# Neutron star cooling by axion emission

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## Contents

# 1 Introduction

The goal of this project is to evaluate the integral

**Emissivity integral**

$$\varepsilon_3 = \int f_a f_b (1 - f_1)(1 - f_2)(1 + f_3)$$

$$\times E_3 \overline{|\mathcal{M}|^2} \, d\Phi_3\big((p_a + p_b)^2; m_1, m_2, m_3\big) \frac{d^3 p_a}{(2\pi)^3 \, 2E_a} \frac{d^3 p_b}{(2\pi)^3 \, 2E_b} \tag{1.1}$$

where,

**Lorentz invariant phase space measure**

$$d\Phi_n\big((p_a + p_b)^2; 1, \ldots, n\big) \equiv \prod_{i=1}^{n} \left[ \delta(p_i^2 - m_i^2) \, \Theta(p_i^0) \frac{d^4 p_i}{(2\pi)^3} \right] (2\pi)^4 \delta^4 \left( p_a + p_b - \sum_{i=1}^{n} p_i \right) \tag{1.2}$$

is the Lorentz-invariant phase space measure, and

$$f_i \equiv f_{FD}(E_i) \equiv \frac{1}{e^{(E_i - \mu_i)/T} + 1} \text{ for } i = a, b, 1, 2 \tag{1.3}$$

$$f_3 \equiv f_{BE}(E_i) \equiv \frac{1}{e^{(E_3 + \mu_3)/T} + 1} \tag{1.4}$$

are thermal distributions which show up due to Pauli blocking and Bose enhancement. $\overline{|\mathcal{M}|^2}$ depends on the axion emission channel: (i) $l + p \to l' + p + a$, (ii) $l + l \to l' + l + a$, (iii) $l + l' \to l + l + a$, where $l, l'$ denote either $e$ or $\mu$, and $l' \neq l$. The matrix elements are

**Spin-summed matrix element**

$$\overline{|\mathcal{M}^{(lp)}|^2} = -\frac{128 \, g_{ae\mu}^2 e^4}{(m_a^2 - m_1^2)^2} \frac{(p_a \cdot p_1 + m_a m_1)(p_b \cdot p_3)(p_2 \cdot p_3)}{(p_b - p_2)^4} \tag{1.5}$$

$$\overline{|\mathcal{M}^{(ll)}|^2} = \overline{|\mathcal{M}^{(lp)}|^2} + (a \leftrightarrow b) + \mathcal{T}^{(ll)} \tag{1.6}$$

$$\overline{|\mathcal{M}^{(ll')}|^2} = \overline{|\mathcal{M}^{(lp)}|^2} + (1 \leftrightarrow 2) + \mathcal{T}^{(ll')} \tag{1.7}$$

$$\mathcal{T}^{(ll)} = \frac{64 g_{ae\mu}^2 e^4}{(m_a^2 - m_1^2)^2} (p_2 \cdot p_3) \times$$

$$\frac{(p_b \cdot p_1 + m_b m_1)(p_a \cdot p_3) + (p_a \cdot p_1 + m_a m_1)(p_b \cdot p_3) - (p_a \cdot p_b + m_a m_b)(p_1 \cdot p_3)}{(p_a - p_2)^2 \, (p_b - p_2)^2}$$

$\mathcal{T}^{(ll')}$ is given by $\mathcal{T}^{(ll)}$ with the replacements $a \leftrightarrow 1$ and $b \leftrightarrow 2$.

> We use a mostly plus metric signature.

Some comments before we move on:

1. The axions (denoted in this introduction by 3) are effectively free-streaming so the Bose-enhancement factor $(1 + f_3)$ will be ignored.

2. The thermal factors $f_i$ are not Lorentz invariant (unlike the spin-summed matrix element and the integration measure) so we emphasize that the energies appearing in eqs. (1.3) and (1.4) are measured in the rest frame of the neutron star. This fact will be very important to remember later.

## 1.1  Integration in terms of momentum variables (didn't work)

*The first strategy I used to evaluate this integral was to do the integration directly in terms of the momentum variables. Below are the notes I wrote detailing my strategy.*

We can use the momentum conserving Dirac delta to evaluate the $\boldsymbol{p}_2$ integral, setting

$$\boldsymbol{p}_2 = \boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3. \tag{1.8}$$

We now choose to align the $z$-axis with $\boldsymbol{p}_a$ and measure angles with respect to this axis. Converting to spherical polar coordinates gives, for example, $\mathrm{d}^3 p_3 = p_3^2 \, \mathrm{d}p_3 \, \mathrm{d}\cos\theta_{13'} \, \mathrm{d}\phi_{13'}$ . Then the energy conserving Dirac delta can be used to evaluate the $\mathrm{d}p_3$ integral in the following way. First, we use $E = \sqrt{p^2 + m^2}$ to rewrite the masses. Then we use momentum conservation to make the replacement $\boldsymbol{p}_2 \rightarrow \boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3$:

$$E_a + E_b = E_1 + E_2 + E_3$$
$$\Rightarrow \sqrt{p_a^2 + m_a^2} + \sqrt{p_b^2 + m_b^2} = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} + \sqrt{p_3^2 + m_3^2}$$
$$\Rightarrow \sqrt{p_a^2 + m_a^2} + \sqrt{p_b^2 + m_b^2} = \sqrt{p_1^2 + m_1^2} + \sqrt{(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3)^2 + m_2^2} + \sqrt{p_3^2 + m_3^2}. \tag{1.9}$$

We would like to solve eqn. (1.9) for $p_3$, but there is a problem coming from the fact that $p_3$ appears under two square root symbols which makes it impossible to get an expression of the form $p_3 = \cdots$. Instead, we will make the approximation that because $p_3 \ll p_i$ for $i \in \{1, 2, 1', 2'\}$ that we can ignore the $p_3$ that shows up in $\sqrt{(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3)^2 + m_2^2}$ . The result we obtain when solving for $p_3$ is then

$$\begin{aligned}
p_3 = \Big[ & m_a^2 + m_1^2 + m_b^2 + m_2^2 - m_3^2 + 2p_a^2 - 2p_a p_1 c_{11'} + 2p_1^2 - 2E_a E_1 \\
& + 2p_a p_b c_{12} - 2p_b p_1 c_{21'} + 2p_b^2 + 2E_a - 2E_1 E_b \\
& - 2E_a E_2(p_a, p_b, p_1, c_{11'}, c_{12}, c_{21'}) + 2E_1 E_2(p_a, p_b, p_1, c_{11'}, c_{12}, c_{21'}) \\
& - 2E_b E_2(p_a, p_b, p_1, c_{11'}, c_{12}, c_{21'}) \Big]^{1/2}
\end{aligned} \tag{1.10}$$

where $E_i \equiv \sqrt{p_i^2 + m_i^2}$, $c_{ij} \equiv \cos\theta_{ij} \equiv \boldsymbol{p}_i \cdot \boldsymbol{p}_j/(p_i p_j)$ and

$$E_2(p_a, p_b, p_1, c_{11'}, c_{12}, c_{21'}) = \sqrt{p_a^2 - 2p_a p_1 c_{11'} + p_1^2 + 2p_a p_b c_{12} - 2p_b p_1 c_{21'} + p_b^2 + m_2^2}.$$
(1.11)

Now that we have an expression for $p_3$ the requirement that $p_3$ must be real and non-negative restricts the domain of integration of the other variables. I am not sure how to derive the new limits so I got stuck here.

## 1.2 A recursive expression for phase space integrals

The problem I ran into at the end of sec. (??) was how to use the energy-momentum delta functions to restrict the bounds of integration. I spent some time searching the literature for useful references and the most promising were refs. [1, 2, 3]. [1] provides an introduction to Monte Carlo methods and discusses the phase space measure in sec. 9, which covers several ways to deal with the Dirac delta in the phase space integration measure by doing appropriate changes of variables. The most useful method is covered in sec. 9.6 and is the same one used in [2]. Ref. [2] extends the treatment in sec. 9.6 of [1] by also introducing momentum transfer variables. In this section we will introduce the formalism without momentum transfer variables and cover the case with momentum transfers in sec. (1.3).

Consider the phase space integral $R_n(p_a + p_b)$ defined by

> **Phase space integral (without $2\pi$ factors)**
>
> $$R_n(p_a + p_b) = \int \cdots \int \prod_{i=1}^{n} \delta(p_i^2 - m_i^2)\, \Theta(p_i^0)\, \mathrm{d}^4 p_i\, \delta^4(p_a + p_b - p_1 - \cdots - p_n). \quad (1.12)$$
>
> where $p_i$ now denotes a four-momentum vector in contrast to the previous section where it represented a three-momentum magnitude. This is manifestly Lorentz invariant and therefore $R_n$ may only be a function of $s \equiv (p_a + p_b)^2$.

We note that $R_n(s)$ is related to $\mathrm{d}\Phi_n(s)$ by

$$R_n(s) = (2\pi)^{3n-4} \int \mathrm{d}\Phi_n(s) \quad \Rightarrow \quad \int \mathrm{d}\Phi_3(s) = (2\pi)^{-5} R_3(s). \quad (1.13)$$

Define the new integration variable $M_{n-1}^2 \equiv (p_a + p_b - p_n)^2$. The physical significance of $M_{n-1}$ is that it is the invariant mass of the first $n-1$ particles, which can be seen using four-momentum conservation: $p_a + p_b - p_n = p_1 + p_2 + \cdots + p_{n-1}$. It is possible to show that the following recursive relation holds (eqn. (3) of [2])

$$R_n(s) = \int_{(m_1 + m_2 + \cdots + m_{n-1})^2}^{(\sqrt{s} - m_n)^2} \mathrm{d}M_{n-1}^2 \int \mathrm{d}\Omega_n \frac{\sqrt{\lambda(s, M_{n-1}^2, m_n^2)}}{8s} R_{n-1}(M_{n-1}^2) \quad (1.14)$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ and the solid angle $\mathrm{d}\Omega_n \equiv \mathrm{d}\cos\theta_n\, \mathrm{d}\varphi_n$ defines the direction of $\boldsymbol{p}_n$ in the frame where $p_a + p_b = (\sqrt{s}, \boldsymbol{0})$. A derivation of eq. (1.14)

can be found in appendix A. The upper limit is obtained by requiring that

$$|\boldsymbol{p}_n|^2 = \frac{\lambda(s, M_{n-1}^2, m_n^2)}{4s} \geq 0 \;, \tag{1.15}$$

whereas the lower limit is given by the threshold below which $R_{n-1}(M_{n-1}^2) = 0$. Both limits can be derived from physical considerations too. Repeated application of eq. (1.14) yields

$$
\begin{aligned}
R_n(s) = &\int_{(m_1+m_2+\cdots+m_{n-1})^2}^{(\sqrt{s}-m_n)^2} \mathrm{d}M_{n-1}^2 \int \mathrm{d}\Omega_n \frac{\sqrt{\lambda(s, M_{n-1}^2, m_n^2)}}{8s} \\
&\times \int_{(m_1+m_2+\cdots+m_{n-2})^2}^{(M_{n-1}-m_{n-1})^2} \mathrm{d}M_{n-2}^2 \int \mathrm{d}\Omega_{n-1} \frac{\sqrt{\lambda(M_{n-1}^2, M_{n-2}^2, m_{n-1}^2)}}{8M_{n-1}^2} \\
&\times \cdots \times \int_{(m_1+m_2)^2}^{(M_3-m_3)^2} \mathrm{d}M_2^2 \int \mathrm{d}\Omega_3 \frac{\sqrt{\lambda(M_3^2, M_2^2, m_3^2)}}{8M_3^2} \times \int \mathrm{d}\Omega_2 \frac{\sqrt{\lambda(M_2^2, m_1^2, m_2^2)}}{8M_2^2} .
\end{aligned}
\tag{1.16}
$$

The important case for me is when $n = 3$:

**Phase space measure for $n = 3$**

$$
\begin{aligned}
R_3(s) &= (2\pi)^5 \int \mathrm{d}\Phi_3(s) \\
&= \int_{(m_1+m_2)^2}^{(\sqrt{s}-m_3)^2} \mathrm{d}M_2^2 \int \mathrm{d}\Omega_3 \frac{\sqrt{\lambda(s, M_2^2, m_3^2)}}{8s} \int \mathrm{d}\Omega_2 \frac{\sqrt{\lambda(M_2^2, m_1^2, m_2^2)}}{8M_2^2} .
\end{aligned}
\tag{1.17}
$$

It's easy to verify that the number of integration variables matches our expectation of $3(3) - 4 = 5$ since we are integrating over one invariant mass, and two pairs of two angles.

In the above discussion we have considered $p_a$ and $p_b$ to be fixed. However, for my purposes I will also be integrating over $p_a$ and $p_b$. This case seems to be similar to the one considered in [3] (cf. the example given between their eqs. (12) and (13)).

## 1.3   Writing the integral in terms of momentum transfers

In sec 1.2 we wrote the phase space integral in terms of invariant masses and angles of the three-momenta $\boldsymbol{p}_i$ defined in the center-of-mass frame where $\sum_{k=1}^{i} p_k = (M_i, \boldsymbol{0})$. The matrix element in eq. (??) is a function of the Lorentz invariants $(p_a \cdot p_1)$, $(p_b \cdot p_3)$, $(p_2 \cdot p_3)$, and $(p_b \cdot p_2) = (-(p_a - p_1 - p_2 - p_3) \cdot p_2)$. It is cumbersome to write this matrix element explicitly in terms of the angles $\{(\theta_i, \varphi_i) : i = 1, 2, \ldots, n\}$ so we would rather use kinematic Lorentz invariants analogous to the Mandelstam variables for $2 \to 2$ scattering. To this end it is convenient to introduce the so-called 'momentum transfers' $t_i \equiv Q_i^2 \equiv (p_a - p_1 - \cdots - p_i)^2 = (p_n + p_{n-1} + \cdots + p_{i+1} - p_b)^2$.

Some of the dot products (but not all) may be written exclusively in terms of the kinematic
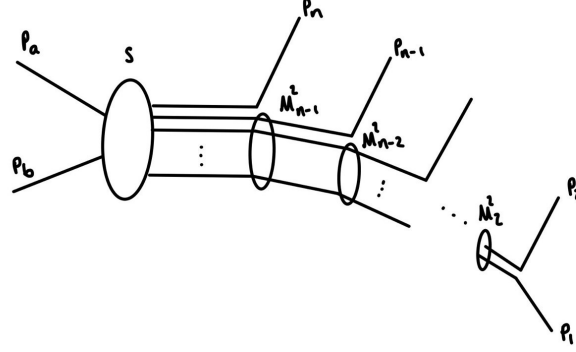
Figure 1: Illustration of the recursion relation as a sequence of effective $2 \to 2$ scattering events (this diagram is largely based on pg. 26 of [1]).

variables. For example,

$$
\begin{aligned}
t_1 &\equiv (p_a - p_1)^2 & \to & & 2p_a \cdot p_1 &= m_a^2 + m_1^2 - t_1 \\
t_2 &\equiv (p_a - (p_1 + p_2))^2 & \to & & 2p_a \cdot p_2 &= M_2^2 - m_1^2 + t_1 - t_2 \\
t_3 &\equiv (p_a - (p_1 + p_2 + p_3))^2 & \to & & 2p_a \cdot p_3 &= t_2 - t_3 - M_2^2 + M_3^2 \\
M_2^2 &\equiv (p_1 + p_2)^2 & \to & & 2p_1 \cdot p_2 &= M_2^2 - m_1^2 - m_2^2 \\
M_3^2 &\equiv (p_1 + p_2 + p_3)^2 & \to & & 2(p_1 + p_2) \cdot p_3 &= M_3^2 - M_2^2 - m_3^2 \\
p_b &= p_1 + p_2 + p_3 - p_a & \to & & 2p_b \cdot p_3 &= m_3^2 + t_3 - t_2 \\
& & & & 2p_b \cdot p_2 &= 2p_2 \cdot p_3 + m_2^2 + t_2 - t_1
\end{aligned}
$$

I could not write $p_2 \cdot p_3$ in terms of the kinematic invariants alone. This is because we need $3(3) - 4 = 5$ integration variables. Right now we only have $t_1$, $t_2$, $M_2^2$ which is not enough to fully describe the system; two more are required. The additional degrees of freedom are given by the azimuthal angles of the $\boldsymbol{Q}_i$ vectors $\varphi_1$, $\varphi_2$ as seen in a frame where $p_a = (m_a, \boldsymbol{0})$ (cf. eq. (1.21) which is eq. (14) of [2].) It is worth describing in more detail the meaning of the angles $\varphi_1$ and $\varphi_2$. I realised on Aug 18 2023 that these azimuthal angles can't be measured from the same axis. The reason for this is the derivation of eq. (1.19) and eq. (1.21) involved a step where we rewrote $\mathrm{d}^3 Q_{n-1}$ as $|\boldsymbol{Q}_{n-1}|^2 \, \mathrm{d}|\boldsymbol{Q}_{n-1}| \, \mathrm{d}\cos\theta_{n-1}\mathrm{d}\varphi_{n-1}$. This transformation requires that $\theta_{n-1}$ be measured from a polar axis and $\varphi_{n-1}$ be measured in the plane perpendicular to that axis. Since $\theta_i$ is defined as the angle between $\boldsymbol{Q}_i$ and $\boldsymbol{Q}_{i+1}$, $\varphi_i$ and $\varphi_{i+1}$ can not be measured in the same plane.

To transform $R_n$ into a form in which the momentum transfers appear as variables we must consider the phase space integral as a function of $p_a$ and $-p_b \equiv Q_n$ separately: $R_n =$

In the margin:

In eq. (1.18) I didn't write the Heaviside function which picks out positive energies, just pretend it is there so that $(Q_{i-1}-Q_i)^0 > 0$.

$R_n(p_a, Q_n)$. Let $Q_i = p_a - p_1 - \cdots - p_i$ so that $p_i = Q_{i-1} - Q_i$. Then, eq. (1.12) becomes

$$R_n(s) = \int \left[ \prod_{i=1}^{n} \mathrm{d}^4 Q_i \, \delta\big((Q_{i-1}-Q_i)^2 - m_i^2\big) \right] \delta^4\Big(p_a + p_b - \sum_{i=1}^{n}(Q_{i-1}-Q_i)\Big)\big|_{Q_0 \equiv p_a}$$

$$= \int \left[ \prod_{i=1}^{n} \mathrm{d}^4 Q_i \, \delta\big((Q_{i-1}-Q_i)^2 - m_i^2\big) \right] \delta^4(Q_n + p_b) \qquad (1.18)$$

$$= \int \mathrm{d}^4 Q_{n-1} \delta\big((Q_{n-2}-Q_{n-1})^2 - m_{n-1}^2\big) \left[ \prod_{i=1}^{n-2} \mathrm{d}^4 Q_i \, \delta\big((Q_{i-1}-Q_i)^2 - m_i^2\big) \right].$$

The recursion relation is then simply

$$R_n(p_a, Q_n) = \int \mathrm{d}^4 Q_{n-1} \, \delta\big((Q_{n-1}-Q_n)^2 - m_n^2\big) \, R_{n-1}(p_a, Q_{n-1}). \qquad (1.19)$$

$R_n$ is now regarded as a function of two invariants $s = s_n = (p_a - Q_n)^2$, and $t_n = Q_n^2$: $R_n = R_n(p_a, Q_n) = R_n(s_n, t_n)$. Introducing these variables in eq. (1.19) the authors of [2] obtain

$$R_n(s_n, t_n, s_{n-1}, t_{n-1}) = \int \mathrm{d}s_{n-1} \, \mathrm{d}t_{n-1} \, K(s_n, t_n, s_{n-1}, t_{n-1}) \, R_{n-1}(s_{n-1}, t_{n-1}), \qquad (1.20)$$

where

$$K(s_n, t_n, s_{n-1}, t_{n-1})$$
$$= \int \mathrm{d}^4 Q_{n-1} \delta(Q_{n-1}^2 - t_{n-1}) \, \delta(s_{n-1} - (p_a - Q_{n-1})^2) \, \delta\big((Q_{n-1}-Q_n)^2 - m_n^2\big)$$
$$= \int_0^{2\pi} \mathrm{d}\varphi_{n-1} \frac{1}{4\sqrt{\lambda(s_n, t_n, m_a^2)}} \, \Theta(-G(t_{n-1}, s_n, s_{n-1}, t_n, m_n^2, m_a^2)) , \qquad (1.21)$$

$\Theta(x)$ is the Heaviside step-function, and $\varphi_{n-1}$ defines the azimuthal angle of $\boldsymbol{Q}_{n-1}$ in the frame $p_a = (m_a, \mathbf{0})$. $G$ is a kinematic function defined by

$$G(x, y, z, u, v, w) = -\frac{1}{2} \begin{vmatrix} 2u & x+u-v & u+w-y \\ & 2x & x-z+w \\ (\text{symm.}) & & 2w \end{vmatrix}$$
$$= x^2 y + xy^2 + z^2 u + vw^2 + v^2 w \qquad (1.22)$$
$$+ xzw + xuv + yzv + yuw$$
$$- xy(z+u+v+w) - zu(x+y+v+w) - vw(x+y+z+u) .$$

In addition, the Dirac deltas impose the following relations hold:

**Formulas for angles and magnitudes in terms of integration variables**

$$|\boldsymbol{Q}_i| = \frac{\sqrt{\lambda_i}}{2m_a} \tag{1.23}$$

$$\varepsilon_i = \frac{\sqrt{\Lambda_i}}{2m_a} \tag{1.24}$$

$$\cos\theta_i \equiv \frac{\boldsymbol{Q}_i \cdot \boldsymbol{Q}_{i+1}}{|\boldsymbol{Q}_i||\boldsymbol{Q}_{i+1}|} = \frac{\xi_i}{\sqrt{\lambda_i \lambda_{i+1}}} \tag{1.25}$$

$$\sin\theta_i = \sqrt{\frac{4m_a^2(-G_i)}{\lambda_i \lambda_{i+1}}} \tag{1.26}$$

where

$$\lambda_i \equiv \lambda(s_i, t_i, m_a^2) \tag{1.27}$$

$$\Lambda_i \equiv \lambda_i + 4m_a^2 t_i = (s_i - t_i - m_a^2)^2 \tag{1.28}$$

$$G_i \equiv G(t_i, s_{i+1}, s_i, t_{i+1}, m_{i+1}^2, m_a^2) < 0 \tag{1.29}$$

$$\xi_i \equiv \sqrt{\Lambda_i \Lambda_{i+1}} - 2m_a^2(t_i + t_{i+1} - m_{i+1}^2) . \tag{1.30}$$

The main result for this section is the phase space measure for the case of 3 outgoing particles, which we highlight below.

$R_n$ **for** $n = 3$

$$R_3(s_3, t_3) = \frac{1}{4\sqrt{\lambda(s_3, t_3, m_a^2)}} \int \frac{\mathrm{d}s_2 \mathrm{d}t_2 \mathrm{d}\varphi_2}{4\sqrt{\lambda(s_2, t_2, m_a^2)}} \Theta(-G_2) \int \mathrm{d}t_1 \mathrm{d}\varphi_1 \Theta(-G_1) \tag{1.31}$$

with

$$G_i = G(t_i, s_{i+1}, s_i, t_{i+1}, m_{i+1}^2, m_a^2) . \tag{1.32}$$

To use this method we rewrite parts the integrand by making the replacements:

$$2p_a \cdot p_1 \to m_a^2 + m_1^2 - t_1 \tag{1.33}$$

$$2p_a \cdot p_2 \to s_2 - m_1^2 + t_1 - t_2 \tag{1.34}$$

$$2p_a \cdot p_3 \to t_2 - t_3 - s_2 + s_3 \tag{1.35}$$

$$2p_1 \cdot p_2 \to s_2 - m_1^2 - m_2^2 \tag{1.36}$$

$$2p_b \cdot p_3 \to m_3^2 + t_3 - t_2 \tag{1.37}$$

$$2p_b \cdot p_2 \to 2p_2 \cdot p_3 + m_2^2 + t_2 - t_1 . \tag{1.38}$$

Then, we will explicitly generate the four-momenta $p_a^{NS}$, $p_b^{NS}$, $p_1^{NS}$, $p_2^{NS}$, $p_3^{NS}$ as seen in the rest frame of the neutron star by

1. Sampling $p_a^{NS}$ and $p_b^{NS}$ from the Fermi-Dirac distribution $f_{FD}(E_a^{NS})$ – this means that we can drop those factors in the integrand.

2. Generating the outgoing particle momenta $p_i$ in the frame $p_a = (m_a, 0)$ and then boosting them into a frame where $p_a = p_a^{NS}$ to obtain $p_i^{NS}$.

## 1.4   Monte Carlo integration using momentum transfers

So far we have rewritten the integral in eq. (1.1) as

$$\varepsilon_3 = \int f_a f_b (1 - f_1)(1 - f_2) E_3 \sum_{\sigma, \sigma'} |\mathcal{M}|^2 \, \mathrm{d}\Phi_3 \frac{\mathrm{d}^3 p_a}{(2\pi)^3 2E_a} \frac{\mathrm{d}^3 p_b}{(2\pi)^3 2E_b} \tag{1.39}$$

where $\mathrm{d}\Phi_3 = (2\pi)^{-5} R_3$ is given by eq. (1.31) and $\sum_{\sigma, \sigma'} |\mathcal{M}|^2$ is given by eq. (??) with the replacements given in eqs. (1.33)–(1.38). Furthermore, it is understood that the energies that are to be plugged into the $f$ distributions are to be evaluated in the rest-frame of the neutron star. Thus, we interpret $\mathrm{d}^3 p_a$ as $\mathrm{d}^3 p_a^{NS}$. The momenta $p_a^{NS}$ and $p_b^{NS}$ will be sampled in the following way. First, rewrite the ingoing momentum integrals

$$\frac{\mathrm{d}^3 p_a}{2E_a} \to \frac{1}{2} \sqrt{E_a^2 - m_a^2} \, \mathrm{d}E_a \, \mathrm{d}\cos\theta_a \mathrm{d}\varphi_a \tag{1.40}$$

$$\frac{\mathrm{d}^3 p_b}{2E_b} \to \frac{1}{2} \sqrt{E_b^2 - m_b^2} \, \mathrm{d}E_b \, \mathrm{d}\cos\theta_b \mathrm{d}\varphi_b \ . \tag{1.41}$$

Next, draw $E_a$ and $E_b$ from the distributions $f_a$ and $f_b$. This is an example of importance sampling, so the factors of $f_a$ and $f_b$ in the integrand will be canceled out by sampling $E_{a,b}$ in this way. Sampling can be accomplished by generating realisations of the random variable $X \sim \mathrm{U}(0, 1)$ and computing $E_a = F^{-1}(X)$ where $F(E_a) = \frac{1}{N} \int_0^{E_a} f(E) \, \mathrm{d}E$ is the cumulative distribution function and $N = \int_0^\infty f(E) \, \mathrm{d}E$ is a normalization factor. These integrals can be performed analytically with the results

**Formulas for sampling ingoing particle energies (as seen in the rest-frame of the neutron star)**

$$N = \int_0^\infty f(E) \, \mathrm{d}E = \int_0^\infty \frac{1}{1 + e^{(E-\mu)/T}} \, \mathrm{d}E = T \ln\left(1 + e^{\mu/T}\right) , \tag{1.42}$$

$$F(E_a) = \frac{1}{N} \int_0^{E_a} f(E) \, \mathrm{d}E = \frac{\ln\left(\frac{1 + e^{\mu/T}}{1 + e^{-(E-\mu)/T}}\right)}{\ln\left(1 + e^{\mu/T}\right)} , \tag{1.43}$$

$$F^{-1}(x) = -T \ln\left(1 + e^{\mu/T}\right) \exp\left\{-x \ln\left(1 + e^{\mu/T}\right) - 1\right\} + \mu \ . \tag{1.44}$$

I verified these formulas by drawing 100,000 samples of $X$, evaluating $E_a = F^{-1}(X)$ for each, and plotting a histogram of the $E_a$ samples against the distribution $f_a/N$ as shown in fig. (2).

### 1.4.1   Momentum reconstruction

To sample $s_i$ we sample a uniform random variable $r_i \sim \mathrm{U}(0, 1)$ and compute

$$\sqrt{s_i} = r_i \left(\sqrt{s_n} - \sum_{j=1}^n m_j\right) + \sum_{j=1}^n m_j \ . \tag{1.45}$$
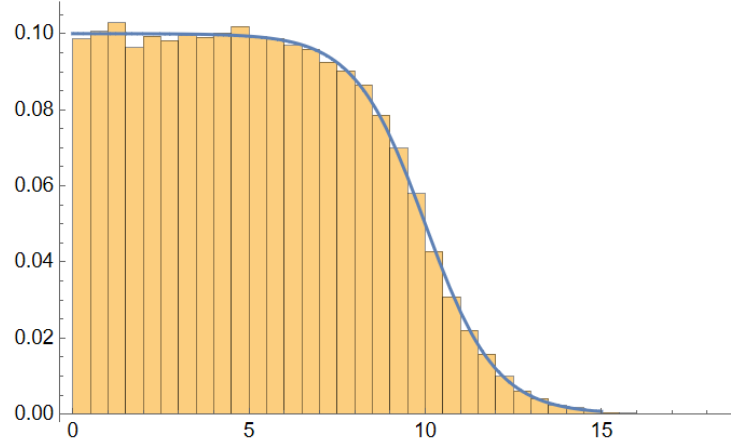
Figure 2: Verification of my method for sampling from a Fermi-Dirac distribution. Histogram is on samples obtained using the method described in sec. (1.4) and blue curve is $f_{FD}(E)$. I chose fiducial values of $\mu = 10$ and $T = 1$.
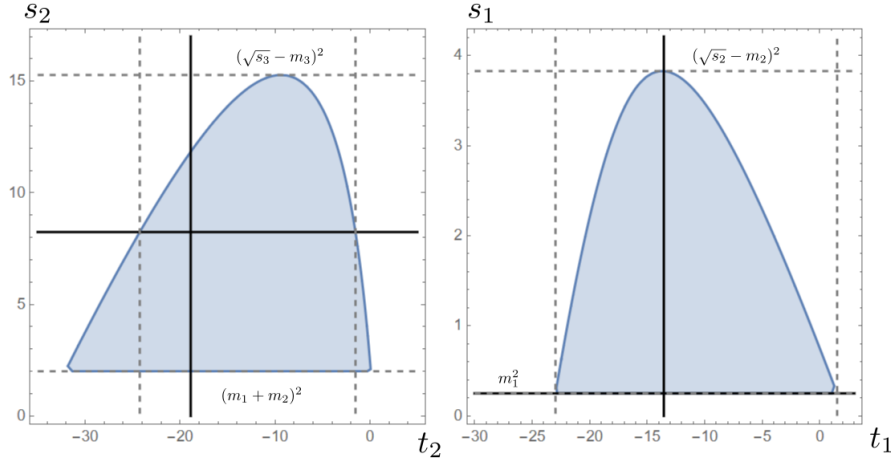


Figure 3: Illustration of sampling from the physical region of parameter space dictated by the $\Theta(-G_i)$ step-functions in eq. (1.31). *(Left:)* Region of the $(t_2, s_2)$ plane where $G_2$ is negative (shaded blue). This corresponds to the physically allowed region of parameter space. The boundary of the region is established by the condition $G_2 = 0$. The lower horizontal dashed line is a lower bound established in the case when the C.M. energy of the effective particle $(p_1 + p_2)$ only has enough energy to create the particle with no kinetic energy $(s_2 = (m_1 + m_2)^2)$. The upper bound is established when the center of mass energy of $(p_1 + p_2 + p_3)$ creates $p_3$ at rest so that $p_3 = (m_3, 0)$ and the remaining energy is used to create $(p_1 + p_2)$ and supply it with kinetic energy. The thick black bars show values of the parameters created using the routine described in the box "Random number generation for Monte Carlo".

Since we are integrating over $s_i$, not $\sqrt{s_i}$, sampling uniformly in $\sqrt{s_i}$ requires us to multiply the integrand by a factor of $2\sqrt{s_i}$ since $ds = d(\sqrt{s})^2 = 2\sqrt{s}\, d(\sqrt{s})$.

Next, we take the generated value of $s_i$ (let's call it $\bar{s}_i$) and use it to calculate the upper and lower bounds from which we generate $t_i$.
To do this we solve $G(t_i, s_{i+1}, \bar{s}_i, t_{i+1}, m_{i+1}^2, m_a^2) = 0$ for $t_i$. This yields,

$$t_i^\pm = t_{i+1} + m_{i+1}^2 +$$
$$\frac{(\bar{s}_i - s_{i+1} - m_{i+1}^2)(t_{i+1} + s_{i+1} - m_a^2) \pm \sqrt{\lambda(\bar{s}_i, s_{i+1}, m_{i+1}^2)\lambda(s_{i+1}, t_{i+1}, m_a^2)}}{2\, s_{i+1}} \quad (1.46)$$

So,

$$t_i^{\text{gen}} = \gamma_i(t_i^+ - t_i^-) + t_i^- \tag{1.47}$$

where $\gamma_i \sim \mathrm{U}(0,1)$ is another uniform random variable. This procedure requires us to weight the generated event by a factor of $(t_i^+ - t_i^-)$. Finally, $\varphi_1$ and $\varphi_2$ are sampled independently from $\mathrm{U}(0, 2\pi)$.

Once $s_2$, $t_1$, $t_2$, $\varphi_1$, and $\varphi_2$ have been generated using the procedure described above we can reconstruct the outgoing momenta $p_i$ in the following way. First, calculate the components of $\boldsymbol{Q}_i$ in the rest frame of $a$. In particular, the components are given by substituting eqs. (1.23–1.30) into the following formulas,

$$\boldsymbol{Q}_2 = |\boldsymbol{Q}_2|(\sin\theta_2\cos\varphi_2,\ \sin\theta_2\sin\varphi_2,\ \cos\theta_2) \tag{1.48}$$
$$\boldsymbol{Q}_1 = |\boldsymbol{Q}_1|R(\theta_2, \varphi_2)(\sin\theta_1\cos\varphi_1,\ \sin\theta_1\sin\varphi_1,\ \cos\theta_1) \tag{1.49}$$

where $R(\theta_2, \varphi_2)$ is rotation matrix that takes $(0,0,1)$ into $(\sin\theta_2\cos\varphi_2, \sin\theta_2\sin\varphi_2, \cos\theta_2)$. Next, demand calculate the energy-component of $Q_i$ as $Q_i^0 = \pm\sqrt{|\boldsymbol{Q}_i|^2 + t_i}$. The sign is chosen by imposing the physical momenta have positive energy so, e.g., $(p_2)^0 = (Q_1^0 - Q_2^0) > 0$. The way I have implemented this is to first compute the positive sign solutions $\varepsilon_i \equiv \sqrt{|\boldsymbol{Q}_i|^2 + t_i}$ and choose $\mathrm{sgn}(Q_i^0) = (-1)^{1+\Theta(\varepsilon_1 - \varepsilon_2)}$, where $\Theta(x)$ is the Heaviside step function. The physical momenta are then given in the rest frame of $a$ frame by

$$p_1 = (p_a - Q_1) \tag{1.50}$$
$$p_2 = (Q_1 - Q_2) \tag{1.51}$$
$$p_3 = (Q_2 + p_b)\ . \tag{1.52}$$

## 2 Monte Carlo integration

### 2.1 The basics
Suppose you are interested in evaluating the integral

$$I = \int_a^b f(x)\, dx. \tag{2.1}$$

The most obvious method of approximating this is with a Riemann sum or trapezoidal integration where you divide the domain into bins of width $\Delta x$ and calculate the area of a rectangle or trapezoid using the values of $f$ at the bin edges.

Monte Carlo integration, on the other hand, takes a a different approach altogether whereby points in the domain are sampled at random and the integral is approximated as an average. To see this, note that the average value of $f$ on the domain is

$$\langle f \rangle = \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x = \frac{I}{b-a}. \tag{2.2}$$

The average $\langle f \rangle$ can be approximated by uniformly sampling $N$ i.i.d random numbers $x_i$ on the domain $x \in [a, b]$ where $i \in 1, 2, ..., N$ and calculating the mean of $f(x_i)$:

$$\langle f \rangle \approx \langle f \rangle_N \equiv \frac{1}{N} \sum_{i=1}^N f(x_i). \tag{2.3}$$

Therefore we can approximate the integral $I$ as

$$I \approx (b-a)\langle f \rangle_N. \tag{2.4}$$

This argument can be extended to higher dimensions, i.e. in $n$ dimensions we have

$$I = \int_V f(\boldsymbol{x})\, \mathrm{d}^n x = V \langle f \rangle. \tag{2.5}$$

One important result is that this strategy always converges as $N^{-1/2}$ in all dimensions, whereas techniques like trapezoidal integration converge rapidly in one dimension $(N^{-2})$, but significantly slower for higher dimensions $(N^{-2/n})$. For example in 5 dimensions, trapezoidal integration converges like $N^{-2/5}$, which is slower than Monte Carlo integration.

## 2.2 Importance sampling

In sec. 2.1 we sampled $x$ uniformly at random on the domain of integration. However, we can choose to sample $x$ from any arbitrary distribution $p(x)$ and compute the same expectation value by weighting the sample $f(x)$ by $1/p(x)$ so that

$$I \approx \frac{V}{N} \sum_{i=1}^N \frac{f(x_i)}{p(x_i)}. \tag{2.6}$$

With an appropriate choice of $p$ our Monte Carlo integration technique can be made to converge significantly faster than by sampling $x$ uniformly at random. To see why this is, suppose $f(x) = e^{-x^2}$ and $[a, b] = [-1000, 1000]$. Sampling uniformly on $[-1000, 1000]$ means that the majority of points we pick will be in the tails of the Gaussian and will hardly affect the value of the integral. If instead we sampled from a standard normal distribution we would mostly choose $x$ values near the peak at $x = 0$ and our result would converge much more rapidly. Clearly the choice of $p$ is heavily dependent on the particular shape of the integrand $f$. The Metropolis-Hastings algorithm is one way to generate samples $x$ from $p(x)$ and can be used for importance sampling.

VEGAS algorithm:

The discussion of the VEGAS algorithm at `https://www.ippp.dur.ac.uk/~krauss/Lectures/QuarksLeptons/Basics/PS_Vegas.html` is quite good. In case the link breaks in the future I reproduce some of the main points below.

"The VEGAS algorithm starts by dividing the $n$-dimensional hypercube $[0,1]^n$ into smaller ones – usually of identical size – and performs a sampling in each of them. The results are then used to refine the integration grid for the next iteration of the sampling procedure."

## 2.3 Stratified sampling

## 2.4 Multichannel sampling

# 3 Tackling a simpler integral

August 28 2023: I spent the last week working on the response to the referee on my non-Gaussianity from axion-string birefringence paper. Now I want to return to this integral project.

I will try doing a simpler integral where the ingoing particle momenta $p_a$ and $p_b$ are fixed and we integrate only over the outgoing states. I will also drop the Lorentz-invariance breaking thermal factors.

$$\int \sum_{\sigma,\sigma'} |\mathcal{M}|^2 \mathrm{d}\Phi_3(s) \tag{3.1}$$

I will take

$$m_a = m_e = 0.5 \text{ MeV} \tag{3.2}$$
$$m_b = m_p = 938.272 \text{ MeV} \tag{3.3}$$
$$m_1 = m_\mu = 106 \text{ MeV} \tag{3.4}$$
$$m_2 = 0 \tag{3.5}$$
$$m_3 = 0 \tag{3.6}$$

and

$$p_a = (m_a, \mathbf{0}) \tag{3.7}$$
$$p_b = (E_b, \boldsymbol{p}_b), \ \boldsymbol{p}_b = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \ E_b = \sqrt{|\boldsymbol{p}_b|^2 + m_b^2} \ . \tag{3.8}$$

## 3.1 Integrating directly over CM energies

In this section I describe a technique for performing phase space integrals directly in terms of the variables $E_1$, $E_2$, $\varphi_{12}$, $\cos\theta_1$, and $\varphi_1$ where all quantities are in the center of mass (CM) frame of the incident particles, $\boldsymbol{p}_a + \boldsymbol{p}_b = 0$.

Consider the LIPS measure for 3 outgoing particles

$$\mathrm{d}\Phi_3(s) = (2\pi)^{-5}\delta^4(p_a + p_b - p_1 - p_2 - p_3)\frac{\mathrm{d}^3 p_1}{2E_1}\frac{\mathrm{d}^3 p_2}{2E_2}\frac{\mathrm{d}^3 p_3}{2E_3} \ . \tag{3.9}$$

We can use the momentum-conserving $\delta$-function to evaluate the $\mathrm{d}^3 p_3$ integral so that in

the CM frame we have

$$d\Phi_3(s) \rightarrow (2\pi)^{-5}\delta(\sqrt{s} - E_1 - E_2 - E_3)\frac{1}{2E_3}\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2} , \tag{3.10}$$

where in the above expression it is understood that $E_3$ is to be replaced with $\sqrt{(-\boldsymbol{p}_1 - \boldsymbol{p}_2)^2 + m_3^2}$. Now rewrite the $d^3p_i$ integrals in spherical polar coordinates,

$$d\Phi_3(s) \rightarrow (2\pi)^{-5}\delta(\sqrt{s} - E_1 - E_2 - E_3)\frac{1}{2E_3}\frac{|\boldsymbol{p}_1|^2 d|\boldsymbol{p}_1| \, d\cos\theta_1 \, d\varphi_1}{2E_1}\frac{|\boldsymbol{p}_2|^2 d|\boldsymbol{p}_2| \, d\cos\theta_{12} \, d\varphi_{12}}{2E_2} . \tag{3.11}$$

$(\theta_1, \varphi_1)$ give the polar and azimuthal angles of $\boldsymbol{p}_1$ in the CM frame, whereas $(\theta_{12}, \varphi_{12})$ give the polar and azimuthal angles of $\boldsymbol{p}_2$ relative to $\boldsymbol{p}_1$. This means that $\varphi_1$ and $\varphi_{12}$ are not measured in the same plane (unless $\theta_1 = 0$). We use the energy-conserving Dirac delta to evaluate the $d\cos\theta_{12}$ integral, obtaining

$$\cos\theta_{12} = \frac{s + 2E_1E_2 - 2\sqrt{s}(E_1 + E_2) + m_1^2 + m_2^2 - m_3^2}{2\sqrt{E_1^2 - m_1^2}\sqrt{E_2^2 - m_2^2}} , \tag{3.12}$$

$$\frac{d}{d\cos\theta_{12}}(\sqrt{s} - E_1 - E_2 - E_3) = -\frac{2|\boldsymbol{p}_1||\boldsymbol{p}_2|}{E_3} , \tag{3.13}$$

so that the $d\cos\theta_{12}$ integral can be done by making the replacement $\delta(\sqrt{s} - E_1 - E_2 - E_3)\,d\cos\theta_{12} \rightarrow E_3/(2|\boldsymbol{p}_1||\boldsymbol{p}_2|)$. Hence,

$$d\Phi_3(s) \rightarrow \frac{(2\pi)^{-5}}{2^3 E_1 E_2 E_3}\frac{E_3}{2|\boldsymbol{p}_1||\boldsymbol{p}_2|}|\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2 \, d|\boldsymbol{p}_1| \, d|\boldsymbol{p}_2| \, d\cos\theta_1 \, d\varphi_1 \, d\varphi_{12} . \tag{3.14}$$

We also have

$$|\boldsymbol{p}_i|^2 = E_i^2 - m_i^2 \Rightarrow |\boldsymbol{p}_i|d|\boldsymbol{p}_i| = E_i dE_i , \tag{3.15}$$

and so

$$d\Phi_3(s) \rightarrow \frac{(2\pi)^{-5}}{2^3} dE_1 \, dE_2 \, d\cos\theta_1 \, d\varphi_1 \, d\varphi_{12} . \tag{3.16}$$

To obtain the limits of integration I found that it's actually easier to introduce the kinematic Lorentz invariants

$$s_1 = (p_a + p_b - p_1)^2 = (p_2 + p_3)^2 \tag{3.17}$$

$$s_2 = (p_a + p_b - p_2)^2 = (p_3 + p_1)^2 \tag{3.18}$$

$$s_3 = (p_a + p_b - p_3)^2 = (p_1 + p_2)^2 \tag{3.19}$$

which are related to the CM frame energies by $E_i = (s + m_1^2 - s_i)/2\sqrt{s}$. The $s_i$ are not independent and satisfy $s_1 + s_2 + s_3 = s + m_1^2 + m_2^2 + m_3^2$. This excercise is done in detail in https://web.physics.utah.edu/~jui/5110/hw/kin_rel.pdf. The key results are as follows. If none of the $s_i$ are fixed then $s_1 \in [(m_2 + m_3)^2, (\sqrt{s} - m_1)^2]$. If $s_1$ is picked from that interval then $s_2$ is bounded by

$$s_2^{\pm} = m_1^2 + m_3^2 + \frac{1}{2s_1}\left[(s - s_1 - m_1^2)(s_1 - m_2^2 + m_3^2) \pm \sqrt{\lambda(s_1, s, m_1^2)\lambda(s_1, m_2^2, m_3^2)}\right] . \tag{3.20}$$

Therefore to sample $(E_1, E_2, \cos\theta_1, \varphi_1, \varphi_{12})$ from the kinematically allowed region of phase space one may make use of the following procedure.

1. Draw $E_1$ uniformly at random from the interval $[m_1, (s + m_1^2 - (m_2 + m_3)^2)/2\sqrt{s}]$.

2. Draw $E_2$ uniformly at random from the interval $[\frac{s + m_2^2 - s_2^+}{2\sqrt{s}}, \frac{s + m_2^2 - s_2^-}{2\sqrt{s}}]$, where $s_2^\pm$ is given by eq. (3.20) and $s_1$ takes the value generated in step 1.

3. Draw $\cos\theta_1$ uniformly from $[-1, 1]$.

4. Independently draw $\varphi_1$ and $\varphi_{12}$ uniformly from $[0, 2\pi]$.

The CM frame momentum vectors for the outgoing particles may then be reconstructed as

$$\boldsymbol{p}_1 = \sqrt{E_1^2 - m_1^2}(\sin\theta_1 \cos\varphi_1 \mathbf{i} + \sin\theta_1 \sin\varphi_1 \mathbf{j} + \cos\theta_1 \mathbf{k}) \tag{3.21}$$

$$\boldsymbol{p}_2 = \sqrt{E_2^2 - m_2^2} \, R_{\theta_1, \varphi_1}^{-1} (\sin\theta_{12} \cos\varphi_{12} \mathbf{i} + \sin\theta_{12} \sin\varphi_{12} \mathbf{j} + \cos\theta_{12} \mathbf{k}) \tag{3.22}$$

$$\boldsymbol{p}_3 = -\boldsymbol{p}_1 - \boldsymbol{p}_2 \,, \tag{3.23}$$

where $\sin\theta$ is given by $\sqrt{1 - \cos\theta^2}$ and $R_{\theta, \varphi}$ is a rotation matrix that takes $\mathbf{k}$ into the $(\theta, \varphi)$ direction (in particular, I used `R[theta_, phi_] := EulerMatrix[{phi, theta, 0}]`).

## 3.2  Preliminary results

**Phase space volume**

The first check I did was to make sure that the phase space volume given by $\mathcal{I} = \int \mathrm{d}\Phi_3(s)$ is the same for both techniques. I calculated the phase space volume using the methods described in sec. (3.1) and sec. (1.3). Performing a Monte Carlo integration over the energies as in sec. (1.3) with chains of length $N = 100,000$, gave $\mathcal{I}_{\mathrm{Energy}} = 98.7 \pm 0.13$. The error is calculated as the standard deviation from 10 independent runs of the integration method. The central value is the mean of the ten chains. It took 35 seconds to run the ten chains for this method. Meanwhile doing the Monte Carlo integration using the momentum transfer technique as in sec. (3.1) gave $\mathcal{I}_{\mathrm{mom.trans.}} = 98.7 \pm 0.15$. It took 60 seconds to run the ten chains for this method. I also verified that the variance goes as $N^{-1/2}$, as expected for Monte Carlo integration.

**$\overline{|\mathcal{M}|^2}$ integral**

The second test I did was to integrate the spin-summed squared matrix element over the outgoing particles,

$$\int \overline{|\mathcal{M}|^2} \, \mathrm{d}\Phi_3(s) \,. \tag{3.24}$$

Below are the results for $N = 100,000$, using 10 runs to estimate the standard deviation:

Energy method: $(1.92 \pm 0.03) \times 10^{-2}$. Took 340 seconds.

Momentum transfer method: $(2.00 \pm 0.02) \times 10^{-2}$. Took 51 seconds.

They converge to different central values but they are still 'close' with an error of $\sim 4\%$.

$E_3$ **integral**

The third test I did was to integrate the outgoing axion neutron-star-rest-frame energy, $E_3$

$$\int E_3 \, \mathrm{d}\Phi_3(s) \ . \tag{3.25}$$

# 4 Lorentz Transformations

I implement Lorentz boosts in arbitrary directions in the following way. A boost in the $+z$-direction that takes the vector $(m, 0, 0, 0)$ into $(E, 0, 0, |\boldsymbol{p}|)$ is given by

$$\Lambda(|\boldsymbol{p}|, m) = \begin{pmatrix} \gamma & 0 & 0 & \sqrt{\gamma^2 - 1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\gamma^2 - 1} & 0 & 0 & \gamma \end{pmatrix} , \tag{4.1}$$

where $\gamma = \sqrt{|\boldsymbol{p}|^2 + m^2}/m$. Meanwhile, a rotation that takes the $z$-axis $(\,\cdot\,, 0, 0, 1)$ into the $(\theta, \phi)$ direction is given by

$$R(\theta, \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\cos\phi & -\sin\phi & \cos\phi\sin\theta \\ 0 & \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} . \tag{4.2}$$

Thus, a boost by momentum $|\boldsymbol{p}|$ in the direction $\hat{\boldsymbol{n}} \equiv (\theta, \phi)$ is given by $\Lambda(|\boldsymbol{p}|, m, \hat{\boldsymbol{n}}) \equiv R(\theta, \phi) \Lambda(|\boldsymbol{p}|, m) R^{-1}(\theta, \phi)$. I had Mathematically symbolically evaluate this matrix and use that to evaluate boosts on four-vectors.

## 4.1 Boosting in-state momenta

Suppose the components of $p_a$ and $p_b$ as seen in the rest-frame of the neutron star are known. We will simply call this frame the NS frame and denote quantities as seen in this frame with a superscript. In order to make use of the momentum transfer method I need to perform a boost and rotation to obtain the components in a frame where $p_a = (m_a, \boldsymbol{0})$ and $\boldsymbol{p}_b \propto -\hat{\mathbf{z}}$. We will refer to this frame as the rest frame of $a$, or simply just the '$a$ frame', denoting quantities as seen in this frame by a superscript a. This is trivial for $p_a$: simply set $p_a = (m_a, \boldsymbol{0})$. For $p_b$ we need only boost it by applying $\Lambda^{-1}(|\boldsymbol{p}_a|, m_b, \hat{\boldsymbol{n}}_a)$ with $\boldsymbol{n}_a = (\theta_a, \phi_a)$ being the spherical coordinates of the direction of $\boldsymbol{p}_a$. Then, we can manually set $p_b = (E_b, 0, 0, -|\boldsymbol{p}_b|)$. Doing things this way helps reduce the room for matrix multiplication to introduce numerical errors in our results.

In addition to this, recall that the out-state momenta $p_i$ will be generated from the integration variables $s_i, t_i, \varphi_i$ in the rest frame so in order to calculate quantities like $E_3^{\mathrm{NS}}$ we need to boost the $p_i$ vectors back into the neutron star rest frame. This can be done by calculating $\Lambda(|\boldsymbol{p}_a|, m_i, \hat{\boldsymbol{n}}_b^{\mathrm{NS}})$ where $\hat{\boldsymbol{n}}_b^{\mathrm{NS}}$ gives the direction of $\boldsymbol{p}_b^{\mathrm{NS}}$. In the $a$ frame,

$$p_1^{\mathrm{a}} = (p_a^{\mathrm{a}} - Q_1) \tag{4.3}$$

$$p_2^{\mathrm{a}} = (Q_1 - Q_2) \tag{4.4}$$

$$p_3^{\mathrm{a}} = (Q_2 + p_b^{\mathrm{a}}) \ . \tag{4.5}$$

These are converted to the NS frame by calculating $\Lambda(|\boldsymbol{p}_a|, m_i, \hat{\boldsymbol{n}}) R(\theta_b^{\mathrm{NS}}, \phi_b^{\mathrm{NS}})$.

# 5 "The dumb method"

Here, I do direct numerical integration over variables in the NS rest frame.

The integral we wish to compute (with all quantities as seen by observers in the NS rest frame) is given by.

$$\varepsilon_3 = \int \frac{\mathrm{d}^3 p_a}{2(2\pi)^3 E_a} \frac{\mathrm{d}^3 p_b}{2(2\pi)^3 E_b} \, \mathrm{d}\Phi[(p_a + p_b)^2] \, f_a f_b (1 - f_1)(1 - f_2) \, E_3 \, \overline{|\mathcal{M}|^2} \quad . \tag{5.1}$$

$\mathrm{d}\Phi(s)$ is given by eq. (1.2). $\overline{|\mathcal{M}|^2}$ is given by eq. (**??**). The thermal factors $f_i$ are given by eq. (1.3). The momentum delta can be used to fix $\boldsymbol{p}_2 = \boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_3$. We then choose to use the energy Dirac delta to fix $E_3 = |\boldsymbol{p}_3|$ assuming (i) the axion is massless: $m_3 = 0$; (ii) the emitted axion's momentum is small: $E_3^2 \ll |\boldsymbol{P}|^2 + m_2^2$ where $\boldsymbol{P} \equiv \boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1$; (iii) we use a coordinate system such that the emitted axion points in the $z$-direction. Under these assumptions energy conservation requires,

$$E_a + E_b = E_1 + E_2 + E_3 \tag{5.2}$$

$$E_a + E_b = E_1 + \sqrt{|\boldsymbol{P} - \boldsymbol{p}_3|^2 + m_2^2} + E_3 \tag{5.3}$$

$$E_a + E_b = E_1 + \sqrt{|\boldsymbol{P}|^2 + m_2^2 - 2 P_z E_3 + E_3^2} + E_3 \tag{5.4}$$

$$\approx E_1 + E_3 + \sqrt{|\boldsymbol{P}|^2 + m_2^2} - \frac{P_z E_3}{\sqrt{|\boldsymbol{P}|^2 + m_2^2}} . \tag{5.5}$$

Solving for $E_3$ gives the formula

$$E_3 = \frac{E_a + E_b - E_1 - \sqrt{|\boldsymbol{P}|^2 + m_2^2}}{1 - P_z/\sqrt{|\boldsymbol{P}|^2 + m_2^2}} . \tag{5.6}$$

Eq. (5.6) restricts the domain of integration such that $E_3 > 0$. This can be enforced with a Heaviside step function. The integrand must then be augmented by dividing by the absolute value of $\mathrm{d}/\mathrm{d}E_3$ of eq. (5.4)

$$\delta(E_a + E_b - E_1 - E_2 - E_3) \to \frac{\delta\left(E_3 - \frac{E_a + E_b - E_1 - \sqrt{|\boldsymbol{P}|^2 + m_2^2}}{1 - P_z/\sqrt{|\boldsymbol{P}|^2 + m_2^2}}\right)}{\left|1 + \frac{E_3 - P_z}{\sqrt{|\boldsymbol{P}|^2 + m_2^2 + E_3^2 - 2 E_3 P_z}}\right|} . \tag{5.7}$$

We rewrite the momentum integrals as energy + direction integrals in spherical coordinates,

$$\frac{\mathrm{d}^3 p_i}{2 E_i} \to \frac{1}{2} |\boldsymbol{p}_i| \, \mathrm{d}E_i \, \mathrm{d}\Omega_i \text{ for } i = a, b, 1, 3. \tag{5.8}$$

Now we have

$$\varepsilon_3 \approx \frac{4\pi}{2^5 (2\pi)^{11}} \int f_a f_b (1 - f_1)(1 - f_2) \, E_3 \, \overline{|\mathcal{M}|^2}$$
$$\times \frac{|\boldsymbol{p}_a| \, |\boldsymbol{p}_b| \, |\boldsymbol{p}_1| \, E_3}{\left|1 + \frac{E_3 - P_z}{\sqrt{|\boldsymbol{P}|^2 + m_2^2 + E_3^2 - 2 E_3 P_z}}\right|} \, \Theta(E_3) \, \frac{1}{E_2} \, \mathrm{d}E_a \, \mathrm{d}E_b \, \mathrm{d}E_1 \, \mathrm{d}\Omega_a \, \mathrm{d}\Omega_b \, \mathrm{d}\Omega_1 \,, \tag{5.9}$$

where $\Theta(x)$ is the Heaviside step function. The factor of $4\pi$ in front of the integral comes from the trivial integration over the direction of $\boldsymbol{p}_3$, since we chose a coordinate system where $\boldsymbol{p}_3 \propto \hat{\mathbf{z}}$.

## 5.1 Extension to $m_3 > 0$

Using the dumb method I was able to reproduce Hong Yi's results. Now I want to extend this method to cover the case when $m_3 > 0$. This means that I can't use $E_3 = |\boldsymbol{p}_3|$ anymore. Consider the expression

$$E_a + E_b - E_1 = \sqrt{|\boldsymbol{p}_3|^2 - 2P_z|\boldsymbol{p}_3| + |\boldsymbol{P}|^2 + m_2^2} + \sqrt{|\boldsymbol{p}_3|^2 + m_3^2}. \tag{5.10}$$

Here, all variables except $|\boldsymbol{p}_3|$ have been fixed so it is in the form of

$$C = \sqrt{x^2 - 2bx + a^2 + c^2} + \sqrt{x^2 + d^2} \tag{5.11}$$

$$= \sqrt{(x - b)^2 + a^2 - b^2 + c^2} + \sqrt{x^2 + d^2}, \tag{5.12}$$

where $|b| \leq a$. I need to find the value of $x$ that makes this equation true, assuming such a value exists. This can be thought of as finding the root of the expression $g(x) = f(x) - C$ where $f(x) = \sqrt{(x - b)^2 + a^2 - b^2 + c^2} + \sqrt{x^2 + d^2}$ and $C$ is the constant. For large $x$ the function $f(x)$ goes to $+\infty$. Therefore a solution only exists if the minimum value of $f(x)$ is less than or equal to $C$. This motivates a study of the global minimum of $f(x)$. Moreover, since $x$ corresponds to $|\boldsymbol{p}_3|$ 'physical' solutions require $x > 0$. The tl;dr is that the minimum of $f(x)$ occurs at $x_* = 0$ unless both $b$ and $d$ are nonzero and positive. In that case,

$$x_* = \begin{cases} \frac{bd(\sqrt{a^2 - b^2 + c^2} - d)}{a^2 - b^2 + c^2 - d^2} & \text{for } d < c \\ \frac{bd(d - \sqrt{a^2 - b^2 + c^2})}{d^2 - (a^2 - b^2 + c^2)} & \text{for } d > \sqrt{a^2 - b^2 + c^2} \end{cases} \tag{5.13}$$

$$= \frac{bd}{\sqrt{a^2 - b^2 + c^2} + d} \tag{5.14}$$

# 6  Making sense of which region of parameter space contributes 'a lot' to the integral

As Hong Yi and I were writing up the final edits for the draft of the paper we added a comment saying that

> "The significance of $T = 10$ MeV can be understood as follows: at low temperatures, the thermal factors in the integrand restricts the phase space to a region where the typical momentum transfer is bigger than the screening scale; at higher temperatures, particles start to become nondegenerate so there is support near the region where $k^2 = 0$ in the photon propagators, where the screening effect becomes crucial."

We are motivated to make a comment like this because we needed to explain why we had to account for medium effects when we extended our numerical integration method to high temperatures. A chronological account of what happened goes like this:

1. I tried dialing up the temperature $T$ from $T = 0.083$ MeV to $T = 83$ MeV without accounting for the electromagnetic screening effect arising from the background of free charged particles.

2. I found that by neglecting this effect, the emissivity I computed was diverging. This manifested as a sharp rise in the emissivity at around $T \sim 1$ MeV as seen in fig. (4). I further checked that if I increased $N_{\mathrm{eval}}$, the value of the integral would continue to increase as the numerical integrator finds points close and closer to the $1/k^2$ pole in the matrix element. This is shown in fig. (5).
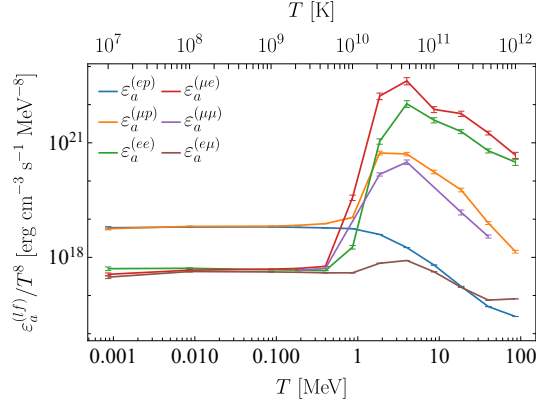


Figure 4: Axion emissivity, calculated without taking into account plasma screening, vs temperature. Notice that the dramatic jump in emissivity seen at $T \sim 1$ MeV. The axion emissivity is normalized by $T^8$ since we are also interested in deviations from $T^8$ scaling – the expected behaviour assuming the Fermi surface approximation.
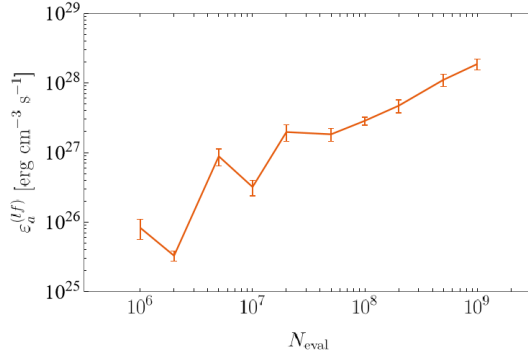


Figure 5: $\varepsilon_a^{(\mu e)}$, calculated without taking into account plasma screening, vs $N_{\mathrm{eval}}$. $T = 4$ MeV, $\beta_{F,\mu} = 0.84$, $m_a = 0$, $n = 10$. If the integral was converging this plot should approach a constant as $N_{\mathrm{eval}}$ is increased. Instead, we observe that $\varepsilon_a^{(\mu e)}$ increases without bound.

3. Based on the observations in the previous point we surmised that at low temperatures

the exponential suppression from the thermal factors prevented my numerical integrator from running into trouble at the pole where $k^2 = 0$. I can check this by sampling points in the integration domain using my trained numerical integrator and checking if there were any points I ran into where $k^2$ was very close to zero.

4. We realised that we needed to account for plasma effects and ended up doing so by making the replacement $k^{-2} \to (k^2 + k_{TF}^2)^{-1}$ in the photon propagators in the matrix element. This gave us a finite emissivity that converged. We believe this is sufficient evidence for the claim that the $k^{-2}$ pole was the problem leading to diverging emissivities.

# 7  What happens if $E_{F,e} \neq E_{F,\mu}$?

Our starting point is eq. (2.5) of Hong Yi's notes which gives the energy integral under the FSA. It is reproduced here for convenience (taking $n = 2$),

$$ J^{(lf)} \equiv \int dE_1 dE_2 dE_1' dE_2' dE_3' \ E_3'^4 \delta(E_1 + E_2 - E_1' - E_2' - E_3') f_1 f_2 (1 - f_1')(1 - f_2') \ . \quad (7.1) $$

The first order of business is to define the variables $x_i \equiv (E_i - F, i)/T$ and $z \equiv E_3'/T$ and rewrite $J^{(lf)}$ in terms of these variables. We have,

$$ E_1 + E_2 - E_1' - E_2' - E_3' = T(x_1 + x_2 - x_1' - x_2' - z) + E_{F,1} + E_{F,2} - E_{F,1}' - E_{F,2}' \ . \quad (7.2) $$

However, 2 is a spectator particle so $E_{F,2} = E_{F,2}'$ (2 and 2' are the same species of particle). Moreover, $E_{F,1} - E_{F,1}'$ is equal to either $T\Delta \equiv E_{F,e} - E_{F,\mu}$ or $-T\Delta = -(E_{F,e} - E_{F,\mu})$ depending on whether ptle 1 is an electron or muon. Hence,

$$ E_1 + E_2 - E_1' - E_2' - E_3' = T(x_1 + x_2 - x_1' - x_2' - z \pm \Delta) \ . \quad (7.3) $$

$$ J^{(lf)}(\Delta) = \int T^5 \, dx_1 dx_2 dx_1' dx_2' dz \ T^4 z^4 \frac{\delta(x_1 + x_2 - x_1' - x_2' - z \pm \Delta)}{T} f_1 f_2 (1 - f_1')(1 - f_2') \ , \quad (7.4) $$

where the upper sign $(+)$ is for $l = e$ and the lower sign $(-)$ is for $l = \mu$. This simplifies to a modified form of Hong Yi's eq. (2.8) when we assume strongly degenerate particles (so that the integrals over $x_i$ are from $-\infty$ to $+\infty$).

$$ J^{(lf)}(\Delta) = \int_{-\infty}^{\infty} dx_1 dx_2 dx_1' dx_2' \int_0^{\infty} dz \frac{T^8 z^4 \, \delta(x_1 + x_2 - x_1' - x_2' - z \pm \Delta)}{(e^{x_1} + 1)(e^{x_2} + 1)(e^{-x_1'} + 1)(e^{-x_2'} + 1)} \ . \quad (7.5) $$

The $x_i$ integrals in eq. (7.5) can be evaluated by doing the change of variables $\tilde{z} = z \mp \Delta$ and using the formula,

$$ \int_{-\infty}^{\infty} dx_1 \cdots dx_4 \frac{\delta(x_1 + x_2 + x_3 + x_4 - \tilde{z})}{(e^{x_1} + 1)(e^{x_2} + 1)(e^{x_3} + 1)(e^{x_4} + 1)} = \frac{1}{6} \frac{\tilde{z}(\tilde{z}^2 + 4\pi^2)}{e^{\tilde{z}} - 1} \ , $$

so that we obtain

$$ J^{(lf)}(\Delta) = \frac{T^8}{6} \int_{\Delta}^{\infty} d\tilde{z} \ (\tilde{z} \pm \Delta)^4 \frac{\tilde{z}(\tilde{z}^2 + 4\pi^2)}{e^{\tilde{z}} - 1} \ . \quad (7.6) $$

Eq. (7.6) can be evaluated analytically for any $\Delta$ but in the special case $\Delta = 0$ the result is $J^{(lf)}(0) = (164\,\pi^8/945)\,T^8$, and is independent of the process. We now consider the cases $J^{(ef)}$ and $J^{(\mu f)}$.

$\underline{ef \to \mu fa}$:

In this case, we evaluate eq. (7.6) with the upper sign (+). Mathematica can do this analytically and we obtain

$$
\begin{aligned}
J^{(ep)}(\Delta) &= \frac{T^8}{6} \int_\Delta^\infty \mathrm{d}\tilde{z}\ (\tilde{z}+\Delta)^4\, \frac{\tilde{z}\,(\tilde{z}^2+4\pi^2)}{e^{\tilde{z}}-1} \\
&= 4\left[210\,\mathrm{Li}_8(e^\Delta) - 90\,\Delta\,\mathrm{Li}_7(e^\Delta) + 5\,(4\pi^2+3\Delta^2)\,\mathrm{Li}_6(e^\Delta) - (4\pi^2\,\Delta+\Delta^3)\,\mathrm{Li}_5(e^\Delta)\right],
\end{aligned}
\tag{7.7}
$$

where $\mathrm{Li}_n(x)$ is the polylogarithm function (see `PolyLog` in Mathematica's documentation). As expected, this coincides with $(164\,\pi^8/945)\,T^8$ when $\Delta = 0$. As shown in fig. (6) (7.7) is well approximated by an exponential decay of the form $e^{-k\Delta}$.
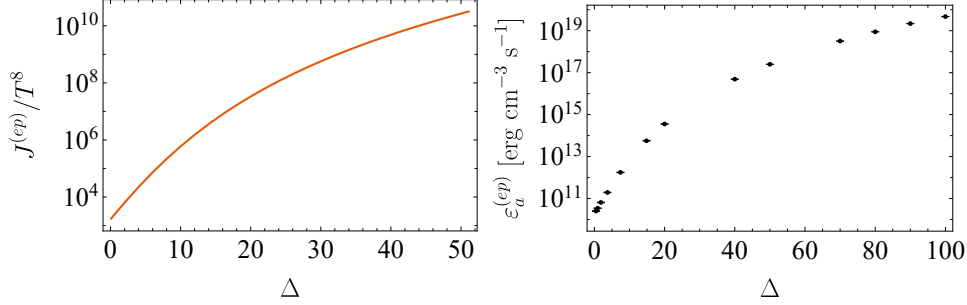


Figure 6: *Left*: Plotting the polylogarithm function given in eq. (7.7) versus $\Delta$. At $\Delta = 0$ this coincides with $(164\,\pi^8/945)\,T^8 \approx 1650\,T^8$. For $\Delta < 10$ this is well-approximated by an exponential enhancement. *Right*: Numerically evaluated emissivity versus $\Delta$. Notice that the behaviour is the same. This gives me confidence that the discussion here is a sufficient explanation.

$\underline{\mu f \to efa}$:

In this case, we evaluate eq. (7.6) with the lower sign (−). Mathematica can do this analytically and we obtain

$$
\frac{J^{(\mu f)}(\Delta)}{T^8} = 4\,T^8\left[(4\pi^2\Delta+\Delta^3)\,\mathrm{Li}_5(e^{-\Delta}) + 5(4\pi^2+3\Delta^2)\,\mathrm{Li}_6(e^{-\Delta}) + 90\Delta\,\mathrm{Li}_7(e^{-\Delta}) + 210\,\mathrm{Li}_8(e^{-\Delta})\right],
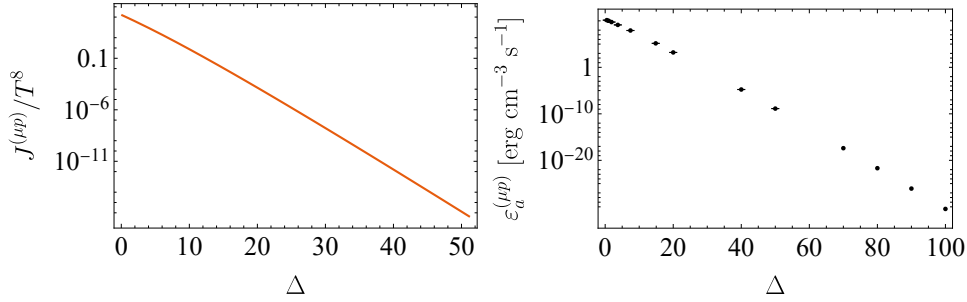\tag{7.8}
$$

Figure 7: *Left*: The polylogarithm function given in eq. (7.8) versus $\Delta$. At $\Delta = 0$ this coincides with $(164\,\pi^8/945)\,T^8 \approx 1650\,T^8$. This is well-approximated by an exponential decay. *Right*: Numerically evaluated emissivity versus $\Delta$. Notice that the behaviour is the same. This gives me confidence that the discussion here is a sufficient explanation.

# A   Derivation of recursive relation

This appendix fills in the details for the derivation of eq. (1.14) which are left out in [2]. Our starting point is

$$
\begin{aligned}
R_n(s) &= \int \prod_{i=1}^{n} \delta(p_i^2 - m_i^2)\,\Theta(p_i^0)\,\mathrm{d}^4 p_i\,\delta^4(p_a + p_b - p_1 - \cdots - p_n) \\
&= \int \delta(p_n^2 - m_n^2)\,\Theta(p_n^0)\,\mathrm{d}^4 p_n \left[\prod_{i=1}^{n-1} \delta(p_i^2 - m_i^2)\,\Theta(p_i^0)\right]\delta^4(p_a + p_b - p_n - p_1 - \cdots - p_{n-1}) \\
&= \int \delta(p_n^2 - m_n^2)\,\Theta(p_n^0)\,\mathrm{d}^4 p_n\,R_{n-1}(p_a + p_b - p_n)
\end{aligned}
$$

Multiply the RHS by $\int \mathrm{d}M_{n-1}^2\,\delta(M_{n-1}^2 - (p_a + p_b - p_n)^2)$.

$$
\int \mathrm{d}M_{n-1}^2\,\delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)\,\delta(p_n^2 - m_n^2)\,\Theta(p_n^0)\,\mathrm{d}^4 p_n\,R_{n-1}\big(p_a + p_b - p_n\big).
$$

Use the on-shell delta function to do the $p_n^0$ integral and write $\mathrm{d}^3 p_n$ in terms of spherical coordinates:

$$
\begin{aligned}
&\int \mathrm{d}M_{n-1}^2\,\delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)\,\frac{\mathrm{d}^3 p_n}{2E_n}\,R_{n-1}\big(p_a + p_b - p_n\big) \\
&= \int \mathrm{d}M_{n-1}^2\,\delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)\,\frac{|\boldsymbol{p}_n|^2 \mathrm{d}|\boldsymbol{p}_n| \mathrm{d}\Omega_n}{2E_n}\,R_{n-1}\big(p_a + p_b - p_n\big).
\end{aligned}
$$

Then use the $M_{n-1}^2$ Dirac delta to do the $\mathrm{d}|\boldsymbol{p}_n|$ integral. This requires us to (i) solve $M_{n-1}^2 - (p_a + p_b - p_n)^2 = 0$ for $|\boldsymbol{p}_n|$; (ii) evaluate $\mathrm{d}/\mathrm{d}|\boldsymbol{p}_n|\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)$; and (iii) impose $|\boldsymbol{p}_n|^2 \geq 0$ to derive bounds of integration for $\int \mathrm{d}M_{n-1}^2$.

(i) We start by expanding the LHS

$$
\begin{aligned}
M_{n-1}^2 - (p_a + p_b - p_n)^2 &= M_{n-1}^2 - (p_a + p_b)^2 - p_n^2 + 2(p_a + p_b) \cdot p_n \\
&= M_{n-1}^2 - s - m_n^2 + 2(E_a + E_b)E_n + 2(\boldsymbol{p}_a + \boldsymbol{p}_b) \cdot \boldsymbol{p}_n.
\end{aligned}
$$

Now we choose the center of mass frame so that $\boldsymbol{p}_a + \boldsymbol{p}_b = 0$. This is valid because the integral is a Lorentz invariant, so the final result is independent of the reference frame.

$$M_{n-1}^2 - s - m_n^2 + 2(E_a + E_b)E_n + 2(\boldsymbol{p}_a + \boldsymbol{p}_b) \cdot \boldsymbol{p}_n$$
$$= M_{n-1}^2 - s - m_n^2 + 2\sqrt{s}\,\sqrt{|\boldsymbol{p}_n|^2 + m_n^2} = 0.$$

Now we can rearrange and solve for $|\boldsymbol{p}_n|$

$$|\boldsymbol{p}_n|^2 = \frac{(s + m_n^2 - M_{n-1}^2)^2}{4s} - m_n^2 = \frac{s^2 + m_n^4 + M_{n-1}^4 - 2sm_n^2 - 2sM_{n-1}^2 - 2m_n^2 M_{n-1}^2}{4s}$$
$$\equiv \boxed{\frac{\lambda(s, M_{n-1}^2, m_n^2)}{4s}}.$$

$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is called the Källen function.

(ii)

$$\frac{\mathrm{d}}{\mathrm{d}|\boldsymbol{p}_n|}\left(M_{n-1}^2 - (p_a + p_b - p_n)^2\right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}|\boldsymbol{p}_n|}\left(M_{n-1}^2 - s - m_n^2 + 2\sqrt{s}\,\sqrt{|\boldsymbol{p}_n|^2 + m_n^2}\right)$$
$$= \boxed{\frac{2\sqrt{s}\,|\boldsymbol{p}_n|}{E_n}}$$

(iii)

$$|\boldsymbol{p}_n|^2 \geq 0$$
$$\Rightarrow \lambda(s, M_{n-1}^2, m_n^2) \geq 0$$
$$\Rightarrow \left[M_{n-1}^2 - (s + m_n^2)\right]^2 - 4sm_n^2 \geq 0$$
$$\Rightarrow \left[M_{n-1}^2 - (s + m_n^2)\right]^2 \geq 4sm_n^2$$
$$\Rightarrow M_{n-1}^2 - (s + m_n^2) \geq 2\sqrt{s\,m_n^2} \;||\; M_{n-1}^2 - (s + m_n^2) \leq -2\sqrt{s\,m_n^2}$$
$$\Rightarrow M_{n-1}^2 \geq (s + m_n^2) + 2\sqrt{s\,m_n^2} \;||\; M_{n-1}^2 \leq (s + m_n^2) - 2\sqrt{s\,m_n^2}$$
$$\Rightarrow \boxed{M_{n-1}^2 \geq (\sqrt{s} + m_n)^2 \;||\; M_{n-1}^2 \leq (\sqrt{s} - m_n)^2.}$$

With these results established we find that

$$\delta\left(M_{n-1}^2 - (p_a + p_b - p_n)^2\right) = \frac{\delta\left(|\boldsymbol{p}_n| - \sqrt{\lambda(s, M_{n-1}^2, m_n^2)}\Big/(2\sqrt{s})\right)}{2\sqrt{s}\,|\boldsymbol{p}_n|/E_n} \quad . \qquad (A.1)$$

Hence,

$$
\int \mathrm{d}M_{n-1}^2 \, \delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big) \, \frac{|\boldsymbol{p}_n|^2 \mathrm{d}|\boldsymbol{p}_n| \mathrm{d}\Omega_n}{2E_n} \, R_{n-1}\big(p_a + p_b - p_n\big)
$$

$$
= \int \mathrm{d}M_{n-1}^2 \, \frac{\delta\Big(|\boldsymbol{p}_n| - \sqrt{\lambda(s, M_{n-1}^2, m_n^2)}\Big/(2\sqrt{s})\Big)}{2\sqrt{s}\,|\boldsymbol{p}_n|/E_n} \, \frac{|\boldsymbol{p}_n|^2 \mathrm{d}|\boldsymbol{p}_n| \mathrm{d}\Omega_n}{2E_n} \, R_{n-1}\big(p_a + p_b - p_n\big)
$$

$$
= \int \mathrm{d}M_{n-1}^2 \, \frac{\sqrt{\lambda(s, M_{n-1}^2, m_n^2)}}{8s} \, \mathrm{d}\Omega_n \, R_{n-1}\big(M_{n-1}^2\big).
$$

# References

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