Geometric Algebra and Electromagnetism

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Contents

0 Introduction

This note explores the fundamentals of classical electrodynamics using geometric algebras. Also known as timespace algebras or Clifford algebras, geometric algebras serve as a powerful mathematical framework, not only as an abstract mathematical structure but also as a natural description of physical laws. Our primary objective is to reformulate Maxwell's equations. Initially, we present them as four equations using ordinary vector analysis such as divergence $\vec{\nabla}\cdot$ and rotation $\vec{\nabla}\times$, where $\cdot$ is the dot-product and $\times$ is the cross-product in three-dimensional real vector space. We then unify these equations into a single equation using geometric algebra. To develop this framework, we first review essential concepts from calculus such as integrals and derivatives. So let us begin with derivatives:

Definition 0.1 (Continuity, Differentiability, and Partial Derivatives)**.** Let $A \subset \mathbb{R}$ be a subset and $f\colon A \to \mathbb{R}$ be a function. We say that f is continuous at a point $w \in A$ iff for each positive $\epsilon > 0$, there exists a positive $\delta > 0$ such that $|f(z) - f(w)| < \epsilon$ whenever $z \in E$ satisfies $|z - w| < \delta$. Schematically, we express the continuity of f at $w \in A$ as "if z tends to w, so as $f(z)$ to $f(w)$:"

$$z \dashrightarrow w \Rightarrow f(z) \dashrightarrow f(w). \tag{1}$$

> The function f is continuous at w if, roughly speaking, $f(z)$ is approximately $f(w)$ when z is near w. The differentiability of f is concerned with finding a more accurate approximation of f by using a polynomial of degree one rather than the constant $f(w)$. [Bea19]

The function $f\colon A \to \mathbb{R}$ is differentiable at $w \in A$ iff there is a constant $\alpha \in \mathbb{R}$ and a function $\eta\colon A \to \mathbb{R}$ satisfying:

$$f(z) = f(w) + \alpha(z - w) + (z - w)\eta(z), \tag{2}$$

for any $z \in A$, where $\eta(z) \dashrightarrow 0$ as $z \dashrightarrow w$ in A. Formally speaking, for each $\epsilon > 0$, there exists $\delta > 0$ such that $|\eta(z)| < \epsilon$ whenever $|z - w| < \delta$ for $z \in A$. In other words, f is differentiable at w iff

$$\frac{f(z) - f(w)}{z - w} \dashrightarrow \alpha \tag{3}$$

as $z \dashrightarrow w$ in A, where η tends to vanish as $z \dashrightarrow w$. If f is differentiable at each point in A, f is differentiable in A; the correspondence $w \mapsto \alpha$ is called the derivative of f:

$$f'(w) = \alpha. \tag{4}$$

There are several notations such as $\dot{f}(w)$ and $f^{(1)}(w)$, since this quantity is commonly referred to as the first derivative of f at w.

Let us consider multivariable cases, for instance, $\mathbb{R}^2$. Suppose $D \subset \mathbb{R}^2$ and $u\colon D \to \mathbb{R}$. We say u is differentiable at $\zeta \in D$ iff there are two real numbers α, β and a continuous function $\epsilon_1\colon D \to \mathbb{R}$ with $\epsilon_1(\zeta) = 0$ such that:

$$u(z) = u(\zeta) + (x - \zeta_1)\alpha + (y - \zeta_2)\beta + |z - \zeta|\,\epsilon_1(z) \tag{5}$$

for $z \in D$, where $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$ and $z = \begin{pmatrix} x \\ y \end{pmatrix}$. If this is the case,

$$\begin{aligned} \frac{\partial u}{\partial x}(\zeta) &= \lim_{h \dashrightarrow 0} \frac{u\left(\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}\right) - u\left(\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}\right)}{h} &= \alpha \\ \frac{\partial u}{\partial y}(\zeta) &= \lim_{h \dashrightarrow 0} \frac{u\left(\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} + \begin{pmatrix} 0 \\ h \end{pmatrix}\right) - u\left(\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}\right)}{h} &= \beta \end{aligned} \tag{6}$$

They are called partial derivatives of u at $\zeta \in D$, namely $\frac{\partial u}{\partial x}(\zeta)$ is the partial derivative of u along "x" direction at ζ.

1 Linear Spaces and Multivectors

Let V be a finite-dimensional linear space over the set $\mathbb{R}$ of scalars. Introducing a set of basis vectors, we can identify V as $\mathbb{R}^n$, where $n = \dim V$.

1.1 Definitions

Definition 1.1 (Vectors as Directed Line Segments)**.** Let $u \in \mathbb{R}^n$ be a vector. With the standard basis, u is expressed as a unique linear combination:

$$u = \sum_{j=1}^{n} u_j e_j, \tag{7}$$

where $\{u_1, \dots, u_n\} \subset \mathbb{R}$ are scalars and $\{e_1, \dots, e_n\} \subset \mathbb{R}^n$ is the set of standard basis vectors:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{8}$$

Thus, each vector in $\mathbb{R}^n$ has a unique coordinate n-tuple. We identify $\sum_{j=1}^{n} u_j e_j$ as a n-tuple $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$. Geometrically, u is expressed as a line segment from origin $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ to the point $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$. The norm of u is given by the following standard distance from 0 to u:

$$|u| := \sqrt{\sum_{j=1}^{n} {u_j}^2} \tag{9}$$

For example, in $\mathbb{R}^3$, $|u| = \sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2}$, where $u \in \mathbb{R}^3$ is expressed as $\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ with respect to the standard basis.

Remark 1 (Span). For the fixed field $\mathbb{R}$, suppose $b_1, \ldots, b_n$ is n elements with no special relation among them. We call

$$\langle b_1, \ldots, b_n \rangle \coloneqq \left\{ \sum_{i=1}^{n} v_i b_i \,\middle|\, \{v_1, \ldots, v_n\} \subset \mathbb{R} \right\} \tag{10}$$

the linear space spanned by the bases $\{b_1, \ldots, b_n\}$ over the set of scalars $\mathbb{R}$; v_i is the i-th component of $\sum_{i=1}^{n} v_i b_i$ with respect to the set of ordered basis $\langle b_1, \ldots, b_n \rangle$. We identify $\langle b_1, \ldots, b_n \rangle$ and $\mathbb{R}^n$.

Remark 2 (Scalar Product). The linear space $\mathbb{R}^n$ is equipped with a scalar product:

$$u \cdot v \coloneqq \sum_{j=1}^{n} u_j v_j. \tag{11}$$

In particular, $u \cdot u = |u|^2$.

Definition 1.2 (Exterior Products and k-vectors)**.** For two vectors $u, v \in \mathbb{R}^n$, let $u \wedge v$ be a directed area that corresponds to the parallelogram specified by the following four points:

$$0, u, u+v, v. \tag{12}$$

The direction, if it is well-defined in $\mathbb{R}^n$, is given by the right-hand rule: along the line segments from 0 to u and from u to $u + v$. As a fundamental property, we assume

$$v \wedge u = -u \wedge v. \tag{13}$$

The norm of $u \wedge v$ is given by $|u \wedge v| = |u|\,|v| \sin\theta$, where θ is the angle from u-direction to v-direction.

- 0-vectors

  We call the set of scalars 0-vectors.

- 1-vectors

  The elements of $\mathbb{R}^n$ are called 1-vectors, or simply vectors.

- 2-vectors

  Directed areas are called 2-vectors.

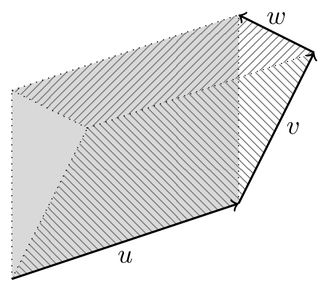
In general, a k-vector is given by a directed k-volume of $\mathbb{R}^n$ for $k \in \{0, 1 \cdots, n\}$.

Theorem 1.1 (Properties of Exterior Product)**.** *For any $u, v, w \in \mathbb{R}^n$:*

1. $u \wedge u = 0$

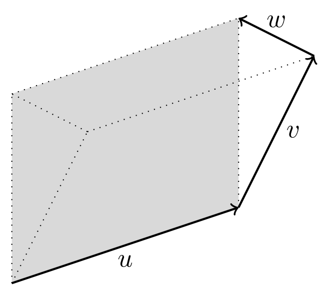
2. *distribution laws*
   - $u \wedge (v + w) = u \wedge v + u \wedge w$
   - $(u + v) \wedge w = u \wedge w + v \wedge w$
3. $(au) \wedge (bv) = (ab) u \wedge v$ *for any scalars* $a, b \in \mathbb{R}$.

Proof. Since the area formed by u and u is zero, we obtain $u \wedge u = 0$; we may apply the antisymmetric property: $u \wedge u = -u \wedge u$. The distribution laws can be represented in the following areas:



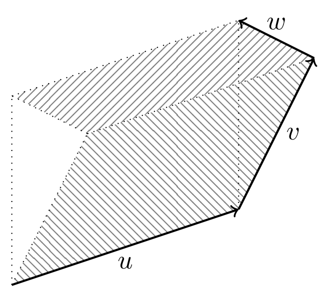
(14)

where the gray area is $u \wedge (v + w)$:



(15)

and two parallelograms are $u \wedge v$ and $u \wedge w$:



(16)

The bilinear property also follows geometrically:

$$(au) \wedge (bv) = (au) \wedge \left(\underbrace{v + \cdots + v}_{b} \right) = \underbrace{((au) \wedge v) + \cdots + ((au) \wedge v)}_{b}. \qquad (17)$$

$\blacksquare$

Remark 3 (Extension by Linearity). The exterior product $\wedge$ can be extended by linearity. We also require the associativity:

$$(u \wedge v) \wedge w = u \wedge (v \wedge w) \tag{18}$$

for a k-vector u, a k'-vector v, and a k''-vector w. Note that for a 0-vector a and a k-vector u, their exterior product is identified as a scalar multiplication of a k-vector:

$$a \wedge u = au. \tag{19}$$

Definition 1.3 (Multivectors and Geometric Product)**.** A multivector of $\mathbb{R}^n$ is a finite linear combinations of k-vectors.

For any 1-vectors $u, v \in \mathbb{R}^n$, we define their geometric product, or simply the product by:

$$uv \coloneqq u \cdot v + u \wedge v. \tag{20}$$

As shown in Remark 3, we relax the conditions u, v being 1-vectors. Recalling $v \cdot u = u \cdot v$ and $v \wedge u = -u \wedge v$, we obain:

$$u \cdot v = \frac{uv + vu}{2}, u \wedge v = \frac{uv - vu}{2} \tag{21}$$

Remark 4 (Fundamental Basis Vectors). Since $e_i \cdot e_j = \delta_{ij}$:

$$e_i \cdot e_j = \begin{cases} 1 & j = i \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

we obtain $e_i e_i = 1$ for any $i \in \{1, \cdots, n\}$ and

$$e_i e_j = e_i \wedge e_j = -e_j e_i, \tag{23}$$

if $j \neq i$. This property is called orthonormal.

Remark 5 (Geometric Algebras). For a real vector space R^n, we call the set of all multivectors with geometric product a geometric algebra.

1.2 Low-Dimensional Examples

Let us examine a few low-dimensional examples of real vector spaces.

1.2.1 $\mathbb{R}^1$ – Line

The set of vectors $\mathbb{R} = \langle e_1 \rangle$ is essentially the set of scalars.

1.2.2 $\mathbb{R}^2$ – Plane

Since there are two basis vectors, $\mathbb{R}^2 = \langle e_1, e_2 \rangle$, the set of 2-vectors is spanned by e_1e_2 only. Moreover, there are no 3-vector or higher vectors. Hence, the general form of a multivector in $\mathbb{R}^2$ is the following:

$$V = s + v_1e_1 + v_2e_2 + pe_1e_2, \tag{24}$$

where $s \in \mathbb{R}$ is the scalar part of V, $\sum_{i=1}^{2} v_ie_i \in \mathbb{R}^2$ is the 1-vector part of V, and pe_1e_2 is called a pseudo scalar of V. That is, the set of multivectors in $\mathbb{R}^2$ forms a four dimensional linear space over $\mathbb{R}$:

$$\langle 1, e_1, e_2, e_1e_2 \rangle \,. \tag{25}$$

Let $I \coloneqq e_1e_2$ be the unit pseudo scalar of $\mathbb{R}^2$, representing a unit oriented area.

Lemma 1.1. *We claim $I^2 = -1$.*

Proof. By a direct calculation, by the anticommutativity $e_2e_1 = -e_1e_2$:

$$I^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -1 \tag{26}$$

it follows. ∎

Remark 6 (Complex Plane $\mathbb{C}$). The subspace spanned by scalars and pseudo scalars $\langle 1, I \rangle$ is essentially the same as $\mathbb{C}$.

1.2.3 $\mathbb{R}^3$ – Space

The set of multivectors in $\mathbb{R}^3 = \langle e_1, e_2, e_3 \rangle$ is given by:

$$\langle 1, e_1, e_2, e_3, e_2e_3, e_3e_1, e_1e_2, e_1e_2e_3 \rangle \tag{27}$$

The general form of a multivector of $\mathbb{R}^3$ is the following:

$$V = s + v_1e_1 + v_2e_2 + v_3e_3 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 + pe_1e_2e_3, \tag{28}$$

where $s \in \mathbb{R}$ is the scalar, $v_1e_1 + v_2e_2 + v_3e_3 \in \mathbb{R}^3$ is the vector, the 2-vector part $a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2$ is called the pseudo vector, and $pe_1e_2e_3$ is the pseudo scalar.

Let $I \coloneqq e_1e_2e_3$ be the unit pseudo scalar of $\mathbb{R}^3$.

Lemma 1.2. *We claim $I^2 = -1$.*

Proof. By a direct calculation:

$$I^2 = e_1e_2e_3e_1e_2e_3 = -e_1e_2e_1e_3e_2e_3 = -e_1e_1e_2e_2e_2e_3 = -1, \tag{29}$$

it follows. ∎

Lemma 1.3. *Let A be an arbitrary multivector of $\mathbb{R}^3$. Then $AI = IA$.*

Proof. It suffices to consider the vector part and pseudo vector part:

- Vector Part

  Consider e_1 and $I = e_1e_2e_3$:

$$e_1e_1e_2e_3 = -e_1e_2e_1e_3 = +e_1e_2e_3e_1. \tag{30}$$

  Similarly, the other cases hold.

- Pseudo Vector Part

  Consider e_2e_3 and $I = e_1e_2e_3$:

$$e_2e_3e_1e_2e_3 = -e_2e_1e_3e_2e_3 = +e_1e_2e_3e_2e_3. \tag{31}$$

  Similarly, the other cases hold.

Hence, the unit pseudo scalar and any multivector are commutative. ∎

Remark 7. Similar calculations such as:

$$Ie_1 = e_1e_2e_3e_1 = +e_2e_1e_1e_3 = e_2e_3 \tag{32}$$

imply that the product of i and any 1-vector is a 2-vector. Moreover, the general multivector of $\mathbb{R}^3$ in (28) is

$$V = s + v + a + p, \tag{33}$$

where $s \in \mathbb{R}$ is a scalar, $v \in \mathbb{R}^3$ is a vector, $a = Iu$ with some vector $u \in \mathbb{R}^3$ is called a pseudo vector, also known as an axial vector, and $p = It$ with some scalar $t \in \mathbb{R}$ is a pseudo scalar.

Lemma 1.4 (Vector Product)**.** *For vectors $u, v \in \mathbb{R}^3$, $u \wedge v = Iu \times v$, where $u \times v$ is defined by:*

$$u \times v = (u_2v_3 - u_3v_2)\, e_1 + (u_3v_1 - u_1v_3)\, e_2 + (u_1v_2 - u_2v_1)\, e_3. \tag{34}$$

Then $(u \times v) \cdot u = 0$ holds.

Proof. Let $u \coloneqq (u_1e_1 + u_2e_2 + u_3e_3)$ and $v \coloneqq (v_1e_1 + v_2e_2 + v_3e_3)$. Since $Ie_1 = e_1I = e_1e_1e_2e_3 = e_2e_3$,

$$\begin{aligned} Iu \times v &= (u_2v_3 - u_3v_2)\, Ie_1 + (u_3v_1 - u_1v_3)\, Ie_2 + (u_1v_2 - u_2v_1)\, Ie_3 \\ &= (u_2v_3 - u_3v_2)\, e_2e_3 + (u_3v_1 - u_1v_3)\, e_3e_1 + (u_1v_2 - u_2v_1)\, e_1e_2 \\ &= (u_1e_1 + u_2e_2 + u_3e_3) \wedge (v_1e_1 + v_2e_2 + v_3e_3) \end{aligned} \tag{35}$$

where we use $e_ie_j = e_i \wedge e_j$. The identity follows by Definition 1.2; we may show directly:

$$\begin{aligned} (u \times v) \cdot u &= (u_2v_3 - u_3v_2)\, u_1 + (u_3v_1 - u_1v_3)\, u_2 + (u_1v_2 - u_2v_1)\, u_3 \\ &= (u_2v_3 - u_3v_2)\, u_1 - u_1v_3u_2 + u_1v_2u_3 + u_3v_1u_2 - u_2v_1u_3 \\ &= 0. \end{aligned} \tag{36}$$

Geometrically, $u \times v \in \mathbb{R}^3$ is a normal vector on the 2-vector $u \wedge v$. ∎

Remark 8 (Quaternion and Vector Product). If we define

$$i := -Ie_1, j := -Ie_2, k := -Ie_3, \tag{37}$$

then

$$i^2 = +Ie_1Ie_1 = +IIe_1e_1 = -1 = j^2 = k^2. \tag{38}$$

Moreover,

$$ijk = -Ie_1Ie_2Ie_3 = Ie_1e_2e_3 = I^2 = -1. \tag{39}$$

The linear space spanned $\langle 1, i, j, k\rangle = \langle 1, e_2e_3, e_3e_1, e_1e_2\rangle$ is the famous Hamlton's quaternions. For $z := z_1i + z_2j + z_3j$ and $w := w_1i + w_2j + w_3j$, their geometric product zw becomes:

$$-\left(z_1w_1 + z_2w_2 + z_3w_3\right) + \left(z_2w_3 - z_3w_2\right)i + \left(z_3w_1 - z_1w_3\right)j + \left(z_1w_2 - z_2w_1\right)k. \tag{40}$$

That is, the real part is essentially the scalar product, and the imaginary part is the vector product of $\mathbb{R}^3$.

Definition 1.4. Let $\nabla := e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}$.

- Three-gradient

  For a scalar-valued function f on $\mathbb{R}^3$,

$$\nabla f = e_1\frac{\partial f}{\partial x} + e_2\frac{\partial f}{\partial y} + e_3\frac{\partial f}{\partial z} \tag{41}$$

- For a vector valued function $V = e_1V_x + e_2V_y + e_3V_z$ on $\mathbb{R}^3$,
  - Divergence

$$\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}. \tag{42}$$

  - Rotation (a.k.a curl)

$$\nabla \times V = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)e_1 + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)e_2 + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)e_3 \tag{43}$$

Remark 9. For simplicity, we may also write $\partial_x := \frac{\partial}{\partial x}$.

Remark 10. For later convenience, consider $\nabla \cdot (\nabla \times V)$:

$$\partial_x\left(\partial_yV_z - \partial_zV_y\right) + \partial_y\left(\partial_zV_x - \partial_xV_z\right) + \partial_z\left(\partial_xV_y - \partial_yV_x\right) \tag{44}$$

If V is smooth enough, we may exchange the order of partial derivatives, so we conclude $\nabla \cdot (\nabla \times V) = 0$. Similarly, $\nabla \cdot (V \times W) = (\nabla \times V) \cdot W - V \cdot (\nabla \times W)$:

$$\begin{aligned}\nabla \cdot (V \times W) &= \partial_x\left(V_yW_z - V_zW_y\right) + \partial_y\left(V_zW_x - V_xW_z\right) + \partial_z\left(V_xW_y - V_yW_x\right)\\ &= \left(\partial_xV_y - \partial_yV_x\right)W_z + V_z\left(-\partial_xW_y + \partial_yW_x\right)\cdots\\ &= (\nabla \times V) \cdot W - V \cdot (\nabla \times W).\end{aligned} \tag{45}$$

Remark 11 (Gradient). If a scalar function $f\colon \mathbb{R}^3 \to \mathbb{R}$ is differentiable, we have

$$f\left(\vec{r}+\vec{h}\right) = f\left(\vec{r}\right) + \vec{h}\cdot\left(\left(\nabla f\right)\left(\vec{r}\right)\right) + \left|\vec{h}\right|\eta\left(\vec{h}\right) \tag{46}$$

for any $\vec{r}\in\mathbb{R}^3$ and $\vec{h}\in\mathbb{R}^3$, where $\eta\left(\vec{h}\right) \dashrightarrow 0$ as $\vec{h} \dashrightarrow 0$. See (2).

1.2.4 $\mathbb{R}_{1,3}$ – Timespace

As an amalgam of a scalar time and 3D space, the dimension of the timespace is four; a point in the timespace is called an event. Let e_0 be the unit time vector with the space $\mathbb{R}^3 = \langle e_1, e_2, e_3\rangle$. Recalling the fundamental asymmetry between time and space, for instance:

$$\left(c(t-t')\right)^2 - \left(x_1 - x_1'\right)^2 - \left(x_2 - x_2'\right)^2 - \left(x_3 - x_3'\right)^2 \tag{47}$$

is invariant under Lorentz transformation for two events $\begin{pmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\begin{pmatrix} ct' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix}$, let e_0 be the unit time vector, satisfying the following anti-commutative property with the space coordinates:

$$e_0e_1 = -e_1e_0, e_0e_2 = -e_2e_0, e_0e_3 = -e_3e_0 \tag{48}$$

with $e_0e_0 = 1$, here we follow [CIA10]. Let $\underline{x}$ be the timespace vector corresponding to an event $\begin{pmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$:

$$\underline{x} \coloneqq \left(ct + x_1e_1 + x_2e_2 + x_3e_3\right)e_0 = \left(ct + \vec{x}\right)e_0, \tag{49}$$

where $\vec{x} \coloneqq x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{R}^3$ is the ordinary space vector. We obtain:

$$\left(\underline{x} - \underline{x'}\right)^2 = \left(c(t-t') + \vec{x} - \vec{x'}\right)e_0\left(c(t-t') + \vec{x} - \vec{x'}\right)e_0 = c^2(t-t')^2 - \left(\vec{x} - \vec{x'}\right)^2. \tag{50}$$

Note that c stands for some constant speed, as a conversion factor from the time coordinate to the space coordinates. See Definition 2.9 for the physical definition of c.

Definition 1.5 (Timespace)**.** Let $\mathbb{R}_{1,3} \coloneqq \langle e_0, e_1e_0, e_2e_0, e_3e_0\rangle$ denote the timespace.

Remark 12. Thanks to the anti-commutative property of e_0, even though ${e_0}^2 = {e_1}^2 = {e_2}^2 = {e_3}^2 = 1$, we obtain the ordinary "almost-minus metric $(+ - - -)$:"

$$e_1e_0e_1e_0 = -e_1e_1e_0e_0 = -1 = e_2e_0e_2e_0 = e_3e_0e_3e_0. \tag{51}$$

Definition 1.6 (D'Alembertian)**.** Following Definition 1.4:

$$\vec{\nabla} := e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z}. \tag{52}$$

We define

$$\Box := \frac{1}{c}\partial_t - \vec{\nabla}, \tag{53}$$

where

$$\partial_t := \frac{\partial}{\partial t} \tag{54}$$

is the ordinary time-derivative. The ordinary 3D Laplacian is given by $\Delta := \vec{\nabla}^2$:

$$\Delta = {\partial_x}^2 + {\partial_y}^2 + {\partial_z}^2. \tag{55}$$

The corresponding operator in the 4D timespace is the following:

$$\underline{\Box} := \Box e_0 = \left(\frac{1}{c}\partial_t - \vec{\nabla}\right) e_0. \tag{56}$$

Then 4D D'Alembertian becomes:

$$\underline{\Box}^2 = \Box e_0 \Box e_0 = \frac{1}{c^2}{\partial_t}^2 - \vec{\nabla}^2. \tag{57}$$

2 Maxwell's Equations

2.1 Principle of Locality – Action through Medium

The principle of locality states that physical interactions – such as force and energy exchange, or in general, information transfer – occur at finite speeds, affecting only neighboring points in space.

- Action at a distance

  As assumed in the early Newtonian gravity and electrostatics, this concept implies instantaneous effects over arbitrary distances. For example, Coulomb's law in electrostatics describes the force between two point charges as if it acts instantly, without any intermediate mechanism.

- Locality principle and the concept of Fields – Action through a medium

  Faraday introduced the concept of electric and magnetic fields, visualizing them through field "lines" whose density is proportional to the field strength, and whose direction represents the orientation of the field vectors. The principle of locality asserts that distortions in these fields affect only their immediate surroundings, leading to a chain reaction that propagates through space.

- Waves propagation

  The concept of fields naturally leads to the idea that interactions are mediated through some medium. In electromagnetism, time-varying electric and magnetic fields generate each other, resulting in wave propagation. Maxwell's equations predict that electromagnetic waves in vacuum travel at a finite speed, reinforcing the principle that interactions are not instantaneous but instead propagate at a finite speed through space.

Locality is guaranteed, in some sense, when the guiding principles are written in terms of differential equations since the infinitesimal comparisons essentially give differentials.

2.2 EM Crash Course

Standard textbooks include [Gri23] and [Jac21].

Definition 2.1 (Test Charges and Static Electric Fields)**.** A test charge q is an object whose size is macroscopically small enough so that we can identify it as a point. Through a test charge as a probe, we define the static electrostatic field $\vec{E}$ at $\vec{x} \in \mathbb{R}^3$:

$$\vec{f} = q\vec{E}(\vec{x}), \tag{58}$$

where $\vec{f}$ is the force on q due to the underlying static electric field $\vec{E}$.

Definition 2.2 (Magnetic Flux Density)**.** Consider $\vec{I}$ of a steady current under some magnetic fields. Across a line segment δl of $\vec{I}$, let $\vec{f}$ be the force on the current $\vec{I}$. We define the underlying magnetic flux density $\vec{B}$ by

$$\vec{f} = \vec{I} \times \vec{B}\delta l. \tag{59}$$

Definition 2.3 (Coulomb's Law)**.** Let Q and q be two charged particles and $r > 0$ be the spacial distance between Q and q. Consider the force $\vec{f}$ acting on q due to Q:

$$\left|\vec{f}\right| = k\frac{|qQ|}{r^2}. \tag{60}$$

This relation, with $k > 0$ is called Coulom's law. For later convenience, we introduce $\epsilon_0 > 0$ via

$$k = \frac{1}{4\pi\epsilon_0}. \tag{61}$$

Remark 13 (Gauss' Law). Suppose $Q > 0$ is at the origin $0 \in \mathbb{R}^3$. Let $\vec{x} \in \mathbb{R}^3$ be a point on the spherical shell with radius $r > 0$, and $\vec{n}(\vec{x})$ be the unit normal vector from in to out direction. Then, since the electric field $\vec{E}$ at x is outward,

$$\vec{E}(\vec{x}) \cdot \vec{n}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}. \tag{62}$$

Summing all the contributions up on the spherical shell, i.e., by integrating over the surface:

$$\int_{|x|=r} \vec{E}(\vec{x}) \cdot \vec{n}(\vec{x}) \mathrm{d}S = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} 4\pi r^2 = \frac{Q}{\epsilon_0}. \tag{63}$$

In general, over the surface S of some volume V:

$$\int_{\partial V} \vec{E}(\vec{x}) \cdot \vec{n}(\vec{x}) \mathrm{d}S = \frac{\text{the total charges in } V}{\epsilon_0}, \tag{64}$$

where $\partial V = S$ represents the surface – the boundary – of the given volume V. It is worth mentioning that the electric filed $\vec{E}$ at $\vec{x} \in \partial V$ contains the contributions from both the charges in V and outside of V.

Remark 14 (Differential form of Gauss's Law)*.* Introducing the charge density ρ, Gauss's law becomes:

$$\int_{\partial V} \vec{E}(\vec{x}) \cdot \vec{n}(\vec{x}) \mathrm{d}S = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}) \mathrm{d}^3 x. \tag{65}$$

Applying Gauss' divergent theorem, we obtain

$$\int_V \vec{\nabla} \cdot \vec{E}(x) \mathrm{d}^3 x = \frac{1}{\epsilon_0} \int_V \rho(\vec{x}) \mathrm{d}^3 x. \tag{66}$$

Since this equation is the case for any volume, we conclude:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho. \tag{67}$$

I.e., for any location $\vec{x} \in \mathbb{R}^3$, $\vec{\nabla} \cdot \vec{E}(\vec{x}) = \frac{1}{\epsilon_0} \rho(\vec{x})$ holds.

If $\vec{E}$ depends only on x, $E(x + \mathrm{d}x) = E(x) + \frac{\rho(x)}{\epsilon_0} \mathrm{d}x$. This is a concrete example of locality: the electric field at $x + \mathrm{d}x$ is determined by the electric field and the source at x. Since we have three unknown functions E_x, E_y, and E_z, with one condition (67), the system is still underdetermined.

Remark 15 (Magnetic Gauss' Law)*.* Similarly:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{68}$$

since no discovery yet of a magnetic monopole.

Definition 2.4 (Time-varying Gauss' Laws)**.** We suppose, for any $\vec{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$:

$$\begin{aligned} \nabla \cdot \vec{E}(\vec{x}, t) &= \frac{1}{\epsilon_0} \rho(\vec{x}, t) \\ \nabla \cdot \vec{B}(\vec{x}, t) &= 0 \end{aligned} \tag{69}$$

for time-varying fields and charge density.

Definition 2.5 (Faraday-Lenz Law)**.** Consider a surface are A with its normal vector $\vec{n}$. Define $\vec{A} := A\vec{n}$. Through this directed are $\vec{A}$, let

$$\Phi := \vec{B} \cdot \vec{A} = \left|\vec{B}\right| \left|\vec{A}\right| \cos\theta, \tag{70}$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. Along the boundary ∂A,

$$emf = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi, \tag{71}$$

where the orientation of ∂A is determined by the RHR. With this important negative sign – Lenz' law – it is called Faraday's law of induction.

Remark 16 (Differential Form). Recalling *emf* is the work along the boundary ∂A per charge and the electric field is the force per charge:

$$emf = \int_{\partial A} \vec{E} \cdot \mathrm{d}\vec{r}. \tag{72}$$

Then, we obtain:

$$\int_{\partial A} \vec{E} \cdot \mathrm{d}\vec{r} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_A \vec{B} \cdot \vec{n}\mathrm{d}S. \tag{73}$$

Applying Stokes' theorem,

$$\int_A \vec{\nabla} \times \vec{E} \cdot \vec{n}\mathrm{d}S = -\int_A \partial_t \vec{B} \cdot \vec{n}\mathrm{d}S. \tag{74}$$

Hence, we conclude $\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = \vec{0}$.

Remark 17. The static electric field is completely determined by:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E}(\vec{x}) &= \frac{1}{\epsilon_0}\rho(\vec{x}) \\ \vec{\nabla} \times \vec{E}(\vec{x}) &= \vec{0} \end{aligned} \tag{75}$$

for each $\vec{x} \in \mathbb{R}^3$. It is worth mentioning that the rotation-free condition ensures that the underlying force is conservative since the work done by an arbitrary closed path is zero.

Definition 2.6 (Ampeȑe's Law)**.** Consider a disc D with radius $r > 0$ at the origin $\vec{0} \in \mathbb{R}^3$. Along with a steady current $\vec{I}$ at the center of D, consider the magnetic flux $\int_{|\vec{x}|=r} \vec{B}(\vec{x})\mathrm{d}\vec{r}$ along the boundary of S. Experimentally, one shows that this integral is proportional to $\left|\vec{I}\right|$:

$$\int_{|\vec{x}|=r} \vec{B}(\vec{x}) \cdot \mathrm{d}\vec{r} = \mu_0 \left|\vec{I}\right|, \tag{76}$$

where the boundary of D is expressed as $|\vec{x}| = r$ with $\mu_0 > 0$. Introducing the current density $\vec{i}$:

$$\left|\vec{I}\right| = \int_{|\vec{x}|\leqq r} \vec{i}(\vec{x}) \cdot \vec{n}(\vec{x})\mathrm{d}S \tag{77}$$

Applying Stokes' theorem,

$$\int_{|\vec{x}|\leqq r} \vec{\nabla} \times \vec{B}(\vec{x}) \cdot \vec{n}(x)\mathrm{d}S = \int_{|\vec{x}|=r} \vec{B}(\vec{x}) \cdot \mathrm{d}\vec{r} = \mu_0 \int_{|\vec{x}|\leqq r} \vec{i}(\vec{x}) \cdot \vec{n}(\vec{x})\mathrm{d}S. \tag{78}$$

Hence, $\vec{\nabla} \times \vec{B}(\vec{x}) = \mu_0\vec{i}(\vec{x})$ for each $\vec{x} \in \mathbb{R}^3$.

Remark 18. The magnetic flux density around $\vec{I}$ is proportional to $\left|\vec{I}\right|$ and inversely proportional to the distance r. Geometrically, it is rather proportional to the circumference:

$$\left|\vec{B}\right| = \mu_0 \frac{\left|\vec{I}\right|}{2\pi r}. \tag{79}$$

Definition 2.7 (Charge-Current Conservation)**.** Since the normal vector is from in to out, for any volume V, the macroscopic relation $\frac{\mathrm{d}Q}{\mathrm{d}t} = -I$ via out-flow I becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \mathrm{d}^3x = -\int_{\partial V} \vec{i} \cdot \vec{n}\mathrm{d}S. \tag{80}$$

Applying Gauss' theorem,

$$\int_V \partial_t\rho(\vec{x})\mathrm{d}^3x = -\int_V \vec{\nabla} \cdot \vec{i}(\vec{x})\mathrm{d}^3x \tag{81}$$

and, hence, we obtain the continuity equation $\partial_t\rho + \vec{\nabla} \cdot \vec{i} = 0$.

Definition 2.8 (Maxwell's Displacement Current)**.** Let $\vec{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$. For the static Ampère's Law $\vec{\nabla} \times \vec{B}(\vec{x}) = \mu_0\vec{i}(\vec{x})$, Maxwell introduced, so-called, displacement current density:

$$\vec{\nabla} \times \vec{B}(\vec{x}, t) = \mu_0\vec{i}(\vec{x}, t) + \mu_0 \underbrace{\epsilon_0\partial_t\vec{E}(\vec{x}, t)} \tag{82}$$

Applying $\vec{\nabla}\cdot$ from their left, by Remark 10, we conclude:

$$\begin{aligned} \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B}(\vec{x}, t)\right) = 0 &= \mu_0 \left(\vec{\nabla} \cdot \vec{i}(\vec{x}, t) + \underbrace{\epsilon_0\partial_t\vec{\nabla} \cdot \vec{E}(\vec{x}, t)}\right) \\ &= \mu_0 \left(\vec{\nabla} \cdot \vec{i}(\vec{x}, t) + \epsilon_0\partial_t\frac{\rho(\vec{x}, t)}{\epsilon_0}\right). \end{aligned} \tag{83}$$

Therefore, we obtain the continuity equation. Without the displacement current, we would have $\vec{\nabla} \cdot \vec{i}(\vec{x}, t) = 0$ and, hence $\partial_t\rho(\vec{x}, t) = 0$, which is in general not the case.

Definition 2.9 (Maxwell-Ampère Law with c)**.** Since

$$[\epsilon_0] = [1/k] = \left(\mathrm{N\,m^2\,C^{-2}}\right)^{-1} = \mathrm{A^2\,s^4\,kg^{-1}\,m^{-3}} \tag{84}$$

and

$$[\mu_0] = \frac{[B]\,\mathrm{m}}{\mathrm{A}} \tag{85}$$

where $[B] = \frac{\mathrm{N}}{\mathrm{C\,m\,s^{-1}}} = \mathrm{kg\,A^{-1}\,s^{-2}}$, we conclude

$$[\mu_0 \epsilon_0] = \frac{\mathrm{kg}}{\mathrm{A\,s^2}} \mathrm{m\,s^{-1}} \frac{\mathrm{A^2\,s^4}}{\mathrm{kg\,m^{-3}}} = \left(\frac{1}{\mathrm{m\,s^{-1}}} \right)^2. \tag{86}$$

That is,

$$c \coloneqq \frac{1}{\sqrt{\mu_0 \epsilon_0}} > 0 \tag{87}$$

has $\mathrm{m\,s^{-1}}$. It is worth mentioning that $[x] = [y] = [z] = [ct] = \mathrm{m}$. With this constant c, Maxwell-Ampère Law is:

$$\vec{\nabla} \times \vec{B}(\vec{x}, t) - \frac{1}{c^2} \partial_t \vec{E}(\vec{x}, t) = \mu_0 \vec{i}(\vec{x}, t). \tag{88}$$

Remark 19 (Tesla). We define $\mathrm{T} = \mathrm{kg\,A^{-1}\,s^{-2}} = \mathrm{Wb\,m^{-2}}$ for the magnetic flux density.

Remark 20. It is worth mentioning that $[E] = [cB]$:

$$\begin{aligned} [E] &= \mathrm{N\,C^{-1}} = \mathrm{kg\,m\,s^{-2}\,A^{-1}\,s^{-1}} \\ &= \mathrm{kg\,m\,s^{-3}\,A^{-1}} \\ [cB] &= \mathrm{m\,s^{-1} kg\,s^{-2}\,A^{-1}} \\ &= \mathrm{kg\,m\,s^{-3}\,A^{-1}} \end{aligned} \tag{89}$$

Definition 2.10 (Maxwell's Equations)**.** Collecting the outcomes from Definition 2.4, Definition 2.5, and Definition 2.9, we obtain:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0} \rho \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= \vec{0} \\ \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \partial_t \vec{E} &= \mu_0 \vec{i}. \end{aligned} \tag{90}$$

As a charge conservation, see Definition 2.7, we have the following continuity equation:

$$\partial_t \rho + \vec{\nabla} \cdot \vec{i} = 0. \tag{91}$$

Remark 21. Applying the identity in Remark 10, we obtain:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) - \mu_0 \epsilon_0 \partial_t \left(\vec{\nabla} \cdot \vec{E} \right) = 0 - \mu_0 \epsilon_0 \partial_t \left(\vec{\nabla} \cdot \vec{E} \right) = \mu_0 \vec{\nabla} \cdot \vec{i} = -\mu_0 \partial_t \rho, \tag{92}$$

where the last equation is due to the charge-current conservation in Definition 2.7. Thus, we obtain:

$$\partial_t \left(\epsilon_0 \vec{\nabla} \cdot \vec{E} - \rho \right) = 0. \tag{93}$$

At the time origin $t = 0$, if $\epsilon_0 \vec{\nabla} \cdot \vec{E}(\vec{x}, 0) - \rho(\vec{x}, 0) = 0$ is chosen as the initial condition, so is $\epsilon_0 \vec{\nabla} \cdot \vec{E}(\vec{x}, t) - \rho(\vec{x}, t) = 0$ for any $t > 0$. Similarly, applying $\vec{\nabla} \cdot$ both sides, we obtain:

$$0 = \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{E} + \partial_t \vec{B} \right) = 0 + \partial_t \vec{\nabla} \cdot \vec{B}. \tag{94}$$

At the time origin $t = 0$, if $\vec{\nabla} \cdot \vec{B}(\vec{x}, 0) = 0$ is chosen as the initial condition, so is $\vec{\nabla} \cdot \vec{B}(\vec{x}, t) - \rho(\vec{x}, t) = 0$ for any $t > 0$.

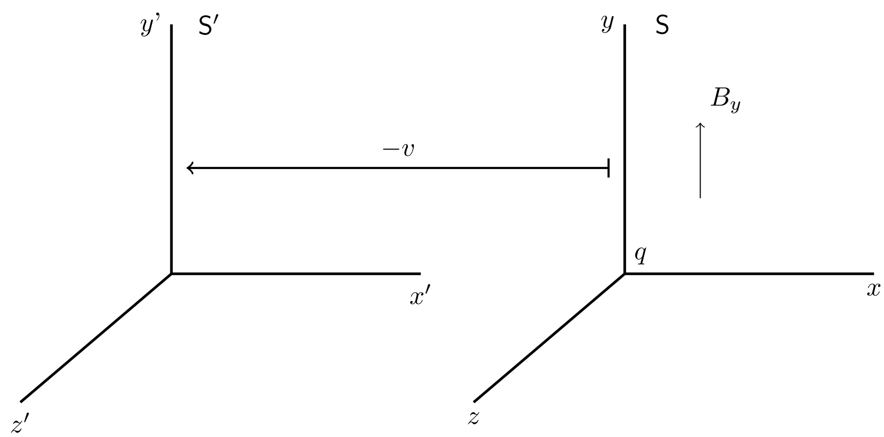
Therefore, the Maxwell equations consist of 2 initial conditions and 6 dynamical equations for 6 unknown functions.

Remark 22. For the charge density ρ and the current density $\vec{i}$ under electromagnetic field $\vec{E}$ and $\vec{B}$, the Lorentz' force density is given by $\vec{f} = \rho \vec{E} + \vec{i} \times \vec{B}$. Maxwell's equations in (90) with the Lorentz' force on a moving charge q with velocity $\vec{v}$:

$$\vec{f}_q = q \left(\vec{E} + \vec{v} \times \vec{B} \right) \tag{95}$$

completely characterize classical electrodynamics.

Consider a charge $q > 0$ at rest on a system S under uniform B_y. Since q does not move, q feels no external force. What if we observe the same q from another system S' that is moving backward along x-direction:



Relative to S', q is under motion with velocity v along x'-direction. Hence, q feels force; according to 95, the acceleration is z'-direction:

- Relative to S:

  No force on z-direction.

- Relative to S':

  Along z'-direction, q feels magnetic force qvB_y.

Such a contradiction leads us to the special theory of relativity; the "assumption" that the magnetic field B_y remains in S' is wrong. Electric fields and magnetic fields are not independent, forming a unified concept of electromagnetic fields.

Definition 2.11 (Electromagnetic Energy Density)**.** We call:

$$u := \frac{\epsilon_0}{2}\left|\vec{E}\right|^2 + \frac{1}{2\mu_0}\left|\vec{B}\right|^2 \tag{96}$$

the electromagnetic energy density.

Remark 23 (Electric Energy Stored in a Capacitor)**.** Consider a parallel plate capacitor with area A and spacial distance d. As the stored charge Q is proportional to the potential difference V, we let C be the capacitance via $Q = CV$. Under the potential difference V, the stored energy is given by:

$$U = \int_{q=0}^{CV} \frac{q}{V}\mathrm{d}q = \frac{1}{2}CV^2. \tag{97}$$

Let E be the electric field strength in the capacitor:

- Gauss's Law

  Across one plate, Gauss's law per unit area implies:

  $$E - 0 = \frac{1}{\epsilon_0}\frac{Q}{A} \tag{98}$$

  Hence, $Q = \epsilon_0 AE$.

- Recalling that voltage is the energy per unit charge:

  $$V = Ed. \tag{99}$$

Then:

$$U = \frac{1}{2}QV = \frac{1}{2}\epsilon_0 AEEd = \frac{\epsilon_0}{2}E^2 Ad, \tag{100}$$

where Ad is the physical volume of the parallel plate capacitor. Hence, the energy density is given by $\frac{\epsilon_0}{2}E^2$.

2.3 Maxwell's Equations in Geometric Algebra

Definition 2.12 (Electromagnetic Field Strength Tensor)**.** Recalling §1.2.4, we define:

$$F := \vec{E} + \sqrt{-1}c\vec{B}, \tag{101}$$

where

$$\sqrt{-1} := e_1 e_2 e_3. \tag{102}$$

Remark 24. Both $\vec{E}$ and $\vec{B}$ are in $\mathbb{R}^3$ are conventionally represented as:

$$\begin{aligned} \vec{E} &= e_1 E_x + e_2 E_y + e_3 E_z \\ \vec{B} &= e_1 B_x + e_2 B_y + e_3 B_z \end{aligned} \tag{103}$$

Since the coordinate system, in particular the right-hand rule $e_3 = e_1 \times e_2$ is our choice, not the nature's, the description should not depend on the specific choice. Under space-inversion, also known as the prity-inversion – the flip in the sign of one spatial coordinate – ordinary vectors must flip its sign, for instance, in (95) of the Lorentz' force, say force, electric fields, and velocity vector:

$$\begin{aligned} \vec{f}_q &\mapsto -\vec{f}_q \\ \vec{E} &\mapsto -\vec{E} \\ \vec{v} &\mapsto -\vec{v} \end{aligned} \tag{104}$$

However, $\vec{B} \mapsto +\vec{B}$ remains, otherwise, we have a relative sign difference between two terms. This property ensures that the appropriate representation for the magnetic flux density is rather via 2-vectors. With $\sqrt{-1} \coloneqq e_1 e_2 e_3$, $\vec{B} \in \mathbb{R}^3$ becomes a 2-vector:

$$\sqrt{-1} c \vec{B} = c \left(e_2 e_3 B_x + e_3 e_1 B_y + e_1 e_2 B_z \right). \tag{105}$$

Since the parity-inversion is $e_1 \mapsto -e_1, e_2 \mapsto -e_2$, and $e_3 \mapsto -e_3$, we obtain the appropriate sign under the parity-inversion, $\sqrt{-1} c \vec{B} \mapsto \sqrt{-1} c \vec{B}$:

$$\vec{f}_q = q \left(\vec{E} - \frac{\vec{v}}{c} \wedge \sqrt{-1} c \vec{B} \right). \tag{106}$$

Lemma 2.1. *We claim:*

$$e_0 \vec{E} = -\vec{E} e_0 \tag{107}$$

and

$$e_0 \sqrt{-1} c \vec{B} = \sqrt{-1} c \vec{B} e_0. \tag{108}$$

Proof. Recalling (48),

$$e_0 \vec{E} = e_0 e_1 E_x + e_0 e_2 E_y + e_0 e_3 E_z = -e_1 e_0 E_x - e_2 e_0 E_y - e_3 e_0 E_z = -\vec{E} e_0, \tag{109}$$

and

$$\begin{aligned} e_0 \sqrt{-1} c \vec{B} &= c e_0 \left(e_2 e_3 B_x + e_3 e_1 B_y + e_1 e_2 B_z \right) \\ &= +c \left(e_2 e_3 B_x + e_3 e_1 B_y + e_1 e_2 B_z \right) e_0 \\ &= \sqrt{-1} c \vec{B} e_0. \end{aligned} \tag{110}$$

■

Theorem 2.1. *Maxwell's equations in* (90) *can be written as:*

$$\underline{\Box} F = c\mu_0 \underline{J}, \tag{111}$$

where $\underline{\Box} = \left(\frac{1}{c}\partial_t - \vec{\nabla}\right) e_0$ *is the d'Alambertian in Definition 1.6 and* $\underline{J} := \left(c\rho + \vec{i}\right) e_0$ *is the charge-current multivector.*

Proof. The right-hand side is:

$$c\mu_0 \underline{J} = \left(c^2\mu_0\rho + c\mu_0\vec{i}\right) e_0 = \left(\frac{1}{\epsilon_0}\rho + c\mu_0\vec{i}\right) e_0. \tag{112}$$

Recalling that it is the geometric product, by Lemma 2.1:

$$\begin{aligned}
\underline{\Box} F &= \left(\frac{1}{c}\partial_t - \vec{\nabla}\right) e_0 \left(\vec{E} + \sqrt{-1}c\vec{B}\right) \\
&= \left(\frac{1}{c}\partial_t - \vec{\nabla}\right) \left(-\vec{E} + \sqrt{-1}c\vec{B}\right) e_0 \\
&= \left(-\frac{1}{c}\partial\vec{E} + \sqrt{-1}\partial_t\vec{B} + \vec{\nabla}\vec{E} - \sqrt{-1}c\vec{\nabla}\vec{B}\right) e_0 \\
&= \left(-\frac{1}{c}\partial\vec{E} + \sqrt{-1}\partial_t\vec{B} + \left(\vec{\nabla}\cdot\vec{E} + \vec{\nabla}\wedge\vec{E}\right) - \sqrt{-1}c\left(\vec{\nabla}\cdot\vec{B} + \vec{\nabla}\wedge\vec{B}\right)\right) e_0.
\end{aligned} \tag{113}$$

By Lemma 1.4,

$$\begin{aligned}
\underline{\Box} F = &\Big(-\frac{1}{c}\partial\vec{E} + \sqrt{-1}\partial_t\vec{B} \\
&+ \left(\vec{\nabla}\cdot\vec{E} + \sqrt{-1}\vec{\nabla}\times\vec{E}\right) - \sqrt{-1}c\left(\vec{\nabla}\cdot\vec{B} + \sqrt{-1}\vec{\nabla}\times\vec{B}\right)\Big) e_0 \\
= &\Big(\vec{\nabla}\cdot\vec{E} - \sqrt{-1}c\vec{\nabla}\cdot\vec{B} \\
&+ \left(-\frac{1}{c}\partial\vec{E} + c\vec{\nabla}\times\vec{B}\right) + \sqrt{-1}\left(\partial_t\vec{B} + \vec{\nabla}\times\vec{E}\right)\Big) e_0.
\end{aligned} \tag{114}$$

Comparing the components, we obtain the desired results, since the right-hand side is:

$$c\mu_0\underline{J} = \left(\frac{1}{\epsilon_0}\rho + \sqrt{-1}0 + c\mu_0\vec{i} + \sqrt{-1}\vec{0}\right) e_0. \tag{115}$$

■

Remark 25 (Continuity Equation). The continuity equation in Definition 2.7 is expressed as the following 4-divergence $\underline{\Box}\cdot\underline{J} = 0$, since

$$\underline{\Box}\cdot\underline{J} = \left(\frac{1}{c}\partial_t - \vec{\nabla}\right) e_0 \cdot \left(c\rho + \vec{i}\right) e_0 = \partial_t\rho - \left(-\vec{\nabla}\cdot\vec{i}\right) = 0, \tag{116}$$

recalling the "almost-minus metric" in Remark 12.

2.4 Some Applications

2.4.1 Energy and Momentum – Poynting's Theorem

Theorem 2.2 (Poynting's Theorem)**.** *The rate of energy loss of fields per unit time $-\partial_t u$ is:*

$$-\frac{\partial u}{\partial t} = \vec{\nabla} \cdot \vec{S} + \vec{E} \cdot \vec{i}, \tag{117}$$

where u is defined in Definition 2.11, $\vec{S} \coloneqq \frac{1}{\mu_0}\vec{E} \times \vec{B}$ is called Poynting vector, $\vec{\nabla} \cdot \vec{S}$ is energy out-flow per unit time, and $\vec{E} \cdot \vec{i}$ represents the rate of the work done by fields.

Proof. The work done by electromagnetic fields per unit time satisfies:

$$P \coloneqq \int \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \rho\vec{v}\mathrm{d}^3x, \tag{118}$$

where $\vec{v}$ represents the velocity vector of $\rho\vec{v}\mathrm{d}^3x$.

- Since $\left(\vec{v} \times \vec{B}\right) \cdot \vec{v} = 0$,

  $$P = \int \vec{E} \cdot \rho\vec{v}\mathrm{d}^3x = \int \vec{E} \cdot \vec{i}\mathrm{d}^3x. \tag{119}$$

  Note that $\vec{B}$ provides no work.

- Recalling Definition 2.9, Remark 10, and Definition 2.5:

  $$\begin{aligned} P &= \int \vec{E} \cdot \frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B} - \mu_0\epsilon_0\partial_t\vec{E}\right)\mathrm{d}^3x \\ &= \int \frac{1}{\mu_0}\left(\left(\vec{\nabla} \times \vec{E}\right) \cdot \vec{B} - \vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right)\right)\mathrm{d}^3x \\ &\quad - \int \epsilon_0\vec{E} \cdot \partial_t\vec{E}\mathrm{d}^3x \\ &= \int \frac{1}{\mu_0}\left(\left(-\partial_t\vec{B}\right) \cdot \vec{B} - \vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right)\right)\mathrm{d}^3x \\ &\quad - \int \epsilon_0\vec{E} \cdot \partial_t\vec{E}\mathrm{d}^3x. \end{aligned} \tag{120}$$

Hence, we obtain the integral form:

$$\int \vec{E}\cdot\vec{i}\mathrm{d}^3x = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int\left(\frac{1}{\mu_0}\left|\vec{B}\right|^2 + \epsilon_0\left|\vec{E}\right|^2\right)\mathrm{d}^3x - \frac{1}{\mu_0}\int\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)\mathrm{d}^3x \tag{121}$$

Since this relation is the case for any volume, we conclude $\vec{E}\cdot\vec{i} = -\partial_t u - \nabla\cdot\vec{S}$. ∎

Remark 26. Observe:

$$\begin{aligned} Fe_0F &= \left(\vec{E}+\sqrt{-1}c\vec{B}\right)e_0\left(\vec{E}+\sqrt{-1}c\vec{B}\right) \\ &= \left(\vec{E}+\sqrt{-1}c\vec{B}\right)\left(-\vec{E}+\left(-\sqrt{-1}\right)c\left(-\vec{B}\right)\right)e_0 \\ &= \left(-\left|\vec{E}\right|^2+\sqrt{-1}c\vec{E}\vec{B}-\sqrt{-1}c\vec{B}\vec{E}-\frac{1}{\mu_0\epsilon_0}\left|\vec{B}\right|^2\right)e_0 \\ &= \left(-\frac{2}{\epsilon_0}\left(\epsilon_0\left|\vec{E}\right|^2+\frac{1}{\mu_0}\left|\vec{B}\right|^2\right)+\sqrt{-1}c2\vec{E}\wedge\vec{B}\right)e_0 \\ &= \left(-\frac{2}{\epsilon_0}u+\sqrt{-1}c2\mu_0\vec{S}\right)e_0, \end{aligned} \tag{122}$$

i.e.,

$$\left(cu+\vec{S}\right)e_0=-\frac{c\epsilon_0}{2}Fe_0F. \tag{123}$$

If we take 4-divergence:

$$\underline{\square}\cdot\left(cu+\vec{S}\right)e_0=\partial_t u+\vec{\nabla}\cdot\vec{S}=-\vec{E}\cdot\vec{i}. \tag{124}$$

Define $T_0 \coloneqq \frac{\epsilon_0}{2}Fe_0F$; if no source exists, we obtain the continuity equation $\underline{\square}\cdot T_0=0$. Recalling Remark 25, the direction of energy transfer is along $\vec{S}$.

2.4.2 Plane Wave Solution and Speed of Light

Consider Maxwell's equation with no source, namely $\rho=0$ and $\vec{i}=\vec{0}$:

$$\underline{\square}F=0. \tag{125}$$

Since

$$\underline{\square}=\left(\frac{1}{c}\partial_t-\vec{\nabla}\right)e_0=e_0\left(\frac{1}{c}\partial_t+\vec{\nabla}\right), \tag{126}$$

it suffices to consider:

$$\left(\frac{1}{c}\partial_t+\vec{\nabla}\right)F=0. \tag{127}$$

Assume the following form of a "trial" solution; adjusting parameters in this solution, we will obtain a specific form of solution, the so-called plane-wave solution:

$$F\begin{pmatrix}ct\\ \vec{r}\end{pmatrix}=A\exp\left(\sqrt{-1}\left(\vec{k}\cdot\vec{r}-\frac{\omega}{c}ct\right)\right), \tag{128}$$

where

- the amplitude A is a multivector of timespace;
- $\begin{pmatrix}ct\\ \vec{r}\end{pmatrix}$ is an event, namely:
  - t is an arbitrary time;

– $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an arbitrary point of the space;

- $\vec{k}$ is a constant vector in $\mathbb{R}^3$;
- ω is a constant scalar.

Applying $\left(\frac{1}{c}\partial_t + \vec{\nabla}\right)$, we obtain:

$$0 = \left(-\frac{\omega}{c} + \vec{k}\right) A \exp\left(\sqrt{-1}\left(\vec{k}\cdot\vec{r} - \omega t\right)\right). \tag{129}$$

Hence, $\left(-\frac{\omega}{c} + \vec{k}\right) A = 0$. Recalling (101), we expect F to have both vector and pseudo vector parts. So we try $A = \vec{V} + \sqrt{-1}c\vec{W}$ form, where both $\vec{V}$ and $\vec{W}$ are vectors in $\mathbb{R}^3$:

$$\begin{aligned} 0 &= \left(-\frac{\omega}{c} + \vec{k}\right)\left(\vec{V} + \sqrt{-1}c\vec{W}\right) \\ &= -\frac{\omega}{c}\vec{V} - \sqrt{-1}\omega\vec{W} + \vec{k}\vec{V} + \sqrt{-1}c\vec{k}\vec{W}. \end{aligned} \tag{130}$$

Collecting the same grade terms, we obtain:

$$0 = \vec{k}\cdot\vec{V} + \left(-\frac{\omega}{c}\vec{V} + \sqrt{-1}c\vec{k}\wedge\vec{W}\right) + \left(\vec{k}\wedge\vec{V} - \sqrt{-1}\omega\vec{W}\right) + \sqrt{-1}\vec{k}\cdot\vec{W}. \tag{131}$$

- Scalar part: $\vec{k}\cdot\vec{V} = 0$. That is, $\vec{k}\perp\vec{V}$ in $\mathbb{R}^3$.
- Pseudo scalar part: $\vec{k}\cdot\vec{W} = 0$. That is, $\vec{k}$ is “tangential” to the 2-vector $\sqrt{-1}\vec{W}$.
- Vector part: $-\frac{\omega}{c}\vec{V} + \sqrt{-1}c\vec{k}\wedge\vec{W} = 0$. That is, $\vec{V} = \sqrt{-1}\frac{c^2}{\omega}\vec{k}\wedge\vec{W} = -\frac{c^2}{\omega}\vec{k}\times\vec{W}$ if $\omega \neq 0$.
- Pseudo vector part: $\vec{k}\wedge\vec{V} - \sqrt{-1}\omega\vec{W} = 0$. That is, the 2-vector is $\sqrt{-1}\vec{W} = \frac{1}{\omega}\vec{k}\wedge\vec{V}$ if $\omega \neq 0$.

See Lemma 1.4. For later analysis, we assume $\omega \neq 0$.

Applying Lemma 1.4, $\vec{W} = \frac{1}{\omega}\vec{k}\times\vec{V}$, we conclude that $\vec{k}, \vec{V}$, and $\vec{W}$ form a right-hand system. In other words, both $\vec{k}$ and $\vec{V}$ are on $\sqrt{-1}\vec{W}$ with $\vec{k}\perp\vec{V}$.

Theorem 2.3. *The phase speed $\frac{\omega}{|\vec{k}|}$ is c.*

Proof. Since $\vec{V} = \frac{c^2}{\omega}\vec{W}\times\vec{k}$ and $\vec{W} = \frac{1}{\omega}\vec{k}\times\vec{V}$, we have

$$\vec{V} = \frac{c^2}{\omega}\left(\frac{1}{\omega}\vec{k}\times\vec{V}\right)\times\vec{k} \tag{132}$$

Since they are perpendicular,

$$\vec{V} = \frac{c^2}{\omega^2} \left|\vec{k}\right|^2 \vec{V}. \tag{133}$$

Hence $\frac{c^2}{\omega^2} \left|\vec{k}\right|^2 = 1$. ∎

Remark 27. For simplicity, if we choose $\vec{k} = \frac{\omega}{c} e_1$, the phase in (128) becomes:

$$\sqrt{-1}\left(\frac{\omega}{c}x - \omega t\right) \tag{134}$$

That is, during $t \mapsto t + \mathrm{d}t$, if $x \mapsto x + \mathrm{d}x$, we obtain $\frac{\omega}{c}\mathrm{d}x - \omega \mathrm{d}t = 0$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c. \tag{135}$$

Theorem 2.4. *The energy and momentum density of the plane wave solution satisfies $u = \epsilon_0 \left|\vec{V}\right|^2$ and $\vec{S} = cu\frac{\vec{k}}{|\vec{k}|}$. The energy density is the amplitude squared, and the momentum is in the propagation direction.*

Proof. The trial function (128) with $A = \vec{V} + \sqrt{-1}c\vec{W}$ becomes

$$F\begin{pmatrix} ct \\ \vec{r} \end{pmatrix} = \left(\vec{V} + \sqrt{-1}c\vec{W}\right)\exp\left(\sqrt{-1}\left(\vec{k}\cdot\vec{r} - \omega t\right)\right) \tag{136}$$

Since e_0 is anti-commutative,

$$\begin{aligned} e_0 F\begin{pmatrix} ct \\ \vec{r} \end{pmatrix} &= \left(-\vec{V} + \sqrt{-1}c\vec{W}\right)\exp\left(-\sqrt{-1}\left(\left(-\vec{k}\right)\cdot(-\vec{r}) - \omega t\right)\right)e_0 \\ &= \left(-\vec{V} + \sqrt{-1}c\vec{W}\right)\exp\left(-\sqrt{-1}\left(\vec{k}\cdot\vec{r} - \omega t\right)\right)e_0. \end{aligned} \tag{137}$$

Hence,

$$\begin{aligned} Fe_0F &= \left(\vec{V} + \sqrt{-1}c\vec{W}\right)\exp\left(\sqrt{-1}\left(\vec{k}\cdot\vec{r} - \omega t\right)\right)\left(-\vec{V} + \sqrt{-1}c\vec{W}\right)\exp\left(-\sqrt{-1}\left(\vec{k}\cdot\vec{r} - \omega t\right)\right)e_0 \\ &= \left(\vec{V} + \sqrt{-1}c\vec{W}\right)\left(-\vec{V} + \sqrt{-1}c\vec{W}\right)e_0 \\ &= \left(-\left|\vec{V}\right|^2 - c^2\left|\vec{W}\right|^2 + \sqrt{-1}c\left(\vec{V}\vec{W} - \vec{W}\vec{V}\right)\right)e_0 \\ &= \left(-\left|\vec{V}\right|^2 - c^2\left|\vec{W}\right|^2 + \sqrt{-1}c2\vec{V}\wedge\vec{W}\right)e_0 \\ &= \left(-\left(\left|\vec{V}\right|^2 + c^2\left|\vec{W}\right|^2\right) - 2c\vec{V}\times\vec{W}\right)e_0. \end{aligned} \tag{138}$$

As shown in Remark 26 in Theorem 2.2:

$$\left(cu+\vec{S}\right)e_0=-\frac{c\epsilon_0}{2}Fe_0F=-\frac{c\epsilon_0}{2}\left(-\left(\left|\vec{V}\right|^2+c^2\left|\vec{W}\right|^2\right)-2c\vec{V}\times\vec{W}\right)e_0 \tag{139}$$

Hence, we conclude:

$$\begin{aligned} u &= \frac{\epsilon_0}{2}\left(\left|\vec{V}\right|^2+c^2\left|\vec{W}\right|^2\right) \\ \vec{S} &= c^2\epsilon_0\vec{V}\times\vec{W}. \end{aligned} \tag{140}$$

Since $\vec{W}=\frac{\left|\vec{k}\right|}{\omega}\frac{\vec{k}}{\left|\vec{k}\right|}\times\vec{V}=\frac{1}{c}\frac{\vec{k}}{\left|\vec{k}\right|}\times\vec{V}$, (128) becomes:

$$F\begin{pmatrix}ct\\ \vec{r}\end{pmatrix}=\left(\vec{V}+\sqrt{-1}\frac{\vec{k}}{\left|\vec{k}\right|}\times\vec{V}\right)\exp\left(\sqrt{-1}\left(\vec{k}\cdot\vec{r}-\omega t\right)\right), \tag{141}$$

Then both energy density and Poyinting vector are given by:

$$\begin{aligned} u &= \epsilon_0\left|\vec{V}\right|^2 \\ \vec{S} &= c\epsilon_0\left|\vec{V}\right|^2\frac{\vec{k}}{\left|\vec{k}\right|}. \end{aligned} \tag{142}$$

■

References

[Vol93a] Terje G. Vold. "An introduction to geometric algebra with an application in rigid body mechanics". In: *American Journal of Physics* 61.6 (June 1993), pp. 491–504. ISSN: 0002-9505. DOI: `10.1119/1.17201`. eprint: `https://pubs.aip.org/aapt/ajp/article-pdf/61/6/491/11864394/491\_1\_online.pdf`. URL: `https://www.researchgate.net/publication/241273951_An_Introduction_to_Geometric_Algebra_with_an_Application_in_Rigid_Body_Mechanics`.

[Vol93b] Terje G. Vold. "An introduction to geometric calculus and its application to electrodynamics". In: *American Journal of Physics* 61.6 (June 1993), pp. 505–513. ISSN: 0002-9505. DOI: `10.1119/1.17202`. eprint: `https://pubs.aip.org/aapt/ajp/article-pdf/61/6/505/11864414/505\_1\_online.pdf`. URL: `https://www.researchgate.net/publication/245345681_An_introduction_to_geometric_calculus_and_its_application_to_electrodynamics`.

[CIA10] James M Chappell, Azhar Iqbal, and Derek Abbott. "A simplified approach to electromagnetism using geometric algebra". In: *arXiv preprint arXiv:1010.4947* (2010).

[Bea19] A.F. Beardon. *Complex Analysis: The Argument Principle in Analysis and Topology*. Dover Books on Mathematics. Dover Publications, 2019. ISBN: 9780486837185. URL: https://store.doverpublications.com/products/9780486837185.

[Jac21] J.D. Jackson. *Classical Electrodynamics*. Wiley, 2021. ISBN: 9781119770763. URL: https://books.google.com/books?id=6VV-EAAAQBAJ.

[Gri23] D.J. Griffiths. *Introduction to Electrodynamics*. Cambridge University Press, 2023. ISBN: 9781009397728. URL: https://books.google.com/books?id=d3fdEAAAQBAJ.