Neutron star cooling by axion emission

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August 18, 2023

Contents

1 Introduction

The goal of this project is to evaluate the integral

Emissivity integral

$$\varepsilon_{3'} = \int f_1 f_2 (1 - f_{1'})(1 - f_{2'})(1 + f_{3'}) \\ \times E_{3'} \sum_{\sigma,\sigma'} |\mathcal{M}|^2 \, \mathrm{d}\Phi_3\big((p_1 + p_2)^2; 1', 2', 3'\big) \, \frac{\mathrm{d}^3 p_1}{(2\pi)^3 \, 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 \, 2E_2} \tag{1.1}$$

where,

Lorentz invariant phase space measure

$$\mathrm{d}\Phi_n\big((p_a + p_b)^2; 1, \ldots, n\big) \equiv \prod_{i=1}^{n} \left[\delta(p_i^2 - m_i^2) \, \Theta(p_i^0) \, \frac{\mathrm{d}^4 p_i}{(2\pi)^3}\right] (2\pi)^4 \delta^4\left(p_a + p_b - \sum_{i=1}^{n} p_i\right) \tag{1.2}$$

is the Lorentz-invariant phase space measure, and

$$f_i \equiv f_{FD}(E_i) \equiv \frac{1}{e^{(E_i - \mu_i)/T} + 1} \quad \text{for } i = 1, 2, 1', 2' \tag{1.3}$$

$$f_{3'} \equiv f_{BE}(E_i) \equiv \frac{1}{e^{(E_{3'} + \mu_{3'})/T} + 1} \tag{1.4}$$

are thermal distributions which show up due to Pauli blocking and Bose enhancement . The spin-summed matrix element squared is given by

Spin-summed matrix element

$$\sum_{\sigma,\sigma'} |\mathcal{M}|^2 = \frac{128 \, g_{ae\mu}^2 e^4}{(m_1^2 - m_{1'}^2)^2} \, \frac{(p_1 \cdot p_{1'} - m_1 m_{1'})(p_2 \cdot p_{3'})(p_{2'} \cdot p_{3'})}{(p_2 - p_{2'})^4} \ . \tag{1.5}$$

1.1 Integration in terms of momentum variables (didn't work)

The first strategy I used to evaluate this integral was to do the integration directly in terms of the momentum variables. Below are the notes I wrote detailing my strategy.

We can use the momentum conserving Dirac delta to evaluate the $\boldsymbol{p}_{2'}$ integral, setting

$$\boldsymbol{p}_{2'} = \boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_{1'} - \boldsymbol{p}_{3'} . \tag{1.6}$$

We now choose to align the z-axis with $\boldsymbol{p}_1$ and measure angles with respect to this axis. Converting to spherical polar coordinates gives, for example, $\mathrm{d}^3 p_{3'} = p_{3'}^2 \, \mathrm{d}p_{3'} \, \mathrm{d}\cos\theta_{13'} \, \mathrm{d}\phi_{13'}$ [1] . Then the energy conserving Dirac delta can be used to evaluate the $\mathrm{d}p_{3'}$ integral in the

[1] The angle $\phi_{13'}$ is not measured with respect to $\boldsymbol{p}_1$, it is measured to some axis orthogonal to $\boldsymbol{p}_1$. We

following way. First, we use $E = \sqrt{p^2 + m^2}$ to rewrite the masses. Then we use momentum conservation to make the replacement $\boldsymbol{p}_{2'} \to \boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_{1'} - \boldsymbol{p}_{3'}$:

$$
\begin{aligned}
&E_1 + E_2 = E_{1'} + E_{2'} + E_{3'} \\
&\Rightarrow \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} = \sqrt{p_{1'}^2 + m_{1'}^2} + \sqrt{p_{2'}^2 + m_{2'}^2} + \sqrt{p_{3'}^2 + m_{3'}^2} \\
&\Rightarrow \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} = \sqrt{p_{1'}^2 + m_{1'}^2} + \sqrt{(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_{1'} - \boldsymbol{p}_{3'})^2 + m_{2'}^2} + \sqrt{p_{3'}^2 + m_{3'}^2}.
\end{aligned}
\tag{1.7}
$$

We would like to solve eqn. (1.7) for $p_{3'}$, but there is a problem coming from the fact that $p_{3'}$ appears under two square root symbols which makes it impossible to get an expression of the form $p_{3'} = \cdots$. Instead, we will make the approximation that because $p_{3'} \ll p_i$ for $i \in \{1, 2, 1', 2'\}$ that we can ignore the $p_{3'}$ that shows up in $\sqrt{(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_{1'} - \boldsymbol{p}_{3'})^2 + m_{2'}^2}$. The result we obtain when solving for $p_{3'}$ is then

$$
\begin{aligned}
p_{3'} = \Big[&m_1^2 + m_{1'}^2 + m_2^2 + m_{2'}^2 - m_{3'}^2 + 2p_1^2 - 2p_1 p_{1'} c_{11'} + 2p_{1'}^2 - 2E_1 E_{1'} \\
&+ 2p_1 p_2 c_{12} - 2p_2 p_{1'} c_{21'} + 2p_2^2 + 2E_1 - 2E_{1'} E_2 \\
&- 2E_1 E_{2'}(p_1, p_2, p_{1'}, c_{11'}, c_{12}, c_{21'}) + 2E_{1'} E_{2'}(p_1, p_2, p_{1'}, c_{11'}, c_{12}, c_{21'}) \\
&- 2E_2 E_{2'}(p_1, p_2, p_{1'}, c_{11'}, c_{12}, c_{21'}) \Big]^{1/2}
\end{aligned}
\tag{1.8}
$$

where $E_i \equiv \sqrt{p_i^2 + m_i^2}$, $c_{ij} \equiv \cos\theta_{ij} \equiv \boldsymbol{p}_i \cdot \boldsymbol{p}_j / (p_i p_j)$ and

$$
E_{2'}(p_1, p_2, p_{1'}, c_{11'}, c_{12}, c_{21'}) = \sqrt{p_1^2 - 2p_1 p_{1'} c_{11'} + p_{1'}^2 + 2p_1 p_2 c_{12} - 2p_2 p_{1'} c_{21'} + p_2^2 + m_{2'}^2}.
\tag{1.9}
$$

Now that we have an expression for $p_{3'}$ the requirement that $p_{3'}$ must be real and non-negative restricts the domain of integration of the other variables. I am not sure how to derive the new limits so I got stuck here.

1.2 Weinberg's wisdom

Update: 11 Aug 2023: I talked to Hong-Yi today and he pointed out page 141 of Weinberg QFT vol. I where the following treatment is done for the outgoing particle phase space measure.

1. Use momentum conservation to eliminate one of the momenta, e.g. $\boldsymbol{p}_1$.

2. Rewrite integrals over other two momenta in spherical polar coords so that $\mathrm{d}^3 p_2\, \mathrm{d}^3 p_3 = |\boldsymbol{p}_2|^2 \mathrm{d}|\boldsymbol{p}_2|\, |\boldsymbol{p}_3|^2 \mathrm{d}|\boldsymbol{p}_3| \mathrm{d}\Omega_3\, \mathrm{d}\phi_{23}\, \mathrm{d}\cos\theta_{23}$.

3. $\cos\theta_{23}$ is then fixed by energy conservation:

$$
\sqrt{|\boldsymbol{p}_2|^2 + 2|\boldsymbol{p}_2||\boldsymbol{p}_3|\cos\theta_{23} + |\boldsymbol{p}_3|^2 + m_1^2} + \sqrt{|\boldsymbol{p}_2|^2 + m_2^2} + \sqrt{|\boldsymbol{p}_3|^2 + m_3^2} = E. \tag{1.10}
$$

don't need to define that axis explicitly as long as the other angles ϕ_{1i} are measured with respect to the same axis.

4. This yields $\delta^4(p_{\text{in}} - p_{\text{out}})\mathrm{d}\beta \to |\boldsymbol{p}_2|\mathrm{d}|\boldsymbol{p}_2||\boldsymbol{p}_3|\mathrm{d}|\boldsymbol{p}_3|E_1\mathrm{d}\Omega_3\mathrm{d}\phi_{23}$. Rewriting in terms of energies gives $E_1E_2E_3\mathrm{d}E_2\mathrm{d}E_3\mathrm{d}\Omega_3\mathrm{d}\phi_{23}$.

1.3 A recursive expression for phase space integrals

The problem I ran into at the end of the last section was how to use the energy-momentum delta functions to restrict the bounds of integration. I spent some time searching the literature for useful references and the most promising were refs. [1, 2, 3]. [1] provides an introduction to Monte Carlo methods and discusses the phase space measure in sec. 9, which covers several ways to deal with the Dirac delta in the phase space integration measure by doing appropriate changes of variables. The most useful method, is covered in sec. 9.6 and is the same one used in [2]. Ref. [2] extends the treatment in sec. 9.6 of [1] by also introducing momentum transfer variables. My understanding is that this is appropriate when your matrix element is flat in terms of the momentum transfers.

Consider the phase space integral $R_n(p_a + p_b)$ defined by

Phase space integral (without 2π factors)

$$R_n(p_a+p_b) = \int \cdots \int \prod_{i=1}^{n} \delta(p_i^2 - m_i^2)\, \Theta(p_i^0)\, \mathrm{d}^4 p_i\, \delta^4(p_a + p_b - p_1 - \cdots - p_n). \quad (1.11)$$

where p_i now denotes a four-momentum vector in constrast to the previous section where it represented a three-momentum magnitude. This is manifestly Lorentz invariant and therefore R_n may only be a function of $s \equiv (p_a + p_b)^2$.

We note that $R_n(s)$ is related to $\mathrm{d}\Phi_n(s)$ by

$$R_n(s) = (2\pi)^{3n-4} \int \mathrm{d}\Phi_n(s). \quad (1.12)$$

Define the new integration variable $M_{n-1}^2 \equiv (p_a + p_b - p_n)^2$. The physical significance of M_{n-1} is that it is the invariant mass of the first $n-1$ particles, which can be seen using four-momentum conservation: $p_a + p_b - p_n = p_1 + p_2 + \cdots + p_{n-1}$. It is possible to show that the following recursive relation holds (eqn. (3) of [2])

$$R_n(s) = \int_{(m_1+m_2+\cdots+m_{n-1})^2}^{(\sqrt{s}-m_n)^2} \mathrm{d}M_{n-1}^2 \int \mathrm{d}\Omega_n \frac{\sqrt{\lambda(s, M_{n-1}^2, m_n^2)}}{8s} R_{n-1}(M_{n-1}^2) \quad (1.13)$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ and the solid angle $\mathrm{d}\Omega_n \equiv \mathrm{d}\cos\theta_n\, \mathrm{d}\varphi_n$ defines the direction of $\boldsymbol{p}_n$ in the frame where $p_a + p_b = (\sqrt{s}, \boldsymbol{0})$. The upper limit is obtained by requiring that

$$|\boldsymbol{p}_n|^2 = \frac{\lambda(s, M_{n-1}^2, m_n^2)}{4s} \quad (1.14)$$

be positive whereas the lower limit is threshold below which $R_{n-1}(M_{n-1}^2) = 0$.[2]Repeated application of eq. (1.13) yields

$$
\begin{aligned}
R_n(s) = &\int_{(m_1+m_2+\cdots+m_{n-1})^2}^{(\sqrt{s}-m_n)^2} \mathrm{d}M_{n-1}^2 \int \mathrm{d}\Omega_n \frac{\sqrt{\lambda(s, M_{n-1}^2, m_n^2)}}{8s} \\
&\times \int_{(m_1+m_2+\cdots+m_{n-2})^2}^{(M_{n-1}-m_{n-1})^2} \mathrm{d}M_{n-2}^2 \int \mathrm{d}\Omega_{n-1} \frac{\sqrt{\lambda(M_{n-1}^2, M_{n-2}^2, m_{n-1}^2)}}{8M_{n-1}^2} \\
&\times \cdots \times \int_{(m_1+m_2)^2}^{(M_3-m_3)^2} \mathrm{d}M_2^2 \int \mathrm{d}\Omega_3 \frac{\sqrt{\lambda(M_3^2, M_2^2, m_3^2)}}{8M_3^2} \times \int \mathrm{d}\Omega_2 \frac{\sqrt{\lambda(M_2^2, m_1^2, m_2^2)}}{8M_2^2}.
\end{aligned}
\tag{1.15}
$$

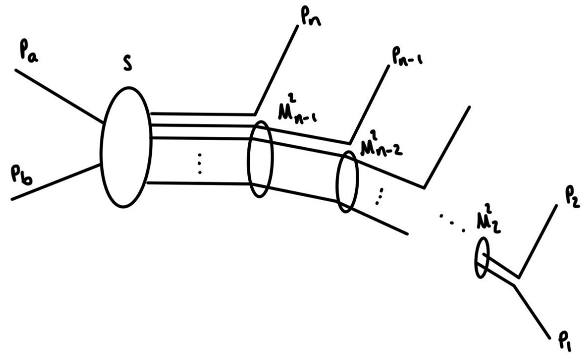

Figure 1: Illustration of the recursion relation as a sequence of effective $2 \to 2$ scattering events (this diagram is largely based on pg. 26 of [1]).

The important case for me is when $n = 3$:

Phase space measure for $n = 3$

$$\begin{aligned} R_3(s) &= (2\pi)^5 \int \mathrm{d}\Phi_3(s) \\ &= \int_{(m_1+m_2)^2}^{(\sqrt{s}-m_3)^2} \mathrm{d}M_2^2 \int \mathrm{d}\Omega_3 \, \frac{\sqrt{\lambda(s, M_2^2, m_3^2)}}{8s} \int \mathrm{d}\Omega_2 \, \frac{\sqrt{\lambda(M_2^2, m_1^2, m_2^2)}}{8M_2^2} \, . \end{aligned} \tag{1.16}$$

It's easy to verify that the number of integration variables matches our expectation of $3(3) - 4 = 5$ since we are integrating over one invariant mass, and two pairs of two angles.

In the above discussion we have considered p_a and p_b to be fixed. However, for my purposes I will also be integrating over p_a and p_b. This case seems to be similar to the one considered in [3] (cf. the example given between their eqs. (12) and (13)).

1.4 Writing the integral in terms of momentum transfers

In sec 1.3 we wrote the phase space integral in terms of invariant masses and angles of the three-momenta $\boldsymbol{p}_i$ defined in the center-of-mass frame where $\sum_{k=1}^{i} p_k = (M_i, \mathbf{0})$. The matrix element in eq. (1.5) is a function of the Lorentz invariants $(p_a \cdot p_1)$, $(p_b \cdot p_3)$, $(p_2 \cdot p_3)$, and $(p_b \cdot p_2) = (-(p_a - p_1 - p_2 - p_3) \cdot p_2)$. It is cumbersome to write this matrix element explicitly in terms of the angles $\{(\theta_i, \varphi_i) : i = 1, 2, \ldots, n\}$ so we would rather use kinematic Lorentz invariants analogous to the Mandelstam variables for $2 \to 2$ scattering[3]. To this end it is convenient to introduce the so-called 'momentum transfers' $t_i \equiv Q_i^2 \equiv (p_a - p_1 - \cdots - p_i)^2 = (p_n + p_{n-1} + \cdots + p_{i+1} - p_b)^2$.

[2]In eq. (3) of [2] the authors allude that the angular $\mathrm{d}\Omega$ integrals can be performed. They do not perform the integrals because *"the variables defining a momentum vector should appear explicitly if a Monte Carlo method is to be applied for the generation of events."* In my case I think that we still can't do these integrals trivially because the matrix element depends on them, although I'm not 100% sure if this is correct.

[3]At least, I think that is the motivation for introducing these variables.

Some of the dot products (but not all) may be written exclusively in terms of the kinematic variables. For example,

$$\begin{aligned}
t_1 &\equiv (p_a - p_1)^2 && \rightarrow && 2p_a \cdot p_1 = m_a^2 + m_1^2 - t_1 \\
t_2 &\equiv (p_a - (p_1 + p_2))^2 && \rightarrow && 2p_a \cdot p_2 = M_2^2 - m_1^2 + t_1 - t_2 \\
t_3 &\equiv (p_a - (p_1 + p_2 + p_3))^2 && \rightarrow && 2p_a \cdot p_3 = t_2 - t_3 - M_2^2 + M_3^2 \\
M_2^2 &\equiv (p_1 + p_2)^2 && \rightarrow && 2p_1 \cdot p_2 = M_2^2 - m_1^2 - m_2^2 \\
M_3^2 &\equiv (p_1 + p_2 + p_3)^2 && \rightarrow && 2(p_1 + p_2) \cdot p_3 = M_3^2 - M_2^2 - m_3^2 \\
p_b &= p_1 + p_2 + p_3 - p_a && \rightarrow && 2p_b \cdot p_3 = m_3^2 + t_3 - t_2 \\
& && && 2p_b \cdot p_2 = 2p_2 \cdot p_3 + m_2^2 + t_2 - t_1
\end{aligned}$$

I could not write $p_2 \cdot p_3$ in terms of the kinematic invariants alone. This is because we need $3(3)-4 = 5$ integration variables. Right now we only have $t_1,\ t_2,\ M_2^2$ which is not enough to fully describe the system; two more are required. The additional degrees of freedom are given by the azimuthal angles of the $\boldsymbol{Q}_i$ vectors $\varphi_1,\ \varphi_2$ as seen in a frame where $p_a = (m_a, \boldsymbol{0})$ (cf. eq. (1.20) which is eq. (14) of [2].) It is worth describing in more detail the meaning of the angles φ_1 and φ_2. I realised on Aug 18 2023 that these azimuthal angles can't be measured from the same axis. The reason for this is the derivation of eq. (1.18) and eq. (1.20) involved a step where we rewrote $\mathrm{d}^3 Q_{n-1}$ as $|\boldsymbol{Q}_{n-1}|^2\,\mathrm{d}|\boldsymbol{Q}_{n-1}|\,\mathrm{d}\cos\theta_{n-1}\mathrm{d}\varphi_{n-1}$. This transformation requires that θ_{n-1} be measured from a polar axis and φ_{n-1} be measured in the plane perpendicular to that axis. Since θ_i is defined as the angle between $\boldsymbol{Q}_i$ and $\boldsymbol{Q}_{i+1}$ we find that φ_i and φ_{i+1} are not measured in the same plane.

The result for $p_2 \cdot p_3$ in terms of the kinematic invariants + the angles is presented in appendix B.

To transform R_n into a form in which the momentum transfers appear as variables we must consider the phase space integral as a function of p_a and $-p_b \equiv Q_n$ separately: $R_n = R_n(p_a, Q_n)$. Let $Q_i = p_a - p_1 - \cdots - p_i$ so that $p_i = Q_{i-1} - Q_i$. Then, eq. (1.11) becomes

In eq. (1.17) I didn't write the Heaviside function which picks out positive energies, just pretend it is there so that $(Q_{i-1} - Q_i)^0 > 0$.

$$\begin{aligned}
R_n(s) &= \int \left[\prod_{i=1}^{n} \mathrm{d}^4 Q_i\, \delta\big((Q_{i-1} - Q_i)^2 - m_i^2\big)\right] \delta^4\Big(p_a + p_b - \sum_{i=1}^{n}(Q_{i-1} - Qi)\Big)\Big|_{Q_0 \equiv p_a} \\
&= \int \left[\prod_{i=1}^{n} \mathrm{d}^4 Q_i \delta\big((Q_{i-1} - Q_i)^2 - m_i^2\big)\right] \delta^4(Q_n + p_b) \qquad (1.17) \\
&= \int \mathrm{d}^4 Q_{n-1} \delta\big((Q_{n-2} - Q_{n-1})^2 - m_{n-1}^2\big) \left[\prod_{i=1}^{n-2} \mathrm{d}^4 Q_i \delta\big((Q_{i-1} - Q_i)^2 - m_i^2\big)\right].
\end{aligned}$$

The recursion relation is then simply

$$R_n(p_a, Q_n) = \int \mathrm{d}^4 Q_{n-1}\, \delta\big((Q_{n-1} - Q_n)^2 - m_n^2\big)\, R_{n-1}(p_a, Q_{n-1}). \qquad (1.18)$$

R_n is now regarded as a function of two invariants $s = s_n = (p_a - Q_n)^2$, and $t_n = Q_n^2$: $R_n = R_n(p_a, Q_n) = R_n(s_n, t_n)$. Introducing these variables in eq. (1.18) the authors of [2] obtain

$$R_n(s_n,t_n,s_{n-1},t_{n-1}) = \int \mathrm{d}s_{n-1}\,\mathrm{d}t_{n-1}\,K(s_n,t_n,s_{n-1},t_{n-1})\,R_{n-1}(s_{n-1},t_{n-1})\,, \tag{1.19}$$

where

$$\begin{aligned} &K(s_n,t_n,s_{n-1},t_{n-1}) \\ &\quad = \int \mathrm{d}^4Q_{n-1}\delta(Q_{n-1}^2 - t_{n-1})\,\delta(s_{n-1} - (p_a - Q_{n-1})^2)\,\delta((Q_{n-1}-Q_n)^2 - m_n^2) \\ &\quad = \int_0^{2\pi} \mathrm{d}\varphi_{n-1}\frac{1}{4\sqrt{\lambda(s_n,t_n,m_a^2)}}\,\Theta(-G(t_{n-1},s_n,s_{n-1},t_n,m_n^2,m_a^2))\,, \end{aligned} \tag{1.20}$$

$\Theta(x)$ is the Heaviside step-function, and φ_{n-1} defines the azimuthal angle of $\boldsymbol{Q}_{n-1}$ in the frame $p_a = (m_a, \boldsymbol{0})$. G is a kinematic function defined by

$$\begin{aligned} G(x,y,z,u,v,w) &= -\frac{1}{2}\begin{vmatrix} 2u & x+u-v & u+w-y \\ & 2x & x-z+w \\ \text{(symm.)} & & 2w \end{vmatrix} \\ &= x^2y + xy^2 + z^2u + vw^2 + v^2w \\ &\quad + xzw + xuv + yzv + yuw \\ &\quad - xy(z+u+v+w) - zu(x+y+v+w) - vw(x+y+z+u)\,. \end{aligned} \tag{1.21}$$

R_n for $n=3$

$$R_3(s_3,t_3) = \frac{1}{4\sqrt{\lambda(s_3,t_3,m_a^2)}}\int \frac{\mathrm{d}s_2\mathrm{d}t_2\mathrm{d}\varphi_2}{4\sqrt{\lambda(s_2,t_2,m_a^2)}}\Theta(-G_2)\int \mathrm{d}t_1\mathrm{d}\varphi_1\Theta(-G_1) \tag{1.22}$$

with

$$G_i = G(t_i, s_{i+1}, s_i, t_{i+1}, m_{i+1}^2, m_a^2)\,. \tag{1.23}$$

To use this method we rewrite the integrand by making the replacements:

This is incorrect. In particular eq. (1.24) doesn't work because the thermal distributions f are not Lorentz invariant. The E_i that shows up in eq. (1.1) is the energy *as seen in the rest frame of the NS*. Furthermore, eq. (1.31) was derived using incorrect assump-

$$E_i \to \frac{s_{i-1} - s_i - m_i^2}{2\sqrt{s_i}} \tag{1.24}$$

$$2p_a \cdot p_1 \to m_a^2 + m_1^2 - t_1 \tag{1.25}$$

$$2p_a \cdot p_2 \to s_2 - m_1^2 + t_1 - t_2 \tag{1.26}$$

$$2p_a \cdot p_3 \to t_2 - t_3 - s_2 + s_3 \tag{1.27}$$

$$2p_1 \cdot p_2 \to s_2 - m_1^2 - m_2^2 \tag{1.28}$$

$$2p_b \cdot p_3 \to m_3^2 + t_3 - t_2 \tag{1.29}$$

$$2p_b \cdot p_2 \to 2p_2 \cdot p_3 + m_2^2 + t_2 - t_1 \tag{1.30}$$

$$2p_2 \cdot p_3 \to \frac{1}{2m_a^2}\Bigg\{\sqrt{\Lambda_1\Lambda_2} - \frac{1}{\sqrt{\lambda_2\lambda_3}}\Bigg[m_a^2\sqrt{G_1G_2}\cos(\varphi_1 - \varphi_2) \tag{1.31}$$

$$+ \big(\sqrt{\Lambda_1\Lambda_2} - 2(t_1 + t_2 - m_2^2)\big)\big(\sqrt{\Lambda_2\Lambda_3} - 2(t_2 + t_3 - m_3^2)\big)\Bigg] \tag{1.32}$$

$$- \left(\sqrt{\Lambda_1\Lambda_3} - \sqrt{\frac{\lambda_3}{\lambda_2}}\big[\sqrt{\Lambda_1\Lambda_2} - 2m_a^2(t_1 + t_2 - m_2^2)\big]\right)\Bigg\} \tag{1.33}$$

$$+ \frac{1}{2}(t_3 - t_2 - m_3^2)\ . \tag{1.34}$$

where Λ_i and λ_i are defined in eq. (B.13).

[3] $M_1^2 = m_1^2$, $M_3^2 = s$ and $t_3 = (-p_b)^2 = m_b^2$ are all constants, not integration variables.

Random number generation for Monte Carlo

To sample s_i we generate a uniform random variable $r_i \sim \mathrm{U}(0,1)$ and compute

$$\sqrt{s_i} = r_i \left(\sqrt{s_n} - \sum_{j=1}^{n} m_j \right) + \sum_{j=1}^{n} m_j \ . \tag{1.35}$$

Next, we take the generated value of s_i (let's call it $\bar{s}_i$) and use it to calculate the upper and lower bounds from which we generate t_i.
To do this we solve $G(t_i, s_{i+1}, \bar{s}_i, t_{i+1}, m_{i+1}^2, m_a^2) = 0$ for t_i. This yields,

$$t_i^{\pm} = t_{i+1} + m_{i+1}^2 + \\ \times \frac{(\bar{s}_i - s_{i+1} - m_{i+1}^2)(t_{i+1} + s_{i+1} - m_a^2) \pm \sqrt{\lambda(\bar{s}_i, s_{i+1}, m_{i+1}^2)\lambda(s_{i+1}, t_{i+1}, m_a^2)}}{2\, s_{i+1}} \ . \tag{1.36}$$

So,

$$t_i^{\mathrm{gen}} = \gamma_i (t_i^+ - t_i^-) + t_i^- \tag{1.37}$$

where $\gamma_i \sim \mathrm{U}(0,1)$ is another uniform random variable. This procedure requires us to weight the generated event by a factor of $(t_i^+ - t_i^-)$.

φ_1 and φ_2 are sampled independently from $\mathrm{U}(0, 2\pi)$.

2 Monte Carlo integration

2.1 The basics

Suppose you are interested in evaluating the integral

$$I = \int_a^b f(x)\,\mathrm{d}x. \tag{2.1}$$

The most obvious method of approximating this is with a Riemann sum or trapezoidal integration where you divide the domain into bins of width Δx and calculate the area of a rectangle or trapezoid using the values of f at the bin edges.

Monte Carlo integration, on the other hand, takes a a different approach altogether whereby points in the domain are sampled at random and the integral is approximated as an average. To see this, note that the average value of f on the domain is

$$\langle f \rangle = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x = \frac{I}{b-a}. \tag{2.2}$$

The average $\langle f \rangle$ can be approximated by uniformly sampling N i.i.d random numbers x_i on the domain $x \in [a, b]$ where $i \in 1, 2, ..., N$ and calculating the mean of $f(x_i)$:

$$\langle f \rangle \approx \langle f \rangle_N \equiv \frac{1}{N} \sum_{i=1}^{N} f(x_i). \tag{2.3}$$

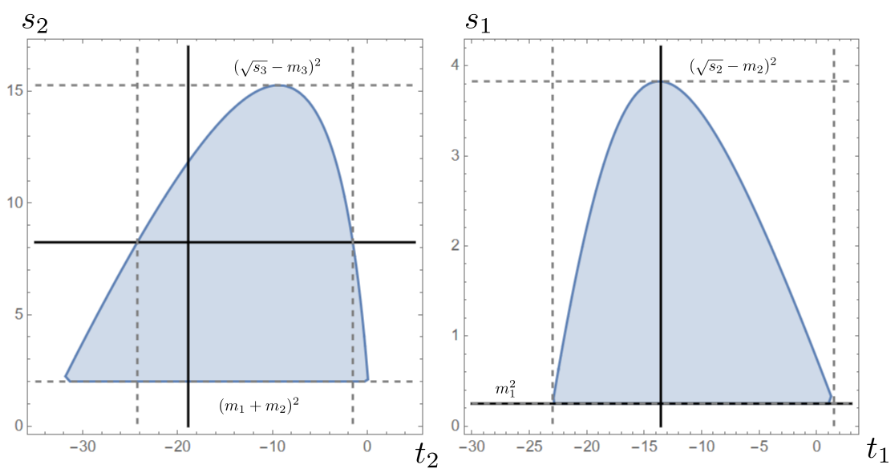

Figure 2: Illustration of sampling from the physical region of parameter space dictated by the $\Theta(-G_i)$ step-functions in eq. (1.22). *(Left:)* Region of the (t_2, s_2) plane where G_2 is negative (shaded blue). This corresponds to the physically allowed region of parameter space. The boundary of the region is established by the condition $G_2 = 0$. The lower horizontal dashed line is a lower bound established in the case when the C.M. energy of the effective particle $(p_1 + p_2)$ only has enough energy to create the particle with no kinetic energy $(s_2 = (m_1 + m_2)^2)$. The upper bound is established when the center of mass energy of $(p_1 + p_2 + p_3)$ creates p_3 at rest so that $p_3 = (m_3, 0)$ and the remaining energy is used to create $(p_1 + p_2)$ and supply it with kinetic energy. The thick black bars show values of the parameters created using the routine described in the box "Random number generation for Monte Carlo".

Therefore we can approximate the integral I as

$$I \approx (b-a)\langle f \rangle_N. \tag{2.4}$$

This argument can be extended to higher dimensions, i.e. in n dimensions we have

$$I = \int_V f(\boldsymbol{x})\,\mathrm{d}^n x = V\langle f \rangle. \tag{2.5}$$

One important result is that this strategy always converges as $N^{-1/2}$ in all dimensions, whereas techniques like trapezoidal integration converge rapidly in one dimension (N^{-2}), but significantly slower for higher dimensions ($N^{-2/n}$). For example in 5 dimensions, trapezoidal integration converges like $N^{-2/5}$, which is slower than Monte Carlo integration.

2.2 Importance sampling

In sec. 2.1 we sampled x uniformly at random on the domain of integration. However, we can choose to sample x from any arbitrary distribution $p(x)$ and compute the same expectation value by weighting the sample $f(x)$ by $1/p(x)$ so that

$$I \approx \frac{V}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}. \tag{2.6}$$

With an appropriate choice of p our Monte Carlo integration technique can be made to converge significantly faster than by sampling x uniformly at random. To see why this is, suppose $f(x) = e^{-x^2}$ and $[a, b] = [-1000, 1000]$. Sampling uniformly on $[-1000, 1000]$ means that the majority of points we pick will be in the tails of the Gaussian and will hardly affect the value of the integral. If instead we sampled from a standard normal distribution we would mostly choose x values near the peak at $x = 0$ and our result would converge much more rapidly. Clearly the choice of p is heavily dependent on the particular shape of the integrand f. The Metropolis-Hastings algorithm is one way to generate samples x from $p(x)$ and can be used for importance sampling.

VEGAS algorithm:

The discussion of the VEGAS algorithm at `https://www.ippp.dur.ac.uk/~krauss/Lectures/QuarksLeptons/Basics/PS_Vegas.html` is quite good. In case the link breaks in the future I reproduce some of the main points below.

> “The VEGAS algorithm starts by dividing the n-dimensional hypercube $[0, 1]^n$ into smaller ones – usually of identical size – and performs a sampling in each of them. The results are then used to refine the integration grid for the next iteration of the sampling procedure.”

2.3 Stratified sampling

2.4 Multichannel sampling

3 CompHEP and MADGRAPH5

3.1 `CompHEP`

While I was looking for ways to do the integral I found a software called `CompHEP` [4] which does a lot of the things I am interested in doing. `CompHEP`is a "a package for evaluation of Feynman diagrams, integration over multi-particle phase space and event generation". It allows the user to specify a Lagrangian and then it can generate tree level Feynman diagrams for a user-specified process, along with C and Fortran code to evaluate matrix elements. It can even compute cross-sections/other observables. `CompHEP`has two main backbones: (i) symbolic calculations (ii) numerical calculations.

The main tasks performed by the symbolic part are

1. Select a process by specifying in and out states for decays into up to 5 outgoing particles or collisions of $2 \to 2,\ 2 \to 3,\ 2 \to 4$.
2. Generate Feynman diagrams.
3. Delete unwanted diagrams.
4. Compute S-matrix elements.
5. Save symbolic results so they can be further analysed in `Mathematica`.
6. Generate C and Fortran code for the matrix elements to be used in numerical studies.

The numerical part, on the other hand, performs Monte Carlo integration and event generation. The main tasks performed are:

1. Choose phase space kinematic variables (idk what this means yet)
2. Introduce kinematic cuts over squared momentum transfers (not sure what the significance of this is)
3. Perform regularization to remove sharp peaks in matrix elements.
4. Calculate distributions, cross-sections, or particle width using Monte Carlo methods.
5. Perform integration, taking into account structure functions for ingoing particles.
6. Event generation.

4 16 Aug 2023: Lorentz transformations of FD distributions

The thermal factors in eq. (1.1) are a source of confusion for me right now. When I was rewriting the LIPS measure I derived identities like

$$|\boldsymbol{p}_i|^2 = \frac{(s_i - s_{i-1} - m_i^2)}{4s_i} \tag{4.1}$$

by boosting into the center of mass frame of the decaying intermediate particle $p_1+\cdots+p_i = (\sqrt{s_i}, \mathbf{0})$. This is inconsequential if the integrand is lorentz invariant. For example if we were just integrating over the spin-summed matrix element this would not matter. However, the thermal factors $f_a f_b(1-f_1)(1-f_2)$ (where I have ignored the $(1+f_3)$ factor since the axion is effectively free streaming and therefore receives negligible Bose-enhancement) are not Lorentz invariant. In other words, I must make sure that the E_a in $f_{FD}(E_a)$ is the energy of particle a in the rest frame of the neutron star; the E_1 in $f_{FD}(E_1)$ is the energy of particle 1 in the rest frame of the neutron star, and so on. This means that eq. (1.24) is not the correct replacement rule since it is valid only in the frame where $p_1+\cdots+p_i = (\sqrt{s_i}, \mathbf{0})$.

$$p_1^{NS} = \Lambda p_1^{p_1=(m_1,\mathbf{0})} \tag{4.2}$$

We know the value of $|\boldsymbol{p}_2|$ in the frame where $\boldsymbol{p}_1+\boldsymbol{p}_2 = 0$. To get E_i^{NS} we must boost into the rest frame of the NS where particles a and b have momenta $\boldsymbol{p}_a^{NS}, \boldsymbol{p}_b^{NS}$.

A Derivation of recursive relation

This appendix fills in the details for the derivation of eq. (1.13) which are left out in [2]. Our starting point is

$$\begin{aligned}
R_n(s) &= \int \prod_{i=1}^{n} \delta(p_i^2 - m_i^2)\,\Theta(p_i^0)\,\mathrm{d}^4 p_i\, \delta^4(p_a+p_b-p_1-\cdots-p_n) \\
&= \int \delta(p_n^2-m_n^2)\,\Theta(p_n^0)\,\mathrm{d}^4 p_n \left[\prod_{i=1}^{n-1}\delta(p_i^2-m_i^2)\,\Theta(p_i^0)\right]\delta^4(p_a+p_b-p_n-p_1-\cdots-p_{n-1}) \\
&= \int \delta(p_n^2-m_n^2)\,\Theta(p_n^0)\,\mathrm{d}^4 p_n\, R_{n-1}(p_a+p_b-p_n)
\end{aligned}$$

Multiply the RHS by $\int \mathrm{d}M_{n-1}^2\,\delta(M_{n-1}^2-(p_a+p_b-p_n)^2)$.

$$\int \mathrm{d}M_{n-1}^2\,\delta\big(M_{n-1}^2-(p_a+p_b-p_n)^2\big)\,\delta(p_n^2-m_n^2)\,\Theta(p_n^0)\,\mathrm{d}^4p_n\,R_{n-1}\big(p_a+p_b-p_n\big).$$

Use the on-shell delta function to do the p_n^0 integral and write d^3p_n in terms of spherical coordinates:

$$\begin{aligned}
&\int \mathrm{d}M_{n-1}^2\,\delta\big(M_{n-1}^2-(p_a+p_b-p_n)^2\big)\,\frac{\mathrm{d}^3p_n}{2E_n}\,R_{n-1}\big(p_a+p_b-p_n\big) \\
&= \int \mathrm{d}M_{n-1}^2\,\delta\big(M_{n-1}^2-(p_a+p_b-p_n)^2\big)\,\frac{|\boldsymbol{p}_n|^2\mathrm{d}|\boldsymbol{p}_n|\mathrm{d}\Omega_n}{2E_n}\,R_{n-1}\big(p_a+p_b-p_n\big).
\end{aligned}$$

Then use the M_{n-1}^2 Dirac delta to do the $\mathrm{d}|\boldsymbol{p}_n|$ integral. This requires us to (i) solve $M_{n-1}^2-(p_a+p_b-p_n)^2=0$ for $|\boldsymbol{p}_n|$; (ii) evaluate $\mathrm{d}/\mathrm{d}|\boldsymbol{p}_n|\big(M_{n-1}^2-(p_a+p_b-p_n)^2\big)$; and (iii) impose $|\boldsymbol{p}_n|^2\geq 0$ to derive bounds of integration for $\int \mathrm{d}M_{n-1}^2$.

(i) We start by expanding the LHS

$$\begin{aligned}
&M_{n-1}^2-(p_a+p_b-p_n)^2 = M_{n-1}^2-(p_a+p_b)^2-p_n^2+2(p_a+p_b)\cdot p_n \\
&= M_{n-1}^2 - s - m_n^2 + 2(E_a+E_b)E_n + 2(\boldsymbol{p}_a+\boldsymbol{p}_b)\cdot\boldsymbol{p}_n.
\end{aligned}$$

Now we choose the center of mass frame so that $\boldsymbol{p}_a + \boldsymbol{p}_b = 0$. This is valid because the integral is a Lorentz invariant, so the final result is independent of the reference frame.

$$\begin{aligned} &M_{n-1}^2 - s - m_n^2 + 2(E_a + E_b)E_n + 2(\boldsymbol{p}_a + \boldsymbol{p}_b)\cdot\boldsymbol{p}_n \\ &= M_{n-1}^2 - s - m_n^2 + 2\sqrt{s}\,\sqrt{|\boldsymbol{p}_n|^2 + m_n^2} = 0. \end{aligned}$$

Now we can rearrange and solve for $|\boldsymbol{p}_n|$

$$\begin{aligned} |\boldsymbol{p}_n|^2 &= \frac{(s + m_n^2 - M_{n-1}^2)^2}{4s} - m_n^2 = \frac{s^2 + m_n^4 + M_{n-1}^4 - 2sm_n^2 - 2sM_{n-1}^2 - 2m_n^2 M_{n-1}^2}{4s} \\ &\equiv \boxed{\frac{\lambda(s, M_{n-1}^2, m_n^2)}{4s}}\,. \end{aligned}$$

$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is called the Källen function.

(ii)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}|\boldsymbol{p}_n|}\Big(M_{n-1}^2 - (p_a + p_b - p_n)^2\Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}|\boldsymbol{p}_n|}\Big(M_{n-1}^2 - s - m_n^2 + 2\sqrt{s}\,\sqrt{|\boldsymbol{p}_n|^2 + m_n^2}\Big) \\ &= \boxed{\frac{2\sqrt{s}\,|\boldsymbol{p}_n|}{E_n}} \end{aligned}$$

(iii)

$$\begin{aligned} &|\boldsymbol{p}_n|^2 \geq 0 \\ &\Rightarrow \lambda(s, M_{n-1}^2, m_n^2) \geq 0 \\ &\Rightarrow \big[M_{n-1}^2 - (s + m_n^2)\big]^2 - 4\,sm_n^2 \geq 0 \\ &\Rightarrow \big[M_{n-1}^2 - (s + m_n^2)\big]^2 \geq 4\,sm_n^2 \\ &\Rightarrow M_{n-1}^2 - (s + m_n^2) \geq 2\sqrt{s\,m_n^2}\ \big|\big|\ M_{n-1}^2 - (s + m_n^2) \leq -2\sqrt{s\,m_n^2} \\ &\Rightarrow M_{n-1}^2 \geq (s + m_n^2) + 2\sqrt{s\,m_n^2}\ \big|\big|\ M_{n-1}^2 \leq (s + m_n^2) - 2\sqrt{s\,m_n^2} \\ &\Rightarrow \boxed{M_{n-1}^2 \geq (\sqrt{s} + m_n)^2\ \big|\big|\ M_{n-1}^2 \leq (\sqrt{s} - m_n)^2.} \end{aligned}$$

With these results established we find that

$$\delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big) = \frac{\delta\Big(|\boldsymbol{p}_n| - \sqrt{\lambda(s, M_{n-1}^2, m_n^2)}\Big/(2\sqrt{s})\Big)}{2\sqrt{s}\,|\boldsymbol{p}_n|/E_n}\ . \tag{A.1}$$

Hence,

$$\begin{aligned}
&\int \mathrm{d}M_{n-1}^2\, \delta\big(M_{n-1}^2 - (p_a + p_b - p_n)^2\big)\, \frac{|\boldsymbol{p}_n|^2 \mathrm{d}|\boldsymbol{p}_n| \mathrm{d}\Omega_n}{2E_n}\, R_{n-1}\big(p_a + p_b - p_n\big) \\
&= \int \mathrm{d}M_{n-1}^2\, \frac{\delta\Big(|\boldsymbol{p}_n| - \sqrt{\lambda(s, M_{n-1}^2, m_n^2)}\Big/(2\sqrt{s})\Big)}{2\sqrt{s}\,|\boldsymbol{p}_n|/E_n}\, \frac{|\boldsymbol{p}_n|^2 \mathrm{d}|\boldsymbol{p}_n| \mathrm{d}\Omega_n}{2E_n}\, R_{n-1}\big(p_a + p_b - p_n\big) \\
&= \int \mathrm{d}M_{n-1}^2\, \frac{\sqrt{\lambda(s, M_{n-1}^2, m_n^2)}}{8s}\, \mathrm{d}\Omega_n\, R_{n-1}\big(M_{n-1}^2\big).
\end{aligned}$$

B Evaluating $2p_2 \cdot p_3$

Let $\varepsilon_i = Q_i^0 = \sqrt{|\boldsymbol{Q}_i|^2 + t_i}$. Then,

$$\begin{aligned}
p_2 \cdot p_3 &= (Q_1 - Q_2) \cdot (Q_2 - Q_3) \\
&= Q_1 \cdot Q_2 - Q_1 \cdot Q_3 - t_2 + Q_2 \cdot Q_3 \\
&= \varepsilon_1 \varepsilon_2 - \boldsymbol{Q}_1 \cdot \boldsymbol{Q}_2 - \varepsilon_1 \varepsilon_3 - \boldsymbol{Q}_1 \cdot \boldsymbol{Q}_3 - t_2 + \varepsilon_2 \varepsilon_3 - \boldsymbol{Q}_2 \cdot \boldsymbol{Q}_3\,.
\end{aligned} \tag{B.1}$$

From eq. (B.1) it is clear that we need to know the angles between vectors $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_3$, and $\boldsymbol{Q}_2$ and $\boldsymbol{Q}_3$ in terms of the integration variables $t_1, t_2, s_2, \varphi_1, \varphi_2$. We already have formulas for all of these (cf. eqs. (B.9-B.12)) except for the angle between $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_3$ defined by $\cos\theta_{13}$. To obtain this formula we first pick to align the z-axis with $\boldsymbol{Q}_3$ so that

$$\boldsymbol{Q}_1 = |\boldsymbol{Q}_1|(\cos\varphi_{13}\sin\theta_{13}\hat{\boldsymbol{\imath}} + \sin\varphi_{13}\sin\theta_{13}\hat{\boldsymbol{\jmath}} + \cos\theta_{13}\hat{\boldsymbol{k}}) \tag{B.2}$$

$$\boldsymbol{Q}_2 = |\boldsymbol{Q}_2|(\cos\varphi_2\sin\theta_2\hat{\boldsymbol{\imath}} + \sin\varphi_2\sin\theta_2\hat{\boldsymbol{\jmath}} + \cos\theta_2\hat{\boldsymbol{k}}) \tag{B.3}$$

$$\boldsymbol{Q}_3 = |\boldsymbol{Q}_3|\hat{\boldsymbol{k}}\,. \tag{B.4}$$

Then,

$$\frac{\boldsymbol{Q}_1 \cdot \boldsymbol{Q}_2}{|\boldsymbol{Q}_1||\boldsymbol{Q}_2|} \equiv \cos\theta_1 = \sin\theta_2\cos(\varphi_1 - \varphi_2)\sin\theta_{13} + \cos\theta_2\cos\theta_{13}. \tag{B.5}$$

This can be rearranged to solve for θ_{13} to obtain

$$\cos\theta_{13} = \frac{\cos\theta_1\cos\theta_2 \pm \sin\theta_2\cos(\varphi_1 - \varphi_2)\sqrt{\cos^2\theta_2 + \cos^2(\varphi_1 - \varphi_2)\sin^2\theta_2 - \cos^2\theta_1}}{\cos^2\theta_2 + \cos^2(\varphi_1 - \varphi_2)\sin^2\theta_2}\,. \tag{B.6}$$

Substituting $\cos\theta_i$ and $\sin\theta_i$ using eqs. (B.11) and (B.12) yields

$$\cos\theta_{13} = \frac{\sqrt{\frac{\lambda_3}{\lambda_1}}\xi_1\xi_2 + \eta_2\sqrt{\eta_2^2 - \frac{\lambda_3}{\lambda_1}\xi_1^2 + \xi_2^2}}{\xi_2^2 + \eta_2^2} \tag{B.7}$$

where

$$\eta_2^2 \equiv (-G_2) m_a^2 \cos^2(\varphi_1 - \varphi_2)\,. \tag{B.8}$$

Note that in [2] there is a typo in their eq. (12b): they are missing a factor of 4 which is present in my eq. (B.12).

Formulas for angles and magnitudes in terms of integration variables

$$|\boldsymbol{Q}_i| = \frac{\sqrt{\lambda_i}}{2m_a} \tag{B.9}$$

$$\varepsilon_i = \frac{\sqrt{\Lambda_i}}{2m_a} \tag{B.10}$$

$$\cos\theta_i \equiv \frac{\boldsymbol{Q}_i \cdot \boldsymbol{Q}_{i+1}}{|\boldsymbol{Q}_i||\boldsymbol{Q}_{i+1}|} = \frac{\xi_i}{\sqrt{\lambda_i \lambda_{i+1}}} \tag{B.11}$$

$$\sin\theta_i = \sqrt{\frac{4m_a^2(-G_i)}{\lambda_i \lambda_{i+1}}} \tag{B.12}$$

where

$$\lambda_i \equiv \lambda(s_i, t_i, m_a^2) \tag{B.13}$$

$$\Lambda_i \equiv \lambda_i + 4m_a^2 t_i = (s_i - t_i - m_a^2)^2 \tag{B.14}$$

$$G_i \equiv G(t_i, s_{i+1}, s_i, t_{i+1}, m_{i+1}^2, m_a^2) < 0 \tag{B.15}$$

$$\xi_i \equiv \sqrt{\Lambda_i \Lambda_{i+1}} - 2m_a^2(t_i + t_{i+1} - m_{i+1}^2)\ . \tag{B.16}$$

Now we can use these identities to evaluate the dot products $Q_2 \cdot Q_3$, $Q_1 \cdot Q_3$, and $Q_1 \cdot Q_2$:

$$\begin{aligned} Q_2 \cdot Q_3 &= \frac{\sqrt{\Lambda_2\Lambda_3}}{4m_a^2} - \frac{\sqrt{\lambda_2\lambda_3}}{4m_a^2} \frac{\sqrt{\Lambda_2\Lambda_3} - 2m_a^2(t_2 + t_3 - m_3^2)}{\sqrt{\lambda_2\lambda_3}} \\ &= \frac{1}{2}(t_2 + t_3 - m_3^2)\ . \end{aligned} \tag{B.17}$$

$$Q_1 \cdot Q_3 = \varepsilon_1\varepsilon_3 - |\boldsymbol{Q}_1||\boldsymbol{Q}_3| \cos\theta_{13} \tag{B.18}$$

$$Q_1 \cdot Q_2 = \varepsilon_1\varepsilon_2 - |\boldsymbol{Q}_1||\boldsymbol{Q}_2|\big(\sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2) + \cos\theta_1 \cos\theta_2\big) \tag{B.19}$$

Main result of this appendix

$$p_2 \cdot p_3 = \text{something} \tag{B.20}$$

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