Neutron star cooling by axion emission

Ray Hagimoto

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Contents

1 Introduction

The goal of this project is to evaluate the integral

Emissivity integral

$$\varepsilon_3 = \int f_a f_b (1-f_1)(1-f_2)(1+f_3) \\ \times E_3 \,\overline{|\mathcal{M}|^2}\, \mathrm{d}\Phi_3\big((p_a+p_b)^2; m_1, m_2, m_3\big) \frac{\mathrm{d}^3 p_a}{(2\pi)^3\, 2E_a} \frac{\mathrm{d}^3 p_b}{(2\pi)^3\, 2E_b} \tag{1.1}$$

where,

Lorentz invariant phase space measure

$$\mathrm{d}\Phi_n\big((p_a+p_b)^2; 1,\ldots,n\big) \equiv \prod_{i=1}^{n} \left[\delta(p_i^2 - m_i^2)\,\Theta(p_i^0)\,\frac{\mathrm{d}^4 p_i}{(2\pi)^3}\right](2\pi)^4 \delta^4\left(p_a + p_b - \sum_{i=1}^{n} p_i\right) \tag{1.2}$$

is the Lorentz-invariant phase space measure, and

$$f_i \equiv f_{FD}(E_i) \equiv \frac{1}{e^{(E_i-\mu_i)/T}+1} \text{ for } i = a, b, 1, 2 \tag{1.3}$$

$$f_3 \equiv f_{BE}(E_i) \equiv \frac{1}{e^{(E_3+\mu_3)/T}+1} \tag{1.4}$$

are thermal distributions which show up due to Pauli blocking and Bose enhancement . The spin-summed matrix element squared is given by

Spin-summed matrix element

$$\overline{|\mathcal{M}|^2} = \frac{128\, g_{ae\mu}^2 e^4}{(m_a^2 - m_1^2)^2} \frac{(p_a \cdot p_1 - m_a m_1)(p_b \cdot p_3)(p_2 \cdot p_3)}{(p_b - p_2)^4} \quad . \tag{1.5}$$

Some comments before we move on:

1. The axions (denoted in this introduction by 3) are effectively free-streaming so the Bose-enhancement factor $(1+f_3)$ will be ignored.

2. The thermal factors f_i are not Lorentz invariant (unlike the spin-summed matrix element and the integration measure) so we emphasize that the energies appearing in eqs. (1.3) and (1.4) are measured in the rest frame of the neutron star. This fact will be very important to remember later.

1.1 Integration in terms of momentum variables (didn't work)

The first strategy I used to evaluate this integral was to do the integration directly in terms of the momentum variables. Below are the notes I wrote detailing my strategy.

We can use the momentum conserving Dirac delta to evaluate the $\boldsymbol{p}_2$ integral, setting

$$\boldsymbol{p}_2 = \boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3. \tag{1.6}$$

We now choose to align the z-axis with $\boldsymbol{p}_a$ and measure angles with respect to this axis. Converting to spherical polar coordinates gives, for example, $\mathrm{d}^3 p_3 = p_3^2 \,\mathrm{d}p_3 \,\mathrm{d}\cos\theta_{13'} \,\mathrm{d}\phi_{13'}$. Then the energy conserving Dirac delta can be used to evaluate the $\mathrm{d}p_3$ integral in the following way. First, we use $E = \sqrt{p^2 + m^2}$ to rewrite the masses. Then we use momentum conservation to make the replacement $\boldsymbol{p}_2 \to \boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3$:

$$\begin{aligned} & E_a + E_b = E_1 + E_2 + E_3 \\ & \Rightarrow \sqrt{p_a^2 + m_a^2} + \sqrt{p_b^2 + m_b^2} = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} + \sqrt{p_3^2 + m_3^2} \\ & \Rightarrow \sqrt{p_a^2 + m_a^2} + \sqrt{p_b^2 + m_b^2} = \sqrt{p_1^2 + m_1^2} + \sqrt{(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3)^2 + m_2^2} + \sqrt{p_3^2 + m_3^2}. \end{aligned} \tag{1.7}$$

We would like to solve eqn. (1.7) for p_3, but there is a problem coming from the fact that p_3 appears under two square root symbols which makes it impossible to get an expression of the form $p_3 = \cdots$. Instead, we will make the approximation that because $p_3 \ll p_i$ for $i \in \{1, 2, 1', 2'\}$ that we can ignore the p_3 that shows up in $\sqrt{(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3)^2 + m_2^2}$. The result we obtain when solving for p_3 is then

$$\begin{aligned} p_3 = \Big[& m_a^2 + m_1^2 + m_b^2 + m_2^2 - m_3^2 + 2p_a^2 - 2p_a p_1 c_{11'} + 2p_1^2 - 2E_a E_1 \\ & + 2p_a p_b c_{12} - 2p_b p_1 c_{21'} + 2p_b^2 + 2E_a - 2E_1 E_b \\ & - 2E_a E_2(p_a, p_b, p_1, c_{11'}, c_{12}, c_{21'}) + 2E_1 E_2(p_a, p_b, p_1, c_{11'}, c_{12}, c_{21'}) \\ & - 2E_b E_2(p_a, p_b, p_1, c_{11'}, c_{12}, c_{21'}) \Big]^{1/2} \end{aligned} \tag{1.8}$$

where $E_i \equiv \sqrt{p_i^2 + m_i^2}$, $c_{ij} \equiv \cos\theta_{ij} \equiv \boldsymbol{p}_i \cdot \boldsymbol{p}_j / (p_i p_j)$ and

$$E_2(p_a, p_b, p_1, c_{11'}, c_{12}, c_{21'}) = \sqrt{p_a^2 - 2p_a p_1 c_{11'} + p_1^2 + 2p_a p_b c_{12} - 2p_b p_1 c_{21'} + p_b^2 + m_2^2}. \tag{1.9}$$

Now that we have an expression for p_3 the requirement that p_3 must be real and non-negative restricts the domain of integration of the other variables. I am not sure how to derive the new limits so I got stuck here.

1.2 A recursive expression for phase space integrals

The problem I ran into at the end of sec. (1.1) was how to use the energy-momentum delta functions to restrict the bounds of integration. I spent some time searching the literature for useful references and the most promising were refs. [1, 2, 3]. [1] provides an introduction to Monte Carlo methods and discusses the phase space measure in sec. 9, which covers several ways to deal with the Dirac delta in the phase space integration measure by doing appropriate changes of variables. The most useful method is covered in sec. 9.6 and is the same one used in [2]. Ref. [2] extends the treatment in sec. 9.6 of [1] by also introducing momentum transfer variables. In this section we will introduce the formalism without momentum transfer variables and cover the case with momentum transfers in sec. (1.3).

Consider the phase space integral $R_n(p_a + p_b)$ defined by

Phase space integral (without 2π factors)

$$R_n(p_a+p_b) = \int \cdots \int \prod_{i=1}^{n} \delta(p_i^2 - m_i^2)\, \Theta(p_i^0)\, \mathrm{d}^4 p_i\, \delta^4(p_a + p_b - p_1 - \cdots - p_n). \quad (1.10)$$

where p_i now denotes a four-momentum vector in constrast to the previous section where it represented a three-momentum magnitude. This is manifestly Lorentz invariant and therefore R_n may only be a function of $s \equiv (p_a + p_b)^2$.

We note that $R_n(s)$ is related to $\mathrm{d}\Phi_n(s)$ by

$$R_n(s) = (2\pi)^{3n-4} \int \mathrm{d}\Phi_n(s) \quad \Rightarrow \quad \int \mathrm{d}\Phi_3(s) = (2\pi)^{-5} R_3(s). \quad (1.11)$$

Define the new integration variable $M_{n-1}^2 \equiv (p_a + p_b - p_n)^2$. The physical significance of M_{n-1} is that it is the invariant mass of the first $n-1$ particles, which can be seen using four-momentum conservation: $p_a + p_b - p_n = p_1 + p_2 + \cdots + p_{n-1}$. It is possible to show that the following recursive relation holds (eqn. (3) of [2])

$$R_n(s) = \int_{(m_1+m_2+\cdots+m_{n-1})^2}^{(\sqrt{s}-m_n)^2} \mathrm{d}M_{n-1}^2 \int \mathrm{d}\Omega_n \frac{\sqrt{\lambda(s, M_{n-1}^2, m_n^2)}}{8s} R_{n-1}(M_{n-1}^2) \quad (1.12)$$

where $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ and the solid angle $\mathrm{d}\Omega_n \equiv \mathrm{d}\cos\theta_n\, \mathrm{d}\varphi_n$ defines the direction of $\boldsymbol{p}_n$ in the frame where $p_a + p_b = (\sqrt{s}, \mathbf{0})$. A derivation of eq. (1.12) can be found in appendix A. The upper limit is obtained by requiring that

$$|\boldsymbol{p}_n|^2 = \frac{\lambda(s, M_{n-1}^2, m_n^2)}{4s} \geq 0\ , \quad (1.13)$$

whereas the lower limit is given by the threshold below which $R_{n-1}(M_{n-1}^2) = 0$. Both limits can be derived from physical considerations too. Repeated application of eq. (1.12) yields

$$\begin{aligned} R_n(s) = &\int_{(m_1+m_2+\cdots+m_{n-1})^2}^{(\sqrt{s}-m_n)^2} \mathrm{d}M_{n-1}^2 \int \mathrm{d}\Omega_n \frac{\sqrt{\lambda(s, M_{n-1}^2, m_n^2)}}{8s} \\ &\times \int_{(m_1+m_2+\cdots+m_{n-2})^2}^{(M_{n-1}-m_{n-1})^2} \mathrm{d}M_{n-2}^2 \int \mathrm{d}\Omega_{n-1} \frac{\sqrt{\lambda(M_{n-1}^2, M_{n-2}^2, m_{n-1}^2)}}{8M_{n-1}^2} \\ &\times \cdots \times \int_{(m_1+m_2)^2}^{(M_3-m_3)^2} \mathrm{d}M_2^2 \int \mathrm{d}\Omega_3 \frac{\sqrt{\lambda(M_3^2, M_2^2, m_3^2)}}{8M_3^2} \times \int \mathrm{d}\Omega_2 \frac{\sqrt{\lambda(M_2^2, m_1^2, m_2^2)}}{8M_2^2}. \end{aligned} \quad (1.14)$$

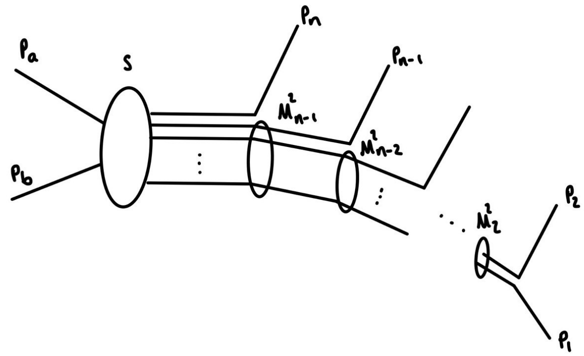

Figure 1: Illustration of the recursion relation as a sequence of effective $2 \to 2$ scattering events (this diagram is largely based on pg. 26 of [1]).

The important case for me is when $n = 3$:

Phase space measure for $n = 3$

$$\begin{aligned} R_3(s) &= (2\pi)^5 \int \mathrm{d}\Phi_3(s) \\ &= \int_{(m_1+m_2)^2}^{(\sqrt{s}-m_3)^2} \mathrm{d}M_2^2 \int \mathrm{d}\Omega_3 \, \frac{\sqrt{\lambda(s, M_2^2, m_3^2)}}{8s} \int \mathrm{d}\Omega_2 \, \frac{\sqrt{\lambda(M_2^2, m_1^2, m_2^2)}}{8M_2^2} \, . \end{aligned} \tag{1.15}$$

It's easy to verify that the number of integration variables matches our expectation of $3(3) - 4 = 5$ since we are integrating over one invariant mass, and two pairs of two angles.

In the above discussion we have considered p_a and p_b to be fixed. However, for my purposes I will also be integrating over p_a and p_b. This case seems to be similar to the one considered in [3] (cf. the example given between their eqs. (12) and (13)).

1.3 Writing the integral in terms of momentum transfers

In sec 1.2 we wrote the phase space integral in terms of invariant masses and angles of the three-momenta $\boldsymbol{p}_i$ defined in the center-of-mass frame where $\sum_{k=1}^{i} p_k = (M_i, \mathbf{0})$. The matrix element in eq. (1.5) is a function of the Lorentz invariants $(p_a \cdot p_1)$, $(p_b \cdot p_3)$, $(p_2 \cdot p_3)$, and $(p_b \cdot p_2) = (-(p_a - p_1 - p_2 - p_3) \cdot p_2)$. It is cumbersome to write this matrix element explicitly in terms of the angles $\{(\theta_i, \varphi_i) : i = 1, 2, \ldots, n\}$ so we would rather use kinematic Lorentz invariants analogous to the Mandelstam variables for $2 \to 2$ scattering. To this end it is convenient to introduce the so-called 'momentum transfers' $t_i \equiv Q_i^2 \equiv (p_a - p_1 - \cdots - p_i)^2 = (p_n + p_{n-1} + \cdots + p_{i+1} - p_b)^2$.

Some of the dot products (but not all) may be written exclusively in terms of the kinematic variables. For example,

$$
\begin{aligned}
t_1 &\equiv (p_a - p_1)^2 && \to && 2p_a \cdot p_1 = m_a^2 + m_1^2 - t_1 \\
t_2 &\equiv (p_a - (p_1 + p_2))^2 && \to && 2p_a \cdot p_2 = M_2^2 - m_1^2 + t_1 - t_2 \\
t_3 &\equiv (p_a - (p_1 + p_2 + p_3))^2 && \to && 2p_a \cdot p_3 = t_2 - t_3 - M_2^2 + M_3^2 \\
M_2^2 &\equiv (p_1 + p_2)^2 && \to && 2p_1 \cdot p_2 = M_2^2 - m_1^2 - m_2^2 \\
M_3^2 &\equiv (p_1 + p_2 + p_3)^2 && \to && 2(p_1 + p_2) \cdot p_3 = M_3^2 - M_2^2 - m_3^2 \\
p_b &= p_1 + p_2 + p_3 - p_a && \to && 2p_b \cdot p_3 = m_3^2 + t_3 - t_2 \\
& && && 2p_b \cdot p_2 = 2p_2 \cdot p_3 + m_2^2 + t_2 - t_1
\end{aligned}
$$

I could not write $p_2 \cdot p_3$ in terms of the kinematic invariants alone. This is because we need $3(3)-4 = 5$ integration variables. Right now we only have $t_1,\ t_2,\ M_2^2$ which is not enough to fully describe the system; two more are required. The additional degrees of freedom are given by the azimuthal angles of the $\boldsymbol{Q}_i$ vectors φ_1, φ_2 as seen in a frame where $p_a = (m_a, \mathbf{0})$ (cf. eq. (1.19) which is eq. (14) of [2].) It is worth describing in more detail the meaning of the angles φ_1 and φ_2. I realised on Aug 18 2023 that these azimuthal angles can't be measured from the same axis. The reason for this is the derivation of eq. (1.17) and eq. (1.19) involved a step where we rewrote $\mathrm{d}^3 Q_{n-1}$ as $|\boldsymbol{Q}_{n-1}|^2\, \mathrm{d}|\boldsymbol{Q}_{n-1}|\, \mathrm{d}\cos\theta_{n-1} \mathrm{d}\varphi_{n-1}$. This transformation requires that θ_{n-1} be measured from a polar axis and φ_{n-1} be measured in the plane perpendicular to that axis. Since θ_i is defined as the angle between $\boldsymbol{Q}_i$ and $\boldsymbol{Q}_{i+1}$, φ_i and φ_{i+1} can not be measured in the same plane.

To transform R_n into a form in which the momentum transfers appear as variables we must consider the phase space integral as a function of p_a and $-p_b \equiv Q_n$ separately: $R_n = R_n(p_a, Q_n)$. Let $Q_i = p_a - p_1 - \cdots - p_i$ so that $p_i = Q_{i-1} - Q_i$. Then, eq. (1.10) becomes

In eq. (1.16) I didn't write the Heaviside function which picks out positive energies, just pretend it is there so that $(Q_{i-1} - Q_i)^0 > 0$.

$$
\begin{aligned}
R_n(s) &= \int \left[\prod_{i=1}^{n} \mathrm{d}^4 Q_i\, \delta\big((Q_{i-1} - Q_i)^2 - m_i^2\big) \right] \delta^4(p_a + p_b - \sum_{i=1}^{n} (Q_{i-1} - Qi))|_{Q_0 \equiv p_a} \\
&= \int \left[\prod_{i=1}^{n} \mathrm{d}^4 Q_i \delta\big((Q_{i-1} - Q_i)^2 - m_i^2\big) \right] \delta^4(Q_n + p_b) \qquad (1.16) \\
&= \int \mathrm{d}^4 Q_{n-1} \delta\big((Q_{n-2} - Q_{n-1})^2 - m_{n-1}^2\big) \left[\prod_{i=1}^{n-2} \mathrm{d}^4 Q_i \delta\big((Q_{i-1} - Q_i)^2 - m_i^2\big) \right].
\end{aligned}
$$

The recursion relation is then simply

$$
R_n(p_a, Q_n) = \int \mathrm{d}^4 Q_{n-1}\, \delta\big((Q_{n-1} - Q_n)^2 - m_n^2\big)\, R_{n-1}(p_a, Q_{n-1}). \qquad (1.17)
$$

R_n is now regarded as a function of two invariants $s = s_n = (p_a - Q_n)^2$, and $t_n = Q_n^2$: $R_n = R_n(p_a, Q_n) = R_n(s_n, t_n)$. Introducing these variables in eq. (1.17) the authors of [2] obtain

$$
R_n(s_n, t_n, s_{n-1}, t_{n-1}) = \int \mathrm{d}s_{n-1}\, \mathrm{d}t_{n-1}\, K(s_n, t_n, s_{n-1}, t_{n-1})\, R_{n-1}(s_{n-1}, t_{n-1})\,, \qquad (1.18)
$$

where

$$
\begin{aligned}
&K(s_n, t_n, s_{n-1}, t_{n-1}) \\
&\quad = \int \mathrm{d}^4 Q_{n-1} \delta(Q_{n-1}^2 - t_{n-1})\, \delta(s_{n-1} - (p_a - Q_{n-1})^2)\, \delta((Q_{n-1} - Q_n)^2 - m_n^2) \\
&\quad = \int_0^{2\pi} \mathrm{d}\varphi_{n-1} \frac{1}{4\sqrt{\lambda(s_n, t_n, m_a^2)}}\, \Theta(-G(t_{n-1}, s_n, s_{n-1}, t_n, m_n^2, m_a^2)) \ ,
\end{aligned}
\tag{1.19}
$$

$\Theta(x)$ is the Heaviside step-function, and φ_{n-1} defines the azimuthal angle of $\boldsymbol{Q}_{n-1}$ in the frame $p_a = (m_a, \mathbf{0})$. G is a kinematic function defined by

$$
\begin{aligned}
G(x, y, z, u, v, w) &= -\frac{1}{2} \begin{vmatrix} 2u & x+u-v & u+w-y \\ & 2x & x-z+w \\ \text{(symm.)} & & 2w \end{vmatrix} \\
&= x^2 y + x y^2 + z^2 u + v w^2 + v^2 w \\
&\quad + xzw + xuv + yzv + yuw \\
&\quad - xy(z+u+v+w) - zu(x+y+v+w) - vw(x+y+z+u) \ .
\end{aligned}
\tag{1.20}
$$

In addition, the Dirac deltas impose the following relations hold:

Formulas for angles and magnitudes in terms of integration variables

$$
|\boldsymbol{Q}_i| = \frac{\sqrt{\lambda_i}}{2m_a} \tag{1.21}
$$

$$
\varepsilon_i = \frac{\sqrt{\Lambda_i}}{2m_a} \tag{1.22}
$$

$$
\cos\theta_i \equiv \frac{\boldsymbol{Q}_i \cdot \boldsymbol{Q}_{i+1}}{|\boldsymbol{Q}_i||\boldsymbol{Q}_{i+1}|} = \frac{\xi_i}{\sqrt{\lambda_i \lambda_{i+1}}} \tag{1.23}
$$

$$
\sin\theta_i = \sqrt{\frac{4m_a^2(-G_i)}{\lambda_i \lambda_{i+1}}} \tag{1.24}
$$

where

$$
\lambda_i \equiv \lambda(s_i, t_i, m_a^2) \tag{1.25}
$$

$$
\Lambda_i \equiv \lambda_i + 4m_a^2 t_i = (s_i - t_i - m_a^2)^2 \tag{1.26}
$$

$$
G_i \equiv G(t_i, s_{i+1}, s_i, t_{i+1}, m_{i+1}^2, m_a^2) < 0 \tag{1.27}
$$

$$
\xi_i \equiv \sqrt{\Lambda_i \Lambda_{i+1}} - 2m_a^2(t_i + t_{i+1} - m_{i+1}^2) \ . \tag{1.28}
$$

The main result for this section is the phase space measure for the case of 3 outgoing particles, which we highlight below.

R_n for $n = 3$

$$
R_3(s_3, t_3) = \frac{1}{4\sqrt{\lambda(s_3, t_3, m_a^2)}} \int \frac{\mathrm{d}s_2 \mathrm{d}t_2 \mathrm{d}\varphi_2}{4\sqrt{\lambda(s_2, t_2, m_a^2)}} \Theta(-G_2) \int \mathrm{d}t_1 \mathrm{d}\varphi_1 \Theta(-G_1) \tag{1.29}
$$

with

$$G_i = G(t_i, s_{i+1}, s_i, t_{i+1}, m_{i+1}^2, m_a^2) \; . \tag{1.30}$$

To use this method we rewrite parts the integrand by making the replacements:

$$2p_a \cdot p_1 \to m_a^2 + m_1^2 - t_1 \tag{1.31}$$
$$2p_a \cdot p_2 \to s_2 - m_1^2 + t_1 - t_2 \tag{1.32}$$
$$2p_a \cdot p_3 \to t_2 - t_3 - s_2 + s_3 \tag{1.33}$$
$$2p_1 \cdot p_2 \to s_2 - m_1^2 - m_2^2 \tag{1.34}$$
$$2p_b \cdot p_3 \to m_3^2 + t_3 - t_2 \tag{1.35}$$
$$2p_b \cdot p_2 \to 2p_2 \cdot p_3 + m_2^2 + t_2 - t_1 \; . \tag{1.36}$$

Then, we will explicitly generate the four-momenta p_a^{NS}, p_b^{NS}, p_1^{NS}, p_2^{NS}, p_3^{NS} as seen in the rest frame of the neutron star by

1. Sampling p_a^{NS} and p_b^{NS} from the Fermi-Dirac distribution $f_{FD}(E_a^{NS})$ – this means that we can drop those factors in the integrand.

2. Generating the outgoing particle momenta p_i in the frame $p_a = (m_a, 0)$ and then boosting them into a frame where $p_a = p_a^{NS}$ to obtain p_i^{NS}.

1.4 Monte Carlo integration using momentum transfers

So far we have rewritten the integral in eq. (1.1) as

$$\varepsilon_3 = \int f_a f_b (1-f_1)(1-f_2)\, E_3 \sum_{\sigma,\sigma'} |\mathcal{M}|^2 \, \mathrm{d}\Phi_3 \, \frac{\mathrm{d}^3 p_a}{(2\pi)^3 2E_a} \, \frac{\mathrm{d}^3 p_b}{(2\pi)^3 2E_b} \tag{1.37}$$

where $\mathrm{d}\Phi_3 = (2\pi)^{-5} R_3$ is given by eq. (1.29) and $\sum_{\sigma,\sigma'} |\mathcal{M}|^2$ is given by eq. (1.5) with the replacements given in eqs. (1.31)–(1.36). Furthermore, it is understood that the energies that are to be plugged into the f distributions are to be evaluated in the rest-frame of the neutron star. Thus, we interpret $\mathrm{d}^3 p_a$ as $\mathrm{d}^3 p_a^{NS}$. The momenta p_a^{NS} and p_b^{NS} will be sampled in the following way. First, rewrite the ingoing momentum integrals

$$\frac{\mathrm{d}^3 p_a}{2E_a} \to \frac{1}{2} \sqrt{E_a^2 - m_a^2} \, \mathrm{d}E_a \, \mathrm{d}\cos\theta_a \mathrm{d}\varphi_a \tag{1.38}$$
$$\frac{\mathrm{d}^3 p_b}{2E_b} \to \frac{1}{2} \sqrt{E_b^2 - m_b^2} \, \mathrm{d}E_b \, \mathrm{d}\cos\theta_b \mathrm{d}\varphi_b \; . \tag{1.39}$$

Next, draw E_a and E_b from the distributions f_a and f_b. This is an example of importance sampling, so the factors of f_a and f_b in the integrand will be canceled out by sampling $E_{a,b}$ in this way. Sampling can be accomplished by generating realisations of the random variable $X \sim \mathrm{U}(0,1)$ and computing $E_a = F^{-1}(X)$ where $F(E_a) = \frac{1}{N}\int_0^{E_a} f(E)\,\mathrm{d}E$ is the cumulative distribution function and $N = \int_0^\infty f(E)\,\mathrm{d}E$ is a normalization factor. These integrals can be performed analytically with the results

Formulas for sampling ingoing particle energies (as seen in the rest-frame of the neutron star)

$$N = \int_0^\infty f(E)\,\mathrm{d}E = \int_0^\infty \frac{1}{1+e^{(E-\mu)/T}}\,\mathrm{d}E = T\ln\left(1+e^{\mu/T}\right)\ , \tag{1.40}$$

$$F(E_a) = \frac{1}{N}\int_0^{E_a} f(E)\,\mathrm{d}E = \frac{\ln\left(\frac{1+e^{\mu/T}}{1+e^{-(E-\mu)/T}}\right)}{\ln\left(1+e^{\mu/T}\right)}\ , \tag{1.41}$$

$$F^{-1}(x) = -T\ln\left(1+e^{\mu/T}\right)\exp\left\{-x\ln\left(1+e^{\mu/T}\right)-1\right\}+\mu\ . \tag{1.42}$$

I verified these formulas by drawing 100,000 samples of X, evaluating $E_a = F^{-1}(X)$ for each, and plotting a histogram of the E_a samples against the distribution f_a/N as shown in fig. (2).

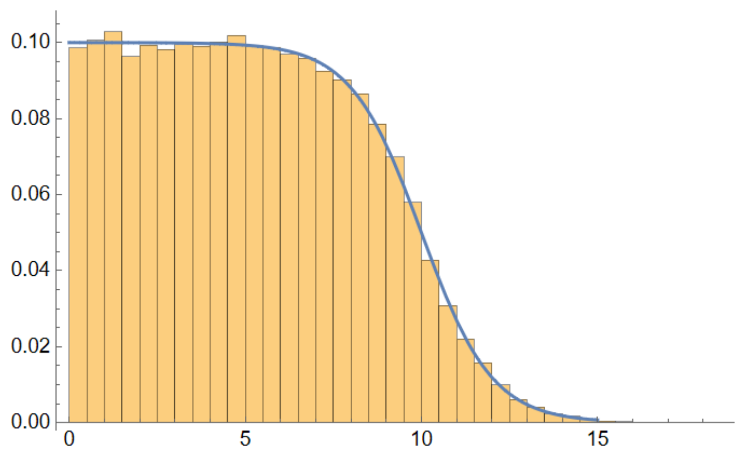

Figure 2: Verification of my method for sampling from a Fermi-Dirac distribution. Histogram is on samples obtained using the method described in sec. (1.4) and blue curve is $f_{FD}(E)$. I chose fiducial values of $\mu = 10$ and $T = 1$.

1.4.1 Momentum reconstruction

To sample s_i we sample a uniform random variable $r_i \sim \mathrm{U}(0,1)$ and compute

$$\sqrt{s_i} = r_i\left(\sqrt{s_n} - \sum_{j=1}^{n} m_j\right) + \sum_{j=1}^{n} m_j\ . \tag{1.43}$$

Since we are integrating over s_i, not $\sqrt{s_i}$, sampling uniformly in $\sqrt{s_i}$ requires us to multiply the integrand by a factor of $2\sqrt{s_i}$ since $\mathrm{d}s = \mathrm{d}(\sqrt{s})^2 = 2\sqrt{s}\ \mathrm{d}(\sqrt{s})$.

Next, we take the generated value of s_i (let's call it $\bar{s}_i$) and use it to calculate the upper and lower bounds from which we generate t_i.

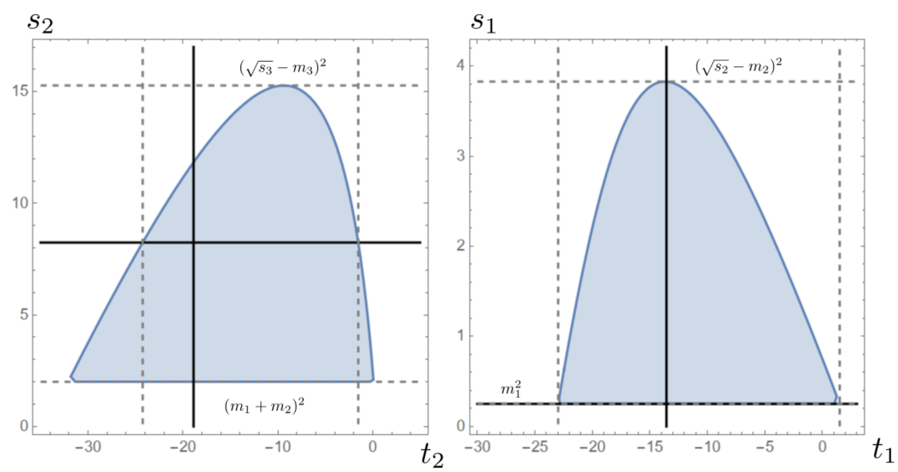

Figure 3: Illustration of sampling from the physical region of parameter space dictated by the $\Theta(-G_i)$ step-functions in eq. (1.29). *(Left:)* Region of the (t_2, s_2) plane where G_2 is negative (shaded blue). This corresponds to the physically allowed region of parameter space. The boundary of the region is established by the condition $G_2 = 0$. The lower horizontal dashed line is a lower bound established in the case when the C.M. energy of the effective particle $(p_1 + p_2)$ only has enough energy to create the particle with no kinetic energy $(s_2 = (m_1 + m_2)^2)$. The upper bound is established when the center of mass energy of $(p_1 + p_2 + p_3)$ creates p_3 at rest so that $p_3 = (m_3, 0)$ and the remaining energy is used to create $(p_1 + p_2)$ and supply it with kinetic energy. The thick black bars show values of the parameters created using the routine described in the box "Random number generation for Monte Carlo".

To do this we solve $G(t_i, s_{i+1}, \bar{s}_i, t_{i+1}, m_{i+1}^2, m_a^2) = 0$ for t_i. This yields,

$$t_i^{\pm} = t_{i+1} + m_{i+1}^2 + \frac{(\bar{s}_i - s_{i+1} - m_{i+1}^2)(t_{i+1} + s_{i+1} - m_a^2) \pm \sqrt{\lambda(\bar{s}_i, s_{i+1}, m_{i+1}^2)\lambda(s_{i+1}, t_{i+1}, m_a^2)}}{2\, s_{i+1}} . \tag{1.44}$$

So,

$$t_i^{\text{gen}} = \gamma_i (t_i^+ - t_i^-) + t_i^- \tag{1.45}$$

where $\gamma_i \sim \mathrm{U}(0,1)$ is another uniform random variable. This procedure requires us to weight the generated event by a factor of $(t_i^+ - t_i^-)$. Finally, φ_1 and φ_2 are sampled independently from $\mathrm{U}(0, 2\pi)$.

Once s_2, t_1, t_2, φ_1, and φ_2 have been generated using the procedure described above we can reconstruct the outgoing momenta p_i in the following way. First, calculate the components of $\boldsymbol{Q}_i$ in the rest frame of a. In particular, the components are given by substituting eqs. (1.21–1.28) into the following formulas,

$$\boldsymbol{Q}_2 = |\boldsymbol{Q}_2| (\sin\theta_2 \cos\varphi_2,\ \sin\theta_2 \sin\varphi_2,\ \cos\theta_2) \tag{1.46}$$

$$\boldsymbol{Q}_1 = |\boldsymbol{Q}_1| R(\theta_2, \varphi_2)(\sin\theta_1 \cos\varphi_1,\ \sin\theta_1 \sin\varphi_1,\ \cos\theta_1) \tag{1.47}$$

where $R(\theta_2, \varphi_2)$ is rotation matrix that takes $(0,0,1)$ into $(\sin\theta_2 \cos\varphi_2, \sin\theta_2 \sin\varphi_2, \cos\theta_2)$. Next, demand calculate the energy-component of Q_i as $Q_i^0 = \pm\sqrt{|\boldsymbol{Q}_i|^2 + t_i}$. The sign is chosen by imposing the physical momenta have positive energy so, e.g., $(p_2)^0 = (Q_1^0 - Q_2^0) > 0$. The way I have implemented this is to first compute the positive sign solutions $\varepsilon_i \equiv \sqrt{|\boldsymbol{Q}_i|^2 + t_i}$ and choose $\mathrm{sgn}(Q_i^0) = (-1)^{1+\Theta(\varepsilon_1 - \varepsilon_2)}$, where $\Theta(x)$ is the Heaviside step function. The physical momenta are then given in the rest frame of a frame by

$$p_1 = (p_a - Q_1) \tag{1.48}$$

$$p_2 = (Q_1 - Q_2) \tag{1.49}$$

$$p_3 = (Q_2 + p_b)\ . \tag{1.50}$$

2 Monte Carlo integration

2.1 The basics

Suppose you are interested in evaluating the integral

$$I = \int_a^b f(x)\,\mathrm{d}x. \tag{2.1}$$

The most obvious method of approximating this is with a Riemann sum or trapezoidal integration where you divide the domain into bins of width Δx and calculate the area of a rectangle or trapezoid using the values of f at the bin edges.

Monte Carlo integration, on the other hand, takes a a different approach altogether whereby points in the domain are sampled at random and the integral is approximated as an average.

To see this, note that the average value of f on the domain is

$$\langle f \rangle = \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x = \frac{I}{b-a}. \tag{2.2}$$

The average $\langle f \rangle$ can be approximated by uniformly sampling N i.i.d random numbers x_i on the domain $x \in [a, b]$ where $i \in 1, 2, ..., N$ and calculating the mean of $f(x_i)$:

$$\langle f \rangle \approx \langle f \rangle_N \equiv \frac{1}{N} \sum_{i=1}^{N} f(x_i). \tag{2.3}$$

Therefore we can approximate the integral I as

$$I \approx (b-a)\langle f \rangle_N. \tag{2.4}$$

This argument can be extended to higher dimensions, i.e. in n dimensions we have

$$I = \int_V f(\boldsymbol{x})\, \mathrm{d}^n x = V \langle f \rangle. \tag{2.5}$$

One important result is that this strategy always converges as $N^{-1/2}$ in all dimensions, whereas techniques like trapezoidal integration converge rapidly in one dimension (N^{-2}), but significantly slower for higher dimensions ($N^{-2/n}$). For example in 5 dimensions, trapezoidal integration converges like $N^{-2/5}$, which is slower than Monte Carlo integration.

2.2 Importance sampling

In sec. 2.1 we sampled x uniformly at random on the domain of integration. However, we can choose to sample x from any arbitrary distribution $p(x)$ and compute the same expectation value by weighting the sample $f(x)$ by $1/p(x)$ so that

$$I \approx \frac{V}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}. \tag{2.6}$$

With an appropriate choice of p our Monte Carlo integration technique can be made to converge significantly faster than by sampling x uniformly at random. To see why this is, suppose $f(x) = e^{-x^2}$ and $[a, b] = [-1000, 1000]$. Sampling uniformly on $[-1000, 1000]$ means that the majority of points we pick will be in the tails of the Gaussian and will hardly affect the value of the integral. If instead we sampled from a standard normal distribution we would mostly choose x values near the peak at $x = 0$ and our result would converge much more rapidly. Clearly the choice of p is heavily dependent on the particular shape of the integrand f. The Metropolis-Hastings algorithm is one way to generate samples x from $p(x)$ and can be used for importance sampling.

<u>VEGAS algorithm:</u>

The discussion of the VEGAS algorithm at `https://www.ippp.dur.ac.uk/~krauss/Lectures/QuarksLeptons/Basics/PS_Vegas.html` is quite good. In case the link breaks in the future I reproduce some of the main points below.

"The VEGAS algorithm starts by dividing the n-dimensional hypercube $[0,1]^n$ into smaller ones – usually of identical size – and performs a sampling in each of them. The results are then used to refine the integration grid for the next iteration of the sampling procedure."

2.3 Stratified sampling

2.4 Multichannel sampling

3 Tackling a simpler integral

August 28 2023: I spent the last week working on the response to the referee on my non-Gaussianity from axion-string birefringence paper. Now I want to return to this integral project.

I will try doing a simpler integral where the ingoing particle momenta p_a and p_b are fixed and we integrate only over the outgoing states. I will also drop the Lorentz-invariance breaking thermal factors.

$$\int \sum_{\sigma,\sigma'} |\mathcal{M}|^2 \mathrm{d}\Phi_3(s) \tag{3.1}$$

I will take

$$m_a = m_e = 0.5 \text{ MeV} \tag{3.2}$$
$$m_b = m_p = 938.272 \text{ MeV} \tag{3.3}$$
$$m_1 = m_\mu = 106 \text{ MeV} \tag{3.4}$$
$$m_2 = 0 \tag{3.5}$$
$$m_3 = 0 \tag{3.6}$$

and

$$p_a = (m_a, \mathbf{0}) \tag{3.7}$$
$$p_b = (E_b, \boldsymbol{p}_b),\ \boldsymbol{p}_b = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k},\ E_b = \sqrt{|\boldsymbol{p}_b|^2 + m_b^2}\ . \tag{3.8}$$

3.1 Integrating directly over CM energies

In this section I describe a technique for performing phase space integrals directly in terms of the variables E_1, E_2, φ_{12}, $\cos\theta_1$, and φ_1 where all quantities are in the center of mass (CM) frame of the incident particles, $\boldsymbol{p}_a + \boldsymbol{p}_b = 0$.

Consider the LIPS measure for 3 outgoing particles

$$\mathrm{d}\Phi_3(s) = (2\pi)^{-5} \delta^4(p_a + p_b - p_1 - p_2 - p_3) \frac{\mathrm{d}^3 p_1}{2E_1} \frac{\mathrm{d}^3 p_2}{2E_2} \frac{\mathrm{d}^3 p_3}{2E_3}\ . \tag{3.9}$$

We can use the momentum-conserving δ-function to evaluate the $\mathrm{d}^3 p_3$ integral so that in the CM frame we have

$$d\Phi_3(s) \to (2\pi)^{-5}\delta(\sqrt{s} - E_1 - E_2 - E_3)\,\frac{1}{2E_3}\,\frac{d^3p_1}{2E_1}\,\frac{d^3p_2}{2E_2}\ , \tag{3.10}$$

where in the above expression it is understood that E_3 is to be replaced with $\sqrt{(-\boldsymbol{p}_1 - \boldsymbol{p}_2)^2 + m_3^2}$. Now rewrite the d^3p_i integrals in spherical polar coordinates,

$$d\Phi_3(s) \to (2\pi)^{-5}\delta(\sqrt{s} - E_1 - E_2 - E_3)\,\frac{1}{2E_3}\,\frac{|\boldsymbol{p}_1|^2 d|\boldsymbol{p}_1|\,d\cos\theta_1\,d\varphi_1}{2E_1}\,\frac{|\boldsymbol{p}_2|^2 d|\boldsymbol{p}_2|\,d\cos\theta_{12}\,d\varphi_{12}}{2E_2}\ . \tag{3.11}$$

(θ_1, φ_1) give the polar and azimuthal angles of $\boldsymbol{p}_1$ in the CM frame, whereas $(\theta_{12}, \varphi_{12})$ give the polar and azimuthal angles of $\boldsymbol{p}_2$ relative to $\boldsymbol{p}_1$. This means that φ_1 and φ_{12} are not measured in the same plane (unless $\theta_1 = 0$). We use the energy-conserving Dirac delta to evaluate the $d\cos\theta_{12}$ integral, obtaining

$$\cos\theta_{12} = \frac{s + 2E_1E_2 - 2\sqrt{s}(E_1 + E_2) + m_1^2 + m_2^2 - m_3^2}{2\sqrt{E_1^2 - m_1^2}\,\sqrt{E_2^2 - m_2^2}}\ , \tag{3.12}$$

$$\frac{d}{d\cos\theta_{12}}(\sqrt{s} - E_1 - E_2 - E_3) = -\frac{2|\boldsymbol{p}_1||\boldsymbol{p}_2|}{E_3}\ , \tag{3.13}$$

so that the $d\cos\theta_{12}$ integral can be done by making the replacement $\delta(\sqrt{s} - E_1 - E_2 - E_3)\,d\cos\theta_{12} \to E_3/(2|\boldsymbol{p}_1||\boldsymbol{p}_2|)$. Hence,

$$d\Phi_3(s) \to \frac{(2\pi)^{-5}}{2^3E_1E_2E_3}\,\frac{E_3}{2|\boldsymbol{p}_1||\boldsymbol{p}_2|}\,|\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2\,d|\boldsymbol{p}_1|\,d|\boldsymbol{p}_2|\,d\cos\theta_1\,d\varphi_1\,d\varphi_{12}\ . \tag{3.14}$$

We also have

$$|\boldsymbol{p}_i|^2 = E_i^2 - m_i^2 \Rightarrow |\boldsymbol{p}_i|d|\boldsymbol{p}_i| = E_i dE_i\ , \tag{3.15}$$

and so

$$d\Phi_3(s) \to \frac{(2\pi)^{-5}}{2^3}\,dE_1\,dE_2\,d\cos\theta_1\,d\varphi_1\,d\varphi_{12}\ . \tag{3.16}$$

To obtain the limits of integration I found that it's actually easier to introduce the kinematic Lorentz invariants

$$s_1 = (p_a + p_b - p_1)^2 = (p_2 + p_3)^2 \tag{3.17}$$

$$s_2 = (p_a + p_b - p_2)^2 = (p_3 + p_1)^2 \tag{3.18}$$

$$s_3 = (p_a + p_b - p_3)^2 = (p_1 + p_2)^2 \tag{3.19}$$

which are related to the CM frame energies by $E_i = (s + m_1^2 - s_i)/2\sqrt{s}$. The s_i are not independent and satisfy $s_1 + s_2 + s_3 = s + m_1^2 + m_2^2 + m_3^2$. This excercise is done in detail in `https://web.physics.utah.edu/~jui/5110/hw/kin_rel.pdf`. The key results are as follows. If none of the s_i are fixed then $s_1 \in [(m_2 + m_3)^2, (\sqrt{s} - m_1)^2]$. If s_1 is picked from that interval then s_2 is bounded by

$$s_2^{\pm} = m_1^2 + m_3^2 + \frac{1}{2s_1}\left[(s - s_1 - m_1^2)(s_1 - m_2^2 + m_3^2) \pm \sqrt{\lambda(s_1, s, m_1^2)\lambda(s_1, m_2^2, m_3^2)}\right]\ . \tag{3.20}$$

Therefore to sample $(E_1, E_2, \cos\theta_1, \varphi_1, \varphi_{12})$ from the kinematically allowed region of phase space one may make use of the following procedure.

1. Draw E_1 uniformly at random from the interval $[m_1, (s + m_1^2 - (m_2 + m_3)^2)/2\sqrt{s}]$.

2. Draw E_2 uniformly at random from the interval $[\frac{s+m_2^2-s_2^+}{2\sqrt{s}}, \frac{s+m_2^2-s_2^-}{2\sqrt{s}}]$, where $s_2^\pm$ is given by eq. (3.20) and s_1 takes the value generated in step 1.

3. Draw $\cos\theta_1$ uniformly from $[-1, 1]$.

4. Independently draw φ_1 and φ_{12} uniformly from $[0, 2\pi]$.

The CM frame momentum vectors for the outgoing particles may then be reconstructed as

$$\boldsymbol{p}_1 = \sqrt{E_1^2 - m_1^2}(\sin\theta_1 \cos\varphi_1 \mathbf{i} + \sin\theta_1\ \sin\varphi_1 \mathbf{j} + \cos\theta_1 \mathbf{k}) \tag{3.21}$$

$$\boldsymbol{p}_2 = \sqrt{E_2^2 - m_2^2}\, R^{-1}_{\theta_1,\varphi_1}(\sin\theta_{12} \cos\varphi_{12} \mathbf{i} + \sin\theta_{12}\ \sin\varphi_{12} \mathbf{j} + \cos\theta_{12} \mathbf{k}) \tag{3.22}$$

$$\boldsymbol{p}_3 = -\boldsymbol{p}_1 - \boldsymbol{p}_2\ , \tag{3.23}$$

where $\sin\theta$ is given by $\sqrt{1-\cos\theta^2}$ and $R_{\theta,\varphi}$ is a rotation matrix that takes $\mathbf{k}$ into the (θ, φ) direction (in particular, I used `R[theta_, phi_] := EulerMatrix[{phi, theta, 0}]`).

3.2 Preliminary results

Phase space volume

The first check I did was to make sure that the phase space volume given by $\mathcal{I} = \int \mathrm{d}\Phi_3(s)$ is the same for both techniques. I calculated the phase space volume using the methods described in sec. (3.1) and sec. (1.3). Performing a Monte Carlo integration over the energies as in sec. (1.3) with chains of length $N = 100,000$, gave $\mathcal{I}_{\text{Energy}} = 98.7 \pm 0.13$. The error is calculated as the standard deviation from 10 independent runs of the integration method. The central value is the mean of the ten chains. It took 35 seconds to run the ten chains for this method. Meanwhile doing the Monte Carlo integration using the momentum transfer technique as in sec. (3.1) gave $\mathcal{I}_{\text{mom.trans.}} = 98.7 \pm 0.15$. It took 60 seconds to run the ten chains for this method. I also verified that the variance goes as $N^{-1/2}$, as expected for Monte Carlo integration.

$\overline{|\mathcal{M}|^2}$ integral

The second test I did was to integrate the spin-summed squared matrix element over the outgoing particles,

$$\int \overline{|\mathcal{M}|^2}\, \mathrm{d}\Phi_3(s)\ . \tag{3.24}$$

Below are the results for $N = 100,000$, using 10 runs to estimate the standard deviation:

Energy method: $(1.92 \pm 0.03) \times 10^{-2}$. Took 340 seconds.

Momentum transfer method: $(2.00 \pm 0.02) \times 10^{-2}$. Took 51 seconds.

They converge to different central values but they are still ‘close’ with an error of $\sim 4\%$.

E_3 **integral**

The third test I did was to integrate the outgoing axion neutron-star-rest-frame energy, E_3

$$\int E_3 \, \mathrm{d}\Phi_3(s) \ . \tag{3.25}$$

4 Lorentz Transformations

I implement Lorentz boosts in arbitrary directions in the following way. A boost in the $+z$-direction that takes the vector $(m, 0, 0, 0)$ into $(E, 0, 0, |\boldsymbol{p}|)$ is given by

$$\Lambda\big(|\boldsymbol{p}|, m\big) = \begin{pmatrix} \gamma & 0 & 0 & \sqrt{\gamma^2-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\gamma^2-1} & 0 & 0 & \gamma \end{pmatrix} \ , \tag{4.1}$$

where $\gamma = \sqrt{|\boldsymbol{p}|^2 + m^2}/m$. Meanwhile, a rotation that takes the z-axis $(\,\cdot\,, 0, 0, 1)$ into the (θ, ϕ) direction is given by

$$R(\theta, \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\cos\phi & -\sin\phi & \cos\phi\sin\theta \\ 0 & \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \ . \tag{4.2}$$

Thus, a boost by momentum $|\boldsymbol{p}|$ in the direction $\hat{\boldsymbol{n}} \equiv (\theta, \phi)$ is given by $\Lambda\big(|\boldsymbol{p}|, m, \hat{\boldsymbol{n}}\big) \equiv R(\theta, \phi)\,\Lambda\big(|\boldsymbol{p}|, m\big)\,R^{-1}(\theta, \phi)$. I had Mathematically symbolically evaluate this matrix and use that to evaluate boosts on four-vectors.

4.1 Boosting in-state momenta

Suppose the components of p_a and p_b as seen in the rest-frame of the neutron star are known. We will simply call this frame the NS frame and denote quantities as seen in this frame with a superscript. In order to make use of the momentum transfer method I need to perform a boost and rotation to obtain the components in a frame where $p_a = (m_a, \mathbf{0})$ and $\boldsymbol{p}_b \propto -\hat{\mathbf{z}}$. We will refer to this frame as the rest frame of a, or simply just the 'a frame', denoting quantities as seen in this frame by a superscript a. This is trivial for p_a: simply set $p_a = (m_a, \mathbf{0})$. For p_b we need only boost it by applying $\Lambda^{-1}\big(|\boldsymbol{p}_a|, m_b, \hat{\boldsymbol{n}}_a\big)$ with $\boldsymbol{n}_a = (\theta_a, \phi_a)$ being the spherical coordinates of the direction of $\boldsymbol{p}_a$. Then, we can manually set $p_b = (E_b, 0, 0, -|\boldsymbol{p}_b|)$. Doing things this way helps reduce the room for matrix multiplication to introduce numerical errors in our results.

In addition to this, recall that the out-state momenta p_i will be generated from the integration variables s_i, t_i, φ_i in the rest frame so in order to calculate quantities like E_3^{NS} we need to boost the p_i vectors back into the neutron star rest frame. This can be done by calculating $\Lambda\big(|\boldsymbol{p}_a, m_i, \hat{\boldsymbol{n}}_b^{\mathrm{NS}}\big)$ where $\hat{\boldsymbol{n}}_b^{\mathrm{NS}}$ gives the direction of $\boldsymbol{p}_b^{\mathrm{NS}}$. In the a frame,

$$p_1^{\mathrm{a}} = (p_a^{\mathrm{a}} - Q_1) \tag{4.3}$$
$$p_2^{\mathrm{a}} = (Q_1 - Q_2) \tag{4.4}$$
$$p_3^{\mathrm{a}} = (Q_2 + p_b^{\mathrm{a}}) \ . \tag{4.5}$$

These are converted to the NS frame by calculating $\Lambda\big(|\boldsymbol{p}_a|, m_i, \hat{\boldsymbol{n}}\big)\,R(\theta_b^{\mathrm{NS}}, \phi_b^{\mathrm{NS}})$.

5 "The dumb method"

Here, I do direct numerical integration over variables in the NS rest frame.

The integral we wish to compute (with all quantities as seen by observers in the NS rest frame) is given by.

$$\varepsilon_3 = \int \frac{\mathrm{d}^3 p_a}{2(2\pi)^3\, E_a}\, \frac{\mathrm{d}^3 p_b}{2(2\pi)^3\, E_b}\, \mathrm{d}\Phi[(p_a+p_b)^2]\, f_a f_b (1-f_1)(1-f_2)\, E_3\, \overline{|\mathcal{M}|^2} \quad . \tag{5.1}$$

$\mathrm{d}\Phi(s)$ is given by eq. (1.2). $\overline{|\mathcal{M}|^2}$ is given by eq. (1.5). The thermal factors f_i are given by eq. (1.3). The momentum delta can be used to fix $\boldsymbol{p}_2 = \boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3$. We then choose to use the energy Dirac delta to fix $E_3 = |\boldsymbol{p}_3|$ assuming (i) the axion is massless: $m_3 = 0$; (ii) the emitted axion's momentum is small: $E_3^2 \ll |\boldsymbol{P}|^2 + m_2^2$ where $\boldsymbol{P} \equiv \boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1$; (iii) we use a coordinate system such that the emitted axion points in the z-direction. Under these assumptions energy conservation requires,

$$\begin{aligned} E_a + E_b &= E_1 + E_2 + E_3 && (5.2)\\ E_a + E_b &= E_1 + \sqrt{|\boldsymbol{P} - \boldsymbol{p}_3|^2 + m_2^2} + E_3 && (5.3)\\ E_a + E_b &= E_1 + \sqrt{|\boldsymbol{P}|^2 + m_2^2 - 2\,P_z\,E_3 + E_3^2} + E_3 && (5.4)\\ &\approx E_1 + E_3 + \sqrt{|\boldsymbol{P}|^2 + m_2^2} - \frac{P_z\,E_3}{\sqrt{|\boldsymbol{P}|^2 + m_2^2}}\,. && (5.5) \end{aligned}$$

Solving for E_3 gives the formula

$$E_3 = \frac{E_a + E_b - E_1 - \sqrt{|\boldsymbol{P}|^2 + m_2^2}}{1 - P_z/\sqrt{|\boldsymbol{P}|^2 + m_2^2}}\,. \tag{5.6}$$

Eq. (5.6) restricts the domain of integration such that $E_3 > 0$. This can be enforced with a Heaviside step function. The integrand must then be augmented by dividing by the absolute value of $\mathrm{d}/\mathrm{d}E_3$ of eq. (5.4)

$$\delta(E_a + E_b - E_1 - E_2 - E_3) \to \frac{\delta\left(E_3 - \frac{E_a+E_b-E_1-\sqrt{|\boldsymbol{P}|^2+m_2^2}}{1-P_z/\sqrt{|\boldsymbol{P}|^2+m_2^2}}\right)}{\left|1 + \frac{E_3 - P_z}{\sqrt{|\boldsymbol{P}|^2+m_2^2+E_3^2-2\,E_3\,P_z}}\right|}\,. \tag{5.7}$$

We rewrite the momentum integrals as energy + direction integrals in spherical coordinates,

$$\frac{\mathrm{d}^3 p_i}{2\,E_i} \to \frac{1}{2}\,|\boldsymbol{p}_i|\,\mathrm{d}E_i\,\mathrm{d}\Omega_i \text{ for } i = a, b, 1, 3. \tag{5.8}$$

Now we have

$$\begin{aligned} \varepsilon_3 \approx \frac{4\pi}{2^5(2\pi)^{11}} &\int f_a f_b (1-f_1)(1-f_2)\, E_3\, \overline{|\mathcal{M}|^2} \\ &\times \frac{|\boldsymbol{p}_a|\,|\boldsymbol{p}_b|\,|\boldsymbol{p}_1|\,E_3}{\left|1 + \frac{E_3 - P_z}{\sqrt{|\boldsymbol{P}|^2+m_2^2+E_3^2-2\,E_3\,P_z}}\right|}\,\Theta(E_3)\,\mathrm{d}E_a\,\mathrm{d}E_b\,\mathrm{d}E_1\,\mathrm{d}\Omega_a\,\mathrm{d}\Omega_b\,\mathrm{d}\Omega_1\ , \end{aligned} \tag{5.9}$$

where $\Theta(x)$ is the Heaviside step function. The factor of 4π in front of the integral comes from the trivial integration over the direction of $\boldsymbol{p}_3$, since we chose a coordinate system where $\boldsymbol{p}_3 \propto \hat{\mathbf{z}}$.

A Derivation of recursive relation

This appendix fills in the details for the derivation of eq. (1.12) which are left out in [2]. Our starting point is

$$\begin{aligned}
R_n(s) &= \int \prod_{i=1}^{n} \delta(p_i^2 - m_i^2)\,\Theta(p_i^0)\,\mathrm{d}^4 p_i\,\delta^4(p_a + p_b - p_1 - \cdots - p_n) \\
&= \int \delta(p_n^2 - m_n^2)\,\Theta(p_n^0)\,\mathrm{d}^4 p_n \left[\prod_{i=1}^{n-1} \delta(p_i^2 - m_i^2)\,\Theta(p_i^0)\right] \delta^4(p_a + p_b - p_n - p_1 - \cdots - p_{n-1}) \\
&= \int \delta(p_n^2 - m_n^2)\,\Theta(p_n^0)\,\mathrm{d}^4 p_n\, R_{n-1}(p_a + p_b - p_n)
\end{aligned}$$

Multiply the RHS by $\int \mathrm{d}M_{n-1}^2\,\delta(M_{n-1}^2 - (p_a + p_b - p_n)^2)$.

$$\int \mathrm{d}M_{n-1}^2\,\delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)\,\delta(p_n^2 - m_n^2)\,\Theta(p_n^0)\,\mathrm{d}^4 p_n\, R_{n-1}\big(p_a + p_b - p_n\big).$$

Use the on-shell delta function to do the p_n^0 integral and write $\mathrm{d}^3 p_n$ in terms of spherical coordinates:

$$\begin{aligned}
&\int \mathrm{d}M_{n-1}^2\,\delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)\,\frac{\mathrm{d}^3 p_n}{2E_n}\, R_{n-1}\big(p_a + p_b - p_n\big) \\
&= \int \mathrm{d}M_{n-1}^2\,\delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)\,\frac{|\boldsymbol{p}_n|^2 \mathrm{d}|\boldsymbol{p}_n| \mathrm{d}\Omega_n}{2E_n}\, R_{n-1}\big(p_a + p_b - p_n\big).
\end{aligned}$$

Then use the M_{n-1}^2 Dirac delta to do the $\mathrm{d}|\boldsymbol{p}_n|$ integral. This requires us to (i) solve $M_{n-1}^2 - (p_a + p_b - p_n)^2 = 0$ for $|\boldsymbol{p}_n|$; (ii) evaluate $\mathrm{d}/\mathrm{d}|\boldsymbol{p}_n|\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)$; and (iii) impose $|\boldsymbol{p}_n|^2 \geq 0$ to derive bounds of integration for $\int \mathrm{d}M_{n-1}^2$.

(i) We start by expanding the LHS

$$\begin{aligned}
M_{n-1}^2 - (p_a + p_b - p_n)^2 &= M_{n-1}^2 - (p_a + p_b)^2 - p_n^2 + 2(p_a + p_b)\cdot p_n \\
&= M_{n-1}^2 - s - m_n^2 + 2(E_a + E_b)E_n + 2(\boldsymbol{p}_a + \boldsymbol{p}_b)\cdot \boldsymbol{p}_n.
\end{aligned}$$

Now we choose the center of mass frame so that $\boldsymbol{p}_a + \boldsymbol{p}_b = 0$. This is valid because the integral is a Lorentz invariant, so the final result is independent of the reference frame.

$$\begin{aligned}
&M_{n-1}^2 - s - m_n^2 + 2(E_a + E_b)E_n + 2(\boldsymbol{p}_a + \boldsymbol{p}_b)\cdot \boldsymbol{p}_n \\
&= M_{n-1}^2 - s - m_n^2 + 2\sqrt{s}\,\sqrt{|\boldsymbol{p}_n|^2 + m_n^2} = 0.
\end{aligned}$$

Now we can rearrange and solve for $|\boldsymbol{p}_n|$

$$\begin{aligned}
|\boldsymbol{p}_n|^2 &= \frac{(s + m_n^2 - M_{n-1}^2)^2}{4s} - m_n^2 = \frac{s^2 + m_n^4 + M_{n-1}^4 - 2sm_n^2 - 2sM_{n-1}^2 - 2m_n^2 M_{n-1}^2}{4s} \\
&\equiv \boxed{\frac{\lambda(s, M_{n-1}^2, m_n^2)}{4s}}.
\end{aligned}$$

$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is called the Källen function.

(ii)

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}|\boldsymbol{p}_n|}\Big(M_{n-1}^2 - (p_a + p_b - p_n)^2\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}|\boldsymbol{p}_n|}\Big(M_{n-1}^2 - s - m_n^2 + 2\sqrt{s}\sqrt{|\boldsymbol{p}_n|^2 + m_n^2}\Big) \\
&= \boxed{\frac{2\sqrt{s}\,|\boldsymbol{p}_n|}{E_n}}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
&|\boldsymbol{p}_n|^2 \geq 0 \\
&\Rightarrow \lambda(s, M_{n-1}^2, m_n^2) \geq 0 \\
&\Rightarrow \big[M_{n-1}^2 - (s + m_n^2)\big]^2 - 4\, s m_n^2 \geq 0 \\
&\Rightarrow \big[M_{n-1}^2 - (s + m_n^2)\big]^2 \geq 4\, s m_n^2 \\
&\Rightarrow M_{n-1}^2 - (s + m_n^2) \geq 2\sqrt{s\, m_n^2} \;||\; M_{n-1}^2 - (s + m_n^2) \leq -2\sqrt{s\, m_n^2} \\
&\Rightarrow M_{n-1}^2 \geq (s + m_n^2) + 2\sqrt{s\, m_n^2} \;||\; M_{n-1}^2 \leq (s + m_n^2) - 2\sqrt{s\, m_n^2} \\
&\Rightarrow \boxed{M_{n-1}^2 \geq (\sqrt{s} + m_n)^2 \;||\; M_{n-1}^2 \leq (\sqrt{s} - m_n)^2.}
\end{aligned}
$$

With these results established we find that

$$
\delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big) = \frac{\delta\Big(|\boldsymbol{p}_n| - \sqrt{\lambda(s, M_{n-1}^2, m_n^2)}\Big/(2\sqrt{s})\Big)}{2\sqrt{s}\,|\boldsymbol{p}_n|/E_n} \quad . \tag{A.1}
$$

Hence,

$$
\begin{aligned}
&\int \mathrm{d}M_{n-1}^2\, \delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)\; \frac{|\boldsymbol{p}_n|^2 \mathrm{d}|\boldsymbol{p}_n| \mathrm{d}\Omega_n}{2E_n}\, R_{n-1}\big(p_a + p_b - p_n\big) \\
&= \int \mathrm{d}M_{n-1}^2\, \frac{\delta\Big(|\boldsymbol{p}_n| - \sqrt{\lambda(s, M_{n-1}^2, m_n^2)}\Big/(2\sqrt{s})\Big)}{2\sqrt{s}\,|\boldsymbol{p}_n|/E_n}\; \frac{|\boldsymbol{p}_n|^2 \mathrm{d}|\boldsymbol{p}_n| \mathrm{d}\Omega_n}{2E_n}\, R_{n-1}\big(p_a + p_b - p_n\big) \\
&= \int \mathrm{d}M_{n-1}^2\, \frac{\sqrt{\lambda(s, M_{n-1}^2, m_n^2)}}{8s}\, \mathrm{d}\Omega_n\, R_{n-1}\big(M_{n-1}^2\big).
\end{aligned}
$$

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