

Jordan Curve Theorem

Ray D. Sameshima

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Contents

0	Abstract	2
1	Preliminaries	3
1.1	Sets and Maps	3
1.1.1	Sets and Maps	3
1.2	Topological Spaces	5
1.2.1	Basic Definitions	5
1.2.2	Separation Axioms	10
1.2.3	Basic Open Sets	10
1.2.4	Continuous Maps	11
1.2.5	Connected Spaces	14
1.2.6	Compact Spaces	16
1.2.7	Product Spaces	17
1.3	Metric Spaces	19
1.3.1	Topological Properties	19
1.3.2	Uniform Continuity and Uniform Limit Theorem	23
2	Complex Analysis 101	27
2.1	Intervals and Curves	27
2.1.1	Real Intervals and Heine-Borel Theorem	27
2.1.2	Curves in \mathbb{C}	30
2.2	Winding Numbers	31
2.3	Boundary-Preserving Maps on Unit Disc	37
2.4	Jordan Curve Theorem	39

Chapter 0

Abstract

In this note, we prove the Jordan curve theorem: a closed curve with no self-intersection in \mathbb{C} divides \mathbb{C} into two connected components – one is unbounded and the other is bounded.

Chapter 1

Preliminaries

1.1 Sets and Maps

We assume some working knowledge of informal set theory including sets and corresponding membership relation \in , subsets, supersets, the empty set \emptyset , union, intersection, set difference, complement, and the like.

1.1.1 Sets and Maps

Definition 1.1.1 (Complement). Let X be a set and $A \subset X$ be a subset. We denote $\neg A = X - A = \{x \in X \mid x \notin A\}$.

Theorem 1.1.1 (Empty Intersection and Empty Union). *Let X be a set, Λ be an index set, and $\{A_\lambda \subset X \mid \lambda \in \Lambda\}$ be a Λ -indexed set of subsets of X . The empty intersection $\bigcap_{\lambda \in \emptyset} A_\lambda$ is the underlying set X and the empty union $\bigcup_{\lambda \in \emptyset} A_\lambda$ is the empty set \emptyset .*

Proof. By definition:

$$\bigcap_{\lambda \in \Lambda} A_\lambda := \{x \in X \mid \forall \lambda \in \Lambda : x \in A_\lambda\}. \quad (1.1)$$

For the empty intersection, the condition is vacuously true. Hence, $\bigcap_{\lambda \in \emptyset} A_\lambda = X$. Similarly:

$$\bigcup_{\lambda \in \Lambda} A_\lambda := \{x \in X \mid \exists \lambda \in \Lambda : x \in A_\lambda\}. \quad (1.2)$$

If the index set is empty, the condition is always false. Hence, $\bigcup_{\lambda \in \emptyset} A_\lambda = \emptyset$. ■

Remark 1. We also have:

$$\neg \bigcap_{\lambda \in \Lambda} A_\lambda := \{x \in X \mid \exists \lambda \in \Lambda : x \notin A_\lambda\} = \bigcup_{\lambda \in \Lambda} \neg A_\lambda \quad (1.3)$$

and

$$\neg \bigcup_{\lambda \in \Lambda} A_\lambda := \{x \in X \mid \forall \lambda \in \Lambda : x \notin A_\lambda\} = \bigcap_{\lambda \in \Lambda} \neg A_\lambda. \quad (1.4)$$

Theorem 1.1.2. *Let X be a set. For $\{V_\alpha \subset X \mid \alpha \in A\}$ and $\{W_\beta \subset X \mid \beta \in B\}$,*

$$\left(\bigcup_{\alpha \in A} V_\alpha \right) \cap \left(\bigcup_{\beta \in B} W_\beta \right) = \bigcup_{(\alpha, \beta) \in A \times B} V_\alpha \cap W_\beta. \quad (1.5)$$

Similarly,

$$\left(\bigcap_{\alpha \in A} V_\alpha \right) \cup \left(\bigcap_{\beta \in B} W_\beta \right) = \bigcap_{(\alpha, \beta) \in A \times B} V_\alpha \cup W_\beta. \quad (1.6)$$

Proof.

$$\begin{aligned} \left(\bigcup_{\alpha \in A} V_\alpha \right) \cap \left(\bigcup_{\beta \in B} W_\beta \right) &= \{x \in X \mid \exists \alpha \in A : x \in V_\alpha\} \\ &\quad \cap \{x \in X \mid \exists \beta \in B : x \in W_\beta\} \\ &= \{x \in X \mid \exists (\alpha, \beta) \in A \times B : x \in V_\alpha \cap W_\beta\} \\ &= \bigcup_{(\alpha, \beta) \in A \times B} V_\alpha \cap W_\beta. \end{aligned} \quad (1.7)$$

Similarly,

$$\begin{aligned} \left(\bigcap_{\alpha \in A} V_\alpha \right) \cup \left(\bigcap_{\beta \in B} W_\beta \right) &= \{x \in X \mid \forall (\alpha, \beta) \in A \times B : x \in V_\alpha \cup W_\beta\} \\ &= \bigcap_{(\alpha, \beta) \in A \times B} V_\alpha \cup W_\beta. \end{aligned} \quad (1.8)$$

■

For a given map $f: X \rightarrow Y$, there are two induced maps:

- Direct image $f: 2^X \rightarrow 2^Y$
- Preimage $f^\leftarrow: 2^Y \rightarrow 2^X$

where $f^\leftarrow W := \{x \in X \mid fx \in W\}$ for any $W \subset Y$.

Theorem 1.1.3 (Properties of Preimage). *Let X and Y be sets and $f: X \rightarrow Y$ be a map. The preimage map f^\leftarrow preserves the following elementary set operations:*

- $f^\leftarrow (\bigcup_{\lambda \in \Lambda} B_\lambda) = \bigcup_{\lambda \in \Lambda} f^\leftarrow B_\lambda$

- $f^{\leftarrow} (\bigcap_{\lambda \in \Lambda} B_\lambda) = \bigcap_{\lambda \in \Lambda} f^{\leftarrow} B_\lambda$
- $f^{\leftarrow} (B_1 - B_2) = f^{\leftarrow} B_1 - f^{\leftarrow} B_2$

where Λ is an arbitrary index set, B_1, B_2, B_λ are all subspaces in Y for each $\lambda \in \Lambda$.

Proof. The first two equations are almost identical:

$$\begin{aligned}
p \in f^{\leftarrow} \left(\bigcup_{\lambda \in \Lambda} B_\lambda \right) &\Leftrightarrow fp \in \bigcup_{\lambda \in \Lambda} B_\lambda \\
&\Leftrightarrow \exists \lambda \in \Lambda : fp \in B_\lambda \\
&\Leftrightarrow \exists \lambda \in \Lambda : p \in f^{\leftarrow} B_\lambda \\
&\Leftrightarrow p \in \bigcup_{\lambda \in \Lambda} f^{\leftarrow} B_\lambda
\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}
p \in f^{\leftarrow} \left(\bigcap_{\lambda \in \Lambda} B_\lambda \right) &\Leftrightarrow fp \in \bigcap_{\lambda \in \Lambda} B_\lambda \\
&\Leftrightarrow \forall \lambda \in \Lambda : fp \in B_\lambda \\
&\Leftrightarrow \forall \lambda \in \Lambda : p \in f^{\leftarrow} B_\lambda \\
&\Leftrightarrow p \in \bigcap_{\lambda \in \Lambda} f^{\leftarrow} B_\lambda
\end{aligned} \tag{1.10}$$

for each $p \in A$.

Recalling $B_1 - B_2 = \{x \in A \mid x \in B_1 \wedge x \in \neg B_2\} = B_1 \cap \neg B_2$, and

$$f^{\leftarrow} (\neg B_2) = \{x \in X \mid fx \in \neg B_2\} = X - f^{\leftarrow} B_2 = \neg f^{\leftarrow} B_2, \tag{1.11}$$

we have

$$\begin{aligned}
f^{\leftarrow} (B_1 - B_2) &= f^{\leftarrow} (B_1 \cap \neg B_2) \\
&= f^{\leftarrow} B_1 \cap f^{\leftarrow} (\neg B_2) \\
&= f^{\leftarrow} B_1 \cap \neg f^{\leftarrow} B_2 \\
&= f^{\leftarrow} B_1 - f^{\leftarrow} B_2.
\end{aligned} \tag{1.12}$$

Thus, the preimage $f^{\leftarrow} : 2^Y \rightarrow 2^X$ preserves union, intersection, and set-difference. ■

1.2 Topological Spaces

1.2.1 Basic Definitions

Definition 1.2.1 (Topological Spaces). Let X be a set. A topology on X is a subset of its subsets $\mathcal{T} \subset 2^X$ that closed under:

- Arbitrary Union

Each union of members in \mathcal{T} is also a member of \mathcal{T} .

- Finite Intersection

Each finite intersection of members of \mathcal{T} is also a member of \mathcal{T} .

As shown in Theorem 1.1.1, the union of an empty family of sets in X is \emptyset , and the intersection of an empty family of sets in X is X . Hence, we may add the following, yet redundant, conditions:

- Both \emptyset and X are members of \mathcal{T} .

The pair (X, \mathcal{T}) is called a topological space. Any member in \mathcal{T} is called an open subset of X . In particular, both \emptyset and X are open subsets in X . A subset $C \subset X$ is called closed iff the complement $\neg C := X - C$ is open, namely $\neg C \in \mathcal{T}$. Since $\emptyset = X - X$ and $X = X - \emptyset$, both \emptyset and X are clopen. Dually, closed subsets are closed under finite union and arbitrary intersections.

Let $Y \subset X$ be a subset of a topological space (X, \mathcal{T}) . The induced topology on Y is

$$\mathcal{T}_Y := \{Y \cap U \mid U \in \mathcal{T}\}. \quad (1.13)$$

The pair (Y, \mathcal{T}_Y) is called a subspace of (X, \mathcal{T}) .

Lemma 1.2.1. *Let (X, \mathcal{T}) be a topological space and $C_1 \subset C_2 \subset X$. If $C_1, C_2 \subset X$ are both closed, then $C_1 \subset C_2$ is closed relative to the subspace topology on C_2 .*

Proof. Let $\neg_2 C_1 := C_2 - C_1$:

$$\neg_2 C_1 = C_2 \cap \neg C_1. \quad (1.14)$$

Since $\neg C_1 \in \mathcal{T}$, i.e., $\neg C_1 \subset X$ is open, $C_2 \cap \neg C_1 \subset C_2$ is open relative to the subspace topology. ■

Definition 1.2.2 (Neighborhoods and Open Subspaces). Let (X, \mathcal{T}) be a topological space, and $p \in X$ be a point. A subspace $U' \subset X$ is called a neighborhood of p iff there exists some $U \in \mathcal{T}$ such that $p \in U$ and $U \subset U'$. Let \mathcal{N}_p be the set of all neighborhoods of p in X relative to \mathcal{T} .

Lemma 1.2.2. *Let (X, \mathcal{T}) be a topological space. A subspace $U \subset X$ is open, $U \in \mathcal{T}$, iff U is a neighborhood of every point in it.*

Proof. (\Rightarrow) Suppose $U \in \mathcal{T}$. Then, for each $p \in U$, U is an open neighborhood of p .

(\Leftarrow) Conversely, suppose U is a neighborhood to its points. For $p \in U$, let $V_p \in \mathcal{T}$ be an open subspace such that $p \in V_p$ and $V_p \subset U$. Then, we conclude $U = \bigcup_{p \in U} V_p$ since:

$$U \subset \bigcup_{p \in U} V_p \subset U. \quad (1.15)$$

Hence U is open. ■

Definition 1.2.3 (Limit Points and Closure). Let $A \subset (X, \mathcal{T})$ be a subspace. A point $p \in X$ is called a limit point of A iff each neighborhood of p contains at least one point of A distinct from p :

$$\forall U' \in \mathcal{N}_p : U' \cap A - \{p\} \neq \emptyset. \quad (1.16)$$

Let A' denote the set of all limit points. We call $\overline{A} := A \cup A'$ the closure of A in X relative to \mathcal{T} .

Lemma 1.2.3. Let $A \subset (X, \mathcal{T})$ be a subspace. For any point $p \in X$, $p \in \overline{A}$ iff

$$\forall U' \in \mathcal{N}_p : U' \cap A \neq \emptyset. \quad (1.17)$$

Proof. (\Rightarrow) Let $p \in \overline{A}$:

- $p \in A$ case

For each neighborhood $U' \in \mathcal{N}_p$, $p \in U' \cap A$.

- $p \notin A$ case

For each neighborhood $U' \in \mathcal{N}_p$, $U' \cap A = U' \cap A - \{p\} \neq \emptyset$ holds.

(\Leftarrow) Let $p \in X$.

- $p \in A$ case

Since $A \subset \overline{A}$, $p \in \overline{A}$.

- $p \notin A$ case

For each $U' \in \mathcal{N}_p$, since $p \in U'$ but $p \notin A$, $p \notin U' \cap A$. Hence, $U' \cap A = U' \cap A - \{p\} \neq \emptyset$, which means p is a limit point of A . ■

Theorem 1.2.1 (Characterization of Closed Subspaces). A subspace $A \subset (X, \mathcal{T})$ is closed iff $A = \overline{A}$.

Proof. (\Rightarrow) Suppose $A \subset (X, \mathcal{T})$ is closed. Then $\neg A \in \mathcal{T}$. Let $p \in \neg A$. Since $\neg A$ is an open neighborhood of p such that $\neg A \cap A = \emptyset$, p is not a limit point of A by Lemma 1.2.3. Therefore $p \notin \overline{A}$. Since $\neg A \subset \neg \overline{A}$ is shown, we obtain $A \supset \overline{A}$; with the inclusion $A \subset \overline{A}$, we conclude $A = \overline{A}$.

(\Leftarrow) Suppose $\overline{A} = A$. We will show $\neg A$ is open. Let $p \in \neg A$. Since $p \in \neg \overline{A}$, p is not a limit point of A . Thus, there is some neighborhood $U' \in \mathcal{N}_p$ with $U' \cap A = \emptyset$ by Lemma 1.2.3. We obtain $U' \subset \neg A$. That is, $\neg A$ is a neighborhood of p . As $p \in \neg A$ is arbitrary, by Lemma 1.2.2, we conclude $\neg A \in \mathcal{T}$. ■

Theorem 1.2.2 (Properties of Closures). Let $A, B \subset (X, \mathcal{T})$ be subspaces.

- The closure \overline{A} is \subset -smallest closed subspace of X containing A :

$$\overline{A} = \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\} \quad (1.18)$$

- $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
- $\overline{\overline{A}} = \overline{A}$, i.e., the closure \overline{A} of A is closed, and the closure-operation is idempotent.
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$
- $\overline{\emptyset} = \emptyset$

Proof. Let $\tilde{A} := \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\}$. Since open subspaces are closed under arbitrary union, the complements, i.e., closed subspaces are closed under arbitrary intersection. Hence, \tilde{A} is closed. To show \tilde{A} is equal to \overline{A} , let us consider their complements:

- ⊂ Let $p \in \neg \tilde{A}$. Since $\neg \tilde{A}$ is an open neighborhood of p such that $\neg \tilde{A} \cap \tilde{A} = \emptyset$, recalling $\tilde{A} \supset A$, we conclude $\neg \tilde{A} \cap A = \emptyset$:

$$\emptyset \subset \neg \tilde{A} \cap A \subset \neg \tilde{A} \cap \tilde{A} = \emptyset. \quad (1.19)$$

Hence, by Lemma 1.2.3, p is not a limit point of A , i.e., $p \in \neg \overline{A}$:

$$\neg \tilde{A} \subset \neg \overline{A}. \quad (1.20)$$

- ⊃ Let $p \in \neg \overline{A}$. Since p is not a limit point of A , there exists an open neighborhood $U \in \mathcal{N}_p \cap \mathcal{T}$ such that $U \cap A - \{p\} = \emptyset$. As p is not in A , $U \cap A = \emptyset$, thus $A \subset \neg U$. Thus, $\neg U$ is a member of the intersection of the right-hand side of (1.18). Hence, we obtain $\tilde{A} \subset \neg U$. Since $p \in U$ and $U \subset \neg \tilde{A}$, we conclude $p \in \neg \tilde{A}$:

$$\neg \tilde{A} \supset \neg \overline{A}. \quad (1.21)$$

Therefore, we obtain $\overline{A} = \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\}$.

- $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
Since any closed subspace containing B also contains A , $\overline{A} \subset \overline{B}$.
- $\overline{\overline{A}} = \overline{A}$
Since \overline{A} is given by an intersection of closed subspaces, \overline{A} is closed. Moreover, $\overline{A} \subset \overline{\overline{A}}$ is the \subset -smallest subspace containing \overline{A} .
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$
 $\overline{A \cup B}$ is closed, and contains both A and B , hence $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. As $\overline{A \cup B}$ is closed, containing $A \cup B$, \subset -smallest property implies $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.
- $\overline{\emptyset} = \emptyset$
Since \emptyset is clopen and $\emptyset \subset \emptyset$, the \subset -smallest property ensures $\overline{\emptyset} = \emptyset$.

■

Remark 2 (Interior and Boundary). Let $A \subset (X, \mathcal{T})$ be a subspace. As a dual concept of closure, the interior A° of A is the \subset -largest open set contained in A :

$$A^\circ = \bigcup \{U \in \mathcal{T} \mid U \subset A\}. \quad (1.22)$$

By Remark 1 in Theorem 1.1.1,

$$\begin{aligned} A^\circ &= \bigcup \{\neg F \in \mathcal{T} \mid \neg F \subset A\} \\ &= \bigcup \{\neg F \subset X \mid \neg F \in \mathcal{T} \wedge F \supset \neg A\} \\ &= \neg \bigcap \{F \subset X \mid \neg F \in \mathcal{T} \wedge F \supset \neg A\} \\ &= \neg \overline{\neg A}. \end{aligned} \quad (1.23)$$

So, a subspace $A \subset X$ is open iff $A = A^\circ$, since $\neg A^\circ = \overline{\neg A}$. We call $\partial A := \overline{A} - A^\circ$ the boundary of A . Moreover, $\partial A = \neg(A^\circ) - \neg(\overline{A})$:

$$\begin{aligned} \neg(A^\circ) - \neg(\overline{A}) &= (X - A^\circ) - (X - \overline{A}) \\ &= \{x \in X \mid x \notin A^\circ \wedge x \notin (X - \overline{A})\} \\ &= \{x \in X \mid x \notin A^\circ \wedge x \in \overline{A}\} \\ &= \overline{A} - A^\circ. \end{aligned} \quad (1.24)$$

Theorem 1.2.3 (Subspaces and Closures). *Let (X, \mathcal{T}) be a topological space and $(Y, \mathcal{T}_Y) \subset (X, \mathcal{T})$ be a subspace. For $A \subset Y$, the closure \overline{A}_Y relative to \mathcal{T}_Y is $Y \cap \overline{A}$, where \overline{A} is the closure of $A \subset X$ relative to \mathcal{T} .*

Proof. It suffices to show $A'_Y = Y \cap A'$ since $\overline{A}_Y = A'_Y \cup A$ and $Y \cap \overline{A} = Y \cup (A \cup A') = (Y \cap A) \cup (Y \cap A') = A \cup (Y \cap A')$.

Let $p \in A'_Y$ and \mathcal{N}_{Yp} be the set of neighborhood of p relative to \mathcal{T}_Y :

$$\forall U' \in \mathcal{N}_{Yp} : \exists U \in \mathcal{T} : p \in (U \cap Y) \subset U'. \quad (1.25)$$

Note that $(U \cap Y) \in \mathcal{T}_Y$ if $U \in \mathcal{T}$. Since $p \in A'_Y$,

$$\forall U' \in \mathcal{N}_{Yp} : U' \cap A - \{p\} \neq \emptyset, \quad (1.26)$$

i.e.,

$$\forall U \in \mathcal{N}_p \cap \mathcal{T} : (U \cap Y) \cap A - \{p\} \neq \emptyset, \quad (1.27)$$

we obtain $p \in (Y \cap A)'$ relative to \mathcal{T} . Recalling $A \subset Y$ and $p \in Y$, we obtain $p \in Y \cap A'$.

Conversely, let $p \in Y \cap A'$ relative to \mathcal{T} :

$$\forall U' \in \mathcal{N}_p : U' \cap A - \{p\} \neq \emptyset. \quad (1.28)$$

Since $A \subset Y$, it is equivalent to

$$\forall U' \in \mathcal{N}_p : U' \cap (A \cap Y) - \{p\} \neq \emptyset. \quad (1.29)$$

Now, $U' \cap Y$ contains an open $(U \cap Y) \in \mathcal{T}_Y$ with $p \in U \cap Y$. That is, $U' \cap Y$ is a neighborhood of p relative to \mathcal{T}_Y , namely $U' \cap Y \in \mathcal{N}_{Yp}$, moreover $p \in A'_Y$.

Hence, we establish $A'_Y = Y \cap A'$, and $\overline{A}_Y = Y \cap \overline{A}$. ■

1.2.2 Separation Axioms

Definition 1.2.4. The following axioms describe how a topology can distinguish points in the underlying set:

T_2 A T_2 space – a Hausdorff space – is a topological space (X, \mathcal{T}) in which each of two distinct points have disjoint neighborhoods, that is, if $p \neq q$, there are $U' \in \mathcal{N}_p$ and $V' \in \mathcal{N}_q$ with $U' \cap V' = \emptyset$.

T_4 A T_4 space is a Hausdorff space in which each disjoint closed subspaces have disjoint neighborhoods.

1.2.3 Basic Open Sets

... we can to an extent preassign the notion of nearness desired. [Dug66]

Definition 1.2.5 (Subbases and Generated Topology). Let X be a set and $\mathcal{S} \subset 2^X$ be a set of subsets in X . As 2^X is a topology of X ,

$$\tau_{\mathcal{S}} := \{\mathcal{T} \subset 2^X \mid \mathcal{T} \text{ is a topology on } X \text{ with } \mathcal{S} \subset \mathcal{T}\} \quad (1.30)$$

is non-empty. Their intersection:

$$\bigcap \tau_{\mathcal{S}} := \bigcap \{\mathcal{T} \in \tau_{\mathcal{S}}\} \quad (1.31)$$

is called the topology generated by \mathcal{S} . It is the \subset -smallest topology containing \mathcal{S} .

For the generated topology, the generating set \mathcal{S} is called the subbasic open set, in short, a subbase.

Remark 3 (Basis). No further conditions for being a subbase of some topology. If \mathcal{S} satisfies:

1. \mathcal{S} covers X

For each $x \in X$, there is a $B \in \mathcal{S}$ with $x \in B$. This condition guarantees that X is open.

2. Binary Intersection

Let $B_1, B_2 \in \mathcal{S}$. If $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{S}$ with $x \in B_3$ and $B_3 \subset B_1 \cap B_2$. This condition guarantees that $B_1 \cap B_2$ is open.

Then \mathcal{S} is called the set of basic open sets, in short, a basis for the topology $\bigcap \tau_{\mathcal{S}}$ of X .

Theorem 1.2.4. Let X be a set, $\mathcal{S} \subset 2^X$ be a basis – \mathcal{S} satisfies both conditions 1 and 2 – and $\mathcal{T}_{\mathcal{S}}$ be the set of all unions of \mathcal{S} . $\mathcal{T}_{\mathcal{S}}$ is a topology on X . Moreover, $\mathcal{T}_{\mathcal{S}} = \bigcap \tau_{\mathcal{S}}$.

Proof. As the condition 1 ensures \mathcal{S} covers X , we have $X \in \mathcal{T}_{\mathcal{S}}$. If we take the empty union, $\emptyset \in \mathcal{T}_{\mathcal{S}}$. By definition, $\mathcal{T}_{\mathcal{S}}$ is closed under arbitrary union. The condition 2 guarantees $\mathcal{T}_{\mathcal{S}}$ is closed under binary, hence any finite intersection. Therefore, $\mathcal{T}_{\mathcal{S}}$ forms a topology on X .

Since $\mathcal{S} \subset \mathcal{T}_{\mathcal{S}}$ holds, $\mathcal{T}_{\mathcal{S}} \in \tau_{\mathcal{S}}$, hence $\bigcap \tau_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{S}}$. To show the other inclusion, let $U \in \mathcal{T}_{\mathcal{S}}$. By construction, there exists $\mathcal{B}_U \subset \mathcal{S}$ with

$$U = \bigcup \mathcal{B}_U = \bigcup \{V \in \mathcal{B}_U\}. \quad (1.32)$$

As $\mathcal{B}_U \subset \mathcal{S}$, and any member $T \in \tau_{\mathcal{S}}$ contains \mathcal{S} , we obtain $\mathcal{B}_U \subset T$ for each $T \in \tau_{\mathcal{S}}$. Thus, $\mathcal{B}_U \subset T$ holds for each $T \in \tau_{\mathcal{S}}$. I.e., $U \in \bigcap \tau_{\mathcal{S}}$. ■

1.2.4 Continuous Maps

For given topological space (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) , and a map between the underlying sets $f: X \rightarrow Y$, we use f^{\leftarrow} to associate the topology since f^{\leftarrow} preserves the elementary set operations as shown in Theorem 1.1.3:

Definition 1.2.6 (Continuous Maps). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f: X \rightarrow Y$ is called continuous iff the preimage of each open subspace in Y is open in X . That is, f^{\leftarrow} maps $\mathcal{T}_Y \subset 2^Y$ into \mathcal{T}_X :

$$f^{\leftarrow}: \mathcal{T}_Y \rightarrow \mathcal{T}_X. \quad (1.33)$$

The set of all continuous maps from X to Y is denoted by $C^0(X, Y)$.

Theorem 1.2.5 (Characterizations of Continuity). *Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, and $f: X \rightarrow Y$ be a map. The following are equivalent:*

1. $f \in C^0(X, Y)$ by means of Definition 1.2.6.
2. For a subbase (or a basis) $\mathcal{S}_Y \subset \mathcal{T}_Y$, $f^{\leftarrow} \mathcal{S}_Y \subset \mathcal{T}_X$.
3. The preimage of a closed subspace in Y is closed in X .
4. For each $x \in X$ and for each neighborhood $V' \in \mathcal{N}_{f_x}$, there exists a neighborhood $U' \in \mathcal{N}_x$ such that $fU' \subset V'$.
5. $f\overline{A} \subset \overline{fA}$ for every $A \subset X$.
6. $\overline{f^{\leftarrow}B} \subset f^{\leftarrow}\overline{B}$ for every $B \subset Y$.

Remark 4 (ϵ - δ). The condition 4 is the topological version of ϵ - δ definition of continuity.

Proof. $(1 \Leftrightarrow 2)$ As $\mathcal{S}_Y \subset \mathcal{T}_Y$, $f^{\leftarrow}|_{\mathcal{S}_Y}: \mathcal{S}_Y \rightarrow \mathcal{T}_X$. Conversely, suppose $f^{\leftarrow} \mathcal{S}_Y \subset \mathcal{T}_X$ is the case. Let $W \in \mathcal{T}_Y$. Since \mathcal{T}_Y is generated by \mathcal{S}_Y , W is given by some, not necessarily finite, union of finite intersections of members in \mathcal{S}_Y :

$$W = \bigcup_{\lambda \in \Lambda} \left(B_1^{(\lambda)} \cap \cdots \cap B_{j_\lambda}^{(\lambda)} \right), \quad (1.34)$$

where $B_1^{(\lambda)} \cdots B_{j_\lambda}^{(\lambda)} \in \mathcal{S}_Y$ for each $\lambda \in \Lambda$. Applying Theorem 1.1.3, we obtain

$$f^{\leftarrow} W = \bigcup_{\lambda \in \Lambda} f^{\leftarrow} (B_1^{(\lambda)} \cap \cdots \cap B_{j_\lambda}^{(\lambda)}) = \bigcup_{\lambda \in \Lambda} (f^{\leftarrow} B_1^{(\lambda)}) \cap \cdots \cap (f^{\leftarrow} B_{j_\lambda}^{(\lambda)}). \quad (1.35)$$

Since $(f^{\leftarrow} B_1^{(\lambda)}) \cap \cdots \cap (f^{\leftarrow} B_{j_\lambda}^{(\lambda)}) \in \mathcal{T}_X$ and W is a union of such open subspaces in X , we conclude $f^{\leftarrow} W \in \mathcal{T}_X$.

(1 \Leftrightarrow 3) By Theorem 1.1.3,

$$f^{\leftarrow} (\neg A) = f^{\leftarrow} (Y - A) = X - f^{\leftarrow} A = \neg f^{\leftarrow} A \quad (1.36)$$

for every $A \subset X$.

(1 \Rightarrow 4) Let $x \in X$, $V' \in \mathcal{N}_{fx}$, and $V \in \mathcal{T}_Y$ s.t., $fx \in V$ and $V \subset V'$. As f is continuous, $f^{\leftarrow} V \in \mathcal{T}_X$. Since $x \in f^{\leftarrow} V$, we may set $U' = f^{\leftarrow} V$.

(4 \Rightarrow 5) Let $A \subset X$ and $x \in \overline{A}$; we will show fx is a member of \overline{fA} . Consider $V' \in \mathcal{N}_{fx}$; as we assume 4, there exists $U' \in \mathcal{N}_x$ with $fU' \subset V'$. Since $x \in \overline{A}$, by Lemma 1.2.3, $U' \cap A \neq \emptyset$ holds. Hence, $fx \in \overline{fA}$:

$$\emptyset \subsetneq f(U' \cap A) \subset fU' \cap fA \subset V' \cap fA. \quad (1.37)$$

(5 \Rightarrow 6) Let $B \subset Y$ and $A := f^{\leftarrow} B$. As we assume 5,

$$f(\overline{f^{\leftarrow} B}) = f\overline{A} \subset \overline{fA} = \overline{f(f^{\leftarrow} B)} \subset \overline{B}. \quad (1.38)$$

Thus, $\overline{f^{\leftarrow} B} \subset f^{\leftarrow} \overline{B}$.

(6 \Rightarrow 3) Let $B \subset Y$ be a closed subspace. As we assume 6, $\overline{f^{\leftarrow} B} \subset f^{\leftarrow} \overline{B}$. Since $\overline{B} = B$, we conclude $\overline{f^{\leftarrow} B} = f^{\leftarrow} B$:

$$\overline{f^{\leftarrow} B} \subset f^{\leftarrow} \overline{B} \subset f^{\leftarrow} B \subset \overline{f^{\leftarrow} B}. \quad (1.39)$$

See Theorem 1.2.1. ■

Lemma 1.2.4 (Universal Property of Relative Topology). *Let $Y \subset (X, \mathcal{T})$ be a subspace. The relative topology \mathcal{T}_Y defined in Definition 1.2.1 can be characterized as the \subset -smallest topology on Y for which the inclusion map:*

$$i: Y \hookrightarrow X; y \mapsto y \quad (1.40)$$

is continuous, namely $i \in C^0(Y, X)$.

Proof. Let \mathcal{T}_Y' be an arbitrary topology on Y . Suppose $i: Y \hookrightarrow X$ is continuous relative to (X, \mathcal{T}) and (Y, \mathcal{T}_Y') . We will show that $\mathcal{T}_Y' \supset \mathcal{T}_Y$.

Let $U \in \mathcal{T}$. As $i \in C^0((Y, \mathcal{T}_Y'), (X, \mathcal{T}))$, the preimage $i^{\leftarrow} U$ is open in (Y, \mathcal{T}_Y') :

$$i^{\leftarrow} U = U \cap Y \in \mathcal{T}_Y'. \quad (1.41)$$

Since U is arbitrary, it follows that any open subspace in Y relative to \mathcal{T}_Y , $U \cap Y \in \mathcal{T}_Y$ is a member of \mathcal{T}_Y' , hence $\mathcal{T}_Y \subset \mathcal{T}_Y'$. ■

Theorem 1.2.6 (Properties of Continuous Maps). *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y), (Z, \mathcal{T}_Z)$ be topological spaces.*

- *If $f \in C^0(X, Y)$ and $g \in C^0(Y, Z)$, the composition $gf \in C^0(X, Z)$.*
- *If $f \in C^0(X, Y)$ and $A \subset X$, the restriction $f|_A : A \rightarrow Y$ is continuous relative to the relative topology on A .*
- *If $f \in C^0(X, Y)$, the corstriction of f on its image is continuous:*

$$f \in C^0(X, fX). \quad (1.42)$$

Proof. Suppose $f \in C^0(X, Y)$, $g \in C^0(Y, Z)$, and $A \subset X$.

- Since $f^\leftarrow : \mathcal{T}_Y \rightarrow \mathcal{T}_X$ and $g^\leftarrow : \mathcal{T}_Z \rightarrow \mathcal{T}_Y$, and $(g \circ f)^\leftarrow = f^\leftarrow \circ g^\leftarrow$, the continuity of the composition $g \circ f$ follows:

$$(g \circ f)^\leftarrow : \mathcal{T}_Z \rightarrow \mathcal{T}_X. \quad (1.43)$$

- Let $i : A \hookrightarrow X$. Since

$$f|_A = f \circ i \quad (1.44)$$

and as shown above $i \in C^0(A, X)$ relative to \mathcal{T}_A , the composition is continuous.

- For each $V \in \mathcal{T}_V$, i.e., for each open subspace $V \cap fX$ in fX ,

$$f^\leftarrow(V \cap fX) = f^\leftarrow V \cap f^\leftarrow(fX) = f^\leftarrow V. \quad (1.45)$$

Since $f^\leftarrow V$ is open in X , the restriction $f : X \rightarrow fX$ is continuous.

■

Definition 1.2.7 (Homeomorphisms and Topological Invariance). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f : X \rightarrow Y$ is called a homeomorphism – a topological isomorphism – iff the following conditions hold:

- The underlying map $f : X \rightarrow Y$ is bijective.
- Both f and f^{-1} are continuous.

If f is a homeomorphism, it is denoted by $f : X \cong Y$. Two spaces X and Y are homeomorphic, written $X \cong Y$, iff there is a homeomorphism between them. It is worth mentioning that a homeomorphism $f : X \cong Y$ is an open map – the image of an open subspace $U \in \mathcal{T}_X$ along f is open $fU \in \mathcal{T}_Y$, since f^{-1} is continuous. Moreover, a homeomorphism $f : X \cong Y$ is a bijection for the underlying set and the associated topologies:

$$\begin{aligned} f : X &\cong Y \\ f^{-1} : \mathcal{T}_Y &\cong \mathcal{T}_X \end{aligned} \quad (1.46)$$

Thus, any topological property about X is mapped to that of Y . We call any property of spaces a topological invariant iff whenever it is true for one space, it is also varied for every homeomorphic space.

Theorem 1.2.7. *Homeomorphism is an equivalence relation in the class of all topological spaces.*

Proof. Observe:

- Reflexive
For any topological space X , $1_X: X \cong X$.
- Symmetric
If $f: X \cong Y$, $Y \cong X$ via f^{-1} .
- Transitive
If $f: X \cong Y$ and $g: Y \cong Z$, then $g \circ f: X \cong Z$.

See Theorem 1.2.6. ■

1.2.5 Connected Spaces

Definition 1.2.8 (Connectedness). A topological space is disconnected iff it is given by the union of two nonempty disjoint open subspaces: a topological space is connected iff it is not disconnected. A subspace is connected iff it is connected relative to its subspace topology. We call a connected open space a domain.

Theorem 1.2.8 (Characteristics of Connectedness). *For a topological space (X, \mathcal{T}) , TFAE:*

1. (X, \mathcal{T}) is connected.
2. The only clopen subspaces of (X, \mathcal{T}) is \emptyset and X .
3. Any $f \in C^0(X, \mathbf{2})$ is constant, where $\mathbf{2}$ is the two points set $\{0, 1\}$ with discrete topology $\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

Proof. (1 \Rightarrow 2) Suppose (X, \mathcal{T}) is a connected space. Let $A \subset X$ be a non-empty clopen subspace of (X, \mathcal{T}) . Then X is expressed as $A \cup \neg A$ of the disjoint union of open subspaces. Since X is connected and $A \neq \emptyset$, $\neg A$ must be empty.

(2 \Rightarrow 3) Assume (X, \mathcal{T}) has only two clopen subspaces \emptyset and X . Let $f \in C^0(X, \mathbf{2})$. Suppose, for contradiction, that f is not constant. Then, both $f^{-1}\{0\}$ and $f^{-1}\{1\}$ are non-empty. Moreover, $\neg f^{-1}\{0\} = f^{-1}\{1\} \neq \emptyset$ implies $f^{-1}\{0\}$ is clopen such that $\emptyset \subsetneq f^{-1}\{0\} \subsetneq X$, which is absurd.

(3 \Rightarrow 1) Assume no continuous non-constant map exists from X to $\mathbf{2}$. Suppose, for contradiction, that (X, \mathcal{T}) is disconnected, i.e., there exists a clopen non-empty subspace $\emptyset \subsetneq A \subsetneq X$. Define $f: X \rightarrow \mathbf{2}$ by $f|_A = 1$ and otherwise zero. By definition, f is non-constant, since $\neg A \neq \emptyset$. Hence $f^{-1}\emptyset = \emptyset$ and $f^{-1}\{0, 1\} = X$. Moreover, both $f^{-1}\{0\} = \neg A$ and $f^{-1}\{1\} = A$ are open. Therefore, such a non-constant f is continuous, which is absurd. ■

Theorem 1.2.9. *The continuous image of a connected space is connected.*

Proof. Let X be a connected space, Y be a topological space, and $f \in C^0(X, Y)$. Suppose, for contradiction, that the continuous image fX is disconnected. By Theorem 1.2.8, there exists a non-constant continuous $g \in C^0(fX, \mathbf{2})$. It follows $g(fX) = \{0, 1\}$. The $g \circ f: X \rightarrow \mathbf{2}$ is continuous by Theorem 1.2.6. Hence, it follows that $(g \circ f)X = \{0, 1\}$ is a non-constant continuous map on a connected space X , which is absurd. ■

Theorem 1.2.10. *Let X be a topological space and $A \subset X$ be a connected subspace. Then, any $B \subset X$ satisfying $A \subset B \subset \overline{A}$ is also connected; particularly, the closure of connected subspace is connected.*

Proof. Let $f \in C^0(B, \mathbf{2})$. Since A is connected, $f|_A \in C^0(A, \mathbf{2})$ becomes constant by Theorem 1.2.6. Let $\{n\} := f|_A \subset \{0, 1\}$; relative to the topology on $\mathbf{2}$, such a singleton $\{n\} \subset \mathbf{2}$ is clopen. Since $B \subset \overline{A}$, we have $B = \overline{A} \cap B$. As shown in Theorem 1.2.3, $\overline{A} \cap B = \overline{A}_B$, we conclude $B = \overline{A}_B$. Since f is continuous, we may apply Theorem 1.2.5 for the relative topology \mathcal{T}_B :

$$fB = f\overline{A}_B \subset \overline{fA} = \overline{\{n\}} = \{n\} = fA. \quad (1.47)$$

Therefore, $f|_B$ is also constant, and hence B is connected by Theorem 1.2.8. ■

Theorem 1.2.11. *For a set of connected spaces, if they share at least one common point, their union is also connected.*

Proof. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a set of connected spaces, and $x \in \bigcap_{\lambda \in \Lambda} X_\lambda$. Consider $f \in C^0(\bigcup_{\lambda \in \Lambda} X_\lambda, \mathbf{2})$. Let $\lambda \in \Lambda$. By Theorem 1.2.6:

$$f|_{X_\lambda} \in C^0(X_\lambda, \mathbf{2}). \quad (1.48)$$

Since X_λ is connected, $f|_{X_\lambda}$ is constant; since $x \in X_\lambda$, $f|_{X_\lambda} x = fx$. Hence, f is constant. By Theorem 1.2.8, we conclude $\bigcup_{\lambda \in \Lambda} X_\lambda$ is connected. ■

Definition 1.2.9 (Connected Components). Let X be a topological space and $x \in X$. The component C_x of x in X is the union of all connected subspaces in X containing x . In other words, C_x is \subset -largest connected subspace in Y containing x . By Theorem 1.2.8, $C_x \subset X$ is a closed subset, because both C_x and $\overline{C_x}$ are connected and its \subset -largest property $\overline{C_x} \subset C_x$ with the trivial inclusion $C_x \subset \overline{C_x}$.

Theorem 1.2.12. *Let X be a topological space. The union of any set of connected subspaces in X having at least one point in common is connected. Hence, the component C_x is connected for each $x \in X$.*

Proof. Let $C := \bigcup_{\lambda \in \Lambda} A_\lambda$ be the union of connected subspace in X and $a \in \bigcap_{\lambda \in \Lambda} A_\lambda$ is a common point. Consider an arbitrary continuous map $f \in C^0(C, \mathbf{2})$. Let $\lambda \in \Lambda$. Since A_λ is connected, the restriction $f|_{A_\lambda}$ is constant by Theorem 1.2.8. Since $a \in A_\lambda$, we obtain $fx = fa$ for each $x \in A_\lambda$. Thus $f|_{A_\lambda} = f(a)$ holds. Since $\lambda \in \Lambda$ is arbitrary, we conclude that f is constant. ■

Theorem 1.2.13. *Let X be a topological space. The set of all distinct components in X forms a partition of X .*

Proof. Let $x, y \in X$. If $C_x \cap C_y \neq \emptyset$, by Theorem 1.2.12, their union $C_x \cup C_y$ is connected. Since $C_x \subset C_x \cup C_y$ and C_x is \subset -largest connected subset containing x , we conclude $C_x = C_x \cup C_y = C_y$. Hence, if $C_x \neq C_y$, then they are disjoint $C_x \cap C_y = \emptyset$. ■

1.2.6 Compact Spaces

Definition 1.2.10 (Open Covers). Let (X, \mathcal{T}) be a topological space and $Y \subset X$ be a subspace. Any set of subspaces $\{A_\lambda \subset X \mid \lambda \in \Lambda\}$ is called a cover of Y iff $Y \subset \bigcup_{\lambda \in \Lambda} A_\lambda$. If a cover $\{A_\lambda \mid \lambda \in \Lambda\}$ consists of open subspaces of X , we call it an open cover.

For a cover $\{A_\lambda \mid \lambda \in \Lambda\}$ of Y , a subcover is a subset $\{A_\lambda \mid \lambda \in \Lambda'\}$, $\Lambda' \subset \Lambda$, that is also a cover of Y .

Definition 1.2.11 (Compact Spaces). A topological space (X, \mathcal{T}) is compact iff each open cover has a finite subcover.

Theorem 1.2.14. *The continuous image of a compact space is compact.*

Proof. Let (X, \mathcal{T}_X) be a compact space, (Y, \mathcal{T}_Y) be a topological space, and $f \in C^0(X, Y)$. Consider an arbitrary open cover $\mathcal{V} \subset \mathcal{T}_Y$ of fX . Then $\{f^{-1}V \mid V \in \mathcal{V}\}$ is an open cover of X ; for every $x \in X$, $fx \in Y$ is covered by some $V \in \mathcal{V}$:

$$x \in f^{-1}V. \quad (1.49)$$

Since X is compact, there exists a finite subcover $X \subset f^{-1}V_1 \cup \dots \cup f^{-1}V_t$. We have the desired finite subcover $\{V_1, \dots, V_t\} \subset \mathcal{V}$, since for each $x \in X$, as $x \in f^{-1}V_s$ for some $s \in \{1, \dots, t\}$, it follows $fx \in V_s$. ■

Theorem 1.2.15. *A closed subspace of a compact space is compact.*

Proof. Let (X, \mathcal{T}_X) be a compact space and $C \subset X$ be a closed subspace. Consider an open cover $\mathcal{U} \subset \mathcal{T}_X$ of C . Since $\neg C \subset X$ is open, we have an open cover of X :

$$\mathcal{U} \cup \{\neg C\}. \quad (1.50)$$

Since X is compact, there is a finite subcover $\{U_1, \dots, U_n\} \subset \mathcal{U} \cup \{\neg C\}$. Since it also covers $C \subset X$, we have the desired finite subcover of C , namely C is covered by $\{U_1, \dots, U_n\} - \{\neg C\}$. ■

Theorem 1.2.16. *A compact subspace of a Hausdorff space is closed.*

Proof. Let (X, \mathcal{T}) be a Hausdorff space and $K \subset X$ be a compact subspace. If $K = X$, $X \subset X$ is clopen. So, suppose $K \subsetneq X$, and let $x \in \neg K$. For each $y \in K$, as $x \neq y$, there are disjoint open subspaces $U_y, V_y \in \mathcal{T}$ such that

$$x \in U_y \wedge y \in V_y. \quad (1.51)$$

Then the open cover $\{V_y \mid y \in K\}$ has a finite subcover:

$$K \subset V := V_{y_1} \cup \cdots \cup V_{y_n}. \quad (1.52)$$

Define $U := U_{y_1} \cap \cdots \cap U_{y_n}$. Both U and V are open in X . Moreover, $U \cap V = \emptyset$, since, if $z \in V$, there is y_p with $z \in V_{y_p}$ but $z \notin U_{y_p} \supset U$. Since $K \subset V$, U and K are disjoint, namely

$$U \subset \neg K. \quad (1.53)$$

Since $x \in U$, we conclude that $\neg K$ is a neighborhood of x . By Lemma 1.2.2, $\neg K \subset X$ is open. ■

Theorem 1.2.17. *A continuous bijection from a compact space to a Hausdorff space is homeomorphic.*

Proof. Let (K, \mathcal{T}_K) be a compact space, (X, \mathcal{T}_X) be a Hausdorff space, and $f \in C^0(K, X)$. Suppose there is a map $g: X \rightarrow K$ with $gf = 1_K$ and $fg = 1_X$. We will show g is continuous. Let $V \in \mathcal{T}_K$. Consider $\neg V := K - V$ of the corresponding closed subspace in K . By Theorem 1.2.15, $\neg V$ is a compact subspace in K ; its continuous image $f\neg V$ is a compact subspace in X . By Theorem 1.2.15, such a compact subspace $f\neg V$ is closed. Now

$$g^\leftarrow \neg V = \{x \in X \mid gx \in \neg V\} = \{x \in X \mid x = fgx \in f\neg V\} = f\neg V \quad (1.54)$$

implies $g^\leftarrow \neg V \subset X$ is closed. By the condition 3 in Theorem 1.2.5, we conclude g is continuous. ■

1.2.7 Product Spaces

Let $\Lambda \neq \emptyset$ be an index set and $\{X_\lambda \mid \lambda \in \Lambda\}$ be a Λ -indexed set of sets. The Cartesian product of $\{X_\lambda \mid \lambda \in \Lambda\}$:

$$\prod_{\lambda \in \Lambda} X_\lambda \quad (1.55)$$

is given by the set of all maps $\{f: \Lambda \rightarrow \bigcup_{\lambda \in \Lambda} X_\lambda \mid \forall \lambda \in \Lambda : f\lambda \in X_\lambda\}$. For instance, $\prod_{\lambda \in \{1,2\}} X_\lambda = X_1 \times X_2$ is given by

$$\{f: \{1,2\} \rightarrow X_1 \cup X_2 \mid f1 \in X_1 \wedge f2 \in X_2\} \quad (1.56)$$

i.e., each member in $X_1 \times X_2$ is essentially a pair (x_1, x_2) , where $x_1 = f1 \in X_1$ and $x_2 = f2 \in X_2$.

There is a natural projection for each $\alpha \in \Lambda$:

$$p_\alpha: \prod_{\lambda \in \Lambda} X_\lambda \rightarrow X_\alpha; f \mapsto f_\alpha. \quad (1.57)$$

Definition 1.2.12 (Product Topologies). Let $\Lambda \neq \emptyset$ be an index set and $\{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in \Lambda\}$ be a Λ -indexed set of topological spaces. For the Cartesian product of the underlying sets $\prod_{\lambda \in \Lambda} X_\lambda$, the topology generated by the following subbase:

$$\bigcup_{\alpha \in \Lambda} \{p_\alpha \leftarrow U \mid U \in \mathcal{T}_\alpha\} \quad (1.58)$$

is called the product topology; with this product topology, we call $\prod_{\lambda \in \Lambda} X_\lambda$ the product space.

Let us consider finite products of topological spaces and compactness.

Theorem 1.2.18. *Let $X \times Y$ be a product of topological spaces. If $X \times Y$ is compact relative to the product topology, then X is also compact.*

Proof. Let $\mathcal{U} \subset \mathcal{T}_X$ be an open cover of X . For each $U \in \mathcal{U}$, consider

$$p_X \leftarrow U = U \times Y. \quad (1.59)$$

Since $p_X \leftarrow U$ is a subbasic open subspace in $X \times Y$, it is open. Then $\{p_X \leftarrow U \mid U \in \mathcal{U}\}$ forms an open cover of the compact $X \times Y$. Therefore, there is a finite subcover:

$$X \times Y = p_X \leftarrow U_1 \cup \cdots \cup p_X \leftarrow U_n. \quad (1.60)$$

Hence, $\{U_1, \dots, U_n\}$ is the desired finite subcover. ■

Theorem 1.2.19 (Finite Tychonoff Theorem). *The product of finite compact spaces is compact.*

Proof. We will show the binary case; let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be compact spaces. Let \mathcal{O} be an open cover of $X \times Y$ and $x \in X$. Since \mathcal{O} covers $X \times Y$:

$$\forall y \in Y : \exists O_{(x,y)} \in \mathcal{O} : (x, y) \in O_{(x,y)}. \quad (1.61)$$

Since $O_{(x,y)} \subset X \times Y$ is open relative to the product topology, there are $U_{(x,y)} \in \mathcal{T}_X$ and $V_{(x,y)} \in \mathcal{T}_Y$ such that

$$(x, y) \in U_{(x,y)} \times V_{(x,y)} \subset O_{(x,y)}, \quad (1.62)$$

where $U_{(x,y)} \times V_{(x,y)}$ is $p_X \leftarrow U_{(x,y)} \cap p_Y \leftarrow V_{(x,y)} = (U_{(x,y)} \times Y) \cap (X \times V_{(x,y)})$. Now $\{V_{(x,y)} \mid y \in Y\}$ covers Y ; there is a finite subcover:

$$Y = V_{(x,y_{j(x,1)})} \cup \cdots \cup V_{(x,y_{j(x,m_x)})}. \quad (1.63)$$

Define:

$$U_x := U_{(x,y_{j(x,1)})} \cap \cdots \cap U_{(x,y_{j(x,m_x)})}. \quad (1.64)$$

Since it is a finite intersection of open subspaces in X , $U_x \in \mathcal{T}_X$. Moreover, U_x is an open neighborhood of x .

We, then, have an open cover of X , $\{U_x \mid x \in X\}$. There exists a finite subcover:

$$X = U_{x_1} \cup \cdots \cup U_{x_n}. \quad (1.65)$$

Consider a finite subset of \mathcal{O} :

$$\{O_{(x,y)} \mid x \in \{x_1, \dots, x_n\}, y \in \{y_{j(x,1)}, \dots, y_{j(x,m_x)}\}\}. \quad (1.66)$$

Note that the indices for y varies as $x \in \{x_1, \dots, x_n\}$. We will show that it is the desired finite subcover of $X \times Y$.

Let $(\xi, \eta) \in X \times Y$. Since (1.65) holds, there is some x_p with $\xi \in U_{x_p}$. For such x_p , since

$$Y = V_{(x_p, y_{j(x_p,1)})} \cup \cdots \cup V_{(x_p, y_{j(x_p, m_{x_p})})} \quad (1.67)$$

there is some $y_{j(x_p, i)}$ with $\eta \in V_{(x_p, y_{j(x_p, i)})}$. For the given pair (ξ, η) , we conclude:

$$(\xi, \eta) \in U_{x_p} \times V_{(x_p, y_{j(x_p, i)})} \subset O_{(x_p, y_{j(x_p, i)})}. \quad (1.68)$$

Hence, (1.66) is the desired finite subcover of $X \times Y$. ■

1.3 Metric Spaces

1.3.1 Topological Properties

Definition 1.3.1 (Metrics and Metric Spaces). Let X be a non-empty set. A metric on X is a real-valued map $d: X \times X \rightarrow \mathbb{R}$ that satisfies the following conditions:

- Non-negative:
For every $x, y \in X$, $d(x, y) \geq 0$.
- Distinguishable:
For every $x, y \in X$, $d(x, y) = 0$ iff $x = y$.
- Symmetric:
For every $x, y \in X$, $d(x, y) = d(y, x)$.
- Triangle Inequality:
For each triple points,

$$d(x, z) \leq d(x, y) + d(y, z). \quad (1.69)$$

We call $d(x, y)$ the distance between two points x and y in X . For a non-empty subset $A \subset X$ and $x \in X$, define the distance between A and x by

$$d(A, x) := \inf \{d(a, x) \mid a \in A\}, \quad (1.70)$$

where \inf stands for the greatest lower bound. Since the possible minimum value of the metric d is zero, $d(A, x) \geq 0$ for each $x \in X$.

Remark 5 (Metric Spaces). Let X be a non-empty set and d be a metric on X . Consider the set of open balls:

$$\mathcal{B}_d := \{B_\epsilon(x) \mid \epsilon > 0 \wedge x \in X\}, \quad (1.71)$$

where

$$B_\epsilon(x) := \{y \in X \mid d(x, y) < \epsilon\}. \quad (1.72)$$

Lemma 1.3.1. *The set of all open balls in X forms a basis.*

Proof. Let X be a set, d be a metric on X , \mathcal{B}_d is the set of all open balls in X defined above. Recalling Definition 1.2.5, we will show that \mathcal{B}_d satisfies the conditions in Remark 3:

1. Since $X \subset \bigcup_{x \in X} B_1(x)$, \mathcal{B}_d covers X .
2. Let $\epsilon_1 > 0, \epsilon_2 > 0$, and $x_1, x_2 \in X$. Consider $B_1 := B_{\epsilon_1}(x_1)$ and $B_2 := B_{\epsilon_2}(x_2)$. Suppose $B_1 \cap B_2 \neq \emptyset$. Let $x \in B_1 \cap B_2$. Define

$$\epsilon := \min\{\epsilon_1 - d(x_1, x), \epsilon_2 - d(x_2, x)\}. \quad (1.73)$$

Let $y \in B_\epsilon(x)$:

$$d(y, x_1) \leq d(y, x) + d(x, x_1) < \epsilon + d(x, x_1) = \epsilon_1. \quad (1.74)$$

Therefore, we obtain $x_1 \in B_1$; dually $x_2 \in B_2$ as well:

$$y \in B_1 \cap B_2. \quad (1.75)$$

We conclude $B_\epsilon(x) \subset B_1 \cap B_2$.

Hence, \mathcal{B}_d forms a basis of a topology on X . ■

With this generated topology, the set X with a metric d forms a topological space. The pair (X, d) is called a metric space with the generated topology.

Remark 6. As an important example of metric space, consider \mathbb{C} of the complex plane, where the metric is induced by the standard Euclid norm:

$$|z| := \sqrt{(\Re z)^2 + (\Im z)^2} \quad (1.76)$$

Lemma 1.3.2. *For two complex numbers z and w , they are equal iff for every $\epsilon > 0$, $|z - w| < \epsilon$ holds.*

Proof. (\Rightarrow) Suppose $z = w$. Then $|z - w| = 0$. Therefore, for every $\epsilon > 0$, $|z - w| < \epsilon$.

(\Leftarrow) Conversely, suppose $z \neq w$. Then, $\epsilon := |z - w| > 0$. Hence, $|z - w| \leq \epsilon$ holds. ■

Lemma 1.3.3. *A metric is continuous.*

Proof. Let (X, d) be a metric space:

$$d: X \times X \rightarrow \mathbb{R}. \quad (1.77)$$

For each $x, x', y, y' \in X$, the triangle inequality $d(x', y) \leq d(x', y') + d(y, y')$ implies:

$$-d(x', y') \leq d(y, y') - d(x', y). \quad (1.78)$$

Hence,

$$d(x, y) - d(x', y') \leq d(x, x') + d(x', y) - d(x', y') \leq d(x, x') + d(y, y'). \quad (1.79)$$

Similarly, $d(x', y') - d(x, y) \leq d(x', x) + d(y', y)$ holds, and

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y'). \quad (1.80)$$

As $(x', y') \rightarrow (x, y)$ i.e., $d(x, x') \rightarrow 0$ and $d(y, y') \rightarrow 0$, we conclude $d(x', y') \rightarrow d(x, y)$. ■

Theorem 1.3.1. *Let (X, d) be a metric space and $A \subset X$ be a non-empty subspace. For each point $p \in X$, $p \in \overline{A}$ iff $d(A, p) = 0$, where \overline{A} is the closure of $A \subset (X, d)$ relative to the topology generated by d via B_d .*

Proof. (\Rightarrow) Suppose $p \in \overline{A}$. Let $\epsilon > 0$. Since $B_\epsilon(p)$ is an open neighborhood around p ,

$$B_\epsilon(p) \cap A - \{p\} \neq \emptyset \quad (1.81)$$

by Definition 1.2.3. We may select $q \in B_\epsilon(p) \cap A - \{p\}$. Since $q \in A$, and $d(A, p)$ is a lower bound of $\{d(a, p) \mid a \in A\}$:

$$d(A, p) \leq d(q, p) < \epsilon. \quad (1.82)$$

Recalling $\epsilon > 0$ is arbitrary and $d(A, p) \geq 0$, by Lemma 1.3.2, we conclude $d(A, p) = 0$.

(\Leftarrow) Consider the complement $\neg \overline{A} = X - \overline{A}$. If $\neg \overline{A} = \emptyset$, nothing has to be proven. Let $p \in \neg \overline{A}$. Since $\neg \overline{A} \subset X$ is open, there is $\epsilon > 0$ such that

$$B_\epsilon(p) \subset \neg \overline{A}. \quad (1.83)$$

For each $a \in A$, since $a \notin B_\epsilon(p)$, $d(a, p) \geq \epsilon$. That is, $\epsilon > 0$ is a lower bound of $\{d(a, p) \mid a \in A\}$:

$$d(A, p) \geq \epsilon > 0. \quad (1.84)$$

Hence, $d(A, p) \neq 0$ if $p \notin \overline{A}$. ■

Theorem 1.3.2. *Metric spaces are T_4 spaces.*

Proof. Let (X, d) be a metric space.

First, we will show (X, d) is a Hausdorff space. Suppose x and y are distinct points in X . Since $x \neq y$,

$$\epsilon := d(x, y) > 0. \quad (1.85)$$

We will show $B_{\epsilon/2}(x) \cap B_{\epsilon/2}(y) = \emptyset$. Suppose, for contradiction, that there exists $p \in B_{\epsilon/2}(x) \cap B_{\epsilon/2}(y)$. Then:

$$\epsilon = d(x, y) \leq d(x, p) + d(p, y) < \epsilon/2 + \epsilon/2 = \epsilon, \quad (1.86)$$

which is absurd.

Consider two non-empty disjoint closed subspaces $F_1, F_2 \subset X$. Let $p \in F_1$. Since $\overline{F_1} = F_1$, by Theorem 1.3.1, $d(F_2, p) > 0$. Define $\delta_p := \frac{1}{3}d(F_2, p)$ and $U_p := B_{\delta_p}(p)$, and

$$G_1 := \bigcup_{p \in F_1} U_p. \quad (1.87)$$

Similarly, $G_2 := \bigcup_{q \in F_2} V_q$, where $\delta_q := \frac{1}{3}d(F_1, q) > 0$ and $V_q := B_{\delta_q}(q)$. By definition, both $G_1 \supset F_1$ and $G_2 \supset F_2$, and they are open in X . We will show G_1 and G_2 are disjoint. Suppose, for contradiction, that there is an $r \in G_1 \cap G_2$. Then, there are some $p \in F_1$ and $q \in F_2$ such that $r \in B_{\delta_p}(p) \cap B_{\delta_q}(q)$. Without loss of generality, $\delta_p \leq \delta_q$:

$$3\delta_p = d(F_1, q) \leq d(p, q) \leq d(p, r) + d(r, q) < \delta_p + \delta_q \leq 2\delta_p, \quad (1.88)$$

which is absurd. ■

Theorem 1.3.3. *Let (X, d) be a metric space and $A \subset X$ be a non-empty subspace. The distance $d(A, \cdot) : X \rightarrow \mathbb{R}$ is continuous.*

Proof. Let $p, q \in X$ and $a \in A$:

$$d(A, p) \leq d(a, p) \leq d(a, q) + d(q, p) \quad (1.89)$$

Therefore, $d(A, p) - d(q, p) \leq d(a, q)$, meaning that $d(A, p) - d(q, p)$ is a lower bound of $\{d(a, q) \mid a \in A\}$:

$$d(A, p) - d(q, p) \leq d(A, q). \quad (1.90)$$

Swapping $p \leftrightarrow q$, we obtain $d(A, q) - d(p, q) \leq d(A, p)$:

$$|d(A, p) - d(A, q)| \leq d(p, q) \quad (1.91)$$

As $q \dashrightarrow p$, i.e., as $d(p, q) \dashrightarrow 0$, $|d(A, p) - d(A, q)| \dashrightarrow 0$.

Formally speaking, for any $\epsilon > 0$, there is a $\delta > 0$ for instance, $\delta := \frac{\epsilon}{2}$ such that

$$|d(A, p) - d(A, q)| \leq d(p, q) < \epsilon \quad (1.92)$$

for any $q \in B_\delta(p)$. By the condition 4 in Theorem 1.2.5, $d(A, \cdot)$ is continuous at $p \in X$. ■

Remark 7 (Lipschitz Continuous). Given two metric spaces X and \mathbb{R} , (1.91) implies $d(A, \cdot)$ is Lipschitz continuous with Lipschitz constant is equal to 1.

1.3.2 Uniform Continuity and Uniform Limit Theorem

Definition 1.3.2 (Uniformly Continuous Maps). A map $f: X \rightarrow Y$ between metric spaces is called uniformly continuous iff for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(fp, fq) < \epsilon \quad (1.93)$$

for each $p, q \in X$ for which $d_X(p, q) < \delta$.

Theorem 1.3.4 (Heine-Cantor Theorem). *A continuous map between two metric spaces is uniformly continuous if the domain space is compact.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f \in C^0(X, Y)$. Suppose (X, d_X) is compact. Let $\epsilon > 0$. For each $x \in X$, since f is continuous, there exists $\delta_x > 0$ such that

$$f(B_{\delta_x}(x)) \subset B_{\epsilon/2}(fx) \quad (1.94)$$

see the condition 4 in Theorem 1.2.5. Since $\{B_{\delta_x/2}(x) \mid x \in X\}$ is an open covering of the given compact space X , there exists a finite subcover:

$$X = B_{\delta_{x_1}/2}(x_1) \cup \cdots \cup B_{\delta_{x_k}/2}(x_k). \quad (1.95)$$

Define $\delta_0 > 0$:

$$\delta_0 := \min \left\{ \frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_k}}{2} \right\}. \quad (1.96)$$

Let $p \in X$; there is some $l \in \{1, \dots, k\}$ such that $p \in B_{\delta_{x_l}/2}(x_l)$. For each $q \in B_{\delta_0}(p)$, namely $d_X(p, q) < \delta_0$:

$$d_X(q, x_l) \leq d_X(q, p) + d_X(p, x_l) < \delta_0 + \frac{\delta_{x_l}}{2} \leq \delta_{x_l}. \quad (1.97)$$

That is, both p and q are in $B_{\delta_{x_l}}(x_l)$. Then, the images fp and fq are in $B_{\epsilon/2}(fx_l)$, hence

$$d_Y(fp, fq) \leq d_Y(fp, fx_l) + d_Y(fx_l, fq) < \frac{\epsilon}{2} + \frac{\epsilon}{2}. \quad (1.98)$$

Since p is arbitrary for preassigned $\epsilon > 0$, we conclude that f is uniformly continuous. ■

Definition 1.3.3 (Uniform Convergence). Let X be a set, (Y, d) be a metric space,

$$\{f_n: X \rightarrow Y \mid n \in \mathbb{N}\} \quad (1.99)$$

be a \mathbb{N} -index set of maps. As a sequence, $\{f_n \mid n \in \mathbb{N}\}$ converges uniformly to a limit f_∞ iff for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$n \geq N \Rightarrow \forall x \in X : d(f_n(x), f_\infty(x)) < \epsilon. \quad (1.100)$$

Theorem 1.3.5 (Uniform Limit Theorem). *Let X be a topological space, (Y, d) be a metric space,*

$$\{f_n: X \rightarrow Y \mid n \in \mathbb{N}\} \quad (1.101)$$

be a sequence of maps converging uniformly to $f_\infty: X \rightarrow Y$. If $\{f_n: X \rightarrow Y \mid n \in \mathbb{N}\}$ is a sequence of continuous maps, then the limit f_∞ is continuous.

Proof. Let $x \in X$. For a given sequence $\{f_n \in C^0(X, Y) \mid n \in \mathbb{N}\}$, we will show that the limit is continuous at x . Consider $\epsilon > 0$.

Since $f_n \dashrightarrow f_\infty$ uniformly as $n \dashrightarrow \infty$, for any $t \in X$, there is some $N_t \in \mathbb{N}$ such that

$$n \geq N_t \Rightarrow d(f_n t, f_\infty t) < \frac{\epsilon}{3}. \quad (1.102)$$

For $n \geq N_x$, since $f_n \in C^0(X, Y)$, there is some neighborhood $U \in \mathcal{N}_x$ such that

$$\forall y \in U : |f_n x - f_n y| < \frac{\epsilon}{3}. \quad (1.103)$$

Let $y \in U$. If $n \geq \max\{N_x, N_y\}$,

$$d(f_\infty x, f_\infty y) \leq d(f_\infty x, f_n x) + d(f_n x, f_n y) + d(f_n y, f_\infty y) < \epsilon. \quad (1.104)$$

Hence as $y \dashrightarrow x$ relative to the topology on X , $f_\infty y \dashrightarrow f_\infty x$. ■

Theorem 1.3.6 (Special Case of Tietze-Urysohn Theorem). *Let (X, d) be a metric space, $F_0, F_1 \subset X$ be non-empty closed subspaces. If F_0 and F_1 are disjoint, then there exists a continuous map $f \in C^0(S, [0, 1])$ such that $f|_{F_0} = 0$ and $f|_{F_1} = 1$.*

Proof. Since $F_0 \cap F_1 = \emptyset$,

$$g := d(F_0, \cdot) + d(F_1, \cdot) \quad (1.105)$$

is continuous and positive definite. Define

$$fp := \frac{d(F_0, p)}{g(p)} = \frac{d(F_0, p)}{d(F_0, p) + d(F_1, p)} \quad (1.106)$$

We will show that f is continuous. For $p, q \in X$,

$$\begin{aligned} fq - fp &= \frac{(d(F_0, p) + d(F_1, p)) d(F_0, q) - d(F_0, p) (d(F_0, q) + d(F_1, q))}{g(p)g(q)} \\ &= \frac{d(F_1, p) (d(F_0, q) - d(F_0, p)) + d(F_0, p) (d(F_1, p) - d(F_1, q))}{g(p)g(q)} \end{aligned} \quad (1.107)$$

By Theorem 1.3.3, we conclude that as $q \dashrightarrow p$, $fq \dashrightarrow fp$. ■

Corollary 1.3.6.1. *With a scaling and a shift, we obtain $\tilde{f} \in C^0(S, [a, b])$:*

$$\tilde{f}x := (b - a)fx + a \quad (1.108)$$

for $a < b$.

Lemma 1.3.4 (Special Case of Tietze's Extension Theorem). *Let (X, d) be a metric space, $F \subset X$ be a closed subspace, and $g \in C^0(F, [-1, 1])$. There exists a continuous extension of g , that is, an $f \in C^0(X, [-1, 1])$ exists such that $f|_F = g$.*

Proof. For closed intervals $[-1, -\frac{1}{3}]$ and $[\frac{1}{3}, 1]$, their preimages:

$$F_{0-} := g^{\leftarrow} \left[-1, -\frac{1}{3} \right], F_{0+} := g^{\leftarrow} \left[\frac{1}{3}, 1 \right] \quad (1.109)$$

are closed in X , see the condition 3 Theorem 1.2.5. Moreover, they are disjoint.

Applying Theorem 1.3.6, there exists

$$f_0 \in C^0 \left(X, \left[-\frac{1}{3}, \frac{1}{3} \right] \right) \quad (1.110)$$

such that $f_0|_{F_{0-}} = -\frac{1}{3}$ and $f_0|_{F_{0+}} = +\frac{1}{3}$. By definition,

$$\forall x \in X : |f_0 x| \leq \frac{1}{3}. \quad (1.111)$$

Since

$$F = \underbrace{g^{\leftarrow} \left[-1, -\frac{1}{3} \right]}_{F_{0-}} \cup g^{\leftarrow} \left[-\frac{1}{3}, \frac{1}{3} \right] \cup \underbrace{g^{\leftarrow} \left[\frac{1}{3}, 1 \right]}_{F_{0+}} \quad (1.112)$$

we conclude $|gx - f_0 x| \leq \frac{2}{3}$ for each $x \in F$:

- $x \in F_{0-}$ case

Since $-1 \leq gx \leq -\frac{1}{3}$ and $f_0 x = -\frac{1}{3}$,

$$-\frac{2}{3} \leq gx - f_0 x \leq 0. \quad (1.113)$$

- $x \in g^{\leftarrow} \left[-\frac{1}{3}, \frac{1}{3} \right]$ case

Since both $-\frac{1}{3} \leq gx, f_0 x \leq +\frac{1}{3}$,

$$-\frac{2}{3} \leq gx - f_0 x \leq \frac{2}{3}. \quad (1.114)$$

- $x \in F_{0+}$ case

Since $\frac{1}{3} \leq gx \leq 1$ and $f_0 x = +\frac{1}{3}$,

$$0 \leq gx - f_0 x \leq \frac{2}{3}. \quad (1.115)$$

Define $g_1 := g - f_0$. As shown above $g_1 \in C^0(F, [-\frac{2}{3}, \frac{2}{3}])$. For

$$F = g_1 \leftarrow \left[-\frac{2}{3}, -\frac{2}{3}\frac{1}{3}\right] \cup g_1 \leftarrow \left[-\frac{2}{3}\frac{1}{3}, \frac{2}{3}\frac{1}{3}\right] \cup g_1 \leftarrow \left[\frac{2}{3}\frac{1}{3}, \frac{2}{3}\right] \quad (1.116)$$

by Theorem 1.3.6, there exists $f_1 \in C^0(X, [-\frac{2}{3}, \frac{2}{3}])$ with

$$\forall x \in F : |g_1x - f_1x| = \left|gx - \sum_{j=0}^1 f_jx\right| \leq \left(\frac{2}{3}\right)^2 \quad (1.117)$$

We can continue this process so that for each $n \in \mathbb{N}$,

$$f_n \in C^0\left(X, \left[-\left(\frac{2}{3}\right)^n \frac{1}{3}, \left(\frac{2}{3}\right)^n \frac{1}{3}\right]\right) \quad (1.118)$$

such that

$$\forall x \in F : \left|gx - \sum_{j=0}^n f_jx\right| \leq \left(\frac{2}{3}\right)^n \quad (1.119)$$

Since $\{f_n \mid n \in \mathbb{N}\}$ is a sequence of bounded maps such that

$$\left\|\sum_{j=0}^n f_j\right\| \leq \sum_{j=0}^n \|f_j\| \leq \sum_{j=0}^n \left(\frac{2}{3}\right)^j \frac{1}{3} < 1, \quad (1.120)$$

the limit $\lim_{n \rightarrow \infty} \sum_{j=0}^n f_j = \sum_{n \in \mathbb{N}} f_n$ exists, where $\|f\| := \sup_{x \in X} |fx|$. Moreover, it is a uniform limit of continuous functions on X ,

$$\sum_{n \in \mathbb{N}} f_n \in C^0(X, [-1, 1]). \quad (1.121)$$

By (1.119), $\sum_{j=0}^n f_j \dashrightarrow g$ as $n \dashrightarrow \infty$ on F :

$$\sum_{n \in \mathbb{N}} f_n \Big|_F = g. \quad (1.122)$$

Hence, $\sum_{n \in \mathbb{N}} f_n$ is the desired continuous extension of g on X . ■

Chapter 2

Complex Analysis 101

We assume some working knowledge of real numbers, particularly the existence of least upper bound: if a subspace $A \subset \mathbb{R}$ of real numbers is non-empty and bounded above, then it has a least upper bound.

2.1 Intervals and Curves

2.1.1 Real Intervals and Heine-Borel Theorem

Definition 2.1.1 (Real Intervals). For $a, b \in \mathbb{R}$, let

$$[a, b] := \{(1 - t)a + tb \mid t \in [0, 1]\}. \quad (2.1)$$

We call $[a, b]$ a real closed interval.

Theorem 2.1.1. *A real closed interval $[a, b] \subset \mathbb{R}$ is connected.*

Proof. Let $F \subset [a, b]$ be a closed proper subinterval:

$$\emptyset \neq F \neq [a, b]. \quad (2.2)$$

We will show that F is not open.

Let $x \in F$ and $y \in \neg F$. Without loss of generality, consider $x < y$ case. Define $F_{<y} := \{t \in F \mid t < y\}$; as $x \in F_{<y}$ and $F_{<y}$ is bounded above, we may set:

$$z := \sup F_{<y}. \quad (2.3)$$

Then $x \leq z \leq y$, since y is an upper bound of $F_{<y}$ and z is the least upper bound.

For any $\epsilon > 0$, $B_\epsilon(z) \cap F \neq \emptyset$, i.e., $z \in \overline{F}$, otherwise any number in $(z - \epsilon, z)$ would be an upper bound of $F_{<y}$, which contradicts the very definition of z . Recalling $F \subset [a, b]$ is closed, we conclude $z \in F$. Therefore, $z < y$. Since the open interval (z, y) does not meet F , $(z, y) \cap F = \emptyset$, for each $\epsilon > 0$, $B_\epsilon(z) \not\subset F$. In other words, F is not a neighborhood of z ; hence, F can not be an open

subspace of $[a, b]$. It follows that no clopen proper subinterval in $[a, b]$. By Theorem 1.2.8, $[a, b] \subset \mathbb{R}$ is connected. ■

Theorem 2.1.2. *A real closed interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. Let \mathcal{O} be an open cover of $[a, b]$. Define

$$S := \{x \in [a, b] \mid [a, x] \text{ is finitely covered by } \mathcal{O}\} \quad (2.4)$$

- S is not empty
 $a \in [a, b]$ is covered by at least one $U \in \mathcal{O}$:

$$[a, a] = \{a\} \subset U. \quad (2.5)$$

Hence, $a \in S$.

- $S \subset [a, b]$ is open
Let $x \in S$ and $\{V_1, \dots, V_n\} \subset \mathcal{O}$ be the finite subcover of $[a, x]$. Since $x \in [a, b]$ is covered by some open $V \in \mathcal{O}$, there exists $\epsilon > 0$ such that:

$$B_\epsilon(x) \subset V, \quad (2.6)$$

where $B_\epsilon(x) := (x - \epsilon, x + \epsilon)$. We will show that $B_\epsilon(x) \subset S$. Let $y \in B_\epsilon(x)$. Since $y \in V$, we have a finite subcover $\{V_1, \dots, V_n, V\}$ of $[a, y]$. Hence, $y \in S$. By Theorem 1.2.2, $S \subset [a, b]$ is open.

- $S \subset [a, b]$ is closed
Let $x \in \overline{S}$. Since $\overline{S} \subset [a, b]$, x is in some open $W \in \mathcal{O}$:

$$x \in W. \quad (2.7)$$

Hence, there is $\epsilon > 0$ with $B_\epsilon(x) \subset W$. Since $x \in \overline{S}$:

$$B_\epsilon(x) \cap S \neq \emptyset. \quad (2.8)$$

There exists, thus, some $y \in B_\epsilon(x) \cap S$ such that $[a, y]$ is finitely covered:

$$[a, y] \subset W_1 \cup \dots \cup W_k. \quad (2.9)$$

Then $[a, x]$ is covered by $\{W_1, \dots, W_k, W\}$, hence $x \in S$. With the trivial inclusion $S \subset \overline{S}$, we conclude $S = \overline{S}$ by Theorem 1.2.1.

As shown, $S \subset [a, b]$ is non-empty and clopen. Since $[a, b] \subset \mathbb{R}$ is connected by Theorem 2.1.1, we conclude $S = [a, b]$. Hence, $[a, b]$ is compact. ■

Theorem 2.1.3 (Heine-Borel Theorem). *Let n be a positive integer. A subspace $K \subset \mathbb{R}^n$ is compact iff it is bounded and closed.*

Proof. (\Rightarrow) Since \mathbb{R}^n is furnished with the standard metric d , \mathbb{R}^n is a Hausdorff space. Thus, if $K \subset \mathbb{R}^n$ is compact, it is closed by Theorem 1.2.16. Consider $\{B_1(x) \mid x \in K\}$ of the set of unit open balls. Since it is an open cover of the compact subspace $K \subset \mathbb{R}^n$, there is a finite subcover:

$$K \subset B_1(x_1) \cup \cdots \cup B_1(x_n). \quad (2.10)$$

Define $M := \max\{|x_1|, \dots, |x_n|\}$. For each $x \in K$, there is some x_p with $x \in B_1(x_p)$:

$$|x| = d(0, x) \leq d(0, x_p) + d(x_p, x) < M + 1. \quad (2.11)$$

Hence, $K \subset B_{M+1}(0)$ i.e., K is bounded.

Conversely, suppose $K \subset \mathbb{R}^n$ is bounded and closed. Since K is bounded, there is $\mu > 0$ with

$$K \subset [-\mu, \mu]^n. \quad (2.12)$$

As shown in Theorem 2.1.2, $[-\mu, \mu] \subset \mathbb{R}$ is compact; by Theorem 1.2.19, the product $[-\mu, \mu]^n$ is a compact subspace in \mathbb{R}^n . By Lemma 1.2.1, since $K \subset [-\mu, \mu]^n$ is closed. By Theorem 1.2.15, the closed subspace $K \subset [-\mu, \mu]^n$ of a compact subspace $[-\mu, \mu]^n \subset \mathbb{R}^n$ is a compact subspace in \mathbb{R}^n . ■

Theorem 2.1.4 (Extreme Value Theorem). *A real valued continuous map f on a compact space K is bounded, and there are $p, q \in K$ such that $fp = \sup_{x \in K} fx$ and $fq = \inf_{x \in K} fx$.*

Proof. Let $f \in C^0(K, \mathbb{R})$ be a continuous map on a compact space K . The image $fK \subset \mathbb{R}$ is compact by Theorem 1.2.14; by Theorem 2.1.3, fK is bounded in \mathbb{R} . Let $M := \sup_{x \in K} fx$. Suppose, for contradiction, that there is no point x on K so that $fx = M$, namely for each $x \in K$, $fx < M$. Then $x \mapsto \frac{1}{M-fx}$ is continuous on K , hence $\frac{1}{M-f}$ is bounded. Let $\epsilon > 0$. There must be some $x_\epsilon \in K$ with $M - \epsilon < fx_\epsilon$, otherwise $M - \epsilon$ would be an upper bound of fK . Hence, $\frac{1}{M-fx_\epsilon} > \frac{1}{\epsilon}$, which means $\frac{1}{M-f}$ is not bounded. ■

Corollary 2.1.4.1. *For a subspace $A \subset \mathbb{C}$, define*

$$\delta A := \sup\{|a - b| \mid a, b \in A\} \quad (2.13)$$

If A is compact, there are $x, y \in A$ with $\delta A = |x - y| < \infty$.

Proof. Let

$$f: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}; (x, y) \mapsto |x - y| \quad (2.14)$$

be the standard metric on \mathbb{C} . By Lemma 1.3.3, f is continuous. If $A \subset \mathbb{C}$ is compact, the product $A \times A$ is also compact by Theorem 1.2.19. Hence, $f|_{A \times A}$ is bounded. Applying Theorem 2.1.4, f has maximum, namely there are $x, y \in A$ with $\delta A = f(x, y) = |x - y|$. ■

2.1.2 Curves in \mathbb{C}

Definition 2.1.2 (Curves and Complex Intervals). Let X be a topological space. A curve in X is a continuous map from some closed interval, namely $\gamma \in C^0([a, b], X)$. We call $\gamma(a)$ the initial point of γ , and $\gamma(b)$ the final point of γ . A closed curve is a curve $\gamma \in C^0([a, b], X)$ with $\gamma(a) = \gamma(b)$. Let $[\gamma] := \gamma[a, b]$ be the image in X of a curve $\gamma \in C^0([a, b], X)$. For a pair of complex numbers $z, w \in \mathbb{C}$, we denote $[w, z] := \{(1-t)w + tz \mid t \in [0, 1]\}$.

Theorem 2.1.5. *The image of a closed curve in \mathbb{C} is compact.*

Proof. Let $\gamma \in C^0([a, b], \mathbb{C})$ be a closed curve, $\gamma a = \gamma b$. By Theorem 1.2.14 and Theorem 2.1.2, the continuous image $[\gamma] \subset \mathbb{C}$ is compact. ■

Theorem 2.1.6. *Let $r > 0$ and $x \in \mathbb{C}$. Both $B_r(x)$ and its complement $\neg B_r(x) = \mathbb{C} - B_r(x)$ are connected.*

Proof. Consider $y \in B_r(x)$ and $[x, y] = \{(1-t)x + ty \mid y \in [0, 1]\}$. Let $p = (1-t)x + ty \in [x, y]$. Since

$$|p - x| = |-tx + ty| = |t| |x - y| \leq |x - y| < r, \quad (2.15)$$

$p \in B_r(x)$. It follows $[x, y] \subset B_r(x)$. Hence,

$$B_r(x) = \bigcup_{y \in B_r(x)} [x, y] \quad (2.16)$$

and each complex interval shares the center x in common, by Theorem 1.2.11, we conclude $B_r(x)$ is connected.

The complement $\neg B_r(x)$ is given by:

$$\{z \in \mathbb{C} \mid |z - x| \geq r\} = C \cup \bigcup_{\theta \in [0, 2\pi]} J_\theta, = \bigcup_{\theta \in [0, 2\pi]} C \cup J_\theta, \quad (2.17)$$

where $C := \partial B_r(x) = \{z \in \mathbb{C} \mid |z - x| = r\}$ and $J_\theta := \{x + t \exp \sqrt{-1}\theta \mid t \geq r\}$. Now, C is the image of a continuous map $\gamma_0 \in C^0([0, 2\pi], \mathbb{C})$:

$$\gamma_0 \theta = \exp \sqrt{-1}\theta. \quad (2.18)$$

Hence, $C = [\gamma]$ is connected, since its domain $[0, 1] \subset \mathbb{R}$ is connected. Similarly, J_θ is also connected for each $\theta \in [0, 1]$ with $C \cap J_\theta = \{r \exp \sqrt{-1}\theta\}$. By Theorem 1.2.11, $C \cup J_\theta$ is connected for each $\theta \in [0, 2\pi]$. Therefore, we conclude $\bigcup_{\theta \in [0, 2\pi]} C \cup J_\theta$ is connected. ■

Definition 2.1.3 (Path-Connectedness). A topological space is called path-connected iff each pair of points can be jointed by a curve.

Lemma 2.1.1. *Each path-connected space is connected.*

Proof. Let X be a path-connected space and $x \in X$. For each $y \in X$, there exists $\gamma_y \in C^0([0, 1], X)$ such that $\gamma_y 0 = x$ and $\gamma_y 1 = y$. Since each γ_y is connected by Theorem 1.2.9, sharing the initial point $\gamma_y 0 = x$,

$$X = \bigcup_{y \in X} [\gamma_y] \quad (2.19)$$

is connected by Theorem 1.2.11. ■

Theorem 2.1.7. *Let X be a topological space. TFAE:*

1. *Each path-component is open.*
2. *Each point of X has a path-connected open neighborhood.*

Proof. (1 \Rightarrow 2) Each point belongs to some path-component. By 1, such a path-component is open, and therefore, it is an open neighborhood of its points.

(2 \Rightarrow 1) Let K be a path-component of X , and $x \in K$. By 2, there is an open and path-connected $U \subset Y$ with $x \in U \subset Y$. By the \subset -largest property of K , $K \subset K \cup U$ implies $U \subset K$. By Lemma 1.2.2, K is open. ■

Remark 8. Let K be a path-component of X . Since $\neg K = X - K$ is given by the union of other open path-components, $\neg K \subset X$ is open. Namely, a path-component of X is clopen.

Theorem 2.1.8. *A topological space is path-connected iff it is connected and each point has a path-connected open neighborhood.*

Proof. (\Rightarrow) Let X be a path-connected space. As shown in Lemma 2.1.1, X is connected, and hence X is clopen. Then, X itself is a path-connected open neighborhood of its points.

(\Leftarrow) Let X be a connected topological space in which each point has a path-connected open neighborhood. Each path-component is open, and hence closed in X . Since X is connected, such a clopen subspace must be X itself. ■

Corollary 2.1.8.1. *An open subspace in \mathbb{R}^n , in particular in \mathbb{C} , is connected iff it is path-connected.*

Proof. Let $U \subset \mathbb{C}$ be an open subspace. Each point $x \in U$ has $\epsilon > 0$ with $B_\epsilon(x) \subset U$. Recall $B_\epsilon(x)$ is path-connected, see the proof in Theorem 2.1.6, via Theorem 2.1.8, the connectedness of $U \subset \mathbb{C}$ is equivalent to the path-connectedness of U . ■

2.2 Winding Numbers

Definition 2.2.1 (Argument). For any $z \in \mathbb{C} - \mathbb{R}_{\leq 0}$, there are unique $\theta \in (-\pi, \pi)$ and $r > 0$ such that $z = r \exp(\sqrt{-1}\theta)$. We call θ the argument of $z = r \exp(\sqrt{-1}\theta)$:

$$\arg: (\mathbb{C} - \mathbb{R}_{\leq 0}) \rightarrow (-\pi, \pi); r \exp(\sqrt{-1}\theta) \mapsto \theta. \quad (2.20)$$

Theorem 2.2.1. *A curve in \mathbb{C} is uniformly continuous.*

Proof. Let $\gamma \in C^0([a, b], \mathbb{C})$ be a curve. As shown in Theorem 2.1.2, the domain $[a, b] \subset \mathbb{R}$ is compact. Hence, it follows. \blacksquare

Definition 2.2.2 (Winding Numbers of Closed Curves). Let $\gamma \in C^0([a, b], \mathbb{R})$ be a closed curve and $z_0 \in \neg[\gamma]$. We will define the winding number $n(\gamma, z_0)$ of the curve γ at z_0 .

Since $[\gamma] \subset \mathbb{C}$ is closed, Theorem 1.3.1 implies

$$\delta_0 := d([\gamma], z_0) > 0 \quad (2.21)$$

Let $\epsilon > 0$ such that

$$0 < \epsilon < \delta_0. \quad (2.22)$$

Since γ is uniformly continuous by Theorem 1.3.4, there exists $\delta > 0$ such that, for each $s, t \in [a, b]$,

$$|s - t| < \delta \Rightarrow |\gamma s - \gamma t| < \epsilon. \quad (2.23)$$

Consider a finite subdivision of $[a, b]$:

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b \quad (2.24)$$

such that $\max\{a_1 - a_0, \dots, a_n - a_{n-1}\} < \delta$. Then, for each pair $(a_{j-1}, a_j), j \in \{1, \dots, n\}$:

$$|\gamma a_j - \gamma a_{j-1}| < \epsilon. \quad (2.25)$$

Moreover, for each $j \in \{1, \dots, n\}$,

$$w_j := \frac{\gamma a_j - z_0}{\gamma a_{j-1} - z_0} \quad (2.26)$$

satisfies $|w_j - 1| < 1$, hence $\Re w_j > 0$:

$$|w_j - 1| = \left| \frac{\gamma a_j - z_0 - (\gamma a_{j-1} - z_0)}{\gamma a_{j-1} - z_0} \right| = \left| \frac{\gamma a_j - \gamma a_{j-1}}{\gamma a_{j-1} - z_0} \right| < \frac{\epsilon}{\delta_0} < 1. \quad (2.27)$$

Thus, for each $j \in \{1, \dots, n\}$,

$$\arg w_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \quad (2.28)$$

Since γ is closed, $\gamma a_0 = \gamma a = \gamma b = \gamma a_n$:

$$\prod_{j=1}^n w_j = \prod_{j=1}^n \frac{\gamma a_j - z_0}{\gamma a_{j-1} - z_0} = \frac{\gamma a_n - z_0}{\gamma a_0 - z_0} = 1 \quad (2.29)$$

we conclude $\sum_{j=1}^n \arg w_j \equiv 0 \pmod{2\pi}$. We define:

$$n(\gamma, z_0) := \frac{1}{2\pi} \sum_{j=1}^n \arg w_j. \quad (2.30)$$

Remark 9. As a trivial example, if a curve is a constant, its winding number is zero.

Lemma 2.2.1. *The winding number is independent of the subdivision.*

Proof. We will show that the winding number based on a new subdivision:

$$a_0 < \cdots < a_{j-1} < \tau < a_j < \cdots < a_n \quad (2.31)$$

is equal to the original $n(\gamma, z_0)$.

Let $\theta_j := \arg w_j$. Since

$$\theta_j = \arg \frac{\gamma a_j - z_0}{\gamma a_{j-1} - z_0} = \arg \frac{\gamma a_j - z_0}{\gamma \tau - z_0} \frac{\gamma \tau - z_0}{\gamma a_{j-1} - z_0} \quad (2.32)$$

if we define $\theta'_j := \arg \frac{\gamma a_j - z_0}{\gamma \tau - z_0}$ and $\theta''_j := \arg \frac{\gamma \tau - z_0}{\gamma a_{j-1} - z_0}$, we have

$$\theta_j \equiv \theta'_j + \theta''_j \pmod{2\pi}. \quad (2.33)$$

Since each argument is in $(-\frac{\pi}{2}, \frac{\pi}{2})$:

$$|\theta_j - (\theta'_j + \theta''_j)| \leq |\theta_j| + |\theta'_j| + |\theta''_j| < \frac{3}{2}\pi, \quad (2.34)$$

we conclude

$$\theta_j = \theta'_j + \theta''_j. \quad (2.35)$$

This means the winding number based on a finer subdivision remains the same. ■

Theorem 2.2.2. *Let γ be a closed curve in \mathbb{C} . Then*

$$n(\gamma, -): \neg[\gamma] \rightarrow \mathbb{Z} \quad (2.36)$$

is constant on each connected component in $\neg[\gamma]$. In particular, $n(\gamma, -)$ is zero on an unbounded connected component.

Proof. Let $\gamma \in C^0([a, b], \mathbb{R})$ be a closed curve, $t \in [a, b]$, and $z_0, z'_0 \in \neg[\gamma]$. We use the same $0 < \epsilon < \delta_0 := d([\gamma], z_0)$ and subdivision $a = a_0 < \cdots < a_n = b$ for z_0 . Since

$$|\gamma t - z_0| \leq |\gamma t - z'_0| + |z_0 - z'_0|. \quad (2.37)$$

we obtain:

$$|\gamma t - z'_0| \geq |\gamma t - z_0| - |z_0 - z'_0| = \delta_0 - |z_0 - z'_0| \quad (2.38)$$

If z_0 and z'_0 are relatively close, namely, if $|z_0 - z'_0| < \delta_0 - \epsilon$,

$$|\gamma t - z'_0| > \epsilon. \quad (2.39)$$

Then, for each $s \in [a, b]$, $|\gamma s - z'_0| > \epsilon > 0$, and

$$d([\gamma], z'_0) \geq \epsilon > 0. \quad (2.40)$$

Hence, for $n(\gamma, z'_0)$, we may use the same subdivision as $n(\gamma, z_0)$:

$$|w'_j - 1| = \left| \frac{\gamma a_j - \gamma a_{j-1}}{\gamma a_{j-1} - z'_0} \right| < \frac{\epsilon}{\epsilon} = 1 \quad (2.41)$$

where

$$w'_j := \frac{\gamma a_j - z'_0}{\gamma a_{j-1} - z'_0}, \quad (2.42)$$

for each $j \in \{1, \dots, n\}$.

We will first show $n(\gamma, -)$ is continuous. Let $j \in \{1, \dots, n\}$. Define:

$$v_j := \frac{\gamma a_j - z_0}{\gamma a_j - z'_0}. \quad (2.43)$$

Since

$$|v_j - 1| = \left| \frac{z'_0 - z_0}{\gamma a_{j-1} - z'_0} \right| < \frac{|z'_0 - z_0|}{\epsilon} \quad (2.44)$$

if z'_0 is sufficiently close to z_0 , namely if

$$|z_0 - z'_0| < \min \{\epsilon, \delta_0 - \epsilon\} \quad (2.45)$$

then we obtain $|v_j - 1| < 1$. Hence

$$\arg v_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \quad (2.46)$$

Since

$$\theta'_j := \arg \frac{\gamma a_j - z'_0}{\gamma a_{j-1} - z'_0} = \arg \frac{\gamma a_j - z'_0}{\gamma a_j - z_0} \frac{\gamma a_j - z_0}{\gamma a_{j-1} - z_0} \frac{\gamma a_{j-1} - z_0}{\gamma a_{j-1} - z'_0}. \quad (2.47)$$

we obtain:

$$\theta'_j \equiv \theta_j - \arg v_j + \arg v_{j-1} \pmod{2\pi}. \quad (2.48)$$

Recalling each angle is in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we conclude

$$\theta'_j = \theta_j - \arg v_j + \arg v_{j-1}. \quad (2.49)$$

Recalling $\gamma a_0 = \gamma a_n$, we have $v_0 = v_n$. Moreover:

$$\sum_{j=1}^n \theta'_j = \sum_{j=1}^n \theta_j. \quad (2.50)$$

Thus, $n(\gamma, -)$ is locally constant, and hence $n(\gamma, -)$ is continuous relative to the discrete topology:

$$n(\gamma, -) \in C^0(\neg[\gamma], \mathbb{Z}). \quad (2.51)$$

Let $\Omega \subset \neg[\gamma]$ be a connected component and $z_0 \in \Omega$. Define

$$\Omega_0 := \{z \in \Omega \mid n(\gamma, z) = n(\gamma, z_0)\} = \Omega \cap n(\gamma, -)^{\leftarrow} n(\gamma, z_0). \quad (2.52)$$

Since $\{n(\gamma, z_0)\} \subset \mathbb{Z}$ is open, its preimage $\Omega_0 \subset \Omega$ is open. Its complement is also open:

$$\Omega_1 := \{z \in \Omega \mid n(\gamma, z) \neq n(\gamma, z_0)\} = \Omega \cap \bigcup_{k \neq n(\gamma, z_0)} n(\gamma, -)^{\leftarrow k} \quad (2.53)$$

By definition, $\Omega_0 \cup \Omega_1 = \Omega$, and these two open subspaces are disjoint:

$$\Omega_0 \cap \Omega_1 = \emptyset. \quad (2.54)$$

Since Ω is connected and $z_0 \in \Omega \cap \Omega_0$, by Theorem 1.2.8, we conclude $\Omega_0 = \Omega$. Hence, $n(\gamma, -)$ is constant on each connected component.

Finally, we will show that $n(\gamma, -)$ is zero on an unbounded connected component. Since \mathbb{C} is Hausdorff, and as shown in Theorem 2.1.5 $[\gamma] \subset \mathbb{C}$ is compact, by Theorem 1.2.16, $[\gamma] \subset \mathbb{C}$ is closed. There exists $R > 0$ with $[\gamma] \subset \overline{B_R(0)} = \{w \in \mathbb{C} \mid |w| \leq R\}$. The complement $\overline{\neg B_R(0)} = \{w \in \mathbb{C} \mid |w| > R\}$ is connected. Let Ω_∞ be an unbounded component of $\neg[\gamma]$:

$$\overline{\neg B_R(0)} \subset \Omega_\infty. \quad (2.55)$$

Consider $z_0 \in \Omega_\infty$ such that $|z_0| > 3R$. Let $s, t \in [a, b]$:

$$\begin{aligned} |\gamma t - z_0| &\geq |z_0| - |\gamma t| > 3R - R = 2R \\ |\gamma s - \gamma t| &\leq |\gamma s| + |\gamma t| \leq 2R \end{aligned} \quad (2.56)$$

Then, we obtain:

$$\left| \frac{\gamma s - \gamma t}{\gamma t - z_0} \right| < 1. \quad (2.57)$$

Since $s, t \in [a, b]$ are arbitrary, we may use the trivial subdivision $a < b$:

$$\arg \frac{\gamma b - z_0}{\gamma a - z_0} = \arg 1 = 0. \quad (2.58)$$

Hence, $n(\gamma, -)|_{\Omega_\infty} = 0$. ■

Theorem 2.2.3. *Let γ_0, γ_1 be closed curves in \mathbb{C} , for simplicity, $\gamma_0, \gamma_1 \in C^0([0, 1], \mathbb{C})$ with $\gamma_0 0 = \gamma_0 1$ and $\gamma_1 0 = \gamma_1 1$. Suppose $\gamma_0 0 = \gamma_1 0$, and there exists $h \in C^0([0, 1] \times [0, 1], \mathbb{C})$ such that*

$$h(0, -) = \gamma_0, h(1, -) = \gamma_1, h(-, 0) = \gamma_0 0 = h(-, 1). \quad (2.59)$$

Then, $n(\gamma_0, z_0) = n(\gamma_1, z_0)$ for $z_0 \in \neg[h]$.

Proof. Note that for each $s \in [0, 1]$, $h(s, 0) = h(s, 1)$, that is $h(s, -) \in C^0([0, 1], \mathbb{C})$ is a closed curve.

Let $z_0 \in \neg[h]$. By Theorem 1.2.14, since h is compact and its domain $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ is compact in \mathbb{C} . Since the underlying \mathbb{C} is a Hausdorff space, by Theorem 1.2.16, $[h] \subset \mathbb{C}$ is closed. Hence,

$$\delta_0 := d([h], z_0) > 0 \quad (2.60)$$

by Theorem 1.3.1. Let $\epsilon > 0$ such that $0 < \epsilon < \delta_0$. Since h is continuous on a compact space $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, by Theorem 1.3.4, h is uniformly continuous. Therefore, there exists $\delta > 0$ such that, for each $s, s', t, t' \in [0, 1]$:

$$|s - s'|, |t - t'| < \delta \Rightarrow |h(s, t) - h(s', t')| < \epsilon. \quad (2.61)$$

Consider subdivisions $0 = s_0 < \dots < s_m = 1$ and $0 = t_0 < \dots < t_n = 1$ such that

$$\max \{s_1 - s_0, \dots, s_m - s_{m-1}, t_1 - t_0, \dots, t_n - t_{n-1}\} < \delta. \quad (2.62)$$

Let $j \in \{0, \dots, m\}$. The condition (2.62) guarantees:

$$2\pi n(h(s_j, -), z_0) = \sum_{k=1}^n \arg \frac{h(s_j, t_k) - z_0}{h(s_j, t_{k-1}) - z_0} \quad (2.63)$$

is well-defined, see the construction in Definition 1.2.16. Moreover, for any $t \in [0, 1]$:

$$\left| \frac{h(s_j, t) - z_0}{h(s_{j-1}, t) - z_0} - 1 \right| = \left| \frac{h(s_j, t) - h(s_{j-1}, t)}{h(s_{j-1}, t) - z_0} \right| < \frac{\epsilon}{\delta_0} < 1 \quad (2.64)$$

holds, where $s_{-1} = s_{m-1}$, and hence, $\left| \arg \frac{h(s_j, t) - z_0}{h(s_{j-1}, t) - z_0} \right| < \frac{\pi}{2}$.

Since

$$\frac{h(s_j, t_k) - z_0}{h(s_j, t_{k-1}) - z_0} \frac{h(s_{j-1}, t_{k-1}) - z_0}{h(s_{j-1}, t_k) - z_0} = \frac{h(s_j, t_k) - z_0}{h(s_{j-1}, t_k) - z_0} \frac{h(s_{j-1}, t_{k-1}) - z_0}{h(s_j, t_{k-1}) - z_0}, \quad (2.65)$$

we obtain:

$$\begin{aligned} & \arg \frac{h(s_j, t_k) - z_0}{h(s_j, t_{k-1}) - z_0} - \arg \frac{h(s_{j-1}, t_k) - z_0}{h(s_{j-1}, t_{k-1}) - z_0} \\ & \equiv \arg \frac{h(s_j, t_k) - z_0}{h(s_{j-1}, t_k) - z_0} - \arg \frac{h(s_j, t_{k-1}) - z_0}{h(s_{j-1}, t_{k-1}) - z_0} \pmod{2\pi}. \end{aligned} \quad (2.66)$$

Since each argument is in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we conclude:

$$\begin{aligned} & \arg \frac{h(s_j, t_k) - z_0}{h(s_j, t_{k-1}) - z_0} - \arg \frac{h(s_{j-1}, t_k) - z_0}{h(s_{j-1}, t_{k-1}) - z_0} \\ & = \arg \frac{h(s_j, t_k) - z_0}{h(s_{j-1}, t_k) - z_0} - \arg \frac{h(s_j, t_{k-1}) - z_0}{h(s_{j-1}, t_{k-1}) - z_0}. \end{aligned} \quad (2.67)$$

Hence, $n(h(s_j, -), z_0) = n(h(s_{j-1}, -), z_0)$:

$$\begin{aligned}
& 2\pi n(h(s_j, -), z_0) - 2\pi n(h(s_{j-1}, -), z_0) \\
&= \sum_{k=1}^n \arg \frac{h(s_j, t_k) - z_0}{h(s_j, t_{k-1}) - z_0} - \sum_{k=1}^n \arg \frac{h(s_{j-1}, t_k) - z_0}{h(s_{j-1}, t_{k-1}) - z_0} \\
&= \sum_{k=1}^n \arg \frac{h(s_j, t_k) - z_0}{h(s_{j-1}, t_k) - z_0} - \sum_{k=1}^n \arg \frac{h(s_j, t_{k-1}) - z_0}{h(s_{j-1}, t_{k-1}) - z_0} \\
&= \arg \frac{h(s_j, t_n) - z_0}{h(s_{j-1}, t_n) - z_0} - \arg \frac{h(s_j, t_0) - z_0}{h(s_{j-1}, t_0) - z_0} \\
&= 0.
\end{aligned} \tag{2.68}$$

Since j is arbitrary, we conclude $n(h(s_0, -), z_0) = \dots = n(h(s_m, -), z_0)$. \blacksquare

2.3 Boundary-Preserving Maps on Unit Disc

Consider $\overline{D} = \overline{B_1(0)} = \{z \in \mathbb{C} \mid |z| \leq 1\}$, its boundary:

$$\partial D = \{z \in \mathbb{C} \mid |z| = 1\} \tag{2.69}$$

and the corresponding closed curve $\gamma_0 \in C^0(I, \partial D)$:

$$\gamma_0 t := \exp(2\pi\sqrt{-1}t), \tag{2.70}$$

where $I := [0, 1]$.

Theorem 2.3.1. *Let $f \in C^0(\overline{D}, \overline{D})$ such that $f\partial D \subset \partial D$. For $z \in \overline{D}$, if $n(f\gamma_0, z) \neq 0$, then $D \subset fD$.*

Proof. Let $z \in D$. Suppose $n(f\gamma, z) \neq 0$ but, for contradiction, $D \not\subset fD$. Then we may select $z_0 \in D - fD$. If we define $\gamma_1 = 1$ of a constant curve and

$$h(s, t) := (1-s)\gamma_0 t + s, \tag{2.71}$$

we obtain $h \in C^0([0, 1] \times [0, 1], \mathbb{C})$ such that

$$h(0, -) = \gamma_0, h(1, -) = \gamma_1, h(-, 0) = 1 = h(-, 1). \tag{2.72}$$

Since $f\partial D \subset \partial D$ and $z_0 \in D - fD$,

$$z_0 \notin f\partial D. \tag{2.73}$$

Hence, $z_0 \notin fD \cup f\partial D$, i.e.,

$$z_0 \notin f\overline{D}. \tag{2.74}$$

Recalling $[h] = \overline{D}$, we conclude $z_0 \notin [f\gamma_0]$. Applying Theorem 2.2.3, $0 \neq n(f\gamma_0, z) = n(f\gamma_1, z) = 0$, which is absurd. \blacksquare

Theorem 2.3.2. *Let $R := R(a, b; c, d)$ be a closed rectangle, $\gamma_1, \gamma_2 \in C^0(I, R)$ be curves in R such that $\Re(\gamma_1 0) = a, \Re(\gamma_1 1) = b, \Im(\gamma_2 0) = c, \Im(\gamma_2 1) = d$, where $I := [0, 1]$. Then there exist $s, t \in I$ such that $\gamma_1 s = \gamma_2 t$. In other words, a curve connecting the top and bottom edges meets another curve connecting the left and right edges.*

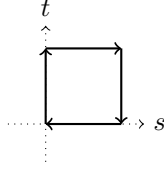
Proof. Suppose, for contradiction, that such a pair of curves never meet, i.e., $\gamma_1 s \neq \gamma_2 t$ for any $s, t \in [0, 1]$. Then, we can define

$$f(s, t) := \frac{\gamma_2 t - \gamma_1 s}{|\gamma_2 t - \gamma_1 s|}. \quad (2.75)$$

Moreover, $f \in C^0(I^2, \overline{D})$ and, since $|f(s, t)| = 1$ for each $(s, t) \in I^2$:

$$[f] \subset \partial D. \quad (2.76)$$

Since $fD \subset [f]$ is in $\partial D = \overline{D} - D$, we have $D \not\subset fD$. Consider a closed path L in I^2 :



- $[(0, 0), (0, 1)]$

Relative to $\gamma_1 0$, the argument of $\gamma_2 - \gamma_1 0: I \rightarrow \mathbb{C}$ moves from $\arg(\gamma_2 0 - \gamma_1 0) \in [-\frac{\pi}{2}, 0]$ to $\arg(\gamma_2 1 - \gamma_1 0) \in [0, \frac{\pi}{2}]$, where

$$\arg: (\mathbb{C} - \mathbb{R}_{\leq 0}) \rightarrow (-\pi, \pi) \quad (2.77)$$

see Definition 2.2.1.

- $[(0, 1), (1, 1)]$

The argument of $\gamma_2 1 - \gamma_1 -: I \rightarrow \mathbb{C}$ moves from $\arg(\gamma_2 1 - \gamma_1 0) \in [0, \frac{\pi}{2}]$ to $\arg(\gamma_2 1 - \gamma_1 1) \in [\frac{\pi}{2}, \pi]$, where

$$\arg: (\mathbb{C} - \sqrt{-1}\mathbb{R}_{\leq 0}) \rightarrow \left(-\frac{\pi}{2}, \frac{3}{2}\pi\right) \quad (2.78)$$

with $\sqrt{-1}\mathbb{R}_{\leq 0} := \{\sqrt{-1}t \mid t \leq 0\}$ so that the argument single-valued and continuous in the corresponding domain.

- $[(1, 1), (1, 0)]$

The argument of $\gamma_2 - \gamma_1 1: I \rightarrow \mathbb{C}$ moves from $\arg(\gamma_2 1 - \gamma_1 1) \in [\frac{\pi}{2}, \pi]$ to $\arg(\gamma_2 1 - \gamma_1 0) \in [\pi, \frac{3}{2}\pi]$, where

$$\arg: (\mathbb{C} - \mathbb{R}_{\geq 0}) \rightarrow (0, 2\pi). \quad (2.79)$$

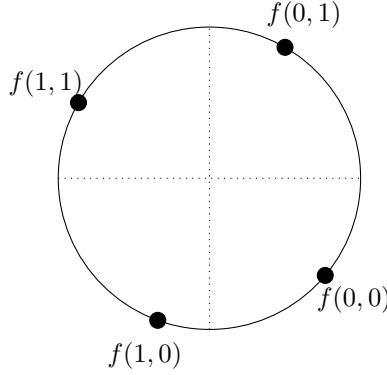
- $[(1, 0), (0, 0)]$

The argument of $\gamma_2 0 - \gamma_1 -$: $I \rightarrow \mathbb{C}$ moves from $\arg(\gamma_2 1 - \gamma_1 0) \in [\pi, \frac{3\pi}{2}]$ to $\arg(\gamma_2 0 - \gamma_1 0) \in [\frac{3\pi}{2}, 2\pi]$, where, with $\sqrt{-1}\mathbb{R}_{\geq 0} := \{\sqrt{-1}t \mid t \geq 0\}$,

$$\arg: (\mathbb{C} - \sqrt{-1}\mathbb{R}_{\geq 0}) \rightarrow \left(\frac{\pi}{2}, \frac{5}{2}\pi\right). \quad (2.80)$$

Let $\gamma_L: [0, 4] \rightarrow L$ be a curve along with $L \subset I^2$:

$$\gamma_L u := \begin{cases} (0, u) & u \in [0, 1] \\ (u - 1, 1) & u \in [1, 2] \\ (1, 3 - u) & u \in [2, 3] \\ (4 - u, 0) & u \in [3, 4] \end{cases} \quad (2.81)$$



Then f circles around the origin once, namely $n(f\gamma_L, 0) = 1$; by Theorem 2.3.1, it follows $D \subset fD$, which is absurd. ■

2.4 Jordan Curve Theorem

We will closely follow [Yan] to show Jordan curve theorem.

Lemma 2.4.1. *Let $F \subset \mathbb{C}$ be a closed subspace and $V \subset \mathbb{C} - F$ be a connected component. Then $\partial V \subset F$.*

Proof. We will first show that a connected component $V \subset \mathbb{C} - F$ is open in \mathbb{C} . Let $x \in V$; since $x \in \mathbb{C} - F$ and $\mathbb{C} - F \subset \mathbb{C}$ is open, there exists $\epsilon > 0$ with $B_\epsilon(x) \subset \mathbb{C} - F$. As shown in Theorem 2.1.6, the open ball $B_\epsilon(x)$ is connected, and V is a connected component with $V \cap B_\epsilon(x) \neq \emptyset$. Since $V \subset V \cup B_\epsilon(x)$, the \subset -largest property, see Definition 1.2.9, implies $B_\epsilon(x) \subset V$. By Lemma 1.2.2, $V \subset \mathbb{C}$ is open.

Let $W \subset \mathbb{C} - F$ be another connected component; as shown above, $W \subset \mathbb{C}$ is open. By Theorem 1.2.13, $W \cap V = \emptyset$. We will show $\partial V \cap W = \emptyset$. Let $x \in W$;

since $W \subset \mathbb{C}$ is open, there is $\epsilon > 0$ with $B_\epsilon(x) \subset W$. If x were also in ∂V , by Lemma 1.2.3, $B_\epsilon(x) \cap V \neq \emptyset$ but $B_\epsilon(x) \cap V \subset W \cap V = \emptyset$, which is absurd.

Since $V \subset \mathbb{C}$ is open, we obtain:

$$\partial V = \overline{V} - V. \quad (2.82)$$

Hence, $\partial V \cap V = \emptyset$. Moreover, for each connected component W of $\mathbb{C} - F$, $\partial V \cap W = \emptyset$:

$$\emptyset = \partial V \cap \bigcup \{W \mid W \subset \mathbb{C} - F \text{ is a connected component}\} = \partial V \cap (\mathbb{C} - F) \quad (2.83)$$

Therefore, $\partial V \subset F$ holds. \blacksquare

Theorem 2.4.1. *Let $\gamma \in C^0([0, 1], \mathbb{C})$ be a simple curve:*

$$\gamma s = \gamma t \Rightarrow s = t \quad (2.84)$$

i.e., a curve with no self-intersection. Then, the complement $\neg[\gamma] = \mathbb{C} - [\gamma]$ is a domain.

Proof. The continuous image $[\gamma] = \gamma[0, 1]$ of a compact interval $[0, 1]$ is compact by Theorem 2.1.5; by Theorem 2.1.3, $[\gamma] \subset \mathbb{C}$ is closed. Hence, $\neg[\gamma]$ is open.

Suppose, for contradiction, that $\neg[\gamma]$ is not connected. Then $\neg[\gamma]$ has at least two connected components. Since $[\gamma]$ is bounded, at least one connected component V_∞ is unbounded; let V be another connected component of $\neg[\gamma]$. Recalling $[\gamma] \subset \mathbb{C}$ is bounded, let $R > 0$ such that $[\gamma] \subset B_R(0)$; let $\gamma_R \theta = R \exp \sqrt{-1} \theta$ be the corresponding closed curve on $\partial B_R(0) = \{z \in \mathbb{C} \mid |z| = R\}$. As shown in Theorem 2.1.6, $\mathbb{C} - B_R(0)$ is connected but $[\gamma] \cap (\mathbb{C} - B_R(0)) = \emptyset$. Hence $\mathbb{C} - B_R(0) \subset V_\infty$, since $\mathbb{C} - B_R(0)$ is unbounded. It follows:

$$B_R(0) \supset \neg V_\infty \supset V. \quad (2.85)$$

Since γ is injective, the corestriction $\gamma : [0, 1] \rightarrow [\gamma]$ is bijective; by Theorem 1.2.6, $\gamma \in C^0([0, 1], [\gamma])$ is a continuous bijection. Applying Theorem 1.2.17, the inverse is also continuous:

$$\gamma^{-1} \in C^0([\gamma], [0, 1]). \quad (2.86)$$

By Lemma 1.2.1, $[\gamma] \subset \overline{B_R(0)}$ is a closed subspace. Hence, γ^{-1} has a continuous extension φ on $\overline{B_R(0)} \supset [\gamma]$ by Lemma 1.3.4:

$$\varphi \in C^0(\overline{B_R(0)}, [0, 1]) \quad (2.87)$$

such that $\varphi|_{[\gamma]} = \gamma^{-1}$. Consider the composition $\gamma \varphi : \overline{B_R(0)} \rightarrow [\gamma]$. Since both are continuous, $\gamma \varphi \in C^0(\overline{B_R(0)}, [\gamma])$. Moreover, the restriction $\gamma \circ \varphi|_{[\gamma]}$ is an identity on $[\gamma]$. Define $f : \overline{B_R(0)} \rightarrow \overline{B_R(0)}$:

$$fz := \begin{cases} z & z \in \overline{B_R(0)} - V \\ \gamma \varphi z & z \in V \end{cases} \quad (2.88)$$

By definition, both $f|_{\overline{B_R(0)}-V}$ and $f|_V$ are both continuous; recalling V is open, $f|_{\partial V=\overline{V}-V}$ is identity, so is continuous. Therefore, $f \in C^0(\overline{B_R(0)}, \overline{B_R(0)})$. Since $f|_{\partial B_R(0)}$ is identity, we obtain:

$$f\partial B_R(0) \subset \partial B_R(0). \quad (2.89)$$

Then, for the curve on $\partial B_R(0)$ $\gamma_R\theta = R \exp \sqrt{-1}\theta$, $\theta \in [0, 2\pi]$ and $z \in B_R(0)$, we obtain $n(f\gamma_R, z) = 1$ since $f\gamma_R$ circles around z once:

$$f\gamma_R\theta = f(R \exp \sqrt{-1}\theta) = R \exp \sqrt{-1}\theta. \quad (2.90)$$

By Theorem 2.3.1, we obtain $B_R(0) \subset fB_R(0)$. Consider the image of $B_R(0)$ over f :

$$fB_R(0) \subset (\overline{B_R(0)} - V) \cup \gamma_R V \subset (\overline{B_R(0)} - V) \cup [\gamma]. \quad (2.91)$$

Recalling $V \subset B_R(0)$, any point in V is not in the image of f , namely $V \not\subset fB_R(0)$. Therefore, we have

$$B_R(0) \not\subset fB_R(0), \quad (2.92)$$

which is absurd. ■

Definition 2.4.1 (Jordan Curves). A curve $\gamma \in C^0([0, 1], \mathbb{C})$ is called a Jordan curve iff it is closed, $\gamma 0 = \gamma 1$, and the restriction $\gamma|_{[0, 1]}$ is a simple curve:

$$\forall s, t \in [0, 1] : \gamma s = \gamma t \Rightarrow s = t. \quad (2.93)$$

Lemma 2.4.2. Let $\gamma \in C^0([0, 1], \mathbb{C})$ be a Jordan curve. If $\neg[\gamma] = \mathbb{C} - [\gamma]$ is not connected, the boundary of each connected component of $\neg[\gamma]$ is $[\gamma]$.

Proof. Since $[\gamma] \subset \mathbb{C}$ is compact – bounded and closed – at least one connected component of $\neg[\gamma]$ is unbounded. Let V_∞ be an unbounded connected component of $\neg[\gamma]$. If $R > 0$ is sufficiently large such that $[\gamma] \subset B_R(0)$, since $\mathbb{C} - B_R(0)$ is unbounded:

$$\mathbb{C} - B_R(0) \subset V_\infty. \quad (2.94)$$

The \subset -largest property implies such an unbounded component is unique.

Since $\neg[\gamma]$ is disconnected, there is at least one bounded connected component, say V . By Lemma 2.4.1, $\partial V_\infty \subset [\gamma]$ and $\partial V \subset [\gamma]$. To show these inclusions are equalities, suppose for contradiction that $\partial V \subsetneq [\gamma]$. Shifting the parameter, we may set

$$\gamma 0 = \gamma 1 \in [\gamma] - \partial V. \quad (2.95)$$

Then, there are $0 < a < b < 1$ such that:

$$\gamma[a, b] \supset \partial V. \quad (2.96)$$

Since $\gamma|_{[a, b]}$ is a simple curve, $\mathbb{C} - \gamma[a, b]$ is connected by Theorem 2.4.1. By Corollary 2.1.8.1 in Theorem 2.1.8, $\mathbb{C} - \gamma[a, b]$ is path-connected. Hence, for

$z \in V$ and $z_\infty \in V_\infty$, there is a curve in $\mathbb{C} - \gamma[a, b] \subset \mathbb{C} - \partial V$. Since $\partial V \cap V = \emptyset = \partial V \cap V_\infty$:

$$V \cup V_\infty \subset \mathbb{C} - \partial V \quad (2.97)$$

It follows $V \cup V_\infty$ is path-connected, and hence connected, which is absurd. ■

Theorem 2.4.2 (Jordan Curve Theorem). *Let γ be a Jordan curve in \mathbb{C} . The open subspace $\neg[\gamma] = \mathbb{C} - [\gamma]$ has exactly two connected components, one is unbounded and the other is bounded. If we let V be the bounded connected component and V_∞ be the unbounded connected component of $\neg[\gamma]$, $\partial V = [\gamma] = \partial V_\infty$ is the case.*

Proof. Since $[\gamma]$ is a compact subspace in \mathbb{C} , by Corollary 2.1.4.1, there are $z_1, z_2 \in [\gamma]$ such that

$$\delta[\gamma] := \sup_{\zeta_1, \zeta_2 \in [\gamma]} |\zeta_1 - \zeta_2| = |z_1 - z_2|. \quad (2.98)$$

Shifting and rotating the curve, we may set $z_1 = -1$ and $z_2 = +1$. Since the diameter of $[\gamma]$ is now 2, from -1 to $+1$,

$$[\gamma] \subset E := \{z \in \mathbb{C} \mid |\Im z| \leq 2 \wedge |\Re z| \leq 1\} \quad (2.99)$$

with

$$[\gamma] \cap \partial E = \{-1, +1\}, \quad (2.100)$$

otherwise, the diameter would be greater than 2. By Theorem 2.3.2, γ and $[-2\sqrt{-1}, 2\sqrt{-1}]$ meet:

$$[\gamma] \cap [-2\sqrt{-1}, 2\sqrt{-1}] \neq \emptyset. \quad (2.101)$$

Since $[\gamma]$ is compact and $[-2\sqrt{-1}, 2\sqrt{-1}] \subset \mathbb{C}$ is closed, by Theorem 1.2.15, $[\gamma] \cap [-2\sqrt{-1}, 2\sqrt{-1}]$ is compact. Since $\Im: \mathbb{C} \rightarrow \mathbb{R}$ is a projection, by Theorem 1.2.18, \Im is continuous; applying Theorem 2.1.4, $\Im([\gamma] \cap [-2\sqrt{-1}, 2\sqrt{-1}])$ has extreme values:

$$l := \max \Im([\gamma] \cap [-2\sqrt{-1}, 2\sqrt{-1}]). \quad (2.102)$$

Then $[2\sqrt{-1}, l\sqrt{-1}] \cap [\gamma] = \emptyset$. Since ± 1 subdivide γ into two simple curves between ± 1 , we let γ_+ be the one that $l\sqrt{-1}$ belongs to:

$$l\sqrt{-1} \in [\gamma_+]. \quad (2.103)$$

Define

$$m := \min \Im([\gamma_+] \cap [-2\sqrt{-1}, 2\sqrt{-1}]). \quad (2.104)$$

It is worth mentioning $l \geq m$. Then $(m\sqrt{-1}, -2\sqrt{-1}) \cap [\gamma_+] = \emptyset$. Let

$$[l\sqrt{-1}, m\sqrt{-1}]_{\gamma_+} \subset [\gamma_+] \quad (2.105)$$

denote the curve segment in γ_+ from $l\sqrt{-1}$ to $m\sqrt{-1}$.

We will show $[\gamma_-] \cap (m\sqrt{-1}, -2\sqrt{-1}) \neq \emptyset$. Consider a curve between $\pm 2\sqrt{-1}$:

$$[2\sqrt{-1}, l\sqrt{-1}] \diamond [l\sqrt{-1}, m\sqrt{-1}]_{\gamma_+} \diamond [m\sqrt{-1}, -2\sqrt{-1}], \quad (2.106)$$

where \diamond stands for the concatenation of two curves. By Theorem 2.3.2, such a curve between $\pm 2\sqrt{-1}$ and γ_- between ± 1 must meet. Since $[\gamma_-] \subset [\gamma]$ does not meet $[2\sqrt{-1}, l\sqrt{-1})$, and $l\sqrt{-1} \in [\gamma_+]$, we conclude:

$$[\gamma_-] \cap [2\sqrt{-1}, l\sqrt{-1}] = \emptyset. \quad (2.107)$$

Moreover, $[l\sqrt{-1}, m\sqrt{-1}]_{\gamma_+} \subset [\gamma_+]$, and $m\sqrt{-1} \in [\gamma_+]$. Hence, $(m\sqrt{-1}, -2\sqrt{-1})$ must meet $[\gamma_-]$:

$$[\gamma_-] \cap (m\sqrt{-1}, -2\sqrt{-1}) \neq \emptyset. \quad (2.108)$$

Since the intersection $[\gamma_-] \cap [m\sqrt{-1}, -2\sqrt{-1}]$ is non-empty and compact:

$$\begin{aligned} p &:= \max \Im ([\gamma_-] \cap [m\sqrt{-1}, -2\sqrt{-1}]) \\ q &:= \min \Im ([\gamma_-] \cap [m\sqrt{-1}, -2\sqrt{-1}]) \end{aligned} \quad (2.109)$$

By definition, $m \geq p$ but $[\gamma_+] \cap [\gamma_-] = \{\pm 1\}$ but the intersection is on the imaginary axis, we have $m \neq p$:

$$m > p. \quad (2.110)$$

Hence $(m\sqrt{-1}, p\sqrt{-1}) \cap [\gamma] = \emptyset$. In particular,

$$z_0 := \frac{m\sqrt{-1} + p\sqrt{-1}}{2} \notin [\gamma]. \quad (2.111)$$

Recalling $[\gamma]$ is compact, its complement $\neg[\gamma]$ should have an unbounded connected component; let V_∞ be such an unbounded component of $\neg[\gamma]$. Let $R > 0$ be sufficiently large $[\gamma] \subset B_R(0)$. Since $\mathbb{C} - B_R(0) \subset \neg[\gamma]$ is connected, see Theorem 2.1.6 and unbounded, we obtain:

$$\mathbb{C} - B_R(0) \subset V_\infty. \quad (2.112)$$

The \subset -largest property of V_∞ implies such an unbounded component of $\neg[\gamma]$ is unique. Then $z_0 \in E^\iota$, since $\Re z_0 = 0$ and

$$\Im z_0 = \frac{m+p}{2} < m \in [-2, 2]. \quad (2.113)$$

We will show that the connected component of $\neg[\gamma]$ around z_0 is not V_∞ . Suppose, for contradiction, that z_0 is in V_∞ . Since V_∞ is connected, there is a curve in V_0 from z_0 to some point in $\neg E$, since $\neg E \subset \neg[\gamma]$ is unbounded:

$$\alpha \in C^0(I, V_\infty), \quad (2.114)$$

where $\alpha 0 = z_0 \in E^\iota$ and $\alpha 1 \in \neg E$. Define

$$t_0 := \inf \{t \in I \mid \alpha t \notin E^\iota\} \quad (2.115)$$

and $w_0 := \alpha t_0$. We will show $w_0 \in E - E^\iota = \partial E$:

- $w_0 \in E$

Let $\epsilon > 0$ and consider $B_\epsilon(w_0)$. Since α is continuous, its preimage $\alpha^\leftarrow B_\epsilon(w_0) \subset I$ is open. Hence, there is $\delta > 0$ with $(t_0 - \delta, t_0 + \delta) \subset \alpha^\leftarrow B_\epsilon(w_0)$:

$$\alpha(t_0 - \delta, t_0 + \delta) \subset B_\epsilon(w_0). \quad (2.116)$$

In particular $t_0 - \frac{\delta}{2} < t_0 = \inf \{t \in I \mid \alpha t \notin E^\iota\}$:

$$\alpha\left(t_0 - \frac{\delta}{2}\right) \neq \alpha t_0 = w_0 \quad (2.117)$$

and $\alpha(t_0 - \frac{\delta}{2}) \in E^\iota \subset E$. Hence, it follows $w_0 \in \overline{E} = E$:

$$B_\epsilon(w_0) \cap E - \{w_0\} \neq \emptyset. \quad (2.118)$$

- $w_0 \notin E^\iota$

Suppose, for contradiction, that w_0 is an interior point of E . Then there is $\epsilon > 0$ with $B_\epsilon(w_0) \subset E^\iota$. Then, around t_0 , there is some $\delta > 0$ with $\alpha(t_0 - \delta, t_0 + \delta) \subset B_\epsilon(w_0)$ since α is continuous. Then $\alpha(t_0 + \frac{\delta}{2}) \in B_\epsilon(w_0) \subset E^\iota$ implies $t_0 + \frac{\delta}{2} > t_0$ would be a lower bound of $\{t \in I \mid \alpha t \notin E^\iota\}$, which is absurd.

Let $\alpha_0 := \alpha|_{[0, t_0]}$ be the curve from z_0 to $w_0 \in \partial E$. Recalling $w_0 \in V_\infty \subset \neg[\gamma]$, $w_0 \neq \pm 1$, hence $\Im w_0 \neq 0$:

- $\Im w_0 < 0$ case

We have $[w_0, -2\sqrt{-1}]_{\partial E} \subset \partial E$, connecting w_0 and $-2\sqrt{-1}$ along with the edge of the rectangle E , without traversing ± 1 . Then, since α_0 is a curve in V_∞ from $z_0 \in E^\iota$ to $w_0 \in \partial E$:

$$[2\sqrt{-1}, l\sqrt{-1}] \diamond [l\sqrt{-1}, m\sqrt{-1}]_{\gamma_+} \diamond [m\sqrt{-1}, z_0] \diamond [\alpha_0] \diamond [w_0, -2\sqrt{-1}]_{\partial E} \quad (2.119)$$

does not meet γ_- , which is absurd.

- $\Im w_0 > 0$

We have $[w_0, 2\sqrt{-1}]_{\partial E} \subset \partial E$, connecting w_0 and $2\sqrt{-1}$ along with the edge of the rectangle E , without traversing ± 1 . Then,

$$[-2\sqrt{-1}, z_0] \diamond [\alpha_0] \diamond [w_0, 2\sqrt{-1}]_{\partial E} \quad (2.120)$$

does not meet γ_+ , which is absurd.

Hence, $z_0 \notin V_\infty$. Let V be a connected component of $\neg[\gamma]$ with $z_0 \in V$:

$$V \cap V_\infty. \quad (2.121)$$

Finally, we will show the unbounded connected component is unique. Suppose $W \subset \neg[\gamma]$ is another unbounded component. Since $\neg[\gamma] \supset \neg E$, we obtain

$$V_\infty \supset \neg E. \quad (2.122)$$

That is, the exterior of E is in V_∞ . Hence, unbounded components are all in E :

$$V \subset E \wedge W \subset E. \quad (2.123)$$

Define a curve $[\beta]$ between $\pm 2\sqrt{-1}$:

$$[2\sqrt{-1}, l\sqrt{-1}] \diamond [l\sqrt{-1}, m\sqrt{-1}]_{\gamma_+} \diamond [m\sqrt{-1}, p\sqrt{-1}] \diamond [p\sqrt{-1}, q\sqrt{-1}]_{\gamma_-} \diamond [q\sqrt{-1}, -2\sqrt{-1}]. \quad (2.124)$$

- $[2\sqrt{-1}, l\sqrt{-1}], [q\sqrt{-1}, -2\sqrt{-1}] \subset V_\infty$
Since $[2\sqrt{-1}, l\sqrt{-1}]$ can be connected with $3\sqrt{-1} \in \neg E \subset V_\infty$, $[2\sqrt{-1}, l\sqrt{-1}] \subset V_\infty$.
- $[l\sqrt{-1}, m\sqrt{-1}]_{\gamma_+}, [p\sqrt{-1}, q\sqrt{-1}]_{\gamma_-} \subset [\gamma]$
By the very definition, they are segments of the original curve γ .
- $[m\sqrt{-1}, p\sqrt{-1}] \subset V$
Since $[m\sqrt{-1}, p\sqrt{-1}]$ contains $z_0 \in V$, $[m\sqrt{-1}, p\sqrt{-1}] \subset V$.

Then $[\beta] \cap W = \emptyset$, since $[\beta] \subset V_\infty \cup [\gamma] \cup V$. Since $\pm 1 \notin [\beta]$, there are open balls $D_\pm \in \mathcal{N}_{\pm 1}$ with $D_\pm \cap [\beta] = \emptyset$, choosing their diameters smaller than $d([\beta], \pm 1)$. Since $\partial W = [\gamma]$ by Lemma 2.4.1, and $\pm 1 \in [\gamma]$, ± 1 are limit points of W :

$$W \cap D_\pm \neq \emptyset. \quad (2.125)$$

Let $a_\pm \in W \cap D_\pm$, c be a curve from a_- to a_+ , and

$$[-1, a_-] \diamond [c] \diamond [a_+, 1] \quad (2.126)$$

be a curve between ± 1 . This curve in E , connecting ± 1 , does not meet β , which is absurd. Hence, the bounded component of $\neg[\gamma]$ must be unique. ■

Definition 2.4.2 (Interior and Exterior of Jordan Curves). For a Jordan curve γ in \mathbb{C} , we call the unbounded connected component V_∞ of $\neg[\gamma]$ the exterior of γ , and the bounded component V the interior of γ . As examined in Theorem 2.2.2, the winding number on V_∞ is zero.

Theorem 2.4.3. *Let γ be a Jordan curve in \mathbb{C} . The winding number of γ satisfies $|n(\gamma, z)| = 1$ for any point z in the interior of γ .*

Proof. We will use the same notation in the proof of Theorem 2.4.2. Assume γ_+ is a curve from $+1$ to -1 ; we will show $n(\gamma, -)|_V = +1$. Let δ_+ be a curve from -1 to $+1$ along ∂E :

$$[\delta_+] = [-1, -1 + 2\sqrt{-1}] \diamond [-1 + 2\sqrt{-1}, +1 + 2\sqrt{-1}] \diamond [+1 + 2\sqrt{-1}, +1] \quad (2.127)$$

Let $\gamma_+ + \delta_+$ be the composite curve from $+1$ to -1 along γ_+ , and from -1 to $+1$ along δ_+ . It follows that $\gamma_+ + \delta_+$ is a Jordan curve. Since $-3\sqrt{-1} \in \neg E$

and $\neg E \subset V_\infty(\gamma_+ + \delta_+)$, the presence of a line segment $[z_0, -3\sqrt{-1}]$ implies z_0 is in the exterior of $\gamma_+ + \delta_+$, namely $z_0 \in V_\infty(\gamma_+ + \delta_+)$:

$$n(\gamma_+ + \delta_+, z_0) = 0. \quad (2.128)$$

Similarly, for

$$[\delta_-] = [+1, +1 - 2\sqrt{-1}] \diamond [+1 - 2\sqrt{-1}, -1 - 2\sqrt{-1}] \diamond [-1 - 2\sqrt{-1}, -1] \quad (2.129)$$

we obtain

$$n(\gamma_- + \delta_-, z_0) = 0 \quad (2.130)$$

since $z_0 \in V_\infty(\gamma_- + \delta_-)$. Recalling Definition 2.2.2, we can write:

$$0 = n(\gamma_+ + \delta_+, z_0) + n(\gamma_- + \delta_-, z_0) = n(\gamma, z_0) + n(\delta_+ + \delta_-, z_0). \quad (2.131)$$

As demonstrated in the proof of Theorem 2.3.2, since $\delta_+ + \delta_-$ cycles around z_0 clockwise once:

$$n(\delta_+ + \delta_-, z_0) = -1, \quad (2.132)$$

we conclude $n(\gamma, z_0) = +1$. ■

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