Options Pricing

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These notes are my attempt at understanding the maths of Black-Scholes options pricing. I want to present a derivation that is intuitive to people who have an understanding of undergraduate level calculus.

1 Futures and no-arbitrage pricing

In this section I start by presenting an argument for the fair price of a contract called a future. The purpose of this example is to provide a simple scenario in which we will see that the requirement of "no arbitrage" enforces a fair price, independent of the random nature of the asset price moves.

> **Definition 1.1**: A *future* is a type of contract between two parties to trade an asset (the "underlying") at a fixed time T in the future for a predetermined price K known as the *strike*.

Futures can be written for any standardisable asset, like oil, corn, coffee, (not tea – apparently it's very difficult to standardise this product), and stocks. A future can be a way for both parties to reduce uncertainty. That being said, what is the *fair* price K for a future? What does it mean for the price to be fair?

To work towards answering these questions let's consider how each party can profit from the future. Suppose I take the buy-side (also called the *long* position) of a future in Apple stock struck at K which expires at time T. If at T, Apple stock is trading on the market for $S_T > K$ then when I buy stock from the seller of the future (who has a *short* position) at price K, I can immediately go to the stock exchange and sell it to another party at S_T, so I profit $S_T - K$. On the other hand, if $S_T < K$ then I must still fulfill my side of the future and purchase the stock for K. If I want to close out the position I have to sell it for its current price for S_T, and I incur a loss of $K - S_T$.

To reiterate, at expiration the net profit for each party is:
- *Long*: $S_T - K$
- *Short*: $K - S_T$

A reasonable guess for the fair value of K is the average value of S_T, which requires us to assume a distribution for S_T. However, there is a more powerful argument which allows us to completely circumvent any need to make assumptions about the probability distribution of S_T. Imagine our financial Universe consists of being able to buy/sell futures and borrow/loan money at a risk-free interest rate r. If the stock is worth S_0 now, when we go long the future we are effectively taking up a long position in the stock without actually needing to hold it. We can actually short the stock to get S_0\$ now and loan it at the risk free rate so that at time T our cash position is $S_0 e^{rT}$. Then if $K < S_0 e^{rT}$ we buy the stock at K and lock in a profit of $S_0 e^{rT} - K$, we then return the stock to close out our initial short position. In this set of trades we have taken on *no risk* and made a profit. Such a set of actions is known as an arbitrage opportunity.

Now suppose $K > S_0 e^{rT}$. From the perspective of the party that is short the future, they could borrow S_0 to buy the stock now, so that when the future expires they owe $(S_0 e^{rT} < K)$\$. Then they lock in a profit of $K - S_0 e^{rT} > 0$.

The price of the future is determined by requiring that neither party can be guaranteed to make a profit. This condition squeezes the fair price to precisely $K = S_0 e^{rT}$. Notice that this doesn't eliminate the ability for either side to profit, it just means that neither party can make a profit with zero risk.

2 Ito Calculus

Suppose you're given the stochastic differential equation

$$\mathrm{d}X_t = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t \,, \tag{2.1}$$

where W_t is a Wiener process and the functions μ_t and σ_t are deterministic functions of time. The mean and variance of X_t are easy to compute:

$$E[X_t] = \int_0^t \mu_s \,\mathrm{d}s \tag{2.2a}$$

$$\mathrm{Var}(X_t) = \int_0^t \sigma_s^2 \,\mathrm{d}s \,. \tag{2.2b}$$

And in general the formal solution of the differential equation is

$$X_t = \int_0^t \mu_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}B_s \,. \tag{2.3}$$

Now suppose that we have a process $Y_t = f(t, X_t)$, where $f(t, x)$ is some function of t and x like $f(t, x) = t + \exp(x)$. How do we write dY_t? We need something like a chain rule for continuous stochastic processes. This is what the Ito lemma provides us. In this section I will sketch the main ideas and provide an intuitive understanding of many of the identities used.

If I had an expression $y = f(t, x)$, where $x = x(t)$ implicitly depends on time, then from multivariable calculus I know I could write

$$\mathrm{d}y = \frac{\partial f}{\partial t} \,\mathrm{d}t + \frac{\partial f}{\partial x} \,\mathrm{d}x \,, \tag{2.4}$$

so a first guess for how to write $\mathrm{d}Y_t$ might look like

$$\mathrm{d}Y_t = \frac{\partial f}{\partial t} \,\mathrm{d}t + \frac{\partial f}{\partial x} \,\mathrm{d}X_t \,. \tag{2.5}$$

This turns out to be kind of close, but not quite. The reason is that we're not being consistent with which terms we're (implicitly) dropping. First, let's agree to keep terms that are $\mathcal{O}(\mathrm{d}t)$. Then, noting that $\mathrm{d}W_t$ is like the continuous-time analog of a random walk, its *variance* (which goes like $(\mathrm{d}W_t)^2$) is expected to be of order $\mathcal{O}(\mathrm{d}t)$, and so $\mathrm{d}W_t \sim \mathcal{O}(\mathrm{d}t^{1/2})$. Since $\mathrm{d}X_t$ has one component of order $\mathrm{d}t$, and another component of order $\mathrm{d}W_t$ then when we Taylor expand f we should keep terms up to $(\mathrm{d}W_t)^2$, which motivates us to do an expansion up to second order in x. To see this in action, let's write

$$\mathrm{d}f(t, X_t) = \frac{\partial f}{\partial t} \,\mathrm{d}t + \frac{\partial f}{\partial x} \,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} (\mathrm{d}X_t)^2 \,, \tag{2.6}$$

where we haven't written cross terms like $\mathrm{d}t\,\mathrm{d}x$ since those will definitely be greater than order $\mathrm{d}t$ anyway. Also note that when I write $\partial f/\partial x$ I mean, "take the partial derivative of $f(t,x)$ wrt x, then evaluate that expression at X_t". Upon substituting eq. (2.1) into eq. (2.6) we obtain

$$\mathrm{d}f(t,X_t) = \frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial x}(\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t)^2\,. \tag{2.7}$$

Now if we expand the term that's second order in $\mathrm{d}X_t$ and drop terms like $(\mathrm{d}t)^2$ and $\mathrm{d}t\,\mathrm{d}W_t$ (since they are of higher order than $\mathrm{d}t$) then the only surviving term will be $(\mathrm{d}W_t)^2$ and we get

$$\mathrm{d}f(t,X_t) = \frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial x}(\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2(\mathrm{d}W_t)^2\,. \tag{2.8}$$

Now here's an argument I find pretty neat. The expectation value of $\mathrm{d}W_t$ is zero and its variance is $\mathrm{d}t$, which is obviously $\mathcal{O}(\mathrm{d}t)$, so the randomness of $\mathrm{d}W_t$ is relevant at order $\mathrm{d}t$. What about $(\mathrm{d}W_t)^2$? It's expectation value is $\mathrm{d}t$ and it's variance is order $(\mathrm{d}t)^2$, which is *much* smaller than $\mathrm{d}t$, so the fact that $(\mathrm{d}W_t)^2$ is random is irrelevant at this scale, and we can simply replace it with its expectation value. This permits us to write $(\mathrm{d}W_t)^2 \to \mathrm{d}t$ which yields the expression,

Ito Lemma

$$\mathrm{d}f(t,X_t) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2\right]\mathrm{d}t + \frac{\partial f}{\partial x}\sigma_t\,\mathrm{d}W_t\,. \tag{2.9}$$

Example:

Let $\mathrm{d}X_t = \tilde{\mu}\,\mathrm{d}t + \sigma W_t$ where $\tilde{\mu}$ and σ are constant, and let $S_t = \exp(X_t)$. Then, by Ito's lemma we have

$$\mathrm{d}S_t = \left[\tilde{\mu}\exp(X_t) + \frac{1}{2}\sigma^2\exp(X_t)\right]\mathrm{d}t + \sigma\exp(X_t)\,\mathrm{d}W_t\,. \tag{2.10}$$

Since $X_t = \ln S_t$ we can equally as well write

$$\mathrm{d}S_t = \left[\tilde{\mu}S_t + \frac{1}{2}\sigma^2 S_t\right]\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t\,. \tag{2.11}$$

Now define $\mu \equiv \tilde{\mu} + \frac{1}{2}\sigma^2$ so that we can simplify this expression to

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t\,. \tag{2.12}$$

Any stochastic process satisfying eq. (2.12) is said to follow a *geometric Brownian motion*. It basically says that if $\ln S_t$ is a random walk, then S_t satisfies eq. (2.12).

3 Black-Scholes

3.1 Derivation

We're now armed with some of the tools needed to construct the Black-Scholes equation. We know that the requirement of no-arbitrage singles out a particular price which prevents savvy investors from "winning every trade". We saw how in the binomial tree model this could be achieved for pricing a European call option by constructing a replicating portfolio of cash bond and stock whose weights are updated at each discrete time step. And now we know of Ito's lemma and of geometric Brownian motion. It's time to tie these ideas together and write down the Black-Scholes equation.

Denote by $V(t, S_t)$ the value of a European call option at time t. We will suppose it expires at $t = T$ and has a strike of K. Let's start with some intuitive statements. I wrote $V(t, S_t)$ because it emphasises that V could explicitly depend on t. But do we expect that it should? The answer is yes. Imagine you have an option which expires 3 months from now versus an option that expires in 3 seconds. If both have the same strike, clearly the longer-dated option is worth more because there is *more uncertainty* at the 3-month time horizon. Therefore we anticipate that $V(t, S_t)$'s "exposure to time", quantified by $\partial V/\partial t$ is nonzero and negative.

Suppose S_t is a geometric Brownian motion so that $\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t$. We apply Ito's lemma to $V(t, S_t)$. This gives us

$$\begin{aligned}
\mathrm{d}V(t, S_t) &= \frac{\partial V}{\partial t}\,\mathrm{d}t + \frac{\partial V}{\partial S}\,\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\mathrm{d}S_t)^2 \\
&= \frac{\partial V}{\partial t}\,\mathrm{d}t + \frac{\partial V}{\partial S}(\mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t)^2 \\
&= \left[\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right]\mathrm{d}t + \sigma S_t \frac{\partial V}{\partial S}\,\mathrm{d}W_t \qquad (3.1.1)
\end{aligned}$$

Now suppose we have a portfolio consisting of Δ stock and B cash bond. Denote its value at time t by Π_t.

$$\Pi_t = \Delta S_t + B\,. \qquad (3.1.2)$$

Suppose that we want to choose Δ and B so that the change in the value after waiting a time interval $\mathrm{d}t$ exactly matches the change in the value of the option over that same infinitesimal period. In other words, we're looking for Δ and B such that

$$\mathrm{d}\Pi_t = \mathrm{d}V(t, S_t)\,. \qquad (3.1.3)$$

So long as eq. (3.1.3) is satisfied we must have $\Pi_t = V(t, S_t)$ for all t.

Otherwise, if $\Pi_t \neq V(t, S_t)$, then there would exist an opportunity to construct a risk-free profit. Specifically, if $\Pi_t > V(t, S_t)$, one could sell the portfolio and buy the option, locking in a risk-free gain. Conversely, if $\Pi_t < V(t, S_t)$, one could sell the option and buy the replicating portfolio, again guaranteeing a profit without risk. Such arbitrage opportunities cannot persist in an efficient market, meaning that the equality

$$\Pi_t = V(t, S_t) \qquad (3.1.4)$$

must hold at all times.

On one hand, computing the differential of eq. (3.1.3) yields

$$\mathrm{d}\Pi_t = \Delta\,\mathrm{d}S_t + rB\,\mathrm{d}t\,. \qquad (3.1.5)$$

Setting this equal to $\mathrm{d}V(t, S_t)$ which is given by eq. (3.1.1) and using the fact that S_t is a geometric Brownian motion, we get

$$\Delta(\mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t) + rB\,\mathrm{d}t \equiv \left[\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right]\mathrm{d}t + \sigma S_t \frac{\partial V}{\partial S}\,\mathrm{d}W_t. \qquad (3.1.6)$$

In order to eliminate the randomness from this expression we set

$$\Delta = \frac{\partial V}{\partial S} \,. \tag{3.1.7}$$

Eq. (3.1.6) then reduces to

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rB\right] \mathrm{d}t = 0 \,. \tag{3.1.8}$$

Solving eq. (3.1.8) for B and substituting back into eq. (3.1.2) we have

$$\Pi_t = \frac{\partial V}{\partial S} S_t + \frac{1}{r}\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right] . \tag{3.1.9}$$

Moreover, if we impose the boundary condition that at time T the portfolio has the same payoff as the option, that is $\Pi_T = \max(S_T - K, 0)$, then the prescription that $\mathrm{d}\Pi_t = \mathrm{d}V(t, S_t)$ enforces $\Pi_t = V(t, S_t)$ for all t so we can make the replacement $\Pi_t \to V(t, S_t)$ on the lhs of eq. (3.1.9) .

Black Scholes Equation

$$rV(t, S_t) = \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \,. \tag{3.1.10}$$

Assuming the "initial" condition $V(T, S_T) = \max(S_T - K, 0)$ one obtains the solution

$$V(S, K, t, \sigma, r) = S\Phi(d_+) - Ke^{-rt}\Phi(d_-) \tag{3.1.11}$$

where

$$d_\pm = \frac{\ln\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \tag{3.1.12}$$

and Φ is the standard normal CDF $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}\, \mathrm{d}s$.

3.2 Puts and put-call parity

Note: In this section I will change some notation. I will denote the value of a call option by C and the value of a put option by P. These should be interpreted as functions of time even if I don't explicitly write the argument, since they dependend on the time to expiration.

You can do something similar for puts to get a formula for the value of the put. An easier way is to recognise that if you have a portfolio which is long a call and short a put, its payoff is $\max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K$. When the time to expiration is T we can easily construct another portfolio with same payoff by holding one stock and borrowing Ke^{-rT} dollars. By the requirement of no-arbitrage we therefore must have,

$$C - P = S - Ke^{-rT} \,. \tag{3.2.1}$$

This important identity is known as Put-Call parity. Since it's important I'll highlight it one more time.

Put-call parity

$$C - P = S - Ke^{-rT} \tag{3.2.2}$$

This formula holds not just at expiration but at all times. It is enforced by the requirement of no-arbitrage.

3.3 Approximations and other useful identities

ATF/ATM approximation

$$C \approx 0.4 S\sigma\sqrt{T} \tag{3.3.1}$$

3.4 Connection with heat equation

Under the change of variables

$$\tau = T - t, \quad u = Ve^{r\tau}$$
$$x = \ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau$$

eq. (3.1.10) becomes the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} \tag{3.4.1}$$

with the initial condition $u(x,0) = K(e^x - 1)\,\mathbb{1}_{x>0}$. Since the Green's function for the heat equation is a Gaussian, the general solution for this initial condition involves convolving the end state with a Gaussian. From this perspective, we can think of the solution to the Black-Scholes equation as taking the payoff function and smoothing it with a Gaussian whose width depends on the time to expiration (τ) and the volatility σ. This intuition will be useful later when we discuss option risks.

4 Brownian motion

Definition 4.1: *Brownian motion*
The process $W = \{W_t : t \geq 0\}$ is a $\mathbb{P}$-Brownian motion if and only if

1. W_t is continuous, and $W_0 = 0$,
2. the value of W_t is distributed, under $\mathbb{P}$, as a normal random variable $N(0,t)$
3. the increment $W_{s+t} - W_s$ is distributed as a normal $N(0,t)$, under $\mathbb{P}$, and is independent of $\mathcal{F}_s$, the history of what that process did up to time s.

5 The Greeks

In the Black-Scholes model the value of the option is exposed to numerous factors, not just the stock price. These are quantified by the *option Greeks* Δ, Γ, $\mathcal{V}$, Θ, ρ.

The Greeks

$$\Delta \equiv \frac{\partial V}{\partial S} \tag{5.1a}$$

$$\Gamma \equiv \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S} \tag{5.1b}$$

$$\mathcal{V} \equiv \frac{\partial V}{\partial \sigma} \tag{5.1c}$$

$$\Theta \equiv \frac{\partial V}{\partial T} \tag{5.1d}$$

$$\rho \equiv \frac{\partial V}{\partial r} \tag{5.1e}$$

5.1 Approximations for the Greeks

Below are some approximations for the values of the Greeks using the Black-Scholes formula. I use Φ to denote the standard normal CDF and ϕ to denote its PDF. d_+ and d_- are defined in eq. (3.1.12)

Greek	Call Option	Put Option
Delta	$\Phi(d_+)$	$\Phi(d_+) - 1$
Gamma	$\phi(d_+)/S\sigma\sqrt{T}$	Same
Vega	$S\sqrt{T}\phi(d_+)$	Same
Rho	$KTe^{-rT}\Phi(d_-)$	$-KTe^{-rT}\Phi(-d_-)$
Theta	< 0	< 0 (can be > 0)

Delta

$\Delta \sim \Phi(d_+) \approx \left(\frac{1}{2} + 0.4\, d_+\right)$ First, $-1 < \Delta < 1$. An in-the-money call is long the stock. The deeper in the money, the closer $\Delta \to 1$. At the money ($K = S$) or at the forward ($K = Se^{rt}$), $\Delta \approx 0.5$.

Gamma

Same for puts and calls: $\Gamma \sim \frac{\phi(d_+)}{S\sigma\sqrt{T}}$. At-the-money: $\Gamma \sim \frac{0.4}{S\sigma\sqrt{T}}$. Here I'll write about calls, but the same logic works for puts. First of all, remember that we can think of the function $C(S_t, t)$ like a smoothed out version of the option payoff $\max(S_T - K, 0)$. So you can imagine taking the hockey stick function and convolving it with a Gaussian to get a smoothed out profile. As you get closer to expiration, you get closer to the hockey stick curve. At $S = K$ this function has effectively $+\infty$ convexity ($\Gamma \to \infty$). This helps explain the presence of the $\frac{1}{\sqrt{T}}$ in the formula.

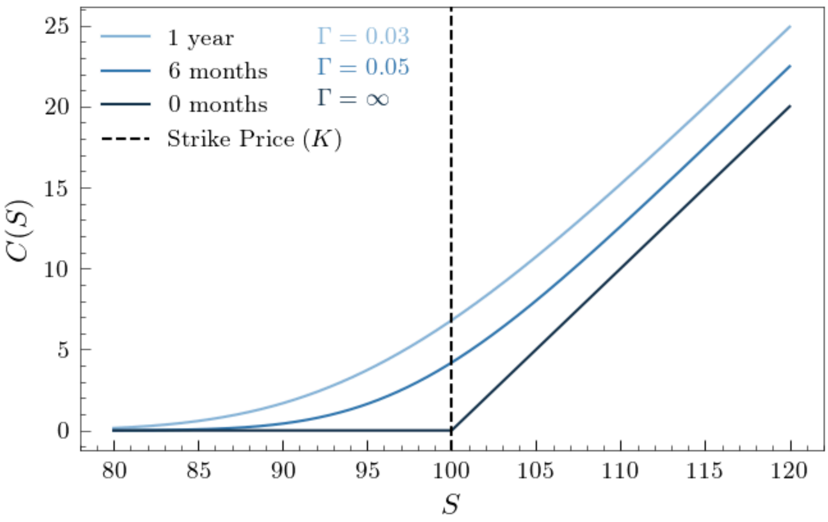

Figure 1: Call option price assuming $K = 100$, $r = 0.05$, $\sigma = 0.1$ versus spot for different DTEs shown in the legend. The legend also shows the Γ evaluated at the strike. It shows that Γ decreases with $\sqrt{T}$.

Rho

Calls: $KTe^{-rT}\Phi(d\_)$. For small r, and ATM, this is approximately $0.5\,KT(1 - rT)$. Therefore longer dated calls (larger T), increase in value more when interest rates increase. Why? If you think of a call as a leveraged position in a stock, then imagine if the stock is worth \$100 right now. If the call costs \$10 then you can buy the call and effectively free up \$90 to invest at the risk-free rate, which is good for you. Another way to look at it is that options are kind of like a delayed purchase with a fixed price of $\$K$ (which you are not obligated to exercise). If you have $\$Ke^{-rT}$ dollars today, then you can afford to exercise the option at expiration. The higher r is, the less cash you need to have now to do this, which makes the call more attractive. Therefore *calls gain value when interest rates increase.*

Puts: $P = KTe^{-rT}\Phi(-d_2)$ For small r, and at-the-money (ATM), this is approximately $P \approx 0.5KT(1 - rT)$. Since the last term $-rT$ reduces the value of the put, we see that *puts lose value when interest rates increase.* Why? Unlike calls, a put option behaves more like a prepaid insurance policy against a stock decline. If you want to short a stock, one alternative is simply borrowing and selling the stock today, which costs nothing upfront and gives you cash in hand that you can invest at the risk-free rate.

However, buying a put requires *paying upfront*, which means you have *less money to invest elsewhere.* When interest rates rise, this opportunity cost becomes larger, making puts *less attractive.* Another way to see this:
- A put option gives you the right to *sell at a fixed price (K) in the future.*
- If interest rates are high, the present value of receiving (K) in the future *decreases.*
- This makes the put less valuable today because the strike price effectively "shrinks" in real terms as rates rise.

5.2 Option portfolios

1. Suppose you're long one ATM call with expiration at T_1 and short another ATM call in the same underlying and the same strike but expiration at $T_2 > T_1$. Is your position
   a. long ρ?

b. long $\mathcal{V}$?
c. long the underlying?

Since calls increase in value with r the long call is long Rho and the short call is short Rho. However, the effect of interest is greater when there is more time to expiration so the portfolio is net short Rho. Something similar can be said for $\mathcal{V}$. The position is net short $\mathcal{V}$. Delta is a bit tricky but we can figure it out in a few ways. First of all, we can approximate $\Delta \approx \Phi(d_1) \sim \left(\frac{r}{\sigma} + \sigma\right)\sqrt{T}$ (where I've ignored a term that decays like $1/\sqrt{T}$ as well as constant terms, because I only care about the dominant scaling with T). This suggests that Δ will generally grow with time to expiration. So the position is short the stock. Another way to see this is to look at the slopes of the curves in 1. As DTE gets smaller the change in slope becomes more dramatic and asymptotes to $\Delta = 0.5$ for very small T.