Adjunctions in Topology
2: An Object Sitting in Two Categories

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Chapter 0

Abstract

No one can over-exaggerate the importance of the empty set $\emptyset$ in mathematics, just as is the case with singleton sets such as $\{\emptyset\}$. This note explores $\{0, 1\}$, the two-element set, where 0 is identified with the empty set $\emptyset$ and 1 is identified with the singleton set $\{\emptyset\}$. Such a simple yet profound set is the foundation for various mathematical concepts, including Boolean algebra, logic, set theory, and general topology. This note examines some examples of adjoint functors with a lens through $\{0, 1\}$.

Chapter 1

Preliminaries

1.1 Sets, Maps, and Orders

We assume some working knowledge of informal set theory including sets, subsets, supersets, the empty set $\emptyset$, union, intersection, set difference, complement.

1.1.1 Sets and Maps

Definition 1.1.1 (Complement)**.** Let X be a set and $A \subset X$ be a subset. We denote $\neg A = X - A = \{x \in X \mid x \notin A\}$.

Theorem 1.1.1 (Empty Intersection and Empty Union)**.** *Let X be a set and $\{A_\lambda \subset X \mid \lambda \in \Lambda\}$ be a Λ-indexed set of subsets of X. The empty intersection $\bigcap_{\lambda \in \emptyset} A_\lambda$ is the underlying set X and the empty union $\bigcup_{\lambda \in \emptyset} A_\lambda$ is the empty set $\emptyset$.*

Proof. By definition:

$$\bigcap_{\lambda \in \Lambda} A_\lambda \coloneqq \{x \in X \mid \forall \lambda \in \Lambda : x \in A_\lambda\}. \tag{1.1}$$

For the empty intersection, the condition is vacuously true. Hence, $\bigcap_{\lambda \in \emptyset} A_\lambda = X$. Similarly:

$$\bigcup_{\lambda \in \Lambda} A_\lambda \coloneqq \{x \in X \mid \exists \lambda \in \Lambda : x \in A_\lambda\}. \tag{1.2}$$

If the index set is empty, the condition is always false. Hence, $\bigcup_{\lambda \in \emptyset} A_\lambda = \emptyset$. ■

Remark 1. We also have:

$$\neg \bigcap_{\lambda \in \Lambda} A_\lambda \coloneqq \{x \in X \mid \exists \lambda \in \Lambda : x \notin A_\lambda\} = \bigcup_{\lambda \in \Lambda} \neg A_\lambda \tag{1.3}$$

and

$$\neg \bigcup_{\lambda \in \Lambda} A_\lambda \coloneqq \{x \in X \mid \forall \lambda \in \Lambda : x \notin A_\lambda\} = \bigcap_{\lambda \in \Lambda} \neg A_\lambda. \tag{1.4}$$

Theorem 1.1.2. *Let X be a set. For $\{V_\alpha \subset X \mid \alpha \in A\}$ and $\{W_\beta \subset X \mid \beta \in B\}$,*

$$\left(\bigcup_{\alpha\in A} V_\alpha\right) \cap \left(\bigcup_{\beta\in B} W_\beta\right) = \bigcup_{(\alpha,\beta)\in A\times B} V_\alpha \cap W_\beta. \tag{1.5}$$

Similarly,

$$\left(\bigcap_{\alpha\in A} V_\alpha\right) \cup \left(\bigcap_{\beta\in B} W_\beta\right) = \bigcap_{(\alpha,\beta)\in A\times B} V_\alpha \cup W_\beta. \tag{1.6}$$

Proof.

$$\begin{aligned}\left(\bigcup_{\alpha\in A} V_\alpha\right) \cap \left(\bigcup_{\beta\in B} W_\beta\right) &= \{x \in X \mid \exists\alpha \in A : x \in V_\alpha\} \\ &\quad \cap \{x \in X \mid \exists\beta \in B : x \in W_\beta\} \\ &= \{x \in X \mid \exists(\alpha,\beta) \in A \times B : x \in V_\alpha \cap W_\beta\} \\ &= \bigcup_{(\alpha,\beta)\in A\times B} V_\alpha \cap W_\beta.\end{aligned} \tag{1.7}$$

Similarly,

$$\begin{aligned}\left(\bigcap_{\alpha\in A} V_\alpha\right) \cup \left(\bigcap_{\beta\in B} W_\beta\right) &= \{x \in X \mid \forall(\alpha,\beta) \in A \times B : x \in V_\alpha \cup W_\beta\} \\ &= \bigcap_{(\alpha,\beta)\in A\times B} V_\alpha \cup W_\beta.\end{aligned} \tag{1.8}$$

■

For a given map $f\colon X \to Y$, there are two induced maps:

- Direct image $f\colon 2^X \to 2^Y$
- Preimage $f^{\leftarrow}\colon 2^Y \to 2^X$

where $f^{\leftarrow}W \coloneqq \{x \in X \mid fx \in W\}$ for any $W \subset Y$.

Theorem 1.1.3 (Properties of Preimage)**.** *Let X and Y be sets and $f\colon X \to Y$ be a map. The preimage map $f^{\leftarrow}$ preserves the following elementary set operations:*

- $f^{\leftarrow}\left(\bigcup_{\lambda\in\Lambda} B_\lambda\right) = \bigcup_{\lambda\in\Lambda} f^{\leftarrow}B_\lambda$
- $f^{\leftarrow}\left(\bigcap_{\lambda\in\Lambda} B_\lambda\right) = \bigcap_{\lambda\in\Lambda} f^{\leftarrow}B_\lambda$
- $f^{\leftarrow}\left(B_1 - B_2\right) = f^{\leftarrow}B_1 - f^{\leftarrow}B_2$

where Λ is an arbitrary index set, B_1, B_2, B_λ are all subspaces in Y for each $\lambda \in \Lambda$.

Proof. The first two equations are almost identical:

$$\begin{aligned} p \in f^{\leftarrow}\left(\bigcup_{\lambda\in\Lambda} B_\lambda\right) &\Leftrightarrow fp \in \bigcup_{\lambda\in\Lambda} B_\lambda \\ &\Leftrightarrow \exists\lambda \in \Lambda : fp \in B_\lambda \\ &\Leftrightarrow \exists\lambda \in \Lambda : p \in f^{\leftarrow} B_\lambda \\ &\Leftrightarrow p \in \bigcup_{\lambda\in\Lambda} f^{\leftarrow} B_\lambda \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} p \in f^{\leftarrow}\left(\bigcap_{\lambda\in\Lambda} B_\lambda\right) &\Leftrightarrow fp \in \bigcap_{\lambda\in\Lambda} B_\lambda \\ &\Leftrightarrow \forall\lambda \in \Lambda : fp \in B_\lambda \\ &\Leftrightarrow \forall\lambda \in \Lambda : p \in f^{\leftarrow} B_\lambda \\ &\Leftrightarrow p \in \bigcap_{\lambda\in\Lambda} f^{\leftarrow} B_\lambda \end{aligned} \tag{1.10}$$

for each $p \in A$.

Recalling $B_1 - B_2 = \{x \in A \mid x \in B_1 \wedge x \in \neg B_2\} = B_1 \cap \neg B_2$, and

$$f^{\leftarrow}(\neg B_2) = \{x \in X \mid fx \in \neg B_2\} = X - f^{\leftarrow} B_2 = \neg f^{\leftarrow} B_2, \tag{1.11}$$

we have

$$\begin{aligned} f^{\leftarrow}(B_1 - B_2) &= f^{\leftarrow}(B_1 \cap \neg B_2) \\ &= f^{\leftarrow} B_1 \cap f^{\leftarrow}(\neg B_2) \\ &= f^{\leftarrow} B_1 \cap \neg f^{\leftarrow} B_2 \\ &= f^{\leftarrow} B_1 - f^{\leftarrow} B_2. \end{aligned} \tag{1.12}$$

Thus, the preimage $f^{\leftarrow} : 2^Y \to 2^X$ preserves union, intersection, and set-difference. ∎

1.1.2 Orders

For a set X, we consider binary relations on it, where a binary relation is represented as a subset of the product set $X \times X := \{(x, y) \mid x \in X \wedge y \in X\}$.

Definition 1.1.2 (Pre-orders and Presets)**.** A pre-order $\leqq$ on a set X is a binary relation $\leqq$ such that:

- Reflexive

  For each $x \in X$, $x \leqq x$ holds.

- Transitive

  If $x \leqq y$ and $y \leqq z$, then $x \leqq z$ holds.

Recalling $\leqq \subset X \times X$, $x \leqq y$ stands for $(x, y) \in \leqq$. We call the pair $(X, \leqq)$ the pre-ordered set, in short, a preset.

Definition 1.1.3 (Posets)**.** A preset $(X, \leqq)$ is called a partially ordered set, in short, a poset, iff the pre-order $\leqq$ is also antisymmetric:

- Antisymmetric

  If $x \leqq y$ and $y \leqq x$, then $x = y$.

1.2 General Topology

General topology, in short, topology is a brunch of mathematics concerned with spaces that are invariant under continuous maps.

1.2.1 Basic Definitions

Definition 1.2.1 (Topological Spaces)**.** Let X be a set. A topology on X is a subset of its subsets $\mathcal{T} \subset 2^X$ that closed under:

- Arbitrary Union

  Each union of members in $\mathcal{T}$ is also a member of $\mathcal{T}$.

- Finite Intersection

  Each finite intersection of members of $\mathcal{T}$ is also a member of $\mathcal{T}$.

Since the union of an empty family of sets in X is $\emptyset$, the intersection of an empty family of sets in X is X, we may add the following, yet redundant, conditions:

- Both $\emptyset$ and X are members of $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space. Any member in $\mathcal{T}$ is called an open subspace of X. In particular, both $\emptyset$ and X are open. A subset $C \subset X$ is called closed iff the complement $\neg C \coloneqq X - C$ is open, namely $\neg C \in \mathcal{T}$. Since $\emptyset = X - X$ and $X = X - \emptyset$, we conclude that both $\emptyset$ and X are clopen.

For a subset $Y \subset X$ of a topological space $(X, \mathcal{T})$, the induced topology is

$$\mathcal{T}_Y \coloneqq \{Y \cap U \mid U \in \mathcal{T}\}. \tag{1.13}$$

The pair $(Y, \mathcal{T}_Y)$ is called a subspace of $(X, \mathcal{T})$.

Definition 1.2.2 (Neighborhoods and Open Subspaces)**.** Let $(X, \mathcal{T})$ be a topological space, and $p \in X$. A subspace $U' \subset X$ is called a neighborhood of p iff there exists some $U \in \mathcal{T}$ such that $p \in U$ and $U \subset U'$. Let $\mathcal{N}_p$ be the set of all neighborhoods of p in X relative to $\mathcal{T}$.

Lemma 1.2.1. *Let $(X, \mathcal{T})$ be a topological space, $U \subset X$ be a subspace. U is open, $U \in \mathcal{T}$, iff U is a neighborhood of every point in it.*

Proof. ($\Rightarrow$) Suppose $U \in \mathcal{T}$. Then, for each $p \in U$, U is an open neighborhood of p.

($\Leftarrow$) Conversely, suppose U is a neighborhood to its points. For $p \in U$, let $V_p \in \mathcal{T}$ be an open subspace such that $p \in V_p$ and $V_p \subset U$. Then, we conclude $U = \bigcup_{p \in U} V_p$ since:

$$U \subset \bigcup_{p \in U} V_p \subset U. \tag{1.14}$$

U is given by a union of open subspaces in X, hence U is open. ∎

Definition 1.2.3 (Limit Points and Closure)**.** Let $A \subset (X, \mathcal{T})$ be a subspace. A point $p \in X$ is called a limit point of A iff each neighborhood of p contains at least one point of A distinct from p:

$$\forall U' \in \mathcal{N}_p : U' \cap A - \{p\} \neq \emptyset. \tag{1.15}$$

Let A' denote the set of all limit points. We call $\overline{A} \coloneqq A \cup A'$ the closure of A.

Lemma 1.2.2. *Let $A \subset (X, \mathcal{T})$ be a subspace. For any point $p \in X$, $p \in \overline{A}$ iff*

$$\forall U' \in \mathcal{N}_p : U' \cap A \neq \emptyset. \tag{1.16}$$

Proof. ($\Rightarrow$) Let $p \in \overline{A}$:

- $p \in A$ case

  For each neighborhood $U' \in \mathcal{N}_p$, $p \in U' \cap A$.

- $p \notin A$ case

  For each neighborhood $U' \in \mathcal{N}_p$, $U' \cap A = U' \cap A - \{p\} \neq \emptyset$ holds.

($\Leftarrow$) Suppose for each neighborhood $U' \in \mathcal{N}_p$, $U' \cap A \neq \emptyset$. Nothing has to be shown if $p \in A$, as $A \subset \overline{A}$. Hence, we may assume $p \notin A$. Then, as $A = A - \{p\}$, $U' \cap A = U' \cap A - \{p\} \neq \emptyset$ is the case for each neighborhood $U' \in \mathcal{N}_p$. ∎

Theorem 1.2.1 (Characterization of Closed Subspaces)**.** *A subspace $A \subset (X, \mathcal{T})$ is closed iff $A = \overline{A}$.*

Proof. ($\Rightarrow$) Suppose that A is closed, i.e., $\neg A \in \mathcal{T}$. Each $p \in \neg A$ has an open neighborhood, namely $\neg A$, which does not meet A since $A \cup \neg A = \emptyset$. So, each $p \in \neg A$ does not belong to $\overline{A}$. We have $\neg A \subset \neg \overline{A}$, and $A \supset \overline{A}$. Since $A \subset \overline{A}$, we conclude $\overline{A} = A$.

($\Leftarrow$) Suppose $\overline{A} = A$. We will show $\neg A$ is open. Let $p \in \neg A$. Since $p \in \neg \overline{A}$, p is not a limit point of A. Thus, there is some neighborhood $U' \in \mathcal{N}_p$ with $U' \cap A = \emptyset$ by Lemma 1.2.2. We obtain $U' \subset \neg A$. That is, $\neg A$ is a neighborhood of p. As $p \in \neg A$ is arbitrary, by Lemma 1.2.1, we conclude $\neg A \in \mathcal{T}$. ∎

Theorem 1.2.2 (Properties of Closures)**.** *Let $A, B \subset (X, \mathcal{T})$ be subspaces.*

- *The closure $\overline{A}$ is $\subset$-smallest closed subspace of X containing A:*

$$\overline{A} = \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\} \tag{1.17}$$

- $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
- $\overline{\overline{A}} = \overline{A}$*, i.e., the closure* $\overline{A}$ *of* A *is closed, and the closure-operation is idempotent.*
- $\overline{A} \cup \overline{B} = \overline{A \cup B}$
- $\overline{\emptyset} = \emptyset$

Proof. Let $\widetilde{A} := \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\}$. Since open subspaces are closed under arbitrary union, the complements, i.e., closed subspaces are closed under arbitrary intersection. Hence, $\widetilde{A}$ is closed. To show $\widetilde{A}$ is equal to $\overline{A}$, let us consider their complements:

$\neg\widetilde{A} \subset \neg\overline{A}$ Let $p \in \neg\widetilde{A}$. $\neg\widetilde{A}$ is an open neighborhood of p with $\neg\widetilde{A} \cap \widetilde{A} = \emptyset$. Since $\widetilde{A} \supset A$, A does not meet $\widetilde{A}$. Thus $\neg\widetilde{A} \cap A = \emptyset$. By Lemma 1.2.2, $p \in \neg\overline{A}$ holds.

$\neg\widetilde{A} \supset \neg\overline{A}$ Let $p \in \neg\overline{A}$. Since p is not a limit point of A, there exists an open neighborhood $U \in \mathcal{N}_p \cap \mathcal{T}$ such that $U \cap A - \{p\} = \emptyset$. As p is not in A, $U \cap A = \emptyset$, thus $A \subset \neg U$. Thus, $\neg U$ is a member of the right-hand side of (1.17), we obtain $\widetilde{A} \subset \neg U$. Since $p \in U$ and $U \subset \neg\widetilde{A}$, we conclude $p \in \neg\widetilde{A}$.

Hence, we obtain $\overline{A} = \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\}$.

- $A \subset B \Rightarrow \overline{A} \subset \overline{B}$

  Since any closed subspace containing B also contains A, $\overline{A} \subset \overline{B}$.

- $\overline{\overline{A}} = \overline{A}$

  Since $\overline{A}$ is given by an intersection of closed subspaces, $\overline{A}$ is closed. Moreover, $\overline{A} \subset \overline{A}$ is the $\subset$-smallest subspace containing $\overline{A}$.

- $\overline{A} \cup \overline{B} = \overline{A \cup B}$

  $\overline{A \cup B}$ is closed, and contains both A and B, hence $\overline{A} \cup \overline{A} \subset \overline{A \cup B}$. As $\overline{A} \cup \overline{B}$ is closed, containing $A \cup B$, $\subset$-smallest property implies $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

- $\overline{\emptyset} = \emptyset$

  Since $\emptyset$ is clopen and $\emptyset \subset \emptyset$, the $\subset$-smallest property ensures $\overline{\emptyset} = \emptyset$.

$\blacksquare$

Theorem 1.2.3 (Subspaces and Closures)**.** *Let $(X, \mathcal{T})$ be a topological space and $(Y, \mathcal{T}_Y) \subset (X, \mathcal{T})$ be a subspace. For $A \subset Y$, the closure $\overline{A}_Y$ relative to $\mathcal{T}_Y$ is $Y \cap \overline{A}$, where $\overline{A}$ is the closure of $A \subset X$ relative to $\mathcal{T}$.*

Proof. It suffices to show $A'_Y = Y \cap A'$ since $\overline{A}_Y = A'_Y \cup A$ and $Y \cap \overline{A} = Y \cup (A \cup A') = (Y \cap A) \cup (Y \cap A') = A \cup (Y \cap A')$.

Let $p \in A'_Y$ and $\mathcal{N}_{Y\,p}$ be the set of neighborhood of p relative to $\mathcal{T}_Y$:

$$\forall U' \in \mathcal{N}_{Y\,p} : \exists U \in \mathcal{T} : p \in (U \cap Y) \subset U'. \tag{1.18}$$

Note that $(U \cap Y) \in \mathcal{T}_Y$ if $U \in \mathcal{T}$. Since $p \in A'_Y$,

$$\forall U' \in \mathcal{N}_{Y\,p} : U' \cap A - \{p\} \neq \emptyset, \tag{1.19}$$

i.e.,

$$\forall U \in \mathcal{N}_p \cap \mathcal{T} : (U \cap Y) \cap A - \{p\} \neq \emptyset, \tag{1.20}$$

we obtain $p \in (Y \cap A)'$ relative to $\mathcal{T}$. Recalling $A \subset Y$ and $p \in Y$, we obtain $p \in Y \cap A'$.

Conversely, let $p \in Y \cap A'$ relative to $\mathcal{T}$:

$$\forall U' \in \mathcal{N}_p : U' \cap A - \{p\} \neq \emptyset. \tag{1.21}$$

Since $A \subset Y$, it is equivalent to

$$\forall U' \in \mathcal{N}_p : U' \cap (A \cap Y) - \{p\} \neq \emptyset. \tag{1.22}$$

Now, $U' \cap Y$ contains an open $(U \cap Y) \in \mathcal{T}_Y$ with $p \in U \cap Y$. That is, $U' \cap Y$ is a neighborhood of p relative to $\mathcal{T}_Y$, namely $U' \cap Y \in \mathcal{N}_{Y\,p}$, moreover $p \in A'_Y$.

Hence, we establish $A'_Y = Y \cap A'$, and $\overline{A}_Y = Y \cap \overline{A}$. ∎

1.2.2 Separation Axioms

Definition 1.2.4. The following axioms describe how a topology can distinguish points in the underlying set:

T_0 A T_0 space – a Kolmogorov space – is a topological space in which every pair of distinct points is topologically distinguishable, i.e., there exists an open subspace that contains one of them and not the other.

T_1 A T_1 space – a Fréchet space – is a topological space in which for every pair of distinct points, each has a neighborhood not containing the other. In other words, each has an open subspace that contains it but not the other.

T_2 A T_2 space – a Hausdorff space – is a topological space $(X, \mathcal{T})$ in which each of two distinct points have disjoint neighborhoods, that is, if $p \neq q$, there are $U' \in \mathcal{N}_p$ and $V' \in \mathcal{N}_q$ with $U' \cap V' = \emptyset$.

1.2.3 Basic Open Sets

...we can to an extent preassign the notion of nearness desired. [Dug66]

Definition 1.2.5 (Subbases and Generated Topology)**.** Let X be a set and $\mathcal{S} \subset 2^X$ be a set of subsets in X. As 2^X is a topology of X,

$$\tau_{\mathcal{S}} \coloneqq \left\{\mathcal{T} \subset 2^X \mid \mathcal{T} \text{ is a topology on } X \text{ with } \mathcal{S} \subset \mathcal{T}\right\} \tag{1.23}$$

is non-empty. Their intersection:

$$\bigcap \tau_{\mathcal{S}} \coloneqq \bigcap \{\mathcal{T} \in \tau_{\mathcal{S}}\} \tag{1.24}$$

is called the topology generated by $\mathcal{S}$. It is the $\subset$-smallest topology containing $\mathcal{S}$.

For the generated topology, the generating set $\mathcal{S}$ is called the subbbasic open set, in short, a subbase.

Remark 2 (Basis). No further conditions for being a subbase of some topology. If $\mathcal{S}$ satisfies:

1. $\mathcal{S}$ covers X

   For each $x \in X$, there is a $B \in \mathcal{S}$ with $x \in B$. This condition guarantees that X is open.

2. Binary Intersection

   Let $B_1, B_2 \in \mathcal{S}$. If $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{S}$ with $x \in B_3$ and $B_3 \subset B_1 \cap B_2$. This condition guarantees that $B_1 \cap B_2$ is open.

Then $\mathcal{S}$ is called the set of basic open sets, in short, a basis for the topology $\bigcap \tau_{\mathcal{S}}$ of X.

Theorem 1.2.4. *Let X be a set, $\mathcal{S} \subset 2^X$ be a basis – $\mathcal{S}$ satisfies both conditions 1 and 2 – and $\mathcal{T}_{\mathcal{S}}$ be the set of all unions of $\mathcal{S}$. $\mathcal{T}_{\mathcal{S}}$ is a topology on X. Moreover, $\mathcal{T}_{\mathcal{S}} = \bigcap \tau_{\mathcal{S}}$.*

Proof. As the condition 1 ensures $\mathcal{S}$ covers X, we have $X \in \mathcal{T}_{\mathcal{S}}$. If we take the empty union, $\emptyset \in \mathcal{T}_{\mathcal{S}}$. By definition, $\mathcal{T}_{\mathcal{S}}$ is closed under arbitrary union. The condition 2 guarantees $\mathcal{T}_{\mathcal{S}}$ is closed under binary, hence any finite intersection. Therefore, $\mathcal{T}_{\mathcal{S}}$ forms a topology on X.

Since $\mathcal{S} \subset \mathcal{T}_{\mathcal{S}}$ holds, $\mathcal{T}_{\mathcal{S}} \in \tau_{\mathcal{S}}$, hence $\bigcap \tau_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{S}}$. To show the other inclusion, let $U \in \mathcal{T}_{\mathcal{S}}$. By construction, there exists $\mathcal{B}_U \subset \mathcal{S}$ with

$$U = \bigcup \mathcal{B}_U = \bigcup \{V \in \mathcal{B}_U\}. \tag{1.25}$$

As $\mathcal{B}_U \subset \mathcal{S}$, and any member $T \in \tau_{\mathcal{S}}$ contains $\mathcal{S}$, we obtain $\mathcal{B}_U \subset T$ for each $T \in \tau_{\mathcal{S}}$. Thus, $\mathcal{B}_U \subset T$ holds for each $T \in \tau_{\mathcal{S}}$. I.e., $U \in \bigcap \tau_{\mathcal{S}}$. ∎

1.2.4 Continuous Maps

For given topological space $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and a map between the underlying sets $f\colon X \to Y$, we use $f^{\leftarrow}$ to associate the topology since $f^{\leftarrow}$ preserves the elementary set operations as shown in Theorem 1.1.3:

Definition 1.2.6 (Continuous Maps)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f\colon X \to Y$ is called continuous iff the preimage of each open subspace in Y is open in X. That is, $f^{\leftarrow}$ maps $\mathcal{T}_Y \subset 2^Y$ into $\mathcal{T}_X$:

$$f^{\leftarrow}\colon \mathcal{T}_Y \to \mathcal{T}_X. \tag{1.26}$$

The set of all continuous maps from X to Y is denoted by $C^0(X, Y)$.

Theorem 1.2.5 (Characterizations of Continuity)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f\colon X \to Y$ be a map. The following are equivalent:*

1. *$f \in C^0(X, Y)$ by means of Definition 1.2.6.*
2. *For a subbase (or a basis) $\mathcal{S}_Y \subset \mathcal{T}_Y$, $f^{\leftarrow}\mathcal{S}_Y \subset \mathcal{T}_X$.*
3. *The preimage of a closed subspace in Y is closed in X.*
4. *For each $x \in X$ and for each neighborhood $V' \in \mathcal{N}_{fx}$, there exists a neighborhood $U' \in \mathcal{N}_x$ s.t., $fU' \subset V'$.*
5. *$f\overline{A} \subset \overline{fA}$ for every $A \subset X$.*
6. *$\overline{f^{\leftarrow}B} \subset f^{\leftarrow}\overline{B}$ for every $B \subset Y$.*

Proof. $(1 \Leftrightarrow 2)$ As $\mathcal{S}_Y \subset \mathcal{T}_Y$, $f^{\leftarrow}|_{\mathcal{S}_Y}\colon \mathcal{S}_Y \to \mathcal{T}_X$. Conversely, suppose $f^{\leftarrow}\mathcal{S}_Y \subset \mathcal{T}_X$ is the case. Let $W \in \mathcal{T}_Y$. Since $\mathcal{T}_Y$ is generated by $\mathcal{S}_Y$, W is given by some, not necessarily finite, union of finite intersections of members in $\mathcal{S}_Y$:

$$W = \bigcup_{\lambda\in\Lambda} \left(B_1^{(\lambda)} \cap \cdots \cap B_{j_\lambda}^{(\lambda)}\right), \tag{1.27}$$

where $B_1^{(\lambda)} \cdots B_{j_\lambda}^{(\lambda)} \in \mathcal{S}_Y$ for each $\lambda \in \Lambda$. Applying Theorem 1.1.3, we obtain

$$f^{\leftarrow}W = \bigcup_{\lambda\in\Lambda} f^{\leftarrow}\left(B_1^{(\lambda)} \cap \cdots \cap B_{j_\lambda}^{(\lambda)}\right) = \bigcup_{\lambda\in\Lambda} \left(f^{\leftarrow}B_1^{(\lambda)}\right)\cap\cdots\cap\left(f^{\leftarrow}B_{j_\lambda}^{(\lambda)}\right). \tag{1.28}$$

Since $\left(f^{\leftarrow}B_1^{(\lambda)}\right)\cap\cdots\cap\left(f^{\leftarrow}B_{j_\lambda}^{(\lambda)}\right) \in \mathcal{T}_X$ and W is a union of such open subspaces in X, we conclude $f^{\leftarrow}W \in \mathcal{T}_X$.

$(1 \Leftrightarrow 3)$ By Theorem 1.1.3,

$$f^{\leftarrow}(\neg A) = f^{\leftarrow}(Y - A) = X - f^{\leftarrow}A = \neg f^{\leftarrow}A \tag{1.29}$$

for every $A \subset X$.

$(1 \Rightarrow 4)$ Let $x \in X$, $V' \in \mathcal{N}_{fx}$, and $V \in \mathcal{T}_Y$ s.t., $fx \in V$ and $V \subset V'$. As f is continuous, $f^{\leftarrow}V \in \mathcal{T}_X$. Since $x \in f^{\leftarrow}V$, we may set $U' = f^{\leftarrow}V$.

(4 ⇒ 5) Let $A \subset X$ and $x \in \overline{A}$; we will show fx is a member of $\overline{fA}$. Consider $V' \in \mathcal{N}_{fx}$; as we assume 4, there exists $U' \in \mathcal{N}_x$ with $fU' \subset V'$. Since $x \in \overline{A}$, by Lemma 1.2.2, $U' \cap A \neq \emptyset$ holds. Hence, $fx \in \overline{fA}$:

$$\emptyset \subsetneq f(U' \cap A) \subset fU' \cap fA \subset V' \cap fA. \tag{1.30}$$

(5 ⇒ 6) Let $B \subset Y$ and $A := f^{\leftarrow}B$. As we assume 5,

$$f\left(\overline{f^{\leftarrow}B}\right) = f\overline{A} \subset \overline{fA} = \overline{f\left(f^{\leftarrow}B\right)} \subset \overline{B}. \tag{1.31}$$

Thus, $\overline{f^{\leftarrow}B} \subset f^{\leftarrow}\overline{B}$.

(6 ⇒ 3) Let $B \subset Y$ be a closed subspace. As we assume 6, $\overline{f^{\leftarrow}B} \subset f^{\leftarrow}\overline{B}$. Since $\overline{B} = B$, we conclude $\overline{f^{\leftarrow}B} = f^{\leftarrow}B$:

$$\overline{f^{\leftarrow}B} \subset f^{\leftarrow}\overline{B} \subset f^{\leftarrow}B \subset \overline{f^{\leftarrow}B}. \tag{1.32}$$

See Theorem 1.2.1. ■

Lemma 1.2.3 (Universal Property of Relative Topology)**.** *Let $Y \subset (X, \mathcal{T})$ be a subspace. The relative topology $\mathcal{T}_Y$ defined in Definition 1.2.1 can be characterized as the $\subset$-smallest topology on Y for which the inclusion map:*

$$i \colon Y \hookrightarrow X; y \mapsto y \tag{1.33}$$

is continuous, namely $i \in C^0(Y, X)$.

Proof. Let ${\mathcal{T}_Y}'$ be an arbitrary topology on Y. Suppose $i \colon Y \hookrightarrow X$ is continuous relative to $(X, \mathcal{T})$ and $\left(Y, {\mathcal{T}_Y}'\right)$. We will show that ${\mathcal{T}_Y}' \supset \mathcal{T}_Y$.

Let $U \in \mathcal{T}$. As $i \in C^0\left(\left(Y, {\mathcal{T}_Y}'\right), (X, \mathcal{T})\right)$, the preimage $i^{\leftarrow}U$ is open in $\left(Y, {\mathcal{T}_Y}'\right)$:

$$i^{\leftarrow}U = U \cap Y \in {\mathcal{T}_Y}'. \tag{1.34}$$

Since U is arbitrary, it follows that any open subspace in Y relative to $\mathcal{T}_Y$, $U \cap Y \in \mathcal{T}_Y$ is a member of ${\mathcal{T}_Y}'$, hence $\mathcal{T}_Y \subset {\mathcal{T}_Y}'$. ■

Theorem 1.2.6 (Properties of Continuous Maps)**.** *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y), (Z, \mathcal{T}_Z)$ be topological spaces.*

- *If $f \in C^0(X, Y)$ and $g \in C^0(Y, Z)$, the composition $gf \in C^0(X, Z)$.*
- *If $f \in C^0(X, Y)$ and $A \subset X$, the restriction $f|_A : A \to Y$ is continuous relative to the relative topology on A.*
- *If $f \in C^0(X, Y)$, the coristriction of f on its image is continuous:*

$$f \in C^0(X, fX). \tag{1.35}$$

Proof. Suppose $f \in C^0((X, Y), g \in C^0(Y, Z)$, and $A \subset X$.

- Since $f^{\leftarrow}\colon \mathcal{T}_Y \to \mathcal{T}_X$ and $g^{\leftarrow}\colon \mathcal{T}_Z \to \mathcal{T}_Y$, and $(g \circ f)^{\leftarrow} = f^{\leftarrow} \circ g^{\leftarrow}$, the continuity of the composition $g \circ f$ follows:

$$(g \circ f)^{\leftarrow}\colon \mathcal{T}_Z \to \mathcal{T}_X. \tag{1.36}$$

- Let $i\colon A \hookrightarrow X$. Since

$$f|_A = f \circ i \tag{1.37}$$

  and as shown above $i \in C^0(A, X)$ relative to $\mathcal{T}_A$, the composition is continuous.

- For each $V \in \mathcal{T}_V$, i.e., for each open subspace $V \cap fX$ in fX,

$$f^{\leftarrow}(V \cap fX) = f^{\leftarrow}V \cap f^{\leftarrow}(fX) = f^{\leftarrow}V. \tag{1.38}$$

  Since $f^{\leftarrow}V$ is open in X, the restriction $f\colon X \to fX$ is continuous.

■

Definition 1.2.7 (Homeomorphisms and Topological Invariance)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f\colon X \to Y$ is called a homeomorphism – a topological isomorphism – iff the following conditions hold:

- The underlying map $f : X \to Y$ is bijective.
- Both f and f^{-1} are continuous.

If f is a homeomorphism, it is denoted by $f\colon X \cong Y$. Two spaces X and Y are homeomorphic, written $X \cong Y$, iff there is a homeomorphism between them. It is worth mentioning that a homeomorphism $f\colon X \cong Y$ is an open map – the image of an open subspace $U \in \mathcal{T}_X$ along f is open $fU \in \mathcal{T}_Y$, since f^{-1} is continuous. Moreover, a homeomorphism $f\colon X \cong Y$ is a bijection for the underlying set and the associated topologies:

$$\begin{aligned} f\colon X &\cong Y \\ f^{-1}\colon \mathcal{T}_Y &\cong \mathcal{T}_X \end{aligned} \tag{1.39}$$

Thus, any topological property about X is mapped to that of Y. We call any property of spaces a topological invariant iff whenever it is true for one space, it is also varied for every homeomorphic space.

Theorem 1.2.7. *Homeomorphism is an equivalence relation in the class of all topological spaces.*

Proof. Observe:

- Reflexive

  For any topological space X, $1_X\colon X \cong X$.

- Symmetric

  If $f\colon X \cong Y$, $Y \cong X$ via f^{-1}.

- Transitive

  If $f\colon X \cong Y$ and $g\colon Y \cong Z$, then $g \circ f\colon X \cong Z$.

See Theorem 1.2.6. ■

1.3 Category Theory

Category theory offers a general theory of mathematical structures and relations.

1.3.1 Basic Definitions

Definition 1.3.1 (Categories)**.** A category $\mathcal{C}$ consists of a class of objects $|\mathcal{C}|$ and, for each pair of objects $A, B \in |\mathcal{C}|$, a set of arrows from A to B, denoted as $\mathcal{C}(A, B)$, such that:

- Each arrow ϕ in $\mathcal{C}$ has unique domain and codomain, namely $X \xrightarrow{\phi} Y$ with $X, Y \in |\mathcal{C}|$.
- Each object $X \in |\mathcal{C}|$ has a unique arrow $X \xrightarrow{1_X} X$.
- For any pair of arrows f, g in $\mathcal{C}$, if the domain of g is equal to the codomain of f, their composite arrow $gf = g \circ f$ exists, namely if $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, their composition is $A \xrightarrow{gf} C$.

These arrows in $\mathcal{C}$ also satisfy the following axioms:

- For any arrow $A \xrightarrow{f} B$, both $f1_A$ and $1_B f$ are f.
- If $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, and $C \xrightarrow{h} D$, the compositions $h(gf)$ and $(hg)f$ are both equal to $A \xrightarrow{hgf} D$.

Remark 3 (Small Categories)*.* A category $\mathcal{C}$ is called small iff $|\mathcal{C}|$ is a set.

Definition 1.3.2 (Isomorphisms)**.** Let $\mathcal{C}$ be a category. An arrow $f \in \mathcal{C}(A, B)$ is called an isomorphism iff there is $f' \in \mathcal{C}(B, A)$ such that $f'f = 1_A$ and $ff' = 1_B$.

Definition 1.3.3 (Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor, in short a functor F from $\mathcal{C}$ and $\mathcal{D}$, denoted $F\colon \mathcal{C} \to \mathcal{D}$, consists of the following correspondences:

- For each object $C \in |\mathcal{C}|$, there exists $FC \in |\mathcal{D}|$.
- For an arrow $f \in \mathcal{C}(X, Y)$, there exists $Ff \in \mathcal{D}(FX, FY)$.

These correspondences satisfy the following axioms:

- For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, $FgFf = F(gf)$ holds. That is, the composition $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ$ in $\mathcal{D}$ is equal to $FX \xrightarrow{F(gf)} FZ$.
- For each $X \in |\mathcal{C}|$, $F1_X = 1_{FX}$.

We denote $\mathcal{D}^{\mathcal{C}}$ the class of functors from $\mathcal{C}$ to $\mathcal{D}$.

Remark 4 (Opposite Categories and Contravariant Functors). Let $\mathcal{C}$ be a category. The opposite $\mathcal{C}^{op}$ is given by:

- The same class of objects $|\mathcal{C}^{op}| = |\mathcal{C}|$.
- An arrow $f^{op} \in \mathcal{C}^{op}(X, Y)$ is an arrow in $\mathcal{C}$ so that the domain and codomain are swapped, $f \in \mathcal{C}(Y, X)$.

The correspondence $\mathcal{C} \to \mathcal{C}^{op}$ preserves the categorical structure, exchanging domains and codomains:

- For each object $X \in |\mathcal{C}|$, $1_X \mapsto {1_X}^{op} = 1_X$.
- For $f^{op} \in \mathcal{C}^{op}(X, Y)$ and $g^{op} \in \mathcal{C}^{op}(Y, Z)$, we define $g^{op}f^{op}$ to be $(fg)^{op}$. That is, $X \xrightarrow{f^{op}} Y \xrightarrow{g^{op}} Z$ is $\left(Z \xrightarrow{g} Y \xrightarrow{f} X\right)^{op} = \left(Z \xrightarrow{fg} X\right)^{op}$.

Hence, $\mathcal{C}^{op}$ forms a category – the opposite category. A contravariant functor F from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F\colon \mathcal{C}^{op} \to \mathcal{D}$.

Theorem 1.3.1. *Functors preserve isomorphisms.*

Proof. Let $f \in \mathcal{C}(A, B)$ be an isomorphism and $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Since f is an isomorphism, there is an arrow $f' \in \mathcal{C}(B, A)$ with $f'f = 1_A$ and $ff' = 1_B$. Then, $Ff \in \mathcal{D}(FA, FB)$ has an inverse Ff', since

$$\begin{aligned} Ff' \circ Ff &= F(f'f) = F1_A = 1_{FA} \\ Ff \circ Ff' &= F(ff') = F1_B = 1_{FB} \end{aligned} \tag{1.40}$$

Hence, Ff is an isomorphism if f is an isomorphism. ∎

Definition 1.3.4 (Natural Transformations)**.** Let $\mathcal{C}, \mathcal{D}$ be two categories, and $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation θ from F to G, denoted as $\theta\colon F \Rightarrow G$ is given by a $|\mathbb{C}|$-indexed class of arrows in $\mathbb{D}$, namely $\{\theta_C \in \mathcal{D}(FC, GC) \mid C \in |\mathcal{C}|\}$, such that $Gc \circ \theta_{C_1} = \theta_{C_2} \circ Fc$ for each $c \in \mathcal{C}(C_1, C_2)$. That is, the following diagram is commutative:

$$\begin{tikzcd} FC_1 \arrow[r, "Fc"] \arrow[d, "\theta_{C_1}"'] & FC_2 \arrow[d, "\theta_{C_2}"] \\ GC_1 \arrow[r, "Gc"'] & GC_2 \end{tikzcd} \tag{1.41}$$

for each $c \in \mathcal{C}(C_1, C_2)$. We call $\theta_C \in \mathcal{D}(FC, GC)$ C-component of $\theta\colon F \Rightarrow G$.

Remark 5 (Curien's Promotion [Cur08]). Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(A, B)$. With the terminal category $\mathbf{1}$ of a singleton set $\{\star\}$ with the identity map on it, we may identify $A \in |\mathcal{C}|$ as a functor $\widetilde{A}\colon \mathbf{1} \to \mathcal{C}$ and $f \in \mathcal{C}(A, B)$ as a natural transformation $\widetilde{f}\colon A \Rightarrow B$.

$$\begin{tikzcd} \mathbf{1} \arrow[r, bend left, "\widetilde{A}"] \arrow[r, bend right, "\widetilde{B}"'] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \star \arrow[loop left, "1_\star"] & & A \arrow[loop right, "1_A"] \arrow[d, "f"'] \\ & & B \arrow[loop right, "1_B"] \end{tikzcd} \tag{1.42}$$

Here, $\widetilde{A}1_\star = 1_{\widetilde{A}\star} = 1_A$, $\widetilde{B}1_\star = 1_B$, and $\widetilde{f}_\star = f$. If no confusion is expected, we omit the $\widetilde{\ }$ symbol.

Theorem 1.3.2 (Functor Category)**.** *Let $\mathcal{C}, \mathcal{D}$ be categories, $\mathcal{D}^{\mathcal{C}}$ be the class of functors. Then $\mathcal{D}^{\mathcal{C}}$ and natural transformations among them form a category if $\mathcal{C}$ is small.*

Proof. We will show that when $\mathcal{C}$ is small, $\mathcal{D}^{\mathcal{C}}$ is locally small, namely for each pair $F, G \in \mathcal{D}^{\mathcal{C}}$, $\mathcal{D}^{\mathcal{C}}(F, G)$ forms a set.

Let $F, G \in \mathcal{D}^{\mathcal{C}}$ be functors. Consider the class of natural transformations $\mathcal{D}^{\mathcal{C}}(F, G)$. Let $\theta \in \mathcal{D}^{\mathcal{C}}(F, G)$. Recall the very definition, θ is indeed a set of $\mathcal{C}$-indexed set of maps in $\mathcal{D}$, $\{\theta_C \in \mathcal{D}(FC, GC) \mid C \in |\mathcal{C}|\}$, such that (1.41) is commutative for each $c \in \mathcal{C}(C_1, C_2)$.

Next, consider a correspondence $C \overset{\delta}{\mapsto} \mathcal{D}(FC, GC)$. This defines a class-valued map $\delta\colon |\mathcal{C}| \to 2^{\mathcal{D}}$, where $2^{\mathcal{D}}$ is the power class of arrows in $\mathcal{D}$. Since $|\mathcal{C}|$ is a set, the image $\delta\,|\mathcal{C}|$ is a set. Moreover, the union of the image $\cup\delta\,|\mathcal{C}| := \bigcup_{C \in |\mathcal{C}|} \delta C$ is a set, containing θ:

$$\mathcal{D}^{\mathcal{C}}(F, G) \subset \cup\delta\,|\mathcal{C}|\,. \tag{1.43}$$

Hence, $\mathcal{D}^{\mathcal{C}}(F, G)$ is a set. ∎

Remark 6. Recalling Remark 5, since we may identify $A, B \in |\mathcal{C}|$ as $A\colon \mathbf{1} \to \mathcal{C}$ and $B\colon \mathbf{1} \to \mathcal{C}$, we have $f \in \mathcal{C}^{\mathbf{1}}(A, B)$.

Definition 1.3.5 (Vertical Composition and Horizontal Composition)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. For $\theta \in \mathcal{D}^{\mathcal{C}}(F, G)$ and $\tau \in \mathcal{D}^{\mathcal{C}}(G, H)$, their vertical composition $\tau \circ \theta \in \mathcal{D}^{\mathcal{C}}(F, H)$ is given by

$$\{\tau_C \circ \theta_C \in \mathcal{D}(FC, HC) \mid C \in |\mathcal{C}|\} \tag{1.44}$$

since

$$(\tau \circ \theta)_{C_2} \circ Fc = \tau_{C_2} \circ \theta_{C_2} \circ Fc = \tau_{C_2} \circ Gc \circ \theta_{C_1} = Hc \circ \tau_{C_1} \circ \theta_{C_1}. \tag{1.45}$$

For natural transformations $\theta\colon F \Rightarrow G$ and $\sigma\colon H \Rightarrow K$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] & \mathcal{E} \end{tikzcd} \tag{1.46}$$

we define their horizontal composition $\theta * \sigma$ via the following lemma:

Lemma 1.3.1 (Godement Product)**.** *Consider:*

- $H\theta\colon HF \Rightarrow HG$, $\sigma G\colon HG \Rightarrow KG$, *and*
$$\sigma G \circ H\theta\colon HF \Rightarrow KG. \tag{1.47}$$
- $\sigma F\colon HF \Rightarrow KF$, $K\theta\colon KF \Rightarrow KG$, *and*
$$K\theta \circ \sigma F\colon HF \Rightarrow KG. \tag{1.48}$$

Then, $\sigma G \circ H\theta = K\theta \circ \sigma F$. We define $\theta * \sigma$ by the corresponding commutative diagram:

$$\begin{tikzcd} HF \arrow[r, Rightarrow, "H\theta"] \arrow[d, Rightarrow, "\sigma F"'] & HG \arrow[d, Rightarrow, "\sigma G"] \\ KF \arrow[r, Rightarrow, "K\theta"'] & KG \end{tikzcd} \qquad K\theta \circ \sigma F = \sigma G \circ H\theta. \tag{1.49}$$

Proof. We will first show that $H\theta$ is a natural transformation. Let $f \in \mathbb{C}(A, B)$. Consider:

$$\begin{tikzcd} HFA \arrow[r, "HFf"] \arrow[d, "H\theta_A"'] & HFB \arrow[d, "H\theta_B"] \\ HGA \arrow[r, "HGf"'] & HGB \end{tikzcd} \tag{1.50}$$

Since $\theta\colon F \Rightarrow G$ is a natural transformation and $H\colon \mathcal{C} \to \mathcal{D}$ is a functor,

$$H\theta_B \circ HFf = H\left(\theta_B \circ Ff\right) = H\left(Gf \circ \theta_A\right) = HGf \circ H\theta_A, \tag{1.51}$$

i.e., the above diagram is commutative. Hence $H\theta\colon HF \Rightarrow HG$ is a natural transformation. Similarly, σG, σF, and $K\theta$ are also natural transformations, and both $\sigma G \circ H\theta$ and $K\theta \circ \sigma F$ are natural transformations from HF to KG.

Let $C \in |\mathcal{C}|$. For $\theta_C \in \mathcal{D}(FC, GC)$, since $\sigma\colon H \Rightarrow K$ is a natural transformation, C-components of these natural transformations satisfy:

$$\begin{tikzcd} HFC \arrow[r, "H\theta_C"] \arrow[d, "\sigma_{FC}"'] & KGC \arrow[d, "\sigma_{GC}"] \\ KFC \arrow[r, "K\theta_C"'] & KGC \end{tikzcd} \qquad K\theta_C \circ \sigma_{FC} = \sigma_{GC} \circ H\theta_C \tag{1.52}$$

Hence, $\{K\theta_C \circ \sigma_{FC} \mid C \in |\mathcal{C}|\}$ and $\{\sigma_{GC} \circ H\theta_C \mid C \in |\mathcal{C}|\}$ define the same natural transformation. ■

Remark 7. The commutative diagram in (1.41) defines $c * \theta$ for $c \in \mathcal{C}(C_1, C_2)$ and $\theta\colon F \Rightarrow G$, see Remark 5, where $c \in \mathcal{C}^{\mathbf{1}}(C_1, C_2)$ with $\theta \in \mathcal{D}^{\mathcal{C}}(F, G)$.

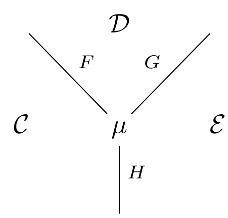
1.3.2 String Diagrams

Following [Cur08], we introduce string diagrams as pictorial representations of arrows in categorical calculations.

Definition 1.3.6 (String Diagrams)**.** We represent a natural transformation:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[rr, bend right, "H"'] & \mathcal{D} \arrow[r, bend left, "G"] & \mathcal{E} \end{tikzcd} \qquad \mu\colon GF \Rightarrow H \tag{1.53}$$

as follows:



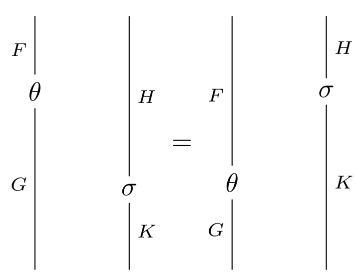
(1.54)

- Poincaré dual

  In this representation, categories are 2-dimensional areas separated by lines of functors, which are 1-dimensional; natural transformations are 0-dimensional. This correspondence is Poincaré dual to the ordinary diagrams.

- Elevator Rule – Godement's Product

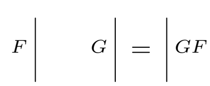
  Godement' product $\theta * \sigma$ in Lemma 1.3.1 is expressed as:



(1.55)

  This is a key axiom of this notation. The natural transformations can freely move up and down as long as they keep the ambient algebraic structures, particularly the domains and codomains of functors.

Remark 8 (Composition Rules and Identities). Functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ can be composed $GF\colon \mathcal{C} \to \mathcal{E}$:



(1.56)

Relative to this composition rule, the identities are expressed as follows:

- $\mu\colon 1_{\mathcal{C}} \Rightarrow F$:

$$\begin{array}{c} \vdots\ {\scriptstyle 1_{\mathcal{C}}} \\ \mu \\ \Big| {\scriptstyle F} \end{array} \quad = \quad \begin{array}{c} \mu \\ \Big| {\scriptstyle F} \end{array} \tag{1.57}$$

- $1_G\colon G \Rightarrow G$:

$$\begin{array}{c} \Big| {\scriptstyle G} \\ 1_G \\ \Big| {\scriptstyle G} \end{array} \quad = \quad \Big| {\scriptstyle G} \tag{1.58}$$

Among functors and natural transformations, we have

- $H\theta\colon HF \Rightarrow HG$, and $\sigma G\colon HG \Rightarrow KG$:

$$\begin{array}{c} \begin{array}{c} \Big| {\scriptstyle HF} \\ H\theta \\ \Big| {\scriptstyle HG} \end{array} \quad = \quad \begin{array}{c} {\scriptstyle F} \Big| \\ \theta \\ {\scriptstyle G} \Big| \end{array} \quad \Bigg| {\scriptstyle H} \quad = \quad \begin{array}{c} \Big| {\scriptstyle HF} \\ 1_H * \theta \\ \Big| {\scriptstyle HG} \end{array} \\ \\ \begin{array}{c} \Big| {\scriptstyle HG} \\ \sigma G \\ \Big| {\scriptstyle KG} \end{array} \quad = \quad {\scriptstyle G} \Bigg| \quad \begin{array}{c} \Big| {\scriptstyle H} \\ \sigma \\ \Big| {\scriptstyle K} \end{array} \quad = \quad \begin{array}{c} \Big| {\scriptstyle HG} \\ \sigma * 1_G \\ \Big| {\scriptstyle KG} \end{array} \end{array} \tag{1.59}$$

With Remark 5, we obtain:

$$\begin{array}{c} \Big| {\scriptstyle FA} \\ Ff \\ \Big| {\scriptstyle FB} \end{array} \quad = \quad \begin{array}{c} {\scriptstyle A} \Big| \\ f \\ {\scriptstyle B} \Big| \end{array} \quad \Bigg| {\scriptstyle F} \quad = \quad \begin{array}{c} \Big| {\scriptstyle FA} \\ 1_F * f \\ \Big| {\scriptstyle FB} \end{array} \tag{1.60}$$

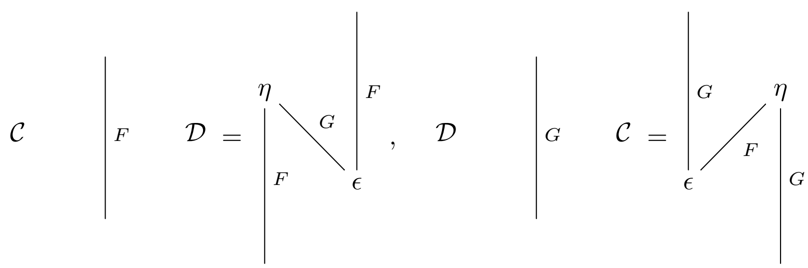
1.3.3 Adjunctions and Kan Extensions

Definition 1.3.7 (Adjunctions)**.** An adjuction – a pair of adjoint functors – is a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ with natural transformations $\eta\colon 1_{\mathcal{C}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow 1_{\mathcal{D}}$ that satisfy the following zig-zag identities:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, equal] & FGF \arrow[d, Rightarrow, "\epsilon F"] \\ & F \end{tikzcd} \quad 1_F = \epsilon F \circ F\eta, \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal] & GFG \arrow[d, Rightarrow, "G\epsilon"] \\ & G \end{tikzcd} \quad 1_G = G\epsilon \circ \eta G. \tag{1.61}$$

We denote $F \dashv G$, and call F the right adjoint and G the left adjoint. The associated natural transformations η and ϵ are called unit and counit, respectively.

Remark 9 (Zig-Zag in String Diagrams).



(1.62)

As a useful characterization of adjunctions, we have the following:

Theorem 1.3.3 (Natural Bijection)**.** *A pair of functors* $\begin{tikzcd} \mathcal{C} \arrow[d, bend right, "F"'] \\ \mathcal{D} \arrow[u, bend right, "G"'] \end{tikzcd}$ *forms an adjunction* $\begin{tikzcd} \mathcal{C} \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{D} \arrow[u, bend right, "G"'] \end{tikzcd}$ *with unit* $\eta\colon 1_{\mathcal{C}} \Rightarrow GF$ *and counit* $\epsilon\colon FG \Rightarrow 1_{\mathcal{D}}$ *iff there is a bijection* $\zeta_{C,D}\colon \mathcal{D}(FC, D) \to \mathcal{C}(C, GD)$ *for each* $C \in |\mathcal{C}|$ *and* $D \in |\mathcal{D}|$ *such that* $\zeta_{C,D}$ *is natural in* C *and* D*, where the naturality is expressed as:*

- *For* $FC \xrightarrow{g} D \xrightarrow{d} D'$ *in* $\mathcal{D}$,

$$\begin{tikzcd} C \arrow[r, "\zeta g"] \arrow[rr, bend right, "\zeta(gd)"'] & GD \arrow[r, "Gd"] & GD' \end{tikzcd} \qquad \zeta_{C,D'}(d \circ g) = Gd \circ (\zeta_{C,D}\, g)\,. \tag{1.63}$$

- *For* $C \xrightarrow{c} C' \xrightarrow{f'} GD'$ *in* $\mathcal{C}$,

$$\begin{tikzcd} FC \arrow[r, "Fc"] \arrow[rr, bend right, "\zeta^{-1}(f'c)"'] & FC' \arrow[r, "\zeta^{-1} f'"] & D' \end{tikzcd} \qquad {\zeta_{C,D'}}^{-1}(f' \circ c) = \left({\zeta_{C',D'}}^{-1} f'\right) \circ Fc. \tag{1.64}$$

Proof. ($\Rightarrow$) Suppose $F \dashv G$ with unit η and counit ϵ. Let $g \in \mathcal{D}(FC, D)$ and $f \in \mathcal{C}(C, GD)$. Define $\zeta_{C,D} g \coloneqq Gg \circ \eta_C$ and $\zeta'_{C,D} f \coloneqq \epsilon_D \circ Ff$. They form an inverse pair:

$$\begin{aligned} \zeta_{C,D}\left(\zeta'_{C,D} f\right) &= G\left(\epsilon_D \circ Ff\right) \circ \eta_C = G\epsilon_D \circ \eta_{GD} \circ f = f \\ \zeta'_{C,D}\left(\zeta_{C,D} g\right) &= \epsilon_D \circ F\left(Gg \circ \eta_C\right) = g \circ \epsilon_{FC} \circ F\eta_C = g, \end{aligned} \tag{1.65}$$

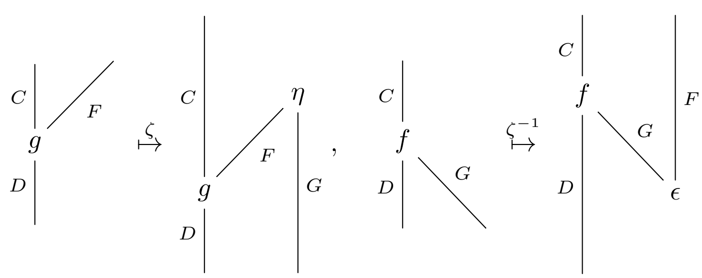
where $G\epsilon_D \circ \eta_{GD} = (G\epsilon \circ \eta G)\, D = 1_{GD}$ and $\epsilon_{FC} \circ F\eta_C = (\epsilon F \circ F\eta)_C = 1_{FC}$. Hence, $\zeta' = \zeta^{-1}$. The naturality follows as both η and ϵ are natural transformations.

($\Leftarrow$) Conversely, for a given natural bijection ζ, define $\eta_C \coloneqq \zeta_{C,FC} 1_{FC}$ and $\epsilon_D \coloneqq {\zeta_{GD,D}}^{-1} 1_{GD}$ for each $C \in |\mathcal{C}|$ and $D \in |\mathcal{D}|$. Let $C \in |\mathcal{C}|$ and $D \in |\mathcal{D}|$:

$$\begin{aligned} (G\epsilon \circ \eta G)_D &= \left(G\epsilon_D \circ \zeta_{GD,FGD}\right) 1_{FGD} \\ &= \zeta_{GD,D}\left(\epsilon_D \circ 1_{FGD}\right) \\ &= \left(\zeta_{GD,D} \circ {\zeta_{GD,D}}^{-1}\right) 1_{GD} \\ &= 1_{GD} \\ (\epsilon F \circ F\eta)_C &= \left({\zeta_{GFC,FC}}^{-1} 1_{GFC}\right) \circ F\eta_C \\ &= {\zeta_{C,FC}}^{-1}\left(1_{GFC} \circ \eta_C\right) \\ &= \left({\zeta_{C,FC}}^{-1} \circ \zeta_{C,FC}\right) 1_{FC} \\ &= 1_{FC}. \end{aligned} \tag{1.66}$$

Hence, we conclude $G\epsilon \circ \eta G = 1_G$ and $\epsilon F \circ F\eta = 1_F$. ■

Remark 10. The natural bijections are represented as the following:



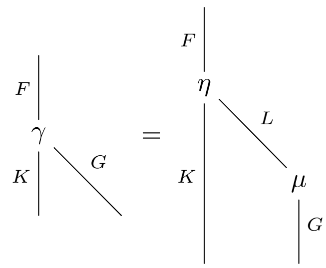
(1.67)

Definition 1.3.8 (Kan Extensions)**.** Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, and $F\colon \mathcal{C} \to \mathcal{E}$ and $K\colon \mathcal{C} \to \mathcal{D}$ be functors.

- A left Kan extension of F along K is a pair (L, η) of a functor $L\colon \mathcal{D} \to \mathcal{E}$, and a natural transformation:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ \mathcal{D} \arrow[ur, "L"'] & \end{tikzcd} \qquad \eta\colon F \Rightarrow LK \tag{1.68}$$

such that for any other pair $(G\colon \mathcal{D} \to \mathcal{E}, \gamma\colon F \Rightarrow GK)$, there exists a unique mediator $\mu\colon L \Rightarrow G$ with $\gamma = \mu K \circ \eta$, where $LK \xRightarrow{\mu K} GK$ is $LK \xRightarrow{\mu * 1_K} GK$, see Lemma 1.3.1:

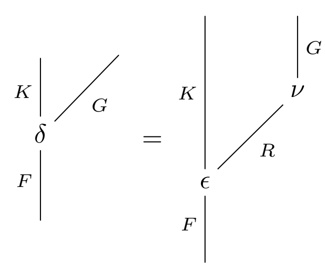


(1.69)

- A right Kan extension of F along K is a pair (R, ϵ) of a functor $R\colon \mathcal{D} \to \mathcal{E}$, and a natural transformation:

$$\begin{tikzcd} \mathcal{D} \arrow[dr, "R"] & \\ \mathcal{C} \arrow[u, "K"] \arrow[r, "F"'] & \mathcal{E} \end{tikzcd} \qquad \epsilon\colon KR \Rightarrow F \tag{1.70}$$

such that for any other pair $(G\colon \mathcal{D} \to \mathcal{E}, \delta\colon GK \Rightarrow F)$, there exists a unique mediator $\nu\colon G \Rightarrow R$ with $\delta = \epsilon \circ \nu K$, where $GK \xRightarrow{\nu K} RK$ is $GK \xRightarrow{\nu * 1_K} RK$, see Lemma 1.3.1:



(1.71)

Remark 11 (Limits as Kan Extensions). Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Suppose a right Kan extension of F along the unique functor $\mathcal{C} \to \mathbf{1}$, where $\mathbf{1}$ is the terminal category, see Remark 5.

We will show such a right Kan extension is a limit cone. Let (R, ϵ) be a right Kan extension of F along $\mathcal{C} \xrightarrow{!} \mathbf{1}$:

- (R, ϵ) is a cone.

  As $R\colon \mathbf{1} \to \mathcal{D}$ is essentially an object in $\mathcal{D}$, the composition $\mathcal{C} \xrightarrow{!} \mathbf{1} \xrightarrow{R} \mathcal{D}$ is a constant functor on $R \in |\mathcal{D}|$. Since $\epsilon\colon R! \Rightarrow F$ is a natural transformation, for each $c \in \mathcal{C}(C_1, C_2)$ the following diagram is commutative:

$$\begin{tikzcd} R \arrow[d, "\epsilon_{C_1}"'] \arrow[dr, "\epsilon_{C_2}"] & \\ FC_1 \arrow[r, "Fc"'] & FC_2 \end{tikzcd} \qquad Fc \circ \epsilon_{C_1} = \epsilon_{C_2}. \tag{1.72}$$

I.e., (R, ϵ) forms a cone in $\mathcal{D}$.

- (R, ϵ) is a limit cone.

  Due to the universal property of (R, ϵ) being a right Kan extension, for any cone (D, θ) such that

$$\begin{tikzcd} D \arrow[d, "\theta_{C_1}"'] \arrow[dr, "\theta_{C_2}"] & \\ FC_1 \arrow[r, "Fc"'] & FC_2 \end{tikzcd} \qquad Fc \circ \theta_{C_1} = \theta_{C_2}, \tag{1.73}$$

  as $\theta \colon D! \Rightarrow F$ is a natural transformation, there exists a unique mediator $\mu \colon D \Rightarrow R$ with $\theta = \epsilon \circ \nu!$, i.e., for each $C \in |\mathcal{C}|$, $\theta_C = \epsilon_C \circ \mu$ holds.

Conversely, if $F \colon \mathcal{C} \to \mathcal{D}$ has a limit (R, ϵ), it defines a right Kan extension of F along $! \colon \mathcal{C} \to \mathbf{1}$.

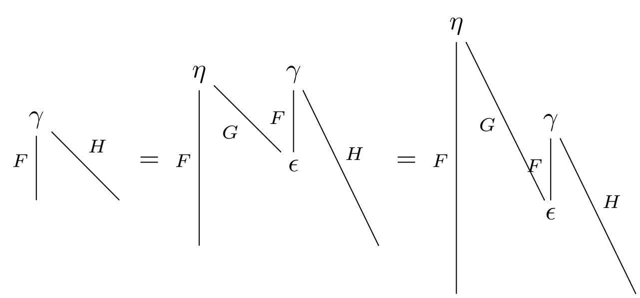
Remark 12 (Adjoints as Kan Extension). Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ be functors. Suppose $F \dashv G$ with unit $\eta \colon 1_{\mathcal{C}} \Rightarrow GF$ and counit $\epsilon \colon FG \Rightarrow 1_{\mathcal{D}}$. Then

- (G, η) is a left Kan extension of $1_{\mathcal{C}}$ along F.

  Consider $(H \colon \mathcal{D} \to \mathcal{E}, \gamma \colon 1_{\mathcal{C}} \Rightarrow HF)$. γ becomes

$$1_{HF}\gamma = H\left(\epsilon F \circ F\eta\right) \circ \gamma = H\epsilon F \circ HF\eta \circ \gamma = H\epsilon F \circ \gamma GF \circ \eta = \left(H\epsilon \circ \gamma G\right) F \circ \eta. \tag{1.74}$$

  That is, $H\epsilon \circ \gamma G \colon G \Rightarrow H$ is the desired mediator:



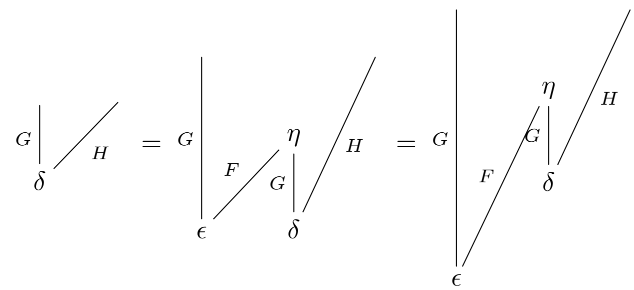
(1.75)

- (F, ϵ) is a right Kan extension of $1_{\mathcal{D}}$ along G.

Consider $(H\colon \mathcal{D} \to \mathcal{E}, \delta\colon HG \Rightarrow 1_{\mathcal{C}})$. δ becomes

$$\delta 1_{HG} = \delta \circ H\,(G\epsilon \circ \eta G) = \delta \circ HG\epsilon \circ H\eta G = \epsilon \circ \delta FG \circ H\eta G = \epsilon \circ (\delta F \circ H\eta)\, G. \tag{1.76}$$

That is, $\delta F \circ H\eta\colon H \Rightarrow G$ is the desired mediator:



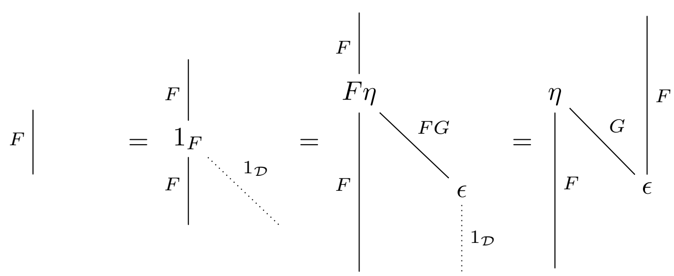
(1.77)

Conversely, if the following two conditions hold:

- $(G\colon \mathcal{D} \to \mathcal{C}, \eta\colon 1_{\mathcal{C}} \Rightarrow GF)$ is a left Kan extension of $1_{\mathcal{C}}$ along $F\colon \mathcal{C} \to \mathcal{D}$.
- F preserves this Kan extension.

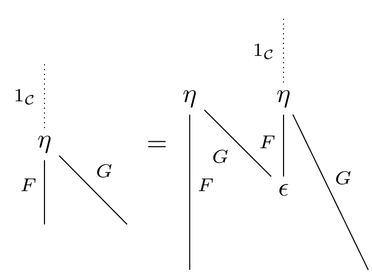
Then $F \dashv G$ with unit η.

We first find the counit. Since $(FG, F\eta)$ is a left Kan extension of F along F, there exists a unique mediator $\epsilon\colon FG \Rightarrow 1_{\mathcal{D}}$ such that $1_F = \epsilon F \circ F\eta$:



(1.78)

Hence, it suffices to show the other zig-zag identity. Now



(1.79)

implies:

$$\eta = 1_{GF}\eta = G\epsilon F \circ GF\eta \circ \eta = G\epsilon F \circ \eta GF \circ \eta = (G\epsilon \circ \eta G)F \circ \eta \tag{1.80}$$

Since (G, η) is a left Kan extension of $1_{\mathcal{C}}$ along F, the unique mediator 1_G must be $G\epsilon \circ \eta G$.

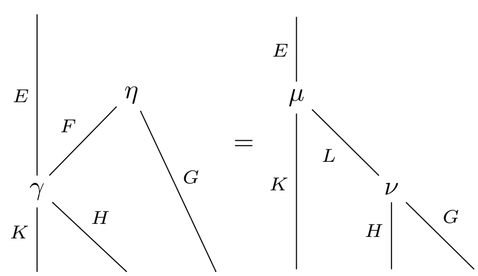
Theorem 1.3.4. *Left adjoints preserve left Kan extensions.*

Proof. Consider an adjunction $\begin{tikzcd} \mathcal{C} \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{D} \arrow[u, bend right, "G"'] \end{tikzcd}$ with unit $\eta\colon 1_{\mathcal{C}} \Rightarrow GF$ and counit $\epsilon\colon FG \Rightarrow 1_{\mathcal{D}}$, and a left Kan extension $(L_K E\colon \mathcal{B} \to \mathcal{C}, \mu\colon E \Rightarrow L_K E \circ K)$ of $E\colon \mathcal{A} \to \mathcal{C}$ along $K\colon \mathcal{A} \to \mathcal{B}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "E"] \arrow[d, "K"'] & \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \\ \mathcal{B} \arrow[ur, bend right, "L_K E"'] & & \end{tikzcd} \qquad \mu\colon E \Rightarrow L_K E \circ K \tag{1.81}$$

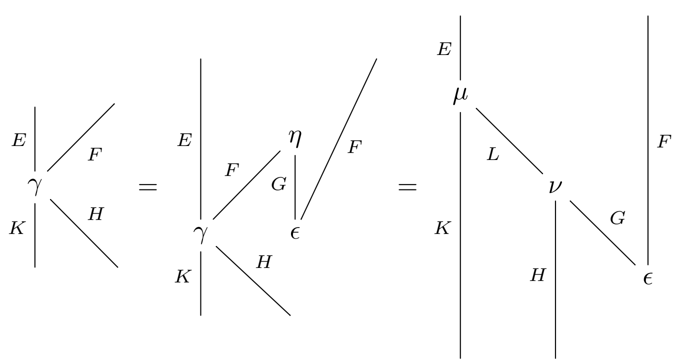
We will show $(F \circ L_K E, F\mu)$ is a left Kan extension of FE along K, in other words, $L_K(FE) = FL_K E$.

For simplicity, let $L \coloneqq L_K E$. Consider $H\colon \mathcal{B} \to \mathcal{D}$ and $\gamma\colon FE \Rightarrow HK$. Applying η, there exists a unique mediator $\nu\colon L \Rightarrow GH$ such that



(1.82)

since $(L_K E, \mu)$ is a left Kan extension of E along K. By a zig-zag identity,



(1.83)

we obtain $\gamma = (\epsilon H \circ F\nu)\, K \circ F\mu$. This $\epsilon H \circ F\nu \colon FL \to H$ is the desired mediator for $(FL, F\mu)$ being a left Kan extension of FE along K. $\blacksquare$

Chapter 2

Adjunctions in Topology

2.1 Spaces, Presets, and Posets

Definition 2.1.1 (Concrete Categories)**.** Here are some categories of structured sets with structure-preserving maps:

- **Set** of sets with maps
- **Pre** of pre-ordered sets with monotone maps, and **Pos** of posets with monotone maps
- **Top** of topological spaces with continuous maps

2.1.1 Presets and Alexandroff Topology

Definition 2.1.2 (Upper Section)**.** Let $(A, \leqq)$ be a preset. An upper section of A is a subset $U \subset A$ such that for all $a, b \in A$:

$$a \in U \wedge a \leqq b \Rightarrow b \in U. \tag{2.1}$$

Let Γ_A denote the set of all upper sections of $(A, \leqq)$.

Theorem 2.1.1 (Alexandroff Topology)**.** *Let $(A, \leqq)$ be a preset. The set of upper sections Γ_A is a topology on A. We call Γ_A the Alexandroff topology of $(A, \leqq)$.*

Proof. $A \in \Gamma_A$ holds. $\emptyset \in \Gamma_A$ is, vacuously, true.

Let $U, V \in \Gamma_A$. If they do not meet $U \cap V = \emptyset$, as shown above, $\emptyset \in \Gamma_A$. Suppose $a \in U \cap V$. For $b \in A$, if $a \leqq b$, then $b \in U$ and $b \in V$ since both U and V are upper sections. Hence, $b \in U \cap V$, and $U \cap V \in \Gamma_A$.

Let $\Gamma' \subset \Gamma_A$ and $a \in \bigcup \Gamma'$. Then, there exists at least one $W \in \Gamma'$ with $a \in W$. For $b \in A$, if $a \leqq b$, then $b \in W \subset \bigcup \Gamma'$. Hence, $\bigcup \Gamma' \in \Gamma_A$. ∎

Theorem 2.1.2 (Upgrading)**.** *For a preset* $(A, \leqq)$*, let* $\Uparrow (A, \leqq) := (A, \Gamma_A)$*. This object assignment induces the corresponding arrow assignment. Hence,* $\Uparrow\colon \mathbf{Pre} \to \mathbf{Top}$ *is a functor.*

Proof. Let $f \in \mathbf{Pre}(A, B)$ be a monotone map, namely $a_1 \leqq a_2 \Rightarrow fa_1 \leqq fa_2$. Relative to the Alexandroff topologies of $(A, \leqq)$ and $(B, \leqq)$, we will show $\Uparrow f := f$ is continuous. Let $W \in \Gamma_B$, and $a_1, a_2 \in A$. Suppose $a_1 \leqq a_2$ and $a_1 \in (\Uparrow f)^{\leftarrow}W = f^{\leftarrow}W$, i.e., $fa_1 \in W$. Since f is order-preserving, $fa_1 \leqq fa_2$, and $W \in \Gamma_B$ is an upper section of B, we conclude $fa_2 \in W$. Hence, $a_2 \in (\Uparrow f)^{\leftarrow}W$. We conclude $(\Uparrow f)^{\leftarrow}W$ is an upper section of A, i.e., $(\Uparrow f)^{\leftarrow}W \in \Gamma_A$. Hence, $(\Uparrow f)^{\leftarrow}\colon \Gamma_B \to \Gamma_A$. ∎

2.1.2 Specialization Preorder and Separation Axioms

> You should remember that a topological space need not be hausdorff.
> The separation properties T_0 and T_1 play a minor role here. [Sim11]

Definition 2.1.3 (Specialization Preorder)**.** For a topological space $(S, \mathcal{T}_S)$, the specialization order $\leqq$ of $(S, \mathcal{T}_S)$ is the following comparison on S:

$$r \leqq s :\Leftrightarrow \forall U \in \mathcal{T}_S : r \in U \Rightarrow s \in U. \tag{2.2}$$

Lemma 2.1.1. *Specialization orders are preorders.*

Proof. Let $(S, \mathcal{T}_S)$ be a topological space and $\leqq$ is the specialization order of $(S, \mathcal{T}_S)$. It suffices to show that $\leqq$ is transitive.

Suppose $r \leqq s$ and $s \leqq t$. Let $U \in \mathcal{T}_S$ such that $r \in U$. Since $r \leqq s$, $s \in U$, which implies $t \in U$. Hence, $r \leqq t$. ∎

Theorem 2.1.3. *Let* $(S, \mathcal{T}_S)$ *be a topological space and* $\leqq$ *is the specialization order of* $(S, \mathcal{T}_S)$*. For* $r, s \in S$*,* $r \leqq s$ *iff* $r \in \overline{\{s\}}$*, that is,* r *is a member of the closure of the singleton subspace* $\{s\} \subset S$ *relative to* $\mathcal{T}_S$*.*

Proof. Since $r \leqq s$ is equivalent to:

$$\forall U \in \mathcal{T}_S : s \in \neg U \Rightarrow r \in \neg U \tag{2.3}$$

In other words, any closed subspace in S that contains s also contains r. Hence, r must be in the $\subset$-smallest closed subspace that contains s. By Theorem 1.2.2, we conclude $r \in \overline{\{s\}}$. ∎

Theorem 2.1.4. *Let* $(S, \mathcal{T}_S)$ *be a topological space and* $\leqq$ *is the specialization order of* $(S, \mathcal{T}_S)$*.*

- $(S, \mathcal{T}_S)$ *is a* T_0 *space iff* $\leqq$ *is a partial order.*
- $(S, \mathcal{T}_S)$ *is a* T_1 *space iff* $\leqq$ *is equality.*

Proof. Let $(S, \mathcal{T}_S)$ be a T_0 space, $\leqq$ be the specialization preorder of $(S, \mathcal{T}_S)$, and $s, t \in S$. Suppose $s \leqq t$ and $t \leqq s$, but $s \neq t$ for contradiction. Since $(S, \mathcal{T}_S)$ is T_0, there exists an open $O \in \mathcal{T}_S$ that contains only one of $\{s, t\}$; without loss of generality, $s \in O$ and $t \notin O$. $t \in \neg O$ implies $s \in \neg O$ since $s \leqq t$, which is absurd. Thus, $s = t$ holds.

Conversely, suppose the specialization preorder $\leqq$ of a topological space $(S, \mathcal{T}_S)$ is a partial order. Consider two distinct points $s \neq t$ in S. As $(S, \leqq)$ is a poset, $s \neq t$ implies either $s \nleqq t$ or $t \nleqq s$. Without loss of generality, we may set $s \nleqq t$. By Theorem 2.1.3, we obtain $s \in \neg\overline{\{t\}}$. Since $\neg\overline{\{t\}} \in \mathcal{T}_S$, it is the desired open subspace, since $\{t\} \subset \overline{\{t\}}$ implies $t \in \overline{\{t\}}$, i.e., $t \notin \neg\overline{\{t\}}$.

Let $(S, \mathcal{T}_S)$ be a T_1-space, $\leqq$ be the specialization preorder of $(S, \mathcal{T}_S)$, and $s, t \in S$. Suppose $s \leqq t$ but $s \neq t$ for contradiction. Since $(S, \mathcal{T}_S)$ is T_1, there are open $U, V \in \mathcal{T}_S$ with $s \in U$, $t \in V$, but $s \notin V$ and $t \notin U$. Since $s \leqq t$ and $s \in U$, we obtain $t \in U$, which is absurd.

Conversely, suppose the specialization preorder $\leqq$ of a topological space $(S, \mathcal{T}_S)$ is merely the equality $=$. Let $s \neq t$ be two distinct points in S. That is, $s \nleqq t$ and $t \nleqq s$:

$$s \in \neg\overline{\{t\}} \wedge t \in \neg\overline{\{s\}}. \tag{2.4}$$

Since both $\neg\overline{\{t\}}$ and $\neg\overline{\{s\}}$ are open, we obtain the desired open neighborhoods:

$$t \notin \neg\overline{\{t\}} \wedge s \notin \neg\overline{\{s\}}. \tag{2.5}$$

since $s \in \overline{\{s\}}$ and $t \in \overline{\{t\}}$. ■

Theorem 2.1.5 (Downgrading)**.** *For a topological space $(S, \mathcal{T}_S)$, let $\Downarrow (S, \mathcal{T}_S) :=$ $(S, \leqq)$, where $\leqq$ is the specialization order. This object assignment induces the corresponding arrow assignment. Hence, $\Downarrow$:* **Top** $\to$ **Pre** *is a functor.*

Proof. Let $f \in \mathbf{Top}(A, B)$ be a continuous map. We will show $\Downarrow f := f$ is monotone. Let $a_1, a_2 \in A$. Assume $a_1 \leqq a_2$. Suppose, for contradiction, that $(\Downarrow f)a_1 \nleqq (\Downarrow f)a_2$. By Theorem 2.1.3, this condition is equivalent to $fa_1 \in \neg\overline{\{fa_2\}}$, and

$$a_1 \in f^{\leftarrow}\left(\neg\overline{\{fa_2\}}\right). \tag{2.6}$$

Since $\neg\overline{\{fa_2\}} \in \mathcal{T}_B$, its preimage is also open $f^{\leftarrow}\left(\neg\overline{\{fa_2\}}\right) \in \mathcal{T}_A$. Recalling $a_1 \leqq a_2$, we conclude $a_2 \in f^{\leftarrow}\left(\neg\overline{\{fa_2\}}\right)$, i.e., $fa_2 \in \neg\overline{\{fa_2\}}$, which is absurd. Hence, $\Downarrow f$ is monotone. ■

Lemma 2.1.2. *Let $(A, \leqq)$ be a preset and Γ_A be the Alexandroff topology on A. For $a, b \in A$, if $a \leqq b$ then $a \in \overline{\{b\}}$, where the closure $\overline{\{b\}}$ is relative to $(A, \Gamma_A) = \Uparrow (A, \leqq)$.*

Proof. Let $a, b \in A$. Suppose $a \leqq b$. Let $U \in \Gamma_A$. Since U is an upper section of A, if $a \in U$ then $b \in U$. It is equivalent to:

$$b \in \neg U \Rightarrow a \in \neg U. \tag{2.7}$$

In other words, any closed subspace relative to Γ_A that contains b contains also a. Hence, $a \in \overline{\{b\}}$. ■

Theorem 2.1.6. *Let $(A, \leqq)$ be a preset, Γ_A be the Alexandroff topology on A, and $\prec$ be the specialization preorder of the topological space (A, Γ_A). We claim $\leqq = \prec$. In other words, $\Downarrow\Uparrow (A, \leqq) = (A, \leqq)$.*

Proof. Recalling $\leqq \subset A \times A$, let $(a_1, a_2) \in \leqq$:

$$a_1 \leqq a_2. \tag{2.8}$$

If $U \in \Gamma_A$ contains $a_1 \in U$, since U is an upper section of A, $a_2 \in U$:

$$a_1 \prec a_2. \tag{2.9}$$

Thus, as subsets of $A \times A$, we conclude $\leqq \subset \prec$.

Suppose, for contradiction, that this inclusion is strict. Then, there exists at least one pair $(s, t) \in A \times A$ such that $s \prec t$ but $s \nleqq t$.

- Since $s \prec t$,
$$\forall U \in \Gamma_A : s \in U \Rightarrow t \in U. \tag{2.10}$$

- Since $s \nleqq t$, by Lemma 2.1.2:
$$s \in \neg\overline{\{t\}}. \tag{2.11}$$

Now, $\neg\overline{\{t\}} \in \Gamma_A$ and $t \notin \neg\overline{\{t\}}$, we have a contradiction. ■

Corollary 2.1.6.1. *The converse of Lemma 2.1.2 is also the case, namely for a preset $(A, \leqq)$, $a \leqq b$ iff $a \in \overline{\{b\}}$, where $\overline{\{b\}}$ is relative to $\Uparrow (A, \leqq)$.*

Proof. Suppose $a \in \overline{\{b\}}$. By Theorem 2.1.3, it is equivalent to $a \prec b$, where $\prec$ is the specialization preorder of (A, Γ_A). As shown above, in Theorem 2.1.6, $a \prec b$ iff $a \leqq b$. ■

Theorem 2.1.7. *Let $(S, \mathcal{T}_S)$ be a topological space, $(S, \leqq) :=\Downarrow (S, \mathcal{T}_S)$ be the preset with the specialization preorder, and $(S, \Gamma_S) :=\Uparrow\Downarrow (S, \mathcal{T}_S)$. We claim $\mathcal{T}_S \subset \Gamma_S$.*

Proof. We will show that any member in $\mathcal{T}_S$ is an upper section relative to the specialization preorder $\leqq$.

Let $U \in \mathcal{T}_S$, and $s, t \in S$. Suppose $s \in U$ and $s \leqq t$. By the very definition of $\leqq$, see Definition 2.1.3, we conclude $t \in U$. Hence U is an upper section, $U \in \Gamma_A$. ■

Now we have a pair of functors:

$$\begin{tikzcd} \mathbf{Pre} \arrow[d, bend right, "\Uparrow"'] \\ \mathbf{Top} \arrow[u, bend right, "\Downarrow"'] \end{tikzcd} \tag{2.12}$$

To show that they form an adjuction, by Theorem 1.3.3, it suffices to show that $\mathbf{Top}\,(\Uparrow(A,\leqq),(S,\mathcal{T}_S))$ and $\mathbf{Pre}\,((A,\leqq),\Downarrow(S,\mathcal{T}_S))$ are naturally bijective for any preset $(A,\leqq)$ and any topological space $(S,\mathcal{T}_S)$:

Theorem 2.1.8. *Let $(A,\leqq)$ be a preset, $(S,\mathcal{T}_S)$ be a topological space, and*

$$\theta\colon A\to S \tag{2.13}$$

be a map between the underlying sets. We claim that θ is monotone relative to $\Downarrow(S,\mathcal{T}_S)$ iff it is continuous relative to $\Uparrow(A,\leqq)$. In other words, as sets of mappings, $\mathbf{Top}\,(\Uparrow(A,\leqq),(S,\mathcal{T}_S))$ and $\mathbf{Pre}\,((A,\leqq),\Downarrow(S,\mathcal{T}_S))$ are the same.

Proof. Suppose $(A,\leqq)\xrightarrow{\theta}(S,\prec)$ is monotone, where $\prec$ is the specialization preorder of $(S,\mathcal{T}_S)$:

$$s\prec t :\Leftrightarrow \forall U\in\mathcal{T}_S : s\in U\Rightarrow t\in U. \tag{2.14}$$

We will show $\theta^{\leftarrow}\colon\mathcal{T}_S\to\Gamma_A$, where Γ_A is the Alexandroff topology. Let $U\in\mathcal{T}_S$ and $a,b\in A$. Suppose $a\leqq b$; since θ is monotone, $\theta a\prec\theta b$. If $a\in\theta^{\leftarrow}U$, i.e., $\theta a\in U$, since $\theta a\prec\theta b$, we obtain $\theta b\in U$. Hence, $b\in\theta^{\leftarrow}U$. We conclude that $\theta^{\leftarrow}U$ is an upper section of A:

$$\theta^{\leftarrow}U\in\Gamma_A. \tag{2.15}$$

Thus, θ is continuous.

Conversely, suppose θ is continuous. We will show θ is monotone relative to the specialization preorder $\prec$. Let $a,b\in A$. Suppose $a\leqq b$. For an arbitrary $U\in\mathcal{T}_S$, $\theta^{\leftarrow}U\in\Gamma_A$ is an upper section, $a\in\theta^{\leftarrow}U\Rightarrow b\in\theta^{\leftarrow}U$. That is, $\theta a\in U\Rightarrow\theta a\in U$:

$$\theta a\prec\theta b. \tag{2.16}$$

Thus, θ is monotone. ■

By Theorem 2.1.8, we obtain the following adjunction:

$$\begin{tikzcd} \mathbf{Pre} \arrow[d, bend right, "\Uparrow"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{Top} \arrow[u, bend right, "\Downarrow"'] \end{tikzcd} \tag{2.17}$$

2.1.3 Topological Spaces and Posets – A Natural Isomorphism

Theorem 2.1.9. *For a topological space* $(X, \mathcal{T}_X)$*, let* $\mathcal{O}(X, \mathcal{T}_X) := (\mathcal{T}_X, \subset)$*. This object assignment induces the corresponding arrow assignment, namely* $\mathcal{O}(f) := f^{\leftarrow}$ *for* $f \in C^0((X, \mathcal{T}_X), (Y, \mathcal{T}_Y))$*. Hence,* $\mathcal{O}\colon \mathbf{Top} \to \mathbf{Pos}$ *is a contravariant functor.*

Proof. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces, and $f \in C^0(X, Y)$. Note that any set with set inclusion $\subset$ forms a poset, hence $(\mathcal{T}_X, \subset)$ is an object of **Pos**. For $V, W \in \mathcal{T}_Y$, if $V \subset W$, we obtain $f^{\leftarrow}V \subset f^{\leftarrow}W$, since

$$x \in f^{\leftarrow}V \Leftrightarrow fx \in V \Rightarrow fx \in W \Leftrightarrow x \in f^{\leftarrow}W \tag{2.18}$$

for each $x \in X$. Hence, $\mathcal{O}f$ is monotone.

Since $\mathcal{O}(1_X) = 1_X{}^{\leftarrow}\colon \mathcal{T}_X \to \mathcal{T}_X$, $\mathcal{O}$ preserves identities. We will show $\mathcal{O}$ passes across compositions. For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in **Top**, and $U \in \mathcal{T}_Z$,

$$\begin{aligned} \mathcal{O}(gf)\,U &:= (gf)^{\leftarrow}\,U \\ &= \{x \in X \mid gfx \in U\} \\ &= \{x \in X \mid fx \in g^{\leftarrow}U\} \\ &= \{x \in X \mid x \in f^{\leftarrow}(g^{\leftarrow}U)\} \end{aligned} \tag{2.19}$$

Recalling $f^{\leftarrow} = \mathcal{O}f$, we obtain $\mathcal{O}(gf) = \mathcal{O}f \circ \mathcal{O}g$. ∎

Definition 2.1.4 (Sierpiński Space)**.** Let $\mathbf{2}'$ be $2 := \{0, 1\}$ with the following topology:

$$\{\emptyset, \{1\}, \{0, 1\}\}\,. \tag{2.20}$$

We call $\mathbf{2}'$ Sierpiński space, and the associated topology Sierpiński topology.

Theorem 2.1.10 (Continuous Characters)**.** *For a topological space* $(X, \mathcal{T}_X)$*, let* $\Xi(X, \mathcal{T}_X) := C^0((X, \mathcal{T}_X), \mathbf{2}')$*. We call* $\Xi(X, \mathcal{T}_X)$ *the set of continuous characters of* $(X, \mathcal{T}_X)$*. For* $f \in C^0((X, \mathcal{T}_X), (Y, \mathcal{T}_Y))$*, define* $\Xi f := \_ \circ f$ *with the pointwise partial order* $\leqq$*:*

$$p \leqq q :\Leftrightarrow \forall y \in Y : py \leqq qy, \tag{2.21}$$

where $p, q \in \Xi(X, \mathcal{T}_X)$*, and* $0 \leqq 0, 0 \leqq 1$*, and* $1 \leqq 1$*. We claim* $\Xi\colon \mathbf{Top} \to \mathbf{Pos}$ *is a contravariant functor.*

Proof. We will first show that Ξf converts continuous characters of Y into continuous characters of X. Let $p\colon Y \to \{0, 1\}$ be a map. Since the following preimages are both open in Y:

$$p^{\leftarrow}\{0, 1\} = Y \wedge p^{\leftarrow}\emptyset = \emptyset, \tag{2.22}$$

the map p is continuous relative to Sierpiński topology iff $p^{\leftarrow}\{1\}$ is open in Y:

$$\Xi f\colon C^0((Y, \mathcal{T}_Y), \mathbf{2}') \to C^0((X, \mathcal{T}_X), \mathbf{2}'). \tag{2.23}$$

If $p \in C^0\left((Y, \mathcal{T}_Y), \mathbf{2}'\right) = \Xi\,(Y, \mathcal{T}_Y)$, $p^{\leftarrow}\{1\} \in \mathcal{T}_Y$ holds. Then $(\Xi f)p = pf$ satisfies:

$$((\Xi f)p)^{\leftarrow}\{1\} = (pf)^{\leftarrow}\{1\} = f^{\leftarrow}\left(p^{\leftarrow}\{1\}\right) \in \mathcal{T}_X. \tag{2.24}$$

Hence, $(\Xi f)p \in C^0\left((X, \mathcal{T}_X), \mathbf{2}'\right) = \Xi\,(X, \mathcal{T}_X)$.

Next, we will show that $\Xi f\colon \Xi\,(Y, \mathcal{T}_Y) \to \Xi\,(X, \mathcal{T}_X)$ is monotone. Let $p, q \in \Xi\,(Y, \mathcal{T}_Y)$ be continuous characters of Y. Suppose $p \leqq q$. Then, we obtain:

$$(\Xi f)p = pf \leqq qf = (\Xi f)q. \tag{2.25}$$

Finally, consider identities and compositions:

$$\Xi 1_X = \_ \circ 1_X \tag{2.26}$$

For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in **Top**,

$$\Xi(gf) = \_ \circ (gf) = (\_ \circ g) \circ f = \Xi f \circ \Xi g. \tag{2.27}$$

Hence, Ξ: **Top** $\to$ **Pos** is a contravariant functor. ■

Definition 2.1.5 (Characteristic Functions)**.** Let X be a set and $U \subset X$ be a subset. We call:

$$\chi_X U\colon X \to \{0,1\}; x \mapsto \begin{cases} 1 & x \in U \\ 0 & \text{otherwise} \end{cases} \tag{2.28}$$

the characteristic function of $U \subset X$.

Lemma 2.1.3. *Let $(X, \mathcal{T}_X)$ be a topological space and $U \in \mathcal{T}_X$ be open. The characteristic function of U is a continuous character of X relative to Sierpiński topology:*

$$\chi_X U \in C^0\,(X, 2)\,. \tag{2.29}$$

Remark 13. If no confusion is expected, we simply denote $C^0(X, Y)$ for the set of continuous maps between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$.

Proof. Since $(\chi_X U)^{\leftarrow} \emptyset = \emptyset$, $(\chi_X U)^{\leftarrow} 2 = X$, and

$$(\chi_X U)^{\leftarrow}\{1\} = \{x \in X \mid (\chi_X U)\,x = 1\} = U, \tag{2.30}$$

we conclude $\chi_X U$ is continuous. ■

Theorem 2.1.11. *Let $(X, \mathcal{T}_X)$ be a topological space. Recalling $\mathcal{O}\,(X, \mathcal{T}_X) = C^0(X, 2)$, we obtain*

$$\chi_X\colon \mathcal{O}\,(X, \mathcal{T}_X) \to \Xi\,(X, \mathcal{T}_X) \tag{2.31}$$

of an assignment between two posets. We claim that χ_X is an isomorphism. Moreover, it is a natural transformation between $\mathcal{O}$ and Ξ.

Proof. For $U, V \in \mathcal{O}(X, \mathcal{T}_X) = (\mathcal{T}_X, \subset)$, suppose $\chi_X U = \chi_X V$. Then

$$U = (\chi_X U)^{\leftarrow}\{1\} = (\chi_X V)^{\leftarrow}\{1\} = V. \tag{2.32}$$

Thus, χ_X is injective.

For a given $\chi' \in \Xi(X, \mathcal{T}_X)$, define $U' \coloneqq \chi'^{\leftarrow}\{1\}$. Since $\chi' \in C^0(X, 2)$, such the preimage U' is open in X. Hence, $\chi' = \chi_X U'$, and χ_X is subjective.

Next, we will show that χ is monotone. For $U, V \in \mathcal{O}(X, \mathcal{T}_X) = (\mathcal{T}_X, \subset)$, suppose $U \subset V$:

- $\chi_X U|_U = 1 = \chi_X V|_U$
- $\chi_X U|_{V-U} = 0 \leqq 1 = \chi_X V|_{V-U}$
- Otherwise, both $\chi_X U$ and $\chi_X V$ are zero.

Thus, $\chi_X U \leqq \chi_X V$.

Finally, we will show $\chi\colon \mathcal{O} \Rightarrow \Xi$. For $f \in C^0(X, Y)$, namely $X \xrightarrow{f} Y$ in **Top**, consider:

$$\begin{tikzcd} \mathcal{O}X \arrow[d, "\chi_X"'] & \mathcal{O}Y \arrow[l, "f^{\leftarrow}"'] \arrow[d, "\chi_Y"] \\ \Xi X & \Xi Y \arrow[l, "\_\circ f"] \end{tikzcd} \tag{2.33}$$

We will show that

$$\begin{aligned} \chi_X \circ f^{\leftarrow}\colon \mathcal{T}_Y &\to C^0(X, \mathbf{2}') \\ (\_\circ f) \circ \chi_Y\colon \mathcal{T}_Y &\to C^0(X, \mathbf{2}') \end{aligned} \tag{2.34}$$

are equal. Let $W \in \mathcal{T}_Y$ and $x \in X$,

$$\begin{aligned} (\chi_X \circ f^{\leftarrow} W)\, x &= \chi_X (f^{\leftarrow} W)\, x \\ &= \begin{cases} 1 & x \in f^{\leftarrow} W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & fx \in W \\ 0 & \text{otherwise} \end{cases} \\ &= (\chi_Y W)\, fx \\ &= (\chi_Y W \circ f)\, x \\ &= ((\_\circ f) \circ \chi_Y W)\, x. \end{aligned} \tag{2.35}$$

Hence, $\chi_X \circ f^{\leftarrow} = (\_\circ f) \circ \chi_Y$ holds. ■

Remark 14. We conclude that $\begin{tikzcd} \mathbf{Top} \arrow[r, bend left, "\mathcal{O}"] \arrow[r, bend right, "\Xi"'] & \mathbf{Pos} \end{tikzcd}$ are naturally isomorphic via χ:

$$\chi\colon \mathcal{O} \overset{\cong}{\Rightarrow} C^0(\_, 2) = \mathbf{Top}(\_, 2), \tag{2.36}$$

where $2 = \{0, 1\}$ is associated with the Sierpiński topology $\mathbf{2}' = (2, \{\emptyset, \{1\}, 2\})$.

2.2 Compact-Open Topology and Locally Compact Spaces

2.2.1 Compact Spaces – Closed Maps

For a topological space $(X, \mathcal{T}_X)$ and its subspace $A \subset X$, an open covering of A is a set of open subspaces $\mathcal{U} \subset \mathcal{T}_X$ such that $A \subset \cup\mathcal{U} = \bigcup_{U\in\mathcal{U}} U$. A finite subcover of an open covering $\mathcal{U}$ of A is a finite subset of $\mathcal{U}$ that also covers A.

Definition 2.2.1 (Compact Spaces)**.** A topological space is called a compact space iff every open covering of the space contains a finite subcover.

For a topological space $(X, \mathcal{T}_X)$, let $\mathcal{K}_X$ be the set of all compact subspaces in X.

Definition 2.2.2 (Closed Maps)**.** A map between topological spaces is called a closed map iff a direct image of a closed space in its domain is a closed subspace in the codomain space.

Theorem 2.2.1 (Compactness via Closed Projections)**.** *A topological space K is compact iff for every topological space X, the canonical projection:*

$$\pi_X : K \times X \to X \tag{2.37}$$

is a closed map relative to the product topology.

Remark 15 (Product Topology). For $\{(X_\lambda, \mathcal{T}_{X_\lambda}) \mid \lambda \in \Lambda\}$ of a set of topological spaces, the product topology is the generated topology of the following subbase:

$$\{\pi_\lambda{}^{\leftarrow} U \mid \lambda \in \Lambda \wedge U \in \mathcal{T}_{X_\lambda}\}, \tag{2.38}$$

where $\pi_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \to X_\lambda$ is a projection for each $\lambda \in \Lambda$. By definition, as this subbase makes the projections continuous, the product topology is $\subset$-smallest topology on which $\pi_\lambda \in C^0\left(\prod_{\lambda\in\Lambda} X_\lambda, X_\lambda\right)$ for each $\lambda \in \Lambda$.

Proof. ($\Rightarrow$) Let X be a topological space and K be a compact space. We will show π_X is a closed map; if $X = \emptyset$, nothing has to prove. Let $C \subset K \times X$ be a closed subspace; if $\pi_X C = X$, as $X \subset X$ is a clopen subspace in Y, done. So we may suppose $\pi_X C \subsetneq X$.

Select $x \in \neg\pi_X C$. Since π_X is a surjection, there is at least one $k \in K$ with $(k, x) \overset{\pi_X}{\mapsto} x$. Then such a pair $(k, x) \in \neg C$, otherwise (k, x) would be be in C, so $x = \pi_X(k, x) \in \pi_X C$, which is absurd. Thus, $x = \pi_X(k, x) \in \pi(\neg C)$, and hence $\neg\pi_X C \subset \pi_X(\neg C)$. However, it implies $\pi_X C \supset \neg\pi_X(\neg C) = \pi_X C$. So, we conclude $\pi_X C = \neg\pi(\neg C)$ and $\neg\pi_X C = \pi_C(\neg C)$.

The preimage $\pi_X{}^{\leftarrow}(x) = K \times \{x\}$ does not meet C, for otherwise $(k, x) \in K \times \{x\} \cap E$, we obtain $\pi_X(k, x) = x \in \pi_Y C$, which is absurd. Hence, $K \times \{x\} \subset \neg C$. Since $\neg C \subset K \times X$ is open, for each point $(k, x) \in K \times \{x\}$, there are open neighborhoods $U_k \in \mathcal{N}_k \cap \mathcal{T}_K$ and $V_{k,x} \in \mathcal{N}_x \cap \mathcal{T}_X$ such that $(k, x) \in U_k \times V_{k,x}$ and

$$U_k \times V_{k,x} \subset \neg C. \tag{2.39}$$

Since $\{U_k \mid k \in K\}$ is an open cover of the compact space K, there is a finite subcover:

$$K \subset U_{k_1} \cup \cdots \cup U_{k_n}. \tag{2.40}$$

Define $W_x \coloneqq W_{k_1,x} \cap \cdots \cap W_{k_n,x}$. Since, for each $k_j \in \{k_1, \ldots, k_n\}$,

$$W_x \times U_{k_j} \subset W_{k_j,x} \times U_{k_j} \subset \neg C, \tag{2.41}$$

we conclude:

$$W_x \times K \subset W_x \times \bigcup_{j=1}^{n} U_{k_j} = \bigcup_{j=1}^{n} \left(W_x \times U_{k_j}\right) \subset \neg C. \tag{2.42}$$

Hence, $W_x \subset \pi_X (\neg C) = \neg \pi_X C$. This W_x is the desired open neighborhood of x; applying Lemma 1.2.1, we conclude that $\neg \pi_X C$ is open.

($\Leftarrow$) Let $(X, \mathcal{T}_X)$ be an arbitrary topological space and $\mathcal{U} \subset \mathcal{T}_X$ be an arbitrary open covering of X. Define $X_\infty \coloneqq X \cup \{\infty\}$, where $\infty \notin X$. For an arbitrary subset $A \subset X_\infty$, we call A closed iff either

$$\infty \in A \vee A \text{ is finitely covered by } \mathcal{U}. \tag{2.43}$$

This relation defines a topology on X_∞:

- Since $\infty \in X_\infty$, the complement $\emptyset$ is open.
- Since $\emptyset$ is vacuously covered by $\emptyset \subset \mathcal{U}$, its complement X_∞ is open.
- Arbitrary Union

  For an arbitrary subset of open subspaces $\{V_\lambda \subset X \mid \lambda \in \Lambda\}$, if at least one $\neg V_{\lambda_0}$ is finitely covered by $\mathcal{U}$:

  $$\neg \bigcup_{\lambda \in \Lambda} V_\lambda = \bigcap_{\lambda \in \Lambda} \neg V_\lambda \subset \neg V_{\lambda_0} \tag{2.44}$$

  so as $\neg \bigcup_{\lambda \in \Lambda} V_\lambda$. Otherwise, every $\neg V_\lambda$ contains ∞:

  $$\infty \in \bigcap_{\lambda \in \Lambda} \neg V_\lambda = \neg \bigcup_{\lambda \in \Lambda} V_\lambda \tag{2.45}$$

  Hence, $\neg \bigcup_{\lambda \in \Lambda} V_\lambda$ is closed and its complement $\bigcup_{\lambda \in \Lambda} V_\lambda$ is open.
- Binary Intersection

  Let U, V be open. Consider $\neg (U \cap V)$:

  $$\begin{aligned} \neg U \cup \neg V &= \{x \in X \mid x \notin U \vee x \notin V\} \\ &= \{x \in X \mid \neg (x \in U \wedge x \in V)\} \\ &= \neg (U \cap V). \end{aligned} \tag{2.46}$$

  - If at least one of $\neg U$ and $\neg V$ contains ∞, then $\infty \in \neg U \cup \neg V$. Hence, $\neg U \cup \neg V = \neg (U \cap V)$ is closed.

– Otherwise, both $\neg U$ and $\neg V$ are finitely covered by $\mathcal{U}$. Then $\neg U \cup \neg V = \neg(U \cap V)$ is also finitely covered by $\mathcal{U}$.

Hence, the complement $U \cap V$ is open.

By hypothesis, the canonical projection $\pi_{X_\infty} : X \times X_\infty \to X_\infty$ is a closed map.

Clearly, $X \subsetneqq X_\infty$. Within the product space $X \times X_\times$, consider a subspace:

$$X \times X \subset X \times X_\infty \tag{2.47}$$

and its closure $\overline{X \times X} \subset X \times X_\infty$ relative to the product topology. We will show that ∞ is not in $\pi_{X_\infty}\left(\overline{X \times X} \subset X \times X_\infty\right)$. Suppose, for contradiction, there is some $x \in X$ with $(x, \infty) \in \overline{X \times X} \subset X \times X_\infty$. Since $\mathcal{U}$ is an open cover of X, there is some $U \in \mathcal{U}$ with $x \in U$. Let $\neg_\infty U \coloneqq X_\infty - U$. Since $\infty \in \neg_\infty U \subset X_\infty$, $U \subset X_\infty$ is open; $U \subset U$ is covered by itself, $U \subset X_\infty$ is closed as well. Then $\neg_\infty U$ is open with $\infty \in \neg_\infty U$. The product subspace $U \times \neg_\infty U \subset X \times X_\infty$ is an open neighborhood of (x, ∞) with

$$(U \times \neg_\infty U) \cap (X \times X) = \emptyset. \tag{2.48}$$

By Lemma 1.2.2, we have a contradiction.

Since $\infty \notin \pi_{X_\infty}\left(\overline{X \times X}\right)$:

$$\pi_{X_\infty}\left(\overline{X \times X}\right) = X \tag{2.49}$$

is closed, by hypothesis, in X_∞. As $\infty \notin X$, X must be finitely covered by $\mathcal{U}$. Hence, X is compact. ■

2.2.2 Compact Open Topology and Locally Compact Spaces

> The idea of topologizing the set of all continuous maps of one space into another plays an important role in modern topology. [Dug66]

Definition 2.2.3 (Compact-Open Topology)**.** Let $(I, \mathcal{T}_I)$ and $(X, \mathcal{T}_Y)$ be topological spaces. For $K \in \mathcal{K}_I$ and $V \in \mathcal{T}_Y$, let

$$\langle K, V \rangle \coloneqq \left\{\theta \in C^0(I, Y) \mid \theta K \subset V\right\}. \tag{2.50}$$

The compact topology on the set of continuous maps $C^0(I, Y)$ is the generated topology by the following subbase:

$$\{\langle K, V \rangle \mid K \in \mathcal{K}_I \wedge V \in \mathcal{T}_Y\}. \tag{2.51}$$

See Definition 1.2.5. Let $I \multimap Y$ denote the space of continuous maps from I to Y with the compact-open topology.

Theorem 2.2.2 (Currying)**.** *Let $(I, \mathcal{T}_I)$ be a topological space. For a pair of topological spaces, $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, if $X \times I \xrightarrow{\psi} Y$ is continuous relative to the product topology, then its curried form is also continuous:*

$$X \xrightarrow{\psi_\flat} (I \multimap Y), \tag{2.52}$$

where $(\psi_\flat x)i \coloneqq \psi(x,i)$ for $x \in X$ and $i \in I$, and $I \multimap Y$ is equipped with the compact-open topology.

Proof. As the compact-open topology on $I \multimap Y$ is generated by $\langle K, V\rangle$ for $K \in \mathcal{K}_I$ and $V \in \mathcal{T}_Y$, consider a subbasic open subspace $\langle K, V\rangle \subset (I \multimap Y)$ and $x \in {\psi_\flat}^{\leftarrow}\langle K, V\rangle$:

$$\psi_\flat x \in \langle K, V\rangle \Leftrightarrow \forall k \in K : (\psi_\flat x)k = \psi(x,k) \in V \tag{2.53}$$

That is, $(x,k) \in \psi^{\leftarrow}V$ for each $k \in K$. Since ψ is continuous and $V \in \mathcal{T}_Y$, $\psi^{\leftarrow}V$ is open in $X \times I$. Thus, there are open neighborhoods $U_{x,k} \in \mathcal{N}_x \cap \mathcal{T}_X$ and $W_k \in \mathcal{N}_k \cap \mathcal{T}_I$ for each $k \in K$ such that

$$(x,k) \in U_{x,k} \times W_k \subset \psi^{\leftarrow}V. \tag{2.54}$$

Since $\{W_k \mid k \in K\}$ covers the compact subspace $K \subset I$, there is a finite subcover:

$$K \subset W \coloneqq W_{k_1} \cup \cdots \cup W_{k_n}. \tag{2.55}$$

Define $U_x \coloneqq U_{x,k_1} \cap \cdots \cap U_{x,k_n}$. Then, we have $x \in U_x \in \mathcal{T}_X$, $K \subset W$, and

$$U_x \times W = U_x \times \bigcup_{j=1}^{n} W_{k_j} = \bigcup_{j=1}^{n} U_x \times W_{k_j} \subset \bigcup_{j=1}^{n} U_{x,k_j} \times W_{k_j} \subset \psi^{\leftarrow}V. \tag{2.56}$$

Moreover, for each $x' \in X$,

$$\begin{aligned} x' \in U_x &\Rightarrow \forall w \in W : (x', w) \in \psi^{\leftarrow}V \\ &\Leftrightarrow \forall w \in W : \psi(x', w) = (\psi_\flat x')w \in V \\ &\Leftrightarrow \forall w \in W : w \in (\psi_\flat x')^{\leftarrow}V \\ &\Rightarrow \forall w \in K : w \in (\psi_\flat x')^{\leftarrow}V \\ &\Leftrightarrow \psi_\flat x' \in \langle K, V\rangle \\ &\Leftrightarrow x' \in {\psi_\flat}^{\leftarrow}\langle K, V\rangle. \end{aligned} \tag{2.57}$$

Hence, we have $U_x \subset {\psi_\flat}^{\leftarrow}\langle K, V\rangle$ with $x \in U_x$. By Lemma 1.2.1, we conclude ${\psi_\flat}^{\leftarrow}\langle K, V\rangle \in \mathcal{T}_X$. By Theorem 1.2.5, $\psi_\flat$ is continuous. ■

> Many of the important spaces occurring in analysis are not compact, but have instead a local version of compactness. [Dug66]

Definition 2.2.4 (Locally Compact Spaces)**.** A topological space $(I, \mathcal{T}_I)$ is locally compact iff for each point $i \in I$ and its open neighborhood $U \in \mathcal{N}_i \cap \mathcal{T}_I$, there are open $W \in \mathcal{T}_I$ and a compact $K \in \mathcal{K}_I$ such that:

$$i \in W \subset K \subset U. \tag{2.58}$$

In other words, a locally compact space is a topological space where each point has a compact neighborhood. In particular, a compact space is locally compact.

Theorem 2.2.3 (Uncurrying)**.** *Let $(I, \mathcal{T}_I)$ be a locally compact space. For a pair of topological spaces, $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, if $X \xrightarrow{\phi} (I \multimap Y)$ is continuous relative to the compact-open topology, then its uncurried form is also continuous:*

$$X \times I \xrightarrow{\phi^\sharp} Y , \tag{2.59}$$

where $\phi^\sharp(x,i) \coloneqq (\phi x)i$ for $x \in X$ and $i \in I$.

Proof. Let $V \in \mathcal{T}_Y$, and $(x,i) \in \phi^{\sharp\leftarrow}V$. By definition, $\phi^\sharp(x,i) = (\phi x)i \in V$, we have

$$i \in (\phi x)^{\leftarrow}V. \tag{2.60}$$

Since $\phi x \in I \multimap Y$ is continuous and $V \in \mathcal{T}_Y$ is open, $(\phi x)^{\leftarrow}V \in \mathcal{T}_I$. Moreover, as I is locally compact, there are $W \in \mathcal{T}_I$ and $K \in \mathcal{K}_I$ such that

$$i \in W \subset K \subset (\phi x)^{\leftarrow}V. \tag{2.61}$$

For each $k \in I$,

$$k \in K \Rightarrow k \in (\phi x)^{\leftarrow}V \Leftrightarrow (\phi x)k \in V. \tag{2.62}$$

Hence, we obtain $(\phi x)K \subset V$:

$$\phi x \in \langle K, V \rangle. \tag{2.63}$$

Since ϕ is continuous, we conclude:

$$x \in \phi^{\leftarrow}\langle K, V \rangle \in \mathcal{T}_X. \tag{2.64}$$

For each $(x', i') \in X \times I$, we obtain:

$$\begin{aligned} (x', i') \in \phi^{\leftarrow}\langle K, V \rangle \times W &\Rightarrow (x', i') \in \phi^{\leftarrow}\langle K, V \rangle \times K \\ &\Rightarrow \phi^\sharp(x', i') = (\phi x')i' \in V \\ &\Leftrightarrow (x', i') \in \phi^{\sharp\leftarrow}V. \end{aligned} \tag{2.65}$$

Hence, $\phi^{\leftarrow}\langle K, V \rangle \times W \subset \phi^{\sharp\leftarrow}V$ is the desired open neighborhood of (x,i); by Lemma 1.2.1 and Theorem 1.2.5, $\phi^\sharp$ is continuous. ∎

Definition 2.2.5. Let $(I, \mathcal{T}_I)$ be a locally compact space. We denote, for a topological space $(X, \mathcal{T}_X)$:

$$\begin{aligned} LX &\coloneqq X \times I \\ RX &\coloneqq I \multimap X \end{aligned} \tag{2.66}$$

These object assignments induce the corresponding arrow assignments:

$$\begin{aligned} L\left(X \xrightarrow{f} X'\right) &= X \times I \xrightarrow{f \times 1_I} X' \times I \\ R\left(Y \xrightarrow{g} Y'\right) &= I \multimap Y \xrightarrow{g \circ \_} I \multimap Y' \end{aligned} \tag{2.67}$$

where

$$\begin{aligned} (f \times 1_I)(x,i) &= (fx, i) \\ (g \circ \_)\, p &= g \circ p \end{aligned} \tag{2.68}$$

Theorem 2.2.4. *Let $(I, \mathcal{T}_I)$ be a locally compact space. With the product topology and the compact open topology, we have the following adjoint endo functors:*

$$\begin{tikzcd} \mathbf{Top} \arrow[d, bend right, "L=\_\times I"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{Top} \arrow[u, bend right, "R=I\multimap\_"'] \end{tikzcd} \tag{2.69}$$

Proof. By Theorem 2.2.2 and Theorem 2.2.3, we have currying-uncurrying bijection. By Theorem 1.3.3, it suffices to show the naturality:

- For $LX \xrightarrow{\psi} Y \xrightarrow{g} Y'$, consider:

$$\begin{tikzcd} X \arrow[r, "\psi_\flat"] \arrow[rr, bend right, "(g\circ\psi)_\flat"'] & RY \arrow[r, "Rg"] & RY' \end{tikzcd} \tag{2.70}$$

  Let $x \in X$ and $i \in I$:

$$\begin{aligned} (Rg \circ \psi_\flat x)\, i &= (g \circ \_)\,(\psi_\flat x)\, i \\ &= (g \circ \_)\,\psi(x,i) \\ &= g \circ \psi(x,i) \\ &= (g \circ \psi_\flat x)\, i \end{aligned} \tag{2.71}$$

  Hence, we conclude $(g \circ \psi)_\flat = Rg \circ \psi_\flat$.

- For $X \xrightarrow{f} X' \xrightarrow{\phi'} LY'$, consider:

$$\begin{tikzcd} LX \arrow[r, "Lf"] \arrow[rr, bend right, "(\phi'\circ f)^\sharp"'] & LX' \arrow[r, "\phi'^\sharp"] & Y' \end{tikzcd} \tag{2.72}$$

  Let $x \in X$ and $i \in I$:

$$\begin{aligned} \phi'^\sharp \circ Lf(x,i) &= \phi'^\sharp(fx, i) \\ &= \phi'(fx)i \\ &= ((\phi' \circ f)\, x)\, i \\ &= (\phi' \circ f)^\sharp\,(x,i) \end{aligned} \tag{2.73}$$

  Hence, we conclude $\phi'^\sharp \circ Lf = (\phi' \circ f)^\sharp$.

Therefore, these two endo functors form an adjunction. ■

2.3 Ambivalent Objects

Consider a contravariant adjunction:

$$\begin{tikzcd} \mathcal{A} \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{S} \arrow[u, bend right, "G"'] \end{tikzcd} \tag{2.74}$$

In the covariant form $\begin{tikzcd} \mathcal{A} \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{S}^{\mathrm{op}} \arrow[u, bend right, "G"'] \end{tikzcd}$, we have the unit $\eta\colon 1_{\mathcal{A}} \Rightarrow GF$ and the counit $\epsilon\colon FG \Rightarrow 1_{\mathcal{S}^{\mathrm{op}}}$, and the natural bijection in Theorem 1.3.3 is $\zeta_{A,S}\colon \mathcal{S}^{\mathrm{op}}(FA, S) \cong \mathcal{A}(A, GS)$ for $A \in |\mathcal{A}|$ and $S \in |\mathcal{S}|$.

Hence, the bijection becomes

$$\zeta_{A,S}\colon \mathcal{S}(S, FA) \cong \mathcal{A}(A, GS) \tag{2.75}$$

with the following conditions:

- For $S' \xrightarrow{s} S \xrightarrow{\phi} FA$ in $\mathcal{S}$,

$$\begin{tikzcd} GS' & GS \arrow[l, "Gs"'] & A \arrow[l, "\zeta\phi"'] \arrow[ll, bend left, "\zeta(\phi s)"] \end{tikzcd} \qquad \zeta_{A,S'}\,(\phi s) = Gs \circ (\zeta_{A,S}\phi) \tag{2.76}$$

- For $A' \xrightarrow{a} A \xrightarrow{f} GS$ in $\mathcal{A}$,

$$\begin{tikzcd} FA' & FA \arrow[l, "Fa"'] & S \arrow[l, "\zeta^{-1}f"'] \arrow[ll, bend left, "\zeta^{-1}(fa)"] \end{tikzcd} \qquad {\zeta_{A',S}}^{-1}(fa) = Fa \circ \left({\zeta_{A,S}}^{-1} f\right) \tag{2.77}$$

The unit and counit become:

$$\begin{aligned} \eta_A &= \zeta_{A,GFA} 1_{FA} \in \mathcal{A}(A, GFA) \\ \epsilon_S &= {\zeta_{FGS,S}}^{-1} 1_{GS} \in \mathcal{S}(S, FGS) \end{aligned} \tag{2.78}$$

for $A \in |\mathcal{A}|$ and $S \in |\mathcal{S}|$.

2.3.1 Posets and Spaces – A Contravariant Adjunction

Theorem 2.3.1 (A Topology on Upper Sections)**.** *Let $(A, \leqq)$ be a poset and Γ_A be the set of upper sections of A, see Definition 2.1.2. Note that, by Definition 1.1.3, a poset is a preset with the antisymmetric order $\leqq$. For each finite subset $a \subset A$, let $\langle a \rangle$ be a subset of Γ_A given by*

$$U \in \langle a \rangle :\Leftrightarrow a \subset U \tag{2.79}$$

for $U \in \Gamma_A$. These subsets $\{\langle a \rangle \mid a \subset A \text{ is finite}\}$ form a basis of some topology, say $\mathcal{T}_{\Gamma_A}$, on Γ_A.

Proof. We will show the conditions in Remark 2 in Definition 1.2.5:

1. $\{\langle a\rangle \mid a \subset A \text{ is finite}\}$ covers Γ_A

   Let $U \in \Gamma_A$ be an upper section of A. Since $\emptyset$ is a finite subset of A, and $\emptyset \subset U$, we conclude $U \in \langle\emptyset\rangle$.

2. Binary Intersection

   Let $a, a' \subset A$ be finite subsets. Consider $\langle a\rangle \cap \langle a'\rangle$. For $U \in \Gamma_A$, we have

   $$U \in \langle a\rangle \cap \langle a'\rangle \Leftrightarrow a \subset p \wedge a' \subset U \Leftrightarrow a \cup a' \subset U \Leftrightarrow U \in \langle a \cup a'\rangle, \tag{2.80}$$

   Hence, we conclude $\langle a\rangle \cap \langle a'\rangle = \langle a \cup a'\rangle$.

Therefore, we may apply Theorem 1.2.4 to obtain the generated topology $\mathcal{T}_{\Gamma_A}$ as the set of all unions of the basis $\{\langle a\rangle \mid a \subset A \text{ is finite}\}$. ■

Theorem 2.3.2. *For a poset $(A, \leqq)$, let $\Upsilon(A, \leqq) := (\Gamma_A, \mathcal{T}_{\Gamma_A})$. If we define $\Upsilon f := f^{\leftarrow}$ for a monotone $f \in \mathbf{Pos}(A, B)$, then $\Upsilon f\colon (\Gamma_B, \mathcal{T}_{\Gamma_B}) \to (\Gamma_A, \mathcal{T}_{\Gamma_A})$ is continuous. We obtain a contravariant functor $\Upsilon\colon \mathbf{Pos} \to \mathbf{Top}$.*

Proof. Consider $(\Upsilon f)^{\leftarrow}$ and $a = \{a_1, \dots, a_n\} \subset A$ of a finite subset. For $V \in \Gamma_B$ of an upper section in B,

$$\begin{aligned} V \in (\Upsilon f)^{\leftarrow}\langle a\rangle &\Leftrightarrow (\Upsilon f)V \in \langle a\rangle \\ &\Leftrightarrow \{a_1, \dots, a_n\} \subset (\Upsilon f)V = f^{\leftarrow}V \\ &\Leftrightarrow a_1 \in f^{\leftarrow}V \wedge \cdots \wedge a_n \in f^{\leftarrow}V \\ &\Leftrightarrow fa_1 \in V \wedge \cdots \wedge fa_n \in V \\ &\Leftrightarrow \{fa_1, \dots, fa_n\} \subset V \\ &\Leftrightarrow fa \subset V \\ &\Leftrightarrow V \in \langle fa\rangle. \end{aligned} \tag{2.81}$$

Hence, $(\Upsilon f)^{\leftarrow}\langle a\rangle = \langle fa\rangle$ is a member of the basis for $\mathcal{T}_{\Gamma_B}$. By Theorem 1.2.6, $\Upsilon f \in C^0(\Gamma_B, \Gamma_A)$. ■

Theorem 2.3.3. *Now we have a pair of contravariant functors:*

$$\begin{tikzcd} \mathbf{Pos} \arrow[d, bend right, "\Upsilon"'] \\ \mathbf{Top} \arrow[u, bend right, "\mathcal{O}"'] \end{tikzcd} \tag{2.82}$$

They form a contravariant adjunction.

Proof. Let $(X, \mathcal{T}_X)$ be a topological space, $(A, \leqq)$ be a poset, and $(\Gamma_A, \mathcal{T}_{\Gamma_A}) := \Upsilon(A, \leqq)$. For $\phi \in \mathbf{Top}(X, \Gamma_A)$, define ϕ^α by

$$x \in \phi^\alpha a :\Leftrightarrow a \in \phi x \tag{2.83}$$

for each $x \in X$ and $a \in A$.

- $\phi^\alpha \colon A \to \mathcal{T}_X$

  Let $\{a\} \subset A$ be a singleton subset. For each $x \in X$,

$$x \in \phi^\alpha a \Leftrightarrow \{a\} \subset \phi x \Leftrightarrow \phi x \in \langle\{a\}\rangle \Leftrightarrow x \in \phi^{\leftarrow}\langle\{a\}\rangle. \tag{2.84}$$

  Hence, we conclude $\phi^\alpha a = \phi^{\leftarrow}\langle\{a\}\rangle \in \mathcal{T}_X$.

- ϕ^α is an arrow in **Pos**

  Let $a \leqq b$ in $(A, \leqq)$. Since $\phi a \in \Gamma_A$ is an upper section of A, if $a \in A$ then $b \in A$ holds. For each $x \in X$,

$$x \in \phi^\alpha a :\Leftrightarrow a \in \phi x \Rightarrow b \in \phi x \Leftrightarrow x \in \phi^\alpha b. \tag{2.85}$$

  Hence, ϕ^α is monotone $\phi^\alpha a \subset \phi^\alpha b$.

We then obtain

$$\zeta_{A,X} \colon \mathbf{Top}\,(X, \Gamma_A) \to \mathbf{Pos}\,(A, \mathcal{T}_X)\,; \phi \mapsto \phi^\alpha. \tag{2.86}$$

Let $(\mathcal{T}_X, \subset) \coloneqq \mathcal{O}\,(X, \mathcal{T}_X)$. For $f \in \mathbf{Pos}\,(A, \mathcal{T}_X)$, define f^σ by

$$a \in f^\sigma x :\Leftrightarrow x \in fa \tag{2.87}$$

for each $a \in A$ and $x \in X$.

- $f^\sigma \colon X \to \Gamma_A$

  Let $x \in X$. Suppose $a \leqq b$ in A. Since f is monotone, $fa \subset fb$ holds. Then

$$a \in f^\sigma x :\Leftrightarrow x \in fa \Rightarrow x \in fb \Leftrightarrow: b \in f^\sigma x. \tag{2.88}$$

  Hence, $f^\sigma x$ is an upper section in A.

- f^σ is an arrow in **Top**

  Consider the preimage $f^{\sigma\leftarrow}$ and a finite subset $a = \{a_1, \ldots, a_n\} \subset A$. For each $x \in X$,

$$\begin{aligned} x \in f^{\sigma\leftarrow}\langle a\rangle &\Leftrightarrow f^\sigma x \in \langle a\rangle \\ &\Leftrightarrow \{a_1, \ldots, a_n\} \subset f^\sigma x \\ &\Leftrightarrow a_1 \in f^\sigma x \wedge \cdots \wedge a_n \in f^\sigma x \\ &\Leftrightarrow x \in fa_1 \wedge \cdots \wedge x \in fa_n \\ &\Leftrightarrow x \in fa_1 \cap \cdots \cap fa_n \end{aligned} \tag{2.89}$$

  Hence, $f^{\sigma\leftarrow}\langle a\rangle = fa_1 \cap \cdots \cap fa_n \in \mathcal{T}_X$. We conclude $f^{\sigma\leftarrow}$ is continuous.

We obtain

$$\zeta'_{A,X} \colon \mathbf{Pos}\,(A, \mathcal{T}_X) \to \mathbf{Top}\,(X, \Gamma_A)\,; f \mapsto f^\sigma. \tag{2.90}$$

They are inverse pair:

- $\phi \mapsto \phi^{\alpha} \mapsto \phi^{\alpha\sigma} = \phi$

  For each $x \in X$ and $a \in A$, we have:

$$a \in \phi^{\alpha\sigma} x \Leftrightarrow x \in \phi^{\alpha} a \Leftrightarrow a \in \phi x. \tag{2.91}$$

- $f \mapsto f^{\sigma} \mapsto f^{\sigma\alpha} = f$

  For each $a \in A$ and $x \in X$, we have:

$$x \in f^{\sigma\alpha} a \Leftrightarrow a \in f^{\sigma} x \Leftrightarrow x \in fa. \tag{2.92}$$

Hence, $\zeta'_{A,X} = {\zeta_{A,X}}^{-1}$ and

$$\begin{tikzcd} \mathbf{Pos} \arrow[d, bend right, "\Upsilon"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{Top} \arrow[u, bend right, "\mathcal{O}"'] \end{tikzcd} \tag{2.93}$$

form a contravariant adjunction. ■

Remark 16 (Unit and Counit). Let $A \in |\mathbf{Pos}|$ and $X \in |\mathbf{Top}|$. The unit $\eta_A = (1_{\Gamma_A})^{\alpha}$ and the counit $\epsilon_X = (1_{\mathcal{T}_X})^{\sigma}$ are the following arrows:

$$\begin{array}{ll} \mathbf{Pos} & A \xrightarrow{\eta_A} \mathcal{T}_{\Gamma_A} \\ \\ \mathbf{Top} & X \xrightarrow[\epsilon_X]{} \Gamma_{\mathcal{T}_X} \end{array} \tag{2.94}$$

2.3.2 Ambivalent Objects

Let $(A, \leqq) \in |\mathbf{Pos}|$ and $\Pi(A, \leqq) \coloneqq \mathbf{Pos}(A, 2)$, where $2 = \{0, 1\}$ is directed by $<$ with $0 < 1$. We call $\mathbf{Pos}(A, 2)$ the set of monotone characters of A, where $p \leqq q$ for $p, q \in \mathbf{Pos}(A, 2)$ iff $\forall a \in A : pa \leqq qa$.

As demonstrated in Theorem 2.1.11, $\chi_A \colon \Gamma_A \cong \mathbf{Pos}(A, 2); U \mapsto \chi_A U$ is a bijection between the underlying sets, where $\chi_A U$ is the characteristic function on $U \subset A$:

$$(\chi_A U)\, a = \begin{cases} 1 & a \in U \\ 0 & \text{otherwise} \end{cases} \tag{2.95}$$

Lemma 2.3.1. *For each upper section $U \in \Gamma_A$ of A, its characteristic function $\chi_A U \colon A \to 2$ is continuous.*

Proof. Let $(A, \leqq) \in |\mathbf{Pos}|$, $U \in \Gamma_A$, and $\chi_A U \in \mathbf{Pos}(A, 2)$. We will show that relative to Alexandroff topology Γ_A of A and Sierpiński topology, $\chi_A U \in \mathbf{Top}(A, 2)$. Recalling $U \in \Gamma_A$ is open in A, we obtain $\chi_A U^{\leftarrow}\emptyset = \emptyset$, $\chi_A U^{\leftarrow} 2 = A$, and

$$\chi_A U^{\leftarrow}\{1\} = \{a \in A \mid (\chi_A U) a = 1\} = U. \tag{2.96}$$

Hence, $\chi_A(U) \in C^0(A, 2) = \mathbf{Top}(A, 2)$. ■

Therefore, $\chi_A\colon \Gamma_A \cong \mathbf{Pos}(A,2)$ returns an object in **Top**. With this bijection, we may topologize $C^0(A,2) = \mathbf{Pos}(A,2)$:

Theorem 2.3.4. *For each poset $(A, \leqq) \in |\mathbf{Pos}|$, $\chi_A \in C^0\left(\Gamma_A, \mathbf{Pos}(A,2)\right)$.*

Proof. Let $(A, \leqq) \in |\mathbf{Pos}|$. Consider the uncurried form:

$$\chi_A{}^\sharp\colon \Gamma_A \times A \to 2 \tag{2.97}$$

We will show $\chi_A{}^\sharp \in C^0\left(\Gamma_A \times A, 2\right)$ relative to the product topology and Sierpiński topology. It suffices to consider $\{1\} \subset \mathbf{2}'$ and its preimage:

$$\chi_A{}^{\sharp\leftarrow}\{1\} \coloneqq \left\{(U,a) \in \Gamma_A \times A \mid (\chi_A U)a = 1\right\}. \tag{2.98}$$

For each $(U,a) \in \Gamma_A \times A$,

$$(U,a) \in \chi_A{}^{\sharp\leftarrow}\{1\} \Leftrightarrow 1 = (\chi_A U)a \Leftrightarrow a \in U \Leftrightarrow \{a\} \subset U \Leftrightarrow U \in \langle a \rangle \tag{2.99}$$

Hence, we conclude:

$$\chi_A{}^{\sharp\leftarrow}\{1\} = \langle a \rangle \times U. \tag{2.100}$$

Since it is the product of a basic open subspace of $(\Gamma_A, \mathcal{T}_{\Gamma_A})$ and an open subspace of (A, Γ_A), $\chi_A{}^{\sharp\leftarrow}\{1\} \subset \Gamma_A \times A$ is open:

$$\chi_A{}^\sharp \in C^0\left(\Gamma_A \times A, 2\right). \tag{2.101}$$

As shown in Theorem 2.2.2, the original χ_A is continuous if the uncurried form $\chi_A{}^\sharp$ is continuous. Hence, $\chi_A \in C^0\left(\Gamma_A, A \multimap 2\right)$ is continuous, where $A \multimap 2 = C^0(A,2)$, see Definition 2.2.3. ■

Moreover, $\chi\colon \Upsilon \Rightarrow \Pi$ is a natural isomorphism, since

$$\begin{tikzcd}
\Gamma_A \arrow[d, "\chi_A"'] & \Gamma_B \arrow[l, "f^\leftarrow"'] \arrow[d, "\chi_B"] \\
\mathbf{Pos}(A,2) & \mathbf{Pos}(B,2) \arrow[l, "_\circ f"]
\end{tikzcd} \tag{2.102}$$

is commutative in **Top** for $A \xrightarrow{f} B$ in **Pos**. It is worth mentioning that the naturality is essentially shown in (2.33) of Theorem 2.1.11.

Recalling Remark 14, $\mathcal{O} \cong \mathbf{Top}(\_,2)$, we obtain:

$$\chi\colon \Upsilon \overset{\cong}{\Rightarrow} \Pi = \mathbf{Pos}(\_,2). \tag{2.103}$$

Hence,

$$\begin{tikzcd}
\mathbf{Pos} \arrow[d, bend right, "\mathbf{Pos}(_,2)"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{Top} \arrow[u, bend right, "\mathbf{Top}(_,2)"']
\end{tikzcd} \tag{2.104}$$

> The object 2 lives in both categories. It is both a poset and topological space. ...Furthermore, it induces both of the functors. [Sim11]

Such an object, sitting in two different categories, is called an ambivalent object, a dualizing object, etc.

Canonical Identification

Let $\mathcal{A}$ and $\mathcal{S}$ be **Set**-based categories, given by the following **Set**-valued functors:

$$U\colon \mathcal{A} \to \mathbf{Set}, \quad V\colon \mathcal{S} \to \mathbf{Set} \tag{2.105}$$

Consider a contravariant adjunction with $\eta\colon 1_{\mathcal{A}} \Rightarrow GF$ and $\epsilon\colon 1_{\mathcal{S}} \Rightarrow FG$:

$$\begin{tikzcd} \mathcal{A} \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{S} \arrow[u, bend right, "G"'] \end{tikzcd} \tag{2.106}$$

such that both $VF\colon \mathcal{A} \to \mathbf{Set}$ and $UG\colon \mathcal{S} \to \mathbf{Set}$ are representable:

- There are an object $* \in |\mathcal{A}|$ and a natural isomorphism $\alpha\colon \mathcal{A}(\_, *) \overset{\cong}{\Rightarrow} VF$, with a representing element:

$$1_* \mapsto \alpha_* 1_* \in VF* \tag{2.107}$$

- There are an object $\star \in |\mathcal{S}|$ and a natural isomorphism $\sigma\colon \mathcal{S}(\_, \star) \overset{\cong}{\Rightarrow} UG$, with a representing element:

$$1_\star \mapsto \sigma_\star 1_\star \in UG\star \tag{2.108}$$

Note that $* =$ `\ast` and $\star =$ `\star`.

Theorem 2.3.5. *The representing elements $\widetilde{\alpha} \coloneqq \alpha_* 1_* \in VF*$ and $\widetilde{\sigma} \coloneqq \sigma_\star 1_\star \in UG\star$ induce a canonical isomorphism between two sets $U*$ and $V\star$.*

Proof. Consider $\eta_* \in \mathcal{A}(*, GF*)$:

$$U\eta_* \in \mathbf{Set}(U*, UGF*) \tag{2.109}$$

Let $x \in U*$:

$$U\eta_* x \in UGF* \tag{2.110}$$

For $F* \in |\mathcal{S}|$,

$$\sigma_{F*}\colon \mathcal{S}(F*, \star) \cong UGF* \tag{2.111}$$

is a bijection between two sets. Hence, for the given $U\eta_* x \in UGF*$, there exists a unique $g \in \mathcal{S}(F*, \star)$ with

$$\sigma_{F*} g = U\eta_* x \tag{2.112}$$

For this $g \in \mathcal{S}(F*, \star)$, recalling G is a contravariant functor, we obtain:

$$UGg \in \mathbf{Set}(UG\star, UGF*). \tag{2.113}$$

Their uniqueness implies $UGg\widetilde{\sigma} = \sigma_{F*} g = U\eta_* x \in UGF*$. Define $\omega\colon U* \to V\star$:

$$U* \xrightarrow{U\eta_*} UGF* \xrightarrow{{\sigma_{F*}}^{-1}} \mathcal{S}(F*, \star) \xrightarrow{V} \mathbf{Set}(VF*, V\star) \xrightarrow{\text{-}(\widetilde{\alpha})} V\star \tag{2.114}$$

by

$$x \xmapsto{U\eta_*} \sigma_{F*}g \xmapsto{{\sigma_{F*}}^{-1}} g \xrightarrow{V} Vg \xmapsto{\_(\widetilde{\alpha})} Vg\,(\widetilde{\alpha}) \tag{2.115}$$

Note that $\_\,(\widetilde{\alpha})$ is the evaluation at $\widetilde{\alpha}$. Similarly, for $y \in V\star$, we define $\omega' \colon V\star \to U*$ by $\omega' y \coloneqq Uf\,(\widetilde{\sigma})$ via:

$$V\star \xrightarrow{V\epsilon_\star} VFG* \xrightarrow{{\alpha_{G\star}}^{-1}} \mathcal{A}(G\star, *) \xrightarrow{U} \mathbf{Set}(UG\star, U*) \xrightarrow{\_(\widetilde{\sigma})} U* \tag{2.116}$$

where $f \in \mathcal{A}(G\star, *)$ is a unique arrow such that

$$\alpha_{G\star} f = VFf\widetilde{\alpha} = V\epsilon_\star y \in VFG\star\,. \tag{2.117}$$

We will show $\omega \circ \omega' = 1_{V\star}$; the other equation follows due to symmetry. For $y \in V\star$, set $x \coloneqq \omega' y = Uf\widetilde{\sigma}$ and consider $\omega x = Vg\widetilde{\alpha}$. For $\sigma_{FG\star} \colon \mathcal{S}(FG\star, \star) \cong UFFG\star$ with $U\eta_{G\star}\widetilde{\sigma} \in UGFG\star$, let

$$s \coloneqq {\sigma_{FG\star}}^{-1} U\eta_{G\star}\widetilde{\sigma} \in \mathcal{S}(FG\star, \star). \tag{2.118}$$

Now we have the following parallels arrows in $\mathcal{S}$:

$$\begin{tikzcd} F* \arrow[r, "Ff"] \arrow[rr, bend right, "g"'] & FG\star \arrow[r, "s"] & \star \end{tikzcd} \tag{2.119}$$

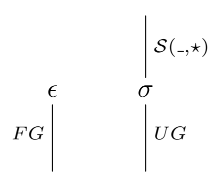
Their uniqueness implies $g = s \circ Ff$. If we apply V:

$$\begin{tikzcd} VF* \arrow[r, "VFf"] \arrow[rr, bend right, "Vg"'] & VFG\star \arrow[r, "Vs"] & V\star \end{tikzcd} \tag{2.120}$$

Along with $Vg = Vs \circ VFf$, $\widetilde{\alpha} \in VF*$ becomes:

$$(Vg)\widetilde{\alpha} = Vs\,(VFf\widetilde{\alpha}) = (Vs \circ V\epsilon_\star)\,y = V\,(s \circ \epsilon_\star)\,y. \tag{2.121}$$

Then the elevator-rule for



(2.122)

gurantees the following diagram commutative:

$$\begin{tikzcd} UGFG\star \arrow[r, "UG\epsilon_\star"] & UG\star \\ \mathcal{S}(FG\star, \star) \arrow[u, "\cong"', "\sigma_{FG\star}"] \arrow[r, "\mathcal{S}(\epsilon_\star, \star)", "\_\circ\epsilon_\star"'] & \mathcal{S}(\star, \star) \arrow[u, "\cong"', "\sigma_\star"] \end{tikzcd} \tag{2.123}$$

Evaluating at $s \in \mathcal{S}(FG\star, \star)$, we obtain:

$$(UG\epsilon_\star \circ \sigma_{FG\star})\, s = \sigma_\star \circ s \circ \epsilon_\star. \tag{2.124}$$

Now, the left-hand side becomes

$$UG\epsilon_\star\,(\sigma_{FG\star} s) = (UG\epsilon_\star \circ U\eta_{G\star})\,\widetilde{\sigma} = U\,(G\epsilon \circ \eta_G)_\star\,\widetilde{\sigma} \tag{2.125}$$

According to a zig-zag identity, see Definition 1.3.7, we obtain:

$$(UG\epsilon_\star \circ \sigma_{FG\star})\, s = \widetilde{\sigma} = \sigma_\star 1_\star. \tag{2.126}$$

Hence, we have $\sigma_\star \circ s \circ \epsilon_\star = \sigma_\star 1_\star$. Since $\sigma_\star$ is an isomorphism, we conclude $s \circ \epsilon_\star = 1_\star$. Hence

$$VG\widetilde{\alpha} = V1_\star y = 1_{V\star} y = y. \tag{2.127}$$

Recalling $\omega\omega' y = VG\widetilde{\alpha}$, we obtain the desired result $\omega\omega' = 1_{V\star}$. ■

Remark 17 (Lift). This canonical identification $\omega\colon U* \cong V\star$ can be seen as an object sitting in two different categories, namely $* \in |\mathcal{A}|$ and $\star \in |\mathcal{S}|$. Moreover, the contravariant functor $G\colon \mathcal{S} \to \mathcal{A}$ is a lift of the representable functor $\mathcal{S}(\_,\star)\colon \mathcal{S} \to \mathbf{Set}$ through U via $\sigma\colon \mathcal{S}(\_,\star)\colon \mathcal{S} \overset{\cong}{\Rightarrow} UG$:

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "U"] \\ \mathcal{S} \arrow[ur, bend left, "G"] \arrow[r, "\mathcal{S}(\_,\star)"'] & \mathbf{Set} \end{tikzcd} \qquad \sigma\colon \mathcal{S}(\_,\star) \overset{\cong}{\Rightarrow} UG. \tag{2.128}$$

Similarly, F is a lift of $\mathcal{A}(\_,*)$ through V:

$$\begin{tikzcd} & \mathcal{S} \arrow[d, "V"] \\ \mathcal{A} \arrow[ur, bend left, "F"] \arrow[r, "\mathcal{A}(\_,*)"'] & \mathbf{Set} \end{tikzcd} \qquad \alpha\colon \mathcal{A}(\_,*) \overset{\cong}{\Rightarrow} VF. \tag{2.129}$$

Remark 18 (Ambivalent Objects). For two **Set**-based categories $\mathcal{A}$ and $\mathcal{S}$, an ambivalent object is a set $\bullet$ that can be furnished in two ways to produce an object in $|\mathcal{A}|$ or an object in $|\mathcal{S}|$. As observed, $2 = \{0, 1\}$ can be seen as a post $(2, \leqq)$ or a topological space $\mathbf{2'} = (2, \{\emptyset, \{1\}, 2\})$.

For each $A \in |\mathcal{A}|$ and $S \in |\mathcal{S}|$,

$$\mathcal{A}(A, \bullet), \quad \mathcal{S}(S, \bullet) \tag{2.130}$$

are both sets. Hence, we have the corresponding contravariant hom-functors:

$$\begin{aligned} \mathcal{A}(\_,\bullet)&\colon \mathcal{A} \to \mathbf{Set} \\ \mathcal{S}(\_,\bullet)&\colon \mathcal{S} \to \mathbf{Set} \end{aligned} \tag{2.131}$$

Suppose the "nature" of $\bullet$ enables us to enrich:

$$\begin{aligned} \mathcal{A}(\_,\bullet)&\colon \mathcal{A} \to \mathcal{S} \\ \mathcal{S}(\_,\bullet)&\colon \mathcal{S} \to \mathcal{A} \end{aligned} \tag{2.132}$$

> This step is not routine ... When the construction works these enrichments are compatible with composition, to give a pair of contravariant functors ... between the categories. [Sim11]

We, then, have the following natural bijection:

$$\mathcal{S}\left(S, \mathcal{A}(A, \bullet)\right) \cong \mathcal{A}\left(A, \mathcal{S}(S, \bullet)\right) \tag{2.133}$$

where each $f \in \mathcal{A}\left(A, \mathcal{S}(S, \bullet)\right)$ is mapped to $\phi \in \mathcal{S}\left(S, \mathcal{A}(A, \bullet)\right)$ defined by:

$$(\phi s)a \coloneqq (fa)s. \tag{2.134}$$

It follows that they form a contravariant adjunction:

$$\begin{tikzcd} \mathcal{A} \arrow[d, bend right, "\mathcal{A}(\_,\bullet)"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{S} \arrow[u, bend right, "\mathcal{S}(\_,\bullet)"'] \end{tikzcd} \tag{2.135}$$

The corresponding unit and counit are both "evaluations:"

- The unit $\eta\colon 1_{\mathcal{A}} \Rightarrow \mathcal{S}\left(\mathcal{A}(\_, \bullet), \bullet\right)$
  Let $A \in |\mathcal{A}|$:
  $$\eta_A\colon A \to \mathcal{S}\left(\mathcal{A}(A, \bullet), \bullet\right) \tag{2.136}$$
  For each $a \in A$ and $p \in \mathcal{A}(A, \bullet)$,
  $$(\eta_A a)\, p = pa. \tag{2.137}$$

- The counit $\epsilon\colon 1_{\mathcal{S}} \Rightarrow \mathcal{A}\left(\mathcal{S}(\_, \bullet), \bullet\right)$
  Let $S \in |\mathcal{S}|$:
  $$\epsilon_S\colon S \to \mathcal{A}\left(\mathcal{S}(S, \bullet), \bullet\right) \tag{2.138}$$
  For each $s \in S$ and $f \in \mathcal{S}(S, \bullet)$,
  $$(\epsilon_S s)\, f = fs. \tag{2.139}$$

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