Jordan Curve Theorem

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Abstract

In this note, we prove the Jordan curve theorem: a closed curve with no self-intersection in the complex plane $\mathbb{C}$ divides $\mathbb{C}$ into exactly two connected components – one is unbounded and the other is bounded.

Chapter 1

Preliminaries

1.1 Sets and Maps

We assume some working knowledge of informal set theory including sets and corresponding membership relation $\in$, subsets, supersets, the empty set $\emptyset$, union, intersection, set difference, complement, and the like.

1.1.1 Sets and Maps

Definition 1.1.1 (Complement)**.** Let X be a set and $A \subset X$ be a subset. We denote $\neg A = X - A = \{x \in X \mid x \notin A\}$.

Theorem 1.1.1 (Empty Intersection and Empty Union)**.** *Let X be a set, Λ be an index set, and $\{A_\lambda \subset X \mid \lambda \in \Lambda\}$ be a Λ-indexed set of subsets of X. The empty intersection $\bigcap_{\lambda\in\emptyset} A_\lambda$ is the underlying set X and the empty union $\bigcup_{\lambda\in\emptyset} A_\lambda$ is the empty set $\emptyset$.*

Proof. By definition:

$$\bigcap_{\lambda\in\Lambda} A_\lambda \coloneqq \{x \in X \mid \forall\lambda \in \Lambda : x \in A_\lambda\}\,. \tag{1.1}$$

For the empty intersection, the condition is vacuously true. Hence, $\bigcap_{\lambda\in\emptyset} A_\lambda = X$. Similarly:

$$\bigcup_{\lambda\in\Lambda} A_\lambda \coloneqq \{x \in X \mid \exists\lambda \in \Lambda : x \in A_\lambda\}\,. \tag{1.2}$$

If the index set is empty, the condition is always false. Hence, $\bigcup_{\lambda\in\emptyset} A_\lambda = \emptyset$. ■

Remark 1. We also have:

$$\neg\bigcap_{\lambda\in\Lambda} A_\lambda \coloneqq \{x \in X \mid \exists\lambda \in \Lambda : x \notin A_\lambda\} = \bigcup_{\lambda\in\Lambda} \neg A_\lambda \tag{1.3}$$

and

$$\neg \bigcup_{\lambda\in\Lambda} A_\lambda \coloneqq \{x \in X \mid \forall\lambda \in \Lambda : x \notin A_\lambda\} = \bigcap_{\lambda\in\Lambda} \neg A_\lambda. \tag{1.4}$$

Theorem 1.1.2. *Let X be a set. For $\{V_\alpha \subset X \mid \alpha \in A\}$ and $\{W_\beta \subset X \mid \beta \in B\}$,*

$$\left(\bigcup_{\alpha\in A} V_\alpha\right) \cap \left(\bigcup_{\beta\in B} W_\beta\right) = \bigcup_{(\alpha,\beta)\in A\times B} V_\alpha \cap W_\beta. \tag{1.5}$$

Similarly,

$$\left(\bigcap_{\alpha\in A} V_\alpha\right) \cup \left(\bigcap_{\beta\in B} W_\beta\right) = \bigcap_{(\alpha,\beta)\in A\times B} V_\alpha \cup W_\beta. \tag{1.6}$$

Proof.

$$\begin{aligned}\left(\bigcup_{\alpha\in A} V_\alpha\right) \cap \left(\bigcup_{\beta\in B} W_\beta\right) &= \{x \in X \mid \exists\alpha \in A : x \in V_\alpha\} \\ &\quad \cap \{x \in X \mid \exists\beta \in B : x \in W_\beta\} \\ &= \{x \in X \mid \exists(\alpha,\beta) \in A \times B : x \in V_\alpha \cap W_\beta\} \\ &= \bigcup_{(\alpha,\beta)\in A\times B} V_\alpha \cap W_\beta.\end{aligned} \tag{1.7}$$

Similarly,

$$\begin{aligned}\left(\bigcap_{\alpha\in A} V_\alpha\right) \cup \left(\bigcap_{\beta\in B} W_\beta\right) &= \{x \in X \mid \forall(\alpha,\beta) \in A \times B : x \in V_\alpha \cup W_\beta\} \\ &= \bigcap_{(\alpha,\beta)\in A\times B} V_\alpha \cup W_\beta.\end{aligned} \tag{1.8}$$

∎

For a given map $f\colon X \to Y$, there are two induced maps:

- Direct image $f\colon 2^X \to 2^Y; U \mapsto \{y \in Y \mid \exists u \in U : y = fu\}$
- Preimage $f^{\leftarrow}\colon 2^Y \to 2^X; W \mapsto \{x \in X \mid fx \in W\}$

Theorem 1.1.3 (Properties of Preimage)**.** *Let X and Y be sets and $f\colon X \to Y$ be a map. The preimage map $f^{\leftarrow}$ preserves the following elementary set operations:*

- $f^{\leftarrow}\left(\bigcup_{\lambda\in\Lambda} B_\lambda\right) = \bigcup_{\lambda\in\Lambda} f^{\leftarrow} B_\lambda$
- $f^{\leftarrow}\left(\bigcap_{\lambda\in\Lambda} B_\lambda\right) = \bigcap_{\lambda\in\Lambda} f^{\leftarrow} B_\lambda$

- $f^{\leftarrow}(B_1 - B_2) = f^{\leftarrow}B_1 - f^{\leftarrow}B_2$

where Λ is an arbitrary index set, B_1, B_2, B_λ are all subspaces in Y for each $\lambda \in \Lambda$.

Proof. The first two equations are almost identical:

$$\begin{aligned} p \in f^{\leftarrow}\left(\bigcup_{\lambda\in\Lambda} B_\lambda\right) &\Leftrightarrow fp \in \bigcup_{\lambda\in\Lambda} B_\lambda \\ &\Leftrightarrow \exists\lambda \in \Lambda : fp \in B_\lambda \\ &\Leftrightarrow \exists\lambda \in \Lambda : p \in f^{\leftarrow}B_\lambda \\ &\Leftrightarrow p \in \bigcup_{\lambda\in\Lambda} f^{\leftarrow}B_\lambda \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} p \in f^{\leftarrow}\left(\bigcap_{\lambda\in\Lambda} B_\lambda\right) &\Leftrightarrow fp \in \bigcap_{\lambda\in\Lambda} B_\lambda \\ &\Leftrightarrow \forall\lambda \in \Lambda : fp \in B_\lambda \\ &\Leftrightarrow \forall\lambda \in \Lambda : p \in f^{\leftarrow}B_\lambda \\ &\Leftrightarrow p \in \bigcap_{\lambda\in\Lambda} f^{\leftarrow}B_\lambda \end{aligned} \tag{1.10}$$

for each $p \in A$.

Recalling $B_1 - B_2 = \{x \in A \mid x \in B_1 \wedge x \in \neg B_2\} = B_1 \cap \neg B_2$, and

$$f^{\leftarrow}(\neg B_2) = \{x \in X \mid fx \in \neg B_2\} = X - f^{\leftarrow}B_2 = \neg f^{\leftarrow}B_2, \tag{1.11}$$

we have

$$\begin{aligned} f^{\leftarrow}(B_1 - B_2) &= f^{\leftarrow}(B_1 \cap \neg B_2) \\ &= f^{\leftarrow}B_1 \cap f^{\leftarrow}(\neg B_2) \\ &= f^{\leftarrow}B_1 \cap \neg f^{\leftarrow}B_2 \\ &= f^{\leftarrow}B_1 - f^{\leftarrow}B_2. \end{aligned} \tag{1.12}$$

Thus, the preimage $f^{\leftarrow} : 2^Y \to 2^X$ preserves union, intersection, and set-difference. ■

1.2 Topological Spaces

A topological space is a structured set in which the concept of convergence can be defined.

1.2.1 Basic Definitions

Definition 1.2.1 (Topological Spaces)**.** Let X be a set. A topology on X is a subset of its subsets $\mathcal{T} \subset 2^X$ that closed under:

- Arbitrary Union

  Each union of members in $\mathcal{T}$ is also a member of $\mathcal{T}$.

- Finite Intersection

  Each finite intersection of members of $\mathcal{T}$ is also a member of $\mathcal{T}$.

As shown in Theorem 1.1.1, the union of an empty family of sets in X is $\emptyset$, and the intersection of an empty family of sets in X is X. Hence, we may add the following, yet redundant, conditions:

- Both $\emptyset$ and X are members of $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space. Any member in $\mathcal{T}$ is called an open subset of X. In particular, both $\emptyset$ and X are open subsets in X. A subset $C \subset X$ is called closed iff the complement $\neg C := X - C$ is open, namely $\neg C \in \mathcal{T}$. Since $\emptyset = X - X$ and $X = X - \emptyset$, both $\emptyset$ and X are clopen. Dually, closed subsets are closed under finite union and arbitrary intersections.

Let $Y \subset X$ be a subset of a topological space $(X, \mathcal{T})$. The induced topology on Y is

$$\mathcal{T}_Y := \{Y \cap U \mid U \in \mathcal{T}\}. \tag{1.13}$$

The pair $(Y, \mathcal{T}_Y)$ is called a subspace of $(X, \mathcal{T})$.

Lemma 1.2.1. *Let $(X, \mathcal{T})$ be a topological space and $C_1 \subset C_2 \subset X$. If $C_1, C_2 \subset X$ are both closed, then $C_1 \subset C_2$ is closed relative to the subspace topology on C_2.*

Proof. Let $\neg_2 C_1 := C_2 - C_1$:

$$\neg_2 C_1 = C_2 \cap \neg C_1. \tag{1.14}$$

Since $\neg C_1 \in \mathcal{T}$, i.e., $\neg C_1 \subset X$ is open, $C_2 \cap \neg C_1 \subset C_2$ is open relative to the subspace topology. ∎

Definition 1.2.2 (Neighborhoods and Open Subspaces)**.** Let $(X, \mathcal{T})$ be a topological space, and $p \in X$ be a point. A subspace $U' \subset X$ is called a neighborhood of p iff there exists some $U \in \mathcal{T}$ such that $p \in U$ and $U \subset U'$. Let $\mathcal{N}_p$ be the set of all neighborhoods of p in X relative to $\mathcal{T}$.

Lemma 1.2.2. *Let $(X, \mathcal{T})$ be a topological space. A subspace $U \subset X$ is open, $U \in \mathcal{T}$, iff U is a neighborhood of every point in it.*

Proof. (⇒) Suppose $U \in \mathcal{T}$. Then, for each $p \in U$, U is an open neighborhood of p.

(⇐) Conversely, suppose U is a neighborhood to its points. For $p \in U$, let $V_p \in \mathcal{T}$ be an open subspace such that $p \in V_p$ and $V_p \subset U$. Then, we conclude $U = \bigcup_{p \in U} V_p$ since:

$$U \subset \bigcup_{p \in U} V_p \subset U. \tag{1.15}$$

Hence U is open. ∎

Definition 1.2.3 (Limit Points and Closure)**.** Let $A \subset (X, \mathcal{T})$ be a subspace. A point $p \in X$ is called a limit point of A iff each neighborhood of p contains at least one point of A distinct from p:

$$\forall U' \in \mathcal{N}_p : U' \cap A - \{p\} \neq \emptyset. \tag{1.16}$$

Let A' denote the set of all limit points. We call $\overline{A} := A \cup A'$ the closure of A in X relative to $\mathcal{T}$.

Lemma 1.2.3. *Let $A \subset (X, \mathcal{T})$ be a subspace. For any point $p \in X$, $p \in \overline{A}$ iff*

$$\forall U' \in \mathcal{N}_p : U' \cap A \neq \emptyset. \tag{1.17}$$

Proof. (⇒) Let $p \in \overline{A}$:

- $p \in A$ case

  For each neighborhood $U' \in \mathcal{N}_p$, $p \in U' \cap A$.

- $p \notin A$ case

  For each neighborhood $U' \in \mathcal{N}_p$, $U' \cap A = U' \cap A - \{p\} \neq \emptyset$ holds.

(⇐) Let $p \in X$. Suppose $U' \cap A \neq \emptyset$ whenever U' is a neighborhood of p.

- $p \in A$ case

  Since $A \subset \overline{A}$, $p \in \overline{A}$.

- $p \notin A$ case

  Let $U' \in \mathcal{N}_p$. Since $p \notin A$ but $p \in U'$, $p \notin U' \cap A$. Hence, $U' \cap A = U' \cap A - \{p\} \neq \emptyset$, which means p is a limit point of A.

∎

Theorem 1.2.1 (Characterization of Closed Subspaces)**.** *A subspace $A \subset (X, \mathcal{T})$ is closed iff $A = \overline{A}$.*

Proof. (⇒) Suppose $A \subset (X, \mathcal{T})$ is closed. Then $\neg A \in \mathcal{T}$. Let $p \in \neg A$. Since $\neg A$ is an open neighborhood of p such that $\neg A \cap A = \emptyset$, p is not a limit point of A by Lemma 1.2.3. Therefore $p \notin \overline{A}$. Since $\neg A \subset \neg\overline{A}$ is shown, we obtain $A \supset \overline{A}$; with the inclusion $A \subset \overline{A}$, we conclude $A = \overline{A}$.

($\Leftarrow$) Suppose $\overline{A} = A$. We will show $\neg A$ is open. Let $p \in \neg A$. Since $p \in \neg\overline{A}$, p is not a limit point of A. Thus, there is some neighborhood $U' \in \mathcal{N}_p$ with $U' \cap A = \emptyset$ by Lemma 1.2.3. We obtain $U' \subset \neg A$. That is, $\neg A$ is a neighborhood of p. As $p \in \neg A$ is arbitrary, by Lemma 1.2.2, we conclude $\neg A \in \mathcal{T}$. ■

Theorem 1.2.2 (Properties of Closures)**.** *Let $A, B \subset (X, \mathcal{T})$ be subspaces.*

- *The closure $\overline{A}$ is $\subset$-smallest closed subspace of X containing A:*

$$\overline{A} = \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\} \tag{1.18}$$

- $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
- $\overline{\overline{A}} = \overline{A}$*, i.e., the closure $\overline{A}$ of A is closed, and the closure-operation is idempotent.*
- $\overline{A} \cup \overline{B} = \overline{A \cup B}$
- $\overline{\emptyset} = \emptyset$

Proof. Let $\widetilde{A} := \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\}$. Since open subspaces are closed under arbitrary union, the complements, i.e., closed subspaces are closed under arbitrary intersection. Hence, $\widetilde{A}$ is closed. To show $\widetilde{A}$ is equal to $\overline{A}$, let us consider their complements:

$\subset$ Let $p \in \neg\widetilde{A}$. Since $\neg\widetilde{A}$ is an open neighborhood of p such that $\neg\widetilde{A} \cap \widetilde{A} = \emptyset$, recalling $\widetilde{A} \supset A$, we conclude $\neg\widetilde{A} \cap A = \emptyset$:

$$\emptyset \subset \neg\widetilde{A} \cap A \subset \neg\widetilde{A} \cap \widetilde{A} = \emptyset. \tag{1.19}$$

Hence, by Lemma 1.2.3, p is not a limit point of A, i.e., $p \in \neg\overline{A}$:

$$\neg\widetilde{A} \subset \neg\overline{A}. \tag{1.20}$$

$\supset$ Let $p \in \neg\overline{A}$. Since p is not a limit point of A, there exists an open neighborhood $U \in \mathcal{N}_p \cap \mathcal{T}$ such that $U \cap A - \{p\} = \emptyset$. As p is not in A, $U \cap A = \emptyset$, thus $A \subset \neg U$. Thus, $\neg U$ is a member of the intersection of the right-hand side of (1.18). Hence, we obtain $\widetilde{A} \subset \neg U$. Since $p \in U$ and $U \subset \neg\widetilde{A}$, we conclude $p \in \neg\widetilde{A}$:

$$\neg\widetilde{A} \supset \neg\overline{A}. \tag{1.21}$$

Therefore, we obtain $\overline{A} = \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\}$.

- $A \subset B \Rightarrow \overline{A} \subset \overline{B}$

  Since any closed subspace containing B also contains A, $\overline{A} \subset \overline{B}$.

- $\overline{\overline{A}} = \overline{A}$

  Since $\overline{A}$ is given by an intersection of closed subspaces, $\overline{A}$ is closed. Moreover, $\overline{A} \subset \overline{A}$ is the $\subset$-smallest subspace containing $\overline{A}$.

- $\overline{A} \cup \overline{B} = \overline{A \cup B}$

  $\overline{A \cup B}$ is closed, and contains both A and B, hence $\overline{A} \cup \overline{A} \subset \overline{A \cup B}$. As $\overline{A} \cup \overline{B}$ is closed, containing $A \cup B$, $\subset$-smallest property implies $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

- $\overline{\emptyset} = \emptyset$

  Since $\emptyset$ is clopen and $\emptyset \subset \emptyset$, the $\subset$-smallest property ensures $\overline{\emptyset} = \emptyset$.

■

Remark 2 (Interior and Boundary). Let $A \subset (X, \mathcal{T})$ be a subspace. As a dual concept of closure, the interior A^ι of A is the $\subset$-largest open set contained in A:

$$A^\iota = \bigcup \{U \in \mathcal{T} \mid U \subset A\}. \tag{1.22}$$

By Remark 1 in Theorem 1.1.1,

$$\begin{aligned} A^\iota &= \bigcup \{\neg F \in \mathcal{T} \mid \neg F \subset A\} \\ &= \bigcup \{\neg F \subset X \mid \neg F \in \mathcal{T} \wedge F \supset \neg A\} \\ &= \neg \bigcap \{F \subset X \mid \neg F \in \mathcal{T} \wedge F \supset \neg A\} \\ &= \neg \overline{\neg A}. \end{aligned} \tag{1.23}$$

So, a subspace $A \subset X$ is open iff $A = A^\iota$, since $\neg A^\iota = \overline{\neg A}$. We call $\partial A := \overline{A} - A^\iota$ the boundary of A. Moreover, $\partial A = \neg(A^\iota) - \neg\left(\overline{A}\right)$:

$$\begin{aligned} \neg(A^\iota) - \neg\left(\overline{A}\right) &= (X - A^\iota) - (X - \overline{A}) \\ &= \left\{x \in X \mid x \notin A^\iota \wedge x \notin (X - \overline{A})\right\} \\ &= \left\{x \in X \mid x \notin A^\iota \wedge x \in \overline{A}\right\} \\ &= \overline{A} - A^\iota. \end{aligned} \tag{1.24}$$

Theorem 1.2.3 (Subspaces and Closures)**.** *Let $(X, \mathcal{T})$ be a topological space and $(Y, \mathcal{T}_Y) \subset (X, \mathcal{T})$ be a subspace. For $A \subset Y$, the closure $\overline{A}_Y$ relative to $\mathcal{T}_Y$ is $Y \cap \overline{A}$, where $\overline{A}$ is the closure of $A \subset X$ relative to $\mathcal{T}$.*

Proof. It suffices to show $A'_Y = Y \cap A'$ since $\overline{A}_Y = A'_Y \cup A$ and $Y \cap \overline{A} = Y \cup (A \cup A') = (Y \cap A) \cup (Y \cap A') = A \cup (Y \cap A')$.

Let $p \in A'_Y$ and $\mathcal{N}_{Y\,p}$ be the set of neighborhood of p relative to $\mathcal{T}_Y$:

$$\forall U' \in \mathcal{N}_{Y\,p} : \exists U \in \mathcal{T} : p \in (U \cap Y) \subset U'. \tag{1.25}$$

Note that $(U \cap Y) \in \mathcal{T}_Y$ if $U \in \mathcal{T}$. Since $p \in A'_Y$,

$$\forall U' \in \mathcal{N}_{Y\,p} : U' \cap A - \{p\} \neq \emptyset, \tag{1.26}$$

i.e.,

$$\forall U \in \mathcal{N}_p \cap \mathcal{T} : (U \cap Y) \cap A - \{p\} \neq \emptyset, \tag{1.27}$$

we obtain $p \in (Y \cap A)'$ relative to $\mathcal{T}$. Recalling $A \subset Y$ and $p \in Y$, we obtain $p \in Y \cap A'$.

Conversely, let $p \in Y \cap A'$ relative to $\mathcal{T}$:

$$\forall U' \in \mathcal{N}_p : U' \cap A - \{p\} \neq \emptyset. \tag{1.28}$$

Since $A \subset Y$, it is equivalent to

$$\forall U' \in \mathcal{N}_p : U' \cap (A \cap Y) - \{p\} \neq \emptyset. \tag{1.29}$$

Now, $U' \cap Y$ contains an open $(U \cap Y) \in \mathcal{T}_Y$ with $p \in U \cap Y$. That is, $U' \cap Y$ is a neighborhood of p relative to $\mathcal{T}_Y$, namely $U' \cap Y \in \mathcal{N}_{Y\,p}$, moreover $p \in A'_Y$.

Hence, we establish $A'_Y = Y \cap A'$, and $\overline{A}_Y = Y \cap \overline{A}$. ■

1.2.2 Separation Axioms

Definition 1.2.4. The following axioms describe how a topology can distinguish points in the underlying set:

T_2 A T_2 space – a Hausdorff space – is a topological space $(X, \mathcal{T})$ in which each of two distinct points have disjoint neighborhoods, that is, if $p \neq q$, there are $U' \in \mathcal{N}_p$ and $V' \in \mathcal{N}_q$ with $U' \cap V' = \emptyset$.

T_4 A T_4 space is a Hausdorff space in which each disjoint closed subspaces have disjoint neighborhoods.

1.2.3 Basic Open Sets

> . . . we can to an extent preassign the notion of nearness desired. [Dug66]

Definition 1.2.5 (Subbases and Generated Topology)**.** Let X be a set and $\mathcal{S} \subset 2^X$ be a set of subsets in X. As 2^X is a topology of X,

$$\tau_{\mathcal{S}} := \left\{\mathcal{T} \subset 2^X \mid \mathcal{T} \text{ is a topology on } X \text{ with } \mathcal{S} \subset \mathcal{T}\right\} \tag{1.30}$$

is non-empty. Their intersection:

$$\bigcap \tau_{\mathcal{S}} := \bigcap \{\mathcal{T} \in \tau_{\mathcal{S}}\} \tag{1.31}$$

is called the topology generated by $\mathcal{S}$. It is the $\subset$-smallest topology containing $\mathcal{S}$.

For the generated topology, the generating set $\mathcal{S}$ is called the subbbasic open set, in short, a subbase.

Remark 3 (Basis)*.* No further conditions for being a subbase of some topology. If $\mathcal{S}$ satisfies:

1. $\mathcal{S}$ covers X

   For each $x \in X$, there is a $B \in \mathcal{S}$ with $x \in B$. This condition guarantees that X is open.

2. Binary Intersection

   Let $B_1, B_2 \in \mathcal{S}$. If $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{S}$ with $x \in B_3$ and $B_3 \subset B_1 \cap B_2$. This condition guarantees that $B_1 \cap B_2$ is open.

Then $\mathcal{S}$ is called the set of basic open sets, in short, a basis for the topology $\bigcap \tau_{\mathcal{S}}$ of X.

Theorem 1.2.4. *Let X be a set, $\mathcal{S} \subset 2^X$ be a basis – $\mathcal{S}$ satisfies both conditions 1 and 2 – and $\mathcal{T}_{\mathcal{S}}$ be the set of all unions of $\mathcal{S}$. $\mathcal{T}_{\mathcal{S}}$ is a topology on X. Moreover, $\mathcal{T}_{\mathcal{S}} = \bigcap \tau_{\mathcal{S}}$.*

Proof. As the condition 1 ensures $\mathcal{S}$ covers X, we have $X \in \mathcal{T}_{\mathcal{S}}$. If we take the empty union, $\emptyset \in \mathcal{T}_{\mathcal{S}}$. By definition, $\mathcal{T}_{\mathcal{S}}$ is closed under arbitrary union. The condition 2 guarantees $\mathcal{T}_{\mathcal{S}}$ is closed under binary, hence any finite intersection. Therefore, $\mathcal{T}_{\mathcal{S}}$ forms a topology on X.

Since $\mathcal{S} \subset \mathcal{T}_{\mathcal{S}}$ holds, $\mathcal{T}_{\mathcal{S}} \in \tau_{\mathcal{S}}$, hence $\bigcap \tau_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{S}}$. To show the other inclusion, let $U \in \mathcal{T}_{\mathcal{S}}$. By construction, there exists $\mathcal{B}_U \subset \mathcal{S}$ with

$$U = \bigcup \mathcal{B}_U = \bigcup \{V \in \mathcal{B}_U\}. \tag{1.32}$$

As $\mathcal{B}_U \subset \mathcal{S}$, and any member $T \in \tau_{\mathcal{S}}$ contains $\mathcal{S}$, we obtain $\mathcal{B}_U \subset T$ for each $T \in \tau_{\mathcal{S}}$. Thus, $\mathcal{B}_U \subset T$ holds for each $T \in \tau_{\mathcal{S}}$. I.e., $U \in \bigcap \tau_{\mathcal{S}}$. ■

1.2.4 Continuous Maps

For given topological space $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and a map between the underlying sets $f\colon X \to Y$, we use $f^{\leftarrow}$ to associate the topology since $f^{\leftarrow}$ preserves the elementary set operations as shown in Theorem 1.1.3:

Definition 1.2.6 (Continuous Maps)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f\colon X \to Y$ is called continuous iff the preimage of each open subspace in Y is open in X. That is, $f^{\leftarrow}$ maps $\mathcal{T}_Y \subset 2^Y$ into $\mathcal{T}_X$:

$$f^{\leftarrow}\colon \mathcal{T}_Y \to \mathcal{T}_X. \tag{1.33}$$

The set of all continuous maps from X to Y is denoted by $C^0(X, Y)$.

Theorem 1.2.5 (Characterizations of Continuity)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f\colon X \to Y$ be a map. The following are equivalent:*

1. *$f \in C^0(X, Y)$ by means of Definition 1.2.6.*
2. *For a subbase (or a basis) $\mathcal{S}_Y \subset \mathcal{T}_Y$, $f^{\leftarrow}\mathcal{S}_Y \subset \mathcal{T}_X$.*
3. *The preimage of a closed subspace in Y is closed in X.*
4. *For each $x \in X$ and for each neighborhood $V' \in \mathcal{N}_{fx}$, there exists a neighborhood $U' \in \mathcal{N}_x$ such that $fU' \subset V'$.*
5. *$f\overline{A} \subset \overline{fA}$ for every $A \subset X$.*

6. $\overline{f^{\leftarrow}B} \subset f^{\leftarrow}\overline{B}$ *for every* $B \subset Y$.

Remark 4 ($\epsilon\delta$-Continuity). The condition 4 is the topological version of ϵ-δ definition of continuity.

Proof. ($1 \Leftrightarrow 2$) As $\mathcal{S}_Y \subset \mathcal{T}_Y$, $f^{\leftarrow}|_{\mathcal{S}_Y} : \mathcal{S}_Y \to \mathcal{T}_X$. Conversely, suppose $f^{\leftarrow}\mathcal{S}_Y \subset \mathcal{T}_X$ is the case. Let $W \in \mathcal{T}_Y$. Since $\mathcal{T}_Y$ is generated by $\mathcal{S}_Y$, W is given by some, not necessarily finite, union of finite intersections of members in $\mathcal{S}_Y$:

$$W = \bigcup_{\lambda \in \Lambda} \left(B_1^{(\lambda)} \cap \cdots \cap B_{j_\lambda}^{(\lambda)} \right), \tag{1.34}$$

where $B_1^{(\lambda)} \cdots B_{j_\lambda}^{(\lambda)} \in \mathcal{S}_Y$ for each $\lambda \in \Lambda$. Applying Theorem 1.1.3, we obtain

$$f^{\leftarrow}W = \bigcup_{\lambda \in \Lambda} f^{\leftarrow} \left(B_1^{(\lambda)} \cap \cdots \cap B_{j_\lambda}^{(\lambda)} \right) = \bigcup_{\lambda \in \Lambda} \left(f^{\leftarrow} B_1^{(\lambda)} \right) \cap \cdots \cap \left(f^{\leftarrow} B_{j_\lambda}^{(\lambda)} \right). \tag{1.35}$$

Since $\left(f^{\leftarrow} B_1^{(\lambda)} \right) \cap \cdots \cap \left(f^{\leftarrow} B_{j_\lambda}^{(\lambda)} \right) \in \mathcal{T}_X$ and W is a union of such open subspaces in X, we conclude $f^{\leftarrow}W \in \mathcal{T}_X$.

($1 \Leftrightarrow 3$) By Theorem 1.1.3,

$$f^{\leftarrow}(\neg A) = f^{\leftarrow}(Y - A) = X - f^{\leftarrow}A = \neg f^{\leftarrow}A \tag{1.36}$$

for every $A \subset X$.

($1 \Rightarrow 4$) Let $x \in X$, $V' \in \mathcal{N}_{fx}$, and $V \in \mathcal{T}_Y$ s.t., $fx \in V$ and $V \subset V'$. As f is continuous, $f^{\leftarrow}V \in \mathcal{T}_X$. Since $x \in f^{\leftarrow}V$, we may set $U' = f^{\leftarrow}V$.

($4 \Rightarrow 5$) Let $A \subset X$ and $x \in \overline{A}$; we will show fx is a member of $\overline{fA}$. Consider $V' \in \mathcal{N}_{fx}$; as we assume 4, there exists $U' \in \mathcal{N}_x$ with $fU' \subset V'$. Since $x \in \overline{A}$, by Lemma 1.2.3, $U' \cap A \neq \emptyset$ holds. Hence, $fx \in \overline{fA}$:

$$\emptyset \subsetneq f(U' \cap A) \subset fU' \cap fA \subset V' \cap fA. \tag{1.37}$$

($5 \Rightarrow 6$) Let $B \subset Y$ and $A \coloneqq f^{\leftarrow}B$. As we assume 5,

$$f\left(\overline{f^{\leftarrow}B}\right) = f\overline{A} \subset \overline{fA} = \overline{f(f^{\leftarrow}B)} \subset \overline{B}. \tag{1.38}$$

Thus, $\overline{f^{\leftarrow}B} \subset f^{\leftarrow}\overline{B}$.

($6 \Rightarrow 3$) Let $B \subset Y$ be a closed subspace. As we assume 6, $\overline{f^{\leftarrow}B} \subset f^{\leftarrow}\overline{B}$. Since $\overline{B} = B$, we conclude $\overline{f^{\leftarrow}B} = f^{\leftarrow}B$:

$$\overline{f^{\leftarrow}B} \subset f^{\leftarrow}\overline{B} \subset f^{\leftarrow}B \subset \overline{f^{\leftarrow}B}. \tag{1.39}$$

See Theorem 1.2.1. ∎

Lemma 1.2.4 (Universal Property of Relative Topology)**.** *Let* $Y \subset (X, \mathcal{T})$ *be a subspace. The relative topology* $\mathcal{T}_Y$ *defined in Definition 1.2.1 can be characterized as the* $\subset$*-smallest topology on* Y *for which the inclusion map:*

$$i \colon Y \hookrightarrow X; y \mapsto y \tag{1.40}$$

is continuous, namely $i \in C^0(Y, X)$.

Proof. Let $\mathcal{T}_Y{}'$ be an arbitrary topology on Y. Suppose $i\colon Y \hookrightarrow X$ is continuous relative to $(X, \mathcal{T})$ and $\left(Y, \mathcal{T}_Y{}'\right)$. We will show that $\mathcal{T}_Y{}' \supset \mathcal{T}_Y$.

Let $U \in \mathcal{T}$. As $i \in C^0\left(\left(Y, \mathcal{T}_Y{}'\right), (X, \mathcal{T})\right)$, the preimage $i^{\leftarrow}U$ is open in $\left(Y, \mathcal{T}_Y{}'\right)$:

$$i^{\leftarrow}U = U \cap Y \in \mathcal{T}_Y{}'. \tag{1.41}$$

Since U is arbitrary, it follows that any open subspace in Y relative to $\mathcal{T}_Y$, $U \cap Y \in \mathcal{T}_Y$ is a member of $\mathcal{T}_Y{}'$, hence $\mathcal{T}_Y \subset \mathcal{T}_Y{}'$. ■

Theorem 1.2.6 (Properties of Continuous Maps)**.** *Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y), (Z, \mathcal{T}_Z)$ be topological spaces.*

- *If $f \in C^0(X, Y)$ and $g \in C^0(Y, Z)$, the composition $gf \in C^0(X, Z)$.*
- *If $f \in C^0(X, Y)$ and $A \subset X$, the restriction $f|_A : A \to Y$ is continuous relative to the relative topology on A.*
- *If $f \in C^0(X, Y)$, the coristriction of f on its image is continuous:*

$$f \in C^0(X, fX). \tag{1.42}$$

Proof. Suppose $f \in C^0((X, Y), g \in C^0(Y, Z)$, and $A \subset X$.

- Since $f^{\leftarrow}\colon \mathcal{T}_Y \to \mathcal{T}_X$ and $g^{\leftarrow}\colon \mathcal{T}_Z \to \mathcal{T}_Y$, and $(g \circ f)^{\leftarrow} = f^{\leftarrow} \circ g^{\leftarrow}$, the continuity of the composition $g \circ f$ follows:

$$(g \circ f)^{\leftarrow}\colon \mathcal{T}_Z \to \mathcal{T}_X. \tag{1.43}$$

- Let $i\colon A \hookrightarrow X$. Since

$$f|_A = f \circ i \tag{1.44}$$

  and as shown above $i \in C^0(A, X)$ relative to $\mathcal{T}_A$, the composition is continuous.

- For each $V \in \mathcal{T}_V$, i.e., for each open subspace $V \cap fX$ in fX,

$$f^{\leftarrow}(V \cap fX) = f^{\leftarrow}V \cap f^{\leftarrow}(fX) = f^{\leftarrow}V. \tag{1.45}$$

  Since $f^{\leftarrow}V$ is open in X, the restriction $f\colon X \to fX$ is continuous.

■

Definition 1.2.7 (Homeomorphisms and Topological Invariance)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f\colon X \to Y$ is called a homeomorphism – a topological isomorphism – iff the following conditions hold:

- The underlying map $f : X \to Y$ is bijective.
- Both f and f^{-1} are continuous.

If f is a homeomorphism, it is denoted by $f\colon X \cong Y$. Two spaces X and Y are homeomorphic, written $X \cong Y$, iff there is a homeomorphism between them. It is worth mentioning that a homeomorphism $f\colon X \cong Y$ is an open map – the image of an open subspace $U \in \mathcal{T}_X$ along f is open $fU \in \mathcal{T}_Y$, since f^{-1} is continuous. Moreover, a homeomorphism $f\colon X \cong Y$ is a bijection for the underlying set and the associated topologies:

$$\begin{aligned} f\colon X &\cong Y \\ f^{-1}\colon \mathcal{T}_Y &\cong \mathcal{T}_X \end{aligned} \tag{1.46}$$

Thus, any topological property about X is mapped to that of Y. We call any property of spaces a topological invariant iff whenever it is true for one space, it is also varied for every homeomorphic space.

Theorem 1.2.7. *Homeomorphism is an equivalence relation in the class of all topological spaces.*

Proof. Observe:

- Reflexive

  For any topological space X, $1_X\colon X \cong X$.

- Symmetric

  If $f\colon X \cong Y$, $Y \cong X$ via f^{-1}.

- Transitive

  If $f\colon X \cong Y$ and $g\colon Y \cong Z$, then $g \circ f\colon X \cong Z$.

See Theorem 1.2.6. ■

1.2.5 Connected Spaces

Definition 1.2.8 (Connectedness)**.** A topological space is disconnected iff it is given by the union of two nonempty disjoint open subspaces: a topological space is connected iff it is not disconnected. A subspace is connected iff it is connected relative to its subspace topology. We call a connected open space a domain.

Theorem 1.2.8 (Characteristics of Connectedness)**.** *For a topological space* $(X, \mathcal{T})$*, TFAE:*

1. $(X, \mathcal{T})$ *is connected.*

2. *The only clopen subspaces of* $(X, \mathcal{T})$ *are* $\emptyset$ *and* X*.*

3. *Any* $f \in C^0(X, \mathbf{2})$ *is constant, where* $\mathbf{2}$ *is the two points set* $\{0, 1\}$ *with discrete topology* $\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$*.*

Proof. ($1 \Rightarrow 2$) Suppose $(X, \mathcal{T})$ is a connected space. Let $A \subset X$ be a non-empty clopen subspace of $(X, \mathcal{T})$. Then X is expressed as $A \cup \neg A$ of the disjoint union of open subspaces. Since X is connected and $A \neq \emptyset$, $\neg A$ must be empty.

($2 \Rightarrow 3$) Assume $(X, \mathcal{T})$ has only two clopen subspaces $\emptyset$ and X. Let $f \in C^0(X, \mathbf{2})$. Suppose, for contradiction, that f is not constant. Then, both $f^{\leftarrow}\{0\}$ and $f^{\leftarrow}\{1\}$ are non-empty. Moreover, $\neg f^{\leftarrow}\{0\} = f^{\leftarrow}\{1\} \neq \emptyset$ implies $f^{\leftarrow}\{0\}$ is clopen such that $\emptyset \subsetneq f^{\leftarrow}\{0\} \subsetneq X$, which is absurd.

($3 \Rightarrow 1$) Assume no continuous non-constant map exists from X to $\mathbf{2}$. Suppose, for contradiction, that $(X, \mathcal{T})$ is disconnected, i.e., there exists a clopen non-empty subspace $\emptyset \subsetneq A \subsetneq X$. Define $f\colon X \to \mathbf{2}$ by $f|_A = 1$ and otherwise zero. By definition, f is non-constant, since $\neg A \neq \emptyset$. Hence $f^{\leftarrow}\emptyset = \emptyset$ and $f^{\leftarrow}\{0,1\} = X$. Moreover, both $f^{\leftarrow}\{0\} = \neg A$ and $f^{\leftarrow}\{1\} = A$ are open. Therefore, such a non-constant f is continuous, which is absurd. ■

Theorem 1.2.9. *The continuous image of a connected space is connected.*

Proof. Let X be a connected space, Y be a topological space, and $f \in C^0(X, Y)$. Suppose, for contradiction, that the continuous image fX is disconnected. By Theorem 1.2.8, there exists a non-constant continuous $g \in C^0(fX, \mathbf{2})$. It follows $g(fX) = \{0, 1\}$. The $g \circ f\colon X \to \mathbf{2}$ is continuous by Theorem 1.2.6. Hence, it follows that $(g \circ f)X = \{0, 1\}$ is a non-constant continuous map on a connected space X, which is absurd. ■

Theorem 1.2.10. *Let X be a topological space and $A \subset X$ be a connected subspace. Then, any $B \subset X$ satisfying $A \subset B \subset \overline{A}$ is also connected; particularly, the closure of connected subspace is connected.*

Proof. Let $f \in C^0(B, \mathbf{2})$. Since A is connected, $f|_A \in C^0(A, \mathbf{2})$ becomes constant by Theorem 1.2.6. Let $\{n\} \coloneqq f|_A \subset \{0, 1\}$; relative to the topology on $\mathbf{2}$, such a singleton $\{n\} \subset \mathbf{2}$ is clopen. Since $B \subset \overline{A}$, we have $B = \overline{A} \cap B$. As shown in Theorem 1.2.3, $\overline{A} \cap B = \overline{A}_B$, we conclude $B = \overline{A}_B$. Since f is continuous, we may apply Theorem 1.2.5 for the relative topology $\mathcal{T}_B$:

$$fB = f\overline{A}_B \subset \overline{fA} = \overline{\{n\}} = \{n\} = fA. \tag{1.47}$$

Therefore, $f|_B$ is also constant, and hence B is connected by Theorem 1.2.8. ■

Theorem 1.2.11. *If a set of non-empty connected spaces share at least one common point, their union is also connected.*

Proof. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a set of non-empty connected spaces, and $x \in \bigcap_{\lambda \in \Lambda} X_\lambda$. Consider $f \in C^0\left(\bigcup_{\lambda \in \Lambda} X_\lambda, \mathbf{2}\right)$. Let $\lambda \in \Lambda$. By Theorem 1.2.6:

$$f|_{X_\lambda} \in C^0\left(X_\lambda, \mathbf{2}\right). \tag{1.48}$$

Since X_λ is connected, $f|_{X_\lambda}$ is constant; since $x \in X_\lambda$, $f|_{X_\lambda} x = fx$. Hence, f is constant. By Theorem 1.2.8, we conclude $\bigcup_{\lambda \in \Lambda} X_\lambda$ is connected. ■

Definition 1.2.9 (Connected Components)**.** Let X be a topological space and $x \in X$. The component C_x of x in X is the union of all connected subspaces in X containing x. In other words, C_x is $\subset$-largest connected subspace in Y containing x. By Theorem 1.2.8, $C_x \subset X$ is a closed subset, because both C_x and $\overline{C_x}$ are connected and its $\subset$-largest property $\overline{C_x} \subset C_x$ with the trivial inclusion $C_x \subset \overline{C_x}$.

Theorem 1.2.12. *Let X be a topological space. The union of any set of connected subspaces in X having at least one point in common is connected. Hence, the component C_x is connected for each $x \in X$.*

Proof. Let $C := \bigcup_{\lambda\in\Lambda} A_\lambda$ be the union of connected subspace in X and $a \in \bigcap_{\lambda\in\Lambda} A_\lambda$ is a common point. Consider an arbitrary continuous map $f \in C^0(C, \mathbf{2})$. Let $\lambda \in \Lambda$. Since A_λ is connected, the restriction $f|_{A_\lambda}$ is constant by Theorem 1.2.8. Since $a \in A_\lambda$, we obtain $fx = fa$ for each $x \in A_\lambda$. Thus $f|_{A_\lambda} = f(a)$ holds. Since $\lambda \in \Lambda$ is arbitrary, we conclude that f is constant. ■

Theorem 1.2.13. *Let X be a topological space. The set of all distinct components in X forms a partition of X.*

Proof. Let $x, y \in X$. If $C_x \cap C_y \neq \emptyset$, by Theorem 1.2.12, their union $C_x \cup C_y$ is connected. Since $C_x \subset C_x \cup C_y$ and C_x is $\subset$-largest connected subset containing x, we conclude $C_x = C_x \cup C_y = C_y$. Hence, if $C_x \neq C_y$, then they are disjoint $C_x \cap C_y = \emptyset$. ■

1.2.6 Compact Spaces

Definition 1.2.10 (Open Covers)**.** Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ be a subspace. Any set of subspaces $\{A_\lambda \subset X \mid \lambda \in \Lambda\}$ is called a cover of Y iff $Y \subset \bigcup_{\lambda\in\Lambda} A_\lambda$. If a cover $\{A_\lambda \mid \lambda \in \Lambda\}$ consists of open subspaces of X, we call it an open cover.

For a cover $\{A_\lambda \mid \lambda \in \Lambda\}$ of Y, a subcover is a subset $\{A_\lambda \mid \lambda \in \Lambda'\}$, $\Lambda' \subset \Lambda$, that is also a cover of Y.

Definition 1.2.11 (Compact Spaces)**.** A topological space $(X, \mathcal{T})$ is compact iff each open cover has a finite subcover.

Theorem 1.2.14. *The continuous image of a compact space is compact.*

Proof. Let $(X, \mathcal{T}_X)$ be a compact space, $(Y, \mathcal{T}_Y)$ be a topological space, and $f \in C^0(X, Y)$. Consider an arbitrary open cover $\mathcal{V} \subset \mathcal{T}_Y$ of $fX \subset Y$. Then $\{f^{\leftarrow}V \mid V \in \mathcal{V}\}$ is an open cover of X; for every $x \in X$, $fx \in Y$ is covered by some $V \in \mathcal{V}$:

$$x \in f^{\leftarrow}V. \tag{1.49}$$

Since X is compact, there exists a finite subcover $X \subset f^{\leftarrow}V_1 \cup \cdots \cup f^{\leftarrow}V_t$. We have the desired finite subcover $\{V_1, \cdots, V_t\} \subset \mathcal{V}$, since for each $x \in X$, as $x \in f^{\leftarrow}V_s$ for some $s \in \{1, \cdots, t\}$, it follows $fx \in V_s$. ■

Theorem 1.2.15. *A closed subspace of a compact space is compact.*

Proof. Let $(X, \mathcal{T}_X)$ be a compact space and $C \subset X$ be a closed subspace. Consider an open cover $\mathcal{U} \subset \mathcal{T}_X$ of C. Since $\neg C \subset X$ is open, we have an open cover of X:

$$\mathcal{U} \cup \{\neg C\}. \tag{1.50}$$

Since X is compact, there is a finite subcover $\{U_1, \cdots, U_n\} \subset \mathcal{U} \cup \{\neg C\}$. Since it also covers $C \subset X$, we have the desired finite subcover of C, namely C is covered by $\{U_1, \cdots, U_n\} - \{\neg C\}$. ■

Theorem 1.2.16. *A compact subspace of a Hausdorff space is closed.*

Proof. Let $(X, \mathcal{T})$ be a Hausdorff space and $K \subset X$ be a compact subspace. If $K = X$, $X \subset X$ is clopen. So, suppose $K \subsetneq X$, and let $x \in \neg K$. For each $y \in K$, as $x \neq y$, there are disjoint open subspaces $U_y, V_y \in \mathcal{T}$ such that

$$x \in U_y \wedge y \in V_y. \tag{1.51}$$

Then the open cover $\{V_y \mid y \in K\}$ has a finite subcover:

$$K \subset V := V_{y_1} \cup \cdots \cup V_{y_n}. \tag{1.52}$$

Define $U := U_{y_1} \cap \cdots \cap U_{y_n}$. Both U and V are open in X. Moreover, $U \cap V = \emptyset$, since, if $z \in V$, there is y_p with $z \in V_{y_p}$ but $z \notin U_{y_p} \supset U$. Since $K \subset V$, U and K are disjoint, namely

$$U \subset \neg K. \tag{1.53}$$

Since $x \in U$, we conclude that $\neg K$ is a neighborhood of x. By Lemma 1.2.2, $\neg K \subset X$ is open. ■

Theorem 1.2.17. *A continuous bijection from a compact space to a Hausdorff space is homeomorphic.*

Proof. Let $(K, \mathcal{T}_K)$ be a compact space, $(X, \mathcal{T}_X)$ be a Hausdorff space, and $f \in C^0(K, X)$. Suppose there is a map $g \colon X \to K$ with $gf = 1_K$ and $fg = 1_X$. We will show g is continuous. Let $V \in \mathcal{T}_K$. Consider $\neg V := K - V$ of the corresponding closed subspace in K. By Theorem 1.2.15, $\neg V$ is a compact subspace in K; its continuous image $f\neg V$ is a compact subspace in X. By Theorem 1.2.15, such a compact subspace $f\neg V$ is closed. Now

$$g^{\leftarrow}\neg V = \{x \in X \mid gx \in \neg V\} = \{x \in X \mid x = fgx \in f\neg V\} = f\neg V \tag{1.54}$$

implies $g^{\leftarrow}\neg V \subset X$ is closed. By the condition 3 in Theorem 1.2.5, we conclude g is continuous. ■

1.2.7 Product Spaces

Let $\Lambda \neq \emptyset$ be an index set and $\{X_\lambda \mid \lambda \in \Lambda\}$ be a Λ-indexed set of sets. The Cartesian product of $\{X_\lambda \mid \lambda \in \Lambda\}$:

$$\prod_{\lambda \in \Lambda} X_\lambda \tag{1.55}$$

is given by the set of all maps $\left\{f \colon \Lambda \to \bigcup_{\lambda \in \Lambda} \mid \forall \lambda \in \Lambda : f\lambda \in X_\lambda\right\}$. For instance, $\prod_{\lambda \in \{1,2\}} = X_1 \times X_2$ is given by

$$\{f \colon \{1,2\} \to X_1 \cup X_2 \mid f1 \in X_1 \wedge f2 \in X_2\} \tag{1.56}$$

i.e., each member in $X_1 \times X_2$ is essentially a pair (x_1, x_2), where $x_1 = f1 \in X_1$ and $x_2 = f2 \in X_2$.

There is a natural projection for each $\alpha \in \Lambda$:

$$p_\alpha \colon \prod_{\lambda \in \Lambda} X_\lambda \to X_\alpha; f \mapsto f_\alpha. \tag{1.57}$$

Definition 1.2.12 (Product Topologies)**.** Let $\Lambda \neq \emptyset$ be an index set and $\{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in \Lambda\}$ be a Λ-indexed set of topological spaces. For the Cartesian product of the underlying sets $\prod_{\lambda \in \Lambda} X_\lambda$, the topology generated by the following subbase:

$$\bigcup_{\alpha \in \Lambda} \{p_\alpha{}^{\leftarrow} U \mid U \in \mathcal{T}_\alpha\} \tag{1.58}$$

is called the product topology; with this product topology, we call $\prod_{\lambda \in \Lambda} X_\lambda$ the product space.

Let us consider finite products of topological spaces and compactness.

Theorem 1.2.18. *Let $X \times Y$ be a product of topological spaces. If $X \times Y$ is compact relative to the product topology, then X is also compact.*

Proof. Let $\mathcal{U} \subset \mathcal{T}_X$ be an open cover of X. For each $U \in \mathcal{U}$, consider

$$p_X{}^{\leftarrow} U = U \times Y. \tag{1.59}$$

Since $p_X{}^{\leftarrow} U$ is a subbasic open subspace in $X \times Y$, it is open. Then $\{p_X{}^{\leftarrow} U \mid U \in \mathcal{U}\}$ forms an open cover of the compact $X \times Y$. Therefore, there is a finite subcover:

$$X \times Y = p_X{}^{\leftarrow} U_1 \cup \cdots \cup p_X{}^{\leftarrow} U_n. \tag{1.60}$$

Hence, $\{U_1, \ldots, U_n\}$ is the desired finite subcover. ■

Theorem 1.2.19 (Finite Tychonoff Theorem)**.** *The product of finite compact spaces is compact.*

Proof. We will show the binary case; let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be compact spaces. Let $\mathcal{O}$ be an open cover of $X \times Y$ and $x \in X$. Since $\mathcal{O}$ covers $X \times Y$:

$$\forall y \in Y : \exists O_{(x,y)} \in \mathcal{O} : (x, y) \in O_{(x,y)}. \tag{1.61}$$

Since $O_{(x,y)} \subset X \times Y$ is open relative to the product topology, there are $U_{(x,y)} \in \mathcal{T}_X$ and $V_{(x,y)} \in \mathcal{T}_Y$ such that

$$(x, y) \in U_{(x,y)} \times V_{(x,y)} \subset O_{(x,y)}, \tag{1.62}$$

where $U_{(x,y)} \times V_{(x,y)}$ is $p_X{}^{\leftarrow}U_{(x,y)} \cap p_Y{}^{\leftarrow}V_{(x,y)} = \big(U_{(x,y)} \times Y\big) \cap \big(X \times V_{(x,y)}\big)$. Now $\big\{V_{(x,y)} \mid y \in Y\big\}$ covers Y; there is a finite subcover:

$$Y = V_{\left(x, y_{j(x,1)}\right)} \cup \cdots \cup V_{\left(x, y_{j(x,m_x)}\right)}. \tag{1.63}$$

Define:

$$U_x \coloneqq U_{\left(x, y_{j(x,1)}\right)} \cap \cdots \cap U_{\left(x, y_{j(x,m_x)}\right)}. \tag{1.64}$$

Since it is a finite intersection of open subspaces in X, $U_x \in \mathcal{T}_X$. Moreover, U_x is an open neighborhood of x.

We, then, have an open cover of X, $\{U_x \mid x \in X\}$. There exists a finite subcover:

$$X = U_{x_1} \cup \cdots \cup X_{x_n}. \tag{1.65}$$

Consider a finite subset of $\mathcal{O}$:

$$\big\{O_{(x,y)} \mid x \in \{x_1, \cdots, x_n\}, y \in \big\{y_{j(x,1)}, \cdots, y_{j(x,m_x)}\big\}\big\}. \tag{1.66}$$

Note that the indices for y varies as $x \in \{x_1, \cdots, x_n\}$. We will show that it is the desired finite subcover of $X \times Y$.

Let $(\xi, \eta) \in X \times Y$. Since (1.65) holds, there is some x_p with $\xi \in U_{x_p}$. For such x_p, since

$$Y = V_{\left(x_p, y_{j(x_p,1)}\right)} \cup \cdots \cup V_{\left(x_p, y_{j(x_p,m_{x_p})}\right)} \tag{1.67}$$

there is some $y_{j(x_p,i)}$ with $\eta \in V_{\left(x_p, y_{j(x_p,i)}\right)}$. For the given pair (ξ, η), we conclude:

$$(\xi, \eta) \in U_{x_p} \times V_{\left(x_p, y_{j(x_p,i)}\right)} \subset O_{\left(x_p, y_{j(x_p,i)}\right)}. \tag{1.68}$$

Hence, (1.66) is the desired finite subcover of $X \times Y$. ■

1.3 Metric Spaces

1.3.1 Topological Properties

Definition 1.3.1 (Metrics and Metric Spaces)**.** Let X be a non-empty set. A metric on X is a real-valued map $d\colon X \times X \to \mathbb{R}$ that satisfies the following conditions:

- Non-negative:

  For every $x, y \in X$, $d(x, y) \geqq 0$.

- Distinguishable:

  For every $x, y \in X$, $d(x, y) = 0$ iff $x = y$.

- Symmetric:

  For every $x, y \in X$, $d(x, y) = d(y, x)$.

- Triangle Inequality:

  For each triple points,

$$d(x, z) \leqq d(x, y) + d(y, z). \tag{1.69}$$

We call $d(x, y)$ the distance between two points x and y in X. For a non-empty subset $A \subset X$ and $x \in X$, define the distance between A and x by

$$d(A, x) \coloneqq \inf \{d(a, x) \mid a \in A\}, \tag{1.70}$$

where inf stands for the greatest lower bound. Since the possible minimum value of the metric d is zero, $d(A, x) \geqq 0$ for each $x \in X$.

Remark 5 (Metric Spaces). Let X be a non-empty set and d be a metric on X. Consider the set of open balls:

$$\mathcal{B}_d \coloneqq \{B_\epsilon(x) \mid \epsilon > 0 \wedge x \in X\}, \tag{1.71}$$

where

$$B_\epsilon(x) \coloneqq \{y \in X \mid d(x, y) < \epsilon\}. \tag{1.72}$$

Lemma 1.3.1. *The set of all open balls in X forms a basis.*

Proof. Let X be a set, d be a metric on X, $\mathcal{B}_d$ is the set of all open balls in X defined above. Recalling Definition 1.2.5, we will show that $\mathcal{B}_d$ satisfies the conditions in Remark 3:

1. Since $X \subset \bigcup_{x \in X} B_1(x)$, $\mathcal{B}_d$ covers X.

2. Let $\epsilon_1 > 0, \epsilon_2 > 0$, and $x_1, x_2 \in X$. Consider $B_1 \coloneqq B_{\epsilon_1}(x_1)$ and $B_2 \coloneqq B_{\epsilon_2}(x_2)$. Suppose $B_1 \cap B_2 \neq \emptyset$. Let $x \in B_1 \cap B_2$. Define

$$\epsilon \coloneqq \min \{\epsilon_1 - d(x_1, x), \epsilon_2 - d(x_2, x)\}. \tag{1.73}$$

   Let $y \in B_\epsilon(x)$:

$$d(y, x_1) \leqq d(y, x) + d(x, x_1) < \epsilon_1 - d(x, x_1) + d(x, x_1) = \epsilon_1. \tag{1.74}$$

   Therefore, we obtain $x_1 \in B_1$; dually $x_2 \in B_2$ as well:

$$y \in B_1 \cap B_2. \tag{1.75}$$

   We conclude $B_\epsilon(x) \subset B_1 \cap B_2$.

Hence, $\mathcal{B}_d$ forms a basis of a topology on X. ■

With this generated topology, the set X with a metric d forms a topological space. The pair (X, d) is called a metric space with the generated topology.

Remark 6. As an important example of metric space, consider $\mathbb{C}$ of the complex plane, where the metric is induced by the standard Euclid norm:

$$|z| \coloneqq \sqrt{(\Re z)^2 + (\Im z)^2} \tag{1.76}$$

Lemma 1.3.2. *For two complex numbers z and w, they are equal iff for every $\epsilon > 0$, $|z - w| < \epsilon$ holds.*

Proof. ($\Rightarrow$) Suppose $z = w$. Then $|z - w| = 0$. Therefore, for every $\epsilon > 0$, $|z - w| < \epsilon$.

($\Leftarrow$) Conversely, suppose $z \neq w$. Then, $\epsilon \coloneqq |z - w| > 0$. Hence, $|z - w| \leqq \epsilon$ holds. ■

Lemma 1.3.3. *A metric is continuous.*

Proof. Let (X, d) be a metric space:

$$d\colon X \times X \to \mathbb{R}. \tag{1.77}$$

For the product $X \times X$, the subbase of the product topology is given by

$$\{U \times X \mid U \in \mathcal{T}_X\} \cup \{X \times V \mid V \in \mathcal{T}_X\} \tag{1.78}$$

where $\mathcal{T}_X$ is the topology generated by the metric d on X, see Definition 1.2.12 and Lemma 1.3.1. Let $0 < s < t$; for further discussion, let

$$(s < t) \coloneqq \{x \in \mathbb{R} \mid s < x < t\} \tag{1.79}$$

be an open interval. We will show that the following preimage is open:

$$d^{\leftarrow}(s < t) = \{(x, y) \in X \times X \mid s < d(x, y) < t\}. \tag{1.80}$$

Let $(x, y) \in d^{\leftarrow}(s < t)$. Select a positive $\epsilon > 0$ such that $s < d(x, y) \pm 2\epsilon < t$. Consider $B_\epsilon(x) \times B_\epsilon(y)$. For any $(x', y') \in B_\epsilon(x) \times B_\epsilon(y)$,

$$d(x', y') \leqq d(x', x) + d(x, y) + d(y, y') < d(x, y) + 2\epsilon < t \tag{1.81}$$

and $s < d(x, y) - 2\epsilon < d(x', y')$ since

$$d(x, y) \leqq d(x, x') + d(x', y') + d(y', y) < d(x', y') + 2\epsilon. \tag{1.82}$$

It follows $(x', y') \in d^{\leftarrow}(s < t)$ and, hence,

$$B_\epsilon(x) \times B_\epsilon(y) \subset d^{\leftarrow}(s < t). \tag{1.83}$$

By Lemma 1.2.2, the preimage of an open interval $d^{\leftarrow}(s < t)$ is open in $X \times X$ relative to the product topology. ■

Remark 7 ($\epsilon\delta$-Continuity). Intuitively speaking, the above proof can be expressed as follows.

For each $x, x', y, y' \in X$, the triangle inequality $d(x', y) \leqq d(x', y') + d(y, y')$ implies $-d(x', y') \leqq d(y, y') - d(x', y)$. Hence,

$$d(x, y) - d(x', y') \leqq d(x, x') + d(x', y) - d(x', y') \leqq d(x, x') + d(y, y'). \quad (1.84)$$

Similarly, $d(x', y') - d(x, y) \leqq d(x', x) + d(y', y)$ holds. Thus,

$$|d(x, y) - d(x', y')| \leqq d(x, x') + d(y, y'). \quad (1.85)$$

As $(x', y') \dashrightarrow (x, y)$ i.e., $d(x, x') \dashrightarrow 0$ and $d(y, y') \dashrightarrow 0$, we conclude d is continuous $d(x', y') \dashrightarrow d(x, y)$.

Theorem 1.3.1. *Let (X, d) be a metric space and $A \subset X$ be a non-empty subspace. For each point $p \in X$, $p \in \overline{A}$ iff $d(A, p) = 0$, where $\overline{A}$ is the closure of $A \subset (X, d)$ relative to the topology generated by d via $\mathcal{B}_d$.*

Proof. ($\Rightarrow$) Suppose $p \in \overline{A}$. Let $\epsilon > 0$. Since $B_\epsilon(x)$ is an open neighborhood around p,

$$B_\epsilon(p) \cap A - \{p\} \neq \emptyset \quad (1.86)$$

by Definition 1.2.3. We may select $q \in B_\epsilon(p) \cap A - \{p\}$. Since $q \in A$, and $d(A, p)$ is a lower bound of $\{d(a, p) \mid a \in A\}$:

$$d(A, p) \leqq d(q, p) < \epsilon. \quad (1.87)$$

Recalling $\epsilon > 0$ is arbitrary and $d(A, p) \geqq 0$, by Lemma 1.3.2, we conclude $d(A, p) = 0$.

($\Leftarrow$) Consider the complement $\neg\overline{A} = X - \overline{A}$. If $\neg\overline{A} = \emptyset$, nothing has to be proven. Let $p \in \neg\overline{A}$. Since $\neg\overline{A} \subset X$ is open, there is $\epsilon > 0$ such that

$$B_\epsilon(p) \subset \neg\overline{A}. \quad (1.88)$$

For each $a \in A$, since $a \notin B_\epsilon(p)$, $d(a, p) \geqq \epsilon$. That is, $\epsilon > 0$ is a lower bound of $\{d(a, p) \mid a \in A\}$:

$$d(A, p) \geqq \epsilon > 0. \quad (1.89)$$

Hence, $d(A, p) \neq 0$ if $p \notin \overline{A}$. ■

Theorem 1.3.2. *Metric spaces are T_4 spaces.*

Proof. Let (X, d) be a metric space.

First, we will show (X, d) is a Hausdorff space. Suppose x and y are distinct points in X. Since $x \neq y$,

$$\epsilon \coloneqq d(x, y) > 0. \quad (1.90)$$

We will show $B_{\epsilon/2}(x) \cap B_{\epsilon/2}(y) = \emptyset$. Suppose, for contradiction, that there exists $p \in B_{\epsilon/2}(x) \cap B_{\epsilon/2}(y)$. Then:

$$\epsilon = d(x, y) \leqq d(x, p) + d(p, q) < \epsilon/2 + \epsilon/2 = \epsilon, \quad (1.91)$$

which is absurd.

Consider two non-empty disjoint closed subspaces $F_1, F_2 \subset X$. Let $p \in F_1$. Since $\overline{F_1} = F_1$, by Theorem 1.3.1, $d(F_2, p) > 0$. Define $\delta_p \coloneqq \frac{1}{3} d(F_2, p)$ and $U_p \coloneqq B_{\delta_p}(p)$, and

$$G_1 \coloneqq \bigcup_{p \in F_1} U_p. \tag{1.92}$$

Similarly, $G_2 \coloneqq \bigcup_{q \in F_2} V_q$, where $\delta_q \coloneqq \frac{1}{3} d(F_1, q) > 0$ and $V_q \coloneqq B_{\delta_q}(q)$. By definition, both $G_1 \supset F_1$ and $G_2 \supset F_2$, and they are open in X. We will show G_1 and G_2 are disjoint. Suppose, for contradiction, that there is an $r \in G_1 \cap G_2$. Then, there are some $p \in F_1$ and $q \in F_2$ such that $r \in B_{\delta_p}(p) \cap B_{\delta_q}(q)$. Without loss of generality, $\delta_p \leqq \delta_q$:

$$3\delta_p = d(F_1, q) \leqq d(p, q) \leqq d(p, r) + d(r, q) < \delta_p + \delta_q \leqq 2\delta_p, \tag{1.93}$$

which is absurd. ■

Theorem 1.3.3. *Let (X, d) be a metric space and $A \subset X$ be a non-empty subspace. The distance $d(A, \_) : X \to \mathbb{R}$ is continuous.*

Proof. Let $p, q \in X$ and $a \in A$:

$$d(A, p) \leqq d(a, p) \leqq d(a, q) + d(q, p) \tag{1.94}$$

Therefore, $d(A, p) - d(q, p) \leqq d(a, q)$, meaning that $d(A, p) - d(q, p)$ is a lower bound of $\{d(a, q) \mid a \in A\}$:

$$d(A, p) - d(q, p) \leqq d(A, q). \tag{1.95}$$

Swapping $p \leftrightarrow q$, we obtain $d(A, q) - d(p, q) \leqq d(A, p)$:

$$|d(A, p) - d(A, q)| \leqq d(p, q) \tag{1.96}$$

As $q \dashrightarrow p$, i.e., as $d(p, q) \dashrightarrow 0$, $|d(A, p) - d(A, q)| \dashrightarrow 0$.

Formally speaking, for any $\epsilon > 0$, there is a $\delta > 0$ for instance, $\delta \coloneqq \frac{\epsilon}{2}$ such that

$$|d(A, p) - d(A, q)| \leqq d(p, q) < \epsilon \tag{1.97}$$

for any $q \in B_\delta(p)$. By the condition 4 in Theorem 1.2.5, $d(A, \_)$ is continuous at $p \in X$. ■

Remark 8 (Lipschitz Continuous). Given two metric spaces X and $\mathbb{R}$, (1.96) implies $d(A, \_)$ is Lipschitz continuous with Lipschitz constant is equal to 1.

1.3.2 Uniform Continuity and Uniform Limit Theorem

Definition 1.3.2 (Uniformly Continuous Maps)**.** A map $f\colon X \to Y$ between metric spaces is called uniformly continuous iff for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(fp, fq) < \epsilon \tag{1.98}$$

for each $p, q \in X$ such that $d_X(p,q) < \delta$.

Theorem 1.3.4 (Heine-Cantor Theorem)**.** *A continuous map between two metric spaces is uniformly continuous if the domain space is compact.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f \in C^0(X,Y)$. Suppose (X, d_X) is compact. Let $\epsilon > 0$. For each $x \in X$, since f is continuous, there exists $\delta_x > 0$ such that

$$f\left(B_{\delta_x}(x)\right) \subset B_{\epsilon/2}(fx) \tag{1.99}$$

see the condition 4 in Theorem 1.2.5. Since $\left\{B_{\delta_x/2}(x) \mid x \in X\right\}$ is an open covering of the given compact space X, there exists a finite subcover:

$$X = B_{\delta_{x_1}/2}(x_1) \cup \cdots \cup B_{\delta_{x_k}/2}(x_k). \tag{1.100}$$

Define $\delta_0 > 0$:

$$\delta_0 := \min\left\{\frac{\delta_{x_1}}{2}, \ldots, \frac{\delta_{x_k}}{2}\right\}. \tag{1.101}$$

Let $p \in X$; there is some $l \in \{1, \ldots, k\}$ such that $p \in B_{\delta_{x_l}/2}(x_l)$. For each $q \in B_{\delta_0}(p)$, namely $d_X(p,q) < \delta_0$:

$$d_X(q, x_l) \leqq d_X(q,p) + d_X(p, x_l) < \delta_0 + \frac{\delta_{x_l}}{2} \leqq \delta_{x_l}. \tag{1.102}$$

That is, both p and q are in $B_{\delta_{x_l}}(x_l)$. Then, the images fp and fq are in $B_{\epsilon/2}(fx_l)$, hence

$$d_Y(fp, fq) \leqq d_Y(fp, fx_l) + d_Y(fx_l, fq) < \frac{\epsilon}{2} + \frac{\epsilon}{2}. \tag{1.103}$$

Since p is arbitrary for the preassigned $\epsilon > 0$, we conclude that f is uniformly continuous. ∎

Definition 1.3.3 (Uniform Convergence)**.** Let X be a set, (Y, d) be a metric space,

$$\{f_n\colon X \to Y \mid n \in \mathbb{N}\} \tag{1.104}$$

be a $\mathbb{N}$-index set of maps. As a sequence, $\{f_n \mid n \in \mathbb{N}\}$ converges uniformly to a limit f_∞ iff for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$n \geqq N \Rightarrow \forall x \in X : d\left(f_n(x), f_\infty(x)\right) < \epsilon. \tag{1.105}$$

Theorem 1.3.5 (Uniform Limit Theorem). *Let X be a topological space, (Y,d) be a metric space,*

$$\{f_n \colon X \to Y \mid n \in \mathbb{N}\} \tag{1.106}$$

be a sequence of maps converging uniformly to $f_\infty \colon X \to Y$. If $\{f_n \colon X \to Y \mid n \in \mathbb{N}\}$ is a sequence of continuous maps, then the limit f_∞ is continuous.

Proof. Let $x \in X$. For a given sequence $\{f_n \in C^0(X,Y) \mid n \in \mathbb{N}\}$, we will show that the limit is continuous at x. Let $\epsilon > 0$ be arbitrary.

Since $f_n \dashrightarrow f_\infty$ uniformly as $n \dashrightarrow \infty$, for any $t \in X$, there is some $N_t \in \mathbb{N}$ such that

$$n \geqq N_t \Rightarrow d\left(f_n t, f_\infty t\right) < \frac{\epsilon}{3}. \tag{1.107}$$

For $n \geqq N_x$, since $f_n \in C^0(X,Y)$, there is some neighborhood $U \in \mathcal{N}_x$ such that

$$\forall y \in U : |f_n x - f_n y| < \frac{\epsilon}{3}. \tag{1.108}$$

Let $y \in U$. If $n \geqq \max\{N_x, N_y\}$,

$$d\left(f_\infty x, f_\infty y\right) \leqq d\left(f_\infty x, f_n x\right) + d\left(f_n x, f_n y\right) + d\left(f_n y, f_\infty y\right) < \epsilon. \tag{1.109}$$

Hence as $y \dashrightarrow x$ relative to the topology on X, $f_\infty y \dashrightarrow f_\infty x$. ∎

Theorem 1.3.6 (Special Case of Tietze-Urysohn Theorem). *Let (X,d) be a metric space, $F_0, F_1 \subset X$ be non-empty closed subspaces. If F_0 and F_1 are disjoint, then there exists a continuous map $f \in C^0(S,[0,1])$ such that $f|_{F_0} = 0$ and $f|_{F_1} = 1$.*

Proof. Since $F_0 \cap F_1 = \emptyset$,

$$g \coloneqq d(F_0, \_) + d(F_1, \_) \tag{1.110}$$

is continuous and positive definite. Define

$$fp \coloneqq \frac{d(F_0,p)}{g(p)} = \frac{d(F_0,p)}{d(F_0,p) + d(F_1,p)} \tag{1.111}$$

We will show that f is continuous. For $p, q \in X$,

$$\begin{aligned} fq - fp &= \frac{\left(d(F_0,p) + d(F_1,p)\right) d(F_0,q) - d(F_0,p)\left(d(F_0,q) + d(F_1,q)\right)}{g(p)g(q)} \\ &= \frac{d(F_1,p)\left(d(F_0,q) - d(F_0,p)\right) + d(F_0,p)\left(d(F_1,p) - d(F_1,q)\right)}{g(p)g(q)} \end{aligned} \tag{1.112}$$

By Theorem 1.3.3, we conclude that as $q \dashrightarrow p$, $fq \dashrightarrow fp$. ∎

Corollary 1.3.6.1. *With a scaling and a shift, we obtain $\tilde{f} \in C^0(S,[a,b])$:*

$$\tilde{f}x \coloneqq (b-a)fx + a \tag{1.113}$$

for $a < b$.

Lemma 1.3.4 (Special Case of Tietze's Extension Theorem)**.** *Let (X, d) be a metric space, $F \subset X$ be a closed subspace, and $g \in C^0(F, [-1, 1])$. There exists a continuous extension of g, that is, an $f \in C^0(X, [-1, 1])$ exists such that $f|_F = g$.*

Proof. For closed intervals $\left[-1, -\frac{1}{3}\right]$ and $\left[+\frac{1}{3}, 1\right]$, their preimages:

$$F_{0-} \coloneqq g^{\leftarrow}\left[-1, -\frac{1}{3}\right], F_{0+} \coloneqq g^{\leftarrow}\left[\frac{1}{3}, 1\right] \tag{1.114}$$

are closed in X, see the condition 3 Theorem 1.2.5. Moreover, they are disjoint.

Applying Theorem 1.3.6, there exists

$$f_0 \in C^0\left(X, \left[-\frac{1}{3}, \frac{1}{3}\right]\right) \tag{1.115}$$

such that $f_0|_{F_{0-}} = -\frac{1}{3}$ and $f_0|_{F_{0+}} = +\frac{1}{3}$. By definition,

$$\forall x \in X : |f_0 x| \leqq \frac{1}{3}. \tag{1.116}$$

Since

$$F = \underbrace{g^{\leftarrow}\left[-1, -\frac{1}{3}\right]}_{F_{0-}} \cup g^{\leftarrow}\left[-\frac{1}{3}, \frac{1}{3}\right] \cup \underbrace{g^{\leftarrow}\left[\frac{1}{3}, 1\right]}_{F_{0+}} \tag{1.117}$$

we conclude $|gx - f_0 x| \leqq \frac{2}{3}$ for each $x \in F$:

- $x \in F_{0-}$ case
  Since $-1 \leqq gx \leqq -\frac{1}{3}$ and $f_0 x = -\frac{1}{3}$,
  $$-\frac{2}{3} \leqq gx - f_0 x \leqq 0. \tag{1.118}$$
- $x \in g^{\leftarrow}\left[-\frac{1}{3}, \frac{1}{3}\right]$ case
  Since both $-\frac{1}{3} \leqq gx, f_0 x \leqq +\frac{1}{3}$,
  $$-\frac{2}{3} \leqq gx - f_0 x \leqq \frac{2}{3}. \tag{1.119}$$
- $x \in F_{0+}$ case
  Since $\frac{1}{3} \leqq gx \leqq 1$ and $f_0 x = +\frac{1}{3}$,
  $$0 \leqq gx - f_0 x \leqq \frac{2}{3}. \tag{1.120}$$

Define $g_1 \coloneqq g - f_0$. As shown above $g_1 \in C^0\left(F, \left[-\frac{2}{3}, \frac{2}{3}\right]\right)$. For

$$F = g_1{}^{\leftarrow}\left[-\frac{2}{3}, -\frac{2}{3}\frac{1}{3}\right] \cup g_1{}^{\leftarrow}\left[-\frac{2}{3}\frac{1}{3}, \frac{2}{3}\frac{1}{3}\right] \cup g_1{}^{\leftarrow}\left[\frac{2}{3}\frac{1}{3}, \frac{2}{3}\right] \tag{1.121}$$

by Theorem 1.3.6, there exists $f_1 \in C^0\left(X, \left[-\frac{2}{3}, \frac{2}{3}\right]\right)$ with

$$\forall x \in F : |g_1 x - f_1 x| = \left| gx - \sum_{j=0}^{1} f_j x \right| \leqq \left(\frac{2}{3}\right)^2 \tag{1.122}$$

We can continue this process so that for each $n \in \mathbb{N}$,

$$f_n \in C^0\left(X, \left[-\left(\frac{2}{3}\right)^n \frac{1}{3}, \left(\frac{2}{3}\right)^n \frac{1}{3}\right]\right) \tag{1.123}$$

such that

$$\forall x \in F : \left| gx - \sum_{j=0}^{n} f_j x \right| \leqq \left(\frac{2}{3}\right)^n \tag{1.124}$$

Since $\{f_n \mid n \in \mathbb{N}\}$ is a sequence of bounded maps such that

$$\left\| \sum_{j=0}^{n} f_j \right\| \leqq \sum_{j=0}^{n} \|f_j\| \leqq \sum_{j=0}^{n} \left(\frac{2}{3}\right)^n \frac{1}{3} < 1, \tag{1.125}$$

the limit $\lim_{n\to\infty} \sum_{j=0}^{n} = \sum_{n\in\mathbb{N}} f_n$ exists, where $\|f\| \coloneqq \sup_{x\in X} |fx|$. Moreover, it is a uniform limit of continuous functions on X,

$$\sum_{n\in\mathbb{N}} f_n \in C^0\left(X, [-1, 1]\right). \tag{1.126}$$

By (1.124), $\sum_{j=0}^{n} f_j \dashrightarrow g$ as $n \dashrightarrow \infty$ on F:

$$\left. \sum_{n\in\mathbb{N}} f_n \right|_F = g. \tag{1.127}$$

Hence, $\sum_{n\in\infty} f_n$ is the desired continuous extension of g on X. ■

Chapter 2

Complex Analysis 101

We assume some working knowledge of real numbers, particularly the existence of lease upper bound: if a subspace $A \subset \mathbb{R}$ of real numbers is non-empty and bounded above, then it has a least upper bound. Such an upper bound, if it exists, is unique.

2.1 Intervals and Curves

2.1.1 Real Intervals and Heine-Borel Theorem

Definition 2.1.1 (Real Intervals)**.** For $a, b \in \mathbb{R}$, let

$$[a, b] \coloneqq \{(1-t)a + tb \mid t \in [0, 1]\}. \tag{2.1}$$

We call $[a, b]$ a real closed interval.

Theorem 2.1.1. *A real closed interval $[a, b] \subset \mathbb{R}$ is connected.*

Proof. Let $F \subset [a, b]$ be a closed proper subspace:

$$\emptyset \subsetneq F \subsetneq [a, b]. \tag{2.2}$$

We will show that F is not open.

Let $x \in F$ and $y \in \neg F$. Without loss of generality, consider $x < y$ case. Define $F_{<y} \coloneqq \{t \in F \mid t < y\}$; as $x \in F_{<y}$ and $F_{<y}$ is bounded above, we may set:

$$z \coloneqq \sup F_{<y}. \tag{2.3}$$

Then $x \leqq z \leqq y$, since y is an upper bound of $F_{<y}$ and z is the least upper bound.

For any $\epsilon > 0$, $B_\epsilon(z) \cap F \neq \emptyset$, i.e., $z \in \overline{F}$, where $B_\epsilon(x) \coloneqq (x - \epsilon, x + \epsilon)$. Otherwise, any number in $(z - \epsilon, z)$ would be an upper bound of $F_{<y}$, which contradicts the very definition of z.

Recalling $F \subset [a,b]$ is closed, we conclude $z \in F$. Therefore, $z < y$. Since the open interval (z,y) does not meet F, $(z,y) \cap F = \emptyset$, for each $\epsilon > 0$, $B_\epsilon(z) \not\subset F$. In other words, F is not a neighborhood of z; hence, F can not be an open subspace of $[a,b]$. It follows that no clopen proper subspace in $[a,b]$. By Theorem 1.2.8, $[a,b] \subset \mathbb{R}$ is connected. ■

Theorem 2.1.2. *A real closed interval $[a,b] \subset \mathbb{R}$ is compact.*

Proof. Let $\mathcal{O}$ be an open cover of $[a,b]$. Define

$$S := \{x \in [a,b] \mid [a,x] \text{ is finitely covered by } \mathcal{O}\} \tag{2.4}$$

- S is not empty

  Since $a \in [a,b]$ is covered by at least one $U \in \mathcal{O}$, $[a,a] = \{a\} \subset U$. Hence, $a \in S$.

- $S \subset [a,b]$ is open

  Let $x \in S$ and $\{V_1, \ldots, V_n\} \subset \mathcal{O}$ be the finite subcover of $[a,x]$. Since $x \in [a,b]$ is covered by some open $V \in \mathcal{O}$, there exists a positive $\epsilon > 0$ such that:

  $$B_\epsilon(x) \subset V. \tag{2.5}$$

  We will show that $B_\epsilon(x) \subset S$. Let $y \in B_\epsilon(x)$. Since $y \in V$, we have a finite subcover $\{V_1, \ldots, V_n, V\}$ of $[a,y]$. Hence, $y \in S$. By Lemma 1.2.2, $S \subset [a,b]$ is open.

- $S \subset [a,b]$ is closed

  Let $x \in \overline{S}$, where the closure $\overline{S}$ is relative to the topology of $[a,b]$. Since $\overline{S} \subset [a,b]$, x is in some open $W \in \mathcal{O}$:

  $$x \in W. \tag{2.6}$$

  Hence, there is a positive $\epsilon > 0$ with $B_\epsilon(x) \subset W$. Since $x \in \overline{S}$:

  $$B_\epsilon(x) \cap S \neq \emptyset. \tag{2.7}$$

  There exists, thus, some $y \in B_\epsilon(x) \cap S$ such that $[a,y]$ is finitely covered:

  $$[a,y] \subset W_1 \cup \cdots W_k. \tag{2.8}$$

  Then $[a,x]$ is covered by $\{W_1, \ldots, W_k, W\}$, since the interval between x and y is covered by W and $x \in W$. Therefore, we conclude $x \in S$. With the trivial inclusion $S \subset \overline{S}$, we conclude $S = \overline{S}$ by Theorem 1.2.1.

As shown, $S \subset [a,b]$ is non-empty and clopen. Since $[a,b] \subset \mathbb{R}$ is connected by Theorem 2.1.1, we conclude $S = [a,b]$. Hence, $[a,b]$ is compact. ■

Theorem 2.1.3 (Heine-Borel Theorem)**.** *Let n be a positive integer. A subspace $K \subset \mathbb{R}^n$ is compact iff it is bounded and closed.*

Proof. ($\Rightarrow$) Since $\mathbb{R}^n$ is furnished with the standard metric d, as shown in Theorem 1.3.2, $\mathbb{R}^n$ is a Hausdorff space. Thus, if $K \subset \mathbb{R}^n$ is compact, it is closed by Theorem 1.2.16. Consider $\{B_1(x) \mid x \in K\}$ of the set of unit open balls. Since it is an open cover of the compact subspace $K \subset \mathbb{R}^n$, there is a finite subcover:

$$K \subset B_1(x_1) \cup \cdots \cup B_1(x_n). \tag{2.9}$$

Define $M := \max\{|x_1|, \cdots, |x_n|\}$. For each $x \in K$, there is some x_p with $x \in B_1(x_p)$:

$$|x| = d(0,x) \leqq d(0,x_p) + d(x_p,x) < M + 1. \tag{2.10}$$

Hence, $K \subset B_{M+1}(0)$ i.e., K is bounded.

Conversely, suppose $K \subset \mathbb{R}^n$ is bounded and closed. Since K is bounded, there is $\mu > 0$ with

$$K \subset [-\mu, \mu]^n. \tag{2.11}$$

As shown in Theorem 2.1.2, $[-\mu, \mu] \subset \mathbb{R}$ is compact; by Theorem 1.2.19, the product $[-\mu,\mu]^n$ is a compact subspace in $\mathbb{R}^n$. By Lemma 1.2.1, since $K \subset [-\mu,\mu]^n$ is closed. By Theorem 1.2.15, the closed subspace $K \subset [-\mu,\mu]^n$ of a compact subspace $[-\mu,\mu]^n \subset \mathbb{R}^n$ is a compact subspace in $\mathbb{R}^n$. ■

Theorem 2.1.4 (Extreme Value Theorem). *A real valued continuous map f on a compact space K is bounded, and there are $p, q \in K$ such that $fp = \sup_{x\in K} fx$ and $fq = \inf_{x \in K} fx$.*

Proof. Let $f \in C^0(K, \mathbb{R})$ be a continuous map on a compact space K. The image $fK \subset \mathbb{R}$ is compact by Theorem 1.2.14; by Theorem 2.1.3, fK is bounded in $\mathbb{R}$. Let $M := \sup_{x\in K} fx$. Suppose, for contradiction, that there is no point x on K so that $fx = M$, namely for each $x \in K$, $fx < M$. Then $x \mapsto \frac{1}{M - fx} > 0$ is continuous on K, hence $\frac{1}{M-f}$ is bounded. Let $\epsilon > 0$ be arbitrary. There must be some $x_\epsilon \in K$ with $M - \epsilon < fx_\epsilon \leqq M$, otherwise $M - \epsilon$ would be an upper bound of fK. Hence, $\frac{1}{M - fx_\epsilon} > \frac{1}{\epsilon}$, which means $\frac{1}{M-f}$ is not bounded, a contradiction. ■

Corollary 2.1.4.1. *For a subspace $A \subset \mathbb{C}$, define*

$$\delta A := \sup\{|a - b| \mid a, b \in A\} \tag{2.12}$$

If A is compact, there are $x, y \in A$ with $\delta A = |x - y| < \infty$.

Proof. Let

$$f\colon \mathbb{C} \times \mathbb{C} \to \mathbb{R}; (x,y) \mapsto |x - y| \tag{2.13}$$

be the standard metric on $\mathbb{C}$. By Lemma 1.3.3, f is continuous. If $A \subset \mathbb{C}$ is compact, the product $A \times A$ is also compact by Theorem 1.2.19. Hence, $f|_{A\times A}$ is bounded. Applying Theorem 2.1.4, f has maximum, namely there are $x, y \in A$ with $\delta A = f(x,y) = |x - y|$. ■

2.1.2 Curves in $\mathbb{C}$

Definition 2.1.2 (Curves and Complex Intervals)**.** Let X be a topological space. A curve in X is a continuous map from some closed interval, namely $\gamma \in C^0([a,b], X)$. We call $\gamma(a)$ the initial point of γ, and $\gamma(b)$ the final point of γ. A closed curve is a curve $\gamma \in C^0([a,b], X)$ with $\gamma(a) = \gamma(b)$. Let $[\gamma] \coloneqq \gamma[a,b]$ be the image in X of a curve $\gamma \in C^0([a,b], X)$. In other words, a closed curve is a curve with no endpoints. For a pair of complex numbers $z, w \in \mathbb{C}$, we denote $[w,z] \coloneqq \{(1-t)w + tz \mid t \in [0,1]\}$.

Theorem 2.1.5. *The image of a curve in $\mathbb{C}$ is compact.*

Proof. Let $\gamma \in C^0([a,b], \mathbb{C})$ be a curve. By Theorem 1.2.14 and Theorem 2.1.2, the continuous image $[\gamma]$ is compact.. ■

Theorem 2.1.6. *Let $r > 0$ and $x \in \mathbb{C}$. Both $B_r(x) \subset \mathbb{C}$ and its complement $\neg B_r(x) = \mathbb{C} - B_r(x)$ are connected.*

Proof. Consider $y \in B_r(x)$ and $[x,y] = \{(1-t)x + ty \mid y \in [0,1]\}$. Let $p = (1-t)x + ty \in [x,y]$. Then $p \in B_r(x)$ since

$$|p - x| = |-tx + ty| = |t|\,|x - y| \leqq |x - y| < r. \tag{2.14}$$

It follows $[x,y] \subset B_r(x)$. Hence,

$$B_r(x) = \bigcup_{y \in B_r(x)} [x,y] \tag{2.15}$$

and each complex interval shares the center x in common. By Theorem 1.2.11, we conclude $B_r(x)$ is connected.

The complement $\neg B_r(x)$ is given by:

$$\{z \in \mathbb{C} \mid |z - x| \geqq r\} = C \cup \bigcup_{\theta \in [0,2\pi]} J_\theta, = \bigcup_{\theta \in [0,2\pi]} C \cup J_\theta, \tag{2.16}$$

where $C \coloneqq \partial B_r(x) = \{z \in \mathbb{C} \mid |z - x| = r\}$ and $J_\theta \coloneqq \left\{x + t \exp\sqrt{-1}\theta \mid t \geqq r\right\}$. Now, C is the image of a continuous map $\gamma_0 \in C^0([0, 2\pi], \mathbb{C})$:

$$\gamma_0\theta = \exp\sqrt{-1}\theta. \tag{2.17}$$

Hence, $C = [\gamma]$ is connected since it is the continuous image of the connected interval $[0,1] \subset \mathbb{R}$; see Theorem 1.2.9 and Theorem 2.1.1. Similarly, J_θ is also connected for each $\theta \in [0,1]$ with $C \cap J_\theta = \left\{r \exp\sqrt{-1}\theta\right\}$. By Theorem 1.2.11, $C \cup J_\theta$ is connected for each $\theta \in [0, 2\pi]$. Therefore, we conclude $\bigcup_{\theta \in [0,2\pi]} C \cup J_\theta$ is connected. ■

Definition 2.1.3 (Path-Connectedness)**.** A topological space is called path-connected iff each pair of points can be joined by a curve.

Lemma 2.1.1. *Each path-connected space is connected.*

Proof. Let X be a path-connected non-empty space and $x \in X$. For each $y \in X$, there exists $\gamma_y \in C^0([0,1], X)$ such that $\gamma_y 0 = x$ and $\gamma_y 1 = y$. Since each γ_y is connected by Theorem 1.2.9, sharing the initial point $\gamma_y 0 = x$,

$$X = \bigcup_{y \in X} [\gamma_y] \tag{2.18}$$

is connected by Theorem 1.2.11. ■

Theorem 2.1.7. *Let X be a topological space. TFAE:*

1. *Each path-component is open.*
2. *Each point of X has a path-connected open neighborhood.*

Proof. $(1 \Rightarrow 2)$ Each point belongs to some path-component. By 1, such a path-component is open, and therefore, it is an open neighborhood of its points.

$(2 \Rightarrow 1)$ Let K be a path-component of X, and $x \in K$. By 2, there is an open and path-connected $U \subset Y$ with $x \in U \subset Y$. By the $\subset$-largest property of K, $K \subset K \cup U$ implies $U \subset K$. By Lemma 1.2.2, K is open. ■

Remark 9. Let K be a path-component of X. Since $\neg K = X - K$ is given by the union of other open path-components, $\neg K \subset X$ is open. Namely, a path-component of X is clopen.

Theorem 2.1.8. *A topological space is path-connected iff it is connected and each point has a path-connected open neighborhood.*

Proof. $(\Rightarrow)$ Let X be a path-connected space. As shown in Lemma 2.1.1, X is connected, and hence X is clopen. Then, X itself is a path-connected open neighborhood of its points.

$(\Leftarrow)$ Let X be a connected topological space in which each point has a path-connected open neighborhood. Each path-component is open and, hence, closed in X. Since X is connected, such a clopen subspace must be X itself. ■

Corollary 2.1.8.1. *An open subspace in $\mathbb{R}^n$, in particular in $\mathbb{C}$, is connected iff it is path-connected.*

Proof. Let $U \subset \mathbb{C}$ be an open subspace. Each point $x \in U$ has $\epsilon > 0$ with $B_\epsilon(x) \subset U$. Recall $B_\epsilon(x)$ is path-connected, see the proof in Theorem 2.1.6, via Theorem 2.1.8, the connectedness of $U \subset \mathbb{C}$ is equivalent to the path-connectedness of U. ■

2.2 Winding Numbers

The winding number of a closed curve is the number of times the curve winds around a given point on the complex plane $\mathbb{C}$.

Definition 2.2.1 (Argument)**.** For any $z \in \mathbb{C} - \mathbb{R}_{\leqq 0}$, there are unique $\theta \in (-\pi, \pi)$ and $r > 0$ such that $z = r \exp\left(\sqrt{-1}\theta\right)$. We call θ the argument of $z = r \exp\left(\sqrt{-1}\theta\right)$:

$$\arg\colon \left(\mathbb{C} - \mathbb{R}_{\leqq 0}\right) \to (-\pi, \pi); r \exp\left(\sqrt{-1}\theta\right) \mapsto \theta. \tag{2.19}$$

Theorem 2.2.1. *A curve in* $\mathbb{C}$ *is uniformly continuous.*

Proof. Let $\gamma \in C^0([a, b], \mathbb{C})$ be a curve. As shown in Theorem 2.1.2, the domain $[a, b] \subset \mathbb{R}$ is compact. By Theorem 1.3.4, it follows. ■

Definition 2.2.2 (Winding Numbers of Closed Curves)**.** Let $\gamma \in C^0([a, b], \mathbb{R})$ be a closed curve and $z_0 \in \neg[\gamma]$. We will define the winding number $n(\gamma, z_0)$ of the curve γ at z_0.

Since $[\gamma] \subset \mathbb{C}$ is closed, Theorem 1.3.1 implies

$$\delta_0 \coloneqq d\left([\gamma], z_0\right) > 0 \tag{2.20}$$

Let $\epsilon > 0$ such that

$$0 < \epsilon < \delta_0. \tag{2.21}$$

Since γ is uniformly continuous by Theorem 1.3.4, there exists $\delta > 0$ such that, for each $s, t \in [a, b]$,

$$|s - t| < \delta \Rightarrow |\gamma s - \gamma t| < \epsilon. \tag{2.22}$$

Consider a finite subdivision of $[a, b]$:

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b \tag{2.23}$$

such that $\max\{a_1 - a_0, \cdots, a_n - a_{n-1}\} < \delta$. Then, for each pair (a_{j-1}, a_j), $j \in \{1, \cdots, n\}$:

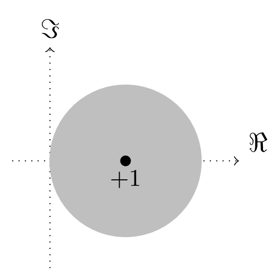
$$|\gamma a_j - \gamma a_{j-1}| < \epsilon. \tag{2.24}$$

Moreover, for each $j \in \{1, \cdots, n\}$,

$$w_j \coloneqq \frac{\gamma a_j - z_0}{\gamma a_{j-1} - z_0} \tag{2.25}$$

satisfies $|w_j - 1| < 1$, hence $\Re w_j > 0$:

$$|w_j - 1| = \left|\frac{\gamma a_j - z_0 - (\gamma a_{j-1} - z_0)}{\gamma a_{j-1} - z_0}\right| = \left|\frac{\gamma a_j - \gamma a_{j-1}}{\gamma a_{j-1} - z_0}\right| < \frac{\epsilon}{\delta_0} < 1. \tag{2.26}$$



Thus, for each $j \in \{1, \cdots, n\}$,

$$\arg w_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.27}$$

Since γ is closed, $\gamma a_0 = \gamma a = \gamma b = \gamma a_n$:

$$\prod_{j=1}^{n} w_j = \prod_{j=1}^{n} \frac{\gamma a_j - z_0}{\gamma a_{j-1} - z_0} = \frac{\gamma a_n - z_0}{\gamma a_0 - z_0} = 1, \tag{2.28}$$

we conclude $\sum_{j=1}^{n} \arg w_j \equiv 0 \mod 2\pi$. We define:

$$n(\gamma, z_0) \coloneqq \frac{1}{2\pi} \sum_{j=1}^{n} \arg w_j. \tag{2.29}$$

Remark 10. As a trivial example, if a curve is a constant, its winding number is zero.

Lemma 2.2.1. *The winding number is independent of the subdivision.*

Proof. We will show that the winding number based on a new subdivision:

$$a_0 < \cdots < a_{j-1} < \tau < a_j < \cdots < a_n \tag{2.30}$$

is equal to the original $n(\gamma, z_0)$ via the subdivision in (2.23), using the same notation in Definition 2.2.2.

Let $\theta_j \coloneqq \arg w_j$. Since

$$\theta_j = \arg \frac{\gamma a_j - z_0}{\gamma a_{j-1} - z_0} = \arg \frac{\gamma a_j - z_0}{\gamma \tau - z_0} \frac{\gamma \tau - z_0}{\gamma a_{j-1} - z_0} \tag{2.31}$$

if we define $\theta_j' \coloneqq \arg \frac{\gamma a_j - z_0}{\gamma \tau - z_0}$ and $\theta_j'' \coloneqq \arg \frac{\gamma \tau - z_0}{\gamma a_{j-1} - z_0}$, we have

$$\theta_j \equiv \theta_j' + \theta_j'' \mod 2\pi. \tag{2.32}$$

Since each argument is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\left|\theta_j - \left(\theta_j' + \theta_j''\right)\right| \leqq |\theta_j| + \left|\theta_j'\right| + \left|\theta_j''\right| < \frac{3}{2}\pi, \tag{2.33}$$

we conclude $\theta_j = \theta_j' + \theta_j''$. This means the winding number based on a finer subdivision remains the same. ■

Theorem 2.2.2. *Let γ be a closed curve in $\mathbb{C}$. Then*

$$n(\gamma, \_)\colon \neg[\gamma] \to \mathbb{Z} \tag{2.34}$$

is constant on each connected component in $\neg[\gamma]$. In particular, $n(\gamma, \_)$ is zero on an unbounded connected component.

Proof. Let $\gamma \in C^0([a,b],\mathbb{R})$ be a closed curve, $t \in [a,b]$, and $z_0, z_0' \in \neg[\gamma]$. We use the same $0 < \epsilon < \delta_0 := d([\gamma], z_0)$ and subdivision $a = a_0 < \cdots < a_n = b$ for z_0. Since

$$|\gamma t - z_0| \leqq |\gamma t - z_0'| + |z_0 - z_0'| . \tag{2.35}$$

we obtain:

$$|\gamma t - z_0'| \geqq |\gamma t - z_0| - |z_0 - z_0'| = \delta_0 - |z_0 - z_0'| \tag{2.36}$$

If z_0 and z_0' are relatively close, namely, if $|z_0 - z_0'| < \delta_0 - \epsilon$,

$$|\gamma t - z_0'| > \epsilon. \tag{2.37}$$

Then, for each $s \in [a,b]$, $|\gamma s - z_0'| > \epsilon > 0$, and

$$d([\gamma], z_0') \geqq \epsilon > 0. \tag{2.38}$$

Hence, for $n(\gamma, z_0')$, we may use the same subdivision as $n(\gamma, z_0)$:

$$\left|w_j' - 1\right| = \left|\frac{\gamma a_j - \gamma a_{j-1}}{\gamma a_{j-1} - z_0'}\right| < \frac{\epsilon}{\epsilon} = 1 \tag{2.39}$$

where

$$w_j' := \frac{\gamma a_j - z_0'}{\gamma a_{j-1} - z_0'}, \tag{2.40}$$

for each $j \in \{1, \cdots, n\}$.

We will first show $n(\gamma, \_)$ is continuous. Let $j \in \{1, \cdots, n\}$. Define:

$$v_j := \frac{\gamma a_j - z_0}{\gamma a_j - z_0'}. \tag{2.41}$$

Since

$$|v_j - 1| = \left|\frac{z_0' - z_0}{\gamma a_{j-1} - z_0'}\right| < \frac{|z_0' - z_0|}{\epsilon} \tag{2.42}$$

if z_0' is sufficiently close to z_0, namely if

$$|z_0 - z_0'| < \min\{\epsilon, \delta_0 - \epsilon\} \tag{2.43}$$

then we obtain $|v_j - 1| < 1$. Hence

$$\arg v_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.44}$$

Since

$$\theta_j' := \arg\frac{\gamma a_j - z_0'}{\gamma a_{j-1} - z_0'} = \arg\frac{\gamma a_j - z_0'}{\gamma a_j - z_0}\frac{\gamma a_j - z_0}{\gamma a_{j-1} - z_0}\frac{\gamma a_{j-1} - z_0}{\gamma a_{j-1} - z_0'}. \tag{2.45}$$

we obtain:

$$\theta_j' \equiv \theta_j - \arg v_j + \arg v_{j-1} \mod 2\pi. \tag{2.46}$$

Recalling each angle is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we conclude

$$\theta'_j = \theta_j - \arg v_j + \arg v_{j-1}. \tag{2.47}$$

Recalling $\gamma a_0 = \gamma a_n$, we have $v_0 = v_n$. Moreover:

$$\sum_{j=1}^{n} \theta'_j = \sum_{j=1}^{n} \theta_j. \tag{2.48}$$

Thus, $n(\gamma, \_)$ is locally constant, and hence $n(\gamma, \_)$ is continuous relative to the discrete topology:

$$n(\gamma, \_) \in C^0\left(\neg[\gamma], \mathbb{Z}\right). \tag{2.49}$$

Let $\Omega \subset \neg[\gamma]$ be a connected component and $z_0 \in \Omega$. Define

$$\Omega_0 := \{z \in \Omega \mid n(\gamma, z) = n(\gamma, z_0)\} = \Omega \cap n(\gamma, \_)^{\leftarrow} n(\gamma, z_0). \tag{2.50}$$

Since the singleton set $\{n(\gamma, z_0)\} \subset \mathbb{Z}$ is open, its preimage $\Omega_0 \subset \Omega$ is open. Moreover, its complement is also open:

$$\Omega_1 := \{z \in \Omega \mid n(\gamma, z) \neq n(\gamma, z_0)\} = \Omega \cap \bigcup_{k \neq n(\gamma, z_0)} n(\gamma, \_)^{\leftarrow} k \tag{2.51}$$

By definition, $\Omega_0 \cup \Omega_1 = \Omega$, and these two open subspaces are disjoint:

$$\Omega_0 \cap \Omega_1 = \emptyset. \tag{2.52}$$

Since Ω is connected and $z_0 \in \Omega \cap \Omega_0$, by Theorem 1.2.8, we conclude $\Omega_0 = \Omega$. Hence, $n(\gamma, \_)$ is constant on each connected component.

Finally, we will show that $n(\gamma, \_)$ is zero on an unbounded connected component. Since $\mathbb{C}$ is Hausdorff, and as shown in Theorem 2.1.5 $[\gamma] \subset \mathbb{C}$ is compact, by Theorem 1.2.16, $[\gamma] \subset \mathbb{C}$ is closed. There exists $R > 0$ with $[\gamma] \subset \overline{B_R(0)} = \{w \in \mathbb{C} \mid |w| \leqq R\}$. The complement $\neg\overline{B_R(0)} = \{w \in \mathbb{C} \mid |w| > R\}$ is connected, as shown in Theorem 2.1.6. Let Ω_∞ be an unbounded component of $\neg[\gamma]$:

$$\neg\overline{B_R(0)} \subset \Omega_\infty. \tag{2.53}$$

Consider $z_0 \in \Omega_\infty$ such that $|z_0| > 3R$. Let $s, t \in [a, b]$:

$$\begin{aligned} |\gamma t - z_0| &\geqq |z_0| - |\gamma t| > 3R - R = 2R \\ |\gamma s - \gamma t| &\leqq |\gamma s| + |\gamma t| \leqq 2R \end{aligned} \tag{2.54}$$

Then, we obtain:

$$\left| \frac{\gamma s - \gamma t}{\gamma t - z_0} \right| < 1. \tag{2.55}$$

Since $s, t \in [a, b]$ are arbitrary, we may use the trivial subdivision $a < b$:

$$\arg \frac{\gamma b - z_0}{\gamma a - z_0} = \arg 1 = 0. \tag{2.56}$$

Hence, $n(\gamma, \_)|_{\Omega_\infty} = 0$. ■

Theorem 2.2.3. *Let γ_0, γ_1 be closed curves in $\mathbb{C}$, for simplicity, $\gamma_0, \gamma_1 \in C^0([0,1], \mathbb{C})$ with $\gamma_0 0 = \gamma_0 1$ and $\gamma_1 0 = \gamma_1 1$. Suppose $\gamma_0 0 = \gamma_1 0$, and there exists $h \in C^0([0,1] \times [0,1], \mathbb{C})$ such that*

$$h(0, \_) = \gamma_0, h(1, \_) = \gamma_1, h(\_, 0) = \gamma_0 0 = h(\_, 1). \tag{2.57}$$

Then, $n(\gamma_0, z_0) = n(\gamma_1, z_0)$ for $z_0 \in \neg[h]$.

Proof. Note that for each $s \in [0,1]$, $h(s, 0) = h(s, 1)$, that is $h(s, \_) \in C^0([0,1], \mathbb{C})$ is a closed curve.

Let $z_0 \in \neg[h]$. By Theorem 1.2.14, since h is compact and its domain $[0,1] \times [0,1] \subset \mathbb{R}^2$ is compact in $\mathbb{C}$. Since the underlying $\mathbb{C}$ is a Hausdorff space, by Theorem 1.2.16, $[h] \subset \mathbb{C}$ is closed. Hence,

$$\delta_0 \coloneqq d([h], z_0) > 0 \tag{2.58}$$

by Theorem 1.3.1. Let $\epsilon > 0$ such that $0 < \epsilon < \delta_0$. Since h is continuous on a compact space $[0,1] \times [0,1] \subset \mathbb{R}^2$, by Theorem 1.3.4, h is uniformly continuous. Therefore, there exists $\delta > 0$ such that, for each $s, s', t, t' \in [0,1]$:

$$|s - s'|, |t - t'| < \delta \Rightarrow |h(s,t) - h(s', t')| < \epsilon. \tag{2.59}$$

Consider subdivisions $0 = s_0 < \cdots < s_m = 1$ and $0 = t_0 < \cdots < t_n = 1$ such that

$$\max\{s_1 - s_0, \cdots, s_m - s_{m-1}, t_1 - t_0, \cdots, t_n - t_{n-1}\} < \delta. \tag{2.60}$$

Let $j \in \{0, \cdots, m\}$. The condition (2.60) guarantees:

$$2\pi n(h(s_j, \_), z_0) = \sum_{k=1}^{n} \arg \frac{h(s_j, t_k) - z_0}{h(s_j, t_{k-1}) - z_0} \tag{2.61}$$

is well-defined; see the construction in Definition 1.2.16. Moreover, for any $t \in [0,1]$:

$$\left| \frac{h(s_j, t) - z_0}{h(s_{j-1}, t) - z_0} - 1 \right| = \left| \frac{h(s_j, t) - h(s_{j-1}, t)}{h(s_{j-1}, t) - z_0} \right| < \frac{\epsilon}{\delta_0} < 1 \tag{2.62}$$

holds, where we set $s_{-1} = s_{m-1}$, and hence, $\left| \arg \frac{h(s_j,t) - z_0}{h(s_{j-1},t) - z_0} \right| < \frac{\pi}{2}$.

Since

$$\frac{h(s_j, t_k) - z_0}{h(s_j, t_{k-1}) - z_0} \frac{h(s_{j-1}, t_{k-1}) - z_0}{h(s_{j-1}, t_k) - z_0} = \frac{h(s_j, t_k) - z_0}{h(s_{j-1}, t_k) - z_0} \frac{h(s_{j-1}, t_{k-1}) - z_0}{h(s_j, t_{k-1}) - z_0}, \tag{2.63}$$

we obtain:

$$\begin{aligned} &\arg \frac{h(s_j, t_k) - z_0}{h(s_j, t_{k-1}) - z_0} - \arg \frac{h(s_{j-1}, t_k) - z_0}{h(s_{j-1}, t_{k-1}) - z_0} \\ &\equiv \arg \frac{h(s_j, t_k) - z_0}{h(s_{j-1}, t_k) - z_0} - \arg \frac{h(s_j, t_{k-1}) - z_0}{h(s_{j-1}, t_{k-1}) - z_0} \mod 2\pi. \end{aligned} \tag{2.64}$$

Since each argument is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we conclude:

$$\begin{aligned} &\arg\frac{h(s_j,t_k)-z_0}{h(s_j,t_{k-1})-z_0} - \arg\frac{h(s_{j-1},t_k)-z_0}{h(s_{j-1},t_{k-1})-z_0} \\ &= \arg\frac{h(s_j,t_k)-z_0}{h(s_{j-1},t_k)-z_0} - \arg\frac{h(s_j,t_{k-1})-z_0}{h(s_{j-1},t_{k-1})-z_0}. \end{aligned} \tag{2.65}$$

Hence, $n\left(h(s_j,\_), z_0\right) = n\left(h(s_{j-1},\_), z_0\right)$:

$$\begin{aligned} &2\pi n\left(h(s_j,\_), z_0\right) - 2\pi n\left(h(s_{j-1},\_), z_0\right) \\ &= \sum_{k=1}^{n}\arg\frac{h(s_j,t_k)-z_0}{h(s_j,t_{k-1})-z_0} - \sum_{k=1}^{n}\arg\frac{h(s_{j-1},t_k)-z_0}{h(s_{j-1},t_{k-1})-z_0} \\ &= \sum_{k=1}^{n}\arg\frac{h(s_j,t_k)-z_0}{h(s_{j-1},t_k)-z_0} - \sum_{k=1}^{n}\arg\frac{h(s_j,t_{k-1})-z_0}{h(s_{j-1},t_{k-1})-z_0} \\ &= \arg\frac{h(s_j,t_n)-z_0}{h(s_{j-1},t_n)-z_0} - \arg\frac{h(s_j,t_0)-z_0}{h(s_{j-1},t_0)-z_0} \\ &= 0. \end{aligned} \tag{2.66}$$

Since j is arbitrary, we conclude $n\left(h(s_0,\_), z_0\right) = \cdots = n\left(h(s_m,\_), z_0\right)$. ∎

Remark 11. The continuous map h is called a homotopy of γ_0 to γ_1. The homotopy h represents, intuitively speaking, a continuous deformation of γ_0 into γ_1. This theorem shows that the winding number is homotopy invariant.

2.3 Boundary-Preserving Maps on Unit Disc

Consider $\overline{D} = \overline{B_1(0)} = \{z \in \mathbb{C} \mid |z| \leqq 1\}$, its boundary:

$$\partial D = \{z \in \mathbb{C} \mid |z| = 1\} \tag{2.67}$$

and the corresponding closed curve $\gamma_0 \in C^0\left(I, \partial D\right)$:

$$\gamma_0 t := \exp\left(2\pi\sqrt{-1}t\right), \tag{2.68}$$

where $I := [0,1]$.

Theorem 2.3.1. *Let $f \in C^0\left(\overline{D}, \overline{D}\right)$ such that $f\partial D \subset \partial D$. If $n\left(f\gamma_0, \_\right)|_D \neq 0$, then $D \subset fD$.*

Proof. Suppose $n\left(f\gamma_0, \_\right)|_D \neq 0$ but, for contradiction, $D \not\subset fD$. Then, we may select z_0 in $D - fD$, and $n\left(f\gamma_0, z_0\right) \neq 0$. If we define $\gamma_1 = 1$ of a constant curve and

$$h(s,t) := (1-s)\gamma_0 t + s, \tag{2.69}$$

we obtain $h \in C^0\left([0,1]\times[0,1], \mathbb{C}\right)$ such that

$$h(0,\_) = \gamma_0, h(1,\_) = \gamma_1, h(\_,0) = 1 = h(\_,1). \tag{2.70}$$

Since $f\partial D \subset \partial D$ and $z_0 \in D - fD \subset D = \overline{D} - \partial D \subset \overline{D} - f\partial D$,

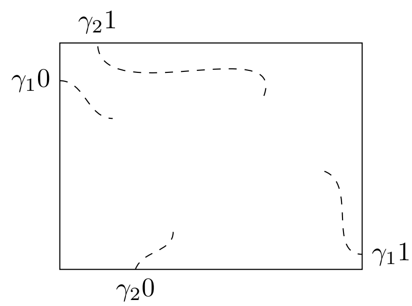
$$z_0 \notin f\partial D. \tag{2.71}$$

Hence, $z_0 \notin fD \cup f\partial D$, i.e.,

$$z_0 \notin f\overline{D}. \tag{2.72}$$

Recalling $[h] = \overline{D}$, we conclude $z_0 \notin [f\gamma_0]$. Applying Theorem 2.2.3, $n\left(f\gamma_0, z_0\right) = n\left(f\gamma_1, z_0\right) = 0$, which is absurd. ∎

Theorem 2.3.2. *Let $R = R(a,b;c,d) := \{z \in \mathbb{C} \mid a \leqq \Im z \leqq b \wedge c \leqq \Im z \leqq d\}$ be a closed rectangle, $\gamma_1, \gamma_2 \in C^0(I, R)$ be curves in R such that $\Re(\gamma_1 0) = a, \Re(\gamma_1 1) = b, \Im(\gamma_2 0) = c, \Im(\gamma_2 1) = d$, where $I := [0,1]$. Then there exist $s, t \in I$ such that $\gamma_1 s = \gamma_2 t$. In other words, a curve connecting the left and right edges meets another curve connecting the top and bottom edges.*



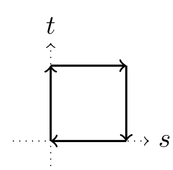
Proof. Suppose, for contradiction, that such a pair of curves never meet, i.e., $\gamma_1 s \neq \gamma_2 t$ for any $s, t \in [0,1]$. Then, we can define

$$f(s,t) := \frac{\gamma_2 t - \gamma_1 s}{|\gamma_2 t - \gamma_1 s|}. \tag{2.73}$$

Moreover, $f \in C^0\left(I^2, \overline{D}\right)$ and, since $|f(s,t)| = 1$ for each $(s,t) \in I^2$:

$$[f] \subset \partial D. \tag{2.74}$$

Since $fD \subset [f]$ is in $\partial D = \overline{D} - D$, we have $D \not\subset fD$. Consider a closed path L in I^2:



- $[(0,0),(0,1)]$

  Relative to $\gamma_1 0$, the argument of $\gamma_{2}\text{-} - \gamma_1 0 \colon I \to \mathbb{C}$ moves from $\arg\left(\gamma_2 0 - \gamma_1 0\right) \in \left[-\frac{\pi}{2}, 0\right]$ to $\arg\left(\gamma_2 1 - \gamma_1 0\right) \in \left[0, \frac{\pi}{2}\right]$, where

$$\arg\colon \left(\mathbb{C} - \mathbb{R}_{\leqq 0}\right) \to (-\pi, \pi) \tag{2.75}$$

see Definition 2.2.1.

- $[(0,1),(1,1)]$
  The argument of $\gamma_2 1 - \gamma_{1-}\colon I \to \mathbb{C}$ moves from $\arg(\gamma_2 1 - \gamma_1 0) \in \left[0, \frac{\pi}{2}\right]$ to $\arg(\gamma_2 1 - \gamma_1 1) \in \left[\frac{\pi}{2}, \pi\right]$, where

$$\arg\colon \left(\mathbb{C} - \sqrt{-1}\mathbb{R}_{\leqq 0}\right) \to \left(-\frac{\pi}{2}, \frac{3}{2}\pi\right) \tag{2.76}$$

  with $\sqrt{-1}\mathbb{R}_{\leqq 0} \coloneqq \left\{\sqrt{-1}t \mid t \leqq 0\right\}$ so that the argument single-valued and continuous in the corresponding domain.

- $[(1,1),(1,0)]$
  The argument of $\gamma_{2-} - \gamma_1 1\colon I \to \mathbb{C}$ moves from $\arg(\gamma_2 1 - \gamma_1 1) \in \left[\frac{\pi}{2}, \pi\right]$ to $\arg(\gamma_2 1 - \gamma_1 0) \in \left[\pi, \frac{3}{2}\pi\right]$, where

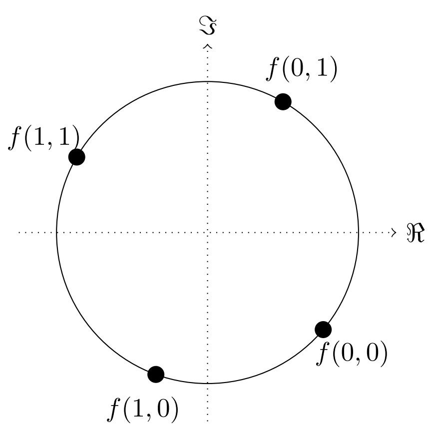
$$\arg\colon \left(\mathbb{C} - \mathbb{R}_{\geqq 0}\right) \to (0, 2\pi). \tag{2.77}$$

- $[(1,0),(0,0)]$
  The argument of $\gamma_2 0 - \gamma_{1-}\colon I \to \mathbb{C}$ moves from $\arg(\gamma_2 1 - \gamma_1 0) \in \left[\pi, \frac{3\pi}{2}\right]$ to $\arg(\gamma_2 0 - \gamma_1 0) \in \left[\frac{3\pi}{2}, 2\pi\right]$, where, with $\sqrt{-1}\mathbb{R}_{\geqq 0} \coloneqq \left\{\sqrt{-1}t \mid t \geqq 0\right\}$,

$$\arg\colon \left(\mathbb{C} - \sqrt{-1}\mathbb{R}_{\geqq 0}\right) \to \left(\frac{\pi}{2}, \frac{5}{2}\pi\right). \tag{2.78}$$

Let $\gamma_L\colon [0,4] \to L$ be a curve along with $L \subset I^2$:

$$\gamma_L u \coloneqq \begin{cases} (0,u) & u \in [0,1] \\ (u-1,1) & u \in [1,2] \\ (1,3-u) & u \in [2,3] \\ (4-u,0) & u \in [3,4] \end{cases} \tag{2.79}$$



Then f circles around the origin once, namely $n(f\gamma_L, 0) = 1$; by Theorem 2.3.1, it follows $D \subset fD$, which is absurd. ■

2.4 Jordan Curve Theorem

We will closely follow [Yan] to show Jordan curve theorem.

Lemma 2.4.1. *Let $F \subset \mathbb{C}$ be a closed subspace and $V \subset \mathbb{C} - F$ be a connected component. Then $\partial V \subset F$.*

Proof. We will first show that a connected component $V \subset \mathbb{C} - F$ is open in $\mathbb{C}$. Let $x \in V$; since $x \in \mathbb{C} - F$ and $\mathbb{C} - F \subset \mathbb{C}$ is open, there exists $\epsilon > 0$ with $B_\epsilon(x) \subset \mathbb{C} - F$. As shown in Theorem 2.1.6, the open ball $B_\epsilon(x)$ is connected, and V is a connected component with $V \cap B_\epsilon(x) \neq \emptyset$. Since $V \subset V \cup B_\epsilon(x)$, the $\subset$-largest property, see Definition 1.2.9, implies $B_\epsilon(x) \subset V$. By Lemma 1.2.2, $V \subset \mathbb{C}$ is open.

Let $W \subset \mathbb{C} - F$ be another connected component; as shown above, $W \subset \mathbb{C}$ is open. By Theorem 1.2.13, $W \cap V = \emptyset$. We will show $\partial V \cap W = \emptyset$. Let $x \in W$; since $W \subset \mathbb{C}$ is open, there is $\epsilon > 0$ with $B_\epsilon(x) \subset W$. If x were also in ∂V, by Lemma 1.2.3, $B_\epsilon(x) \cap V \neq \emptyset$ but $B_\epsilon(x) \cap V \subset W \cap V = \emptyset$, which is absurd.

Since $V \subset \mathbb{C}$ is open, we obtain:

$$\partial V = \overline{V} - V. \tag{2.80}$$

Hence, $\partial V \cap V = \emptyset$. Moreover, for each connected component W of $\mathbb{C} - F$, $\partial V \cap W = \emptyset$:

$$\emptyset = \partial V \cap \bigcup \{W \mid W \subset \mathbb{C} - F \text{ is a connected component}\} = \partial V \cap (\mathbb{C} - F) \tag{2.81}$$

Therefore, $\partial V \subset F$ holds. ■

Theorem 2.4.1. *Let $\gamma \in C^0([0,1], \mathbb{C})$ be a simple curve:*

$$\gamma s = \gamma t \Rightarrow s = t \tag{2.82}$$

i.e., a curve with no self-intersection. Then, the complement $\neg[\gamma] = \mathbb{C} - [\gamma]$ is a domain.

Proof. The continuous image $[\gamma] = \gamma[0,1]$ of a compact interval $[0,1]$ is compact by Theorem 2.1.5; by Theorem 2.1.3, $[\gamma] \subset \mathbb{C}$ is closed. Hence, $\neg[\gamma]$ is open.

Suppose, for contradiction, that $\neg[\gamma]$ is not connected. Then $\neg[\gamma]$ has at least two connected components. Since $[\gamma]$ is bounded, at least one connected component V_∞ is unbounded; let V be another connected component of $\neg[\gamma]$. Recalling $[\gamma] \subset \mathbb{C}$ is bounded, let $R > 0$ such that $[\gamma] \subset B_R(0)$; let $\gamma_R\theta = R \exp \sqrt{-1}\theta$ be the corresponding closed curve on $\partial B_R(0) = \{z \in \mathbb{C} \mid |z| = R\}$. As shown in Theorem 2.1.6, $\mathbb{C} - B_R(0)$ is connected but $[\gamma] \cap (\mathbb{C} - B_R(0)) = \emptyset$. Hence $\mathbb{C} - B_R(0) \subset V_\infty$, since $\mathbb{C} - B_R(0)$ is unbounded. It follows:

$$B_R(0) \supset \neg V_\infty \supset V. \tag{2.83}$$

Since γ is injective, the corestriction $\gamma : [0,1] \to [\gamma]$ is bijective; by Theorem 1.2.6, $\gamma \in C^0([0,1],[\gamma])$ is a continuous bijection. Applying Theorem 1.2.17, the inverse is also continuous:

$$\gamma^{-1} \in C^0([\gamma],[0,1]). \tag{2.84}$$

By Lemma 1.2.1, $[\gamma] \subset \overline{B_R(0)}$ is a closed subspace. Hence, γ^{-1} has a continuous extension φ on $\overline{B_R(0)} \supset [\gamma]$ by Lemma 1.3.4:

$$\varphi \in C^0\left(\overline{B_R(0)},[0,1]\right) \tag{2.85}$$

such that $\varphi|_{[\gamma]} = \gamma^{-1}$. Consider the composition $\gamma\varphi\colon \overline{B_R(0)} \to [\gamma]$. Since both are continuous, $\gamma\varphi \in C^0\left(\overline{B_R(0)},[\gamma]\right)$. Moreover, the restriction $\gamma \circ \varphi|_{[\gamma]}$ is an identity on $[\gamma]$. Define $f\colon \overline{B_R(0)} \to \overline{B_R(0)}$:

$$fz := \begin{cases} z & z \in \overline{B_R(0)} - V \\ \gamma\varphi z & z \in V \end{cases} \tag{2.86}$$

By definition, both $f|_{\overline{B_R(0)}-V}$ and $f|_V$ are both continuous; recalling V is open, $f|_{\partial V = \overline{V} - V}$ is identity, so is continuous. Therefore, $f \in C^0\left(\overline{B_R(0)},\overline{B_R(0)}\right)$. Since $f|_{\partial B_R(0)}$ is identity, we obtain:

$$f\partial B_R(0) \subset \partial B_R(0). \tag{2.87}$$

Then, for the curve on $\partial B_R(0)$ $\gamma_R\theta = R\exp\sqrt{-1}\theta, \theta \in [0,2\pi]$ and $z \in B_R(0)$, we obtain $n(f\gamma_R, z) = 1$ since $f\gamma_R$ circles around z once:

$$f\gamma_R\theta = f\left(R\exp\sqrt{-1}\theta\right) = R\exp\sqrt{-1}\theta. \tag{2.88}$$

By Theorem 2.3.1, we obtain $B_R(0) \subset fB_R(0)$. Consider the image of $B_R(0)$ over f:

$$fB_R(0) \subset \left(\overline{B_R(0)} - V\right) \cup \gamma\varphi V \subset \left(\overline{B_R(0)} - V\right) \cup [\gamma]. \tag{2.89}$$

Recalling $V \subset B_R(0)$, any point in V is not in the image of f, namely $V \not\subset fB_R(0)$. Therefore, we have

$$B_R(0) \not\subset fB_R(0), \tag{2.90}$$

which is absurd. ■

Definition 2.4.1 (Jordan Curves)**.** A curve $\gamma \in C^0([0,1],\mathbb{C})$ is called a Jordan curve iff it is closed, $\gamma 0 = \gamma 1$, and the restriction $\gamma|_{[0,1)}$ is a simple curve:

$$\forall s,t \in [0,1) : \gamma s = \gamma t \Rightarrow s = t. \tag{2.91}$$

Lemma 2.4.2. *Let $\gamma \in C^0([0,1],\mathbb{C})$ be a Jordan curve. If $\neg[\gamma] = \mathbb{C} - [\gamma]$ is not connected, the boundary of each connected component of $\neg[\gamma]$ is $[\gamma]$.*

Proof. Since $[\gamma] \subset \mathbb{C}$ is compact – bounded and closed – at least one connected component of $\neg[\gamma]$ is unbounded. Let V_∞ be an unbounded connected component of $\neg[\gamma]$. If $R > 0$ is sufficiently large such that $[\gamma] \subset B_R(0)$, since $\mathbb{C} - B_R(0)$ is unbounded:

$$\mathbb{C} - B_R(0) \subset V_\infty. \tag{2.92}$$

The $\subset$-largest property implies such an unbounded component is unique.

Since $\neg[\gamma]$ is disconnected, there is at least one bounded connected component, say V. By Lemma 2.4.1, $\partial V_\infty \subset [\gamma]$ and $\partial V \subset [\gamma]$. To show these inclusions are equalities, suppose for contradiction that $\partial V \subsetneq [\gamma]$. Shifting the parameter, we may set

$$\gamma 0 = \gamma 1 \in [\gamma] - \partial V. \tag{2.93}$$

Then, there are $0 < a < b < 1$ such that:

$$\gamma[a, b] \supset \partial V. \tag{2.94}$$

Since $\gamma|_{[a,b]}$ is a simple curve, $\mathbb{C} - \gamma[a, b]$ is connected by Theorem 2.4.1. By Corollary 2.1.8.1 in Theorem 2.1.8, $\mathbb{C} - \gamma[a, b]$ is path-connected. Hence, for $z \in V$ and $z_\infty \in V_\infty$, there is a curve in $\mathbb{C} - \gamma[a, b] \subset \mathbb{C} - \partial V$. Since $\partial V \cap V = \emptyset = \partial V \cap V_\infty$:

$$V \cup V_\infty \subset \mathbb{C} - \partial V \tag{2.95}$$

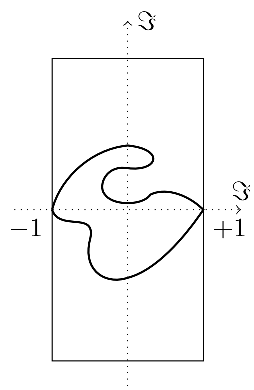
It follows $V \cup V_\infty$ is path-connected, and hence connected, which is absurd. ■

Theorem 2.4.2 (Jordan Curve Theorem)**.** *Let γ be a Jordan curve in $\mathbb{C}$. The open subspace $\neg[\gamma] = \mathbb{C} - [\gamma]$ has exactly two connected components, one is unbounded and the other is bounded. If we let V be the bounded connected component and V_∞ be the unbounded connected component of $\neg[\gamma]$, $\partial V = [\gamma] = \partial V_\infty$ is the case.*

Proof. Since $[\gamma]$ is a compact subspace in $\mathbb{C}$, by Corollary 2.1.4.1, there are $z_1, z_2 \in [\gamma]$ such that

$$\delta[\gamma] := \sup_{\zeta_1, \zeta_2 \in [\gamma]} |\zeta_1 - \zeta_2| = |z_1 - z_2|. \tag{2.96}$$

Shifting and rotating the curve, we may set $z_1 = -1$ and $z_2 = +1$:



(2.97)

Since the diameter of $[\gamma]$ is now 2, from -1 to $+1$,

$$[\gamma] \subset E := \{z \in \mathbb{C} \mid |\Im z| \leqq 2 \land |\Re z| \leqq 1\} \tag{2.98}$$

with

$$[\gamma] \cap \partial E = \{-1, +1\}, \tag{2.99}$$

otherwise, the diameter would be greater than 2. By Theorem 2.3.2, γ and $[-2\sqrt{-1}, 2\sqrt{-1}]$ meet:

$$[\gamma] \cap [-2\sqrt{-1}, 2\sqrt{-1}] \neq \emptyset. \tag{2.100}$$

Since $[\gamma]$ is compact and $[-2\sqrt{-1}, 2\sqrt{-1}] \subset \mathbb{C}$ is closed, by Theorem 1.2.15, $[\gamma] \cap [-2\sqrt{-1}, 2\sqrt{-1}]$ is compact. Since $\Im\colon \mathbb{C} \to \mathbb{R}$ is a projection, by Theorem 1.2.18, $\Im$ is continuous; applying Theorem 2.1.4, $\Im\left([\gamma] \cap [-2\sqrt{-1}, 2\sqrt{-1}]\right)$ has extreme values:

$$l := \max \Im\left([\gamma] \cap [-2\sqrt{-1}, 2\sqrt{-1}]\right). \tag{2.101}$$

Then $[2\sqrt{-1}, l\sqrt{-1}) \cap [\gamma] = \emptyset$. Since ± 1 subdivide γ into two simple curves between ± 1, we let γ_+ be the one that $l\sqrt{-1}$ belongs to:

$$l\sqrt{-1} \in [\gamma_+]. \tag{2.102}$$

Define

$$m := \min \Im\left([\gamma_+] \cap [-2\sqrt{-1}, 2\sqrt{-1}]\right). \tag{2.103}$$

It is worth mentioning $l \geqq m$. Then $(m\sqrt{-1}, -2\sqrt{-1}] \cap [\gamma_+] = \emptyset$. Let

$$\left[l\sqrt{-1}, m\sqrt{-1}\right]_{\gamma_+} \subset [\gamma_+] \tag{2.104}$$

denote the curve segment in γ_+ from $l\sqrt{-1}$ to $m\sqrt{-1}$.

We will show $[\gamma_-] \cap (m\sqrt{-1}, -2\sqrt{-1}] \neq \emptyset$. Consider a curve between $\pm 2\sqrt{-1}$:

$$[2\sqrt{-1}, l\sqrt{-1}] \diamond \left[l\sqrt{-1}, m\sqrt{-1}\right]_{\gamma_+} \diamond [m\sqrt{-1}, -2\sqrt{-1}], \tag{2.105}$$

where $\diamond$ stands for the concatenation of two curves. By Theorem 2.3.2, such a curve between $\pm 2\sqrt{-1}$ and γ_- between ± 1 must meet. Since $[\gamma_-] \subset [\gamma]$ does not meet $[2\sqrt{-1}, l\sqrt{-1})$, and $l\sqrt{-1} \in [\gamma_+]$, we conclude:

$$[\gamma_-] \cap [2\sqrt{-1}, l\sqrt{-1}] = \emptyset. \tag{2.106}$$

Moreover, $\left[l\sqrt{-1}, m\sqrt{-1}\right]_{\gamma_+} \subset [\gamma_+]$, and $m\sqrt{-1} \in [\gamma_+]$. Hence, $(m\sqrt{-1}, -2\sqrt{-1}]$ must meet $[\gamma_-]$:

$$[\gamma_-] \cap (m\sqrt{-1}, -2\sqrt{-1}] \neq \emptyset. \tag{2.107}$$

Since the intersection $[\gamma_-] \cap [m\sqrt{-1}, -2\sqrt{-1}]$ is non-empty and compact:

$$\begin{aligned} p &:= \max \Im\left([\gamma_-] \cap [m\sqrt{-1}, -2\sqrt{-1}]\right) \\ q &:= \min \Im\left([\gamma_-] \cap [m\sqrt{-1}, -2\sqrt{-1}]\right) \end{aligned} \tag{2.108}$$

By definition, $m \geqq p$ but $[\gamma_+] \cap [\gamma_-] = \{\pm 1\}$ but the intersection is on the imaginary axis, we have $m \neq p$:

$$m > p. \tag{2.109}$$

Hence $(m\sqrt{-1}, p\sqrt{-1}) \cap [\gamma] = \emptyset$. In particular,

$$z_0 \coloneqq \frac{m\sqrt{-1} + p\sqrt{-1}}{2} \notin [\gamma]. \tag{2.110}$$

Recalling $[\gamma]$ is compact, its complement $\neg[\gamma]$ should have an unbounded connected component; let V_∞ be such an unbounded component of $\neg[\gamma]$. Let $R > 0$ be sufficiently large $[\gamma] \subset B_R(0)$. Since $\mathbb{C} - B_R(0) \subset \neg[\gamma]$ is connected, see Theorem 2.1.6 and unbounded, we obtain:

$$\mathbb{C} - B_R(0) \subset V_\infty. \tag{2.111}$$

The $\subset$-largest property of V_∞ implies such an unbounded component of $\neg[\gamma]$ is unique. Then $z_0 \in E^\iota$, since $\Re z_0 = 0$ and

$$\Im z_0 = \frac{m+p}{2} < m \in [-2, 2]. \tag{2.112}$$

We will show that the connected component of $\neg[\gamma]$ around z_0 is not V_∞. Suppose, for contradiction, that z_0 is in V_∞. Since V_∞ is connected, there is a curve in V_0 from z_0 to some point in $\neg E$, since $\neg E \subset \neg[\gamma]$ is unbounded:

$$\alpha \in C^0(I, V_\infty), \tag{2.113}$$

where $\alpha 0 = z_0 \in E^\iota$ and $\alpha 1 \in \neg E$. Define

$$t_0 \coloneqq \inf \{t \in I \mid \alpha t \notin E^\iota\} \tag{2.114}$$

and $w_0 \coloneqq \alpha t_0$. We will show $w_0 \in E - E^\iota = \partial E$:

- $w_0 \in E$

  Let $\epsilon > 0$ and consider $B_\epsilon(w_0)$. Since α is continuous, its preimage $\alpha^{\leftarrow} B_\epsilon(w_0) \subset I$ is open. Hence, there is $\delta > 0$ with $(t_0 - \delta, t_0 + \delta) \subset \alpha^{\leftarrow} B_\epsilon(w_0)$:

  $$\alpha(t_0 - \delta, t_0 + \delta) \subset B_\epsilon(w_0). \tag{2.115}$$

  In particular $t_0 - \frac{\delta}{2} < t_0 = \inf \{t \in I \mid \alpha t \notin E^\iota\}$:

  $$\alpha\left(t_0 - \frac{\delta}{2}\right) \neq \alpha t_0 = w_0 \tag{2.116}$$

  and $\alpha\left(t_0 - \frac{\delta}{2}\right) \in E^\iota \subset E$. Hence, it follows $w_0 \in \overline{E} = E$:

  $$B_\epsilon(w_0) \cap E - \{w_0\} \neq \emptyset. \tag{2.117}$$

- $w_0 \notin E^\iota$

  Suppose, for contradiction, that w_0 is an interior point of E. Then there is $\epsilon > 0$ with $B_\epsilon(w_0) \subset E^\iota$. Then, around t_0, there is some $\delta > 0$ with $\alpha(t_0 - \delta, t_0 + \delta) \subset B_\epsilon(w_0)$ since α is continuous. Then $\alpha\left(t_0 + \frac{\delta}{2}\right) \in B_\epsilon(w_0) \subset E^\iota$ implies $t_0 + \frac{\delta}{2} > t_0$ would be a lower bound of $\{t \in I \mid \alpha t \notin E^\iota\}$, which is absurd.

Let $\alpha_0 := \alpha|_{[0,t_0]}$ be the curve from z_0 to $w_0 \in \partial E$. Recalling $w_0 \in V_\infty \subset \neg[\gamma]$, $w_0 \neq \pm 1$, hence $\Im w_0 \neq 0$:

- $\Im w_0 < 0$ case

  We have $[w_0, -2\sqrt{-1}]_{\partial E} \subset \partial E$, connecting w_0 and $-2\sqrt{-1}$ along with the edge of the rectangle E, without traversing ± 1. Then, since α_0 is a curve in V_∞ from $z_0 \in E^\iota$ to $w_0 \in \partial E$:

$$[2\sqrt{-1}, l\sqrt{-1}] \diamond [l\sqrt{-1}, m\sqrt{-1}]_{\gamma_+} \diamond [m\sqrt{-1}, z_0] \diamond [\alpha_0] \diamond [w_0, -2\sqrt{-1}]_{\partial E} \tag{2.118}$$

  does not meet γ_-, which is absurd.

- $\Im w_0 > 0$

  We have $[w_0, 2\sqrt{-1}]_{\partial E} \subset \partial E$, connecting w_0 and $2\sqrt{-1}$ along with the edge of the rectangle E, without traversing ± 1. Then,

$$[-2\sqrt{-1}, z_0] \diamond [\alpha_0] \diamond [w_0, 2\sqrt{-1}]_{\partial E} \tag{2.119}$$

  does not meet γ_+, which is absurd.

Hence, $z_0 \notin V_\infty$. Let V be a connected component of $\neg[\gamma]$ with $z_0 \in V$:

$$V \cap V_\infty. \tag{2.120}$$

Finally, we will show the unbounded connected component is unique. Suppose $W \subset \neg[\gamma]$ is another unbounded component. Since $\neg[\gamma] \supset \neg E$, we obtain

$$V_\infty \supset \neg E. \tag{2.121}$$

That is, the exterior of E is in V_∞. Hence, unbounded components are all in E:

$$V \subset E \wedge W \subset E. \tag{2.122}$$

Define a curve $[\beta]$ between $\pm 2\sqrt{-1}$:

$$[2\sqrt{-1}, l\sqrt{-1}] \diamond [l\sqrt{-1}, m\sqrt{-1}]_{\gamma_+} \diamond [m\sqrt{-1}, p\sqrt{-1}] \diamond [p\sqrt{-1}, q\sqrt{-1}]_{\gamma_-} \diamond [q\sqrt{-1}, -2\sqrt{-1}]. \tag{2.123}$$

- $[2\sqrt{-1}, l\sqrt{-1}], [q\sqrt{-1}, -2\sqrt{-1}] \subset V_\infty$

  Since $[2\sqrt{-1}, l\sqrt{-1}]$ can be connected with $3\sqrt{-1} \in \neg E \subset V_\infty$, $[2\sqrt{-1}, l\sqrt{-1}] \subset V_\infty$.

- $[l\sqrt{-1}, m\sqrt{-1}]_{\gamma_+}, [p\sqrt{-1}, q\sqrt{-1}]_{\gamma_-} \subset [\gamma]$
  By the very definition, they are segments of the original curve γ.
- $[m\sqrt{-1}, p\sqrt{-1}] \subset V$
  Since $[m\sqrt{-1}, p\sqrt{-1}]$ contains $z_0 \in V$, $[m\sqrt{-1}, p\sqrt{-1}] \subset V$.

Then $[\beta] \cap W = \emptyset$, since $[\beta] \subset V_\infty \cup [\gamma] \cup V$. Since $\pm 1 \notin [\beta]$, there are open balls $D_\pm \in \mathcal{N}_{\pm 1}$ with $D_\pm \cap [\beta] = \emptyset$, choosing their diameters smaller than $d([\beta], \pm 1)$. Since $\partial W = [\gamma]$ by Lemma 2.4.1, and $\pm 1 \in [\gamma]$, ± 1 are limit points of W:

$$W \cap D_\pm \neq \emptyset. \tag{2.124}$$

Let $a_\pm \in W \cap D_\pm$, c be a curve from a_- to a_+, and

$$[-1, a_-] \diamond [c] \diamond [a_+, 1] \tag{2.125}$$

be a curve between ± 1. This curve in E, connecting ± 1, does not meet β, which is absurd. Hence, the bounded component of $\neg[\gamma]$ must be unique. ■

Definition 2.4.2 (Interior and Exterior of Jordan Curves)**.** For a Jordan curve γ in $\mathbb{C}$, we call the unbounded connected component V_∞ of $\neg[\gamma]$ the exterior of γ, and the bounded component V the interior of γ. As examined in Theorem 2.2.2, the winding number on V_∞ is zero.

Theorem 2.4.3. *Let γ be a Jordan curve in $\mathbb{C}$. The winding number of γ satisfies $|n(\gamma, z)| = 1$ for any point z in the interior of γ.*

Proof. We will use the same notation in the proof of Theorem 2.4.2. Assume γ_+ is a curve from $+1$ to -1; we will show $n(\gamma, \_)|_V = +1$, where V is the interior of γ. Let δ_+ be the line segments from -1 to $+1$ along ∂E:

$$[\delta_+] = [-1, -1 + 2\sqrt{-1}] \diamond [-1 + 2\sqrt{-1}, +1 + 2\sqrt{-1}] \diamond [+1 + 2\sqrt{-1}, +1] \tag{2.126}$$

Let $\gamma_+ + \delta_+$ be the composite curve from $+1$ to -1 along γ_+, and from -1 to $+1$ along δ_+. It follows that $\gamma_+ + \delta_+$ is a Jordan curve. Since $-3\sqrt{-1} \in \neg E$ and $\neg E \subset V_\infty(\gamma_+ + \delta_+)$, the presence of a line segment $[z_0, -3\sqrt{-1}]$ implies z_0 is in the exterior of $\gamma_+ + \delta_+$, namely $z_0 \in V_\infty(\gamma_+ + \delta_+)$:

$$n(\gamma_+ + \delta_+, z_0) = 0. \tag{2.127}$$

Similarly, for

$$[\delta_-] = [+1, +1 - 2\sqrt{-1}] \diamond [+1 - 2\sqrt{-1}, -1 - 2\sqrt{-1}] \diamond [-1 - 2\sqrt{-1}, -1] \tag{2.128}$$

we obtain

$$n(\gamma_- + \delta_-, z_0) = 0 \tag{2.129}$$

since $z_0 \in V_\infty(\gamma_- + \delta_-)$. Recalling Definition 2.2.2, we can write:

$$0 = n(\gamma_+ + \delta_+, z_0) + n(\gamma_- + \delta_-, z_0) = n(\gamma, z_0) + n(\delta_+ + \delta_-, z_0). \tag{2.130}$$

As demonstrated in the proof of Theorem 2.3.2, since $\delta_+ + \delta_-$ cycles around z_0 clockwise once:

$$n\left(\delta_+ + \delta_-, z_0\right) = -1, \tag{2.131}$$

we conclude $n\left(\gamma, z_0\right) = +1$. ■

Remark 12. We can use Theorem 2.2.3 to show this claim.

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