_\lambda$, and consider gfg^{-1} . There is a finite subset $I \subset \Lambda$ such that $i \in I$ iff $f_i \neq 1$. For any $i \in \Lambda$,

$$(gfg^{-1})_i = \begin{cases} 1 & i \in \Lambda - I \\ g_i f_i g_i^{-1} & \text{otherwise.} \end{cases} \quad (52)$$

Therefore, $(gfg^{-1})_i$ is not 1 at most finitely many $i \in I$. Hence, $gfg^{-1} \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$. □

Theorem 2.5. For the weak direct product $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ of a set of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, the canonical injection $\iota_k: G_k \rightarrow \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ defined by

$$(\iota_k a_k)_\lambda := \begin{cases} 1_{G_\lambda} & \lambda \neq k \\ a_k & \lambda = k \end{cases} \quad (53)$$

for $a_k \in G_k$, is a mono for each $k \in \Lambda$.

Proof. Let $k \in \Lambda$, and $a_k, b_k \in G_k$. For each $\lambda \in \Lambda$, we have

$$(\iota_k(b_k a_k))_\lambda = \begin{cases} 1_{G_\lambda} & \lambda \neq k \\ b_k a_k & \lambda = k \end{cases} \quad (54)$$

and

$$(\iota_k b_k) (\iota_k a_k)_\lambda = \begin{cases} 1_{G_\lambda} 1_{G_\lambda} & \lambda \neq k \\ b_k a_k & \lambda = k. \end{cases} \quad (55)$$

Hence, $\iota_k(b_k a_k) = (\iota_k b_k) (\iota_k a_k)$.

If we suppose $\iota_k b_k = \iota_k a_k$, then

$$\forall \lambda \in \Lambda : (\iota_k b_k)_\lambda = (\iota_k a_k)_\lambda \quad (56)$$

In particular, when $\lambda = k$, we obtain $b_k = a_k$. \square

Theorem 2.6. *For a set of groups $\{G_\lambda \mid \lambda \in \Lambda\}$ and each $k \in \Lambda$, $\iota_k G_k \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$.*

Proof. Let $k \in \Lambda$. By Theorem 2.5, ι_k is a group homomorphism. Hence, its image is a subgroup, $\iota_k G_k < \prod_{\lambda \in \Lambda} G_\lambda$ by Theorem 0.5.

Let $h \in \iota_k G_k$; there is some $g_k \in G_k$ such that $h = \iota_k g_k$. For each $\lambda \in \Lambda$, and $f \in \prod_{\lambda \in \Lambda} G_\lambda$,

$$\pi_\lambda(f h f^{-1}) = f_\lambda \pi_\lambda(\iota_k g_k) f_\lambda^{-1} = \begin{cases} f_k g_k f_k^{-1} & \lambda = k \\ 1 & \text{otherwise.} \end{cases} \quad (57)$$

Recalling $f_k g_k f_k^{-1} \in G_k$, we conclude $f h f^{-1} \in \iota_k G_k$. \square

Theorem 2.7. *Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there is a given group homomorphism $\theta_\lambda : G_\lambda \rightarrow H_\lambda$. Let $\theta := \prod_{\lambda \in \Lambda} \theta_\lambda$ be a map from $\prod_{\lambda \in \Lambda} G_\lambda$ to $\prod_{\lambda \in \Lambda} H_\lambda$, given by*

$$(\theta g)_\lambda := \theta_\lambda g_\lambda \quad (58)$$

for each $\lambda \in \Lambda$. Then, θ is a group homomorphism, $\theta(\prod_{\lambda \in \Lambda} G_\lambda) \subset \prod_{\lambda \in \Lambda}^{\text{weak}} H_\lambda$, $\theta^{-1} = \prod_{\lambda \in \Lambda}(\theta_\lambda^{-1})$, and $\theta(\prod_{\lambda \in \Lambda} G_\lambda) = \prod_{\lambda \in \Lambda} \theta_\lambda G_\lambda$. Moreover, θ is monic iff θ_λ is monic for each $\lambda \in \Lambda$, and θ is epic iff θ_λ is epic for each $\lambda \in \Lambda$.

Proof. Let us first show $\theta : \prod_{\lambda \in \Lambda} G_\lambda \rightarrow \prod_{\lambda \in \Lambda} H_\lambda$ is a group homomorphism. Suppose $f, g \in \prod_{\lambda \in \Lambda} G_\lambda$, and $k \in \Lambda$. Then

$$(\theta(gf))_k = \theta_k(g_k f_k) = (\theta_k g_k)(\theta_k f_k) = (\theta g)_k (\theta f)_k. \quad (59)$$

Since $k \in \Lambda$ is arbitrary, we conclude $\theta(gf) = (\theta g)(\theta f)$.

Next, $\theta(\prod_{\lambda \in \Lambda} G_\lambda) \subset \prod_{\lambda \in \Lambda}^{\text{weak}} H_\lambda$. Let $h \in \theta(\prod_{\lambda \in \Lambda} G_\lambda)$; there is a $g \in \prod_{\lambda \in \Lambda} G_\lambda$ such that $h = \theta g$, where g satisfies

$$g_\lambda = 1 \text{ for all but finitely many } \lambda \in \Lambda. \quad (60)$$

There exists a finite subset $I_h \subset \Lambda$ such that $i \in I_h$ iff $g_i \neq 1$. Then $h = fg$ satisfies

$$\lambda \in \Lambda - I_h \Rightarrow h_\lambda - (\theta 1)_\lambda = 1. \quad (61)$$

Therefore, h_λ is not 1 at most finitely many $\lambda \in \Lambda$. That is, $h \in \prod_{\lambda \in \Lambda}^{\text{weak}} H_\lambda$.

The kernel of θ is

$$\begin{aligned} \theta^{-1} &= \left\{ g \in \prod_{\lambda \in \Lambda} G_\lambda \mid \forall \lambda \in \Lambda : \theta_\lambda g_\lambda = 1_{H_\lambda} \right\} \\ &= \left\{ g \in \prod_{\lambda \in \Lambda} G_\lambda \mid \forall \lambda \in \Lambda : g_\lambda \in \theta_\lambda^{-1} \right\} \\ &= \prod_{\lambda \in \Lambda} (\theta_\lambda^{-1}) \end{aligned} \quad (62)$$

and the image is

$$\begin{aligned} \theta \left(\prod_{\lambda \in \Lambda} G_\lambda \right) &= \left\{ h \in \prod_{\lambda \in \Lambda} H_\lambda \mid \exists g \in \prod_{\lambda \in \Lambda} G_\lambda : h = \theta g \right\} \\ &= \left\{ h: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} H_\lambda \mid \forall \lambda \in \Lambda : h_\lambda \in H_\lambda \wedge \exists g_\lambda \in G_\lambda : h_\lambda = \theta_\lambda g_\lambda \right\} \\ &= \left\{ h: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} H_\lambda \mid \forall \lambda \in \Lambda : \exists g_\lambda \in G_\lambda : h_\lambda = \theta_\lambda g_\lambda \right\} \\ &= \left\{ h: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} H_\lambda \mid \forall \lambda \in \Lambda : h_\lambda \in \theta_\lambda G_\lambda \right\} \\ &= \prod_{\lambda \in \Lambda} \theta_\lambda G_\lambda. \end{aligned} \quad (63)$$

Recall that θ is monic iff $\theta^{-1} = \{1\}$ by Lemma 0.4. As shown above, $\theta^{-1} = \prod_{\lambda \in \Lambda} (\theta_\lambda^{-1})$, θ is monic iff

$$\forall \lambda \in \Lambda : \theta_\lambda^{-1} = \{1\}. \quad (64)$$

Thus, θ is monic iff each θ_λ is monic. As shown above, $\theta(\prod_{\lambda \in \Lambda} G_\lambda) = \prod_{\lambda \in \Lambda} \theta_\lambda G_\lambda$, θ is epic iff $\theta(\prod_{\lambda \in \Lambda} G_\lambda) = \prod_{\lambda \in \Lambda} H_\lambda$, i.e.,

$$\forall \lambda \in \Lambda : \theta_\lambda G_\lambda = H_\lambda. \quad (65)$$

Therefore, θ is epic iff each θ_λ is epic. \square

Theorem 2.8. *Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there are a given group and its subgroup $N_\lambda \triangleleft G_\lambda$. Then*

- $\prod_{\lambda \in \Lambda} N_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$, and $(\prod_{\lambda \in \Lambda} G_\lambda) / (\prod_{\lambda \in \Lambda} N_\lambda) \cong \prod_{\lambda \in \Lambda} (G_\lambda / N_\lambda)$.
- $\prod_{\lambda \in \Lambda}^{weak} N_\lambda \triangleleft \prod_{\lambda \in \Lambda}^{weak} G_\lambda$, and $(\prod_{\lambda \in \Lambda}^{weak} G_\lambda) / (\prod_{\lambda \in \Lambda}^{weak} N_\lambda) \cong \prod_{\lambda \in \Lambda}^{weak} (G_\lambda / N_\lambda)$.

Proof. Let $\lambda \in \Lambda$, and

$$\pi_\lambda: G_\lambda \rightarrow G_\lambda / N_\lambda; g \mapsto gN_\lambda \quad (66)$$

be the canonical epi. By Theorem 1.6, the kernel is N_λ , $\pi_\lambda^{-1} 1 = N_\lambda$. Hence, $p := \prod_{\lambda \in \Lambda} \pi_\lambda$ is an epi:

$$p: \prod_{\lambda \in \Lambda} G_\lambda \rightarrow \prod_{\lambda \in \Lambda} (G_\lambda / N_\lambda) \quad (67)$$

with the kernel $p^{-1} 1 = \prod_{\lambda \in \Lambda} (\theta_\lambda^{-1} 1) = \prod_{\lambda \in \Lambda} N_\lambda$ by Theorem 2.7. Since the kernel $p^{-1} 1$ is a normal subgroup of $\prod_{\lambda \in \Lambda} G_\lambda$ by Lemma 1.1, we conclude $\prod_{\lambda \in \Lambda} N_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$. Since p is epic, its image is the codomain, $p(\prod_{\lambda \in \Lambda} G_\lambda) = \prod_{\lambda \in \Lambda} (G_\lambda / N_\lambda)$. Applying Theorem 1.8, we conclude

$$\prod_{\lambda \in \Lambda} (G_\lambda / N_\lambda) = p \left(\prod_{\lambda \in \Lambda} G_\lambda \right) \cong \left(\prod_{\lambda \in \Lambda} G_\lambda \right) / (p^{-1} 1) = \left(\prod_{\lambda \in \Lambda} G_\lambda \right) / \left(\prod_{\lambda \in \Lambda} N_\lambda \right). \quad (68)$$

Let p^{weak} be the restriction of p on $\prod_{\lambda \in \Lambda}^{weak} G_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$. By Theorem 2.7, p^{weak} is $(\prod_{\lambda \in \Lambda}^{weak} (G_\lambda / N_\lambda))$ -valued:

$$p^{weak}: \prod_{\lambda \in \Lambda}^{weak} G_\lambda \rightarrow \prod_{\lambda \in \Lambda}^{weak} (G_\lambda / N_\lambda). \quad (69)$$

Since the original p is a group homomorphism, the restriction is also a group homomorphism, since

$$p^{weak}(gf) = p(gf) = (pg)(pf) = (p^{weak}g)(p^{weak}f) \quad (70)$$

for each $f, g \in \prod_{\lambda \in \Lambda}^{weak} (G_\lambda / N_\lambda)$.

Next, we will show p^{weak} is an epi. Let $[g] \in \prod_{\lambda \in \Lambda}^{weak} (G_\lambda / N_\lambda)$. Since the original p is an epi, there is $g' \in \prod_{\lambda \in \Lambda} G_\lambda$ such that $pg' = [g]$, that is

$$\forall \lambda \in \Lambda : g'_\lambda N_\lambda = [g]_\lambda. \quad (71)$$

Since $[g] \in \prod_{\lambda \in \Lambda}^{weak} (G_\lambda / N_\lambda)$, $[g]_\lambda = N_\lambda$ for all but finitely many $\lambda \in \Lambda$. Hence, we conclude $g' \in \prod_{\lambda \in \Lambda}^{weak} G_\lambda$.

Finally, let us consider the kernel:

$$\begin{aligned} p^{weak}{}^{-1} 1 &= \left\{ g \in \prod_{\lambda \in \Lambda}^{weak} G_\lambda \mid p^{weak}g = 1 \in \prod_{\lambda \in \Lambda}^{weak} (G_\lambda / N_\lambda) \right\} \\ &= \left\{ g \in \prod_{\lambda \in \Lambda}^{weak} G_\lambda \mid g_\lambda N_\lambda = N_\lambda \text{ for all but finitely many } \lambda \in \Lambda \right\} \end{aligned} \quad (72)$$

Since $g_\lambda N_\lambda = N_\lambda$ iff $g_\lambda \in N_\lambda$, we obtain

$$p^{\text{weak}} \dashv 1 = \left\{ g \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \mid g_\lambda \in N_\lambda \text{ for all but finitely many } \lambda \in \Lambda \right\} = \prod_{\lambda \in \Lambda}^{\text{weak}} N_\lambda. \quad (73)$$

Therefore, if we apply Theorem 1.8 for the epic $p^{\text{weak}}: \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \rightarrow \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda / N_\lambda)$ with the kernel $p^{\text{weak}} \dashv 1 = \prod_{\lambda \in \Lambda}^{\text{weak}} N_\lambda$, we conclude

$$\prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda / N_\lambda) \cong \left(\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \right) / \left(\prod_{\lambda \in \Lambda}^{\text{weak}} N_\lambda \right). \quad (74)$$

□

2.3 Direct Sums

Let us begin with the direct sum version of Theorem 2.8:

Theorem 2.9. *Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there are given an abelian group and its subgroup $N_\lambda \triangleleft G_\lambda$. Then, $(\sum_{\lambda \in \Lambda} G_\lambda) / (\sum_{\lambda \in \Lambda} N_\lambda) \cong \sum_{\lambda \in \Lambda} (G_\lambda / N_\lambda)$.*

Theorem 2.10 (Universal Property of Direct Sums). *Let B be an abelian group and Λ be a set, and suppose that for each $\lambda \in \Lambda$, there is given an abelian group homomorphism $\psi_\lambda: A_\lambda \rightarrow B$. Then, there exists a unique mediator homomorphism $\psi: \sum_{\lambda \in \Lambda} A_\lambda \rightarrow B$ such that the following diagram for each $k \in \Lambda$ is commutative:*

$$\begin{array}{ccc} & B & \\ \exists! \psi \uparrow & \swarrow \psi_k & \\ \sum_{\lambda \in \Lambda} A_\lambda & \xleftarrow{\iota_k} & A_k \end{array} \quad \psi_k = \psi \circ \iota_k. \quad (75)$$

Proof. Recall $a \in \sum_{\lambda \in \Lambda} A_\lambda$ iff $\pi_\lambda a = 0$ for all but finitely many $\lambda \in \Lambda$. For each $a \in \sum_{\lambda \in \Lambda} A_\lambda$, there exists a finite subset $I_a \subset \Lambda$ such that $\pi_i a \neq 0$ iff $i \in I_a$. Define $\psi: \sum_{\lambda \in \Lambda} A_\lambda \rightarrow B$ by

$$\begin{aligned} \psi 0 &:= 0 \\ \psi a &:= \sum_{i \in I_a} \psi_i (\pi_i a). \end{aligned} \quad (76)$$

Let $a, a' \in \sum_{\lambda \in \Lambda} A_\lambda$. Since, if both are zero, so is their sum, namely $\neg I_{a+a'} \supset (\neg I_a) \cap (\neg I_{a'})$, we have that

$$I_{a+a'} \subset I_a \cup I_{a'}, \quad (77)$$

where $\neg I := \Lambda - I$ for any subset $I \subset \Lambda$. Hence,

$$\psi(a + a') = \sum_{i \in I_{a+a'}} \psi_i (\pi_i (a + a')) \quad (78)$$

is a sum over finite set $I_{a+a'}$. Since it is a finite sum, we can expand the right-hand side:

$$\psi(a + a') = \sum_{i \in I_{a+a'}} \psi_i(\pi_i a) + \sum_{i \in I_{a+a'}} \psi_i(\pi_i a') \quad (79)$$

Let $i \in I_{a+a'}$, and consider the first term $\sum_{i \in I_{a+a'}} \psi_i(\pi_i a)$.

- If $\pi_i a' = 0$, $\pi_i a \neq 0$ must be the case. So $i \in I_a$.
- Otherwise, $\pi_i a' \neq 0$. Therefore, either $\pi_i a = 0$ or $\pi_i a \neq 0$ is the case:
 - $\pi_i a' \neq 0$ and $\pi_i a = 0$ case. The corresponding term in $\sum_{i \in I_{a+a'}} \psi_i(\pi_i a)$ is zero.
 - $\pi_i a' \neq 0$ and $\pi_i a \neq 0$ case. Then $i \in I_a$ is the case.

Thus, the first term is

$$\sum_{i \in I_{a+a'}} \psi_i(\pi_i a) = 0 + \sum_{i \in I_a} \psi_i(\pi_i a) = \psi a. \quad (80)$$

Therefore, we conclude $\psi(a + a') = \psi a + \psi a'$.

Let $j \in \Lambda$ and $a_j \in A_j$. Since $\iota_j a_j \in \sum_{\lambda \in \Lambda} A_\lambda$ and $I_{\iota_j a_j} = \{j\}$, see Theorem 2.5, we have

$$\psi(\iota_j a_j) = \sum_{i \in I_{\iota_j a_j}} \psi_i(\pi_i(\iota_j a_j)) = \psi_j a_j. \quad (81)$$

Since $a_j \in A_j$ is arbitrary, we conclude $\psi \circ \iota_j = \psi_j$.

Finally, let us show the uniqueness. Suppose an abelian group homomorphism $\phi: \sum_{\lambda \in \Lambda} A_\lambda \rightarrow B$ also satisfies

$$\forall \lambda \in \Lambda : \phi \circ \iota_\lambda = \pi_\lambda. \quad (82)$$

Then, $\phi = \psi$ since

$$\phi a = \phi \sum_{i \in I_a} \iota_j a_j = \sum_{i \in I_a} \phi(\iota_j a_j) = \sum_{i \in I_a} \psi_j a_j = \psi a \quad (83)$$

for all $a \in \sum_{\lambda \in \Lambda} A_\lambda$. \square

Remark. This theorem becomes false if we remove the restriction that the groups are abelian.

Consider S_3 – the symmetric group on a set of three elements:

$$\begin{cases} () \\ (12), (23), (31), \\ (123), (132), \end{cases} \quad (84)$$

where $()$ denotes the identity, and for example (12) is 1 ↔ 2 swap, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \xrightarrow{(12)} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, and (123) is a cyclic permutation $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \xrightarrow{(123)} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.
 S_3 is generated by $\{(12), (123)\}$, since

$$\begin{aligned} () &= (12) \circ (12) \\ (23) &= (12) \circ (123) \\ (31) &= (123) \circ (12) \\ (132) &= (123) \circ (123). \end{aligned} \tag{85}$$

Moreover, S_3 is not abelian:

$$(12) \circ (123) \neq (123) \circ (12). \tag{86}$$

Consider the weak product of \mathbb{Z}_2 and \mathbb{Z}_3 , where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ are modular arithmetic. Since it is a finite product case, their weak product is the ordinary product $\mathbb{Z}_2 \times \mathbb{Z}_3$. Let ι_2 and ι_3 be the corresponding canonical injections:

$$\begin{aligned} \iota_2: \mathbb{Z}_2 &\rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3 \\ \iota_3: \mathbb{Z}_3 &\rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3. \end{aligned} \tag{87}$$

Consider the following group homomorphisms

$$\begin{aligned} \phi_2: \mathbb{Z}_2 &\rightarrow S_3 \\ \phi_3: \mathbb{Z}_3 &\rightarrow S_3 \end{aligned} \tag{88}$$

defined by

$$\phi_2 0 = (), \phi_2 1 = (12), \tag{89}$$

and

$$\phi_3 0 = (), \phi_3 1 = (123), \phi_3 2 = (132). \tag{90}$$

It is worth mentioning that $\phi_3(1+1) = (\phi_3 1) \circ (\phi_3 1) = (123) \circ (123)$. Suppose, for contradiction, that there is a mediator $\phi: \mathbb{Z}_2 \times \mathbb{Z}_3 \rightarrow S_3$. As shown above, (12) and (123) generate S_3 , and $\{(12), (123)\} \subset \phi(\mathbb{Z}_2 \times \mathbb{Z}_3)$. Hence, ϕ is an epi, $\phi(\mathbb{Z}_2 \times \mathbb{Z}_3) = S_3$. Recalling the image forms a subgroup, $\phi(\mathbb{Z}_2 \times \mathbb{Z}_3) < S_3$, this equality is a group isomorphism. As shown above, S_3 is non-abelian but $\phi(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is the image of the product of two abelian groups, so it is abelian, which is absurd.

Theorem 2.11. *Let G be an abelian group, Λ be a set, and suppose that for each $\lambda \in \Lambda$, there is a given subgroup $N_\lambda < G$. If we further assume each $g \in G$ has a unique representation $g = \sum_{\lambda \in \Lambda} g_\lambda$, where $g_\lambda \in N_\lambda$ is zero for all but finitely many $\lambda \in \Lambda$. Then G is isomorphic to $\sum_{\lambda \in \Lambda} N_\lambda$.*

Proof. Since G is abelian, its subgroup is always normal. Let $n \in \sum_{\lambda \in \Lambda} N_\lambda$. Then $n_\lambda = 0$ for all but finitely many indices, say $I_n \subset \Lambda$:

$$i \in I_n \Leftrightarrow n_i \neq 0. \quad (91)$$

Then, $\sum_{i \in I_n} n_i$ is a finite sum of elements in G , hence $\sum_{i \in I_n} n_i \in G$. Define $\varphi: \sum_{\lambda \in \Lambda} N_\lambda \rightarrow G; n \mapsto \sum_{i \in I_n} n_i$.

- φ is a group homomorphism.

Let $f, g \in \sum_{\lambda \in \Lambda} N_\lambda$. Then $\varphi(f + g) = \sum_{i \in I_{f+g}} (f + g)_i$. If we let $I_{f+g} = \{i_1, \dots, i_k\}$, we have

$$\varphi(f + g) = (f + g)_{i_1} + \dots + (f + g)_{i_k} = f_{i_1} + \dots + f_{i_k} + g_{i_1} + \dots + g_{i_k}, \quad (92)$$

since they are all members in the abelian group G . For each $i \in I_{f+g}$, either $f_i = 0$ or $f_i \neq 0$:

- If $f_i = 0, i \in \Lambda - I_f$.
- If $f_i \neq 0, i \in I_f$.

Therefore, $\sum_{i \in I_{f+g}} f_i = \sum_{i \in I_f} f_i = \varphi f$. We conclude $\varphi(f + g) = \varphi f + \varphi g$.

- φ is epic.

Let $g \in G$. By hypothesis, there is a unique representation $g = \sum_{\lambda \in \Lambda} g_\lambda$, where $g_\lambda \in N_\lambda$ is zero for all but finitely many $\lambda \in \Lambda$. In other words, G is a free abelian group with basis $\bigcup_{\lambda \in \Lambda} N_\lambda$. Let $I_g \subset \Lambda$ is the corresponding subset: $i \in I_g$ iff $g_i \neq 0$. Since $\sum_{i \in I_g} g_i \in \sum_{\lambda \in \Lambda} N_\lambda$ satisfies $\varphi \sum_{i \in I_g} g_i = g$, φ is surjective.

- φ is monic.

Consider the kernel:

$$\varphi^{\leftarrow} 0 := \left\{ g \in \sum_{\lambda \in \Lambda} N_\lambda \mid \varphi g = 0 \right\} \quad (93)$$

Let $z \in \varphi^{\leftarrow} 0$. By hypothesis, the corresponding unique representation of $\varphi z = 0$ is given by

$$\forall \lambda \in \Lambda : z_\lambda = 0. \quad (94)$$

It follows $z = 0$, namely $\varphi^{\leftarrow} 0 = \{0\}$. By Lemma 0.4, φ is monic.

Hence, $\varphi: \sum_{\lambda \in \Lambda} N_\lambda \rightarrow G$ is an isomorphism. \square

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