

# Adjunctions in Topology

## 2: An Object Sitting in Two Categories

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# Chapter 0

## Abstract

No one can over-exaggerate the importance of the empty set  $\emptyset$  in mathematics, just as is the case with singleton sets such as  $\{\emptyset\}$ . This note explores  $\{0, 1\}$ , the two-element set, where 0 is identified with the empty set  $\emptyset$  and 1 is identified with the singleton set  $\{\emptyset\}$ . Such a simple yet profound set is the foundation for various mathematical concepts, including Boolean algebra, logic, set theory, and general topology. This note examines some examples of adjoint functors with a lens through  $\{0, 1\}$ .

# Chapter 1

## Preliminaries

### 1.1 Sets, Maps, and Orders

We assume some working knowledge of informal set theory including sets, subsets, supersets, the empty set  $\emptyset$ , union, intersection, set difference, complement.

#### 1.1.1 Sets and Maps

**Definition 1.1.1** (Complement). Let  $X$  be a set and  $A \subset X$  be a subset. We denote  $\neg A = X - A = \{x \in X \mid x \notin A\}$ .

**Theorem 1.1.1** (Empty Intersection and Empty Union). *Let  $X$  be a set and  $\{A_\lambda \subset X \mid \lambda \in \Lambda\}$  be a  $\Lambda$ -indexed set of subsets of  $X$ . The empty intersection  $\bigcap_{\lambda \in \emptyset} A_\lambda$  is the underlying set  $X$  and the empty union  $\bigcup_{\lambda \in \emptyset} A_\lambda$  is the empty set  $\emptyset$ .*

*Proof.* By definition:

$$\bigcap_{\lambda \in \Lambda} A_\lambda := \{x \in X \mid \forall \lambda \in \Lambda : x \in A_\lambda\}. \quad (1.1)$$

For the empty intersection, the condition is vacuously true. Hence,  $\bigcap_{\lambda \in \emptyset} A_\lambda = X$ . Similarly:

$$\bigcup_{\lambda \in \Lambda} A_\lambda := \{x \in X \mid \exists \lambda \in \Lambda : x \in A_\lambda\}. \quad (1.2)$$

If the index set is empty, the condition is always false. Hence,  $\bigcup_{\lambda \in \emptyset} A_\lambda = \emptyset$ . ■

*Remark 1.* We also have:

$$\neg \bigcap_{\lambda \in \Lambda} A_\lambda := \{x \in X \mid \exists \lambda \in \Lambda : x \notin A_\lambda\} = \bigcup_{\lambda \in \Lambda} \neg A_\lambda \quad (1.3)$$

and

$$\neg \bigcup_{\lambda \in \Lambda} A_\lambda := \{x \in X \mid \forall \lambda \in \Lambda : x \notin A_\lambda\} = \bigcap_{\lambda \in \Lambda} \neg A_\lambda. \quad (1.4)$$

**Theorem 1.1.2.** Let  $X$  be a set. For  $\{V_\alpha \subset X \mid \alpha \in A\}$  and  $\{W_\beta \subset X \mid \beta \in B\}$ ,

$$\left( \bigcup_{\alpha \in A} V_\alpha \right) \cap \left( \bigcup_{\beta \in B} W_\beta \right) = \bigcup_{(\alpha, \beta) \in A \times B} V_\alpha \cap W_\beta. \quad (1.5)$$

Similarly,

$$\left( \bigcap_{\alpha \in A} V_\alpha \right) \cup \left( \bigcap_{\beta \in B} W_\beta \right) = \bigcap_{(\alpha, \beta) \in A \times B} V_\alpha \cup W_\beta. \quad (1.6)$$

*Proof.*

$$\begin{aligned} \left( \bigcup_{\alpha \in A} V_\alpha \right) \cap \left( \bigcup_{\beta \in B} W_\beta \right) &= \{x \in X \mid \exists \alpha \in A : x \in V_\alpha\} \\ &\quad \cap \{x \in X \mid \exists \beta \in B : x \in W_\beta\} \\ &= \{x \in X \mid \exists (\alpha, \beta) \in A \times B : x \in V_\alpha \cap W_\beta\} \\ &= \bigcup_{(\alpha, \beta) \in A \times B} V_\alpha \cap W_\beta. \end{aligned} \quad (1.7)$$

Similarly,

$$\begin{aligned} \left( \bigcap_{\alpha \in A} V_\alpha \right) \cup \left( \bigcap_{\beta \in B} W_\beta \right) &= \{x \in X \mid \forall (\alpha, \beta) \in A \times B : x \in V_\alpha \cup W_\beta\} \\ &= \bigcap_{(\alpha, \beta) \in A \times B} V_\alpha \cup W_\beta. \end{aligned} \quad (1.8)$$

■

For a given map  $f: X \rightarrow Y$ , there are two induced maps:

- Direct image  $f: 2^X \rightarrow 2^Y$
- Preimage  $f^\leftarrow: 2^Y \rightarrow 2^X$

where  $f^\leftarrow W := \{x \in X \mid fx \in W\}$  for any  $W \subset Y$ .

**Theorem 1.1.3** (Properties of Preimage). *Let  $X$  and  $Y$  be sets and  $f: X \rightarrow Y$  be a map. The preimage map  $f^\leftarrow$  preserves the following elementary set operations:*

- $f^\leftarrow (\bigcup_{\lambda \in \Lambda} B_\lambda) = \bigcup_{\lambda \in \Lambda} f^\leftarrow B_\lambda$
- $f^\leftarrow (\bigcap_{\lambda \in \Lambda} B_\lambda) = \bigcap_{\lambda \in \Lambda} f^\leftarrow B_\lambda$
- $f^\leftarrow (B_1 - B_2) = f^\leftarrow B_1 - f^\leftarrow B_2$

where  $\Lambda$  is an arbitrary index set,  $B_1, B_2, B_\lambda$  are all subspaces in  $Y$  for each  $\lambda \in \Lambda$ .

*Proof.* The first two equations are almost identical:

$$\begin{aligned}
p \in f^{\leftarrow} \left( \bigcup_{\lambda \in \Lambda} B_\lambda \right) &\Leftrightarrow fp \in \bigcup_{\lambda \in \Lambda} B_\lambda \\
&\Leftrightarrow \exists \lambda \in \Lambda : fp \in B_\lambda \\
&\Leftrightarrow \exists \lambda \in \Lambda : p \in f^{\leftarrow} B_\lambda \\
&\Leftrightarrow p \in \bigcup_{\lambda \in \Lambda} f^{\leftarrow} B_\lambda
\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}
p \in f^{\leftarrow} \left( \bigcap_{\lambda \in \Lambda} B_\lambda \right) &\Leftrightarrow fp \in \bigcap_{\lambda \in \Lambda} B_\lambda \\
&\Leftrightarrow \forall \lambda \in \Lambda : fp \in B_\lambda \\
&\Leftrightarrow \forall \lambda \in \Lambda : p \in f^{\leftarrow} B_\lambda \\
&\Leftrightarrow p \in \bigcap_{\lambda \in \Lambda} f^{\leftarrow} B_\lambda
\end{aligned} \tag{1.10}$$

for each  $p \in A$ .

Recalling  $B_1 - B_2 = \{x \in A \mid x \in B_1 \wedge x \in \neg B_2\} = B_1 \cap \neg B_2$ , and

$$f^{\leftarrow}(\neg B_2) = \{x \in X \mid fx \in \neg B_2\} = X - f^{\leftarrow} B_2 = \neg f^{\leftarrow} B_2, \tag{1.11}$$

we have

$$\begin{aligned}
f^{\leftarrow}(B_1 - B_2) &= f^{\leftarrow}(B_1 \cap \neg B_2) \\
&= f^{\leftarrow} B_1 \cap f^{\leftarrow}(\neg B_2) \\
&= f^{\leftarrow} B_1 \cap \neg f^{\leftarrow} B_2 \\
&= f^{\leftarrow} B_1 - f^{\leftarrow} B_2.
\end{aligned} \tag{1.12}$$

Thus, the preimage  $f^{\leftarrow} : 2^Y \rightarrow 2^X$  preserves union, intersection, and set-difference. ■

### 1.1.2 Orders

For a set  $X$ , we consider binary relations on it, where a binary relation is represented as a subset of the product set  $X \times X := \{(x, y) \mid x \in X \wedge y \in X\}$ .

**Definition 1.1.2** (Pre-orders and Presets). A pre-order  $\leq$  on a set  $X$  is a binary relation  $\leq$  such that:

- Reflexive

For each  $x \in X$ ,  $x \leq x$  holds.

- Transitive

If  $x \leq y$  and  $y \leq z$ , then  $x \leq z$  holds.

Recalling  $\leq \subset X \times X$ ,  $x \leq y$  stands for  $(x, y) \in \leq$ . We call the pair  $(X, \leq)$  the pre-ordered set, in short, a preset.

**Definition 1.1.3** (Posets). A preset  $(X, \leq)$  is called a partially ordered set, in short, a poset, iff the pre-order  $\leq$  is also antisymmetric:

- Antisymmetric

If  $x \leq y$  and  $y \leq x$ , then  $x = y$ .

## 1.2 General Topology

General topology, in short, topology is a brunch of mathematics concerned with spaces that are invariant under continuous maps.

### 1.2.1 Basic Definitions

**Definition 1.2.1** (Topological Spaces). Let  $X$  be a set. A topology on  $X$  is a subset of its subsets  $\mathcal{T} \subset 2^X$  that closed under:

- Arbitrary Union

Each union of members in  $\mathcal{T}$  is also a member of  $\mathcal{T}$ .

- Finite Intersection

Each finite intersection of members of  $\mathcal{T}$  is also a member of  $\mathcal{T}$ .

Since the union of an empty family of sets in  $X$  is  $\emptyset$ , the intersection of an empty family of sets in  $X$  is  $X$ , we may add the following, yet redundant, conditions:

- Both  $\emptyset$  and  $X$  are members of  $\mathcal{T}$ .

The pair  $(X, \mathcal{T})$  is called a topological space. Any member in  $\mathcal{T}$  is called an open subspace of  $X$ . In particular, both  $\emptyset$  and  $X$  are open. A subset  $C \subset X$  is called closed iff the complement  $\neg C := X - C$  is open, namely  $\neg C \in \mathcal{T}$ . Since  $\emptyset = X - X$  and  $X = X - \emptyset$ , we conclude that both  $\emptyset$  and  $X$  are clopen.

For a subset  $Y \subset X$  of a topological space  $(X, \mathcal{T})$ , the induced topology is

$$\mathcal{T}_Y := \{Y \cap U \mid U \in \mathcal{T}\}. \quad (1.13)$$

The pair  $(Y, \mathcal{T}_Y)$  is called a subspace of  $(X, \mathcal{T})$ .

**Definition 1.2.2** (Neighborhoods and Open Subspaces). Let  $(X, \mathcal{T})$  be a topological space, and  $p \in X$ . A subspace  $U' \subset X$  is called a neighborhood of  $p$  iff there exists some  $U \in \mathcal{T}$  such that  $p \in U$  and  $U \subset U'$ . Let  $\mathcal{N}_p$  be the set of all neighborhoods of  $p$  in  $X$  relative to  $\mathcal{T}$ .

**Lemma 1.2.1.** *Let  $(X, \mathcal{T})$  be a topological space,  $U \subset X$  be a subspace.  $U$  is open,  $U \in \mathcal{T}$ , iff  $U$  is a neighborhood of every point in it.*

*Proof.* ( $\Rightarrow$ ) Suppose  $U \in \mathcal{T}$ . Then, for each  $p \in U$ ,  $U$  is an open neighborhood of  $p$ .

( $\Leftarrow$ ) Conversely, suppose  $U$  is a neighborhood to its points. For  $p \in U$ , let  $V_p \in \mathcal{T}$  be an open subspace such that  $p \in V_p$  and  $V_p \subset U$ . Then, we conclude  $U = \bigcup_{p \in U} V_p$  since:

$$U \subset \bigcup_{p \in U} V_p \subset U. \quad (1.14)$$

$U$  is given by a union of open subspaces in  $X$ , hence  $U$  is open.  $\blacksquare$

**Definition 1.2.3** (Limit Points and Closure). Let  $A \subset (X, \mathcal{T})$  be a subspace. A point  $p \in X$  is called a limit point of  $A$  iff each neighborhood of  $p$  contains at least one point of  $A$  distinct from  $p$ :

$$\forall U' \in \mathcal{N}_p : U' \cap A - \{p\} \neq \emptyset. \quad (1.15)$$

Let  $A'$  denote the set of all limit points. We call  $\bar{A} := A \cup A'$  the closure of  $A$ .

**Lemma 1.2.2.** *Let  $A \subset (X, \mathcal{T})$  be a subspace. For any point  $p \in X$ ,  $p \in \bar{A}$  iff*

$$\forall U' \in \mathcal{N}_p : U' \cap A \neq \emptyset. \quad (1.16)$$

*Proof.* ( $\Rightarrow$ ) Let  $p \in \bar{A}$ :

- $p \in A$  case

For each neighborhood  $U' \in \mathcal{N}_p$ ,  $p \in U' \cap A$ .

- $p \notin A$  case

For each neighborhood  $U' \in \mathcal{N}_p$ ,  $U' \cap A = U' \cap A - \{p\} \neq \emptyset$  holds.

( $\Leftarrow$ ) Suppose for each neighborhood  $U' \in \mathcal{N}_p$ ,  $U' \cap A \neq \emptyset$ . Nothing has to be shown if  $p \in A$ , as  $A \subset \bar{A}$ . Hence, we may assume  $p \notin A$ . Then, as  $A = A - \{p\}$ ,  $U' \cap A = U' \cap A - \{p\} \neq \emptyset$  is the case for each neighborhood  $U' \in \mathcal{N}_p$ .  $\blacksquare$

**Theorem 1.2.1** (Characterization of Closed Subspaces). *A subspace  $A \subset (X, \mathcal{T})$  is closed iff  $A = \bar{A}$ .*

*Proof.* ( $\Rightarrow$ ) Suppose that  $A$  is closed, i.e.,  $\neg A \in \mathcal{T}$ . Each  $p \in \neg A$  has an open neighborhood, namely  $\neg A$ , which does not meet  $A$  since  $A \cup \neg A = \emptyset$ . So, each  $p \in \neg A$  does not belong to  $\bar{A}$ . We have  $\neg A \subset \neg \bar{A}$ , and  $A \supset \bar{A}$ . Since  $A \subset \bar{A}$ , we conclude  $\bar{A} = A$ .

( $\Leftarrow$ ) Suppose  $\bar{A} = A$ . We will show  $\neg A$  is open. Let  $p \in \neg A$ . Since  $p \in \neg \bar{A}$ ,  $p$  is not a limit point of  $A$ . Thus, there is some neighborhood  $U' \in \mathcal{N}_p$  with  $U' \cap A = \emptyset$  by Lemma 1.2.2. We obtain  $U' \subset \neg A$ . That is,  $\neg A$  is a neighborhood of  $p$ . As  $p \in \neg A$  is arbitrary, by Lemma 1.2.1, we conclude  $\neg A \in \mathcal{T}$ .  $\blacksquare$

**Theorem 1.2.2** (Properties of Closures). *Let  $A, B \subset (X, \mathcal{T})$  be subspaces.*



- The closure  $\overline{A}$  is  $\subset$ -smallest closed subspace of  $X$  containing  $A$ :

$$\overline{A} = \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\} \quad (1.17)$$

- $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
- $\overline{\overline{A}} = \overline{A}$ , i.e., the closure  $\overline{A}$  of  $A$  is closed, and the closure-operation is idempotent.
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$
- $\overline{\emptyset} = \emptyset$

*Proof.* Let  $\tilde{A} := \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\}$ . Since open subspaces are closed under arbitrary union, the complements, i.e., closed subspaces are closed under arbitrary intersection. Hence,  $\tilde{A}$  is closed. To show  $\tilde{A}$  is equal to  $\overline{A}$ , let us consider their complements:

$\neg \tilde{A} \subset \neg \overline{A}$  Let  $p \in \neg \tilde{A}$ .  $\neg \tilde{A}$  is an open neighborhood of  $p$  with  $\neg \tilde{A} \cap \tilde{A} = \emptyset$ . Since  $\tilde{A} \supset A$ ,  $A$  does not meet  $\neg \tilde{A}$ . Thus  $\neg \tilde{A} \cap A = \emptyset$ . By Lemma 1.2.2,  $p \in \neg \overline{A}$  holds.

$\neg \tilde{A} \supset \neg \overline{A}$  Let  $p \in \neg \overline{A}$ . Since  $p$  is not a limit point of  $A$ , there exists an open neighborhood  $U \in \mathcal{N}_p \cap \mathcal{T}$  such that  $U \cap A - \{p\} = \emptyset$ . As  $p$  is not in  $A$ ,  $U \cap A = \emptyset$ , thus  $A \subset \neg U$ . Thus,  $\neg U$  is a member of the right-hand side of (1.17), we obtain  $\tilde{A} \subset \neg U$ . Since  $p \in U$  and  $U \subset \neg \tilde{A}$ , we conclude  $p \in \neg \tilde{A}$ .

Hence, we obtain  $\overline{A} = \bigcap \{F \subset X \mid F \supset A \wedge \neg F \in \mathcal{T}\}$ .

- $A \subset B \Rightarrow \overline{A} \subset \overline{B}$   
Since any closed subspace containing  $B$  also contains  $A$ ,  $\overline{A} \subset \overline{B}$ .
- $\overline{\overline{A}} = \overline{A}$   
Since  $\overline{A}$  is given by an intersection of closed subspaces,  $\overline{A}$  is closed. Moreover,  $\overline{A} \subset \overline{\overline{A}}$  is the  $\subset$ -smallest subspace containing  $\overline{A}$ .
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$   
 $\overline{A \cup B}$  is closed, and contains both  $A$  and  $B$ , hence  $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ . As  $\overline{A \cup B}$  is closed, containing  $A \cup B$ ,  $\subset$ -smallest property implies  $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ .
- $\overline{\emptyset} = \emptyset$   
Since  $\emptyset$  is clopen and  $\emptyset \subset \emptyset$ , the  $\subset$ -smallest property ensures  $\overline{\emptyset} = \emptyset$ .

■

**Theorem 1.2.3** (Subspaces and Closures). *Let  $(X, \mathcal{T})$  be a topological space and  $(Y, \mathcal{T}_Y) \subset (X, \mathcal{T})$  be a subspace. For  $A \subset Y$ , the closure  $\overline{A}_Y$  relative to  $\mathcal{T}_Y$  is  $Y \cap \overline{A}$ , where  $\overline{A}$  is the closure of  $A \subset X$  relative to  $\mathcal{T}$ .*

*Proof.* It suffices to show  $A'_Y = Y \cap A'$  since  $\overline{A}_Y = A'_Y \cup A$  and  $Y \cap \overline{A} = Y \cup (A \cup A') = (Y \cap A) \cup (Y \cap A') = A \cup (Y \cap A')$ .

Let  $p \in A'_Y$  and  $\mathcal{N}_{Yp}$  be the set of neighborhood of  $p$  relative to  $\mathcal{T}_Y$ :

$$\forall U' \in \mathcal{N}_{Yp} : \exists U \in \mathcal{T} : p \in (U \cap Y) \subset U'. \quad (1.18)$$

Note that  $(U \cap Y) \in \mathcal{T}_Y$  if  $U \in \mathcal{T}$ . Since  $p \in A'_Y$ ,

$$\forall U' \in \mathcal{N}_{Yp} : U' \cap A - \{p\} \neq \emptyset, \quad (1.19)$$

i.e.,

$$\forall U \in \mathcal{N}_p \cap \mathcal{T} : (U \cap Y) \cap A - \{p\} \neq \emptyset, \quad (1.20)$$

we obtain  $p \in (Y \cap A)'$  relative to  $\mathcal{T}$ . Recalling  $A \subset Y$  and  $p \in Y$ , we obtain  $p \in Y \cap A'$ .

Conversely, let  $p \in Y \cap A'$  relative to  $\mathcal{T}$ :

$$\forall U' \in \mathcal{N}_p : U' \cap A - \{p\} \neq \emptyset. \quad (1.21)$$

Since  $A \subset Y$ , it is equivalent to

$$\forall U' \in \mathcal{N}_p : U' \cap (A \cap Y) - \{p\} \neq \emptyset. \quad (1.22)$$

Now,  $U' \cap Y$  contains an open  $(U \cap Y) \in \mathcal{T}_Y$  with  $p \in U \cap Y$ . That is,  $U' \cap Y$  is a neighborhood of  $p$  relative to  $\mathcal{T}_Y$ , namely  $U' \cap Y \in \mathcal{N}_{Yp}$ , moreover  $p \in A'_Y$ .

Hence, we establish  $A'_Y = Y \cap A'$ , and  $\overline{A}_Y = Y \cap \overline{A}$ .  $\blacksquare$

## 1.2.2 Separation Axioms

**Definition 1.2.4.** The following axioms describe how a topology can distinguish points in the underlying set:

$T_0$  A  $T_0$  space – a Kolmogorov space – is a topological space in which every pair of distinct points is topologically distinguishable, i.e., there exists an open subspace that contains one of them and not the other.

$T_1$  A  $T_1$  space – a Fréchet space – is a topological space in which for every pair of distinct points, each has a neighborhood not containing the other. In other words, each has an open subspace that contains it but not the other.

$T_2$  A  $T_2$  space – a Hausdorff space – is a topological space  $(X, \mathcal{T})$  in which each of two distinct points have disjoint neighborhoods, that is, if  $p \neq q$ , there are  $U' \in \mathcal{N}_p$  and  $V' \in \mathcal{N}_q$  with  $U' \cap V' = \emptyset$ .

### 1.2.3 Basic Open Sets

... we can to an extent preassign the notion of nearness desired. [Dug66]

**Definition 1.2.5** (Subbases and Generated Topology). Let  $X$  be a set and  $\mathcal{S} \subset 2^X$  be a set of subsets in  $X$ . As  $2^X$  is a topology of  $X$ ,

$$\tau_{\mathcal{S}} := \{\mathcal{T} \subset 2^X \mid \mathcal{T} \text{ is a topology on } X \text{ with } \mathcal{S} \subset \mathcal{T}\} \quad (1.23)$$

is non-empty. Their intersection:

$$\bigcap \tau_{\mathcal{S}} := \bigcap \{\mathcal{T} \in \tau_{\mathcal{S}}\} \quad (1.24)$$

is called the topology generated by  $\mathcal{S}$ . It is the  $\subset$ -smallest topology containing  $\mathcal{S}$ .

For the generated topology, the generating set  $\mathcal{S}$  is called the subbasic open set, in short, a subbase.

*Remark 2* (Basis). No further conditions for being a subbase of some topology. If  $\mathcal{S}$  satisfies:

1.  $\mathcal{S}$  covers  $X$

For each  $x \in X$ , there is a  $B \in \mathcal{S}$  with  $x \in B$ . This condition guarantees that  $X$  is open.

2. Binary Intersection

Let  $B_1, B_2 \in \mathcal{S}$ . If  $x \in B_1 \cap B_2$ , there is a  $B_3 \in \mathcal{S}$  with  $x \in B_3$  and  $B_3 \subset B_1 \cap B_2$ . This condition guarantees that  $B_1 \cap B_2$  is open.

Then  $\mathcal{S}$  is called the set of basic open sets, in short, a basis for the topology  $\bigcap \tau_{\mathcal{S}}$  of  $X$ .

**Theorem 1.2.4.** Let  $X$  be a set,  $\mathcal{S} \subset 2^X$  be a basis –  $\mathcal{S}$  satisfies both conditions 1 and 2 – and  $\mathcal{T}_{\mathcal{S}}$  be the set of all unions of  $\mathcal{S}$ .  $\mathcal{T}_{\mathcal{S}}$  is a topology on  $X$ . Moreover,  $\mathcal{T}_{\mathcal{S}} = \bigcap \tau_{\mathcal{S}}$ .

*Proof.* As the condition 1 ensures  $\mathcal{S}$  covers  $X$ , we have  $X \in \mathcal{T}_{\mathcal{S}}$ . If we take the empty union,  $\emptyset \in \mathcal{T}_{\mathcal{S}}$ . By definition,  $\mathcal{T}_{\mathcal{S}}$  is closed under arbitrary union. The condition 2 guarantees  $\mathcal{T}_{\mathcal{S}}$  is closed under binary, hence any finite intersection. Therefore,  $\mathcal{T}_{\mathcal{S}}$  forms a topology on  $X$ .

Since  $\mathcal{S} \subset \mathcal{T}_{\mathcal{S}}$  holds,  $\mathcal{T}_{\mathcal{S}} \in \tau_{\mathcal{S}}$ , hence  $\bigcap \tau_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{S}}$ . To show the other inclusion, let  $U \in \mathcal{T}_{\mathcal{S}}$ . By construction, there exists  $\mathcal{B}_U \subset \mathcal{S}$  with

$$U = \bigcup \mathcal{B}_U = \bigcup \{V \in \mathcal{B}_U\}. \quad (1.25)$$

As  $\mathcal{B}_U \subset \mathcal{S}$ , and any member  $T \in \tau_{\mathcal{S}}$  contains  $\mathcal{S}$ , we obtain  $\mathcal{B}_U \subset T$  for each  $T \in \tau_{\mathcal{S}}$ . Thus,  $\mathcal{B}_U \subset T$  holds for each  $T \in \tau_{\mathcal{S}}$ . I.e.,  $U \in \bigcap \tau_{\mathcal{S}}$ . ■

### 1.2.4 Continuous Maps

For given topological space  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$ , and a map between the underlying sets  $f: X \rightarrow Y$ , we use  $f^\leftarrow$  to associate the topology since  $f^\leftarrow$  preserves the elementary set operations as shown in Theorem 1.1.3:

**Definition 1.2.6** (Continuous Maps). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A map  $f: X \rightarrow Y$  is called continuous iff the preimage of each open subspace in  $Y$  is open in  $X$ . That is,  $f^\leftarrow$  maps  $\mathcal{T}_Y \subset 2^Y$  into  $\mathcal{T}_X$ :

$$f^\leftarrow: \mathcal{T}_Y \rightarrow \mathcal{T}_X. \quad (1.26)$$

The set of all continuous maps from  $X$  to  $Y$  is denoted by  $C^0(X, Y)$ .

**Theorem 1.2.5** (Characterizations of Continuity). *Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces, and  $f: X \rightarrow Y$  be a map. The following are equivalent:*

1.  $f \in C^0(X, Y)$  by means of Definition 1.2.6.
2. For a subbase (or a basis)  $\mathcal{S}_Y \subset \mathcal{T}_Y$ ,  $f^\leftarrow \mathcal{S}_Y \subset \mathcal{T}_X$ .
3. The preimage of a closed subspace in  $Y$  is closed in  $X$ .
4. For each  $x \in X$  and for each neighborhood  $V' \in \mathcal{N}_{fx}$ , there exists a neighborhood  $U' \in \mathcal{N}_x$  s.t.,  $fU' \subset V'$ .
5.  $f\overline{A} \subset \overline{fA}$  for every  $A \subset X$ .
6.  $\overline{f^\leftarrow B} \subset f^\leftarrow \overline{B}$  for every  $B \subset Y$ .

*Proof.*  $(1 \Leftrightarrow 2)$  As  $\mathcal{S}_Y \subset \mathcal{T}_Y$ ,  $f^\leftarrow|_{\mathcal{S}_Y}: \mathcal{S}_Y \rightarrow \mathcal{T}_X$ . Conversely, suppose  $f^\leftarrow \mathcal{S}_Y \subset \mathcal{T}_X$  is the case. Let  $W \in \mathcal{T}_Y$ . Since  $\mathcal{T}_Y$  is generated by  $\mathcal{S}_Y$ ,  $W$  is given by some, not necessarily finite, union of finite intersections of members in  $\mathcal{S}_Y$ :

$$W = \bigcup_{\lambda \in \Lambda} \left( B_1^{(\lambda)} \cap \dots \cap B_{j_\lambda}^{(\lambda)} \right), \quad (1.27)$$

where  $B_1^{(\lambda)} \dots B_{j_\lambda}^{(\lambda)} \in \mathcal{S}_Y$  for each  $\lambda \in \Lambda$ . Applying Theorem 1.1.3, we obtain

$$f^\leftarrow W = \bigcup_{\lambda \in \Lambda} f^\leftarrow \left( B_1^{(\lambda)} \cap \dots \cap B_{j_\lambda}^{(\lambda)} \right) = \bigcup_{\lambda \in \Lambda} \left( f^\leftarrow B_1^{(\lambda)} \right) \cap \dots \cap \left( f^\leftarrow B_{j_\lambda}^{(\lambda)} \right). \quad (1.28)$$

Since  $\left( f^\leftarrow B_1^{(\lambda)} \right) \cap \dots \cap \left( f^\leftarrow B_{j_\lambda}^{(\lambda)} \right) \in \mathcal{T}_X$  and  $W$  is a union of such open subspaces in  $X$ , we conclude  $f^\leftarrow W \in \mathcal{T}_X$ .

$(1 \Leftrightarrow 3)$  By Theorem 1.1.3,

$$f^\leftarrow (\neg A) = f^\leftarrow (Y - A) = X - f^\leftarrow A = \neg f^\leftarrow A \quad (1.29)$$

for every  $A \subset X$ .

$(1 \Rightarrow 4)$  Let  $x \in X$ ,  $V' \in \mathcal{N}_{fx}$ , and  $V \in \mathcal{T}_Y$  s.t.,  $fx \in V$  and  $V \subset V'$ . As  $f$  is continuous,  $f^\leftarrow V \in \mathcal{T}_X$ . Since  $x \in f^\leftarrow V$ , we may set  $U' = f^\leftarrow V$ .

(4  $\Rightarrow$  5) Let  $A \subset X$  and  $x \in \overline{A}$ ; we will show  $fx$  is a member of  $\overline{fA}$ . Consider  $V' \in \mathcal{N}_{fx}$ ; as we assume 4, there exists  $U' \in \mathcal{N}_x$  with  $fU' \subset V'$ . Since  $x \in \overline{A}$ , by Lemma 1.2.2,  $U' \cap A \neq \emptyset$  holds. Hence,  $fx \in \overline{fA}$ :

$$\emptyset \subsetneq f(U' \cap A) \subset fU' \cap fA \subset V' \cap fA. \quad (1.30)$$

(5  $\Rightarrow$  6) Let  $B \subset Y$  and  $A := f^{\leftarrow} B$ . As we assume 5,

$$f(\overline{f^{\leftarrow} B}) = f\overline{A} \subset \overline{fA} = \overline{f(f^{\leftarrow} B)} \subset \overline{B}. \quad (1.31)$$

Thus,  $\overline{f^{\leftarrow} B} \subset f^{\leftarrow} \overline{B}$ .

(6  $\Rightarrow$  3) Let  $B \subset Y$  be a closed subspace. As we assume 6,  $\overline{f^{\leftarrow} B} \subset f^{\leftarrow} \overline{B}$ . Since  $\overline{B} = B$ , we conclude  $f^{\leftarrow} B = f^{\leftarrow} \overline{B}$ :

$$\overline{f^{\leftarrow} B} \subset f^{\leftarrow} \overline{B} \subset f^{\leftarrow} B \subset \overline{f^{\leftarrow} B}. \quad (1.32)$$

See Theorem 1.2.1. ■

**Lemma 1.2.3** (Universal Property of Relative Topology). *Let  $Y \subset (X, \mathcal{T})$  be a subspace. The relative topology  $\mathcal{T}_Y$  defined in Definition 1.2.1 can be characterized as the  $\subset$ -smallest topology on  $Y$  for which the inclusion map:*

$$i: Y \hookrightarrow X; y \mapsto y \quad (1.33)$$

*is continuous, namely  $i \in C^0(Y, X)$ .*

*Proof.* Let  $\mathcal{T}_Y'$  be an arbitrary topology on  $Y$ . Suppose  $i: Y \hookrightarrow X$  is continuous relative to  $(X, \mathcal{T})$  and  $(Y, \mathcal{T}_Y')$ . We will show that  $\mathcal{T}_Y' \supset \mathcal{T}_Y$ .

Let  $U \in \mathcal{T}$ . As  $i \in C^0((Y, \mathcal{T}_Y'), (X, \mathcal{T}))$ , the preimage  $i^{\leftarrow} U$  is open in  $(Y, \mathcal{T}_Y')$ :

$$i^{\leftarrow} U = U \cap Y \in \mathcal{T}_Y'. \quad (1.34)$$

Since  $U$  is arbitrary, it follows that any open subspace in  $Y$  relative to  $\mathcal{T}_Y$ ,  $U \cap Y \in \mathcal{T}_Y$  is a member of  $\mathcal{T}_Y'$ , hence  $\mathcal{T}_Y \subset \mathcal{T}_Y'$ . ■

**Theorem 1.2.6** (Properties of Continuous Maps). *Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y), (Z, \mathcal{T}_Z)$  be topological spaces.*

- *If  $f \in C^0(X, Y)$  and  $g \in C^0(Y, Z)$ , the composition  $gf \in C^0(X, Z)$ .*
- *If  $f \in C^0(X, Y)$  and  $A \subset X$ , the restriction  $f|_A : A \rightarrow Y$  is continuous relative to the relative topology on  $A$ .*
- *If  $f \in C^0(X, Y)$ , the corstriction of  $f$  on its image is continuous:*

$$f \in C^0(X, fX). \quad (1.35)$$

*Proof.* Suppose  $f \in C^0((X, \mathcal{T}_X), (Y, \mathcal{T}_Y))$ ,  $g \in C^0((Y, \mathcal{T}_Y), (Z, \mathcal{T}_Z))$ , and  $A \subset X$ .

- Since  $f^\leftarrow: \mathcal{T}_Y \rightarrow \mathcal{T}_X$  and  $g^\leftarrow: \mathcal{T}_Z \rightarrow \mathcal{T}_Y$ , and  $(g \circ f)^\leftarrow = f^\leftarrow \circ g^\leftarrow$ , the continuity of the composition  $g \circ f$  follows:

$$(g \circ f)^\leftarrow: \mathcal{T}_Z \rightarrow \mathcal{T}_X. \quad (1.36)$$

- Let  $i: A \hookrightarrow X$ . Since

$$f|_A = f \circ i \quad (1.37)$$

and as shown above  $i \in C^0(A, X)$  relative to  $\mathcal{T}_A$ , the composition is continuous.

- For each  $V \in \mathcal{T}_V$ , i.e., for each open subspace  $V \cap fX$  in  $fX$ ,

$$f^\leftarrow(V \cap fX) = f^\leftarrow V \cap f^\leftarrow(fX) = f^\leftarrow V. \quad (1.38)$$

Since  $f^\leftarrow V$  is open in  $X$ , the restriction  $f: X \rightarrow fX$  is continuous.

■

**Definition 1.2.7** (Homeomorphisms and Topological Invariance). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A map  $f: X \rightarrow Y$  is called a homeomorphism – a topological isomorphism – iff the following conditions hold:

- The underlying map  $f: X \rightarrow Y$  is bijective.
- Both  $f$  and  $f^{-1}$  are continuous.

If  $f$  is a homeomorphism, it is denoted by  $f: X \cong Y$ . Two spaces  $X$  and  $Y$  are homeomorphic, written  $X \cong Y$ , iff there is a homeomorphism between them. It is worth mentioning that a homeomorphism  $f: X \cong Y$  is an open map – the image of an open subspace  $U \in \mathcal{T}_X$  along  $f$  is open  $fU \in \mathcal{T}_Y$ , since  $f^{-1}$  is continuous. Moreover, a homeomorphism  $f: X \cong Y$  is a bijection for the underlying set and the associated topologies:

$$\begin{aligned} f: X &\cong Y \\ f^{-1}: \mathcal{T}_Y &\cong \mathcal{T}_X \end{aligned} \quad (1.39)$$

Thus, any topological property about  $X$  is mapped to that of  $Y$ . We call any property of spaces a topological invariant iff whenever it is true for one space, it is also varied for every homeomorphic space.

**Theorem 1.2.7.** *Homeomorphism is an equivalence relation in the class of all topological spaces.*

*Proof.* Observe:

- Reflexive  
For any topological space  $X$ ,  $1_X: X \cong X$ .
- Symmetric  
If  $f: X \cong Y$ ,  $Y \cong X$  via  $f^{-1}$ .

- Transitive

If  $f: X \cong Y$  and  $g: Y \cong Z$ , then  $g \circ f: X \cong Z$ .

See Theorem 1.2.6. ■

## 1.3 Category Theory

Category theory offers a general theory of mathematical structures and relations.

### 1.3.1 Basic Definitions

**Definition 1.3.1** (Categories). A category  $\mathcal{C}$  consists of a class of objects  $|\mathcal{C}|$  and, for each pair of objects  $A, B \in |\mathcal{C}|$ , a set of arrows from  $A$  to  $B$ , denoted as  $\mathcal{C}(A, B)$ , such that:

- Each arrow  $\phi$  in  $\mathcal{C}$  has unique domain and codomain, namely  $X \xrightarrow{\phi} Y$  with  $X, Y \in |\mathcal{C}|$ .
- Each object  $X \in |\mathcal{C}|$  has a unique arrow  $X \xrightarrow{1_X} X$ .
- For any pair of arrows  $f, g$  in  $\mathcal{C}$ , if the domain of  $g$  is equal to the codomain of  $f$ , their composite arrow  $gf = g \circ f$  exists, namely if  $A \xrightarrow{f} B$  and  $B \xrightarrow{g} C$ , their composition is  $A \xrightarrow{gf} C$ .

These arrows in  $\mathcal{C}$  also satisfy the following axioms:

- For any arrow  $A \xrightarrow{f} B$ , both  $f1_A$  and  $1_B f$  are  $f$ .
- If  $A \xrightarrow{f} B$ ,  $B \xrightarrow{g} C$ , and  $C \xrightarrow{h} D$ , the compositions  $h(gf)$  and  $(hg)f$  are both equal to  $A \xrightarrow{hgf} D$ .

*Remark 3* (Small Categories). A category  $\mathcal{C}$  is called small iff  $|\mathcal{C}|$  is a set.

**Definition 1.3.2** (Isomorphisms). Let  $\mathcal{C}$  be a category. An arrow  $f \in \mathcal{C}(A, B)$  is called an isomorphism iff there is  $f' \in \mathcal{C}(B, A)$  such that  $f'f = 1_A$  and  $ff' = 1_B$ .

**Definition 1.3.3** (Functors). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A covariant functor, in short a functor  $F$  from  $\mathcal{C}$  and  $\mathcal{D}$ , denoted  $F: \mathcal{C} \rightarrow \mathcal{D}$ , consists of the following correspondences:

- For each object  $C \in |\mathcal{C}|$ , there exists  $FC \in |\mathcal{D}|$ .
- For an arrow  $f \in \mathcal{C}(X, Y)$ , there exists  $Ff \in \mathcal{D}(FX, FY)$ .

These correspondences satisfy the following axioms:

- For  $X \xrightarrow{f} Y \xrightarrow{g} Z$  in  $\mathcal{C}$ ,  $FgFf = F(gf)$  holds. That is, the composition  $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ$  in  $\mathcal{D}$  is equal to  $FX \xrightarrow{F(gf)} FZ$ .
- For each  $X \in |\mathcal{C}|$ ,  $F1_X = 1_{FX}$ .

We denote  $\mathcal{D}^{\mathcal{C}}$  the class of functors from  $\mathcal{C}$  to  $\mathcal{D}$ .

*Remark 4* (Opposite Categories and Contravariant Functors). Let  $\mathcal{C}$  be a category. The opposite  $\mathcal{C}^{op}$  is given by:

- The same class of objects  $|\mathcal{C}^{op}| = |\mathcal{C}|$ .
- An arrow  $f^{op} \in \mathcal{C}^{op}(X, Y)$  is an arrow in  $\mathcal{C}$  so that the domain and codomain are swapped,  $f \in \mathcal{C}(Y, X)$ .

The correspondence  $\mathcal{C} \rightarrow \mathcal{C}^{op}$  preserves the categorical structure, exchanging domains and codomains:

- For each object  $X \in |\mathcal{C}|$ ,  $1_X \mapsto 1_X^{op} = 1_X$ .
- For  $f^{op} \in \mathcal{C}^{op}(X, Y)$  and  $g^{op} \in \mathcal{C}^{op}(Y, Z)$ , we define  $g^{op}f^{op}$  to be  $(fg)^{op}$ . That is,  $X \xrightarrow{f^{op}} Y \xrightarrow{g^{op}} Z$  is  $\left(Z \xrightarrow{g} Y \xrightarrow{f} X\right)^{op} = \left(Z \xrightarrow{fg} X\right)^{op}$ .

Hence,  $\mathcal{C}^{op}$  forms a category – the opposite category. A contravariant functor  $F$  from  $\mathcal{C}$  to  $\mathcal{D}$  is a functor  $F: \mathcal{C}^{op} \rightarrow \mathcal{D}$ .

**Theorem 1.3.1.** *Functors preserve isomorphisms.*

*Proof.* Let  $f \in \mathcal{C}(A, B)$  be an isomorphism and  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Since  $f$  is an isomorphism, there is an arrow  $f' \in \mathcal{C}(B, A)$  with  $f'f = 1_A$  and  $ff' = 1_B$ . Then,  $Ff \in \mathcal{D}(FA, FB)$  has an inverse  $Ff'$ , since

$$\begin{aligned} Ff' \circ Ff &= F(f'f) = F1_A = 1_{FA} \\ Ff \circ Ff' &= F(ff') = F1_B = 1_{FB} \end{aligned} \tag{1.40}$$

Hence,  $Ff$  is an isomorphism if  $f$  is an isomorphism. ■

**Definition 1.3.4** (Natural Transformations). Let  $\mathcal{C}, \mathcal{D}$  be two categories, and  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{C} \rightarrow \mathcal{D}$  be two functors. A natural transformation  $\theta$  from  $F$  to  $G$ , denoted as  $\theta: F \Rightarrow G$  is given by a  $|\mathcal{C}|$ -indexed class of arrows in  $\mathcal{D}$ , namely  $\{\theta_C \in \mathcal{D}(FC, GC) \mid C \in |\mathcal{C}|\}$ , such that  $Gc \circ \theta_{C_1} = \theta_{C_2} \circ Fc$  for each  $c \in \mathcal{C}(C_1, C_2)$ . That is, the following diagram is commutative:

$$\begin{array}{ccc} FC_1 & \xrightarrow{Fc} & FC_2 \\ \theta_{C_1} \downarrow & & \downarrow \theta_{C_2} \\ GC_1 & \xrightarrow{Gc} & GC_2 \end{array} \tag{1.41}$$

for each  $c \in \mathcal{C}(C_1, C_2)$ . We call  $\theta_C \in \mathcal{D}(FC, GC)$   $C$ -component of  $\theta: F \Rightarrow G$ .



*Remark 5* (Curien's Promotion [Cur08]). Let  $\mathcal{C}$  be a category and  $f \in \mathcal{C}(A, B)$ . With the terminal category  $\mathbf{1}$  of a singleton set  $\{\star\}$  with the identity map on it, we may identify  $A \in |\mathcal{C}|$  as a functor  $\tilde{A}: \mathbf{1} \rightarrow \mathcal{C}$  and  $f \in \mathcal{C}(A, B)$  as a natural transformation  $\tilde{f}: A \Rightarrow B$ .

$$\begin{array}{ccc}
 \mathbf{1} & \xrightleftharpoons[\tilde{B}]{\tilde{A}} & \mathcal{C} \\
 1_\star \circlearrowleft \star & & A \circlearrowleft 1_A \\
 & & \downarrow f \\
 & & B \circlearrowleft 1_B
 \end{array} \tag{1.42}$$

Here,  $\tilde{A}1_\star = 1_{\tilde{A}\star} = 1_A$ ,  $\tilde{B}1_\star = 1_B$ , and  $\tilde{f}_\star = f$ . If no confusion is expected, we omit the  $\sim$  symbol.

**Theorem 1.3.2** (Functor Category). *Let  $\mathcal{C}, \mathcal{D}$  be categories,  $\mathcal{D}^{\mathcal{C}}$  be the class of functors. Then  $\mathcal{D}^{\mathcal{C}}$  and natural transformations among them form a category if  $\mathcal{C}$  is small.*

*Proof.* We will show that when  $\mathcal{C}$  is small,  $\mathcal{D}^{\mathcal{C}}$  is locally small, namely for each pair  $F, G \in \mathcal{D}^{\mathcal{C}}$ ,  $\mathcal{D}^{\mathcal{C}}(F, G)$  forms a set.

Let  $F, G \in \mathcal{D}^{\mathcal{C}}$  be functors. Consider the class of natural transformations  $\mathcal{D}^{\mathcal{C}}(F, G)$ . Let  $\theta \in \mathcal{D}^{\mathcal{C}}(F, G)$ . Recall the very definition,  $\theta$  is indeed a set of  $\mathcal{C}$ -indexed set of maps in  $\mathcal{D}$ ,  $\{\theta_C \in \mathcal{D}(FC, GC) \mid C \in |\mathcal{C}|\}$ , such that (1.41) is commutative for each  $c \in \mathcal{C}(C_1, C_2)$ .

Next, consider a correspondence  $C \mapsto \delta_C \in \mathcal{D}(FC, GC)$ . This defines a class-valued map  $\delta: |\mathcal{C}| \rightarrow 2^{\mathcal{D}}$ , where  $2^{\mathcal{D}}$  is the power class of arrows in  $\mathcal{D}$ . Since  $|\mathcal{C}|$  is a set, the image  $\delta|\mathcal{C}|$  is a set. Moreover, the union of the image  $\cup \delta|\mathcal{C}| := \bigcup_{C \in |\mathcal{C}|} \delta_C$  is a set, containing  $\theta$ :

$$\mathcal{D}^{\mathcal{C}}(F, G) \subset \cup \delta|\mathcal{C}|. \tag{1.43}$$

Hence,  $\mathcal{D}^{\mathcal{C}}(F, G)$  is a set. ■

*Remark 6.* Recalling Remark 5, since we may identify  $A, B \in |\mathcal{C}|$  as  $A: \mathbf{1} \rightarrow \mathcal{C}$  and  $B: \mathbf{1} \rightarrow \mathcal{C}$ , we have  $f \in \mathcal{C}^{\mathbf{1}}(A, B)$ .

**Definition 1.3.5** (Vertical Composition and Horizontal Composition). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. For  $\theta \in \mathcal{D}^{\mathcal{C}}(F, G)$  and  $\tau \in \mathcal{D}^{\mathcal{C}}(G, H)$ , their vertical composition  $\tau \circ \theta \in \mathcal{D}^{\mathcal{C}}(F, H)$  is given by

$$\{\tau_C \circ \theta_C \in \mathcal{D}(FC, HC) \mid C \in |\mathcal{C}|\} \tag{1.44}$$

since

$$(\tau \circ \theta)_{C_2} \circ Fc = \tau_{C_2} \circ \theta_{C_2} \circ Fc = \tau_{C_2} \circ Gc \circ \theta_{C_1} = Hc \circ \tau_{C_1} \circ \theta_{C_1}. \tag{1.45}$$

For natural transformations  $\theta: F \Rightarrow G$  and  $\sigma: H \Rightarrow K$ :

$$\begin{array}{ccc} \mathcal{C} & \xrightleftharpoons[F]{G} & \mathcal{D} \xrightleftharpoons[H]{K} \mathcal{E} \end{array} \quad (1.46)$$

we define their horizontal composition  $\theta * \sigma$  via the following lemma:

**Lemma 1.3.1** (Godement Product). *Consider:*

- $H\theta: HF \Rightarrow HG$ ,  $\sigma G: HG \Rightarrow KG$ , and

$$\sigma G \circ H\theta: HF \Rightarrow KG. \quad (1.47)$$

- $\sigma F: HF \Rightarrow KF$ ,  $K\theta: KF \Rightarrow KG$ , and

$$K\theta \circ \sigma F: HF \Rightarrow KG. \quad (1.48)$$

Then,  $\sigma G \circ H\theta = K\theta \circ \sigma F$ . We define  $\theta * \sigma$  by the corresponding commutative diagram:

$$\begin{array}{ccc} HF & \xrightarrow{H\theta} & HG \\ \sigma F \downarrow & & \downarrow \sigma G \\ KF & \xrightarrow{K\theta} & KG \end{array} \quad K\theta \circ \sigma F = \sigma G \circ H\theta. \quad (1.49)$$

*Proof.* We will first show that  $H\theta$  is a natural transformation. Let  $f \in \mathbb{C}(A, B)$ . Consider:

$$\begin{array}{ccc} HFA & \xrightarrow{HFf} & HFB \\ H\theta_A \downarrow & & \downarrow H\theta_B \\ HGA & \xrightarrow{HGf} & HGB \end{array} \quad (1.50)$$

Since  $\theta: F \Rightarrow G$  is a natural transformation and  $H: \mathcal{C} \rightarrow \mathcal{D}$  is a functor,

$$H\theta_B \circ HFf = H(\theta_B \circ Ff) = H(Gf \circ \theta_A) = HGf \circ H\theta_A, \quad (1.51)$$

i.e., the above diagram is commutative. Hence  $H\theta: HF \Rightarrow HG$  is a natural transformation. Similarly,  $\sigma G$ ,  $\sigma F$ , and  $K\theta$  are also natural transformations, and both  $\sigma G \circ H\theta$  and  $K\theta \circ \sigma F$  are natural transformations from  $HF$  to  $KG$ .

Let  $C \in |\mathcal{C}|$ . For  $\theta_C \in \mathcal{D}(FC, GC)$ , since  $\sigma: H \Rightarrow K$  is a natural transformation,  $C$ -components of these natural transformations satisfy:

$$\begin{array}{ccc} HFC & \xrightarrow{H\theta_C} & KGC \\ \sigma_{FC} \downarrow & & \downarrow \sigma_{GC} \\ KFC & \xrightarrow{K\theta_C} & KGC \end{array} \quad K\theta_C \circ \sigma_{FC} = \sigma_{GC} \circ H\theta_C \quad (1.52)$$

Hence,  $\{K\theta_C \circ \sigma_{FC} \mid C \in |\mathcal{C}|\}$  and  $\{\sigma_{GC} \circ H\theta_C \mid C \in |\mathcal{C}|\}$  define the same natural transformation.  $\blacksquare$

*Remark 7.* The commutative diagram in (1.41) defines  $c * \theta$  for  $c \in \mathcal{C}(C_1, C_2)$  and  $\theta: F \Rightarrow G$ , see Remark 5, where  $c \in \mathcal{C}^1(C_1, C_2)$  with  $\theta \in \mathcal{D}^c(F, G)$ .

### 1.3.2 String Diagrams

Following [Cur08], we introduce string diagrams as pictorial representations of arrows in categorical calculations.

**Definition 1.3.6** (String Diagrams). We represent a natural transformation:

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \quad \mu: GF \Rightarrow H \quad (1.53)$$

as follows:

$$\begin{array}{ccc} & \mathcal{D} & \\ & \swarrow F \quad \searrow G & \\ \mathcal{C} & \mu & \mathcal{E} \\ & \downarrow H & \end{array} \quad (1.54)$$

- Poincaré dual

In this representation, categories are 2-dimensional areas separated by lines of functors, which are 1-dimensional; natural transformations are 0-dimensional. This correspondence is Poincaré dual to the ordinary diagrams.

- Elevator Rule – Godement’s Product

Godement’ product  $\theta * \sigma$  in Lemma 1.3.1 is expressed as:

$$\begin{array}{c} F \\ \downarrow \\ \theta \\ \downarrow \\ G \end{array} \quad \begin{array}{c} \downarrow \\ H \\ \downarrow \\ \sigma \\ \downarrow \\ K \end{array} = \begin{array}{c} \downarrow \\ F \\ \downarrow \\ \theta \\ \downarrow \\ G \end{array} \quad \begin{array}{c} \downarrow \\ H \\ \downarrow \\ \sigma \\ \downarrow \\ K \end{array} \quad (1.55)$$

This is a key axiom of this notation. The natural transformations can freely move up and down as long as they keep the ambient algebraic structures, particularly the domains and codomains of functors.

*Remark 8* (Composition Rules and Identities). Functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{E}$  can be composed  $GF: \mathcal{C} \rightarrow \mathcal{E}$ :

$$\begin{array}{c} F \\ \downarrow \end{array} \quad \begin{array}{c} G \\ \downarrow \end{array} = \begin{array}{c} GF \\ \downarrow \end{array} \quad (1.56)$$

Relative to this composition rule, the identities are expressed as follows:

- $\mu: 1_C \Rightarrow F$ :

$$\begin{array}{c} \mu \\ \left| \right. \\ F \end{array} = \begin{array}{c} \overset{\dots\dots\dots}{1_C} \\ \mu \\ \left| \right. \\ F \end{array} \quad (1.57)$$

- $1_G: G \Rightarrow G$ :

$$\begin{array}{c} \left| \right. \\ G \\ 1_G \\ \left| \right. \\ G \end{array} = \begin{array}{c} \left| \right. \\ G \end{array} \quad (1.58)$$

Among functors and natural transformations, we have

- $H\theta: HF \Rightarrow HG$ , and  $\sigma G: HG \Rightarrow KG$ :

$$\begin{array}{ccc} \begin{array}{c} \left| \right. \\ HF \\ H\theta \\ \left| \right. \\ HG \end{array} & = & \begin{array}{c} F \\ \theta \\ \left| \right. \\ G \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} \left| \right. \\ H \\ = \\ \left| \right. \\ H \end{array} & = & \begin{array}{c} \left| \right. \\ HF \\ 1_H * \theta \\ \left| \right. \\ HG \end{array} \end{array} \quad (1.59)$$

$$\begin{array}{ccc} \begin{array}{c} \left| \right. \\ HG \\ \sigma G \\ \left| \right. \\ KG \end{array} & = & \begin{array}{c} \left| \right. \\ G \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} \left| \right. \\ H \\ \sigma \\ \left| \right. \\ K \end{array} & = & \begin{array}{c} \left| \right. \\ HG \\ \sigma * 1_G \\ \left| \right. \\ KG \end{array} \end{array}$$

With Remark 5, we obtain:

$$\begin{array}{ccc} \begin{array}{c} \left| \right. \\ FA \\ Ff \\ \left| \right. \\ FB \end{array} & = & \begin{array}{c} A \\ f \\ \left| \right. \\ B \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} \left| \right. \\ FA \\ F \\ \left| \right. \\ FB \end{array} & = & \begin{array}{c} \left| \right. \\ FA \\ 1_F * f \\ \left| \right. \\ FB \end{array} \end{array} \quad (1.60)$$

### 1.3.3 Adjunctions and Kan Extensions

**Definition 1.3.7** (Adjunctions). An adjunction – a pair of adjoint functors – is a pair of functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{C}$  with natural transformations  $\eta: 1_{\mathcal{C}} \Rightarrow GF$  and  $\epsilon: FG \Rightarrow 1_{\mathcal{D}}$  that satisfy the following zig-zag identities:

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow & \downarrow \epsilon_F \\ & & F \end{array} \quad 1_F = \epsilon_F \circ F\eta, \quad \begin{array}{ccc} G & \xrightarrow{G\eta} & GFG \\ & \searrow & \downarrow G\epsilon \\ & & G \end{array} \quad 1_G = G\epsilon \circ \eta G. \quad (1.61)$$

We denote  $F \dashv G$ , and call  $F$  the right adjoint and  $G$  the left adjoint. The associated natural transformations  $\eta$  and  $\epsilon$  are called unit and counit, respectively.

*Remark 9* (Zig-Zag in String Diagrams).

$$\begin{array}{c} \mathcal{C} \\ \downarrow F \\ \mathcal{D} \end{array} = \begin{array}{c} \eta \\ \swarrow \quad \searrow \\ F \quad \quad G \\ \downarrow \quad \quad \downarrow \\ F \quad \quad \epsilon \end{array}, \quad \begin{array}{c} \mathcal{D} \\ \downarrow G \\ \mathcal{C} \end{array} = \begin{array}{c} G \quad \quad \eta \\ \swarrow \quad \searrow \\ \epsilon \quad \quad F \\ \downarrow \quad \quad \downarrow \\ \epsilon \quad \quad G \end{array} \quad (1.62)$$

As a useful characterization of adjunctions, we have the following:

**Theorem 1.3.3** (Natural Bijection). A pair of functors  $F \begin{array}{c} \mathcal{C} \\ \uparrow \downarrow \\ \mathcal{D} \end{array} G$  forms an

adjunction  $F \begin{array}{c} \mathcal{C} \\ \uparrow \downarrow \\ \mathcal{D} \end{array} G$  with unit  $\eta: 1_{\mathcal{C}} \Rightarrow GF$  and counit  $\epsilon: FG \Rightarrow 1_{\mathcal{D}}$  iff there is

a bijection  $\zeta_{C,D}: \mathcal{D}(FC, D) \rightarrow \mathcal{C}(C, GD)$  for each  $C \in |\mathcal{C}|$  and  $D \in |\mathcal{D}|$  such that  $\zeta_{C,D}$  is natural in  $C$  and  $D$ , where the naturality is expressed as:

- For  $FC \xrightarrow{g} D \xrightarrow{d} D'$  in  $\mathcal{D}$ ,

$$C \xrightarrow{\zeta g} GD \xrightarrow{Gd} GD' \quad \zeta_{C,D'}(d \circ g) = Gd \circ (\zeta_{C,D}g). \quad (1.63)$$

- For  $C \xrightarrow{c} C' \xrightarrow{f'} GD'$  in  $\mathcal{C}$ ,

$$FC \xrightarrow{Fc} FC' \xrightarrow{\zeta^{-1}f'} D' \quad \zeta_{C,D'}^{-1}(f' \circ c) = (\zeta_{C',D'}^{-1}f') \circ Fc. \quad (1.64)$$

*Proof.* ( $\Rightarrow$ ) Suppose  $F \dashv G$  with unit  $\eta$  and counit  $\epsilon$ . Let  $g \in \mathcal{D}(FC, D)$  and  $f \in \mathcal{C}(C, GD)$ . Define  $\zeta_{C,D}g := Gg \circ \eta_C$  and  $\zeta'_{C,D}f := \epsilon_D \circ Ff$ . They form an inverse pair:

$$\begin{aligned}\zeta_{C,D}(\zeta'_{C,D}f) &= G(\epsilon_D \circ Ff) \circ \eta_C = G\epsilon_D \circ \eta_{GD} \circ f = f \\ \zeta'_{C,D}(\zeta_{C,D}g) &= \epsilon_D \circ F(Gg \circ \eta_C) = g \circ \epsilon_{FC} \circ F\eta_C = g,\end{aligned}\tag{1.65}$$

where  $G\epsilon_D \circ \eta_{GD} = (G\epsilon \circ \eta G)D = 1_{GD}$  and  $\epsilon_{FC} \circ F\eta_C = (\epsilon F \circ F\eta)_C = 1_{FC}$ . Hence,  $\zeta' = \zeta^{-1}$ . The naturality follows as both  $\eta$  and  $\epsilon$  are natural transformations.

( $\Leftarrow$ ) Conversely, for a given natural bijection  $\zeta$ , define  $\eta_C := \zeta_{C,FC}1_{FC}$  and  $\epsilon_D := \zeta_{GD,D}^{-1}1_{GD}$  for each  $C \in |\mathcal{C}|$  and  $D \in |\mathcal{D}|$ . Let  $C \in |\mathcal{C}|$  and  $D \in |\mathcal{D}|$ :

$$\begin{aligned}(G\epsilon \circ \eta G)_D &= (G\epsilon_D \circ \zeta_{GD,FGD})1_{FGD} \\ &= \zeta_{GD,D}(\epsilon_D \circ 1_{FGD}) \\ &= (\zeta_{GD,D} \circ \zeta_{GD,D}^{-1})1_{GD} \\ &= 1_{GD} \\ (\epsilon F \circ F\eta)_C &= (\zeta_{GFC,FC}^{-1}1_{GFC}) \circ F\eta_C \\ &= \zeta_{C,FC}^{-1}(1_{GFC} \circ \eta_C) \\ &= (\zeta_{C,FC}^{-1} \circ \zeta_{C,FC})1_{FC} \\ &= 1_{FC}.\end{aligned}\tag{1.66}$$

Hence, we conclude  $G\epsilon \circ \eta G = 1_G$  and  $\epsilon F \circ F\eta = 1_F$ . ■

*Remark 10.* The natural bijections are represented as the following:

$$\begin{array}{ccc} \begin{array}{c} C \\ | \\ g \\ | \\ D \end{array} \begin{array}{c} \nearrow F \\ \searrow \end{array} & \xrightarrow{\zeta} & \begin{array}{c} C \\ | \\ g \\ | \\ D \end{array} \begin{array}{c} \nearrow \eta \\ \searrow F \\ | \\ G \end{array}, \quad \begin{array}{c} C \\ | \\ f \\ | \\ D \end{array} \begin{array}{c} \nearrow \\ \searrow G \end{array} & \xrightarrow{\zeta^{-1}} & \begin{array}{c} C \\ | \\ f \\ | \\ D \end{array} \begin{array}{c} \nearrow \\ \searrow G \\ | \\ \epsilon \end{array} \begin{array}{c} \nearrow F \\ \searrow \end{array} \end{array}\tag{1.67}$$

**Definition 1.3.8** (Kan Extensions). Let  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  be categories, and  $F: \mathcal{C} \rightarrow \mathcal{E}$  and  $K: \mathcal{C} \rightarrow \mathcal{D}$  be functors.

- A left Kan extension of  $F$  along  $K$  is a pair  $(L, \eta)$  of a functor  $L: \mathcal{D} \rightarrow \mathcal{E}$ , and a natural transformation:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ K \downarrow & \nearrow L & \\ \mathcal{D} & & \end{array} \quad \eta: F \Rightarrow LK \tag{1.68}$$

such that for any other pair  $(G: \mathcal{D} \rightarrow \mathcal{E}, \gamma: F \Rightarrow GK)$ , there exists a unique mediator  $\mu: L \Rightarrow G$  with  $\gamma = \mu K \circ \eta$ , where  $LK \xRightarrow{\mu K} GK$  is  $LK \xRightarrow{\mu * 1_K} GK$ , see Lemma 1.3.1:

$$\begin{array}{c} F \\ | \\ \gamma \\ | \\ K \end{array} \begin{array}{c} \diagdown \\ G \end{array} = \begin{array}{c} F \\ | \\ \eta \\ | \\ K \end{array} \begin{array}{c} \diagdown L \\ \mu \\ | \\ G \end{array} \quad (1.69)$$

- A right Kan extension of  $F$  along  $K$  is a pair  $(R, \epsilon)$  of a functor  $R: \mathcal{D} \rightarrow \mathcal{E}$ , and a natural transformation:

$$\begin{array}{ccc} & \mathcal{D} & \\ K \uparrow & \searrow R & \\ \mathcal{C} & \xrightarrow{F} & \mathcal{E} \end{array} \quad \epsilon: KR \Rightarrow F \quad (1.70)$$

such that for any other pair  $(G: \mathcal{D} \rightarrow \mathcal{E}, \delta: GK \Rightarrow F)$ , there exists a unique mediator  $\nu: G \Rightarrow R$  with  $\delta = \epsilon \circ \nu K$ , where  $GK \xRightarrow{\nu K} RK$  is  $GK \xRightarrow{\nu * 1_K} RK$ , see Lemma 1.3.1:

$$\begin{array}{c} K \\ | \\ \delta \\ | \\ F \end{array} \begin{array}{c} \diagup \\ G \end{array} = \begin{array}{c} K \\ | \\ \epsilon \\ | \\ F \end{array} \begin{array}{c} \diagup R \\ \nu \\ | \\ G \end{array} \quad (1.71)$$

*Remark 11* (Limits as Kan Extensions). Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Suppose a right Kan extension of  $F$  along the unique functor  $\mathcal{C} \rightarrow \mathbf{1}$ , where  $\mathbf{1}$  is the terminal category, see Remark 5.

We will show such a right Kan extension is a limit cone. Let  $(R, \epsilon)$  be a right Kan extension of  $F$  along  $\mathcal{C} \xrightarrow{!} \mathbf{1}$ :

- $(R, \epsilon)$  is a cone.

As  $R: \mathbf{1} \rightarrow \mathcal{D}$  is essentially an object in  $\mathcal{D}$ , the composition  $\mathcal{C} \xrightarrow{!} \mathbf{1} \xrightarrow{R} \mathcal{D}$  is a constant functor on  $R \in |\mathcal{D}|$ . Since  $\epsilon: R! \Rightarrow F$  is a natural transfor-

mation, for each  $c \in \mathcal{C}(C_1, C_2)$  the following diagram is commutative:

$$\begin{array}{ccc}
 R & & \\
 \epsilon_{C_1} \downarrow & \searrow \epsilon_{C_2} & \\
 FC_1 & \xrightarrow{Fc} & FC_2
 \end{array}
 \quad Fc \circ \epsilon_{C_1} = \epsilon_{C_2}. \quad (1.72)$$

I.e.,  $(R, \epsilon)$  forms a cone in  $\mathcal{D}$ .

- $(R, \epsilon)$  is a limit cone.

Due to the universal property of  $(R, \epsilon)$  being a right Kan extension, for any cone  $(D, \theta)$  such that

$$\begin{array}{ccc}
 D & & \\
 \theta_{C_1} \downarrow & \searrow \theta_{C_2} & \\
 FC_1 & \xrightarrow{Fc} & FC_2
 \end{array}
 \quad Fc \circ \theta_{C_1} = \theta_{C_2}, \quad (1.73)$$

as  $\theta: D! \Rightarrow F$  is a natural transformation, there exists a unique mediator  $\mu: D \Rightarrow R$  with  $\theta = \epsilon \circ \nu!$ , i.e., for each  $C \in |\mathcal{C}|$ ,  $\theta_C = \epsilon_C \circ \mu$  holds.

Conversely, if  $F: \mathcal{C} \rightarrow \mathcal{D}$  has a limit  $(R, \epsilon)$ , it defines a right Kan extension of  $F$  along  $!: \mathcal{C} \rightarrow \mathbf{1}$ .

*Remark 12* (Adjoint as Kan Extension). Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{C}$  be functors. Suppose  $F \dashv G$  with unit  $\eta: 1_{\mathcal{C}} \Rightarrow GF$  and counit  $\epsilon: FG \Rightarrow 1_{\mathcal{D}}$ . Then

- $(G, \eta)$  is a left Kan extension of  $1_{\mathcal{C}}$  along  $F$ .

Consider  $(H: \mathcal{D} \rightarrow \mathcal{E}, \gamma: 1_{\mathcal{D}} \Rightarrow HF)$ .  $\gamma$  becomes

$$1_{HF} \gamma = H(\epsilon F \circ F \eta) \circ \gamma = H \epsilon F \circ H F \eta \circ \gamma = H \epsilon F \circ \gamma G F \circ \eta = (H \epsilon \circ \gamma G) F \circ \eta. \quad (1.74)$$

That is,  $H \epsilon \circ \gamma G: G \Rightarrow H$  is the desired mediator:

$$\begin{array}{c}
 \gamma \\
 \downarrow \\
 F \mid \searrow H
 \end{array}
 =
 \begin{array}{c}
 \eta \\
 \downarrow \\
 F \mid \begin{array}{c} \searrow G \\ \downarrow F \end{array} \begin{array}{c} \gamma \\ \downarrow \epsilon \end{array} \searrow H
 \end{array}
 =
 \begin{array}{c}
 \eta \\
 \downarrow \\
 F \mid \begin{array}{c} \searrow G \\ \downarrow F \end{array} \begin{array}{c} \gamma \\ \downarrow \epsilon \end{array} \searrow H
 \end{array} \quad (1.75)$$

- $(F, \epsilon)$  is a right Kan extension of  $1_{\mathcal{D}}$  along  $G$ .



Consider  $(H: \mathcal{D} \rightarrow \mathcal{E}, \delta: HG \Rightarrow 1_{\mathcal{C}})$ .  $\delta$  becomes

$$\delta 1_{HG} = \delta \circ H (G\epsilon \circ \eta G) = \delta \circ HG\epsilon \circ H\eta G = \epsilon \circ \delta FG \circ H\eta G = \epsilon \circ (\delta F \circ H\eta) G. \quad (1.76)$$

That is,  $\delta F \circ H\eta: H \Rightarrow G$  is the desired mediator:

$$\begin{array}{c} G \\ \downarrow \delta \\ \end{array} \begin{array}{c} \nearrow H \\ \end{array} = \begin{array}{c} G \\ \downarrow \epsilon \\ \end{array} \begin{array}{c} \nearrow F \\ \end{array} \begin{array}{c} \eta \\ \downarrow G \\ \end{array} \begin{array}{c} \nearrow H \\ \end{array} = \begin{array}{c} G \\ \downarrow \epsilon \\ \end{array} \begin{array}{c} \nearrow F \\ \end{array} \begin{array}{c} \eta \\ \downarrow G \\ \end{array} \begin{array}{c} \nearrow H \\ \end{array} \begin{array}{c} \delta \\ \downarrow \end{array} \begin{array}{c} \nearrow H \\ \end{array} \quad (1.77)$$

Conversely, if the following two conditions hold:

- $(G: \mathcal{D} \rightarrow \mathcal{C}, \eta: 1_{\mathcal{C}} \Rightarrow GF)$  is a left Kan extension of  $1_{\mathcal{C}}$  along  $F: \mathcal{C} \rightarrow \mathcal{D}$ .
- $F$  preserves this Kan extension.

Then  $F \dashv G$  with unit  $\eta$ .

We first find the counit. Since  $(FG, F\eta)$  is a left Kan extension of  $F$  along  $F$ , there exists a unique mediator  $\epsilon: FG \Rightarrow 1_{\mathcal{D}}$  such that  $1_F = \epsilon F \circ F\eta$ :

$$\begin{array}{c} F \\ \downarrow \end{array} = \begin{array}{c} F \\ \downarrow 1_F \\ \end{array} \begin{array}{c} \nearrow F \\ \end{array} \begin{array}{c} \nearrow 1_{\mathcal{D}} \\ \end{array} = \begin{array}{c} F \\ \downarrow \end{array} \begin{array}{c} \nearrow F\eta \\ \end{array} \begin{array}{c} \nearrow FG \\ \end{array} \begin{array}{c} \nearrow \epsilon \\ \end{array} \begin{array}{c} \nearrow 1_{\mathcal{D}} \\ \end{array} = \begin{array}{c} \eta \\ \downarrow G \\ \end{array} \begin{array}{c} \nearrow G \\ \end{array} \begin{array}{c} \nearrow \epsilon \\ \end{array} \begin{array}{c} \nearrow F \\ \end{array} \quad (1.78)$$

Hence, it suffices to show the other zig-zag identity. Now

$$\begin{array}{c} 1_{\mathcal{C}} \\ \downarrow \eta \\ \end{array} \begin{array}{c} \nearrow G \\ \end{array} = \begin{array}{c} \eta \\ \downarrow G \\ \end{array} \begin{array}{c} \nearrow F \\ \end{array} \begin{array}{c} \nearrow \epsilon \\ \end{array} \begin{array}{c} \nearrow G \\ \end{array} \begin{array}{c} 1_{\mathcal{C}} \\ \downarrow \eta \\ \end{array} \begin{array}{c} \nearrow G \\ \end{array} \quad (1.79)$$

implies:

$$\eta = 1_{GF}\eta = G\epsilon F \circ GF\eta \circ \eta = G\epsilon F \circ \eta GF \circ \eta = (G\epsilon \circ \eta G)F \circ \eta \quad (1.80)$$

Since  $(G, \eta)$  is a left Kan extension of  $1_{\mathcal{C}}$  along  $F$ , the unique mediator  $1_G$  must be  $G\epsilon \circ \eta G$ .

**Theorem 1.3.4.** *Left adjoints preserve left Kan extensions.*

*Proof.* Consider an adjunction  $F \left( \begin{smallmatrix} \mathcal{C} \\ \dashv \\ \mathcal{D} \end{smallmatrix} \right) G$  with unit  $\eta: 1_{\mathcal{C}} \Rightarrow GF$  and counit  $\epsilon: FG \Rightarrow 1_{\mathcal{D}}$ , and a left Kan extension  $(L_K E: \mathcal{B} \rightarrow \mathcal{C}, \mu: E \Rightarrow L_K E \circ K)$  of  $E: \mathcal{A} \rightarrow \mathcal{C}$  along  $K: \mathcal{A} \rightarrow \mathcal{B}$ :

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{E} & \mathcal{C} & \xrightleftharpoons[F]{F} & \mathcal{D} \\ K \downarrow & \nearrow L_K E & & & \\ \mathcal{B} & & & & \end{array} \quad \mu: E \Rightarrow L_K E \circ K \quad (1.81)$$

We will show  $(F \circ L_K E, F\mu)$  is a left Kan extension of  $FE$  along  $K$ , in other words,  $L_K(FE) = FL_K E$ .

For simplicity, let  $L := L_K E$ . Consider  $H: \mathcal{B} \rightarrow \mathcal{D}$  and  $\gamma: FE \Rightarrow HK$ . Applying  $\eta$ , there exists a unique mediator  $\nu: L \Rightarrow GH$  such that

$$\begin{array}{ccc} E & & E \\ \downarrow & \nearrow \eta & \downarrow \mu \\ \gamma & & L \\ \downarrow K & \searrow H & \downarrow K \\ & & \nu \\ & & \downarrow H \\ & & G \end{array} = \begin{array}{ccc} E & & E \\ \downarrow & \nearrow \eta & \downarrow \mu \\ \gamma & & L \\ \downarrow K & \searrow H & \downarrow K \\ & & \nu \\ & & \downarrow H \\ & & G \end{array} \quad (1.82)$$

since  $(L_K E, \mu)$  is a left Kan extension of  $E$  along  $K$ . By a zig-zag identity,

$$\begin{array}{ccc} E & & E \\ \downarrow & \nearrow F & \downarrow \mu \\ \gamma & & L \\ \downarrow K & \searrow H & \downarrow K \\ & & \nu \\ & & \downarrow H \\ & & G \end{array} = \begin{array}{ccc} E & & E \\ \downarrow & \nearrow F & \downarrow \mu \\ \gamma & & L \\ \downarrow K & \searrow H & \downarrow K \\ & & \nu \\ & & \downarrow H \\ & & G \end{array} = \begin{array}{ccc} E & & E \\ \downarrow & \nearrow F & \downarrow \mu \\ \gamma & & L \\ \downarrow K & \searrow H & \downarrow K \\ & & \nu \\ & & \downarrow H \\ & & G \end{array} \quad (1.83)$$

we obtain  $\gamma = (\epsilon H \circ F\nu) K \circ F\mu$ . This  $\epsilon H \circ F\nu: FL \rightarrow H$  is the desired mediator for  $(FL, F\mu)$  being a left Kan extension of  $FE$  along  $K$ . ■

## Chapter 2

# Adjunctions in Topology

### 2.1 Spaces, Presets, and Posets

**Definition 2.1.1** (Concrete Categories). Here are some categories of structured sets with structure-preserving maps:

- **Set** of sets with maps
- **Pre** of pre-ordered sets with monotone maps, and **Pos** of posets with monotone maps
- **Top** of topological spaces with continuous maps

#### 2.1.1 Presets and Alexandroff Topology

**Definition 2.1.2** (Upper Section). Let  $(A, \leq)$  be a preset. An upper section of  $A$  is a subset  $U \subset A$  such that for all  $a, b \in A$ :

$$a \in U \wedge a \leq b \Rightarrow b \in U. \quad (2.1)$$

Let  $\Gamma_A$  denote the set of all upper sections of  $(A, \leq)$ .

**Theorem 2.1.1** (Alexandroff Topology). *Let  $(A, \leq)$  be a preset. The set of upper sections  $\Gamma_A$  is a topology on  $A$ . We call  $\Gamma_A$  the Alexandroff topology of  $(A, \leq)$ .*

*Proof.*  $A \in \Gamma_A$  holds.  $\emptyset \in \Gamma_A$  is, vacuously, true.

Let  $U, V \in \Gamma_A$ . If they do not meet  $U \cap V = \emptyset$ , as shown above,  $\emptyset \in \Gamma_A$ . Suppose  $a \in U \cap V$ . For  $b \in A$ , if  $a \leq b$ , then  $b \in U$  and  $b \in V$  since both  $U$  and  $V$  are upper sections. Hence,  $b \in U \cap V$ , and  $U \cap V \in \Gamma_A$ .

Let  $\Gamma' \subset \Gamma_A$  and  $a \in \bigcup \Gamma'$ . Then, there exists at least one  $W \in \Gamma'$  with  $a \in W$ . For  $b \in A$ , if  $a \leq b$ , then  $b \in W \subset \bigcup \Gamma'$ . Hence,  $\bigcup \Gamma' \in \Gamma_A$ . ■

**Theorem 2.1.2** (Upgrading). *For a preset  $(A, \leq)$ , let  $\uparrow(A, \leq) := (A, \Gamma_A)$ . This object assignment induces the corresponding arrow assignment. Hence,  $\uparrow: \mathbf{Pre} \rightarrow \mathbf{Top}$  is a functor.*

*Proof.* Let  $f \in \mathbf{Pre}(A, B)$  be a monotone map, namely  $a_1 \leq a_2 \Rightarrow fa_1 \leq fa_2$ . Relative to the Alexandroff topologies of  $(A, \leq)$  and  $(B, \leq)$ , we will show  $\uparrow f := f$  is continuous. Let  $W \in \Gamma_B$ , and  $a_1, a_2 \in A$ . Suppose  $a_1 \leq a_2$  and  $a_1 \in (\uparrow f)^{\leftarrow} W = f^{\leftarrow} W$ , i.e.,  $fa_1 \in W$ . Since  $f$  is order-preserving,  $fa_1 \leq fa_2$ , and  $W \in \Gamma_B$  is an upper section of  $B$ , we conclude  $fa_2 \in W$ . Hence,  $a_2 \in (\uparrow f)^{\leftarrow} W$ . We conclude  $(\uparrow f)^{\leftarrow} W$  is an upper section of  $A$ , i.e.,  $(\uparrow f)^{\leftarrow} W \in \Gamma_A$ . Hence,  $(\uparrow f)^{\leftarrow}: \Gamma_B \rightarrow \Gamma_A$ . ■

### 2.1.2 Specialization Preorder and Separation Axioms

You should remember that a topological space need not be hausdorff. The separation properties  $T_0$  and  $T_1$  play a minor role here. [Sim11]

**Definition 2.1.3** (Specialization Preorder). For a topological space  $(S, \mathcal{T}_S)$ , the specialization order  $\leq$  of  $(S, \mathcal{T}_S)$  is the following comparison on  $S$ :

$$r \leq s :\Leftrightarrow \forall U \in \mathcal{T}_S : r \in U \Rightarrow s \in U. \quad (2.2)$$

**Lemma 2.1.1.** *Specialization orders are preorders.*

*Proof.* Let  $(S, \mathcal{T}_S)$  be a topological space and  $\leq$  is the specialization order of  $(S, \mathcal{T}_S)$ . It suffices to show that  $\leq$  is transitive.

Suppose  $r \leq s$  and  $s \leq t$ . Let  $U \in \mathcal{T}_S$  such that  $r \in U$ . Since  $r \leq s$ ,  $s \in U$ , which implies  $t \in U$ . Hence,  $r \leq t$ . ■

**Theorem 2.1.3.** *Let  $(S, \mathcal{T}_S)$  be a topological space and  $\leq$  is the specialization order of  $(S, \mathcal{T}_S)$ . For  $r, s \in S$ ,  $r \leq s$  iff  $r \in \overline{\{s\}}$ , that is,  $r$  is a member of the closure of the singleton subspace  $\{s\} \subset S$  relative to  $\mathcal{T}_S$ .*

*Proof.* Since  $r \leq s$  is equivalent to:

$$\forall U \in \mathcal{T}_S : s \in \neg U \Rightarrow r \in \neg U \quad (2.3)$$

In other words, any closed subspace in  $S$  that contains  $s$  also contains  $r$ . Hence,  $r$  must be in the  $\subseteq$ -smallest closed subspace that contains  $s$ . By Theorem 1.2.2, we conclude  $r \in \overline{\{s\}}$ . ■

**Theorem 2.1.4.** *Let  $(S, \mathcal{T}_S)$  be a topological space and  $\leq$  is the specialization order of  $(S, \mathcal{T}_S)$ .*

- $(S, \mathcal{T}_S)$  is a  $T_0$  space iff  $\leq$  is a partial order.
- $(S, \mathcal{T}_S)$  is a  $T_1$  space iff  $\leq$  is equality.

*Proof.* Let  $(S, \mathcal{T}_S)$  be a  $T_0$  space,  $\leq$  be the specialization preorder of  $(S, \mathcal{T}_S)$ , and  $s, t \in S$ . Suppose  $s \leq t$  and  $t \leq s$ , but  $s \neq t$  for contradiction. Since  $(S, \mathcal{T}_S)$  is  $T_0$ , there exists an open  $O \in \mathcal{T}_S$  that contains only one of  $\{s, t\}$ ; without loss of generality,  $s \in O$  and  $t \notin O$ .  $t \in \neg O$  implies  $s \in \neg O$  since  $s \leq t$ , which is absurd. Thus,  $s = t$  holds.

Conversely, suppose the specialization preorder  $\leq$  of a topological space  $(S, \mathcal{T}_S)$  is a partial order. Consider two distinct points  $s \neq t$  in  $S$ . As  $(S, \leq)$  is a poset,  $s \neq t$  implies either  $s \not\leq t$  or  $t \not\leq s$ . Without loss of generality, we may set  $s \not\leq t$ . By Theorem 2.1.3, we obtain  $s \in \neg\overline{\{t\}}$ . Since  $\neg\overline{\{t\}} \in \mathcal{T}_S$ , it is the desired open subspace, since  $\{t\} \subset \overline{\{t\}}$  implies  $t \in \overline{\{t\}}$ , i.e.,  $t \notin \neg\overline{\{t\}}$ .

Let  $(S, \mathcal{T}_S)$  be a  $T_1$ -space,  $\leq$  be the specialization preorder of  $(S, \mathcal{T}_S)$ , and  $s, t \in S$ . Suppose  $s \leq t$  but  $s \neq t$  for contradiction. Since  $(S, \mathcal{T}_S)$  is  $T_1$ , there are open  $U, V \in \mathcal{T}_S$  with  $s \in U$ ,  $t \in V$ , but  $s \notin V$  and  $t \notin U$ . Since  $s \leq t$  and  $s \in U$ , we obtain  $t \in U$ , which is absurd.

Conversely, suppose the specialization preorder  $\leq$  of a topological space  $(S, \mathcal{T}_S)$  is merely the equality  $=$ . Let  $s \neq t$  be two distinct points in  $S$ . That is,  $s \not\leq t$  and  $t \not\leq s$ :

$$s \in \neg\overline{\{t\}} \wedge t \in \neg\overline{\{s\}}. \quad (2.4)$$

Since both  $\neg\overline{\{t\}}$  and  $\neg\overline{\{s\}}$  are open, we obtain the desired open neighborhoods:

$$t \notin \neg\overline{\{t\}} \wedge s \notin \neg\overline{\{s\}}. \quad (2.5)$$

since  $s \in \overline{\{s\}}$  and  $t \in \overline{\{t\}}$ . ■

**Theorem 2.1.5** (Downgrading). *For a topological space  $(S, \mathcal{T}_S)$ , let  $\Downarrow (S, \mathcal{T}_S) := (S, \leq)$ , where  $\leq$  is the specialization order. This object assignment induces the corresponding arrow assignment. Hence,  $\Downarrow: \mathbf{Top} \rightarrow \mathbf{Pre}$  is a functor.*

*Proof.* Let  $f \in \mathbf{Top}(A, B)$  be a continuous map. We will show  $\Downarrow f := f$  is monotone. Let  $a_1, a_2 \in A$ . Assume  $a_1 \leq a_2$ . Suppose, for contradiction, that  $(\Downarrow f)a_1 \not\leq (\Downarrow f)a_2$ . By Theorem 2.1.3, this condition is equivalent to  $fa_1 \in \neg\overline{\{fa_2\}}$ , and

$$a_1 \in f^{\leftarrow} \left( \neg\overline{\{fa_2\}} \right). \quad (2.6)$$

Since  $\neg\overline{\{fa_2\}} \in \mathcal{T}_B$ , its preimage is also open  $f^{\leftarrow} \left( \neg\overline{\{fa_2\}} \right) \in \mathcal{T}_A$ . Recalling  $a_1 \leq a_2$ , we conclude  $a_2 \in f^{\leftarrow} \left( \neg\overline{\{fa_2\}} \right)$ , i.e.,  $fa_2 \in \neg\overline{\{fa_2\}}$ , which is absurd. Hence,  $\Downarrow f$  is monotone. ■

**Lemma 2.1.2.** *Let  $(A, \leq)$  be a preset and  $\Gamma_A$  be the Alexandroff topology on  $A$ . For  $a, b \in A$ , if  $a \leq b$  then  $a \in \overline{\{b\}}$ , where the closure  $\overline{\{b\}}$  is relative to  $(A, \Gamma_A) = \uparrow(A, \leq)$ .*

*Proof.* Let  $a, b \in A$ . Suppose  $a \leq b$ . Let  $U \in \Gamma_A$ . Since  $U$  is an upper section of  $A$ , if  $a \in U$  then  $b \in U$ . It is equivalent to:

$$b \in \neg U \Rightarrow a \in \neg U. \quad (2.7)$$

In other words, any closed subspace relative to  $\Gamma_A$  that contains  $b$  contains also  $a$ . Hence,  $a \in \overline{\{b\}}$ . ■

**Theorem 2.1.6.** *Let  $(A, \leq)$  be a preset,  $\Gamma_A$  be the Alexandroff topology on  $A$ , and  $\prec$  be the specialization preorder of the topological space  $(A, \Gamma_A)$ . We claim  $\leq = \prec$ . In other words,  $\Downarrow \Uparrow (A, \leq) = (A, \leq)$ .*

*Proof.* Recalling  $\leq \subset A \times A$ , let  $(a_1, a_2) \in \leq$ :

$$a_1 \leq a_2. \quad (2.8)$$

If  $U \in \Gamma_A$  contains  $a_1 \in U$ , since  $U$  is an upper section of  $A$ ,  $a_2 \in U$ :

$$a_1 \prec a_2. \quad (2.9)$$

Thus, as subsets of  $A \times A$ , we conclude  $\leq \subset \prec$ .

Suppose, for contradiction, that this inclusion is strict. Then, there exists at least one pair  $(s, t) \in A \times A$  such that  $s \prec t$  but  $s \not\leq t$ .

- Since  $s \prec t$ ,

$$\forall U \in \Gamma_A : s \in U \Rightarrow t \in U. \quad (2.10)$$

- Since  $s \not\leq t$ , by Lemma 2.1.2:

$$s \in \neg \overline{\{t\}}. \quad (2.11)$$

Now,  $\neg \overline{\{t\}} \in \Gamma_A$  and  $t \notin \neg \overline{\{t\}}$ , we have a contradiction. ■

**Corollary 2.1.6.1.** *The converse of Lemma 2.1.2 is also the case, namely for a preset  $(A, \leq)$ ,  $a \leq b$  iff  $a \in \overline{\{b\}}$ , where  $\overline{\{b\}}$  is relative to  $\Uparrow (A, \leq)$ .*

*Proof.* Suppose  $a \in \overline{\{b\}}$ . By Theorem 2.1.3, it is equivalent to  $a \prec b$ , where  $\prec$  is the specialization preorder of  $(A, \Gamma_A)$ . As shown above, in Theorem 2.1.6,  $a \prec b$  iff  $a \leq b$ . ■

**Theorem 2.1.7.** *Let  $(S, \mathcal{T}_S)$  be a topological space,  $(S, \leq) := \Downarrow (S, \mathcal{T}_S)$  be the preset with the specialization preorder, and  $(S, \Gamma_S) := \Uparrow \Downarrow (S, \mathcal{T}_S)$ . We claim  $\mathcal{T}_S \subset \Gamma_S$ .*

*Proof.* We will show that any member in  $\mathcal{T}_S$  is an upper section relative to the specialization preorder  $\leq$ .

Let  $U \in \mathcal{T}_S$ , and  $s, t \in S$ . Suppose  $s \in U$  and  $s \leq t$ . By the very definition of  $\leq$ , see Definition 2.1.3, we conclude  $t \in U$ . Hence  $U$  is an upper section,  $U \in \Gamma_S$ . ■

Now we have a pair of functors:

$$\begin{array}{c} \mathbf{Pre} \\ \uparrow \left( \begin{array}{c} \downarrow \\ \uparrow \end{array} \right) \downarrow \\ \mathbf{Top} \end{array} \quad (2.12)$$

To show that they form an adjunction, by Theorem 1.3.3, it suffices to show that  $\mathbf{Top}(\uparrow(A, \leq), (S, \mathcal{T}_S))$  and  $\mathbf{Pre}((A, \leq), \downarrow(S, \mathcal{T}_S))$  are naturally bijective for any preset  $(A, \leq)$  and any topological space  $(S, \mathcal{T}_S)$ :

**Theorem 2.1.8.** *Let  $(A, \leq)$  be a preset,  $(S, \mathcal{T}_S)$  be a topological space, and*

$$\theta: A \rightarrow S \quad (2.13)$$

*be a map between the underlying sets. We claim that  $\theta$  is monotone relative to  $\downarrow(S, \mathcal{T}_S)$  iff it is continuous relative to  $\uparrow(A, \leq)$ . In other words, as sets of mappings,  $\mathbf{Top}(\uparrow(A, \leq), (S, \mathcal{T}_S))$  and  $\mathbf{Pre}((A, \leq), \downarrow(S, \mathcal{T}_S))$  are the same.*

*Proof.* Suppose  $(A, \leq) \xrightarrow{\theta} (S, \prec)$  is monotone, where  $\prec$  is the specialization preorder of  $(S, \mathcal{T}_S)$ :

$$s \prec t : \Leftrightarrow \forall U \in \mathcal{T}_S : s \in U \Rightarrow t \in U. \quad (2.14)$$

We will show  $\theta^\leftarrow: \mathcal{T}_S \rightarrow \Gamma_A$ , where  $\Gamma_A$  is the Alexandroff topology. Let  $U \in \mathcal{T}_S$  and  $a, b \in A$ . Suppose  $a \leq b$ ; since  $\theta$  is monotone,  $\theta a \prec \theta b$ . If  $a \in \theta^\leftarrow U$ , i.e.,  $\theta a \in U$ , since  $\theta a \prec \theta b$ , we obtain  $\theta b \in U$ . Hence,  $b \in \theta^\leftarrow U$ . We conclude that  $\theta^\leftarrow U$  is an upper section of  $A$ :

$$\theta^\leftarrow U \in \Gamma_A. \quad (2.15)$$

Thus,  $\theta$  is continuous.

Conversely, suppose  $\theta$  is continuous. We will show  $\theta$  is monotone relative to the specialization preorder  $\prec$ . Let  $a, b \in A$ . Suppose  $a \leq b$ . For an arbitrary  $U \in \mathcal{T}_S$ ,  $\theta^\leftarrow U \in \Gamma_A$  is an upper section,  $a \in \theta^\leftarrow U \Rightarrow b \in \theta^\leftarrow U$ . That is,  $\theta a \in U \Rightarrow \theta b \in U$ :

$$\theta a \prec \theta b. \quad (2.16)$$

Thus,  $\theta$  is monotone. ■

By Theorem 2.1.8, we obtain the following adjunction:

$$\begin{array}{c} \mathbf{Pre} \\ \uparrow \left( \begin{array}{c} \downarrow \\ \uparrow \end{array} \right) \downarrow \\ \mathbf{Top} \end{array} \quad (2.17)$$



### 2.1.3 Topological Spaces and Posets – A Natural Isomorphism

**Theorem 2.1.9.** *For a topological space  $(X, \mathcal{T}_X)$ , let  $\mathcal{O}(X, \mathcal{T}_X) := (\mathcal{T}_X, \subset)$ . This object assignment induces the corresponding arrow assignment, namely  $\mathcal{O}(f) := f^\leftarrow$  for  $f \in C^0((X, \mathcal{T}_X), (Y, \mathcal{T}_Y))$ . Hence,  $\mathcal{O}: \mathbf{Top} \rightarrow \mathbf{Pos}$  is a contravariant functor.*

*Proof.* Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be topological spaces, and  $f \in C^0(X, Y)$ . Note that any set with set inclusion  $\subset$  forms a poset, hence  $(\mathcal{T}_X, \subset)$  is an object of  $\mathbf{Pos}$ . For  $V, W \in \mathcal{T}_Y$ , if  $V \subset W$ , we obtain  $f^\leftarrow V \subset f^\leftarrow W$ , since

$$x \in f^\leftarrow V \Leftrightarrow fx \in V \Rightarrow fx \in W \Leftrightarrow x \in f^\leftarrow W \quad (2.18)$$

for each  $x \in X$ . Hence,  $\mathcal{O}f$  is monotone.

Since  $\mathcal{O}(1_X) = 1_X^\leftarrow: \mathcal{T}_X \rightarrow \mathcal{T}_X$ ,  $\mathcal{O}$  preserves identities. We will show  $\mathcal{O}$  passes across compositions. For  $X \xrightarrow{f} Y \xrightarrow{g} Z$  in  $\mathbf{Top}$ , and  $U \in \mathcal{T}_Z$ ,

$$\begin{aligned} \mathcal{O}(gf)U &:= (gf)^\leftarrow U \\ &= \{x \in X \mid gfx \in U\} \\ &= \{x \in X \mid fx \in g^\leftarrow U\} \\ &= \{x \in X \mid x \in f^\leftarrow (g^\leftarrow U)\} \end{aligned} \quad (2.19)$$

Recalling  $f^\leftarrow = \mathcal{O}f$ , we obtain  $\mathcal{O}(gf) = \mathcal{O}f \circ \mathcal{O}g$ . ■

**Definition 2.1.4** (Sierpiński Space). Let  $\mathbf{2}'$  be  $\mathbf{2} := \{0, 1\}$  with the following topology:

$$\{\emptyset, \{1\}, \{0, 1\}\}. \quad (2.20)$$

We call  $\mathbf{2}'$  Sierpiński space, and the associated topology Sierpiński topology.

**Theorem 2.1.10** (Continuous Characters). *For a topological space  $(X, \mathcal{T}_X)$ , let  $\Xi(X, \mathcal{T}_X) := C^0((X, \mathcal{T}_X), \mathbf{2}')$ . We call  $\Xi(X, \mathcal{T}_X)$  the set of continuous characters of  $(X, \mathcal{T}_X)$ . For  $f \in C^0((X, \mathcal{T}_X), (Y, \mathcal{T}_Y))$ , define  $\Xi f := \_ \circ f$  with the pointwise partial order  $\leq$ :*

$$p \leq q \Leftrightarrow \forall y \in Y : py \leq qy, \quad (2.21)$$

where  $p, q \in \Xi(X, \mathcal{T}_X)$ , and  $0 \leq 0, 0 \leq 1$ , and  $1 \leq 1$ . We claim  $\Xi: \mathbf{Top} \rightarrow \mathbf{Pos}$  is a contravariant functor.

*Proof.* We will first show that  $\Xi f$  converts continuous characters of  $Y$  into continuous characters of  $X$ . Let  $p: Y \rightarrow \{0, 1\}$  be a map. Since the following preimages are both open in  $Y$ :

$$p^\leftarrow \{0, 1\} = Y \wedge p^\leftarrow \emptyset = \emptyset, \quad (2.22)$$

the map  $p$  is continuous relative to Sierpiński topology iff  $p^\leftarrow \{1\}$  is open in  $Y$ :

$$\Xi f: C^0((Y, \mathcal{T}_Y), \mathbf{2}') \rightarrow C^0((X, \mathcal{T}_X), \mathbf{2}'). \quad (2.23)$$

If  $p \in C^0((Y, \mathcal{T}_Y), \mathbf{2}') = \Xi(Y, \mathcal{T}_Y)$ ,  $p^{\leftarrow}\{1\} \in \mathcal{T}_Y$  holds. Then  $(\Xi f)p = pf$  satisfies:

$$((\Xi f)p)^{\leftarrow}\{1\} = (pf)^{\leftarrow}\{1\} = f^{\leftarrow}(p^{\leftarrow}\{1\}) \in \mathcal{T}_X. \quad (2.24)$$

Hence,  $(\Xi f)p \in C^0((X, \mathcal{T}_X), \mathbf{2}') = \Xi(X, \mathcal{T}_X)$ .

Next, we will show that  $\Xi f: \Xi(Y, \mathcal{T}_Y) \rightarrow \Xi(X, \mathcal{T}_X)$  is monotone. Let  $p, q \in \Xi(Y, \mathcal{T}_Y)$  be continuous characters of  $Y$ . Suppose  $p \leq q$ . Then, we obtain:

$$(\Xi f)p = pf \leq qf = (\Xi f)q. \quad (2.25)$$

Finally, consider identities and compositions:

$$\Xi 1_X = \_ \circ 1_X \quad (2.26)$$

For  $X \xrightarrow{f} Y \xrightarrow{g} Z$  in **Top**,

$$\Xi(gf) = \_ \circ (gf) = (\_ \circ g) \circ f = \Xi f \circ \Xi g. \quad (2.27)$$

Hence,  $\Xi: \mathbf{Top} \rightarrow \mathbf{Pos}$  is a contravariant functor. ■

**Definition 2.1.5** (Characteristic Functions). Let  $X$  be a set and  $U \subset X$  be a subset. We call:

$$\chi_X U: X \rightarrow \{0, 1\}; x \mapsto \begin{cases} 1 & x \in U \\ 0 & \text{otherwise} \end{cases} \quad (2.28)$$

the characteristic function of  $U \subset X$ .

**Lemma 2.1.3.** *Let  $(X, \mathcal{T}_X)$  be a topological space and  $U \in \mathcal{T}_X$  be open. The characteristic function of  $U$  is a continuous character of  $X$  relative to Sierpiński topology:*

$$\chi_X U \in C^0(X, 2). \quad (2.29)$$

*Remark 13.* If no confusion is expected, we simply denote  $C^0(X, Y)$  for the set of continuous maps between two topological spaces  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$ .

*Proof.* Since  $(\chi_X U)^{\leftarrow} \emptyset = \emptyset$ ,  $(\chi_X U)^{\leftarrow} 2 = X$ , and

$$(\chi_X U)^{\leftarrow} \{1\} = \{x \in X \mid (\chi_X U)x = 1\} = U, \quad (2.30)$$

we conclude  $\chi_X U$  is continuous. ■

**Theorem 2.1.11.** *Let  $(X, \mathcal{T}_X)$  be a topological space. Recalling  $\mathcal{O}(X, \mathcal{T}_X) = C^0(X, 2)$ , we obtain*

$$\chi_X: \mathcal{O}(X, \mathcal{T}_X) \rightarrow \Xi(X, \mathcal{T}_X) \quad (2.31)$$

*of an assignment between two posets. We claim that  $\chi_X$  is an isomorphism. Moreover, it is a natural transformation between  $\mathcal{O}$  and  $\Xi$ .*

*Proof.* For  $U, V \in \mathcal{O}(X, \mathcal{T}_X) = (\mathcal{T}_X, \subset)$ , suppose  $\chi_X U = \chi_X V$ . Then

$$U = (\chi_X U)^{\leftarrow} \{1\} = (\chi_X V)^{\leftarrow} \{1\} = V. \quad (2.32)$$

Thus,  $\chi_X$  is injective.

For a given  $\chi' \in \Xi(X, \mathcal{T}_X)$ , define  $U' := \chi'^{\leftarrow} \{1\}$ . Since  $\chi' \in C^0(X, 2)$ , such the preimage  $U'$  is open in  $X$ . Hence,  $\chi' = \chi_X U'$ , and  $\chi_X$  is surjective.

Next, we will show that  $\chi$  is monotone. For  $U, V \in \mathcal{O}(X, \mathcal{T}_X) = (\mathcal{T}_X, \subset)$ , suppose  $U \subset V$ :

- $\chi_X U|_U = 1 = \chi_X V|_U$
- $\chi_X U|_{V-U} = 0 \leq 1 = \chi_X V|_{V-U}$
- Otherwise, both  $\chi_X U$  and  $\chi_X V$  are zero.

Thus,  $\chi_X U \leq \chi_X V$ .

Finally, we will show  $\chi: \mathcal{O} \Rightarrow \Xi$ . For  $f \in C^0(X, Y)$ , namely  $X \xrightarrow{f} Y$  in **Top**, consider:

$$\begin{array}{ccc} \mathcal{O}X & \xleftarrow{f^{\leftarrow}} & \mathcal{O}Y \\ \chi_X \downarrow & & \downarrow \chi_Y \\ \Xi X & \xleftarrow{- \circ f} & \Xi Y \end{array} \quad (2.33)$$

We will show that

$$\begin{aligned} \chi_X \circ f^{\leftarrow} &: \mathcal{T}_Y \rightarrow C^0(X, \mathbf{2}') \\ (- \circ f) \circ \chi_Y &: \mathcal{T}_Y \rightarrow C^0(X, \mathbf{2}') \end{aligned} \quad (2.34)$$

are equal. Let  $W \in \mathcal{T}_Y$  and  $x \in X$ ,

$$\begin{aligned} (\chi_X \circ f^{\leftarrow} W) x &= \chi_X (f^{\leftarrow} W) x \\ &= \begin{cases} 1 & x \in f^{\leftarrow} W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & fx \in W \\ 0 & \text{otherwise} \end{cases} \\ &= (\chi_Y W) fx \\ &= (\chi_Y W \circ f) x \\ &= ((- \circ f) \circ \chi_Y W) x. \end{aligned} \quad (2.35)$$

Hence,  $\chi_X \circ f^{\leftarrow} = (- \circ f) \circ \chi_Y$  holds. ■

*Remark 14.* We conclude that **Top**  $\xrightleftharpoons[\Xi]{\mathcal{O}}$  **Pos** are naturally isomorphic via  $\chi$ :

$$\chi: \mathcal{O} \xrightarrow{\cong} C^0(-, 2) = \mathbf{Top}(-, 2), \quad (2.36)$$

where  $2 = \{0, 1\}$  is associated with the Sierpiński topology  $\mathbf{2}' = (2, \{\emptyset, \{1\}, 2\})$ .

## 2.2 Compact-Open Topology and Locally Compact Spaces

### 2.2.1 Compact Spaces – Closed Maps

For a topological space  $(X, \mathcal{T}_X)$  and its subspace  $A \subset X$ , an open covering of  $A$  is a set of open subspaces  $\mathcal{U} \subset \mathcal{T}_X$  such that  $A \subset \cup \mathcal{U} = \bigcup_{U \in \mathcal{U}} U$ . A finite subcover of an open covering  $\mathcal{U}$  of  $A$  is a finite subset of  $\mathcal{U}$  that also covers  $A$ .

**Definition 2.2.1** (Compact Spaces). A topological space is called a compact space iff every open covering of the space contains a finite subcover.

For a topological space  $(X, \mathcal{T}_X)$ , let  $\mathcal{K}_X$  be the set of all compact subspaces in  $X$ .

**Definition 2.2.2** (Closed Maps). A map between topological spaces is called a closed map iff a direct image of a closed space in its domain is a closed subspace in the codomain space.

**Theorem 2.2.1** (Compactness via Closed Projections). *A topological space  $K$  is compact iff for every topological space  $X$ , the canonical projection:*

$$\pi_X: K \times X \rightarrow X \quad (2.37)$$

*is a closed map relative to the product topology.*

**Remark 15** (Product Topology). For  $\{(X_\lambda, \mathcal{T}_{X_\lambda}) \mid \lambda \in \Lambda\}$  of a set of topological spaces, the product topology is the generated topology of the following subbase:

$$\{\pi_\lambda^{-1}U \mid \lambda \in \Lambda \wedge U \in \mathcal{T}_{X_\lambda}\}, \quad (2.38)$$

where  $\pi_\lambda: \prod_{\lambda \in \Lambda} X_\lambda \rightarrow X_\lambda$  is a projection for each  $\lambda \in \Lambda$ . By definition, as this subbase makes the projections continuous, the product topology is  $\subset$ -smallest topology on which  $\pi_\lambda \in C^0(\prod_{\lambda \in \Lambda} X_\lambda, X_\lambda)$  for each  $\lambda \in \Lambda$ .

*Proof.* ( $\Rightarrow$ ) Let  $X$  be a topological space and  $K$  be a compact space. We will show  $\pi_X$  is a closed map; if  $X = \emptyset$ , nothing has to prove. Let  $C \subset K \times X$  be a closed subspace; if  $\pi_X C = X$ , as  $X \subset X$  is a clopen subspace in  $X$ , done. So we may suppose  $\pi_X C \subsetneq X$ .

Select  $x \in \neg \pi_X C$ . Since  $\pi_X$  is a surjection, there is at least one  $k \in K$  with  $(k, x) \xrightarrow{\pi_X} x$ . Then such a pair  $(k, x) \in \neg C$ , otherwise  $(k, x)$  would be in  $C$ , so  $x = \pi_X(k, x) \in \pi_X C$ , which is absurd. Thus,  $x = \pi_X(k, x) \in \pi_X(\neg C)$ , and hence  $\neg \pi_X C \subset \pi_X(\neg C)$ . However, it implies  $\pi_X C \supset \neg \pi_X(\neg C) = \pi_X C$ . So, we conclude  $\pi_X C = \neg \pi_X(\neg C)$  and  $\neg \pi_X C = \pi_X(\neg C)$ .

The preimage  $\pi_X^{-1}(x) = K \times \{x\}$  does not meet  $C$ , for otherwise  $(k, x) \in K \times \{x\} \cap C$ , we obtain  $\pi_X(k, x) = x \in \pi_X C$ , which is absurd. Hence,  $K \times \{x\} \subset \neg C$ . Since  $\neg C \subset K \times X$  is open, for each point  $(k, x) \in K \times \{x\}$ , there are open neighborhoods  $U_k \in \mathcal{N}_k \cap \mathcal{T}_K$  and  $V_{k,x} \in \mathcal{N}_x \cap \mathcal{T}_X$  such that  $(k, x) \in U_k \times V_{k,x}$  and

$$U_k \times V_{k,x} \subset \neg C. \quad (2.39)$$

Since  $\{U_k \mid k \in K\}$  is an open cover of the compact space  $K$ , there is a finite subcover:

$$K \subset U_{k_1} \cup \cdots \cup U_{k_n}. \quad (2.40)$$

Define  $W_x := W_{k_1,x} \cap \cdots \cap W_{k_n,x}$ . Since, for each  $k_j \in \{k_1, \dots, k_n\}$ ,

$$W_x \times U_{k_j} \subset W_{k_j,x} \times U_{k_j} \subset \neg C, \quad (2.41)$$

we conclude:

$$W_x \times K \subset W_x \times \bigcup_{j=1}^n U_{k_j} = \bigcup_{j=1}^n (W_x \times U_{k_j}) \subset \neg C. \quad (2.42)$$

Hence,  $W_x \subset \pi_X(\neg C) = \neg \pi_X C$ . This  $W_x$  is the desired open neighborhood of  $x$ ; applying Lemma 1.2.1, we conclude that  $\neg \pi_X C$  is open.

( $\Leftarrow$ ) Let  $(X, \mathcal{T}_X)$  be an arbitrary topological space and  $\mathcal{U} \subset \mathcal{T}_X$  be an arbitrary open covering of  $X$ . Define  $X_\infty := X \cup \{\infty\}$ , where  $\infty \notin X$ . For an arbitrary subset  $A \subset X_\infty$ , we call  $A$  closed iff either

$$\infty \in A \vee A \text{ is finitely covered by } \mathcal{U}. \quad (2.43)$$

This relation defines a topology on  $X_\infty$ :

- Since  $\infty \in X_\infty$ , the complement  $\emptyset$  is open.
- Since  $\emptyset$  is vacuously covered by  $\emptyset \subset \mathcal{U}$ , its complement  $X_\infty$  is open.
- Arbitrary Union  
For an arbitrary subset of open subspaces  $\{V_\lambda \subset X \mid \lambda \in \Lambda\}$ , if at least one  $\neg V_{\lambda_0}$  is finitely covered by  $\mathcal{U}$ :

$$\neg \bigcup_{\lambda \in \Lambda} V_\lambda = \bigcap_{\lambda \in \Lambda} \neg V_\lambda \subset \neg V_{\lambda_0} \quad (2.44)$$

so as  $\neg \bigcup_{\lambda \in \Lambda} V_\lambda$ . Otherwise, every  $\neg V_\lambda$  contains  $\infty$ :

$$\infty \in \bigcap_{\lambda \in \Lambda} \neg V_\lambda = \neg \bigcup_{\lambda \in \Lambda} V_\lambda \quad (2.45)$$

Hence,  $\neg \bigcup_{\lambda \in \Lambda} V_\lambda$  is closed and its complement  $\bigcup_{\lambda \in \Lambda} V_\lambda$  is open.

- Binary Intersection

Let  $U, V$  be open. Consider  $\neg(U \cap V)$ :

$$\begin{aligned} \neg U \cup \neg V &= \{x \in X \mid x \notin U \vee x \notin V\} \\ &= \{x \in X \mid \neg(x \in U \wedge x \in V)\} \\ &= \neg(U \cap V). \end{aligned} \quad (2.46)$$

- If at least one of  $\neg U$  and  $\neg V$  contains  $\infty$ , then  $\infty \in \neg U \cup \neg V$ . Hence,  $\neg U \cup \neg V = \neg(U \cap V)$  is closed.

- Otherwise, both  $\neg U$  and  $\neg V$  are finitely covered by  $\mathcal{U}$ . Then  $\neg U \cup \neg V = \neg(U \cap V)$  is also finitely covered by  $\mathcal{U}$ .

Hence, the complement  $U \cap V$  is open.

By hypothesis, the canonical projection  $\pi_{X_\infty} : X \times X_\infty \rightarrow X_\infty$  is a closed map.

Clearly,  $X \subsetneq X_\infty$ . Within the product space  $X \times X_\infty$ , consider a subspace:

$$X \times X \subset X \times X_\infty \quad (2.47)$$

and its closure  $\overline{X \times X} \subset X \times X_\infty$  relative to the product topology. We will show that  $\infty$  is not in  $\pi_{X_\infty}(\overline{X \times X} \subset X \times X_\infty)$ . Suppose, for contradiction, there is some  $x \in X$  with  $(x, \infty) \in \overline{X \times X} \subset X \times X_\infty$ . Since  $\mathcal{U}$  is an open cover of  $X$ , there is some  $U \in \mathcal{U}$  with  $x \in U$ . Let  $\neg_\infty U := X_\infty - U$ . Since  $\infty \in \neg_\infty U \subset X_\infty$ ,  $U \subset X_\infty$  is open;  $U \subset U$  is covered by itself,  $U \subset X_\infty$  is closed as well. Then  $\neg_\infty U$  is open with  $\infty \in \neg_\infty U$ . The product subspace  $U \times \neg_\infty U \subset X \times X_\infty$  is an open neighborhood of  $(x, \infty)$  with

$$(U \times \neg_\infty U) \cap (X \times X) = \emptyset. \quad (2.48)$$

By Lemma 1.2.2, we have a contradiction.

Since  $\infty \notin \pi_{X_\infty}(\overline{X \times X})$ :

$$\pi_{X_\infty}(\overline{X \times X}) = X \quad (2.49)$$

is closed, by hypothesis, in  $X_\infty$ . As  $\infty \notin X$ ,  $X$  must be finitely covered by  $\mathcal{U}$ . Hence,  $X$  is compact.  $\blacksquare$

### 2.2.2 Compact Open Topology and Locally Compact Spaces

The idea of topologizing the set of all continuous maps of one space into another plays an important role in modern topology. [Dug66]

**Definition 2.2.3** (Compact-Open Topology). Let  $(I, \mathcal{T}_I)$  and  $(X, \mathcal{T}_X)$  be topological spaces. For  $K \in \mathcal{K}_I$  and  $V \in \mathcal{T}_X$ , let

$$\langle K, V \rangle := \{ \theta \in C^0(I, X) \mid \theta K \subset V \}. \quad (2.50)$$

The compact topology on the set of continuous maps  $C^0(I, X)$  is the generated topology by the following subbase:

$$\{ \langle K, V \rangle \mid K \in \mathcal{K}_I \wedge V \in \mathcal{T}_X \}. \quad (2.51)$$

See Definition 1.2.5. Let  $I \multimap Y$  denote the space of continuous maps from  $I$  to  $Y$  with the compact-open topology.

**Theorem 2.2.2** (Currying). Let  $(I, \mathcal{T}_I)$  be a topological space. For a pair of topological spaces,  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$ , if  $X \times I \xrightarrow{\psi} Y$  is continuous relative to the product topology, then its curried form is also continuous:

$$X \xrightarrow{\psi_\flat} (I \multimap Y), \quad (2.52)$$

where  $(\psi_b x)i := \psi(x, i)$  for  $x \in X$  and  $i \in I$ , and  $I \multimap Y$  is equipped with the compact-open topology.

*Proof.* As the compact-open topology on  $I \multimap Y$  is generated by  $\langle K, V \rangle$  for  $K \in \mathcal{K}_I$  and  $V \in \mathcal{T}_Y$ , consider a subbasic open subspace  $\langle K, V \rangle \subset (I \multimap Y)$  and  $x \in \psi_b^\leftarrow \langle K, V \rangle$ :

$$\psi_b x \in \langle K, V \rangle \Leftrightarrow \forall k \in K : (\psi_b x)k = \psi(x, k) \in V \quad (2.53)$$

That is,  $(x, k) \in \psi^\leftarrow V$  for each  $k \in K$ . Since  $\psi$  is continuous and  $V \in \mathcal{T}_Y$ ,  $\psi^\leftarrow V$  is open in  $X \times I$ . Thus, there are open neighborhoods  $U_{x,k} \in \mathcal{N}_x \cap \mathcal{T}_X$  and  $W_k \in \mathcal{N}_k \cap \mathcal{T}_I$  for each  $k \in K$  such that

$$(x, k) \in U_{x,k} \times W_k \subset \psi^\leftarrow V. \quad (2.54)$$

Since  $\{W_k \mid k \in K\}$  covers the compact subspace  $K \subset I$ , there is a finite subcover:

$$K \subset W := W_{k_1} \cup \cdots \cup W_{k_n}. \quad (2.55)$$

Define  $U_x := U_{x,k_1} \cap \cdots \cap U_{x,k_n}$ . Then, we have  $x \in U_x \in \mathcal{T}_X$ ,  $K \subset W$ , and

$$U_x \times W = U_x \times \bigcup_{j=1}^n W_{k_j} = \bigcup_{j=1}^n U_x \times W_{k_j} \subset \bigcup_{j=1}^n U_{x,k_j} \times W_{k_j} \subset \psi^\leftarrow V. \quad (2.56)$$

Moreover, for each  $x' \in X$ ,

$$\begin{aligned} x' \in U_x &\Rightarrow \forall w \in W : (x', w) \in \psi^\leftarrow V \\ &\Leftrightarrow \forall w \in W : \psi(x', w) = (\psi_b x')w \in V \\ &\Leftrightarrow \forall w \in W : w \in (\psi_b x')^\leftarrow V \\ &\Rightarrow \forall w \in K : w \in (\psi_b x')^\leftarrow V \\ &\Leftrightarrow \psi_b x' \in \langle K, V \rangle \\ &\Leftrightarrow x' \in \psi_b^\leftarrow \langle K, V \rangle. \end{aligned} \quad (2.57)$$

Hence, we have  $U_x \subset \psi_b^\leftarrow \langle K, V \rangle$  with  $x \in U_x$ . By Lemma 1.2.1, we conclude  $\psi_b^\leftarrow \langle K, V \rangle \in \mathcal{T}_X$ . By Theorem 1.2.5,  $\psi_b$  is continuous.  $\blacksquare$

Many of the important spaces occurring in analysis are not compact, but have instead a local version of compactness. [Dug66]

**Definition 2.2.4** (Locally Compact Spaces). A topological space  $(I, \mathcal{T}_I)$  is locally compact iff for each point  $i \in I$  and its open neighborhood  $U \in \mathcal{N}_i \cap \mathcal{T}_I$ , there are open  $W \in \mathcal{T}_I$  and a compact  $K \in \mathcal{K}_I$  such that:

$$i \in W \subset K \subset U. \quad (2.58)$$

In other words, a locally compact space is a topological space where each point has a compact neighborhood. In particular, a compact space is locally compact.

**Theorem 2.2.3** (Uncurrying). *Let  $(I, \mathcal{T}_I)$  be a locally compact space. For a pair of topological spaces,  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$ , if  $X \xrightarrow{\phi} (I \multimap Y)$  is continuous relative to the compact-open topology, then its uncurried form is also continuous:*

$$X \times I \xrightarrow{\phi^\sharp} Y, \quad (2.59)$$

where  $\phi^\sharp(x, i) := (\phi x)i$  for  $x \in X$  and  $i \in I$ .

*Proof.* Let  $V \in \mathcal{T}_Y$ , and  $(x, i) \in \phi^{\sharp\leftarrow} V$ . By definition,  $\phi^\sharp(x, i) = (\phi x)i \in V$ , we have

$$i \in (\phi x)^{\leftarrow} V. \quad (2.60)$$

Since  $\phi x \in I \multimap Y$  is continuous and  $V \in \mathcal{T}_Y$  is open,  $(\phi x)^{\leftarrow} V \in \mathcal{T}_I$ . Moreover, as  $I$  is locally compact, there are  $W \in \mathcal{T}_I$  and  $K \in \mathcal{K}_I$  such that

$$i \in W \subset K \subset (\phi x)^{\leftarrow} V. \quad (2.61)$$

For each  $k \in I$ ,

$$k \in K \Rightarrow k \in (\phi x)^{\leftarrow} V \Leftrightarrow (\phi x)k \in V. \quad (2.62)$$

Hence, we obtain  $(\phi x)K \subset V$ :

$$\phi x \in \langle K, V \rangle. \quad (2.63)$$

Since  $\phi$  is continuous, we conclude:

$$x \in \phi^{\leftarrow} \langle K, V \rangle \in \mathcal{T}_X. \quad (2.64)$$

For each  $(x', i') \in X \times I$ , we obtain:

$$\begin{aligned} (x', i') \in \phi^{\leftarrow} \langle K, V \rangle \times W &\Rightarrow (x', i') \in \phi^{\leftarrow} \langle K, V \rangle \times K \\ &\Rightarrow \phi^\sharp(x', i') = (\phi x')i' \in V \\ &\Leftrightarrow (x', i') \in \phi^{\sharp\leftarrow} V. \end{aligned} \quad (2.65)$$

Hence,  $\phi^{\leftarrow} \langle K, V \rangle \times W \subset \phi^{\sharp\leftarrow} V$  is the desired open neighborhood of  $(x, i)$ ; by Lemma 1.2.1 and Theorem 1.2.5,  $\phi^\sharp$  is continuous.  $\blacksquare$

**Definition 2.2.5.** Let  $(I, \mathcal{T}_I)$  be a locally compact space. We denote, for a topological space  $(X, \mathcal{T}_X)$ :

$$\begin{aligned} LX &:= X \times I \\ RX &:= I \multimap X \end{aligned} \quad (2.66)$$

These object assignments induce the corresponding arrow assignments:

$$\begin{aligned} L(X \xrightarrow{f} X') &= X \times I \xrightarrow{f \times 1_I} X' \times I \\ R(Y \xrightarrow{g} Y') &= I \multimap Y \xrightarrow{g \circ -} I \multimap Y' \end{aligned} \quad (2.67)$$

where

$$\begin{aligned} (f \times 1_I)(x, i) &= (fx, i) \\ (g \circ -)p &= g \circ p \end{aligned} \quad (2.68)$$



**Theorem 2.2.4.** *Let  $(I, \mathcal{T}_I)$  be a locally compact space. With the product topology and the compact open topology, we have the following adjoint endo functors:*

$$\begin{array}{c} \mathbf{Top} \\ L = - \times I \left( \begin{array}{c} \uparrow \\ - \\ \downarrow \end{array} \right) R = I \multimap - \\ \mathbf{Top} \end{array} \quad (2.69)$$

*Proof.* By Theorem 2.2.2 and Theorem 2.2.3, we have currying-uncurrying bijection. By Theorem 1.3.3, it suffices to show the naturality:

- For  $LX \xrightarrow{\psi} Y \xrightarrow{g} Y'$ , consider:

$$X \xrightarrow{\psi_b} RY \xrightarrow{Rg} RY' \quad (2.70)$$

$(g \circ \psi)_b$

Let  $x \in X$  and  $i \in I$ :

$$\begin{aligned} (Rg \circ \psi_b) i &= (g \circ -) (\psi_b x) i \\ &= (g \circ -) \psi(x, i) \\ &= g \circ \psi(x, i) \\ &= (g \circ \psi_b x) i \end{aligned} \quad (2.71)$$

Hence, we conclude  $(g \circ \psi)_b = Rg \circ \psi_b$ .

- For  $X \xrightarrow{f} X' \xrightarrow{\phi'} LY'$ , consider:

$$LX \xrightarrow{Lf} LX' \xrightarrow{\phi'^{\sharp}} Y' \quad (2.72)$$

$(\phi' \circ f)^{\sharp}$

Let  $x \in X$  and  $i \in I$ :

$$\begin{aligned} \phi'^{\sharp} \circ Lf(x, i) &= \phi'^{\sharp}(fx, i) \\ &= \phi'(fx) i \\ &= ((\phi' \circ f) x) i \\ &= (\phi' \circ f)^{\sharp}(x, i) \end{aligned} \quad (2.73)$$

Hence, we conclude  $\phi'^{\sharp} \circ Lf = (\phi' \circ f)^{\sharp}$ .

Therefore, these two endo functors form an adjunction. ■

## 2.3 Ambivalent Objects

Consider a contravariant adjunction:

$$\begin{array}{c} \mathcal{A} \\ \begin{array}{c} \curvearrowright \\ F \downarrow \quad \uparrow G \\ \mathcal{S} \end{array} \end{array} \quad (2.74)$$

In the covariant form  $\begin{array}{c} \mathcal{A} \\ \begin{array}{c} \curvearrowright \\ F \downarrow \quad \uparrow G \\ \mathcal{S}^{\text{op}} \end{array} \end{array}$ , we have the unit  $\eta: 1_{\mathcal{A}} \Rightarrow GF$  and the counit

$\epsilon: FG \Rightarrow 1_{\mathcal{S}^{\text{op}}}$ , and the natural bijection in Theorem 1.3.3 is  $\zeta_{A,S}: \mathcal{S}^{\text{op}}(FA, S) \cong \mathcal{A}(A, GS)$  for  $A \in |\mathcal{A}|$  and  $S \in |\mathcal{S}|$ .

Hence, the bijection becomes

$$\zeta_{A,S}: \mathcal{S}(S, FA) \cong \mathcal{A}(A, GS) \quad (2.75)$$

with the following conditions:

- For  $S' \xrightarrow{s} S \xrightarrow{\phi} FA$  in  $\mathcal{S}$ ,

$$GS' \begin{array}{c} \xleftarrow{Gs} GS \xleftarrow{\zeta\phi} A \\ \zeta(\phi s) \end{array} \quad \zeta_{A,S'}(\phi s) = Gs \circ (\zeta_{A,S}\phi) \quad (2.76)$$

- For  $A' \xrightarrow{a} A \xrightarrow{f} GS$  in  $\mathcal{A}$ ,

$$FA' \begin{array}{c} \xleftarrow{Fa} FA \xleftarrow{\zeta^{-1}f} S \\ \zeta^{-1}(fa) \end{array} \quad \zeta_{A',S}^{-1}(fa) = Fa \circ (\zeta_{A,S}^{-1}f) \quad (2.77)$$

The unit and counit become:

$$\begin{aligned} \eta_A &= \zeta_{A,GFA} 1_{FA} \in \mathcal{A}(A, GFA) \\ \epsilon_S &= \zeta_{FGS,S}^{-1} 1_{GS} \in \mathcal{S}(S, FGS) \end{aligned} \quad (2.78)$$

for  $A \in |\mathcal{A}|$  and  $S \in |\mathcal{S}|$ .

### 2.3.1 Posets and Spaces – A Contravariant Adjunction

**Theorem 2.3.1** (A Topology on Upper Sections). *Let  $(A, \leq)$  be a poset and  $\Gamma_A$  be the set of upper sections of  $A$ , see Definition 2.1.2. Note that, by Definition 1.1.3, a poset is a preset with the antisymmetric order  $\leq$ . For each finite subset  $a \subset A$ , let  $\langle a \rangle$  be a subset of  $\Gamma_A$  given by*

$$U \in \langle a \rangle :\Leftrightarrow a \subset U \quad (2.79)$$

*for  $U \in \Gamma_A$ . These subsets  $\{\langle a \rangle \mid a \subset A \text{ is finite}\}$  form a basis of some topology, say  $\mathcal{T}_{\Gamma_A}$ , on  $\Gamma_A$ .*

*Proof.* We will show the conditions in Remark 2 in Definition 1.2.5:

1.  $\{\langle a \rangle \mid a \subset A \text{ is finite}\}$  covers  $\Gamma_A$

Let  $U \in \Gamma_A$  be an upper section of  $A$ . Since  $\emptyset$  is a finite subset of  $A$ , and  $\emptyset \subset U$ , we conclude  $U \in \langle \emptyset \rangle$ .

2. Binary Intersection

Let  $a, a' \subset A$  be finite subsets. Consider  $\langle a \rangle \cap \langle a' \rangle$ . For  $U \in \Gamma_A$ , we have

$$U \in \langle a \rangle \cap \langle a' \rangle \Leftrightarrow a \subset U \wedge a' \subset U \Leftrightarrow a \cup a' \subset U \Leftrightarrow U \in \langle a \cup a' \rangle, \quad (2.80)$$

Hence, we conclude  $\langle a \rangle \cap \langle a' \rangle = \langle a \cup a' \rangle$ .

Therefore, we may apply Theorem 1.2.4 to obtain the generated topology  $\mathcal{T}_{\Gamma_A}$  as the set of all unions of the basis  $\{\langle a \rangle \mid a \subset A \text{ is finite}\}$ . ■

**Theorem 2.3.2.** *For a poset  $(A, \leq)$ , let  $\Upsilon(A, \leq) := (\Gamma_A, \mathcal{T}_{\Gamma_A})$ . If we define  $\Upsilon f := f^\leftarrow$  for a monotone  $f \in \mathbf{Pos}(A, B)$ , then  $\Upsilon f: (\Gamma_B, \mathcal{T}_{\Gamma_B}) \rightarrow (\Gamma_A, \mathcal{T}_{\Gamma_A})$  is continuous. We obtain a contravariant functor  $\Upsilon: \mathbf{Pos} \rightarrow \mathbf{Top}$ .*

*Proof.* Consider  $(\Upsilon f)^\leftarrow$  and  $a = \{a_1, \dots, a_n\} \subset A$  of a finite subset. For  $V \in \Gamma_B$  of an upper section in  $B$ ,

$$\begin{aligned} V \in (\Upsilon f)^\leftarrow \langle a \rangle &\Leftrightarrow (\Upsilon f)V \in \langle a \rangle \\ &\Leftrightarrow \{a_1, \dots, a_n\} \subset (\Upsilon f)V = f^\leftarrow V \\ &\Leftrightarrow a_1 \in f^\leftarrow V \wedge \dots \wedge a_n \in f^\leftarrow V \\ &\Leftrightarrow fa_1 \in V \wedge \dots \wedge fa_n \in V \\ &\Leftrightarrow \{fa_1, \dots, fa_n\} \subset V \\ &\Leftrightarrow fa \subset V \\ &\Leftrightarrow V \in \langle fa \rangle. \end{aligned} \quad (2.81)$$

Hence,  $(\Upsilon f)^\leftarrow \langle a \rangle = \langle fa \rangle$  is a member of the basis for  $\mathcal{T}_{\Gamma_B}$ . By Theorem 1.2.6,  $\Upsilon f \in C^0(\Gamma_B, \Gamma_A)$ . ■

**Theorem 2.3.3.** *Now we have a pair of contravariant functors:*

$$\begin{array}{ccc} & \mathbf{Pos} & \\ & \Upsilon \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \circ & \\ & \mathbf{Top} & \end{array} \quad (2.82)$$

*They form a contravariant adjunction.*

*Proof.* Let  $(X, \mathcal{T}_X)$  be a topological space,  $(A, \leq)$  be a poset, and  $(\Gamma_A, \mathcal{T}_{\Gamma_A}) := \Upsilon(A, \leq)$ . For  $\phi \in \mathbf{Top}(X, \Gamma_A)$ , define  $\phi^\alpha$  by

$$x \in \phi^\alpha a \Leftrightarrow a \in \phi x \quad (2.83)$$

for each  $x \in X$  and  $a \in A$ .

- $\phi^\alpha: A \rightarrow \mathcal{T}_X$

Let  $\{a\} \subset A$  be a singleton subset. For each  $x \in X$ ,

$$x \in \phi^\alpha a \Leftrightarrow \{a\} \subset \phi x \Leftrightarrow \phi x \in \langle \{a\} \rangle \Leftrightarrow x \in \phi^\leftarrow \langle \{a\} \rangle. \quad (2.84)$$

Hence, we conclude  $\phi^\alpha a = \phi^\leftarrow \langle \{a\} \rangle \in \mathcal{T}_X$ .

- $\phi^\alpha$  is an arrow in **Pos**

Let  $a \leq b$  in  $(A, \leq)$ . Since  $\phi a \in \Gamma_A$  is an upper section of  $A$ , if  $a \in A$  then  $b \in A$  holds. For each  $x \in X$ ,

$$x \in \phi^\alpha a \Leftrightarrow a \in \phi x \Rightarrow b \in \phi x \Leftrightarrow x \in \phi^\alpha b. \quad (2.85)$$

Hence,  $\phi^\alpha$  is monotone  $\phi^\alpha a \subset \phi^\alpha b$ .

We then obtain

$$\zeta_{A,X}: \mathbf{Top}(X, \Gamma_A) \rightarrow \mathbf{Pos}(A, \mathcal{T}_X); \phi \mapsto \phi^\alpha. \quad (2.86)$$

Let  $(\mathcal{T}_X, \subset) := \mathcal{O}(X, \mathcal{T}_X)$ . For  $f \in \mathbf{Pos}(A, \mathcal{T}_X)$ , define  $f^\sigma$  by

$$a \in f^\sigma x \Leftrightarrow x \in fa \quad (2.87)$$

for each  $a \in A$  and  $x \in X$ .

- $f^\sigma: X \rightarrow \Gamma_A$

Let  $x \in X$ . Suppose  $a \leq b$  in  $A$ . Since  $f$  is monotone,  $fa \subset fb$  holds. Then

$$a \in f^\sigma x \Leftrightarrow x \in fa \Rightarrow x \in fb \Leftrightarrow b \in f^\sigma x. \quad (2.88)$$

Hence,  $f^\sigma x$  is an upper section in  $A$ .

- $f^\sigma$  is an arrow in **Top**

Consider the preimage  $f^{\sigma\leftarrow}$  and a finite subset  $a = \{a_1, \dots, a_n\} \subset A$ . For each  $x \in X$ ,

$$\begin{aligned} x \in f^{\sigma\leftarrow} \langle a \rangle &\Leftrightarrow f^\sigma x \in \langle a \rangle \\ &\Leftrightarrow \{a_1, \dots, a_n\} \subset f^\sigma x \\ &\Leftrightarrow a_1 \in f^\sigma x \wedge \dots \wedge a_n \in f^\sigma x \\ &\Leftrightarrow x \in fa_1 \wedge \dots \wedge x \in fa_n \\ &\Leftrightarrow x \in fa_1 \cap \dots \cap fa_n \end{aligned} \quad (2.89)$$

Hence,  $f^{\sigma\leftarrow} \langle a \rangle = fa_1 \cap \dots \cap fa_n \in \mathcal{T}_X$ . We conclude  $f^{\sigma\leftarrow}$  is continuous.

We obtain

$$\zeta'_{A,X}: \mathbf{Pos}(A, \mathcal{T}_X) \rightarrow \mathbf{Top}(X, \Gamma_A); f \mapsto f^\sigma. \quad (2.90)$$

They are inverse pair:

- $\phi \mapsto \phi^\alpha \mapsto \phi^{\alpha\sigma} = \phi$

For each  $x \in X$  and  $a \in A$ , we have:

$$a \in \phi^{\alpha\sigma} x \Leftrightarrow x \in \phi^\alpha a \Leftrightarrow a \in \phi x. \quad (2.91)$$

- $f \mapsto f^\sigma \mapsto f^{\sigma\alpha} = f$

For each  $a \in A$  and  $x \in X$ , we have:

$$x \in f^{\sigma\alpha} a \Leftrightarrow a \in f^\sigma x \Leftrightarrow x \in f a. \quad (2.92)$$

Hence,  $\zeta'_{A,X} = \zeta_{A,X}^{-1}$  and

$$\begin{array}{ccc} & \mathbf{Pos} & \\ & \uparrow \scriptstyle \gamma & \\ \mathbf{Top} & \left( \dashv \right) & \mathcal{O} \end{array} \quad (2.93)$$

form a contravariant adjunction. ■

*Remark 16* (Unit and Counit). Let  $A \in |\mathbf{Pos}|$  and  $X \in |\mathbf{Top}|$ . The unit  $\eta_A = (1_{\Gamma_A})^\alpha$  and the counit  $\epsilon_X = (1_{\mathcal{T}_X})^\sigma$  are the following arrows:

$$\begin{array}{ccc} \mathbf{Pos} & A & \xrightarrow{\eta_A} \mathcal{T}_{\Gamma_A} \\ & & \\ \mathbf{Top} & X & \xrightarrow[\epsilon_X]{} \Gamma_{\mathcal{T}_X} \end{array} \quad (2.94)$$

### 2.3.2 Ambivalent Objects

Let  $(A, \leq) \in |\mathbf{Pos}|$  and  $\Pi(A, \leq) := \mathbf{Pos}(A, 2)$ , where  $2 = \{0, 1\}$  is directed by  $<$  with  $0 < 1$ . We call  $\mathbf{Pos}(A, 2)$  the set of monotone characters of  $A$ , where  $p \leq q$  for  $p, q \in \mathbf{Pos}(A, 2)$  iff  $\forall a \in A : pa \leq qa$ .

As demonstrated in Theorem 2.1.11,  $\chi_A : \Gamma_A \cong \mathbf{Pos}(A, 2); U \mapsto \chi_A U$  is a bijection between the underlying sets, where  $\chi_A U$  is the characteristic function on  $U \subset A$ :

$$(\chi_A U) a = \begin{cases} 1 & a \in U \\ 0 & \text{otherwise} \end{cases} \quad (2.95)$$

**Lemma 2.3.1.** *For each upper section  $U \in \Gamma_A$  of  $A$ , its characteristic function  $\chi_A U : A \rightarrow 2$  is continuous.*

*Proof.* Let  $(A, \leq) \in |\mathbf{Pos}|$ ,  $U \in \Gamma_A$ , and  $\chi_A U \in \mathbf{Pos}(A, 2)$ . We will show that relative to Alexandroff topology  $\Gamma_A$  of  $A$  and Sierpiński topology,  $\chi_A U \in \mathbf{Top}(A, 2)$ . Recalling  $U \in \Gamma_A$  is open in  $A$ , we obtain  $\chi_A U^{\leftarrow} \emptyset = \emptyset$ ,  $\chi_A U^{\leftarrow} 2 = A$ , and

$$\chi_A U^{\leftarrow} \{1\} = \{a \in A \mid (\chi_A U) a = 1\} = U. \quad (2.96)$$

Hence,  $\chi_A(U) \in C^0(A, 2) = \mathbf{Top}(A, 2)$ . ■

Therefore,  $\chi_A: \Gamma_A \cong \mathbf{Pos}(A, 2)$  returns an object in **Top**. With this bijection, we may topologize  $C^0(A, 2) = \mathbf{Pos}(A, 2)$ :

**Theorem 2.3.4.** *For each poset  $(A, \leq) \in |\mathbf{Pos}|$ ,  $\chi_A \in C^0(\Gamma_A, \mathbf{Pos}(A, 2))$ .*

*Proof.* Let  $(A, \leq) \in |\mathbf{Pos}|$ . Consider the uncurried form:

$$\chi_A^\sharp: \Gamma_A \times A \rightarrow 2 \quad (2.97)$$

We will show  $\chi_A^\sharp \in C^0(\Gamma_A \times A, 2)$  relative to the product topology and Sierpiński topology. It suffices to consider  $\{1\} \subset 2'$  and its preimage:

$$\chi_A^{\sharp \leftarrow} \{1\} := \{(U, a) \in \Gamma_A \times A \mid (\chi_A U)a = 1\}. \quad (2.98)$$

For each  $(U, a) \in \Gamma_A \times A$ ,

$$(U, a) \in \chi_A^{\sharp \leftarrow} \{1\} \Leftrightarrow 1 = (\chi_A U)a \Leftrightarrow a \in U \Leftrightarrow \{a\} \subset U \Leftrightarrow U \in \langle a \rangle \quad (2.99)$$

Hence, we conclude:

$$\chi_A^{\sharp \leftarrow} \{1\} = \langle a \rangle \times U. \quad (2.100)$$

Since it is the product of a basic open subspace of  $(\Gamma_A, \mathcal{T}_{\Gamma_A})$  and an open subspace of  $(A, \Gamma_A)$ ,  $\chi_A^{\sharp \leftarrow} \{1\} \subset \Gamma_A \times A$  is open:

$$\chi_A^\sharp \in C^0(\Gamma_A \times A, 2). \quad (2.101)$$

As shown in Theorem 2.2.2, the original  $\chi_A$  is continuous if the uncurried form  $\chi_A^\sharp$  is continuous. Hence,  $\chi_A \in C^0(\Gamma_A, A \multimap 2)$  is continuous, where  $A \multimap 2 = C^0(A, 2)$ , see Definition 2.2.3.  $\blacksquare$

Moreover,  $\chi: \Upsilon \Rightarrow \Pi$  is a natural isomorphism, since

$$\begin{array}{ccc} \Gamma_A & \xleftarrow{f^\leftarrow} & \Gamma_B \\ \chi_A \downarrow & & \downarrow \chi_B \\ \mathbf{Pos}(A, 2) & \xleftarrow{- \circ f} & \mathbf{Pos}(B, 2) \end{array} \quad (2.102)$$

is commutative in **Top** for  $A \xrightarrow{f} B$  in **Pos**. It is worth mentioning that the naturality is essentially shown in (2.33) of Theorem 2.1.11.

Recalling Remark 14,  $\mathcal{O} \cong \mathbf{Top}(-, 2)$ , we obtain:

$$\chi: \Upsilon \xrightarrow{\cong} \Pi = \mathbf{Pos}(-, 2). \quad (2.103)$$

Hence,

$$\begin{array}{ccc} & \mathbf{Pos} & \\ & \uparrow & \\ \mathbf{Pos}(-, 2) & \left( \begin{array}{c} \uparrow \\ - \\ \downarrow \end{array} \right) & \mathbf{Top}(-, 2) \\ & \mathbf{Top} & \end{array} \quad (2.104)$$

The object 2 lives in both categories. It is both a poset and topological space. ... Furthermore, it induces both of the functors. [Sim11]

Such an object, sitting in two different categories, is called an ambivalent object, a dualizing object, etc.

### Canonical Identification

Let  $\mathcal{A}$  and  $\mathcal{S}$  be **Set**-based categories, given by the following **Set**-valued functors:

$$U: \mathcal{A} \rightarrow \mathbf{Set}, \quad V: \mathcal{S} \rightarrow \mathbf{Set} \quad (2.105)$$

Consider a contravariant adjunction with  $\eta: 1_{\mathcal{A}} \Rightarrow GF$  and  $\epsilon: 1_{\mathcal{S}} \Rightarrow FG$ :

$$\begin{array}{c} \mathcal{A} \\ \begin{array}{c} \uparrow F \\ \downarrow G \end{array} \\ \mathcal{S} \end{array} \quad (2.106)$$

such that both  $VF: \mathcal{A} \rightarrow \mathbf{Set}$  and  $UG: \mathcal{S} \rightarrow \mathbf{Set}$  are representable:

- There are an object  $*$  in  $|\mathcal{A}|$  and a natural isomorphism  $\alpha: \mathcal{A}(-, *) \xrightarrow{\cong} VF$ , with a representing element:

$$1_* \mapsto \alpha_* 1_* \in VF* \quad (2.107)$$

- There are an object  $\star$  in  $|\mathcal{S}|$  and a natural isomorphism  $\sigma: \mathcal{S}(-, \star) \xrightarrow{\cong} UG$ , with a representing element:

$$1_{\star} \mapsto \sigma_{\star} 1_{\star} \in UG\star \quad (2.108)$$

Note that  $*$  = `\ast` and  $\star$  = `\star`.

**Theorem 2.3.5.** *The representing elements  $\tilde{\alpha} := \alpha_* 1_* \in VF*$  and  $\tilde{\sigma} := \sigma_{\star} 1_{\star} \in UG\star$  induce a canonical isomorphism between two sets  $U*$  and  $V\star$ .*

*Proof.* Consider  $\eta_* \in \mathcal{A}(*, GF*)$ :

$$U\eta_* \in \mathbf{Set}(U*, UGF*) \quad (2.109)$$

Let  $x \in U*$ :

$$U\eta_* x \in UGF* \quad (2.110)$$

For  $F* \in |\mathcal{S}|$ ,

$$\sigma_{F*}: \mathcal{S}(F*, \star) \cong UGF* \quad (2.111)$$

is a bijection between two sets. Hence, for the given  $U\eta_* x \in UGF*$ , there exists a unique  $g \in \mathcal{S}(F*, \star)$  with

$$\sigma_{F*} g = U\eta_* x \quad (2.112)$$

For this  $g \in \mathcal{S}(F*, \star)$ , recalling  $G$  is a contravariant functor, we obtain:

$$UGg \in \mathbf{Set}(UG\star, UGF*). \quad (2.113)$$

Their uniqueness implies  $UGg\tilde{\sigma} = \sigma_{F*} g = U\eta_* x \in UGF*$ . Define  $\omega: U* \rightarrow V\star$ :

$$U* \xrightarrow{U\eta_*} UGF* \xrightarrow{\sigma_{F*}^{-1}} \mathcal{S}(F*, \star) \xrightarrow{V} \mathbf{Set}(VF*, V\star) \xrightarrow{-(\tilde{\alpha})} V\star \quad (2.114)$$

by

$$x \xrightarrow{U\eta_*} \sigma_{F*} g \xrightarrow{\sigma_{F*}^{-1}} g \xrightarrow{V} Vg \xrightarrow{-(\tilde{\alpha})} Vg(\tilde{\alpha}) \quad (2.115)$$

Note that  $-(\tilde{\alpha})$  is the evaluation at  $\tilde{\alpha}$ . Similarly, for  $y \in V\star$ , we define  $\omega': V\star \rightarrow U\star$  by  $\omega'y := Uf(\tilde{\sigma})$  via:

$$V\star \xrightarrow{V\epsilon_*} VFG\star \xrightarrow{\alpha_{G\star}^{-1}} \mathcal{A}(G\star, *) \xrightarrow{U} \mathbf{Set}(UG\star, U\star) \xrightarrow{-(\tilde{\sigma})} U\star \quad (2.116)$$

where  $f \in \mathcal{A}(G\star, *)$  is a unique arrow such that

$$\alpha_{G\star} f = VFf\tilde{\alpha} = V\epsilon_* y \in VFG\star. \quad (2.117)$$

We will show  $\omega \circ \omega' = 1_{V\star}$ ; the other equation follows due to symmetry. For  $y \in V\star$ , set  $x := \omega'y = Uf\tilde{\sigma}$  and consider  $\omega x = Vg\tilde{\alpha}$ . For  $\sigma_{FG\star}: \mathcal{S}(FG\star, \star) \cong UFG\star$  with  $U\eta_{G\star}\tilde{\sigma} \in UFG\star$ , let

$$s := \sigma_{FG\star}^{-1} U\eta_{G\star}\tilde{\sigma} \in \mathcal{S}(FG\star, \star). \quad (2.118)$$

Now we have the following parallels arrows in  $\mathcal{S}$ :

$$\begin{array}{ccc} F\star & \xrightarrow{Ff} & FG\star \\ & \searrow g & \nearrow s \\ & & \star \end{array} \quad (2.119)$$

Their uniqueness implies  $g = s \circ Ff$ . If we apply  $V$ :

$$\begin{array}{ccc} VF\star & \xrightarrow{VFf} & VFG\star \\ & \searrow Vg & \nearrow Vs \\ & & V\star \end{array} \quad (2.120)$$

Along with  $Vg = Vs \circ VFf$ ,  $\tilde{\alpha} \in VF\star$  becomes:

$$(Vg)\tilde{\alpha} = Vs(VFf\tilde{\alpha}) = (Vs \circ V\epsilon_*)y = V(s \circ \epsilon_*)y. \quad (2.121)$$

Then the elevator-rule for

$$\begin{array}{ccc} & & \mathcal{S}(-, \star) \\ & \epsilon & \uparrow \sigma \\ FG & \left| \right. & UG \end{array} \quad (2.122)$$

guarantees the following diagram commutative:

$$\begin{array}{ccc} UGFG\star & \xrightarrow{UG\epsilon_*} & UG\star \\ \cong \uparrow \sigma_{FG\star} & & \cong \uparrow \sigma_* \\ \mathcal{S}(FG\star, \star) & \xrightarrow[\text{--} \circ \epsilon_*]{\mathcal{S}(\epsilon_*, \star)} & \mathcal{S}(\star, \star) \end{array} \quad (2.123)$$



Evaluating at  $s \in \mathcal{S}(FG\star, \star)$ , we obtain:

$$(UG\epsilon_\star \circ \sigma_{FG\star})s = \sigma_\star \circ s \circ \epsilon_\star. \quad (2.124)$$

Now, the left-hand side becomes

$$UG\epsilon_\star(\sigma_{FG\star}s) = (UG\epsilon_\star \circ U\eta_{G\star})\tilde{\sigma} = U(G\epsilon \circ \eta_G)_\star \tilde{\sigma} \quad (2.125)$$

According to a zig-zag identity, see Definition 1.3.7, we obtain:

$$(UG\epsilon_\star \circ \sigma_{FG\star})s = \tilde{\sigma} = \sigma_\star 1_\star. \quad (2.126)$$

Hence, we have  $\sigma_\star \circ s \circ \epsilon_\star = \sigma_\star 1_\star$ . Since  $\sigma_\star$  is an isomorphism, we conclude  $s \circ \epsilon_\star = 1_\star$ . Hence

$$VG\tilde{\alpha} = V1_\star y = 1_{V\star} y = y. \quad (2.127)$$

Recalling  $\omega\omega'y = VG\tilde{\alpha}$ , we obtain the desired result  $\omega\omega' = 1_{V\star}$ .  $\blacksquare$

*Remark 17 (Lift).* This canonical identification  $\omega: U\star \cong V\star$  can be seen as an object sitting in two different categories, namely  $\star \in |\mathcal{A}|$  and  $\star \in |\mathcal{S}|$ . Moreover, the contravariant functor  $G: \mathcal{S} \rightarrow \mathcal{A}$  is a lift of the representable functor  $\mathcal{S}(-, \star): \mathcal{S} \rightarrow \mathbf{Set}$  through  $U$  via  $\sigma: \mathcal{S}(-, \star): \mathcal{S} \xrightarrow{\cong} UG$ :

$$\begin{array}{ccc} & \mathcal{A} & \\ G \nearrow & \downarrow U & \\ \mathcal{S} & \xrightarrow{\mathcal{S}(-, \star)} \mathbf{Set} & \end{array} \quad \sigma: \mathcal{S}(-, \star) \xrightarrow{\cong} UG. \quad (2.128)$$

Similarly,  $F$  is a lift of  $\mathcal{A}(-, \star)$  through  $V$ :

$$\begin{array}{ccc} & \mathcal{S} & \\ F \nearrow & \downarrow V & \\ \mathcal{A} & \xrightarrow{\mathcal{A}(-, \star)} \mathbf{Set} & \end{array} \quad \alpha: \mathcal{A}(-, \star) \xrightarrow{\cong} VF. \quad (2.129)$$

*Remark 18 (Ambivalent Objects).* For two  $\mathbf{Set}$ -based categories  $\mathcal{A}$  and  $\mathcal{S}$ , an ambivalent object is a set  $\bullet$  that can be furnished in two ways to produce an object in  $|\mathcal{A}|$  or an object in  $|\mathcal{S}|$ . As observed,  $2 = \{0, 1\}$  can be seen as a post  $(2, \leq)$  or a topological space  $\mathbf{2}' = (2, \{\emptyset, \{1\}, 2\})$ .

For each  $A \in |\mathcal{A}|$  and  $S \in |\mathcal{S}|$ ,

$$\mathcal{A}(A, \bullet), \quad \mathcal{S}(S, \bullet) \quad (2.130)$$

are both sets. Hence, we have the corresponding contravariant hom-functors:

$$\begin{aligned} \mathcal{A}(-, \bullet): \mathcal{A} &\rightarrow \mathbf{Set} \\ \mathcal{S}(-, \bullet): \mathcal{S} &\rightarrow \mathbf{Set} \end{aligned} \quad (2.131)$$

Suppose the “nature” of  $\bullet$  enables us to enrich:

$$\begin{aligned} \mathcal{A}(-, \bullet): \mathcal{A} &\rightarrow \mathcal{S} \\ \mathcal{S}(-, \bullet): \mathcal{S} &\rightarrow \mathcal{A} \end{aligned} \quad (2.132)$$

This step is not routine ... When the construction works these enrichments are compatible with composition, to give a pair of contravariant functors ... between the categories. [Sim11]

We, then, have the following natural bijection:

$$\mathcal{S}(S, \mathcal{A}(A, \bullet)) \cong \mathcal{A}(A, \mathcal{S}(S, \bullet)) \quad (2.133)$$

where each  $f \in \mathcal{A}(A, \mathcal{S}(S, \bullet))$  is mapped to  $\phi \in \mathcal{S}(S, \mathcal{A}(A, \bullet))$  defined by:

$$(\phi s)a := (fa)s. \quad (2.134)$$

It follows that they form a contravariant adjunction:

$$\mathcal{A}(-, \bullet) \begin{pmatrix} \mathcal{A} \\ \dashv \\ \mathcal{S} \end{pmatrix} \mathcal{S}(-, \bullet) \quad (2.135)$$

The corresponding unit and counit are both “evaluations:”

- The unit  $\eta: 1_{\mathcal{A}} \Rightarrow \mathcal{S}(\mathcal{A}(-, \bullet), \bullet)$

Let  $A \in |\mathcal{A}|$ :

$$\eta_A: A \rightarrow \mathcal{S}(\mathcal{A}(A, \bullet), \bullet) \quad (2.136)$$

For each  $a \in A$  and  $p \in \mathcal{A}(A, \bullet)$ ,

$$(\eta_A a) p = pa. \quad (2.137)$$

- The counit  $\epsilon: 1_{\mathcal{S}} \Rightarrow \mathcal{A}(\mathcal{S}(-, \bullet), \bullet)$

Let  $S \in |\mathcal{S}|$ :

$$\epsilon_S: S \rightarrow \mathcal{A}(\mathcal{S}(S, \bullet), \bullet) \quad (2.138)$$

For each  $s \in S$  and  $f \in \mathcal{S}(S, \bullet)$ ,

$$(\epsilon_S s) f = fs. \quad (2.139)$$

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