

# Group Theory 101

for Homology Theory

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## 0 Definitions and Basic Properties

### 0.1 Groups – Multiplicative Groups

**Definition 0.1** (Groups). A group is given by

$$(G, \circ, 1, (-)^{-1}), \quad (1)$$

where

- $G$  is a set and  $\circ$  is a binary product

$$\circ: G \times G \rightarrow G; (g_2, g_1) \mapsto g_2 \circ g_1. \quad (2)$$

- $(G, \circ, 1)$  forms a monoid. Namely,  $G$  is a non-empty set,  $\circ$  is an associative binary product, and  $1 \in G$  is a multiplicative identity such that

$$g \circ 1 = g = g \circ 1, \quad (3)$$

for each  $g \in G$ . Note that the associativity is expressed by

$$g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1 \quad (4)$$

for any  $g_1, g_2, g_3 \in G$ .

- The unary operation  $(\_)^{-1}: G \rightarrow G$  returns the multiplicative inverse  $g^{-1}$  of a given  $g \in G$  such that

$$g^{-1} \circ g = 1 = g \circ g^{-1}. \quad (5)$$

**Lemma 0.1.** *Let  $(G, \circ, 1, (\_)^{-1})$  be a group.*

- *If  $e \in G$  satisfies*

$$g \circ e = g = e \circ g \quad (6)$$

*for each  $g \in G$ , then  $e = 1$ . That is, the identity element is unique.*

- *For each  $g \in G$ , the inverse  $g^{-1}$  is unique.*
- *For each  $g \in G$ ,  $(g^{-1})^{-1} = g$ .*

*Proof.* Since  $1 \in G$  is an multiplicative identity,  $1 \circ e = e = e \circ 1$ ;  $e \in G$  satisfies the same properties, we conclude  $e = 1 \circ e = 1$ . Let  $g \in G$ . Suppose  $g' \in G$  satisfies  $g' \circ g = g = g \circ g'$ . Applying  $g^{-1}$ , we obtain  $g' = g^{-1}$ . We, then, have  $g^{-1} \circ g = 1 = g^{-1} \circ g$ , showing  $(g^{-1})^{-1} = g$ .  $\square$

**Definition 0.2** (Subgroups). A subset  $H \subset G$  of a group  $(G, \circ, 1, (\_)^{-1})$  is called a subgroup of  $G$  iff  $H$  forms a group and  $H$  is closed under group operations:

- For any  $h_1, h_2 \in H$ ,  $h_2 \circ h_1 \in H$ .
- The identity  $1 \in G$  is in  $H$ ,  $1 \in H$ .
- For any  $h \in H$ , the inverse is in  $H$ ,  $h^{-1} \in H$ .

A trivial subgroup is the singleton  $\{1\} \subset G$ ; another example is  $G$  itself. We denote a subgroup  $H$  of  $G$  by  $H < G$ .

**Lemma 0.2** (Subgroup-Test). *Let  $(G, \circ, 1, (\_)^{-1})$  be a group. A non-empty subset  $\emptyset \neq H \subset G$  is a subgroup iff  $h_2 \circ h_1^{-1} \in H$  for any  $h_1, h_2 \in H$ .*

*Proof.*  $(\Rightarrow)$  If  $H < G$  and  $h_1, h_2 \in H$ , then  $h_1^{-1} \in H$  and hence  $h_2 \circ h_1^{-1} \in H$ .  $(\Leftarrow)$  There is at least one element in  $H$ . Select  $h \in H$ . Then  $h \circ h^{-1} = 1 \in H$ . For any  $k \in H$ ,  $k^{-1} = 1 \circ k^{-1} \in H$ . Suppose  $h_1, h_2 \in H$ . Then  $h_2^{-1} \in H$ , and hence  $h_1 h_2 = h_1 (h_2^{-1})^{-1} \in H$ . Since the underlying  $\circ$  is an associative binary product of  $G$ ,  $(H, \circ, 1, (\_)^{-1})$  forms an group.  $\square$

From this point, the group operation can be written without the symbol  $\circ$ ; namely, for  $g_1, g_2 \in G$ , we write  $g_2 g_1$  instead of  $g_2 \circ g_1$ .

**Definition 0.3** (Group Homomorphisms). Let  $G$  and  $H$  be groups. A map  $\theta: G \rightarrow H$  is called a group homomorphism iff  $\theta(g_2g_1) = (\theta g_2)(\theta g_1)$ .

- If a group homomorphism  $\theta: G \rightarrow H$  is injective:

$$\theta g_1 = \theta g_2 \Rightarrow g_1 = g_2 \quad (7)$$

for  $g_1, g_2 \in G$ , then  $\theta$  is called a mono.

- If a group homomorphism  $\theta: G \rightarrow H$  is surjective:

$$\theta G = H, \quad (8)$$

then  $\theta$  is called an epi.

A group homomorphism is called an isomorphism iff it is both monic and epic.

**Lemma 0.3.** Any group homomorphism preserves the identity and inverses.

*Proof.* Let  $\theta: G \rightarrow H$  be a group homomorphism. Since  $1_G \circ 1_G = 1_G$ , we have  $(\theta 1_G) \circ (\theta 1_G) = \theta 1_G$ . Applying the inverse  $(\theta 1_G)^{-1}$  of  $\theta 1_G \in H$ , we obtain  $\theta 1_G = 1_H$ . Let  $g \in G$  and consider the inverse  $g^{-1} \in G$ . Since  $g^{-1} \circ g = 1_G = g \circ g^{-1}$ , if we apply  $\theta$ , we have  $(\theta g^{-1}) \circ (\theta g) = 1_H = (\theta g) \circ (\theta g^{-1})$ . Since the inverse is unique by Lemma 0.1,  $\theta g^{-1} = (\theta g)^{-1}$ .  $\square$

**Lemma 0.4.** A group homomorphism  $\theta: G \rightarrow H$  is monic iff the kernel is singleton,  $\theta^{\leftarrow} 1_H = \{1_G\}$ .

*Proof.* ( $\Rightarrow$ ) Recall the very definition:

$$\theta^{\leftarrow} 1_H := \{g \in G \mid \theta g = 1_H\}. \quad (9)$$

The kernel is non-empty,  $1_G \in \theta^{\leftarrow} 1_H$ , since  $\theta 1_G = 1_H$ . Let  $g \in \theta^{\leftarrow} 1_H$ . Then  $\theta g = 1_H = \theta 1_G$ . Since  $\theta$  is injective,  $g = 1_G$ , and hence  $\theta^{\leftarrow} 1_H = \{1_G\}$ .

( $\Leftarrow$ ) Conversely, suppose  $\theta^{\leftarrow} 1_H = \{1_G\}$ . Let  $g_1, g_2 \in G$  and assume  $\theta g_1 = \theta g_2$  for  $g_1, g_2 \in G$ . Then  $1_H = (\theta g_2)(\theta g_1)^{-1} = (\theta g_2)(\theta g_1^{-1}) = \theta(g_2g_1^{-1})$ . Hence,  $g_2g_1^{-1} \in \theta^{\leftarrow} 1_H = \{1_G\}$ , and we have  $g_1 = g_2$  since  $g_2g_1^{-1} = 1_G$ .  $\square$

**Theorem 0.5.** Both image and kernel of a group homomorphism are subgroups.

*Proof.* Let  $\theta: G \rightarrow H$  be a group homomorphism. Let  $h_1, h_2 \in \theta G$ . There are  $g_1, g_2 \in G$  such that  $h_1 = \theta g_1$  and  $h_2 = \theta g_2$ . Since  $g_2g_1^{-1} \in G$ ,

$$h_2h_1^{-1} = \theta g_2(\theta g_1)^{-1} = \theta(g_2g_1^{-1}) \in \theta G. \quad (10)$$

By Lemma 0.2,  $\theta G < H$ .

Let  $g_1, g_2 \in \theta^{\leftarrow} 1$ . Then  $\theta g_1 = \theta g_2$  and, hence

$$\theta(g_2g_1^{-1}) = (\theta g_2)(\theta g_1)^{-1} = 1. \quad (11)$$

By Lemma 0.2,  $\theta^{\leftarrow} 1 < G$ .  $\square$

## 0.2 Abelian Groups – Additive Groups

**Definition 0.4** (Abelian Groups). Let  $(G, \circ, 1, (-)^{-1})$  be a group. Two elements  $a, b \in G$  are commutative iff  $a \circ b = b \circ a$ ;  $G$  is called an abelian group iff any pair is commutative. We use  $+$  sign for the commutative binary operator, and 0 for the additive identity, and  $-g$  for the additive inverse of  $g \in G$ .

**Definition 0.5** (Free Group). An abelian group  $(A, +, 0, -(-))$  is called free iff there is  $B \subset A$  such that every element  $a \in A$  has a unique representation

$$a = \sum_{b \in B} n_b b, \quad (12)$$

where  $\{n_b \in \mathbb{Z} \mid b \in B\}$  are all equal to zero but finitely many. In other words, the “sequence” is eventually zero. Such a subset  $B \subset A$ , which generates the given abelian group  $A$ , is called a basis for  $A$ .

*Remark* (Construction). Let  $S$  be a set. We will construct a free abelian group with basis  $S$ .

Let  $F_S$  be a set of functions from  $S$  to integers  $\mathbb{Z}$  such that  $f \in F_S$  iff

$$fs = 0 \text{ for all but finitely many } s \in S. \quad (13)$$

Then the zero map is in  $F_S$ ,  $0 \in F_S$ . For any  $f, g \in F_S$ ,  $f + g$  is defined componentwisely,  $(f + g)s = fs + gs$  for each  $s \in S$ . The additive inverse of  $f$  is also defined componentwisely,  $(-f)s = -(fs)$ . By construction, the set of characteristic functions forms a basis for  $F_S$ :

$$B := \{\chi_{\{t\}} \mid t \in S\} \cong S, \quad (14)$$

where

$$\chi_{\{t\}}s := \begin{cases} 0 & s \neq t \\ 1 & s = t. \end{cases} \quad (15)$$

For each  $f \in F_S$ , let  $I_f \subset S$  be a finite subset such that  $fs \neq 0$  iff  $s \in I_f$ . Then any  $f \in F_S$  is written as  $f = \sum_{t \in I_f} (ft)\chi_{\{t\}}$ .

We denote  $O$  for the abelian group with empty basis; since the empty sum is the additive identity, the unique element in  $O$  is 0:

$$O = (\{0\}, +, 0, -(-)). \quad (16)$$

*Remark* (Extension “by Linearity”). If an abelian group  $(A, +, 0, -(-))$  is free with basis  $B$  and  $C$  is an abelian group, then any map  $\theta: B \rightarrow C$  can be uniquely extended to an abelian group homomorphism  $\theta: A \rightarrow C$  by linearity, namely for each  $a \in A$  with its unique representation  $\sum_{b \in B} n_b b$ ,  $\theta a := \sum_{b \in B} n_b (\theta b)$ .

# 1 Normal Subgroups and Quotient Groups

## 1.1 Normal Subgroups

**Definition 1.1** (Normal Subgroups). Let  $G$  be a group. A subgroup  $N < G$  is called a normal subgroup iff  $gNg^{-1} \subset N$  for each  $g \in G$ , where

$$gNg^{-1} := \{gng^{-1} \mid n \in N\}. \quad (17)$$

We denote  $N \triangleleft G$  for a normal subgroup  $H$  of  $G$ .

**Lemma 1.1.** *The kernel of a group homomorphism is a normal subgroup.*

*Proof.* Let  $\theta: G \rightarrow H$  be a group homomorphism. By Theorem 0.5,  $\theta^{-1}1 < G$ . Let  $g \in G$  and  $k \in \theta^{-1}1$ . Then  $gkg^{-1} \in \theta^{-1}1$  since

$$\theta(gkg^{-1}) = (\theta g)1(\theta g)^{-1} = 1. \quad (18)$$

Therefore,  $g(\theta^{-1}1)g^{-1} \subset \theta^{-1}1$ .  $\square$

**Lemma 1.2.** *Let  $G$  be a group. If  $N \triangleleft G$  and  $N < M < G$ ,*

$$N \begin{array}{c} \xrightarrow{\triangleleft} \\ \xrightarrow{<} \end{array} M \begin{array}{c} \xrightarrow{\triangleleft} \\ \xrightarrow{<} \end{array} G,$$

*then  $N \triangleleft M$  holds.*

*Proof.* Since  $N < M$ , it suffices to show its normality. Let  $m \in M$  and  $n \in N$ . Since  $N \triangleleft G$  and  $M < G$ ,  $m \in G$  and hence,  $mnm^{-1} \in N$ . That is,  $mNm^{-1} \subset N$ .  $\square$

**Theorem 1.3.** *Let  $N \triangleleft G$  be a normal subgroup. If we define  $\sim$  by*

$$g_1 \sim g_2 :\Leftrightarrow g_2g_1^{-1} \in N \quad (19)$$

*then  $\sim$  is an equivalence relation on  $G$  relative to  $N$ .*

*Remark.* For each  $g \in G$ , let

$$[g] := \{g' \in G \mid g' \sim g\} \quad (20)$$

be the set of equivalent elements represented by  $g$ . Let  $G/\sim = \{[g] \mid g \in G\}$  denote the set of equivalent classes relative to  $N \triangleleft G$ .

*Proof.* Let  $g \in G$ . Since  $gg^{-1} = 1 \in N$ ,  $g \sim g$ . Hence,  $\sim$  is reflexive.

Suppose  $g_1 \sim g_2$  for  $g_1, g_2 \in G$ . Then  $n := g_2g_1^{-1} \in N$ . Since  $N$  is a subgroup of  $G$ ,  $n^{-1} \in N$ , namely  $n^{-1} = g_1g_2^{-1} \in N$ . Hence,  $g_2 \sim g_1$ , and  $\sim$  is symmetric.

Finally, suppose  $g_1 \sim g_2$  and  $g_2 \sim g_3$ . Then  $g_3g_1^{-1} \in K$ , since  $g_2g_1^{-1} \in N$ ,  $g_3g_2^{-1} \in N$  and  $g_3g_1^{-1} = g_3g_2^{-1}g_2g_1^{-1}$ . Therefore,  $\sim$  is transitive.  $\square$

**Corollary 1.3.1.**  *$G/\sim$  with respect to  $G \triangleleft G$  is a singleton.*

*Proof.* All elements are equivalent to  $1 \in G$  since  $g = g1^{-1} \in G$  for each  $g \in G$ . Hence  $[1] = \{g \in G \mid g \sim 1\} = G$  is the unique element.  $\square$

## 1.2 Quotient Groups

**Lemma 1.4.** *Let  $G$  be a group and  $N \triangleleft G$  be a normal subgroup.*

- For each  $g \in G$ ,  $gN = Ng = [g]$ .
- For  $g_1, g_2 \in G$ ,  $[g_1][g_2] = [g_1g_2]$ .

*Remark.* Let  $G/N$  denote the set  $G/\sim = \{[g] \mid g \in G\}$  of equivalent classes with respect to  $N \triangleleft G$  with the binary product  $[g_1][g_2] = [g_1g_2]$  for  $[g_1], [g_2] \in G/N$ .

*Proof.* Let  $g \in G$ . For each  $n \in N$ , since  $N \triangleleft G$ ,  $k := gng^{-1} \in N$ . That is,  $gn = kg \in Ng$ , and hence  $gN \subset Ng$ . Conversely, for each  $n \in N$ , as  $g^{-1} \in G$ ,  $h := g^{-1}ng \in N$ . It follows  $Ng \subset gN$ . Therefore,  $gN = Ng$  holds.

Let  $g \in G$  and consider  $[g] \in G/\sim$ . For each  $g' \in [g]$ , there exists  $n \in N$  with  $n = g'g^{-1}$ . Then,  $[g] \subset Ng$  since for each  $g' \in [g]$ ,  $g' = ng \in Ng$ . Conversely, consider  $gk \in gN$ . As shown above,  $gN = Ng$ , there must be some  $h \in N$  such that  $gk = hg$ . Therefore,  $h = (gk)g^{-1} \in N$ , showing  $gk \sim g$ , and  $gk \in [g]$ . Hence, we conclude  $[g] = Ng$ .

Finally, consider the binary operation. Let  $g_1, g_2 \in G$ .

$$\begin{aligned} [g_2][g_1] &= \{h_2h_1 \mid h_2 \in Ng_2 \wedge h_1 \in Ng_1\} \\ &= \{n_2g_2n_1g_1 \mid n_1, n_2 \in N\} \\ &= \{n_2(g_2n_1g_2^{-1})g_2g_1 \mid n_1, n_2 \in N\} \end{aligned} \tag{21}$$

Since  $N \triangleleft G$ ,  $(g_2n_1g_2^{-1}) \in N$  for each  $g_2 \in G$  and  $n_1 \in N$ . Hence, we obtain

$$[g_2][g_1] = \{ng_2g_1 \mid n \in N\} = N(g_2g_1) = [g_2g_1]. \tag{22}$$

□

**Theorem 1.5** (Quotient Groups). *Let  $G$  be a group and  $N \triangleleft G$  be a normal subgroup. Then  $G/N$  forms a group:*

$$(G/N, \circ, N, [(-)^{-1}]), \tag{23}$$

where the binary product  $\circ$  is defined by  $[g_1] \circ [g_2] = [g_1g_2]$  for  $[g_1], [g_2] \in G/N$ .

*Proof.* Since the binary product is essentially the binary product of  $G$ , it suffices to show the identity and inverses.

- $N = 1_{G/N}$

Consider  $[g]$  for  $g \in G$ . By Lemma 1.4,  $[g] = gN$ , and hence

$$[g]N = gNN = gN = [g]. \tag{24}$$

Similarly, one can show that  $N[g] = [g]$ .

- $[g]^{-1} = [g^{-1}]$

Consider  $[g]$  for  $g \in G$ . By Lemma 1.4,

$$[g] [g^{-1}] = [gg^{-1}] = [1] = N = [g^{-1}] [g]. \quad (25)$$

Since the inverse is unique, by Theorem 0.1,  $[g^{-1}]$  is the desired inverse of  $[g]$ .

We call the group  $G/N$  for  $N \triangleleft G$  the quotient group of  $G$  modulo  $N$ .  $\square$

### 1.3 Group Homomorphisms and Quotient Groups

**Theorem 1.6.** *Let  $G$  and  $H$  be groups. For a group homomorphism  $\theta: G \rightarrow H$ , as shown in Lemma 1.1, the kernel forms a normal subgroup,  $\theta^{-1}1 \triangleleft G$ . Conversely, if  $N \triangleleft G$ , then the canonical projection*

$$\pi: G \rightarrow G/N; g \mapsto [g] \quad (26)$$

*is an epi with  $\pi^{-1}1 = N$ .*

*Proof.* Suppose  $N \triangleleft G$ . For any  $[g] \in G/N$ , if we choose  $g' \in [g]$ ,  $\pi g' = [g'] = [g]$  since  $g' \sim g$ . Hence, the canonical projection  $\pi$  is surjective. For  $g_1, g_2 \in G$ , it follows

$$\pi(g_2 g_1) = [g_2 g_1] = [g_2] [g_1] = (\pi g_2) (\pi g_1) \quad (27)$$

Hence,  $\pi$  is a group homomorphism. Finally, we have  $\pi^{-1}1 = N$  since

$$\{g \in G \mid [g] = N = N1\} = \{g \in G \mid g \sim 1\} = \{g \in G \mid g = g1^{-1} \in N\}. \quad (28)$$

$\square$

**Theorem 1.7.** *Let  $G$  and  $H$  be groups, and  $\theta: G \rightarrow H$  be a group homomorphism. Suppose  $N \triangleleft G$  and  $N \subset \theta^{-1}1$ . Then, there exists a unique mediator  $\bar{\theta}: G/N \rightarrow H$  such that  $\bar{\theta}[g] = \theta g$  for each  $g \in G$ . That is, the following diagram is commutative:*

$$\begin{array}{ccc} G & \xrightarrow{\theta} & H \\ \pi \downarrow & \nearrow \exists! \bar{\theta} & \\ G/N & & \end{array} \quad \bar{\theta} \circ \pi = \theta. \quad (29)$$

*We also have  $\bar{\theta}(G/N) = \theta G$  and  $\bar{\theta}^{-1}1 = (\theta^{-1}1)/N$ .*

*Proof.* If  $g \sim g'$  relative to  $N \triangleleft G$ , there is some  $n \in N$  such that  $n := g'g^{-1}$  and

$$\theta g' = \theta(n g) = (\theta n) (\theta g) = 1 \theta g = \theta g, \quad (30)$$

since  $n \in N \subset \theta^{-1}1$ . Thus  $\bar{\theta}: G/N \rightarrow H; [g] \mapsto \theta g$  is a well-defined map among the corresponding sets. Since  $\bar{\theta}$  is fully determined in terms of the given group homomorphism  $\theta$ , it is unique.

The induced map  $\bar{\theta}: G/N \rightarrow H$  is a group homomorphism, since  $\theta$  is a group homomorphism:

$$\bar{\theta}([g_2][g_1]) = \bar{\theta}[g_2g_1] = \theta(g_2g_1) = (\theta g_2)(\theta g_1) = (\bar{\theta}[g_2])(\bar{\theta}[g_1]) \quad (31)$$

for  $[g_1], [g_2] \in G/N$ . The images coincide:

$$\theta G = \{h \in H \mid \exists g \in G : h = \theta g = \bar{\theta}[g]\} = \bar{\theta}(G/N). \quad (32)$$

Let  $K := \theta^{\leftarrow} 1 \triangleleft G$ . Since  $N < K \triangleleft G$  and  $N \triangleleft G$ , we may apply Lemma 1.2,  $N \triangleleft K$ . Hence,  $K/N = (\theta^{\leftarrow} 1)/N$  is a quotient group. For any  $g \in G$ ,  $[g] = gN \in K/N$  iff  $\bar{\theta}[g] = fg = 1$  iff  $[g] \in \bar{\theta}^{\leftarrow} 1$ . That is,  $(\theta^{\leftarrow} 1)/N = K/N = \bar{\theta}^{\leftarrow} 1$ .  $\square$

**Corollary 1.7.1.** *The induced group homomorphism  $\bar{\theta}: G/N \rightarrow H$  is an isomorphism iff  $\theta$  is an epi and  $N = \theta^{\leftarrow} 1$ .*

*Proof.* ( $\Rightarrow$ ) Suppose  $\bar{\theta}$  is an isomorphism. Since  $\bar{\theta}$  is epic, so is  $\theta$  since

$$\theta G = \bar{\theta}(G/N) = H. \quad (33)$$

By hypothesis,  $N \subset \theta^{\leftarrow} 1$ ; to show the other inclusion, let  $k \in \theta^{\leftarrow} 1$ :

$$\bar{\theta}[k] = \theta k = 1 \quad (34)$$

Since  $\bar{\theta}$  is monic, by Lemma 0.4,  $\bar{\theta}^{\leftarrow} 1 = \{1_{G/N}\}$ , where  $1_{G/N} = N$  by Theorem 1.5. Hence,  $[k] = N$ , i.e.,  $k \in N$ .

( $\Leftarrow$ ) Suppose  $\theta$  is epic and  $N = \theta^{\leftarrow} 1$ . Then  $\bar{\theta}$  is epic since  $\bar{\theta}(G/N) = \theta G = H$ . Recalling  $(\theta^{\leftarrow} 1)/N = \bar{\theta}^{\leftarrow} 1$ , and by hypothesis  $N = \theta^{\leftarrow} 1$ , the kernel is a singleton by Corollary 1.3.1,  $\bar{\theta}^{\leftarrow} 1 = \{N\} = \{1_{G/N}\}$ .  $\square$

**Theorem 1.8** (First Isomorphic Theorem). *A group homomorphism  $\theta: G \rightarrow H$  induces an isomorphism between  $G/\theta^{\leftarrow} 1$  and  $\theta G$ .*

*Proof.* The corestriction  $\theta: G \rightarrow \theta G$  is an epi. Thus, the induced group homomorphism via Corollary 1.7.1:

$$\bar{\theta}: G/N \rightarrow \theta G \quad (35)$$

is an isomorphism, where  $N := \theta^{\leftarrow} 1 \triangleleft G$ .  $\square$

## 1.4 Exact Sequences

**Definition 1.2** (Exact Sequences). A pair of adjacent group homomorphisms  $F \xrightarrow{\theta} G \xrightarrow{\varphi} H$  is called exact iff  $\theta F = \varphi^{\leftarrow} 1$ . A sequence of group homomorphisms is called exact iff every adjacent pair of homomorphisms is exact. An exact sequence of the following form is called short exact:

$$O \longrightarrow F \xrightarrow{\theta} G \xrightarrow{\varphi} H \longrightarrow O \quad (36)$$

where  $O \rightarrow F$  is essentially an inclusion map  $\{1_F\} \subset F$  and  $H \rightarrow O$  is a constant map toward a singleton group.

**Lemma 1.9.** *A group homomorphism  $\theta: F \rightarrow G$  is an isomorphism iff  $O \rightarrow F \xrightarrow{\theta} G \rightarrow O$  is exact.*

*Proof.* Since  $O \rightarrow F$  is essentially an inclusion map  $\{1_F\} \subset F$ , the image is  $\{1_F\}$ . Hence,  $\theta$  is monic, by Lemma 0.4  $\theta^\leftarrow 1 = \{1_F\}$ , iff  $O \rightarrow F \xrightarrow{\theta} G$  is exact. Since  $G \rightarrow O$  is a constant map toward the identity, its kernel is the domain  $G$ . Hence,  $\theta$  is epic,  $\theta F = G$ , iff  $F \xrightarrow{\theta} G \rightarrow O$  is exact.  $\square$

**Theorem 1.10.** *For a short exact sequence  $O \rightarrow F \xrightarrow{\theta} G \xrightarrow{\varphi} H \rightarrow O$ ,  $\theta$  is monic and  $\varphi$  is epic.*

*Proof.* Since  $O \rightarrow F \xrightarrow{\theta} G$  is exact,  $\{1_F\} = \theta^\leftarrow 1$ . By Lemma 0.4,  $\theta$  is monic. Since  $G \xrightarrow{\varphi} H \rightarrow O$  is exact,  $\varphi G = H$ . Hence,  $\varphi$  is epic.  $\square$

*Remark.* The corestriction  $\theta: F \rightarrow \theta F$  is isomorphic. Thus, we may identify  $F \subset G$  through  $F \cong \theta F \subset G$ . Since  $\varphi: G \rightarrow H$  is epic,  $\varphi G = H$ . With the kernel  $\varphi^\leftarrow 1 = \theta F$ , Theorem 1.8 implies  $\varphi G \cong G/(\varphi^\leftarrow 1)$ . Hence, we obtain  $F \cong \theta F \triangleleft G$  and

$$G/F \cong G/(\theta F) = G/(\varphi^\leftarrow 1) \cong \varphi G = H. \quad (37)$$

Hence, the given short exact sequence is equivalent to

$$O \longrightarrow F \xhookrightarrow{\iota} G \xrightarrow{\pi} G/F \longrightarrow O \quad (38)$$

## 2 Direct Products

### 2.1 Direct Products

**Definition 2.1** (Direct Products and Componentwise Binary Product). Let  $\Lambda$  be a set and suppose that for each  $\lambda \in \Lambda$ , there is given a group  $G_\lambda$ . On their Cartesian product

$$\prod_{\lambda \in \Lambda} G_\lambda := \left\{ f: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda \mid \forall \lambda \in \Lambda : f_\lambda \in G_\lambda \right\}, \quad (39)$$

we define a binary operation componentwisely, namely for  $f, g \in \prod_{\lambda \in \Lambda}$ ,

$$gf: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda; \lambda \mapsto g_\lambda f_\lambda. \quad (40)$$

We call  $\prod_{\lambda \in \Lambda} G_\lambda$ , with the associated binary operation, the direct product of  $\{G_\lambda \mid \lambda \in \Lambda\}$ .

**Theorem 2.1.** *For a set of groups  $\{G_\lambda \mid \lambda \in \Lambda\}$ , their direct product  $\prod_{\lambda \in \Lambda} G_\lambda$  forms a group.*

*Proof.* The corresponding binary product is essentially that of each component, hence it is associative. Let  $1: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda; 1 \mapsto 1_{G_\lambda}$ . Then 1 is the multiplicative identity. For each  $f \in \prod_{\lambda \in \Lambda} G_\lambda$ , let

$$f^{-1}: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda; \lambda \mapsto f_\lambda^{-1}. \quad (41)$$

It follows  $f^{-1}f = 1 = ff^{-1}$ , and  $(\cdot)^{-1}: \bigcup_{\lambda \in \Lambda} G_\lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda$  is the desired inverse unary operation.  $\square$

**Theorem 2.2.** *For the direct product  $\prod_{\lambda \in \Lambda} G_\lambda$  of a set of groups  $\{G_\lambda \mid \lambda \in \Lambda\}$ , the canonical projection*

$$\pi_k: \prod_{\lambda \in \Lambda} G_\lambda \rightarrow G_k; f \mapsto f_k \quad (42)$$

*is an epi for each  $k \in \Lambda$ .*

*Proof.* Let  $f, g \in \prod_{\lambda \in \Lambda} G_\lambda$ . For each  $k \in \Lambda$ , by Definition 2.1,

$$\pi_k(gf) = (gf)_k = g_k f_k = (\pi_k g)(\pi_k f). \quad (43)$$

Hence,  $\pi_k$  is a group homomorphism for each  $k \in \Lambda$ .

Let  $g_k \in G_k$  for a given  $k \in \Lambda$ . Define  $h: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda$  by

$$h_\lambda := \begin{cases} 1_{G_\lambda} & \lambda \neq k \\ g_k & \lambda = k \end{cases} \quad (44)$$

Then  $h \in \prod_{\lambda \in \Lambda} G_\lambda$  such that,  $\pi_k h = g_k$ . Hence,  $\pi_k$  is surjective.  $\square$

**Theorem 2.3** (Universal Property of Direct Products). *Let  $H$  be a group and  $\Lambda$  be a set, and suppose that for each  $\lambda \in \Lambda$ , there is given a group homomorphism  $\theta_\lambda: H \rightarrow G_\lambda$ . Then, there exists a unique mediator homomorphism  $\theta: H \rightarrow \prod_{\lambda \in \Lambda} G_\lambda$  such that the following diagram for each  $k \in \Lambda$  is commutative:*

$$\begin{array}{ccc} H & \xrightarrow{\theta_k} & G_k \\ \exists! \theta \downarrow & \searrow & \uparrow \pi_k \\ \prod_{\lambda \in \Lambda} G_\lambda & \xrightarrow{\pi_k} & G_k \end{array} \quad \pi_k \circ \theta = \theta_k. \quad (45)$$

*Proof.* Define

$$\theta_-: H \rightarrow \left( \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda \right); h \mapsto \theta_- h, \quad (46)$$

by, for each  $h \in H$ ,

$$\theta_- h: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} G_\lambda; k \mapsto \theta_k h. \quad (47)$$

For each  $k \in \Lambda$ ,  $\theta_k h \in G_k$  holds, hence  $\theta_- h \in \prod_{\lambda \in \Lambda} G_\lambda$ . Therefore,  $\theta_-: H \rightarrow \prod_{\lambda \in \Lambda} G_\lambda$ . By construction,  $\theta_-$  is entirely given by  $\{\theta_\lambda \mid \lambda \in \Lambda\}$ , so it is unique.

For  $h_1, h_2 \in H$ ,  $\theta_-(h_2 h_1) = (\theta_- h_2)(\theta_- h_1)$  holds since  $\theta_k$  is a group homomorphism and, hence,

$$\theta_k(h_2 h_1) = (\theta_k h_2)(\theta_k h_1) \quad (48)$$

for each  $k \in \Lambda$ .

Let  $k \in \Lambda$ . Then,  $\pi_k \circ \theta_- = \theta_k$  holds, since for each  $h \in H$ ,

$$(\pi_k \circ \theta_-)h = \pi_k(\theta_- h) = (\theta_- h)_k = \theta_k h. \quad (49)$$

□

## 2.2 Weak Direct Products

**Definition 2.2** (Weak Direct Products). Let  $\Lambda$  be a set and suppose that for each  $\lambda \in \Lambda$ , there is a given group  $G_\lambda$ . The weak direct product  $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$  is a subset of  $\prod_{\lambda \in \Lambda} G_\lambda$  of all  $f$  such that  $f_\lambda = 1_{G_\lambda}$  for all but finitely many  $\lambda \in \Lambda$ .

*Remark* (Direct Sums). For additive, i.e., abelian groups  $\{A_\lambda \mid \lambda \in \Lambda\}$ , the corresponding weak direct product is called the direct sum, denoted by  $\sum_{\lambda \in \Lambda} A_\lambda$ .

**Theorem 2.4.** Let  $\Lambda$  be a set and suppose that for each  $\lambda \in \Lambda$ , there is given a group  $G_\lambda$ . Their weak direct product is a normal subgroup of the direct product:

$$\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda. \quad (50)$$

*Proof.* Let  $f, g \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ . There are finite subsets  $I, J \subset \Lambda$  such that  $j \in J$  iff  $f_j \neq 1$ , and  $k \in K$  iff  $g_k \neq 1$ . Consider  $gf^{-1}$ . Recalling  $(gf^{-1})_i = g_i f_i^{-1}$  for each  $i \in \Lambda$ ,

$$\{i \in \Lambda \mid g_i f_i^{-1} \neq 1\} \subset J \cup K. \quad (51)$$

Since  $J \cup K \subset \Lambda$  is finite,  $gf^{-1} \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ . By Lemma 0.2,  $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \subset \prod_{\lambda \in \Lambda} G_\lambda$ .

Suppose  $f \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$  and  $g \in \prod_{\lambda \in \Lambda} G_\lambda$ , and consider  $gfg^{-1}$ . There is a finite subset  $I \subset \Lambda$  such that  $i \in I$  iff  $f_i \neq 1$ . For any  $i \in \Lambda$ ,

$$(gfg^{-1})_i = \begin{cases} 1 & i \in \Lambda - I \\ g_i f_i g_i^{-1} & \text{otherwise.} \end{cases} \quad (52)$$

Therefore,  $(gfg^{-1})_i$  is not 1 at most finitely many  $i \in I$ . Hence,  $gfg^{-1} \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ . □

**Theorem 2.5.** For the weak direct product  $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$  of a set of groups  $\{G_\lambda \mid \lambda \in \Lambda\}$ , the canonical injection  $\iota_k: G_k \rightarrow \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$  defined by

$$(\iota_k a_k)_\lambda := \begin{cases} 1_{G_\lambda} & \lambda \neq k \\ a_k & \lambda = k \end{cases} \quad (53)$$

for  $a_k \in G_k$ , is a mono for each  $k \in \Lambda$ .

*Proof.* Let  $k \in \Lambda$ , and  $a_k, b_k \in G_k$ . For each  $\lambda \in \Lambda$ , we have

$$(\iota_k(b_k a_k))_\lambda = \begin{cases} 1_{G_\lambda} & \lambda \neq k \\ b_k a_k & \lambda = k \end{cases} \quad (54)$$

and

$$(\iota_k b_k)_\lambda (\iota_k a_k)_\lambda = \begin{cases} 1_{G_\lambda} 1_{G_\lambda} & \lambda \neq k \\ b_k a_k & \lambda = k. \end{cases} \quad (55)$$

Hence,  $\iota_k(b_k a_k) = (\iota_k b_k)(\iota_k a_k)$ .

If we suppose  $\iota_k b_k = \iota_k a_k$ , then

$$\forall \lambda \in \Lambda : (\iota_k b_k)_\lambda = (\iota_k a_k)_\lambda \quad (56)$$

In particular, when  $\lambda = k$ , we obtain  $b_k = a_k$ .  $\square$

**Theorem 2.6.** For a set of groups  $\{G_\lambda \mid \lambda \in \Lambda\}$  and each  $k \in \Lambda$ ,  $\iota_k G_k \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$ .

*Proof.* Let  $k \in \Lambda$ . By Theorem 2.5,  $\iota_k$  is a group homomorphism. Hence, its image is a subgroup,  $\iota_k G_k < \prod_{\lambda \in \Lambda} G_\lambda$  by Theorem 0.5.

Let  $h \in \iota_k G_k$ ; there is some  $g_k \in G_k$  such that  $h = \iota_k g_k$ . For each  $\lambda \in \Lambda$ , and  $f \in \prod_{\lambda \in \Lambda} G_\lambda$ ,

$$\pi_\lambda(f h f^{-1}) = f_\lambda \pi_\lambda(\iota_k g_k) f_\lambda^{-1} = \begin{cases} f_k g_k f_k^{-1} & \lambda = k \\ 1 & \text{otherwise.} \end{cases} \quad (57)$$

Recalling  $f_k g_k f_k^{-1} \in G_k$ , we conclude  $f h f^{-1} \in \iota_k G_k$ .  $\square$

**Theorem 2.7.** Let  $\Lambda$  be a set and suppose that for each  $\lambda \in \Lambda$ , there is a given group homomorphism  $\theta_\lambda : G_\lambda \rightarrow H_\lambda$ . Let  $\theta := \prod_{\lambda \in \Lambda} \theta_\lambda$  be a map from  $\prod_{\lambda \in \Lambda} G_\lambda$  to  $\prod_{\lambda \in \Lambda} H_\lambda$ , given by

$$(\theta g)_\lambda := \theta_\lambda g_\lambda \quad (58)$$

for each  $\lambda \in \Lambda$ . Then,  $\theta$  is a group homomorphism,  $\theta(\prod_{\lambda \in \Lambda} G_\lambda) \subset \prod_{\lambda \in \Lambda}^{weak} H_\lambda$ ,  $\theta^{\leftarrow 1} = \prod_{\lambda \in \Lambda} (\theta_\lambda^{\leftarrow 1})$ , and  $\theta(\prod_{\lambda \in \Lambda} G_\lambda) = \prod_{\lambda \in \Lambda} \theta_\lambda G_\lambda$ . Moreover,  $\theta$  is monic iff  $\theta_\lambda$  is monic for each  $\lambda \in \Lambda$ , and  $\theta$  is epic iff  $\theta_\lambda$  is epic for each  $\lambda \in \Lambda$ .

*Proof.* Let us first show  $\theta : \prod_{\lambda \in \Lambda} G_\lambda \rightarrow \prod_{\lambda \in \Lambda} H_\lambda$  is a group homomorphism. Suppose  $f, g \in \prod_{\lambda \in \Lambda} G_\lambda$ , and  $k \in \Lambda$ . Then

$$(\theta(gf))_k = \theta_k(g_k f_k) = (\theta_k g_k)(\theta_k f_k) = (\theta g)_k(\theta f)_k. \quad (59)$$

Since  $k \in \Lambda$  is arbitrary, we conclude  $\theta(gf) = (\theta g)(\theta f)$ .

Next,  $\theta(\prod_{\lambda \in \Lambda} G_\lambda) \subset \prod_{\lambda \in \Lambda}^{weak} H_\lambda$ . Let  $h \in \theta(\prod_{\lambda \in \Lambda} G_\lambda)$ ; there is a  $g \in \prod_{\lambda \in \Lambda} G_\lambda$  such that  $h = \theta g$ , where  $g$  satisfies

$$g_\lambda = 1 \text{ for all but finitely many } \lambda \in \Lambda. \quad (60)$$

There exists a finite subset  $I_h \subset \Lambda$  such that  $i \in I_h$  iff  $g_i \neq 1$ . Then  $h = fg$  satisfies

$$\lambda \in \Lambda - I_h \Rightarrow h_\lambda - (\theta 1)_\lambda = 1. \quad (61)$$

Therefore,  $h_\lambda$  is not 1 at most finitely many  $\lambda \in \Lambda$ . That is,  $h \in \prod_{\lambda \in \Lambda}^{\text{weak}} H_\lambda$ .

The kernel of  $\theta$  is

$$\begin{aligned} \theta^{\leftarrow} 1 &= \left\{ g \in \prod_{\lambda \in \Lambda} G_\lambda \mid \forall \lambda \in \Lambda : \theta_\lambda g_\lambda = 1_{H_\lambda} \right\} \\ &= \left\{ g \in \prod_{\lambda \in \Lambda} G_\lambda \mid \forall \lambda \in \Lambda : g_\lambda \in \theta_\lambda^{\leftarrow} 1 \right\} \\ &= \prod_{\lambda \in \Lambda} (\theta_\lambda^{\leftarrow} 1) \end{aligned} \quad (62)$$

and the image is

$$\begin{aligned} \theta \left( \prod_{\lambda \in \Lambda} G_\lambda \right) &= \left\{ h \in \prod_{\lambda \in \Lambda} H_\lambda \mid \exists g \in \prod_{\lambda \in \Lambda} G_\lambda : h = \theta g \right\} \\ &= \left\{ h : \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} H_\lambda \mid \forall \lambda \in \Lambda : h_\lambda \in H_\lambda \wedge \exists g_\lambda \in G_\lambda : h_\lambda = \theta_\lambda g_\lambda \right\} \\ &= \left\{ h : \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} H_\lambda \mid \forall \lambda \in \Lambda : \exists g_\lambda \in G_\lambda : h_\lambda = \theta_\lambda g_\lambda \right\} \\ &= \left\{ h : \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} H_\lambda \mid \forall \lambda \in \Lambda : h_\lambda \in \theta_\lambda G_\lambda \right\} \\ &= \prod_{\lambda \in \Lambda} \theta_\lambda G_\lambda. \end{aligned} \quad (63)$$

Recall that  $\theta$  is monic iff  $\theta^{\leftarrow} 1 = \{1\}$  by Lemma 0.4. As shown above,  $\theta^{\leftarrow} 1 = \prod_{\lambda \in \Lambda} (\theta_\lambda^{\leftarrow} 1)$ ,  $\theta$  is monic iff

$$\forall \lambda \in \Lambda : \theta_\lambda^{\leftarrow} 1 = \{1\}. \quad (64)$$

Thus,  $\theta$  is monic iff each  $\theta_\lambda$  is monic. As shown above,  $\theta \left( \prod_{\lambda \in \Lambda} G_\lambda \right) = \prod_{\lambda \in \Lambda} \theta_\lambda G_\lambda$ ,  $\theta$  is epic iff  $\theta \left( \prod_{\lambda \in \Lambda} G_\lambda \right) = \prod_{\lambda \in \Lambda} H_\lambda$ , i.e.,

$$\forall \lambda \in \Lambda : \theta_\lambda G_\lambda = H_\lambda. \quad (65)$$

Therefore,  $\theta$  is epic iff each  $\theta_\lambda$  is epic.  $\square$

**Theorem 2.8.** *Let  $\Lambda$  be a set and suppose that for each  $\lambda \in \Lambda$ , there are a given group and its subgroup  $N_\lambda \triangleleft G_\lambda$ . Then*

- $\prod_{\lambda \in \Lambda} N_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$ , and  $(\prod_{\lambda \in \Lambda} G_\lambda) / (\prod_{\lambda \in \Lambda} N_\lambda) \cong \prod_{\lambda \in \Lambda} (G_\lambda / N_\lambda)$ .
- $\prod_{\lambda \in \Lambda}^{\text{weak}} N_\lambda \triangleleft \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ , and  $(\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda) / (\prod_{\lambda \in \Lambda}^{\text{weak}} N_\lambda) \cong \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda / N_\lambda)$ .

*Proof.* Let  $\lambda \in \Lambda$ , and

$$\pi_\lambda : G_\lambda \rightarrow G_\lambda / N_\lambda; g \mapsto gN_\lambda \quad (66)$$

be the canonical epi. By Theorem 1.6, the kernel is  $N_\lambda$ ,  $\pi_\lambda^{\leftarrow} 1 = N_\lambda$ . Hence,  $p := \prod_{\lambda \in \Lambda} \pi_\lambda$  is an epi:

$$p : \prod_{\lambda \in \Lambda} G_\lambda \rightarrow \prod_{\lambda \in \Lambda} (G_\lambda / N_\lambda) \quad (67)$$

with the kernel  $p^{\leftarrow} 1 = \prod_{\lambda \in \Lambda} (\pi_\lambda^{\leftarrow} 1) = \prod_{\lambda \in \Lambda} N_\lambda$  by Theorem 2.7. Since the kernel  $p^{\leftarrow} 1$  is a normal subgroup of  $\prod_{\lambda \in \Lambda} G_\lambda$  by Lemma 1.1, we conclude  $\prod_{\lambda \in \Lambda} N_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$ . Since  $p$  is epic, its image is the codomain,  $p(\prod_{\lambda \in \Lambda} G_\lambda) = \prod_{\lambda \in \Lambda} (G_\lambda / N_\lambda)$ . Applying Theorem 1.8, we conclude

$$\prod_{\lambda \in \Lambda} (G_\lambda / N_\lambda) = p \left( \prod_{\lambda \in \Lambda} G_\lambda \right) \cong \left( \prod_{\lambda \in \Lambda} G_\lambda \right) / (p^{\leftarrow} 1) = \left( \prod_{\lambda \in \Lambda} G_\lambda \right) / \left( \prod_{\lambda \in \Lambda} N_\lambda \right). \quad (68)$$

Let  $p^{\text{weak}}$  be the restriction of  $p$  on  $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$ . By Theorem 2.7,  $p^{\text{weak}}$  is  $(\prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda / N_\lambda))$ -valued:

$$p^{\text{weak}} : \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \rightarrow \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda / N_\lambda). \quad (69)$$

Since the original  $p$  is a group homomorphism, the restriction is also a group homomorphism, since

$$p^{\text{weak}}(gf) = p(gf) = (pg)(pf) = (p^{\text{weak}}g)(p^{\text{weak}}f) \quad (70)$$

for each  $f, g \in \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda / N_\lambda)$ .

Next, we will show  $p^{\text{weak}}$  is an epi. Let  $[g] \in \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda / N_\lambda)$ . Since the original  $p$  is an epi, there is  $g' \in \prod_{\lambda \in \Lambda} G_\lambda$  such that  $pg' = [g]$ , that is

$$\forall \lambda \in \Lambda : g'_\lambda N_\lambda = [g]_\lambda. \quad (71)$$

Since  $[g] \in \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda / N_\lambda)$ ,  $[g]_\lambda = N_\lambda$  for all but finitely many  $\lambda \in \Lambda$ . Hence, we conclude  $g' \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ .

Finally, let us consider the kernel:

$$\begin{aligned} p^{\text{weak}\leftarrow} 1 &= \left\{ g \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \mid p^{\text{weak}}g = 1 \in \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda / N_\lambda) \right\} \\ &= \left\{ g \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \mid g_\lambda N_\lambda = N_\lambda \text{ for all but finitely many } \lambda \in \Lambda \right\} \end{aligned} \quad (72)$$

Since  $g_\lambda N_\lambda = N_\lambda$  iff  $g_\lambda \in N_\lambda$ , we obtain

$$p^{\text{weak} \leftarrow} 1 = \left\{ g \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \mid g_\lambda \in N_\lambda \text{ for all but finitely many } \lambda \in \Lambda \right\} = \prod_{\lambda \in \Lambda}^{\text{weak}} N_\lambda. \quad (73)$$

Therefore, if we apply Theorem 1.8 for the epic  $p^{\text{weak}}: \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \rightarrow \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda/N_\lambda)$  with the kernel  $p^{\text{weak} \leftarrow} 1 = \prod_{\lambda \in \Lambda}^{\text{weak}} N_\lambda$ , we conclude

$$\prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda/N_\lambda) \cong \left( \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \right) / \left( \prod_{\lambda \in \Lambda}^{\text{weak}} N_\lambda \right). \quad (74)$$

□

## 2.3 Direct Sums

Let us begin with the direct sum version of Theorem 2.8:

**Theorem 2.9.** *Let  $\Lambda$  be a set and suppose that for each  $\lambda \in \Lambda$ , there are a given abelian group and its subgroup  $N_\lambda \triangleleft G_\lambda$ . Then,  $(\sum_{\lambda \in \Lambda} G_\lambda) / (\sum_{\lambda \in \Lambda} N_\lambda) \cong \sum_{\lambda \in \Lambda} (G_\lambda/N_\lambda)$ .*

**Theorem 2.10** (Universal Property of Direct Sums). *Let  $B$  be an abelian group and  $\Lambda$  be a set, and suppose that for each  $\lambda \in \Lambda$ , there is given an abelian group homomorphism  $\psi_\lambda: A_\lambda \rightarrow B$ . Then, there exists a unique mediator homomorphism  $\psi: \sum_{\lambda \in \Lambda} A_\lambda \rightarrow B$  such that the following diagram for each  $k \in \Lambda$  is commutative:*

$$\begin{array}{ccc} & B & \\ \exists! \psi \uparrow & \swarrow \psi_k & \\ \sum_{\lambda \in \Lambda} A_\lambda & \xleftarrow{\iota_k} & A_k \end{array} \quad \psi_k = \psi \circ \iota_k. \quad (75)$$

*Proof.* Recall  $a \in \sum_{\lambda \in \Lambda} A_\lambda$  iff  $\pi_\lambda a = 0$  for all but finitely many  $\lambda \in \Lambda$ . For each  $a \in \sum_{\lambda \in \Lambda} A_\lambda$ , there exists a finite subset  $I_a \subset \Lambda$  such that  $\pi_i a \neq 0$  iff  $i \in I_a$ . Define  $\psi: \sum_{\lambda \in \Lambda} A_\lambda \rightarrow B$  by

$$\begin{aligned} \psi 0 &:= 0 \\ \psi a &:= \sum_{i \in I_a} \psi_i (\pi_i a). \end{aligned} \quad (76)$$

Let  $a, a' \in \sum_{\lambda \in \Lambda} A_\lambda$ . Since, if both are zero, so is their sum, namely  $\neg I_{a+a'} \supset (\neg I_a) \cap (\neg I_{a'})$ , we have that

$$I_{a+a'} \subset I_a \cup I_{a'}, \quad (77)$$

where  $\neg I := \Lambda - I$  for any subset  $I \subset \Lambda$ . Hence,

$$\psi(a + a') = \sum_{i \in I_{a+a'}} \psi_i (\pi_i (a + a')) \quad (78)$$

is a sum over finite set  $I_{a+a'}$ . Since it is a finite sum, we can expand the right-hand side:

$$\psi(a + a') = \sum_{i \in I_{a+a'}} \psi_i(\pi_i a) + \sum_{i \in I_{a+a'}} \psi_i(\pi_i a') \quad (79)$$

Let  $i \in I_{a+a'}$ , and consider the first term  $\sum_{i \in I_{a+a'}} \psi_i(\pi_i a)$ .

- If  $\pi_i a' = 0$ ,  $\pi_i a \neq 0$  must be the case. So  $i \in I_a$ .
- Otherwise,  $\pi_i a' \neq 0$ . Therefore, either  $\pi_i a = 0$  or  $\pi_i a \neq 0$  is the case:
  - $\pi_i a' \neq 0$  and  $\pi_i a = 0$  case. The corresponding term in  $\sum_{i \in I_{a+a'}} \psi_i(\pi_i a)$  is zero.
  - $\pi_i a' \neq 0$  and  $\pi_i a \neq 0$  case. Then  $i \in I_a$  is the case.

Thus, the first term is

$$\sum_{i \in I_{a+a'}} \psi_i(\pi_i a) = 0 + \sum_{i \in I_a} \psi_i(\pi_i a) = \psi a. \quad (80)$$

Therefore, we conclude  $\psi(a + a') = \psi a + \psi a'$ .

Let  $j \in \Lambda$  and  $a_j \in A_j$ . Since  $\iota_j a_j \in \sum_{\lambda \in \Lambda} A_\lambda$  and  $I_{\iota_j a_j} = \{j\}$ , see Theorem 2.5, we have

$$\psi(\iota_j a_j) = \sum_{i \in I_{\iota_j a_j}} \psi_i(\pi_i(\iota_j a_j)) = \psi_j a_j. \quad (81)$$

Since  $a_j \in A_j$  is arbitrary, we conclude  $\psi \circ \iota_j = \psi_j$ .

Finally, let us show the uniqueness. Suppose an abelian group homomorphism  $\phi: \sum_{\lambda \in \Lambda} A_\lambda \rightarrow B$  also satisfies

$$\forall \lambda \in \Lambda : \phi \circ \iota_\lambda = \pi_\lambda. \quad (82)$$

Then,  $\phi = \psi$  since

$$\phi a = \phi \sum_{i \in I_a} \iota_i a_i = \sum_{i \in I_a} \phi(\iota_i a_i) = \sum_{i \in I_a} \psi_i a_i = \psi a \quad (83)$$

for all  $a \in \sum_{\lambda \in \Lambda} A_\lambda$ . □

*Remark.* This theorem becomes false if we remove the restriction that the groups are abelian.

Consider  $S_3$  – the symmetric group on a set of three elements:

$$\left\{ \begin{array}{c} () \\ (12), (23), (31) \\ (123), (132), \end{array} \right. \quad (84)$$

where  $()$  denotes the identity, and for example  $(12)$  is  $1 \leftrightarrow 2$  swap,  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \xrightarrow{(12)} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ , and  $(123)$  is a cyclic permutation  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \xrightarrow{(123)} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ .  
 $S_3$  is generated by  $\{(12), (123)\}$ , since

$$\begin{aligned} () &= (12) \circ (12) \\ (23) &= (12) \circ (123) \\ (31) &= (123) \circ (12) \\ (132) &= (123) \circ (123). \end{aligned} \tag{85}$$

Moreover,  $S_3$  is not abelian:

$$(12) \circ (123) \neq (123) \circ (12). \tag{86}$$

Consider the weak product of  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$ , where  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$  are modular arithmetic. Since it is a finite product case, their weak product is the ordinary product  $\mathbb{Z}_2 \times \mathbb{Z}_3$ . Let  $\iota_2$  and  $\iota_3$  be the corresponding canonical injections:

$$\begin{aligned} \iota_2: \mathbb{Z}_2 &\rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3 \\ \iota_3: \mathbb{Z}_3 &\rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3. \end{aligned} \tag{87}$$

Consider the following group homomorphisms

$$\begin{aligned} \phi_2: \mathbb{Z}_2 &\rightarrow S_3 \\ \phi_3: \mathbb{Z}_3 &\rightarrow S_3 \end{aligned} \tag{88}$$

defined by

$$\phi_2 0 = (), \phi_2 1 = (12), \tag{89}$$

and

$$\phi_3 0 = (), \phi_3 1 = (123), \phi_3 2 = (132). \tag{90}$$

It is worth mentioning that  $\phi_3(1+1) = (\phi_3 1) \circ (\phi_3 1) = (123) \circ (123)$ . Suppose, for contradiction, that there is a mediator  $\phi: \mathbb{Z}_2 \times \mathbb{Z}_3 \rightarrow S_3$ . As shown above,  $(12)$  and  $(123)$  generate  $S_3$ , and  $\{(12), (123)\} \subset \phi(\mathbb{Z}_2 \times \mathbb{Z}_3)$ . Hence,  $\phi$  is an epi,  $\phi(\mathbb{Z}_2 \times \mathbb{Z}_3) = S_3$ . Recalling the image forms a subgroup,  $\phi(\mathbb{Z}_2 \times \mathbb{Z}_3) < S_3$ , this equality is a group isomorphism. As shown above,  $S_3$  is non-abelian but  $\phi(\mathbb{Z}_2 \times \mathbb{Z}_3)$  is the image of the product of two abelian groups, so it is abelian, which is absurd.

**Theorem 2.11.** *Let  $G$  be an abelian group,  $\Lambda$  be a set, and suppose that for each  $\lambda \in \Lambda$ , there is a given subgroup  $N_\lambda < G$ . If we further assume each  $g \in G$  has a unique representation  $g = \sum_{\lambda \in \Lambda} g_\lambda$ , where  $g_\lambda \in N_\lambda$  is zero for all but finitely many  $\lambda \in \Lambda$ . Then  $G$  is isomorphic to  $\sum_{\lambda \in \Lambda} N_\lambda$ .*

*Proof.* Since  $G$  is abelian, its subgroup is always normal. Let  $n \in \sum_{\lambda \in \Lambda} N_\lambda$ . Then  $n_\lambda = 0$  for all but finitely many indices, say  $I_n \subset \Lambda$ :

$$i \in I_n \Leftrightarrow n_i \neq 0. \quad (91)$$

Then,  $\sum_{i \in I_n} n_i$  is a finite sum of elements in  $G$ , hence  $\sum_{i \in I_n} n_i \in G$ . Define  $\varphi: \sum_{\lambda \in \Lambda} N_\lambda \rightarrow G; n \mapsto \sum_{i \in I_n} n_i$ .

- $\varphi$  is a group homomorphism.

Let  $f, g \in \sum_{\lambda \in \Lambda} N_\lambda$ . Then  $\varphi(f+g) = \sum_{i \in I_{f+g}} (f+g)_i$ . If we let  $I_{f+g} = \{i_1, \dots, i_k\}$ , we have

$$\varphi(f+g) = (f+g)_{i_1} + \dots + (f+g)_{i_k} = f_{i_1} + \dots + f_{i_k} + g_{i_1} + \dots + g_{i_k}, \quad (92)$$

since they are all members in the abelian group  $G$ . For each  $i \in I_{f+g}$ , either  $f_i = 0$  or  $f_i \neq 0$ :

- If  $f_i = 0$ ,  $i \in \Lambda - I_f$ .
- If  $f_i \neq 0$ ,  $i \in I_f$ .

Therefore,  $\sum_{i \in I_{f+g}} f_i = \sum_{i \in I_f} f_i = \varphi f$ . We conclude  $\varphi(f+g) = \varphi f + \varphi g$ .

- $\varphi$  is epic.

Let  $g \in G$ . By hypothesis, there is a unique representation  $g = \sum_{\lambda \in \Lambda} g_\lambda$ , where  $g_\lambda \in N_\lambda$  is zero for all but finitely many  $\lambda \in \Lambda$ . In other words,  $G$  is a free abelian group with basis  $\bigcup_{\lambda \in \Lambda} N_\lambda$ . Let  $I_g \subset \Lambda$  is the corresponding subset:  $i \in I_g$  iff  $g_i \neq 0$ . Since  $\sum_{i \in I_g} g_i \in \sum_{\lambda \in \Lambda} N_\lambda$  satisfies  $\varphi \sum_{i \in I_g} g_i = g$ ,  $\varphi$  is surjective.

- $\varphi$  is monic.

Consider the kernel:

$$\varphi^{\leftarrow} 0 := \left\{ g \in \sum_{\lambda \in \Lambda} N_\lambda \mid \varphi g = 0 \right\} \quad (93)$$

Let  $z \in \varphi^{\leftarrow} 0$ . By hypothesis, the corresponding unique representation of  $\varphi z = 0$  is given by

$$\forall \lambda \in \Lambda : z_\lambda = 0. \quad (94)$$

It follows  $z = 0$ , namely  $\varphi^{\leftarrow} 0 = \{0\}$ . By Lemma 0.4,  $\varphi$  is monic.

Hence,  $\varphi: \sum_{\lambda \in \Lambda} N_\lambda \rightarrow G$  is an isomorphism.  $\square$