Group Theory 101
for Homology Theory

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Contents

0   Definitions and Basic Properties

This note explores basic group theory, following [1, 3, 2].

0.1   Groups – Multiplicative Groups

Definition 0.1 (Groups). A group is given by

$$\left(G, \circ, 1, (\_)^{-1}\right), \tag{1}$$

where

- the first entry G is a set and $\circ$ is a binary product

$$\circ \colon G \times G \to G; (g_2, g_1) \mapsto g_2 \circ g_1. \tag{2}$$

- The triple $(G, \circ, 1)$ forms a monoid. Namely, G is a non-empty set, $\circ$ is an associative binary product, and $1 \in G$ is a multiplicative identity such that

$$g \circ 1 = g = g \circ 1, \tag{3}$$

  for each $g \in G$. Note that the associativity is expressed by

$$g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1 \tag{4}$$

  for any $g_1, g_2, g_3 \in G$.

- The unary operation $(\_)^{-1} \colon G \to G$ returns the multiplicative inverse g^{-1} of a given $g \in G$ such that

$$g^{-1} \circ g = 1 = g \circ g^{-1}. \tag{5}$$

Lemma 0.1. *Let $\left(G, \circ, 1, (\_)^{-1}\right)$ be a group.*

- *If $e \in G$ satisfies*

$$g \circ e = g = e \circ g \tag{6}$$

  for each $g \in G$, then $e = 1$. That is, the identity element is unique.

- *For each $g \in G$, the inverse g^{-1} is unique.*

- *For each $g \in G$, $\left(g^{-1}\right)^{-1} = g$.*

Proof. Since $1 \in G$ is an multiplicative identity, $1 \circ e = e = e \circ 1$; $e \in G$ satisfies the same properties, we conclude $e = 1 \circ e = 1$. Let $g \in G$. Suppose $g' \in G$ satisfies $g' \circ g = g = g \circ g'$. Applying g^{-1}, we obtain $g' = g^{-1}$. We, then, have $g^{-1} \circ g = 1 = g^{-1} \circ g$, showing $\left(g^{-1}\right)^{-1} = g$. $\qquad\square$

Definition 0.2 (Subgroups). A subset $H \subset G$ of a group $\left(G, \circ, 1, (\_)^{-1}\right)$ is called a subgroup of G iff H forms a group and H is closed under group operations:

- For any $h_1, h_2 \in H$, $h_2 \circ h_1 \in H$.

- The identity $1 \in G$ is in H, $1 \in H$.

- For any $h \in H$, the inverse is in H, $h^{-1} \in H$.

A trivial subgroup is the singleton $\{1\} \subset G$; another example is G itself. We denote a subgroup H of G by $H < G$.

Lemma 0.2 (Subgroup-Test). *Let $\left(G, \circ, 1, (\_)^{-1}\right)$ be a group. A non-empty subset $\emptyset \neq H \subset G$ is a subgroup iff $h_2 \circ h_1{}^{-1} \in H$ for any $h_1, h_2 \in H$.*

Proof. ($\Rightarrow$) If $H < G$ and $h_1, h_2 \in H$, then $h_1{}^{-1} \in H$ and hence $h_2 \circ h_1{}^{-1} \in H$.
  ($\Leftarrow$) There is at least one element in H. Select $h \in H$. Then $h \circ h^{-1} = 1 \in H$. For any $k \in H$, $k^{-1} = 1 \circ k^{-1} \in H$. Suppose $h_1, h_2 \in H$. Then $h_2{}^{-1} \in H$, and hence $h_1 h_2 = h_1 \left(h_2{}^{-1}\right)^{-1} \in H$. Since the underlying $\circ$ is an associative binary product of G, $\left(H, \circ, 1, (\_)^{-1}\right)$ forms an group. $\qquad\square$

From this point, the group operation can be written without the symbol $\circ$. Namely, for $g_1, g_2 \in G$, we write $g_2 g_1$ instead of $g_2 \circ g_1$.

Definition 0.3 (Group Homomorphisms)**.** Let G and H be groups. A map $\theta\colon G \to H$ is called a group homomorphism iff $\theta\left(g_2 g_1\right) = \left(\theta g_2\right)\left(\theta g_1\right)$.

- One to one

  If a group homomorphism $\theta\colon G \to H$ is injective:

  $$\theta g_1 = \theta g_2 \Rightarrow g_1 = g_2 \tag{7}$$

  for $g_1, g_2 \in G$, then θ is called a monomorphism, in short a mono.

- Onto

  If a group homomorphism $\theta\colon G \to H$ is subjective:

  $$\theta G = H, \tag{8}$$

  then θ is called an epimorphism, in short an epi.

A group homomorphism is called an isomorphism iff it is both monic and epic.

Lemma 0.3. *Any group homomorphism preserves the identity and inverses.*

Proof. Let $\theta\colon G \to H$ be a group homomorphism. Since $1_G \circ 1_G = 1_G$, we have $\left(\theta 1_G\right) \circ \left(\theta 1_G\right) = \theta 1_G$. Applying the inverse $\left(\theta 1_G\right)^{-1}$ of $\theta 1_G \in H$, we obtain $\theta 1_G = 1_H$. Let $g \in G$ and consider the inverse $g^{-1} \in G$. Since $g^{-1} \circ g = 1_G = g \circ g^{-1}$, if we apply θ, we have $\left(\theta g^{-1}\right) \circ \left(\theta g\right) = 1_H = \left(\theta g\right) \circ \left(\theta g^{-1}\right)$. Since the inverse is unique by Lemma 0.1, $\theta g^{-1} = \left(\theta g\right)^{-1}$. $\qquad\square$

Lemma 0.4. *A group homomorphism $\theta\colon G \to H$ is monic iff the kernel is singleton, $\theta^{\leftarrow} 1_H = \{1_G\}$.*

Proof. ($\Rightarrow$) Recall the very definition:

$$\theta^{\leftarrow} 1_H := \left\{g \in G \mid \theta g = 1_H\right\}. \tag{9}$$

The kernel is non-empty, $1_G \in \theta^{\leftarrow} 1_H$, since $\theta 1_G = 1_H$. Let $g \in \theta^{\leftarrow} 1_H$. Then $\theta g = 1_H = \theta 1_G$. Since θ is injective, $g = 1_G$, and hence $\theta^{\leftarrow} 1_H = \{1_G\}$.

($\Leftarrow$) Conversely, suppose $\theta^{\leftarrow} 1_H = \{1_G\}$. Let $g_1, g_2 \in G$ and assume $\theta g_1 = \theta g_2$ for $g_1, g_2 \in G$. Then $1_H = \left(\theta g_2\right)\left(\theta g_1\right)^{-1} = \left(\theta g_2\right)\left(\theta g_1^{-1}\right) = \theta\left(g_2 g_1^{-1}\right)$. Hence, $g_2 g_1^{-1} \in \theta^{\leftarrow} 1_H = \{1_G\}$, and we have $g_1 = g_2$ since $g_2 g_1^{-1} = 1_G$. $\qquad\square$

Theorem 0.5. *Both image and kernel of a group homomorphism are subgroups.*

Proof. Let $\theta\colon G \to H$ be a group homomorphism. Let $h_1, h_2 \in \theta G$. There are $g_1, g_2 \in G$ such that $h_1 = \theta g_1$ and $h_2 = \theta g_2$. Since $g_2 g_1^{-1} \in G$,

$$h_2 h_1^{-1} = \theta g_2 \left(\theta g_1\right)^{-1} = \theta\left(g_2 g_1^{-1}\right) \in \theta G. \tag{10}$$

By Lemma 0.2, $\theta G < H$.

Let $g_1, g_2 \in \theta^{\leftarrow}1$. Then $\theta g_1 = \theta g_2$ and, hence

$$\theta\left(g_2 g_1^{-1}\right) = (\theta g_2)(\theta g_1)^{-1} = 1. \tag{11}$$

By Lemma 0.2, $\theta^{\leftarrow}1 < G$. $\qquad\square$

Corollary 0.5.1. *Let $\theta \colon G \to H$ be a group homomorphism. Then its corestriction $\theta \colon G \to \theta G$ is an epimorphism.*

Proof. As shown above, $\theta G < H$. Hence, corestriction $\theta \colon G \to \theta G$ is a group homomorphism. By the very definition, the image θG is the codomain of the corestriction $\theta \colon G \to \theta G$. Thus, $\theta \colon G \to \theta G$ is surjective. $\qquad\square$

0.2 Abelian Groups – Additive Groups

Definition 0.4 (Abelian Groups)**.** Let $\left(G, \circ, 1, (\_)^{-1}\right)$ be a group. Two elements $a, b \in G$ are commutative iff $a \circ b = b \circ a$; G is called an abelian group iff any pair is commutative. We use $+$ sign for the commutative binary operator, and 0 for the additive identity, and $-g$ for the additive inverse of $g \in G$.

Definition 0.5 (Free Group)**.** An abelian group $(A, +, 0, -(\_))$ is called free iff there is $B \subset A$ such that every element $a \in A$ has a unique representation

$$a = \sum_{b \in B} n_b b, \tag{12}$$

where $\{n_b \in \mathbb{Z} \mid b \in B\}$ are all equal to zero but finitely many. In other words, the "sequence" is eventually zero. Such a subset $B \subset A$, which generates the given abelian group A, is called a basis for A.

Remark (Construction)*.* Let S be a set. We will construct a free abelian group with basis S.

Let F_S be a set of functions from S to integers $\mathbb{Z}$ such that $f \in F_S$ iff

$$fs = 0 \text{ for all but finitely many } s \in S. \tag{13}$$

Then the zero map is in F_S, $0 \in F_S$. For any $f, g \in F_S$, $f + g$ is defined componentwisely, $(f + g)s = fs + gs$ for each $s \in S$. The additive inverse of f is also defined componentwisely, $(-f)s = -(fs)$. By construction, the set of characteristic functions forms a basis for F_S:

$$B := \left\{\chi_{\{t\}} \mid t \in S\right\} \cong S, \tag{14}$$

where

$$\chi_{\{t\}}s := \begin{cases} 0 & s \neq t \\ 1 & s = t. \end{cases} \tag{15}$$

For each $f \in F_S$, let $I_f \subset S$ be a finite subset such that $fs \neq 0$ iff $s \in I_f$. Then any $f \in F_S$ is written as $f = \sum_{t \in I_f}(ft)\chi_{\{t\}}$.

We denote O for the abelian group with empty basis; since the empty sum is the additive identity, the unique element in O is 0:

$$O = \left(\{0\}, +, 0, -(\_)\right). \tag{16}$$

Such a singleton group, the subgroup consisting of the identity alone in any group, is also called the trivial, null group. Hence, Lemma 0.4 becomes: a group homomorphism is monic iff its kernel is trivial.

Remark (Extension "by Linearity"). If an abelian group $(A, +, 0, -(\_))$ is free with basis B and C is an abelian group, then any map $\theta \colon B \to C$ can be uniquely extended to an abelian group homomorphism $\theta \colon A \to C$ by linearity, namely for each $a \in A$ with its unique representation $\sum_{b \in B} n_b b$, $\theta a \coloneqq \sum_{b \in B} n_b(\theta b)$.

1 Normal Subgroups and Quotient Groups

1.1 Normal Subgroups

Definition 1.1 (Normal Subgroups)**.** Let G be a group. A subgroup $N < G$ is called a normal subgroup iff $gNg^{-1} \subset N$ for each $g \in G$, where

$$gNg^{-1} \coloneqq \left\{ gng^{-1} \mid n \in N \right\}. \tag{17}$$

We denote $N \lhd G$ for a normal subgroup H of G.

Lemma 1.1. *The kernel of a group homomorphism is a normal subgroup.*

Proof. Let $\theta \colon G \to H$ be a group homomorphism. By Theorem 0.5, $\theta^{\leftarrow}1 < G$. Let $g \in G$ and $k \in \theta^{\leftarrow}1$. Then $gkg^{-1} \in \theta^{\leftarrow}1$ since

$$\theta\left(gkg^{-1}\right) = (\theta g)\, 1\, (\theta g)^{-1} = 1. \tag{18}$$

Therefore, $g\left(\theta^{\leftarrow}1\right)g^{-1} \subset \theta^{\leftarrow}1$. $\qquad\square$

Lemma 1.2. *Let G be a group. If $N \lhd G$, $M < G$, and $N \subset M$, namely*

$$\overset{\lhd}{\underset{<}{N \hookrightarrow M \hookrightarrow G}}\;,\; \text{then } N \lhd M \text{ holds.}$$

Proof. Since $N \lhd G$, i.e., N is closed under group operations in G, so is in M, $N < M$. Hence, it suffices to show its normality. Let $m \in M$ and $n \in N$. Since $N \lhd G$ and $M < G$, $m \in G$ and hence, $mnm^{-1} \in N$. That is, $mNm^{-1} \subset N$. $\quad\square$

Theorem 1.3. *Let $N \lhd G$ be a normal subgroup. If we define $\sim$ by*

$$g_1 \sim g_2 \coloneqq\!\Leftrightarrow g_2 g_1^{-1} \in N \tag{19}$$

then $\sim$ is an equivalence relation on G relative to N.

Remark. For each $g \in G$, let

$$[g] := \{g' \in G \mid g' \sim g\} \tag{20}$$

be the set of equivalent elements represented by g. Let $G/\sim = \{[g] \mid g \in G\}$ denote the set of equivalent classes relative to $N \lhd G$.

Proof. Let $g \in G$. Since $gg^{-1} = 1 \in N$, $g \sim g$. Hence, $\sim$ is reflexive.

Suppose $g_1 \sim g_2$ for $g_1, g_2 \in G$. Then $n := g_2 g_1^{-1} \in N$. Since N is a subgroup of G, $n^{-1} \in N$, namely $n^{-1} = g_1 g_2^{-1} \in N$. Hence, $g_2 \sim g_1$, and $\sim$ is symmetric.

Finally, suppose $g_1 \sim g_2$ and $g_2 \sim g_3$. Then $g_3 g_1^{-1} \in K$, since $g_2 g_1^{-1} \in N$, $g_3 g_2^{-1} \in N$ and $g_3 g_1^{-1} = g_3 g_2^{-1} g_3 g_2^{-1}$. Therefore, $\sim$ is transitive. $\qquad\square$

Corollary 1.3.1. *$G/\sim$ with respect to $G \lhd G$ is a singleton.*

Proof. All elements are equivalent to $1 \in G$ since $g = g 1^{-1} \in G$ for each $g \in G$. Hence $[1] = \{g \in G \mid g \sim 1\} = G$ is the unique element. $\qquad\square$

1.2 Quotient Groups

Lemma 1.4. *Let G be a group and $N \lhd G$ be a normal subgroup.*

- *For each $g \in G$, $gN = Ng = [g]$.*

- *For $g_1, g_2 \in G$, $[g_1][g_2] = [g_1 g_2]$.*

Remark. Let G/N denote the set $G/\sim = \{[g] \mid g \in G\}$ of equivalent classes with respect to $N \lhd G$ with the binary product $[g_1][g_2] = [g_1 g_2]$ for $[g_1], [g_2] \in G/N$.

Proof. Let $g \in G$. For each $n \in N$, since $N \lhd G$, $k := gng^{-1} \in N$. That is, $gn = kg \in Ng$, and hence $gN \subset Ng$. Conversely, for each $n \in N$, as $g^{-1} \in G$, $h := g^{-1}ng \in N$. It follows $Ng \subset gN$. Therefore, $gN = Ng$ holds.

Let $g \in G$ and consider $[g] \in G/\sim$. For each $g' \in [g]$, there exists $n \in N$ with $n = g' g^{-1}$. Then, $[g] \subset Ng$ since for each $g' \in [g]$, $g' = ng \in Ng$. Conversely, consider $gk \in gN$. As shown above, $gN = Ng$, there must be some $h \in N$ such that $gk = hg$. Therefore, $h = (gk)g^{-1} \in N$, showing $gk \sim g$, and $gk \in [g]$. Hence, we conclude $[g] = Ng$

Finally, consider the binary operation. Let $g_1, g_2 \in G$.

$$\begin{aligned}
[g_2][g_1] &= \{h_2 h_1 \mid h_2 \in Ng_2 \wedge h_1 \in Ng_2\} \\
&= \{n_2 g_2 n_1 g_1 \mid n_1, n_2 \in N\} \\
&= \left\{ n_2 \left(g_2 n_1 g_2^{-1} \right) g_2 g_1 \mid n_1, n_2 \in N \right\}
\end{aligned} \tag{21}$$

Since $N \lhd G$, $\left(g_2 n_1 g_2^{-1} \right) \in N$ for each $g_2 \in G$ and $n_1 \in N$. Hence, we obtain

$$[g_2][g_1] = \{n g_2 g_1 \mid n \in N\} = N(g_2 g_1) = [g_2 g_1]. \tag{22}$$

$\qquad\square$

Theorem 1.5 (Quotient Groups)**.** *Let G be a group and $N \triangleleft G$ be a normal subgroup. Then G/N forms a group:*

$$\left(G/N, \circ, N, [(\_)^{-1}]\right), \tag{23}$$

where the binary product $\circ$ is defined by $[g_1] \circ [g_2] = [g_1 g_2]$ for $[g_1], [g_2] \in G/N$.

Proof. Since the binary product is essentially the binary product of G, it suffices to show the identity and inverses.

- $N = 1_{G/N}$

  Consider $[g]$ for $g \in G$. By Lemma 1.4, $[g] = gN$, and hence

  $$[g]N = gNN = gN = [g]. \tag{24}$$

  Similarly, one can show that $N[g] = [g]$.

- $[g]^{-1} = [g^{-1}]$

  Consider $[g]$ for $g \in G$. By Lemma 1.4,

  $$[g]\left[g^{-1}\right] = \left[gg^{-1}\right] = [1] = N = \left[g^{-1}\right][g]. \tag{25}$$

  Since the inverse is unique, by Theorem 0.1, $\left[g^{-1}\right]$ is the desired inverse of $[g]$.

We call the group G/N for $N \triangleleft G$ the quotient group of G modulo N. $\qquad \square$

1.3   Group Homomorphisms and Quotient Groups

Theorem 1.6 (Canonical Projection)**.** *Let G and H be groups. For a group homomorphism $\theta \colon G \to H$, as shown in Lemma 1.1, the kernel forms a normal subgroup, $\theta^{\leftarrow}1 \triangleleft G$. Conversely, if $N \triangleleft G$, then the canonical projection*

$$\pi \colon G \to G/N; g \mapsto [g] \tag{26}$$

is an epimorphism with $\pi^{\leftarrow}1 = N$.

Proof. Suppose $N \triangleleft G$. For any $[g] \in G/N$, if we choose $g' \in [g]$, $\pi g' = [g'] = [g]$ since $g' \sim g$. Hence, the canonical projection π is surjective. For $g_1, g_2 \in G$, it follows

$$\pi(g_2 g_1) = [g_2 g_1] = [g_2][g_1] = (\pi g_2)(\pi g_1) \tag{27}$$

Hence, π is a group homomorphism. Finally, we have $\pi^{\leftarrow}1 = N$ since

$$\{g \in G \mid [g] = N = N1\} = \{g \in G \mid g \sim 1\} = \left\{g \in G \mid g = g1^{-1} \in N\right\}. \tag{28}$$

$\qquad \square$

Remark. From now on, for $N \triangleleft G$, $G \to G/N$ always denotes the canonical epimorphism $\pi \colon G \to G/N; g \mapsto [g]$, unless otherwise stated explicitly.

Theorem 1.7. *Let G and H be groups, and $\theta \colon G \to H$ be a group homomorphism. Suppose $N \lhd G$ and $N \subset \theta^{\leftarrow}1$. Then, there exists a unique mediating group homomorphism $\bar{\theta} \colon G/N \to H$ such that $\bar{\theta}[g] = \theta g$ for each $g \in G$. That is, the following diagram is commutative:*

$$
\begin{array}{ccc}
G & \xrightarrow{\ \theta\ } & H \\
{\scriptstyle \pi}\downarrow & \nearrow & \\
G/N & {\scriptstyle \exists!\bar{\theta}} &
\end{array}
\qquad \bar{\theta} \circ \pi = \theta. \tag{29}
$$

Such a unique mediator satisfies $\bar{\theta}(G/N) = \theta G$ and $\overrightarrow{\bar{\theta}}^{\leftarrow}1 = (\theta^{\leftarrow}1)/N$.

Proof. Let $g, g' \in G$. If $g \sim g'$ relative to $N \lhd G$, there is some $n \in N$ such that $n := g'g^{-1}$ and

$$
\theta g' = \theta\,(ng) = (\theta n)\,(\theta g) = 1\theta g = \theta g, \tag{30}
$$

since $n \in N \subset \theta^{\leftarrow}1$. Thus $\bar{\theta} \colon G/N \to H; [g] \to \theta g$ is a well-defined map among the corresponding sets. Since $\bar{\theta}$ is fully determined in terms of the given group homomorphism θ, it must unique. Indeed, if $\underline{\theta} \colon G/N \to H$ satisfies $\underline{\theta} \circ \pi = \theta$, then for each $[g] \in G/N$, recalling there is some $g \in G$ such that $\pi g - gN = [g]$,

$$
\underline{\theta}[g] = \underline{\theta}\,(\pi g) = (\underline{\theta} \circ \pi)\,g = \theta g = \bar{\theta}[g]. \tag{31}
$$

Hence, $\underline{\theta} = \bar{\theta}$.

The induced map $\bar{\theta} \colon G/N \to H$ is a group homomorphism, since θ is a group homomorphism:

$$
\bar{\theta}\,([g_2][g_1]) = \bar{\theta}\,[g_2 g_1] = \theta\,(g_2 g_1) = (\theta g_2)\,(\theta g_1) = \left(\bar{\theta}[g_2]\right)\left(\bar{\theta}[g_1]\right) \tag{32}
$$

for $[g_1], [g_2] \in G/N$. The images coincide:

$$
\theta G = \left\{ h \in H \mid \exists g \in G : h = \theta g = \bar{\theta}[g] \right\} = \bar{\theta}(G/N). \tag{33}
$$

Let $K := \theta^{\leftarrow}1 \lhd G$. Since $N < K \lhd G$ and $N \lhd G$, we may apply Lemma 1.2, $N \lhd K$. Hence, $K/N = (\theta^{\leftarrow}1)/N$ is a quotient group. For any $g \in G, [g] = gN \in K/N$ iff $\bar{\theta}[g] = \theta g = 1$ iff $[g] \in \overrightarrow{\bar{\theta}}^{\leftarrow}1$. That is, $(\theta^{\leftarrow}1)/N = K/N = \overrightarrow{\bar{\theta}}^{\leftarrow}1$. $\qquad\square$

Corollary 1.7.1. *For a group homomorphism $\theta \colon G \to H$, if N is the kernel, namely $N = \theta^{\leftarrow}1$, then the unique induced meditator $\bar{\theta} \colon G/N \to H$ is monic. Moreover, θ is factored into the following group homomorphisms:*

$$
\begin{array}{ccc}
G & \xrightarrow{\ \theta\ } & H \\
{\scriptstyle \pi}\downarrow & & \uparrow{\scriptstyle j} \\
G/N & \xrightarrow[\ \bar{\theta}\ ]{} & \theta G
\end{array}
\qquad \theta = j \circ \bar{\theta} \circ \pi, \tag{34}
$$

where j is the inclusion $\theta G \subset H$.

Proof. Suppose $\overline{\theta}[g_1] = \overline{\theta}[g_2]$, namely, $\theta g_1 = \theta g_2$ for $g_1, g_2 \in G$. Then $\theta\left(g_2 g_1{}^{-1}\right) = 1$ holds. Therefore, $g_2 \in [g_1] = g_1 N$ since $g_2 g_1{}^{-1} \in \theta^{\leftarrow}1 = N$. Hence, $g_1 \sim g_2$ with respect to $N \lhd G$.

For each $g \in G$,

$$\left(j \circ \overline{\theta} \circ \pi\right) g = \left(j \circ \overline{\theta}\right) gN = j\left(\theta g\right) = \theta g. \tag{35}$$

Hence, $j \circ \overline{\theta} \circ \pi = \theta$. $\qquad\square$

Corollary 1.7.2. *Let G and H be groups, and $\theta \colon G \to H$ be a group homomorphism. Suppose $N \lhd G$ and $N \subset \theta^{\leftarrow}1$. Then the induced group homomorphism $\overline{\theta} \colon G/N \to H$ is an isomorphism iff $\theta \colon G \to H$ is epic and $N = \theta^{\leftarrow}1$.*

Proof. $(\Rightarrow)$ Suppose $\overline{\theta}$ is an isomorphism. Then $\overline{\theta}$ is epic, so is θ since $\theta G = \overline{\theta}(G/N) = H$. By hypothesis, $N \subset \theta^{\leftarrow}1$; to show the other inclusion, let $k \in \theta^{\leftarrow}1$. Then $\overline{\theta}[k] = \theta k = 1$. As shown above, $\overline{\theta}$ is monic. Hence, $\overline{\theta}^{\leftarrow}1 = \left\{1_{G/N}\right\}$ by Lemma 0.4, where $1_{G/N} = N$ by Theorem 1.5. Therefore, $[k] = N$, i.e., $k \in N$.

$(\Leftarrow)$ Suppose θ is epic and $N = \theta^{\leftarrow}1$. By Corollary 1.3, the induced $\overline{\theta}$ is monic. Since $\overline{\theta}(G/N) = \theta G = H$ is the case, $\overline{\theta}$ is epic. $\qquad\square$

Remark. Recalling $\left(\theta^{\leftarrow}1\right)/N = \overline{\theta}^{\leftarrow}1$, and by hypothesis $N = \theta^{\leftarrow}1$, the kernel is a singleton by Corollary 1.3.1, $\overline{\theta}^{\leftarrow}1 = \{N\} = \left\{1_{G/N}\right\}$.

Theorem 1.8 (First Isomorphic Theorem)**.** *A group homomorphism $\theta \colon G \to H$ induces an isomorphism between $G/\theta^{\leftarrow}1$ and θG.*

Proof. The corestriction $\theta \colon G \to \theta G$ is an epi. Thus, the induced group homomorphism via Corollary 1.7.2:

$$\overline{\theta} \colon G/N \to \theta G \tag{36}$$

is an isomorphism, where $N := \theta^{\leftarrow}1 \lhd G$. $\qquad\square$

1.4 Exact Sequences

Consider a group G and a normal subgroup $N \lhd G$. We have two canonical maps: the inclusion $\iota \colon N \hookrightarrow G$ and the projection $\pi \colon G \to G/N$. The inclusion ι is a monomorphism, since its image is $N = \iota N < G$ and, for $n_1, n_2 \in N$, if $\iota n_1 = \iota n_2$, $n_1 = \iota n_1 = \iota n_2 = n_2$. The canonical projection $\pi \colon G \to G/N$ is epic by Theorem 1.6 such that $\pi^{\leftarrow}1 = N$. Since the image of the inclusion ι is N, we have $\iota N = \pi^{\leftarrow}1$. This alignment between the image and the kernel is the foundation for the following definition:

Definition 1.2 (Exact Sequences)**.** A pair of adjacent group homomorphisms $F \xrightarrow{\theta} G \xrightarrow{\varphi} H$ is called exact iff $\theta F = \varphi^{\leftarrow}1$. For a group G and a normal subgroup $N \lhd G$, $N \xrightarrow{\lhd} G \xrightarrow{\pi} G/N$ is exact. A sequence of group homomorphisms

is called exact iff every adjacent pair of homomorphisms is exact. An exact sequence of the following form is called a short exact sequence:

$$O \longrightarrow F \xrightarrow{\;\theta\;} G \xrightarrow{\;\varphi\;} H \longrightarrow O \tag{37}$$

where $O \to F$ is essentially an inclusion map $\{1_F\} \subset F$ and $H \to O$ is a constant map toward a singleton group.

Lemma 1.9. *Let $\theta\colon F \to G$ be a group homomorphism.*

- *A sequence of group homomorphisms $O \to F \xrightarrow{\theta} G$ is exact iff θ is monic.*

- *A sequence of group homomorphisms $F \xrightarrow{\theta} G \to O$ is exact iff θ is epic.*

- *A sequence of group homomorphisms $O \to F \xrightarrow{\theta} G \to O$ is exact iff θ is isomorphic.*

Consequently, a sequence of group homomorphisms $O \to F \xrightarrow{\theta} G \xrightarrow{\varphi} H \to O$ is a short exact sequence iff θ is monic and φ is epic.

Proof. It suffices to show the first two cases.

Suppose $O \to F \xrightarrow{\theta} G$ is exact. Then $\{1_F\} = \theta^{\leftarrow} 1_G$. By Lemma 0.4, θ is monic. Conversely, if θ is monic, its kernel is the trivial group $\{1_F\}$. Since the image of $O \to F$ is $\{1_F\}$, $O \to F \xrightarrow{\theta} G$ is exact.

If $F \xrightarrow{\theta} G \to O$ is exact, then $\theta F = G$ since $G \to O$ is a constant map. Conversely, suppose θ is epic. Since $G \to O$ maps every element in G toward the unique identity in O, its kernel is the domain G. Hence, $F \xrightarrow{\theta} G \to O$ is exact. $\square$

1.5   Isomorphism Theorems

There are several important isomorphism theorems in group theory. One example is Theorem 1.8, which implies the followings:

Theorem 1.10. *If a sequence of group homomorphisms $O \to F \to G \to H \to O$ is exact, then H is isomorphic to G/F.*

Proof. Let $O \to F \xrightarrow{\theta} G \xrightarrow{\varphi} H \to O$. As shown in 0.5.1, the corestriction $\theta\colon F \to \theta F$ is epic, and hence isomorphic, since θ is monic by Lemma 1.9. Thus, we may identify $F \cong \theta F \subset G$ as an inclusion $F \hookrightarrow G$. Since $\varphi\colon G \to H$ is epic, $\varphi G = H$. Moreover, since $F \xrightarrow{\theta} G \xrightarrow{\varphi} H$ is exact, $\varphi^{\leftarrow} 1 = \theta F$ holds. Hence, we can identify F as a normal subgroup of G through $F \cong \theta F = \varphi^{\leftarrow} 1 \triangleleft G$. Theorem 1.8 implies $G/(\varphi^{\leftarrow} 1) \cong \varphi G$, and

$$G/F \cong G/(\theta F) = G/(\varphi^{\leftarrow} 1) \cong \varphi G = H. \tag{38}$$

Thus, the given short exact sequence is equivalent to the following canonical short exact sequence induced by $F \overset{\triangleleft}{\hookrightarrow} G$:

$$O \longrightarrow F \overset{\triangleleft}{\hookrightarrow} G \overset{\pi}{\longrightarrow} G/F \longrightarrow O \tag{39}$$

where $\pi \colon G \to G/F$ is the canonical projection. More precisely,

$$
\begin{array}{ccccccccc}
O & \longrightarrow & F & \overset{\theta}{\longrightarrow} & G & \overset{\varphi}{\longrightarrow} & H & \longrightarrow & O \\
& & \| & & \| & & \downarrow{\scriptstyle\cong} & & \\
O & \longrightarrow & F & \overset{\triangleleft}{\hookrightarrow} & G & \overset{\pi}{\longrightarrow} & G/F & \longrightarrow & O
\end{array}
\tag{40}
$$

is a commutative diagram. $\qquad\square$

Definition 1.3 (Normalizer)**.** Let G be a group and $S \subset G$ be a subset. We call

$$N_S := \left\{ g \in G \mid gSg^{-1} = S \right\}. \tag{41}$$

the normalizer of S in G.

Lemma 1.11. *For a group G and its subset $S \subset G$, the normalizer N_S is a subgroup of G.*

Proof. Let $n_1, n_2 \in N_S$, and consider $n_2 n_1{}^{-1} \in G$. Let $s \in S$. Since $n_1 \in G$, so is $n_1{}^{-1} \in G$ and $n_1{}^{-1} S n_1 = S$. It follows $n_1{}^{-1} \in N_S$ and $n_1{}^{-1} s n_1 \in S$. Thus, $n_2 \left(n_1{}^{-1} s n_1 \right) n_2{}^{-1} \in S$ holds. Since $\left(n_2 n_1{}^{-1} \right)^{-1} = n_1 n_2{}^{-1}$, we have $\left(n_2 n_1{}^{-1} \right) s \left(n_2 n_1{}^{-1} \right)^{-1} \in S$. Hence, $n_2 n_1{}^{-1} \in N_S$. By Lemma 0.2, $N_S < G$. $\qquad\square$

Theorem 1.12. *Let G be a group, and both $H < G$ and $K < G$ be two subgroups. Assume $H \subset N_K$. Then $H \cap K \triangleleft H$, and $HK = KH < G$. Moreover, there is an epimorphism*

$$p \colon H \to HK/K \tag{42}$$

whose kernel is $H \cap K$. Consequently, we have a canonical isomorphism $H/(H \cap K) \cong HK/K$.

Proof. Recalling $N_K := \left\{ g \in G \mid gKg^{-1} = K \right\}$, $H \subset N_K$ implies any $h \in H$ satisfies $hKh^{-1} = K$, and hence $hK = Kh$.

Since $H < G$ and $K < G$, $H \cap K \subset G$ is closed under group operations in both H and K. Hence, $H \cap K < H$. So it suffices to show the normality. Let $k \in H \cap K$ and $h \in H$. Since $k \in H$, $hkh^{-1} \in H$. Recalling $H \subset N_K$ and $k \in K$, $hkh^{-1} \in K$. Therefore, $hkh^{-1} \in H \cap K$.

Since $hK = Kh$ for each $h \in H$, $HK = KH$. We will show that $HK = KH$ is a subgroup of G via the subgroup test in Lemma 0.2. Ket $k_1 h_1, k_2 h_2 \in KH$ and consider $k_2 h_2 \left(k_1 h_1 \right)^{-1} = k_2 h_2 h_1{}^{-1} k_1{}^{-1}$. Since $H < G$, $h_3 := h_2 h_1{}^{-1} \in H$. Since $k_2 h_3 \in Kh_3 = h_3 K$, there is some $k_3 \in K$ such that $k_2 h_3 = $

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$h_3 k_3$. Thus, $k_2 h_2 (k_1 h_1)^{-1} = h_3 k_3 k_1^{-1} \in HK = KH$. As shown, we have

$$H \cap K \overset{\lhd}{\underset{<}{\longrightarrow}} HK = KH \hookrightarrow G$$. By Lemma 1.2, $H \cap K \lhd HK$ holds.

Thus $p : H \to HK/K; h \mapsto (hK)K$ defines a group homomorphism, since

$$(h'K)K(hK)K = (h'KhK)K = (h'hK)K \tag{43}$$

for each $h, h' \in H$ by Lemma 1.4. Since p is a projection, p is an epimorphism.

Consider the kernel $p^\leftarrow 1$. If $k \in H \cap K$, $(kK)K = KK = K = 1_{HK/K}$, hence $H \cap K \subset p^\leftarrow 1$. Conversely, if $h \in p^{-1} 1$, $ph = (hK)K = K$ holds. Since $hK = \{hk' \mid k' \in K\}$, $K = (hK)K$ iff for each $k_1, k_2 \in K$, there is $k_3 \in K$ such that $k_3 = hk_1 k_2$, namely $h = k_3 k_2^{-1} k_1^{-1} \in K$. Recalling $h \in H$, we conclude $h \in H \cap K$. Thus, $p^{-1} 1 = H \cap K$.

If we apply Theorem 1.8 for the canonical epimorphism $p : H \to HK/K$ with $p^\leftarrow 1 = H \cap K$, we have a group isomorphism $\bar{p} : H/(H \cap K) \cong HK/K$. $\square$

Theorem 1.13. *Let G be a group, and both $K \lhd G$ and $N \lhd G$ be two normal subgroups. Suppose $K \subset N$. Then $K \lhd N$, and there is a group epimorphism from G/K to G/N whose kernel is N/K. Consequently, $(G/K)/(N/K) \cong G/N$.*

Proof. Since $K \overset{\lhd}{\underset{\lhd}{\longrightarrow}} H \hookrightarrow G$, by Lemma 1.2, $K \lhd H$ holds. Let $[g_1]_K, [g_2]_K \in$

G/K, where $[g_1]_K = g_1 K$ and $[g_2]_K = g_2 K$. By Lemma 1.4, $[g_2]_K [g_1]_K = [g_2 g_1]_K$ holds. Then $\varphi : G/K \to G/N; gK \to gN$ defines a group homomorphism since

$$\varphi([g_2]_K [g_1]_K) = \varphi[g_2 g_1]_K = g_2 g_1 N = g_2 N g_1 N = \varphi([g_2]_K) \varphi([g_1]_K). \tag{44}$$

Let $gH \in G/H$. There is $g' \in G$ such that $[g']_N = gN$. Then $g'K \in G/K$ satisfies $\varphi g' K = g' N = [g']_N = gN$. Hence, φ is epic. If $nK \in N/K$, applying φ, we have $\varphi n K = nN = N = 1_{G/N}$. Conversely, if $gK \in \varphi^\leftarrow 1$, $gN = N$. That is, for each $n_1 \in N$, there is some $n_2 \in N$ such that $gn_1 = n_2$, namely $g = n_2 n_1^{-1} \in N$. Therefore, $N/K = \varphi^\leftarrow 1$.

Since $N \lhd G$, by Definition 1.2, $O \longrightarrow N \hookrightarrow G \longrightarrow G/N \longrightarrow O$ is a short exact sequence. Define $j : N/K \to G/K; nK \mapsto nK$. Then, j is a group monomorphism since it is essentially an inclusion, and hence, the image of j is N/K, $j(N/K) = N/K = \varphi^\leftarrow 1$. Moreover, by Lemma 1.1, $N/K \lhd G/N$. Thus, we have a short exact sequence $O \longrightarrow N/K \overset{j}{\hookrightarrow} G/K \overset{\varphi}{\longrightarrow} G/N \longrightarrow O$.

More precisely, the following diagram is commutative:

$$
\begin{array}{ccccccccc}
O & \longrightarrow & N & \overset{\lhd}{\hookrightarrow} & G & \longrightarrow & G/N & \longrightarrow & O \\
& & \downarrow & & \downarrow & & \| & & \\
O & \longrightarrow & N/K & \overset{j}{\hookrightarrow} & G/K & \overset{\varphi}{\longrightarrow} & G/N & \longrightarrow & O
\end{array} \tag{45}
$$

By Theorem 1.10, we conclude G/N is isomorphic to $(G/K)/(N/K)$. $\square$

Theorem 1.14. *Let $\theta\colon G \to H$ be a group homomorphism, $K \lhd H$, and $N := \theta^\leftarrow K \subset G$. Then $N \lhd G$, and hence we have a group homomorphism*

$G \xrightarrow{\ \theta\ } H \longrightarrow H/K$ *whose kernel is N. Moreover, we have a group monomorphism $\overline{\theta}\colon G/N \to H/K$.*

If, further, θ is an epimorphism, the induced group homomorphism $\overline{\theta}$ is an isomorphism

Proof. Let $m, n \in N$. Applying θ, since $N = \theta^\leftarrow K$ and $K \lhd H$, $\theta(nm) = (\theta n)(\theta m)^{-1} \in K$. Thus, $nm^{-1} \in \theta^\leftarrow K = N$. By Lemma 0.2, $N < G$. To show the normality, let $g \in G$ and $n \in N$. Consider $gng^{-1} \in G$. Applying θ, since $K \lhd H$, $\theta g \in H$, and $\theta n \in K$,

$$\theta(gng^{-1}) = (\theta g)(\theta n)(\theta g)^{-1} \in K. \tag{46}$$

Hence, $gng^{-1} \in \theta^\leftarrow K = N$.

Let θ' be the composition of $G \xrightarrow{\ \theta\ } H \xrightarrow{\ p\ } H/K$, namely $\theta'\colon G \to H/K; g \mapsto (\theta g)K$. If $n \in \theta'^\leftarrow 1$, $K = \theta' n = p(\theta n) = (\theta n)K$ holds, since $K = 1_{H/K}$. It follows $\theta n \in K$ and hence $\theta'^\leftarrow 1 \subset K$. Conversely, if $n \in N = \theta^\leftarrow K$, $\theta n \in K$ and $\theta' n = p(\theta n) = K = 1_{H/K}$. Hence, by Corollary 1.7.1, there is a unique induced monomorphism $\overline{\theta}\colon G/N \to H/K; gN \mapsto (\theta g)K$. Then, we obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
O & \longrightarrow & N & \overset{\lhd}{\hookrightarrow} & G & \longrightarrow & G/N & \longrightarrow & O \\
& & \downarrow{\scriptstyle \theta|_N} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \overline{\theta}} & & \\
O & \longrightarrow & K & \overset{\lhd}{\hookrightarrow} & H & \overset{p}{\longrightarrow} & H/K & \longrightarrow & O
\end{array}
\tag{47}
$$

If we assume θ is epic $\theta G = H$, so is θ', since

$$\theta' G = (\theta G)/K = H/K. \tag{48}$$

Then, by Corollary 1.7.2, the induced group homomorphism $\overline{\theta}$ is isomorphism.
$\square$

Corollary 1.14.1 (General Theorem on Abelian Group Homomorphism)**.** *If $\theta\colon A \to B$ is an abelian group homomorphism such that*

- *onto B, $\theta A = B$, namely epic, and*

- *$\theta^\leftarrow B_0 = A_0$,*

where $A_0 < A$ and $B_0 < B$, then the induced abelian group homomorphism $\overline{\theta}\colon A/A_0 \to B/B_0; a + A_0 \mapsto \theta a + B_0$ is an isomorphism.

2 Direct Products

2.1 Direct Products

Definition 2.1 (Direct Products and Componentwise Binary Product). Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there is given a group G_λ. On their Cartesian, set-theoretical, product

$$\prod_{\lambda \in \Lambda} G_\lambda := \left\{ f \colon \Lambda \to \bigcup_{\lambda \in \Lambda} G_\lambda \ \middle| \ \forall \lambda \in \Lambda : f_\lambda \in G_\lambda \right\}, \tag{49}$$

we define a binary operation componentwisely, namely for $f, g \in \prod_{\lambda \in \Lambda}$,

$$gf \colon \Lambda \to \bigcup_{\lambda \in \Lambda} G_\lambda; \lambda \mapsto g_\lambda f_\lambda. \tag{50}$$

We call $\prod_{\lambda \in \Lambda} G_\lambda$, with the associated binary operation, the direct product of $\{G_\lambda \mid \lambda \in \Lambda\}$.

Theorem 2.1. *For a set of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, their direct product $\prod_{\lambda \in \Lambda} G_\lambda$ forms a group.*

Proof. The corresponding binary product is essentially that of each component, hence it is associative. Let $1 \colon \Lambda \to \bigcup_{\lambda \in \Lambda} G_\lambda; 1 \mapsto 1_{G_\lambda}$. Then 1 is the multiplicative identity. For each $f \in \prod_{\lambda \in \Lambda} G_\lambda$, let

$$f^{-1} \colon \Lambda \to \bigcup_{\lambda \in \Lambda} G_\lambda; \lambda \mapsto f_\lambda^{-1}. \tag{51}$$

It follows $f^{-1}f = 1 = ff^{-1}$, and $(\_)^{-1} \colon \bigcup_{\lambda \in \Lambda} G_\lambda \to \bigcup_{\lambda \in \Lambda} G_\lambda$ is the desired inverse unary operation. $\qquad\square$

Theorem 2.2. *For the direct product $\prod_{\lambda \in \Lambda} G_\lambda$ of a set of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, the canonical projection*

$$\pi_k \colon \prod_{\lambda \in \Lambda} G_\lambda \to G_k; f \mapsto f_k \tag{52}$$

is an epi for each $k \in \Lambda$.

Proof. Let $f, g \in \prod_{\lambda \in \Lambda} G_\lambda$. For each $k \in \Lambda$, by Definition 2.1,

$$\pi_k(gf) = (gf)_k = g_k f_k = (\pi_k g)(\pi_k f). \tag{53}$$

Hence, π_k is a group homomorphism for each $k \in \Lambda$.

Let $g_k \in G_k$ for a given $k \in \Lambda$. Define $h \colon \Lambda \to \bigcup_{\lambda \in \Lambda} G_\lambda$ by

$$h_\lambda := \begin{cases} 1_{G_\lambda} & \lambda \neq k \\ g_k & \lambda = k \end{cases} \tag{54}$$

Then $h \in \prod_{\lambda \in \Lambda} G_\lambda$ such that, $\pi_k h = g_k$. Hence, π_k is surjective. $\qquad\square$

Theorem 2.3 (Universal Property of Direct Products)**.** *Let H be a group and Λ be a set, and suppose that for each $\lambda \in \Lambda$, there is given a group homomorphism $\theta_\lambda \colon H \to G_\lambda$. Then, there exists a unique mediator homomorphism $\theta \colon H \to \prod_{\lambda \in \Lambda} G_\lambda$ such that the following diagram for each $k \in \Lambda$ is commutative:*

$$
\begin{array}{ccc}
H & & \\
{\scriptstyle \exists ! \theta}\downarrow & \searrow^{\theta_k} & \\
\prod_{\lambda \in \Lambda} G_\lambda & \xrightarrow[\pi_k]{} & G_k
\end{array}
\qquad \pi_k \circ \theta = \theta_k. \tag{55}
$$

Proof. Define

$$
\theta_\_ \colon H \to \left(\Lambda \to \bigcup_{\lambda \in \Lambda} G_\lambda \right); h \to \theta_\_ h, \tag{56}
$$

by, for each $h \in H$,

$$
\theta_\_ h \colon \Lambda \to \bigcup_{\lambda \in \Lambda} G_\lambda; k \mapsto \theta_k h. \tag{57}
$$

For each $k \in \Lambda$, $\theta_k h \in G_k$ holds, hence $\theta_\_ h \in \prod_{\lambda \in \Lambda} G_\lambda$. Therefore, $\theta_\_ \colon H \to \prod_{\lambda \in \Lambda} G_\lambda$. By construction, $\theta_\_$ is entirely given by $\{\theta_\lambda \mid \lambda \in \Lambda\}$, so it is unique.

For $h_1, h_2 \in H$, $\theta_\_ (h_2 h_1) = (\theta_\_ h_2)(\theta_\_ h_1)$ holds since θ_k is a group homomorphism and, hence,

$$
\theta_k (h_2 h_1) = (\theta_k h_2)(\theta_k h_1) \tag{58}
$$

for each $k \in \Lambda$.

Let $k \in \Lambda$. Then, $\pi_k \circ \theta_\_ = \theta_k$ holds, since for each $h \in H$,

$$
(\pi_k \circ \theta_\_)\, h = \pi_k (\theta_\_ h) = (\theta_\_ h)_k = \theta_k h. \tag{59}
$$

$\square$

Here is a useful criterion for a group to be a binary direct product of its subgroups:

Lemma 2.4. *Let G be a group, H, K be two subgroups, namely $H < G$ and $K < G$. If $H \cap K = \{1_G\}$, $HK = G$, and for any $h \in H$ and $k \in K$, $hk = kh$, then $\circ \colon H \times K \to G; (h, k) \mapsto hk$ is an isomorphism.*

Remark. If the underlying group is abelian, we can relax the auxiliary commutative condition. Namely, for an abelian group A with its subgroups $B < A$ and $C < A$, if $B \cap C = \{0_A\}$ and $B + C = A$, then $+ \colon B \times C \to A$ is an isomorphism. We denote $B \oplus C$ for the sum $B + C$ with $B \cap C = \{0\}$.

Proof. For $(h_1, k_1), (h_2, k_2) \in H \times K$, since $h_1 k_2 = k_2 h_1$, and both H and K are subgroups, $h_2 h_1 \in H$ and $k_2 k_1 \in K$, we have that

$$
\circ((h_2, k_2)(h_1, k_1)) = \circ(h_2 h_1, k_2 k_1) = h_2 h_1 k_2 k_1 = h_2 k_2 h_1 k_1 = \circ(h_2, k_2) \circ (h_1, k_1). \tag{60}
$$

Hence, it is homomorphism. Since $HK = G$, it is an epimorphism. Suppose (h, k) is in the kernel, then $hk = 1_G$. Hence, $k = h^{-1} \in H$. That is, $k \in K \cap H = \{1_G\}$. We conclude $k = 1_G = h$. By Lemma 0.4, the map is monic. $\square$

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Remark. Inductively, if $H_1 < G \wedge \ldots H_n < G$ satisfies $H_1 \cdots H_2 = G$ and $H_{i+1} \cap (H_1 \cdots H_i)$ for $i \in \{1, \ldots, n-1\}$, then G is isomorphic to $H_1 \times \cdots \times H_n$.

2.2 Weak Direct Products

Definition 2.2 (Weak Direct Products)**.** Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there is a given group G_λ. The weak direct product $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ is a subset of $\prod_{\lambda \in \Lambda} G_\lambda$ of all f such that $f_\lambda = 1_{G_\lambda}$ for all but finitely many $\lambda \in \Lambda$.

Remark (Direct Sums)*.* For additive, i.e., abelian groups $\{A_\lambda \mid \lambda \in \Lambda\}$, the corresponding weak direct product is called the direct sum, denoted by $\sum_{\lambda \in \Lambda} A_\lambda$.

Theorem 2.5. *Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there is given a group G_λ. Their weak direct product is a normal subgroup of the direct product:*

$$\prod_{\lambda \in \Lambda}^{weak} G_\lambda \lhd \prod_{\lambda \in \Lambda} G_\lambda. \tag{61}$$

Proof. Let $f, g \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$. There are finite subsets $I, J \subset \Lambda$ such that $j \in J$ iff $f_j \neq 1$, and $k \in K$ iff $g_k \neq 1$. Consider gf^{-1}. Recalling $\left(gf^{-1}\right)_i = g_i f_i^{-1}$ for each $i \in \Lambda$,

$$\left\{i \in \Lambda \mid g_i f_i^{-1} \neq 1\right\} \subset J \cup K. \tag{62}$$

Since $J \cup K \subset \Lambda$ is finite, $gf^{-1} \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$. By Lemma 0.2, $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda < \prod_{\lambda \in \Lambda} G_\lambda$.

Suppose $f \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$ and $g \in \prod_{\lambda \in \Lambda} G_\lambda$, and consider gfg^{-1}. There is a finite subset $I \subset \Lambda$ such that $i \in I$ iff $f_i \neq 1$. For any $i \in \Lambda$,

$$\left(gfg^{-1}\right)_i = \begin{cases} 1 & i \in \Lambda - I \\ g_i f_i g_i^{-1} & \text{otherwise.} \end{cases} \tag{63}$$

Therefore, $\left(gfg^{-1}\right)_i$ is not 1 at most finitely many $i \in I$. Hence, $gfg^{-1} \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$. $\qquad\square$

Theorem 2.6. *For the weak direct product $\prod_{\lambda \in \Lambda}^{weak} G_\lambda$ of a set of groups $\{G_\lambda \mid \lambda \in \Lambda\}$, the canonical injection $\iota_k \colon G_k \to \prod_{\lambda \in \Lambda}^{weak} G_\lambda$ defined by*

$$(\iota_k a_k)_\lambda := \begin{cases} 1_{G_\lambda} & \lambda \neq k \\ a_k & \lambda = k \end{cases} \tag{64}$$

for $a_k \in G_k$, is a mono for each $k \in \Lambda$.

Proof. Let $k \in \Lambda$, and $a_k, b_k \in G_k$. For each $\lambda \in \Lambda$, we have

$$(\iota_k(b_k a_k))_\lambda = \begin{cases} 1_{G_\lambda} & \lambda \neq k \\ b_k a_k & \lambda = k \end{cases} \tag{65}$$

and

$$(\iota_k b_k)_\lambda \, (\iota_k a_k)_\lambda = \begin{cases} 1_{G_\lambda} 1_{G_\lambda} & \lambda \neq k \\ b_k a_k & \lambda = k. \end{cases} \tag{66}$$

Hence, $\iota_k (b_k a_k) = (\iota_k b_k)(\iota_k a_k)$.

If we suppose $\iota_k b_k = \iota_k a_k$, then

$$\forall \lambda \in \Lambda : (\iota_k b_k)_\lambda = (\iota_k a_k)_\lambda \tag{67}$$

In particular, when $\lambda = k$, we obtain $b_k = a_k$. $\qquad\square$

Theorem 2.7. *For a set of groups $\{G_\lambda \mid \lambda \in \Lambda\}$ and each $k \in \Lambda$, $\iota_k G_k \lhd \prod_{\lambda \in \Lambda} G_\lambda$.*

Proof. Let $k \in \Lambda$. By Theorem 2.6, ι_k is a group homomorphism. Hence, its image is a subgroup, $\iota_k G_k < \prod_{\lambda \in \Lambda} G_\lambda$ by Theorem 0.5.

Let $h \in \iota_k G_k$; there is some $g_k \in G_k$ such that $h = \iota_k g_k$. For each $\lambda \in \Lambda$, and $f \in \prod_{\lambda \in \Lambda} G_\lambda$,

$$\pi_\lambda \left(f h f^{-1} \right) = f_\lambda \pi_\lambda \left(\iota_k g_k \right) f_\lambda^{-1} = \begin{cases} f_k g_k f_k^{-1} & \lambda = k \\ 1 & \text{otherwise.} \end{cases} \tag{68}$$

Recalling $f_k g_k f_k^{-1} \in G_k$, we conclude $f h f^{-1} \in \iota_k G_k$. $\qquad\square$

Theorem 2.8. *Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there is a given group homomorphism $\theta_\lambda \colon G_\lambda \to H_\lambda$. Let $\theta := \prod_{\lambda \in \Lambda} \theta_\lambda$ be a map from $\prod_{\lambda \in \Lambda} G_\lambda$ to $\prod_{\lambda \in \Lambda} H_\lambda$, given by*

$$(\theta g)_\lambda := \theta_\lambda g_\lambda \tag{69}$$

for each $\lambda \in \Lambda$. Then, θ is a group homomorphism, $\theta \left(\prod_{\lambda \in \Lambda} G_\lambda \right) \subset \prod_{\lambda \in \Lambda}^{weak} H_\lambda$, $\theta^{\leftarrow} 1 = \prod_{\lambda \in \Lambda} (\theta_\lambda^{\leftarrow} 1)$, and $\theta \left(\prod_{\lambda \in \Lambda} G_\lambda \right) = \prod_{\lambda \in \Lambda} \theta_\lambda G_\lambda$. Moreover, θ is monic iff θ_λ is monic for each $\lambda \in \Lambda$, and θ is epic iff θ_λ is epic for each $\lambda \in \Lambda$.

Proof. Let us first show $\theta \colon \prod_{\lambda \in \Lambda} G_\lambda \to \prod_{\lambda \in \Lambda} H_\lambda$ is a group homomorphism. Suppose $f, g \in \prod_{\lambda \in \Lambda} G_\lambda$, and $k \in \Lambda$. Then

$$(\theta(gf))_k = \theta_k \left(g_k f_k \right) = \left(\theta_k g_k \right) \left(\theta_k f_k \right) = (\theta g)_k \, (\theta f)_k . \tag{70}$$

Since $k \in \Lambda$ is arbitrary, we conclude $\theta(gf) = (\theta g)(\theta f)$.

Next, $\theta \left(\prod_{\lambda \in \Lambda} G_\lambda \right) \subset \prod_{\lambda \in \Lambda}^{\text{weak}} H_\lambda$. Let $h \in \theta \left(\prod_{\lambda \in \Lambda} G_\lambda \right)$; there is a $g \in \prod_{\lambda \in \Lambda} G_\lambda$ such that $h = \theta g$, where g satisfies

$$g_\lambda = 1 \text{ for all but finitely many } \lambda \in \Lambda. \tag{71}$$

There exists a finite subset $I_h \subset \Lambda$ such that $i \in I_h$ iff $g_i \neq 1$. Then $h = fg$ satisfies

$$\lambda \in \Lambda - I_h \Rightarrow h_\lambda - (\theta 1)_\lambda = 1. \tag{72}$$

Therefore, h_λ is not 1 at most finitely many $\lambda \in \Lambda$. That is, $h \in \prod_{\lambda \in \Lambda}^{\text{weak}} H_\lambda$.

The kernel of θ is

$$
\begin{aligned}
\theta^{\leftarrow}1 &= \left\{ g \in \prod_{\lambda \in \Lambda} G_\lambda \,\middle|\, \forall \lambda \in \Lambda : \theta_\lambda g_\lambda = 1_{H_\lambda} \right\} \\
&= \left\{ g \in \prod_{\lambda \in \Lambda} G_\lambda \,\middle|\, \forall \lambda \in \Lambda : g_\lambda \in \theta_\lambda{}^{\leftarrow}1 \right\} \qquad (73) \\
&= \prod_{\lambda \in \Lambda} \left(\theta_\lambda{}^{\leftarrow}1 \right)
\end{aligned}
$$

and the image is

$$
\begin{aligned}
\theta \left(\prod_{\lambda \in \Lambda} G_\lambda \right) &= \left\{ h \in \prod_{\lambda \in \Lambda} H_\lambda \,\middle|\, \exists g \in \prod_{\lambda \in \Lambda} G_\lambda : h = \theta g \right\} \\
&= \left\{ h : \Lambda \to \bigcup_{\lambda \in \Lambda} H_\lambda \,\middle|\, \forall \lambda \in \Lambda : h_\lambda \in H_\lambda \wedge \exists g_\lambda \in G_\lambda : h_\lambda = \theta_\lambda g_\lambda \right\} \\
&= \left\{ h : \Lambda \to \bigcup_{\lambda \in \Lambda} H_\lambda \,\middle|\, \forall \lambda \in \Lambda : \exists g_\lambda \in G_\lambda : h_\lambda = \theta_\lambda g_\lambda \right\} \\
&= \left\{ h : \Lambda \to \bigcup_{\lambda \in \Lambda} H_\lambda \,\middle|\, \forall \lambda \in \Lambda : h_\lambda \in \theta_\lambda G_\lambda \right\} \\
&= \prod_{\lambda \in \Lambda} \theta_\lambda G_\lambda .
\end{aligned}
$$

$$(74)$$

Recall that θ is monic iff $\theta^{\leftarrow}1 = \{1\}$ by Lemma 0.4. As shown above, $\theta^{\leftarrow}1 = \prod_{\lambda \in \Lambda} \left(\theta_\lambda{}^{\leftarrow}1 \right)$, θ is monic iff

$$\forall \lambda \in \Lambda : \theta_\lambda{}^{\leftarrow}1 = \{1\}. \qquad (75)$$

Thus, θ is monic iff each θ_λ is monic. As shown above, $\theta \left(\prod_{\lambda \in \Lambda} G_\lambda \right) = \prod_{\lambda \in \Lambda} \theta_\lambda G_\lambda$, θ is epic iff $\theta \left(\prod_{\lambda \in \Lambda} G_\lambda \right) = \prod_{\lambda \in \Lambda} H_\lambda$, i.e.,

$$\forall \lambda \in \Lambda : \theta_\lambda G_\lambda = H_\lambda. \qquad (76)$$

Therefore, θ is epic iff each θ_λ is epic. $\qquad \square$

Theorem 2.9. *Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there are a given group and its subgroup $N_\lambda \lhd G_\lambda$. Then*

- *$\prod_{\lambda \in \Lambda} N_\lambda \lhd \prod_{\lambda \in \Lambda} G_\lambda$, and $\left(\prod_{\lambda \in \Lambda} G_\lambda \right) / \left(\prod_{\lambda \in \Lambda} N_\lambda \right) \cong \prod_{\lambda \in \Lambda} \left(G_\lambda / N_\lambda \right)$.*

- *$\prod_{\lambda \in \Lambda}^{weak} N_\lambda \lhd \prod_{\lambda \in \Lambda}^{weak} G_\lambda$, and $\left(\prod_{\lambda \in \Lambda}^{weak} G_\lambda \right) / \left(\prod_{\lambda \in \Lambda}^{weak} N_\lambda \right) \cong \prod_{\lambda \in \Lambda}^{weak} \left(G_\lambda / N_\lambda \right)$.*

Proof. Let $\lambda \in \Lambda$, and

$$\pi_\lambda \colon G_\lambda \to G_\lambda/N_\lambda; g \mapsto gN_\lambda \tag{77}$$

be the canonical epi. By Theorem 1.6, the kernel is N_λ, $\pi_\lambda^{\leftarrow}1 = N_\lambda$. Hence, $p \coloneqq \prod_{\lambda \in \Lambda} \pi_\lambda$ is an epi:

$$p \colon \prod_{\lambda \in \Lambda} G_\lambda \to \prod_{\lambda \in \Lambda} (G_\lambda/N_\lambda) \tag{78}$$

with the kernel $p^{\leftarrow}1 = \prod_{\lambda \in \Lambda} (\theta_\lambda^{\leftarrow}1) = \prod_{\lambda \in \Lambda} N_\lambda$ by Theorem 2.8. Since the kernel $p^{\leftarrow}1$ is a normal subgroup of $\prod_{\lambda \in \Lambda} G_\lambda$ by Lemma 1.1, we conclude $\prod_{\lambda \in \Lambda} N_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$. Since p is epic, its image is the codomain, $p\left(\prod_{\lambda \in \Lambda} G_\lambda\right) = \prod_{\lambda \in \Lambda} (G_\lambda/N_\lambda)$. Applying Theorem 1.8, we conclude

$$\prod_{\lambda \in \Lambda} (G_\lambda/N_\lambda) = p\left(\prod_{\lambda \in \Lambda} G_\lambda\right) \cong \left(\prod_{\lambda \in \Lambda} G_\lambda\right) / (p^{\leftarrow}1) = \left(\prod_{\lambda \in \Lambda} G_\lambda\right) / \left(\prod_{\lambda \in \Lambda} N_\lambda\right). \tag{79}$$

Let p^{weak} be the restriction of p on $\prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \triangleleft \prod_{\lambda \in \Lambda} G_\lambda$. By Theorem 2.8, p^{weak} is $\left(\prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda/N_\lambda)\right)$-valued:

$$p^{\text{weak}} \colon \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \to \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda/N_\lambda). \tag{80}$$

Since the original p is a group homomorphism, the restriction is also a group homomorphism, since

$$p^{\text{weak}}(gf) = p(gf) = (pg)(pf) = \left(p^{\text{weak}}g\right)\left(p^{\text{weak}}f\right) \tag{81}$$

for each $f, g \in \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda/N_\lambda)$.

Next, we will show p^{weak} is an epi. Let $[g] \in \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda/N_\lambda)$. Since the original p is an epi, there is $g' \in \prod_{\lambda \in \Lambda} G_\lambda$ such that $pg' = [g]$, that is

$$\forall \lambda \in \Lambda : g'_\lambda N_\lambda = [g]_\lambda. \tag{82}$$

Since $[g] \in \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda/N_\lambda)$, $[g]_\lambda = N_\lambda$ for all but finitely many $\lambda \in \Lambda$. Hence, we conclude $g' \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda$.

Finally, let us consider the kernel:

$$
\begin{aligned}
p^{\text{weak}\leftarrow}1 &= \left\{ g \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \,\middle|\, p^{\text{weak}}g = 1 \in \prod_{\lambda \in \Lambda}^{\text{weak}} (G_\lambda/N_\lambda) \right\} \\
&= \left\{ g \in \prod_{\lambda \in \Lambda}^{\text{weak}} G_\lambda \,\middle|\, g_\lambda N_\lambda = N_\lambda \text{ for all but finitely many } \lambda \in \Lambda \right\}
\end{aligned}
\tag{83}
$$

Since $g_\lambda N_\lambda = N_\lambda$ iff $g_\lambda \in N_\lambda$, we obtain

$$p^{\mathrm{weak}\leftarrow}1 = \left\{ g \in \prod_{\lambda\in\Lambda}^{\mathrm{weak}} G_\lambda \;\middle|\; g_\lambda \in N_\lambda \text{ for all but finitely many } \lambda \in \Lambda \right\} = \prod_{\lambda\in\Lambda}^{\mathrm{weak}} N_\lambda.$$
(84)

Therefore, if we apply Theorem 1.8 for the epic $p^{\mathrm{weak}}\colon \prod_{\lambda\in\Lambda}^{\mathrm{weak}} G_\lambda \to \prod_{\lambda\in\Lambda}^{\mathrm{weak}} (G_\lambda/N_\lambda)$ with the kernel $p^{\mathrm{weak}\leftarrow}1 = \prod_{\lambda\in\Lambda}^{\mathrm{weak}} N_\lambda$, we conclude

$$\prod_{\lambda\in\Lambda}^{\mathrm{weak}} (G_\lambda/N_\lambda) \cong \left(\prod_{\lambda\in\Lambda}^{\mathrm{weak}} G_\lambda \right) \Big/ \left(\prod_{\lambda\in\Lambda}^{\mathrm{weak}} N_\lambda \right).$$
(85)

$\square$

2.3   Direct Sums

Let us begin with the direct sum version of Theorem 2.9:

Theorem 2.10. *Let Λ be a set and suppose that for each $\lambda \in \Lambda$, there are a given abelian group and its subgroup $N_\lambda \lhd G_\lambda$. Then, $\left(\sum_{\lambda\in\Lambda} G_\lambda\right) / \left(\sum_{\lambda\in\Lambda} N_\lambda\right) \cong \sum_{\lambda\in\Lambda} (G_\lambda/N_\lambda)$.*

Theorem 2.11 (Universal Property of Direct Sums)**.** *Let B be an abelian group and Λ be a set, and suppose that for each $\lambda \in \Lambda$, there is given an abelian group homomorphism $\psi_\lambda\colon A_\lambda \to B$. Then, there exists a unique mediator homomorphism $\psi\colon \sum_{\lambda\in\Lambda} A_\lambda \to B$ such that the following diagram for each $k \in \Lambda$ is commutative:*

$$\psi_k = \psi \circ \iota_k.$$
(86)

Proof. Recall $a \in \sum_{\lambda\in\Lambda} A_\lambda$ iff $\pi_\lambda a = 0$ for all but finitely manly $\lambda \in \Lambda$. For each $a \in \sum_{\lambda\in\Lambda} A_\lambda$, there exists a finite subset $I_a \subset \Lambda$ such that $\pi_i a \neq 0$ iff $i \in I_a$. Define $\psi\colon \sum_{\lambda\in\Lambda} A_\lambda \to B$ by

$$\psi 0 := 0$$
$$\psi a := \sum_{i\in I_a} \psi_i (\pi_i a).$$
(87)

Let $a, a' \in \sum_{\lambda\in\Lambda} A_\lambda$. Since, if both are zero, so is their sum, namely $\neg I_{a+a'} \supset (\neg I_a) \cap (\neg I_{a'})$, we have that

$$I_{a+a'} \subset I_a \cup I_{a'},$$
(88)

where $\neg I := \Lambda - I$ for any subset $I \subset \Lambda$. Hence,

$$\psi(a + a') = \sum_{i\in I_{a+a'}} \psi_i (\pi_i(a + a'))$$
(89)

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is a sum over finite set $I_{a+a'}$. Since it is a finite sum, we can expand the right-hand side:

$$\psi(a + a') = \sum_{i \in I_{a+a'}} \psi_i(\pi_i a) + \sum_{i \in I_{a+a'}} \psi_i(\pi_i a') \tag{90}$$

Let $i \in I_{a+a'}$, and consider the first term $\sum_{i \in I_{a+a'}} \psi_i(\pi_i a)$.

- If $\pi_i a' = 0$, $\pi_i a \neq 0$ must be the case. So $i \in I_a$.

- Otherwise, $\pi_i a' \neq 0$. Therefore, either $\pi_i a = 0$ or $\pi_i a \neq 0$ is the case:

  - $\pi_i a' \neq 0$ and $\pi_i a = 0$ case. The corresponding term in $\sum_{i \in I_{a+a'}} \psi_i(\pi_i a)$ is zero.

  - $\pi_i a' \neq 0$ and $\pi_i a \neq 0$ case. Then $i \in I_a$ is the case.

Thus, the first term is

$$\sum_{i \in I_{a+a'}} \psi_i(\pi_i a) = 0 + \sum_{i \in I_a} \psi_i(\pi_i a) = \psi a. \tag{91}$$

Therefore, we conclude $\psi(a + a') = \psi a + \psi a'$.

Let $j \in \Lambda$ and $a_j \in A_j$. Since $\iota_j a_j \in \sum_{\lambda \in \Lambda} A_\lambda$ and $I_{\iota_j a_j} = \{j\}$, see Theorem 2.6, we have

$$\psi(\iota_j a_j) = \sum_{i \in I_{\iota_j a_j}} \psi_i(\pi_i(\iota_j a_j)) = \psi_j a_j. \tag{92}$$

Since $a_j \in A_j$ is arbitrary, we conclude $\psi \circ \iota_j = \psi_j$.

Finally, let us show the uniqueness. Suppose an abelian group homomorphism $\phi \colon \sum_{\lambda \in \Lambda} A_\lambda \to B$ also satisfies

$$\forall \lambda \in \Lambda : \phi \circ \iota_\lambda = \pi_\lambda. \tag{93}$$

Then, $\phi = \psi$ since

$$\phi a = \phi \sum_{i \in I_a} \iota_j a_j = \sum_{i \in I_a} \phi(\iota_j a_j) = \sum_{i \in I_a} \psi_j a_j = \psi a \tag{94}$$

for all $a \in \sum_{\lambda \in \Lambda} A_\lambda$. $\qquad\square$

Remark. This theorem becomes false if we remove the restriction that the groups are abelian.

Consider S_3 – the symmetric group on a set of three elements:

$$\begin{cases} (), \\ (12), (23), (31), \\ (123), (132), \end{cases} \tag{95}$$

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where () denotes the identity, and for example (12) is $1 \leftrightarrow 2$ swap, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \xrightarrow{(12)}$

$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, and (123) is a cyclic permutation $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \xrightarrow{(123)} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

S_3 is generated by $\{(12), (123)\}$, since

$$
\begin{aligned}
() &= (12) \circ (12) \\
(23) &= (12) \circ (123) \\
(31) &= (123) \circ (12) \\
(132) &= (123) \circ (123).
\end{aligned}
\tag{96}
$$

Moreover, S_3 is not abelian:

$$
(12) \circ (123) \neq (123) \circ (12).
\tag{97}
$$

Consider the weak product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ are modular arithmetic. Since it is a finite product case, their weak product is the ordinary product $\mathbb{Z}_2 \times \mathbb{Z}_3$. Let ι_2 and ι_3 be the corresponding canonical injections:

$$
\begin{aligned}
\iota_2 &: \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_3 \\
\iota_3 &: \mathbb{Z}_3 \to \mathbb{Z}_2 \times \mathbb{Z}_3.
\end{aligned}
\tag{98}
$$

Consider the following group homomorphisms

$$
\begin{aligned}
\phi_2 &: \mathbb{Z}_2 \to S_3 \\
\phi_3 &: \mathbb{Z}_3 \to S_3
\end{aligned}
\tag{99}
$$

defined by

$$
\phi_2 0 = (), \phi_2 1 = (12),
\tag{100}
$$

and

$$
\phi_3 0 = (), \phi_3 1 = (123), \phi_3 2 = (132).
\tag{101}
$$

It is worth mentioning that $\phi_3(1+1) = (\phi_3 1) \circ (\phi_3 1) = (123) \circ (123)$. Suppose, for contradiction, that there is a mediator $\phi: \mathbb{Z}_2 \times \mathbb{Z}_3 \to S_3$. As shown above, (12) and (123) generate S_3, and $\{(12), (123)\} \subset \phi(\mathbb{Z}_2 \times \mathbb{Z}_3)$. Hence, ϕ is an epi, $\phi(\mathbb{Z}_2 \times \mathbb{Z}_3) = S_3$. Recalling the image forms a subgroup, $\phi(\mathbb{Z}_2 \times \mathbb{Z}_3) < S_3$, this equality is a group isomorphism. As shown above, S_3 is non-abelian but $\phi(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is the image of the product of two abelian groups, so it is abelian, which is absurd.

Theorem 2.12. *Let G be an abelian group, Λ be a set, and suppose that for each $\lambda \in \Lambda$, there is a given subgroup $N_\lambda < G$. If we further assume each $g \in G$ has a unique representation $g = \sum_{\lambda \in \Lambda} g_\lambda$, where $g_\lambda \in N_\lambda$ is zero for all but finitely many $\lambda \in \Lambda$. Then G is isomorphic to $\sum_{\lambda \in \Lambda} N_\lambda$.*

Proof. Since G is abelian, its subgroup is always normal. Let $n \in \sum_{\lambda \in \Lambda} N_\lambda$. Then $n_\lambda = 0$ for all but finitely many indices, say $I_n \subset \Lambda$:

$$i \in I_n \Leftrightarrow n_i \neq 0. \tag{102}$$

Then, $\sum_{i \in I_n} n_i$ is a finite sum of elements in G, hence $\sum_{i \in I_n} n_i \in G$. Define $\varphi \colon \sum_{\lambda \in \Lambda} N_\lambda \to G; n \mapsto \sum_{i \in I_n} n_i$.

- φ is a group homomorphism.

  Let $f, g \in \sum_{\lambda \in \Lambda} N_\lambda$. Then $\varphi(f+g) = \sum_{i \in I_{f+g}}(f+g)_i$. If we let $I_{f+g} = \{i_1, \ldots, i_k\}$, we have

  $$\varphi(f+g) = (f+g)_{i_1} + \cdots + (f+g)_{i_k} = f_{i_1} + \cdots + f_{i_k} + g_{i_1} + \cdots + g_{i_k}, \tag{103}$$

  since they are all members in the abelian group G. For each $i \in I_{f+g}$, either $f_i = 0$ or $f_i \neq 0$:

  - If $f_i = 0$, $i \in \Lambda - I_f$.
  - If $f_i \neq 0$, $i \in I_f$.

  Therefore, $\sum_{i \in I_{f+g}} f_i = \sum_{i \in I_f} f_i = \varphi f$. We conclude $\varphi(f+g) = \varphi f + \varphi g$.

- φ is epic.

  Let $g \in G$. By hypothesis, there is a unique representation $g = \sum_{\lambda \in \Lambda} g_\lambda$, where $g_\lambda \in N_\lambda$ is zero for all but finitely many $\lambda \in \Lambda$. In other words, G is a free abelian group with basis $\bigcup_{\lambda \in \Lambda} N_\lambda$. Let $I_g \subset \Lambda$ is the corresponding subset: $i \in I_g$ iff $g_i \neq 0$. Sine $\sum_{i \in I_g} g_i \in \sum_{\lambda \in \Lambda} N_\lambda$ satisfies $\varphi \sum_{i \in I_g} g_i = g$, φ is surjective.

- φ is monic.

  Consider the kernel:

  $$\varphi^{\leftarrow} 0 := \left\{ g \in \sum_{\lambda \in \Lambda} N_\lambda \,\middle|\, \varphi g = 0 \right\} \tag{104}$$

  Let $z \in \varphi^{\leftarrow} 0$. By hypothesis, the corresponding unique representation of $\varphi z = 0$ is given by

  $$\forall \lambda \in \Lambda : z_\lambda = 0. \tag{105}$$

  It follows $z = 0$, namely $\varphi^{\leftarrow} 0 = \{0\}$. By Lemma 0.4, φ is monic.

Hence, $\varphi \colon \sum_{\lambda \in \Lambda} N_\lambda \to G$ is an isomorphism. $\qquad\square$

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