1 What happens if $E_{F,e} \neq E_{F,\mu}$?

Our starting point is eq. (2.5) of Hong Yi's notes which gives the energy integral under the FSA. It is reproduced here for convenience (taking $n = 2$),

$$J^{(lf)} \equiv \int dE_1 dE_2 dE'_1 dE'_2 dE'_3 \; {E'_3}^4 \delta(E_1 + E_2 - E'_1 - E'_2 - E'_3) f_1 f_2 (1 - f'_1)(1 - f'_2) \; . \quad (1.1)$$

The first order of business is to define the variables $x_i \equiv (E_i - E_{F,i})/T$ and $z \equiv E'_3/T$ and rewrite the J in terms of these variables. We have,

$$E_1 + E_2 - E'_1 - E'_2 - E'_3 = T\,(x_1 + x_2 - x'_1 - x'_2 - z) + E_{F,1} + E_{F,2} - E'_{F,1} - E'_{F,2} \; . \quad (1.2)$$

However, 2 is a spectator particle so $E_{F,2} = E'_{F,2}$ (2 and $2'$ are the same species of particle). Moreover, $E_{F,1} - E'_{F,1}$ is equal to either $T\Delta \equiv E_{F,e} - E_{F,\mu}$ or $-T\Delta = -E_{F,e} - E_{F,\mu}$ depending on whether ptle 1 is an electron or muon. Hence,

$$E_1 + E_2 - E'_1 - E'_2 - E'_3 = T\,(x_1 + x_2 - x'_1 - x'_2 - z \pm \Delta) \; . \quad (1.3)$$

$$J^{(lf)}(\Delta) = \int T^5 \, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x'_1 \mathrm{d}x'_2 \mathrm{d}z \; T^4 z^4 \frac{\delta(x_1 + x_2 - x'_1 - x'_2 - z \pm \Delta)}{T} f_1 f_2 (1 - f'_1)(1 - f'_2) \; , \quad (1.4)$$

where the upper sign $(+)$ is for $l = e$ and the lower sign $(-)$ is for $l = \mu$. This simplifies to a modified form of Hong Yi's eq. (2.8) when we assume strongly degenerate particles (so that the integrals over x_i are from $-\infty$ to $+\infty$).

$$J^{(lf)}(\Delta) = \int_{-\infty}^{\infty} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x'_1 \mathrm{d}x'_2 \int_0^{\infty} \mathrm{d}z \frac{T^8 z^4 \, \delta(x_1 + x_2 - x'_1 - x'_2 - z \pm \Delta)}{(e^{x_1} + 1)(e^{x_2} + 1)(e^{-x'_1} + 1)(e^{-x'_2} + 1)} \; . \quad (1.5)$$

The x_i integrals in eq. (1.5) can be evaluated by doing the change of variables $\tilde{z} = z \mp \Delta$ and using the formula,

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_4 \, \frac{\delta(x_1 + x_2 + x_3 + x_4 - \tilde{z})}{(e^{x_1} + 1)(e^{x_2} + 1)(e^{x_3} + 1)(e^{x_4} + 1)} = \frac{1}{6} \frac{\tilde{z}\,(\tilde{z}^2 + 4\pi^2)}{e^{\tilde{z}} - 1} \; ,$$

so that we obtain

$$J^{(lf)}(\Delta) = \frac{T^8}{6} \int_{\Delta}^{\infty} \mathrm{d}\tilde{z} \; (\tilde{z} \pm \Delta)^4 \, \frac{\tilde{z}\,(\tilde{z}^2 + 4\pi^2)}{e^{\tilde{z}} - 1} \; . \quad (1.6)$$

Eq. (1.6) can be evaluated analytically for any Δ but in the special case $\Delta = 0$ the result is $J^{(lf)}(0) = (164\,\pi^8/945)\,T^8$, and is independent of the process. We now consider the cases $J^{(ef)}$ and $J^{(\mu f)}$.

$\underline{ef \to \mu f a}$:

In this case, we evaluate eq. (1.6) with the upper sign $(+)$. Mathematica can do this analytically and we obtain

$$\begin{aligned} J^{(ep)}(\Delta) &= \frac{T^8}{6} \int_{\Delta}^{\infty} \mathrm{d}\tilde{z} \; (\tilde{z} + \Delta)^4 \, \frac{\tilde{z}\,(\tilde{z}^2 + 4\pi^2)}{e^{\tilde{z}} - 1} \\ &= 4\left[210\,\mathrm{Li}_8(e^{\Delta}) - 90\,\Delta\,\mathrm{Li}_7(e^{\Delta}) + 5\,(4\pi^2 + 3\Delta^2)\,\mathrm{Li}_6(e^{\Delta}) - (4\pi^2\,\Delta + \Delta^3)\,\mathrm{Li}_5(e^{\Delta})\right] \; , \end{aligned} \quad (1.7)$$

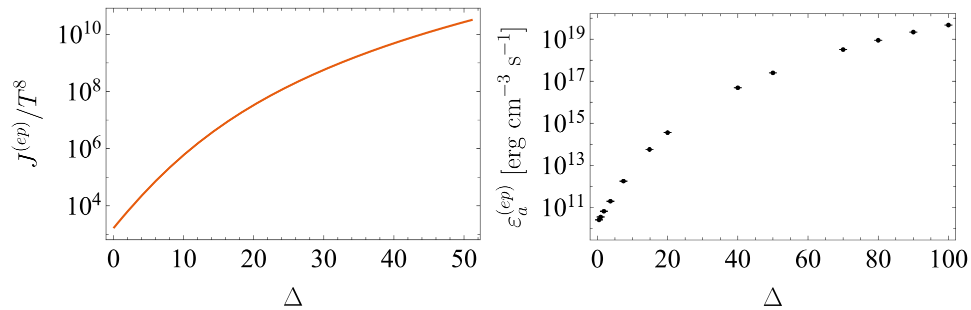

Figure 1: *Left*: Plotting the polylogarithm function given in eq. (1.7) versus Δ. At $\Delta = 0$ this coincides with $(164\,\pi^8/945)\,T^8 \approx 1650\,T^8$. For $\Delta < 10$ this is well-approximated by an exponential enhancement. *Right*: Numerically evaluated emissivity versus Δ. Notice that the behaviour is the same. This gives me confidence that the discussion here is a sufficient explanation.

where $\mathrm{Li}_n(x)$ is the polylogarithm function (see `PolyLog` in Mathematica's documentation). As expected, this coincides with $(164\,\pi^8/945)\,T^8$ when $\Delta = 0$. As shown in fig. (1) (1.7) is well approximated by an exponential decay of the form $e^{-k\Delta}$.

<u>$\mu p \to epa$:</u>

We can also apply this technique to the process $\mu p \to epa$. The only difference is that $2'$ is now the electron. This essentially flips the sign in front of the Δ so that we define $\tilde{z} = z - \Delta$ instead. Then we obtain,

$$\frac{J^{(\mu p)}(\Delta)}{T^8} = 4\,T^8\left[(4\pi^2\Delta + \Delta^3)\,\mathrm{Li}_5(e^{-\Delta}) + 5(4\pi^2 + 3\Delta^2)\,\mathrm{Li}_6(e^{-\Delta}) + 90\Delta\,\mathrm{Li}_7(e^{-\Delta}) + 210\,\mathrm{Li}_8(e^{-\Delta})\right]\,, \tag{1.8}$$

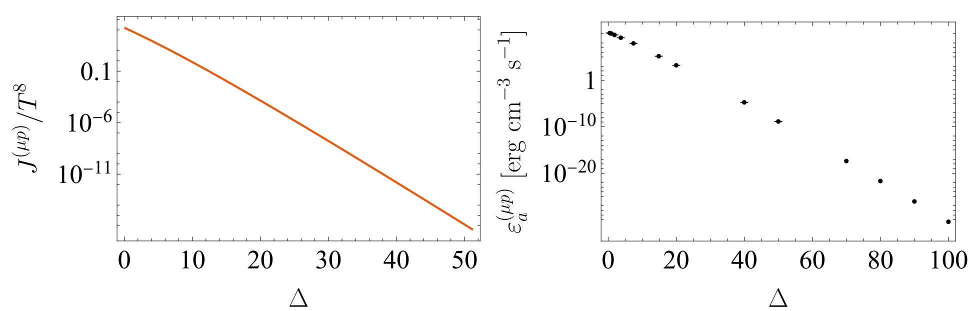

Figure 2: *Left*: The polylogarithm function given in eq. (1.8) versus Δ. At $\Delta = 0$ this coincides with $(164\,\pi^8/945)\,T^8 \approx 1650\,T^8$. This is well-approximated by an exponential decay. *Right*: Numerically evaluated emissivity versus Δ. Notice that the behaviour is the same. This gives me confidence that the discussion here is a sufficient explanation.