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Prof. Ta-Te Lin
Dept. of Biomechatroincs Engineering, National Taiwan University

WAVELETS AND OTHER IMAGE TRANSFORMS

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Prof. Ta-Te Lin
Dept. of Biomechatroincs Engineering, National Taiwan University

7.1 Preliminaries

- Vector Space and Inner Product Space
  - A ***vector space*** is a set of mathematical objects or entities, called *vectors*, that can be added together and multiplied by ***scalars***.
  - An ***inner product space*** is an abstract vector space over a field of numbers, together with an ***inner product function*** that maps two vectors of the vector space to a scalar of the number field such that

(a) $\langle u, v\rangle = \langle u, v\rangle^*$

(b) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

(c) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$

(d) $\langle v, v\rangle \geq 0 \; and \; \langle v, v\rangle = 0 \; if \; and \; only \; if \; v = 0$

7.1 Preliminaries

- Inner Product Space of Particular Interest
  1. Euclidean space $\mathbf{R}^N$ over real number field R with dot or scalar inner product

$$\langle \mathrm{u,v} \rangle = \mathbf{u}^T\mathbf{v} = u_0v_0 + u_1v_1 + .. + u_{N-1}v_{N-1} = \sum_{i=0}^{N-1} u_iv_i \quad (7-1)$$

  2. Unitary space $\mathbf{C}^N$ over complex number field C with inner product function

$$\langle \mathrm{u,v} \rangle = \mathbf{u}^{*T}\mathbf{v} = \sum_{i=0}^{N-1} u_i^* v_i = \langle \boldsymbol{v}, \boldsymbol{u} \rangle^* \qquad (7-2)$$

  3. Inner product space C([*a*,*b*]), where the vectors are continuous functions on the interval $a \leq x \leq b$

$$\langle f(x), g(x) \rangle = \int_a^b f^*(x)g(x)\,dx \qquad (7-3)$$

7.1 Preliminaries

- Orthogonal and Biorthonormal Basis
  1. The ***norm*** or length of vector z

$$\|z\| = \sqrt{\langle z, z \rangle} \qquad (7-4)$$

  2. The angle between two nonzero vector z and w is

$$\theta = \cos^{-1} \frac{\langle z, w \rangle}{\|z\| \|w\|} \qquad (7-5)$$

  3. A set of nonzero vectors $w_{0,}\, w_{1,}\, w_{2,} \ldots$ is mutually or pairwise ***orthogonal*** if and only if

$$\langle w_k, w_l \rangle = 0 \ \text{ for } k \neq l \qquad (7-6)$$

  4. If basis vectors are normalized, they are ***orthonormal basis*** and

$$\langle w_k, w_l \rangle = \delta_{kl} = \begin{cases} 0 \ \text{ for } k \neq l \\ 1 \text{ for } k = l \end{cases} \qquad (7-7)$$

7.1 Preliminaries

Orthogonal and Biorthonormal Basis

5. A set of vectors $w_{0,}\, w_{1,}\, w_{2,} \ldots$ and a complementary set of dual vectors $\widetilde{w_1}, \widetilde{w_2}, \widetilde{w_3}, \ldots$ are said to be ***biorthogonal*** and a biorthogonal basis of the vector space that they span if

$$\langle \widetilde{w}_k, w_l \rangle = 0 \text{ for } k \neq l \qquad (7-8)$$

6. They are ***biorthonormal basis*** if and only if

$$\langle \widetilde{w}_k, w_l \rangle = \delta_{kl} = \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases} \qquad (7-9)$$

7. Vector $z \in V$ (inner product space) can then be expressed as the following combination of basis vectors

$$z = \alpha_0 w_0 + \alpha_1 w_1 + \alpha_2 w_2 + \ldots \qquad (7-10)$$

7.1 Preliminaries

- Orthogonal and Biorthonormal Basis

8. The inner product with basis vector *w*, is

$$\langle w_i, z\rangle = \langle w_i, \alpha_0 w_0 + \alpha_1 w_1 + \alpha_2 w_2 + \ldots\rangle$$
$$= \alpha_0\langle w_i, w_0\rangle + \alpha_1\langle w_i, w_1\rangle + \ldots + \alpha_i\langle w_i, w_i\rangle + \ldots \quad (7-11)$$

Eliminating the zero terms and dividing both sides of the equation by $\langle w_i, w_i\rangle$ gives

$$\alpha_i = \frac{\langle w_i, z\rangle}{\langle w_i, w_i\rangle} \quad (7-12)$$

If the norms of the basis vectors are 1

$$\alpha_i = \langle w_i, z\rangle \quad (7-13)$$

For biorthogonal and biorthonormal basis vectors

$$\alpha_i = \frac{\langle \widetilde{w}_i, z\rangle}{\langle \widetilde{w}_i, w_i\rangle} \quad (7-14)$$

$$\alpha_i = \langle \widetilde{w}_i, z\rangle \quad (7-15)$$

7.1 Preliminaries

- Orthogonal and Biorthonormal Basis
  - Example 7.1: Vector norms and angles

The norm of vector $f(x) = \cos x$ of inner product space $C([0,2\pi])$ is

$$\|f(x)\| = \sqrt{\langle f(x), f(x)\rangle} = \left[\int_0^{2\pi} cos^2 x dx\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{2}x + \frac{1}{4}\sin(2x)\Big|_0^{2\pi}\right]^{\frac{1}{2}} = \sqrt{\pi}$$

The angle between vectors $\boldsymbol{z} = [1\ 1]^T$ and $\boldsymbol{w} = [1\ 0]^T$ of Euclidean inner product space $\boldsymbol{R}^2$ is

$$\theta = \cos^{-1}\left(\frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle}{\|\boldsymbol{z}\|\|\boldsymbol{w}\|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

7.2 Matrix-Based Transforms

- Forward and Inverse Transform

$$T(u) = \sum_{x=0}^{N-1} f(x)r(x,u) \qquad (7-16)$$

$$f(x) = \sum_{u=0}^{N-1} T(u)s(x,u) \qquad (7-17)$$

Linear expansion

$$f(x) = T(0)s(x,0) + T(1)s(x,1) + \cdots + T(N-1)s(x,N-1) \quad (7-18)$$

$$T(u) = \langle s(x,u), f(x) \rangle \qquad (7-19)$$

$s(x,u)$ in Eq. (7-18) are orthonormal basis vectors of an inner product space

7.2 Matrix-Based Transforms

- Forward and Inverse Transform

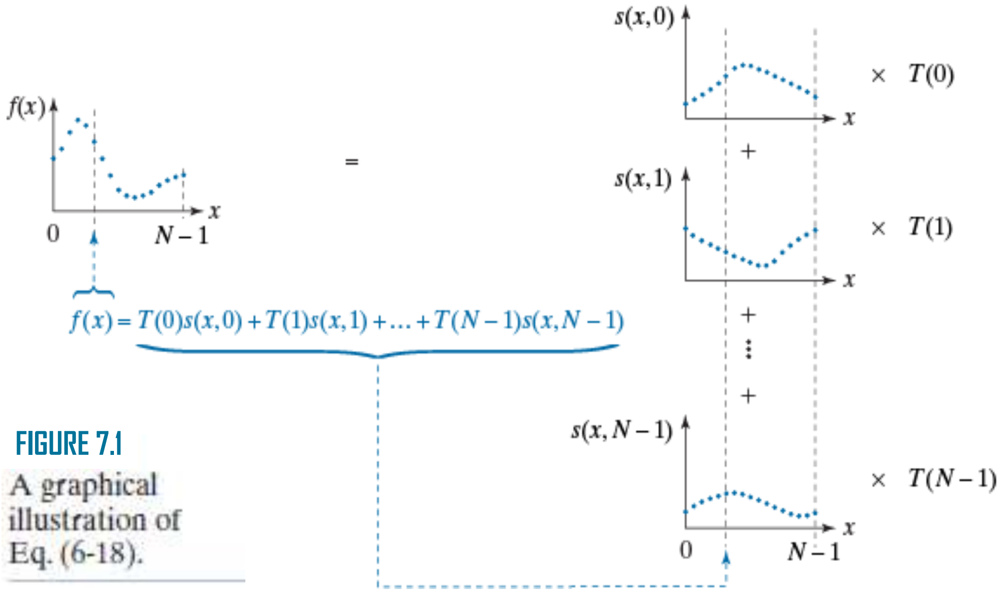


FIGURE 7.1
A graphical illustration of Eq. (6-18).

7.2 Matrix-Based Transforms

- Linear Transform in Matrix Form

$$\mathbf{f} = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix} \qquad (7-20)$$

$$\mathbf{t} = \begin{bmatrix} T(0) \\ T(1) \\ \vdots \\ T(N-1) \end{bmatrix} = \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_{N-1} \end{bmatrix} \qquad (7-21)$$

$$\mathbf{s}_u = \begin{bmatrix} s(0,u) \\ s(1,u) \\ \vdots \\ s(N-1,u) \end{bmatrix} = \begin{bmatrix} s_{u,0} \\ s_{u,1} \\ \vdots \\ s_{u,N-1} \end{bmatrix} \quad \text{for } u = 0,1,\ldots,N{-}1 \qquad (7{-}22)$$

And using the above equations to rewrite Eq. (7-19)

$$T(u) = \langle \mathbf{s}_u, \mathbf{f} \rangle \ \text{ for } \ u = 0,1,\ldots,\mathrm{N}{-}1 \qquad (7-23)$$

7.2 Matrix-Based Transforms

- Linear Transform in Matrix Form

$$\boldsymbol{A} = \begin{bmatrix} s_0^T \\ s_1^T \\ \vdots \\ s_{N-1}^T \end{bmatrix} = [s_0 \;\; s_1 \; \ldots \; s_{N-1}]^T \qquad (7-24)$$

$$\mathrm{t} = \begin{bmatrix} \langle \mathrm{s}_0, \mathrm{f} \rangle \\ \langle \mathrm{s}_1, \mathrm{f} \rangle \\ \vdots \\ \langle \mathrm{s}_{N-1}, \mathrm{f} \rangle \end{bmatrix} = \begin{bmatrix} \mathrm{s}_{0,0} f_0 + \mathrm{s}_{1,0} f_1 + .. + \mathrm{s}_{N-1,0} f_{N-1} \\ \mathrm{s}_{0,1} f_0 + \mathrm{s}_{1,1} f_1 + .. + \mathrm{s}_{N-1,1} f_{N-1} \\ \vdots \\ \mathrm{s}_{0,N-1} f_0 + \mathrm{s}_{1,N-1} f_1 + .. + \mathrm{s}_{N-1,N-1} f_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{s}_{0,0} & \mathrm{s}_{1,0} & \cdots & \mathrm{s}_{N-1,0} \\ \mathrm{s}_{0,1} & \mathrm{s}_{1,1} & & \\ \vdots & & \ddots & \vdots \\ \mathrm{s}_{0,N-1} & & \cdots & \mathrm{s}_{N-1,N-1} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix} \qquad (7-25)$$

or

$$\mathrm{t} = \mathrm{Af} \qquad (7-26)$$

7.2 Matrix-Based Transforms

- Linear Transform in Matrix Form

The inverse of the equation follows from the observation

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} s_0^T \\ s_1^T \\ \vdots \\ s_{N-1}^T \end{bmatrix} [s_0 \quad s_1 \quad \cdots \quad s_{N-1}] = \begin{bmatrix} s_0^T s_0 & s_0^T s_1 & \cdots & s_0^T s_{N-1} \\ s_1^T s_0 & s_1^T s_1 & & \\ \vdots & & \ddots & \vdots \\ s_{N-1}^T s_0 & & \cdots & s_{N-1}^T s_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} \langle s_0, s_0 \rangle & \langle s_0, s_1 \rangle & \cdots & \langle s_0, s_{N-1} \rangle \\ \langle s_1, s_0 \rangle & \langle s_1, s_1 \rangle & & \\ \vdots & & \ddots & \vdots \\ \langle s_{N-1}, s_0 \rangle & & \cdots & \langle s_{N-1}, s_{N-1} \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 1 \end{bmatrix} = \mathbf{I} \quad (7-27)$$

Thus, Eqs. (7-16) and (7-17) become the matrix-based transform pair

$$\mathrm{t} = \mathrm{Af} \qquad (7-28)$$

$$\mathrm{f} = \mathrm{A}^T \mathrm{t} \qquad (7-29)$$

$$\langle \mathrm{s}_k, \mathrm{s}_l \rangle = s_k^T s_l = \delta_{kl} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases} \qquad (7-30)$$

7.2 Matrix-Based Transforms

- Linear Transform in Matrix Form

Because the basis vectors of **A** are real and orthonormal, the transform defined in Eq. (7-28) is called an ***orthonormal transform.*** It preserves inner products – i.e., $\langle \mathbf{f}_1, \mathbf{f}_2 \rangle = \langle \mathbf{t}_1, \mathbf{t}_2 \rangle = \langle \mathbf{A}\mathbf{f}_1, \mathbf{A}\mathbf{f}_2 \rangle$ - and thus the distances and angles between vectors before and after transformation

$$T(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) r(x,y,u,v) \qquad (7-31)$$

$$f(x,y) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} T(u,v) s(x,y,u,v) \qquad (7-32)$$

$$r(x,y,u,v) = r_1(x,u) r_2(y,v) \qquad (7-33)$$

$$r(x,y,u,v) = r_1(x,u) r_1(y,v) \qquad (7-34)$$

$$\mathrm{T} = \mathrm{A}\mathrm{F}\mathrm{A}^T \qquad (7-35)$$

$$\mathrm{F} = \mathrm{A}^T \mathrm{T}\mathrm{A} \qquad (7-36)$$

7.2 Matrix-Based Transforms

- Rectangular Arrays

$$\mathbf{T} = \mathbf{A}_M \mathbf{F} \mathbf{A}_N^T \qquad (7-38)$$

$$\mathbf{F} = \mathbf{A}_M^T \mathbf{T} \mathbf{A}_N \qquad (7-39)$$

– Example 7.3: Computing the transform of a rectangular array

$$\mathbf{T} = \mathbf{A}_M \mathbf{F} \mathbf{A}_N^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 100 & 44 \\ 6 & 103 & 40 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0.366 & -1.366 \\ 1 & -1.366 & 0.366 \end{bmatrix}^T$$

$$= \frac{1}{\sqrt{6}}\begin{bmatrix} 11 & 203 & 84 \\ -1 & -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0.366 & -1.366 \\ 1 & -1.366 & 0.366 \end{bmatrix}$$

$$= \begin{bmatrix} 121.6580 & -12.0201 & -96.1657 \\ 0 & -3.0873 & 1.8624 \end{bmatrix}$$

7.2 Matrix-Based Transforms

- Complex Orthonormal Basis Vectors
  - Complex-valued basis vectors are orthonormal if and only if

$$\langle \mathbf{s}_k, \mathbf{s}_l \rangle = \langle \mathbf{s}_l, \mathbf{s}_k \rangle^* = s_k^{*T} s_l = \delta_{kl} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases} \qquad (7-40)$$

$$\mathbf{T} = \mathbf{A}\mathbf{F}\mathbf{A}^{\mathrm{T}} \qquad (7-41)$$

$$\mathbf{F} = \mathbf{A}^{*\mathrm{T}}\mathbf{T}\mathbf{A} \qquad (7-42)$$

  - Transformation matrix **A** is then called a unitary matrix. An important property of matrix **A** is that $\mathbf{A}^{*T}\mathbf{A}=\mathbf{A}\ \mathbf{A}^{*T}=\mathbf{A}^{*}\mathbf{A}^{T} = \mathbf{A}^{T}\mathbf{A}^{*}=\mathbf{I}$
  - The 1-D counterpart of Eqs. (7-41) and (7-42)

$$\mathbf{t} = \mathbf{A}\mathbf{f} \qquad (7-43)$$

$$\mathbf{f} = \mathbf{A}^{*\mathrm{T}}\mathbf{t} \qquad (7-44)$$

7.2 Matrix-Based Transforms

- Biorthogonal Basis Vectors
  - Expansion functions $s_{0,}\, s_{1,}\, s_{2,}\, \ldots, s_{N-1,}$ in Eq. (7-24) are ***biorthonormal*** if there exists a set of ***dual expansion functions*** $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \ldots, \tilde{s}_{N-1}$ such that

$$\langle \widetilde{S_k}, S_l \rangle = \delta_{kl} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases} \qquad (7-46)$$

$$\mathbf{T} = \widetilde{\mathbf{A}}\mathbf{F}\widetilde{\mathbf{A}}^{\mathrm{T}} \qquad (7-47)$$

$$\mathbf{F} = \mathbf{A}^{\mathrm{T}}\mathbf{T}\mathbf{A} \qquad (7-48)$$

  - When $\tilde{s}_u = s_u$, Eqs. (7-47) and (7-48) reduce to Eqs. (7-35) and (7-36).
  - The 1-D counterparts of Eqs. (7-47) and (7-48)

$$\mathbf{t} = \widetilde{\mathbf{A}}\mathbf{f} \qquad (7-49)$$

$$\mathbf{f} = \mathbf{A}^{\mathrm{T}}\mathbf{t} \qquad (7-50)$$

7.2 Matrix-Based Transforms

- Continuous Expansion

$$f(x) = \sum_{u=-\infty}^{\infty} \alpha_u s_u(x) \qquad (7-51)$$

$$\alpha_u = \langle s_u(x), f(x) \rangle \qquad (7-52)$$

  – Example 7.6: The Fourier series and Fourier transform

$$s_u = \frac{1}{\sqrt{T}} e^{j2\pi ux/T} \quad \text{for } u = 0, \pm 1, \pm 2 \ldots \qquad (7-53)$$

$$f(x) = \sum_{u=-\infty}^{\infty} \alpha_u [\frac{1}{\sqrt{T}} e^{j2\pi ux/T}]$$
$$= \frac{1}{\sqrt{T}} \sum_{u=-\infty}^{\infty} \alpha_u e^{j2\pi ux/T} \qquad (7-54)$$

7.2 Matrix-Based Transforms

- Continuous Expansion

$$\alpha_u = \langle s_u(x), f(x) \rangle$$
$$= \int_{-T/2}^{T/2} f(x) e^{j2\pi ux/T} \qquad (7-55)$$

$$s(x,u) = \frac{1}{\sqrt{N}} e^{j2\pi ux/N} \quad for\ u = 0, 1, \dots, N-1 \qquad (7-56)$$

$$f(x) = \frac{1}{\sqrt{N}} \sum_{u=0}^{N-1} T(u)\, e^{j2\pi ux/N} \qquad (7-57)$$

$$T(u) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x)\, e^{-j2\pi ux/N} \qquad (7-58)$$

– For Fourier transform of $f(x) = \sin(2\pi x)$ of period T=1

$$f(x) = j0.5e^{-j2\pi x} - j0.5e^{j2\pi x} \qquad (7-59)$$

$$T(u) = \begin{cases} -j1.414 & u = 1 \\ +j1.414 & u = 7 \\ 0 & otherwise \end{cases} \qquad (7-60)$$

7.2 Matrix-Based Transforms

- Continuous Expansion

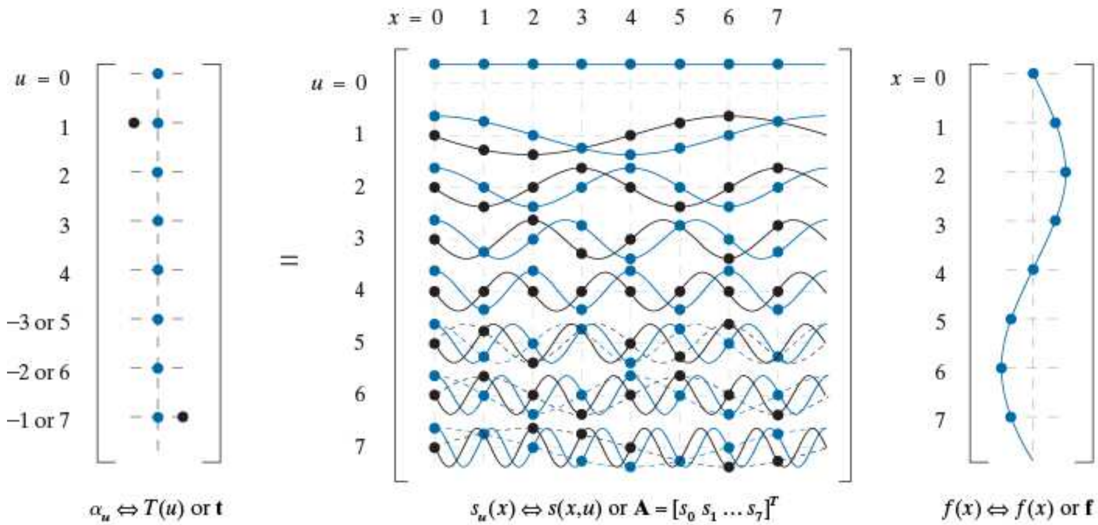


FIGURE 7.2 Depicting the continuous Fourier series and 8-point DFT of $f(x) = \sin(2\pi x)$ as "matrix multiplications." The real and imaginary parts of all complex quantities are shown in blue and black, respectively. Continuous and discrete functions are represented using lines and dots, respectively. Dashed lines are included to show that $\mathbf{s}_5 = \mathbf{s}_3^*$, $\mathbf{s}_6 = \mathbf{s}_2^*$, and $\mathbf{s}_7 = \mathbf{s}_1^*$, effectively cutting the maximum frequency of the DFT in half. The negative indices to the left of $\mathbf{t}$ are for the Fourier series computation alone.

7.3 Correlation

- Definition of Correlation
  - Given two continuous functions *f*(*x*) and *g*(*x*), the ***correlation*** of *f* and *g* is defined as

$$f \star g(\Delta x) = \int_{-\infty}^{\infty} f^*(x) g(x+\Delta x) dx$$
$$= \langle f(x), g(x+\Delta x) \rangle \qquad (7-61)$$

  - Correlation measures the similarity of *f*(*x*) and *g*(*x*) as a function of their relative displacement Δx. If $\Delta x = 0$,

$$f \star g(0) = \langle f(x), g(x) \rangle \qquad (7-62)$$

$$\alpha_u = \langle f, s_u \rangle = f \star s_u(0) \qquad (7-63)$$

$$\mathbf{f} \star \mathbf{g}(m) = \sum_{x=-\infty}^{\infty} f_n{}^* g_{n+m} \qquad (7-64)$$

$$\mathbf{f} \star \mathbf{g(0)} = \langle \mathbf{f}, \mathbf{g} \rangle \qquad (7-65)$$

$$T(u) = \langle \mathbf{s}_u, \mathbf{f} \rangle = \mathbf{s}_u \star \mathbf{f}(0) \qquad (7-66)$$

7.4 Basis Functions in the Time-Frequency Plane

- Basis Functions

  - Each element of an orthogonal transform [i.e. transform coefficient $\boldsymbol{T}(u)$ of Eq. (7-23)] is a single-point correlation that measures the similarity of $\mathbf{f}$ and vector $\mathbf{s}_u$.
  - Transforms measure the degree to which a function resembles a selected set of basis vectors.

7.4 Basis Functions in the Time-Frequency Plane

- Basis Functions

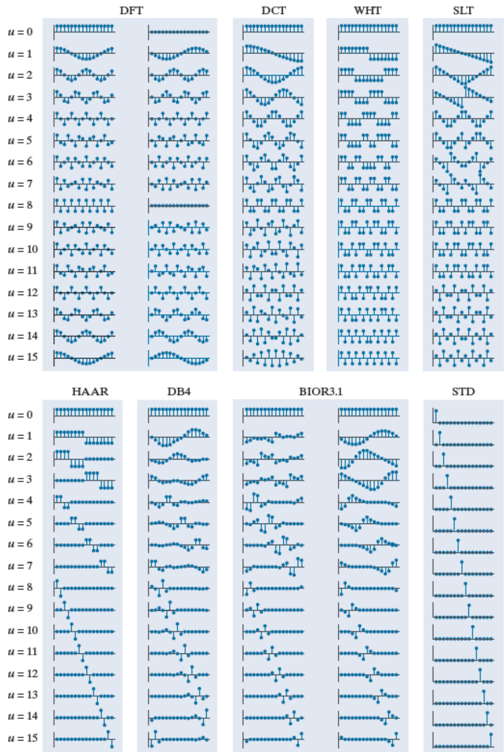


FIGURE 7.3
Basis vectors (for $N = 16$) of some commonly encountered transforms: (a) Fourier basis (real and imaginary parts), (b) discrete Cosine basis, (c) Walsh-Hadamard basis, (d) Slant basis, (e) Haar basis, (f) Daubechies basis, (g) Biorthogonal B-spline basis and its dual, and (h) the standard basis, which is included for reference only (i.e., not used as the basis of a transform).

7.4 Basis Functions in the Time-Frequency Plane

- Time-Frequency Plane
  - Let $p_h(t) = |h(t)|^2/\|h(t)\|^2$ be a probability density function with mean and variance

$$u_t = \frac{1}{\|h(t)^2\|}\int_{-\infty}^{\infty} t|h(t)|^2 dt \qquad (7-67)$$

$$\sigma_t{}^2 = \frac{1}{\|h(t)^2\|}\int_{-\infty}^{\infty} (t-u_t)^2|h(t)|^2 dt \qquad (7-68)$$

  - Let $p_H(f) = |H(f)|^2/\|H(f)\|^2$ be a probability density function with mean and variance

$$u_f = \frac{1}{\|H(f)^2\|}\int_{-\infty}^{\infty} f|H(f)|^2 df \qquad (7-69)$$

$$\sigma_t{}^2 = \frac{1}{\|H(f)^2\|}\int_{-\infty}^{\infty} (f-u_f)^2|H(f)|^2 df \qquad (7-70)$$

7.4 Basis Functions in the Time-Frequency Plane

- Time-Frequency Plane
  - The energy of basis function *h*, as illustrated in Fig. 7.4(a), is concentrated at (μ_t, μ_f) on the time-frequency plane. The majority of the energy falls in a rectangular region, called a ***Heisenberg box*** or ***cell***, of area $4\sigma_t\sigma_f$ such that

$$\sigma_t{}^2\sigma_f{}^2 \geq \frac{1}{16\pi^2} \qquad (7-71)$$

  - Since the ***support*** of a function can be defined as the set of points where the function is nonzero, ***Heisenberg's uncertainty principle*** tells us that it is impossible for function to have finite support in both time and frequency. Eq. (7-71), called the ***Heisenberg-Gabor inequality***, places a lower bound on the area of the Heisenberg cell.

7.4 Basis Functions in the Time-Frequency Plane

- Time-Frequency Plane

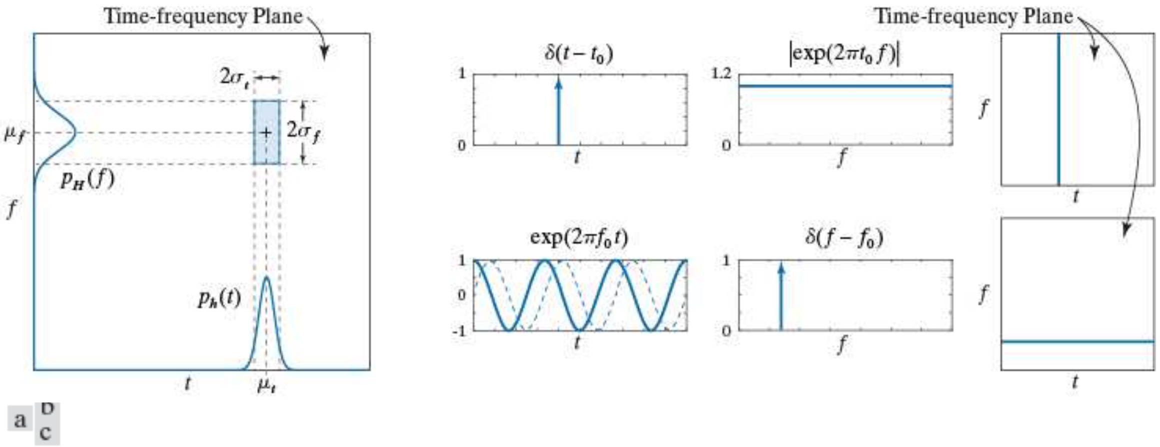


FIGURE 7.4 (a) Basis function localization in the time-frequency plane. (b) A standard basis function, its spectrum, and location in the time-frequency plane. (c) A complex sinusoidal basis function (with its real and imaginary parts shown as solid and dashed lines, respectively), its spectrum, and location in the time-frequency plane.

7.4 Basis Functions in the Time-Frequency Plane

- Wavelet Basis Functions

$$\psi_{s,\tau}(t) = 2^{\frac{s}{2}}\psi(2^s t - \tau) \qquad (7-72)$$

$$\Im\{\psi(2^s t)\} = \frac{1}{|2^s|}\Psi\left(\frac{f}{2^s}\right) \qquad (7-73)$$

$$\Im\{\psi(t-\tau)\} = e^{-j2\pi\tau f}\Psi(f) \qquad (7-74)$$

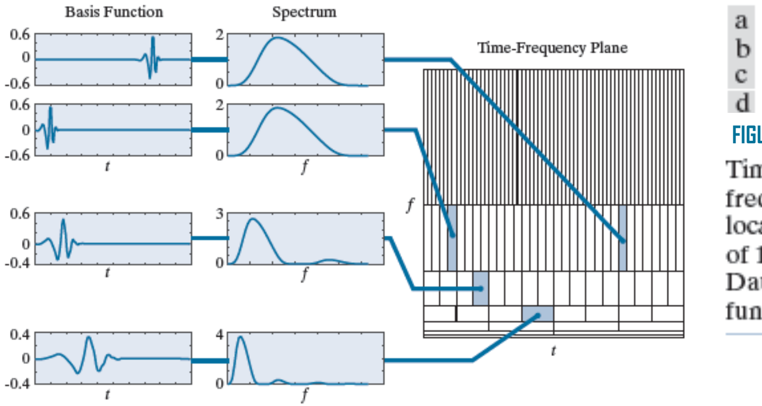


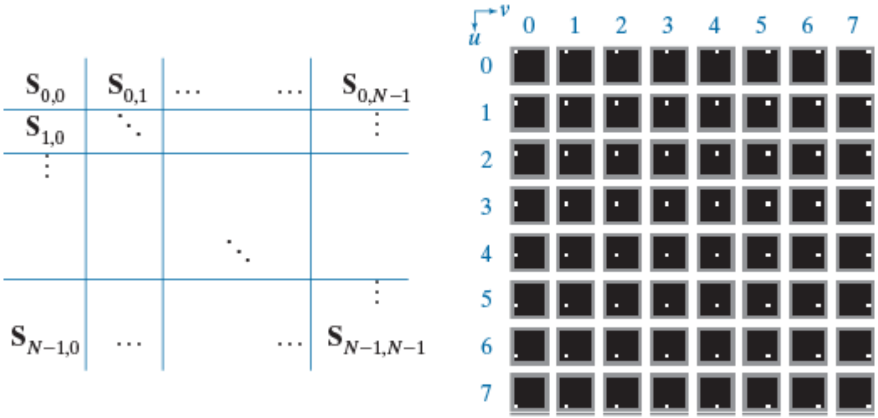
FIGURE 7.5 Time and frequency localization of 128-point Daubechies basis functions.

7.5 Basis Images

$$\mathbf{S}_{u,v} = \boldsymbol{s}_u \boldsymbol{s}_t{}^T \qquad (7-77)$$

$$\mathbf{F} = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} T(u,v)\mathbf{S}_{u,v} \qquad (7-75)$$

$$\mathbf{S}_{u,v} = \begin{bmatrix} s(0,0,u,v) & s(0,1,u,v) & \cdots & s(0,N-1,u,v) \\ s(1,0,u,v) & \ddots & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ s(N-1,0,u,v) & s(N-1,1,u,v) & \cdots & s(N-1,N-1,u,v) \end{bmatrix} \qquad (7-76)$$

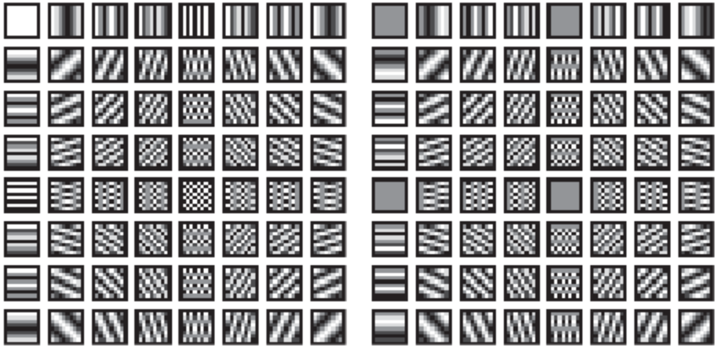


a b

FIGURE 7.6 (a) Basis image organization and (b) a standard basis of size 8 × 8. For clarity, a gray border has been added around each basis image. The origin of each basis image (i.e., $x = y = 0$) is at its top left.

7.5 Basis Images

$$\frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & -j & -j\omega & -1 & -\omega & j & j\omega \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -j\omega & j & \omega & -1 & j\omega & -j & -\omega \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & -j & j\omega & -1 & \omega & j & -j\omega \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & j\omega & j & -\omega & -1 & -j\omega & -j & \omega \end{bmatrix}$$



a b c

FIGURE 7.7 (a) Tranformation matrix $\mathbf{A}_F$ of the discrete Fourier transform for $N = 8$, where $\omega = e^{-j2\pi/8}$ or $(1-j)/\sqrt{2}$. (b) and (c) The real and imaginary parts of the DFT basis images of size 8×8. For clarity, a black border has been added around each basis image. For 1-D transforms, matrix $\mathbf{A}_F$ is used in conjunction with Eqs. (6-43) and (6-44); for 2-D transforms, it is used with Eqs. (6-41) and (6-42).

7.6 Fourier-Related Transforms

- The Discrete Hartley Transform

$$s(x,u)=\frac{1}{\sqrt{N}}cas\left(\frac{2\pi ux}{N}\right)$$
$$=\frac{1}{\sqrt{N}}\left[\cos\left(\frac{2\pi ux}{N}\right)+\sin(\frac{2\pi ux}{N})\right] \qquad (7-78)$$

$$s(x,y,u,v)=\left[\frac{1}{\sqrt{N}}cas\left(\frac{2\pi ux}{N}\right)\right]\left[\frac{1}{\sqrt{N}}cas\left(\frac{2\pi vy}{N}\right)\right] \qquad (7-79)$$

$$\mathbf{A}_{HY}=\mathrm{Real}\{\mathbf{A}_F\}-\mathrm{Imag}\{\mathbf{A}_F\}$$
$$=\mathrm{Real}\{(1+j)\mathbf{A}_F\} \qquad (7-80)$$

– Where $\mathbf{A}_\mathrm{F}$ denotes the unitary transformation matrix of the DFT

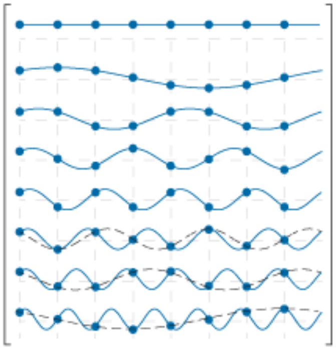
$$\mathrm{Re}\{s_F(x,u)\}=\mathrm{Re}\{\frac{1}{\sqrt{N}}e^{j2\pi ux/N}\}=\frac{1}{\sqrt{N}}\cos(\frac{2\pi ux}{N}) \qquad (7-81)$$

$$s_\mathrm{H}(x,u)=\sqrt{\frac{2}{N}}\cos\left(\frac{2\pi ux}{N}-\frac{\pi}{4}\right) \qquad (7-82)$$

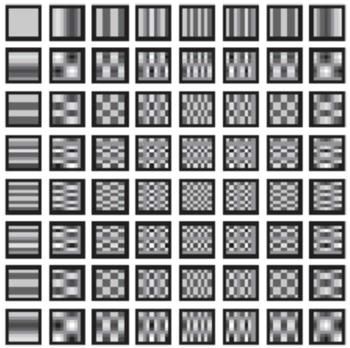
– The basis functions of DFT and DHT are scaled and shifted versions of one another

7.6 Fourier-Related Transforms

- The Discrete Hartley Transform



$$\begin{bmatrix} 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 \\ 0.35 & 0.50 & 0.35 & 0 & -0.35 & -0.50 & -0.35 & 0 \\ 0.35 & 0.35 & -0.35 & -0.35 & 0.35 & 0.35 & -0.35 & -0.35 \\ 0.35 & 0 & -0.35 & 0.50 & -0.35 & 0 & 0.35 & -0.50 \\ 0.35 & -0.35 & 0.35 & -0.35 & 0.35 & -0.35 & 0.35 & -0.35 \\ 0.35 & -0.50 & 0.35 & 0 & -0.35 & 0.50 & -0.35 & 0 \\ 0.35 & -0.35 & -0.35 & 0.35 & 0.35 & -0.35 & -0.35 & 0.35 \\ 0.35 & 0 & -0.35 & -0.50 & -0.35 & 0 & 0.35 & 0.50 \end{bmatrix}$$

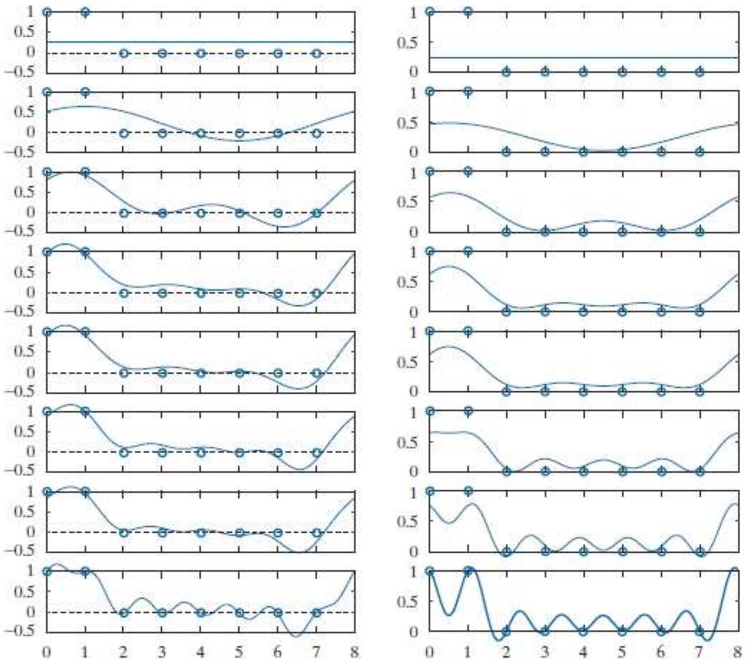


a b c

FIGURE 7.8 The transformation matrix and basis images of the discrete Hartley transform for $N = 8$: (a) Graphical representation of orthogonal transformation matrix $\mathbf{A}_{HY}$, (b) $\mathbf{A}_{HY}$ rounded to two decimal places, and (c) 2-D basis images. For 1-D transforms, matrix $\mathbf{A}_{HY}$ is used in conjunction with Eqs. (6-28) and (6-29); for 2-D transforms, it is used with Eqs. (6-35) and (6-36).

7.6 Fourier-Related Transforms

- The Discrete Hartley Transform



a b

FIGURE 7.9 Reconstructions of a discrete function by the addition of progressively higher frequency components: (a) DHT and (b) DFT.

7.6 Fourier-Related Transforms

- The Discrete Cosine Transform

$$s(x,u) = \alpha(u)\cos\left(\frac{(2x+1)u\pi}{2N}\right) \qquad (7-83)$$

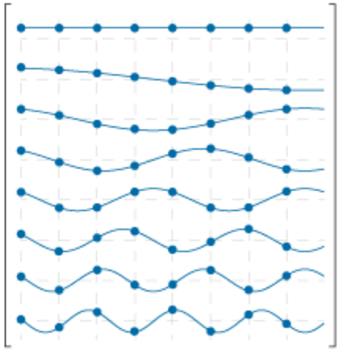
$$\alpha(u) = \begin{cases} \sqrt{\dfrac{1}{N}} & for\ u = 0 \\ \sqrt{\dfrac{2}{N}} & for\ u = 1, 2, \dots, N-1 \end{cases} \qquad (7-84)$$

$$s(x,y,u,v) = \alpha(u)\alpha(v)\cos\left(\frac{(2x+1)u\pi}{2N}\right)\cos\left(\frac{(2y+1)v\pi}{2N}\right) \qquad (7-85)$$

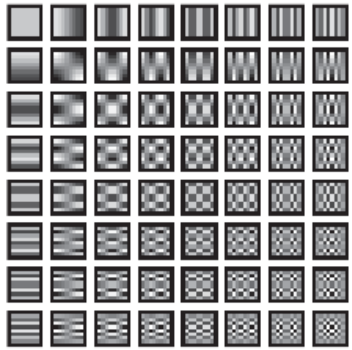
  - Rather than *N*-point periodicity, the underlying assumption of the DFT, the DCT transform assumes 2*N*-point periodicity and even symmetry.
  - The DCT of *N*-point function *f*(*x*) can be obtained from the DFT of a 2*N*-point symmetrically extended *f*(*x*)

7.6 Fourier-Related Transforms

- The Discrete Cosine Transform



$$
\begin{bmatrix}
0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 \\
0.49 & 0.42 & 0.28 & 0.10 & -0.10 & -0.28 & -0.42 & -0.49 \\
0.46 & 0.19 & -0.19 & -0.46 & -0.46 & -0.19 & 0.19 & 0.46 \\
0.42 & -0.10 & -0.49 & -0.28 & 0.28 & 0.49 & 0.10 & -0.42 \\
0.35 & -0.35 & -0.35 & 0.35 & 0.35 & -0.35 & -0.35 & 0.35 \\
0.28 & -0.49 & 0.10 & 0.42 & -0.42 & -0.10 & 0.49 & -0.28 \\
0.19 & -0.46 & 0.46 & -0.19 & -0.19 & 0.46 & -0.46 & 0.19 \\
0.10 & -0.28 & 0.42 & -0.49 & 0.49 & -0.42 & 0.28 & -0.10
\end{bmatrix}
$$

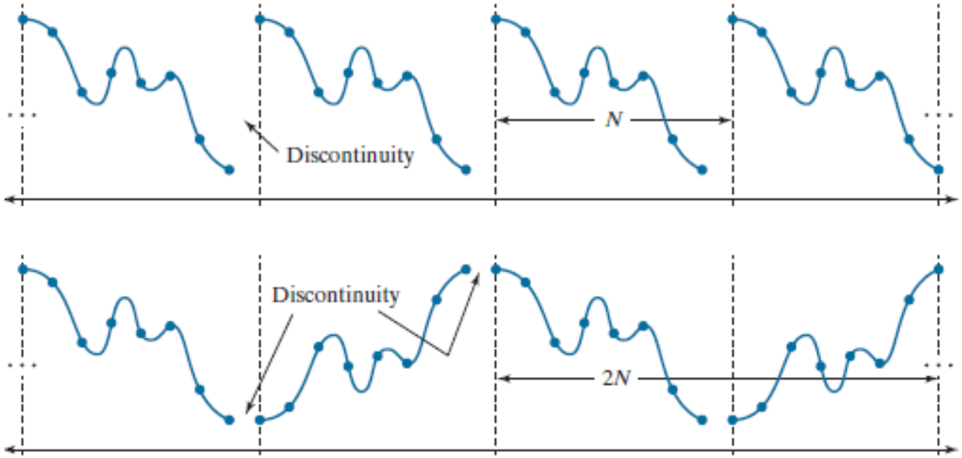


a b c

FIGURE 7.10 The transformation matrix and basis images of the discrete cosine transform for $N = 8$. (a) Graphical representation of orthogonal transformation matrix $\mathbf{A}_C$, (b) $\mathbf{A}_C$ rounded to two decimal places, and (c) basis images. For 1-D transforms, matrix $\mathbf{A}_C$ is used in conjunction with Eqs. (6-28) and (6-29); for 2-D transforms, it is used with Eqs. (6-35) and (6-36).

7.6 Fourier-Related Transforms

- The Discrete Cosine Transform



a
b

FIGURE 7.11

The periodicity implicit in the 1-D (a) DFT and (b) DCT.

7.6 Fourier-Related Transforms

- The Discrete Cosine Transform

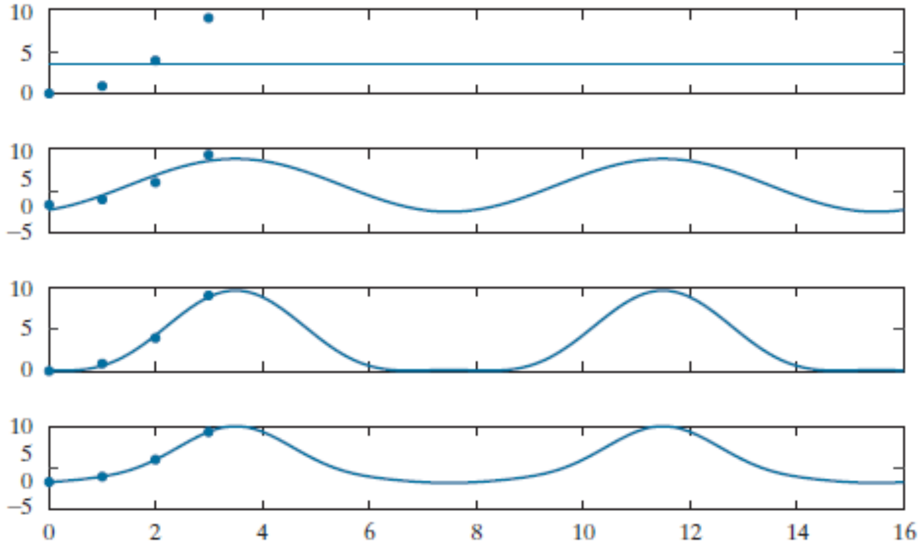


FIGURE 7.12 DCT reconstruction of a discrete function by the addition of progressively higher frequency components. Note the 2N-point periodicity and even symmetry imposed by the DCT.

7.6 Fourier-Related Transforms

- The Discrete Sine Transform

$$s(x,u) = \sqrt{\frac{2}{N+1}} \sin\left(\frac{(x+1)(u+1)\pi}{N+1}\right) \qquad (7-90)$$

$$s(x,y,u,v) = \frac{2}{N+1} \sin\left(\frac{(x+1)(u+1)\pi}{N+1}\right) \sin\left(\frac{(y+1)(v+1)\pi}{N+1}\right) \quad (7-91)$$

  – The DST of N-point function $f(x)$ can be obtained from the DFT of a $2(N+1)$-point symmetrically extended $f(x)$ with odd symmetry.

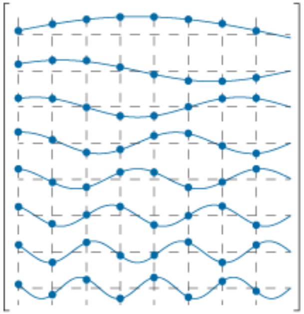
$$g(x) = \begin{cases} 0 & for\ x = 0 \\ f(x-1) & for\ 1 \le x \le N \\ 0 & for\ x = N+1 \\ -f(2N-x+1) & for\ N+2 \le x \le 2N+2 \end{cases} \qquad (7-92)$$

$$\mathbf{t}_\mathrm{F} = \mathbf{A}_\mathrm{F}\mathbf{g} = \begin{bmatrix} 0 \\ \mathbf{t}_1 \\ 0 \\ \mathbf{t}_2 \end{bmatrix} \qquad (7-93)$$

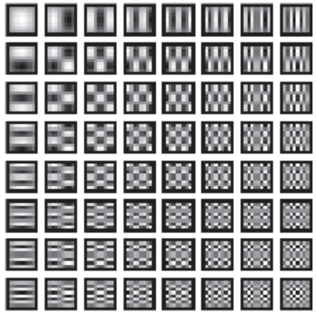
$$\mathbf{t}_S = -\mathrm{Imag}\{\mathbf{t}_1\} \qquad (7-94)$$

7.6 Fourier-Related Transforms

- The Discrete Sine Transform



0.16	0.30	0.41	0.46	0.46	0.41	0.30	0.16
0.30	0.46	0.41	0.16	−0.16	−0.41	−0.46	−0.30
0.41	0.41	0.00	−0.41	−0.41	0.00	0.41	0.41
0.46	0.16	−0.41	−0.30	0.30	0.41	−0.16	−0.46
0.46	−0.16	−0.41	0.30	0.30	−0.41	−0.16	0.46
0.41	−0.41	0.00	0.41	−0.41	−0.00	0.41	−0.41
0.30	−0.46	0.41	−0.16	−0.16	0.41	−0.46	0.30
0.16	−0.30	0.41	−0.46	0.46	−0.41	0.30	−0.16



a b c

FIGURE 7.13 The transformation matrix and basis images of the discrete sine transform for $N = 8$. (a) Graphical representation of orthogonal transformation matrix $\mathbf{A}_S$, (b) $\mathbf{A}_S$ rounded to two decimal places, and (c) basis images. For 1-D transforms, matrix $\mathbf{A}_S$ is used in conjunction with Eqs. (6-28) and (6-29); for 2-D transforms, it is used with Eqs. (6-35) and (6-36).

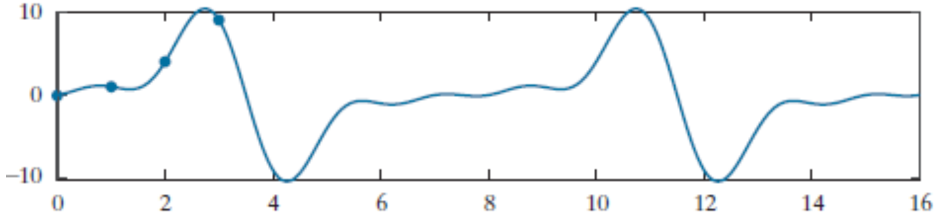
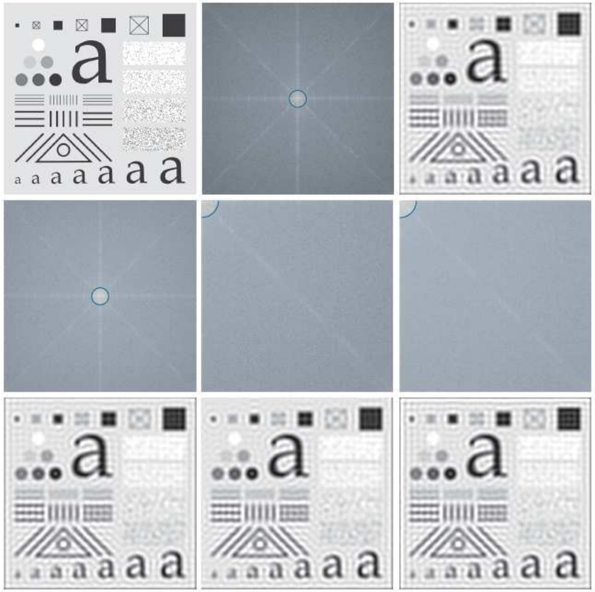


FIGURE 7.14 A reconstruction of the DST of the function defined in Example 6.10.

7.6 Fourier-Related Transforms

- The Discrete Sine Transform



a b c
d e f
g h i

FIGURE 7.15 (a) Original image of the 688 × 688 test pattern from Fig. 4.41(a). (b) Discrete Fourier transform (DFT) of the test pattern in (a) after padding to size 1376 × 1376. The blue overlay is an ideal lowpass filter (ILPF) with a radius of 60. (c) Result of Fourier filtering. (d)–(f) Discrete Hartley transform, discrete cosine transform (DCT), and discrete sine transform (DST) of the test pattern in (a) after padding. The blue overlay is the same ILPF in (b), but appears bigger in (e) and (f) because of the higher frequency resolution of the DCT and DST. (g)–(i) Results of filtering for the Hartley, cosine, and sine transforms, respectively.

7.7 Walsh-Hadamard Transforms

– ***Walsh-Hadamard transforms*** (WHTs) are non-sinusoidal transformations that decompose a function into a linear combination of rectangular basis functions, called ***Walsh functions***, of value +1 and -1.

$$s(x,u)=\frac{1}{\sqrt{N}}(-1)^{\sum_{i=0}^{n-1} b_i(x)b_i(u)} \qquad (7-95)$$

$$\mathbf{A}_W=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \qquad (7-96)$$

$$\mathbf{A}_W=\frac{1}{\sqrt{N}}\mathbf{H}_N \qquad (7-97)$$

$$\mathbf{H}_{2N}=\begin{bmatrix}\mathbf{H}_N & \mathbf{H}_N\\ \mathbf{H}_N & -\mathbf{H}_N\end{bmatrix} \qquad (7-98)$$

$$\mathbf{H}_2=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \qquad (7-99)$$

7.7 Walsh-Hadamard Transforms

$$\mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad (7-100)$$

$$\mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 \\ \mathbf{H}_4 & -\mathbf{H}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \qquad (7-101)$$

– The number of sign changes along a row of a Hadarmard matrix is known as the ***sequency*** of the row. Like frequency, sequency measure the rate of change of a function.

7.7 Walsh-Hadamard Transforms

– The transformation matrix of the resulting sequency-ordered Walsh Hadamard transform is obtained by substituting the inverse transformation kernel

$$s(x,u) = \frac{1}{\sqrt{N}}(-1)^{\sum_{i=0}^{n-1} b_i(x)p_i(u)} \qquad (7-102)$$

$$\begin{aligned}
p_0(u) &= b_{n-1}(u) \\
p_1(u) &= b_{n-1}(u) + b_{n-2}(u) \\
p_2(u) &= b_{n-2}(u) + b_{n-3}(u) \\
&\vdots \\
p_{n-1}(u) &= b_1(u) + b_0(u)
\end{aligned} \qquad (7-103)$$

$$\mathbf{H}_8' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \qquad (7-104)$$

7.7 Walsh-Hadamard Transforms

– An alternate way to generate H_8' is to rearrange the rows of Hadamard-ordered H_8, noting that the rows of H_8' corresponds to the rows of H_8 that is the bit-reversed ***gray code*** of *s*.

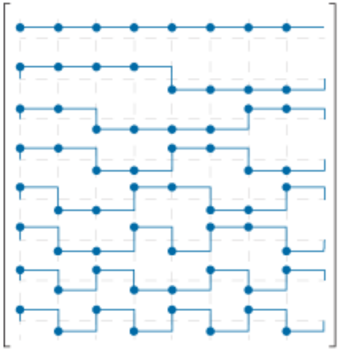
$$\begin{aligned} g_i &= s_i \oplus s_{i+1} \quad for\ 0 \le i \le n-2 \\ g_{n-1} &= s_{n-1} \quad for\ i = n-1 \end{aligned} \qquad (7-105)$$

$$\mathbf{A}_{W'} = \frac{1}{\sqrt{N}} \mathbf{H}_8' \qquad (7-106)$$

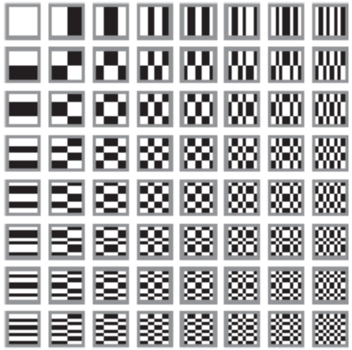
$$s(x,y,u,v) = \frac{1}{\sqrt{N}} (-1)^{\sum_{i=0}^{n-1}[b_i(x)p_i(u)+b_i(y)p_i(v)]} \qquad (7-107)$$

Row of H_4'	Binary Code	Gray Code	Bit-Reversed Gray Code	Row of H_4
0	00	00	00	0
1	01	01	10	2
2	10	11	11	3
3	11	10	01	1

7.7 Walsh-Hadamard Transforms



$$\begin{bmatrix} 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 \\ 0.35 & 0.35 & 0.35 & 0.35 & -0.35 & -0.35 & -0.35 & -0.35 \\ 0.35 & 0.35 & -0.35 & -0.35 & -0.35 & -0.35 & 0.35 & 0.35 \\ 0.35 & 0.35 & -0.35 & -0.35 & 0.35 & 0.35 & -0.35 & -0.35 \\ 0.35 & -0.35 & -0.35 & 0.35 & 0.35 & -0.35 & -0.35 & 0.35 \\ 0.35 & -0.35 & -0.35 & 0.35 & -0.35 & 0.35 & 0.35 & -0.35 \\ 0.35 & -0.35 & 0.35 & -0.35 & -0.35 & 0.35 & -0.35 & 0.35 \\ 0.35 & -0.35 & 0.35 & -0.35 & 0.35 & -0.35 & 0.35 & -0.35 \end{bmatrix}$$



a b c

FIGURE 7.16 The transformation matrix and basis images of the sequency-ordered Walsh-Hadamard transform for $N = 8$. (a) Graphical representation of orthogonal transformation matrix $\mathbf{A}_{W'}$, (b) $\mathbf{A}_{W'}$ rounded to two decimal places, and (c) basis images. For 1-D transforms, matrix $\mathbf{A}_{W'}$ is used in conjunction with Eqs. (6-28) and (6-29); for 2-D transforms, it is used with Eqs. (6-35) and (6-36).

7.8 Slant Transform

The transformation matrix of the slant transform of order $N \times N$ where $N = 2^n$ is generated recursively using

$$\mathbf{A}_{\mathrm{Sl}} = \frac{1}{\sqrt{N}}\mathbf{S}_N \qquad (7-108)$$

$$\mathbf{S}_N = \begin{bmatrix} \begin{matrix} 1 & 0 \\ a_N & b_N \end{matrix} & \mathbf{0} & \begin{matrix} 1 & 0 \\ -a_N & b_N \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{(N/2)-2} & \mathbf{0} & \mathbf{I}_{(N/2)-2} \\ \begin{matrix} 0 & 1 \\ -b_N & a_N \end{matrix} & \mathbf{0} & \begin{matrix} 0 & -1 \\ b_N & a_N \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{(N/2)-2} & \mathbf{0} & -\mathbf{I}_{(N/2)-2} \end{bmatrix} \begin{bmatrix} \mathbf{S}_{N/2} & 0 \\ 0 & \mathbf{S}_{N/2} \end{bmatrix} \qquad (7-109)$$

$$\mathbf{S}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad (7-110)$$

7.8 Slant Transform

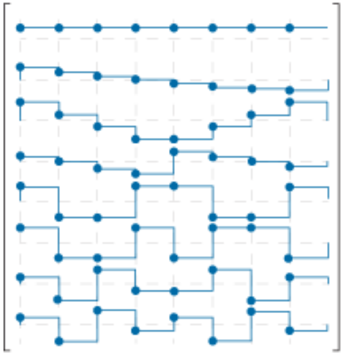
$$a_N = \left[\frac{3N^2}{4(N^2-1)}\right]^{1/2} \qquad (7-111)$$

$$b_N = \left[\frac{N^2-4}{4(N^2-1)}\right]^{1/2} \qquad (7-112)$$

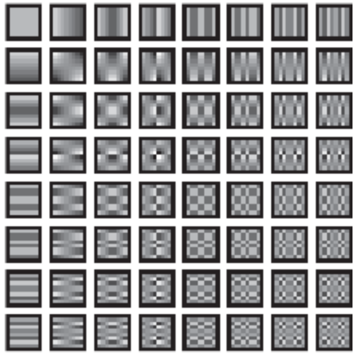
$$\mathbf{A}_{\mathrm{Sl}} = \frac{1}{\sqrt{4}}\mathbf{S}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 & -1 \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} \\ 1 & -1 & -1 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \qquad (7-113)$$

7.8 Slant Transform



$$
\begin{bmatrix}
0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 \\
0.54 & 0.39 & 0.23 & 0.08 & -0.08 & -0.23 & -0.39 & -0.54 \\
0.47 & 0.16 & -0.16 & -0.47 & -0.47 & -0.16 & 0.16 & 0.47 \\
0.24 & -0.04 & -0.31 & -0.59 & 0.59 & 0.31 & 0.04 & -0.24 \\
0.35 & -0.35 & -0.35 & 0.35 & 0.35 & -0.35 & -0.35 & 0.35 \\
0.35 & -0.35 & -0.35 & 0.35 & -0.35 & 0.35 & 0.35 & -0.35 \\
0.16 & -0.47 & 0.47 & -0.16 & -0.16 & 0.47 & -0.47 & 0.16 \\
0.16 & -0.47 & 0.47 & -0.16 & 0.16 & -0.47 & 0.47 & -0.16
\end{bmatrix}
$$



a b c

FIGURE 7.17 The transformation matrix and basis images of the slant transform for $N = 8$. (a) Graphical representation of orthogonal transformation matrix $\mathbf{A}_{SI'}$, (b) $\mathbf{A}_{SI'}$ rounded to two decimal places, and (c) basis images. For 1-D transforms, matrix $\mathbf{A}_{SI'}$ is used in conjunction with Eqs. (6-28) and (6-29); for 2-D transforms, it is used with Eqs. (6-35) and (6-36).

7.9 Haar Transform

The Haar transform is based on Haar functions, $h_u(x)$, that are defined over the continuous half-open interval $x \in [0,1)$

$$u = 2^p + q \qquad (7-114)$$

Where *p* is the largest power of 2 contained in *u* and *q* is the remainder

$$h_u(x) = \begin{cases} 1 & u = 0 \ and \ 0 \le x < 1 \\ 2^{p/2} & u > 0 \ and \ q/2^p \le x < (q + 0.5/2^p) \\ -2^{p/2} & u > 0 \ and \ (q + 0.5/2^p) \le x < (q+1)/2^p \\ 0 & otherwise \end{cases} \qquad (7-115)$$

$$s(x,u) = \frac{1}{\sqrt{N}} h_u(x/N) \quad for \ x = 0, 1, \dots, N-1 \qquad (7-116)$$

7.9 Haar Transform

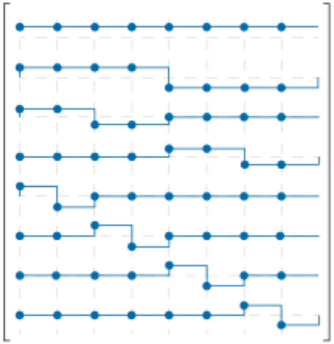
$$\mathbf{H}_N = \begin{bmatrix} h_0(0/N) & h_0(0/N) & \cdots & h_0(N-1/N) \\ h_1(0/N) & h_1(0/N) & & \\ \vdots & & \ddots & \vdots \\ h_{N-1}(0/N) & & \cdots & h_{N-1}(N-1/N) \end{bmatrix} \quad (7-117)$$

$$\mathbf{A}_H = \frac{1}{\sqrt{N}} \mathbf{H}_N \quad (7-118)$$

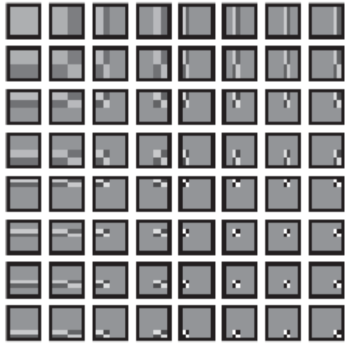
$$\mathbf{A}_H = \frac{1}{\sqrt{2}} \begin{bmatrix} h_0(0) & h_0(1/2) \\ h_1(0) & h_1(1/2) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (7-119)$$

$$\mathbf{A}_\mathrm{H} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \quad (7-120)$$

7.9 Haar Transform



$$\begin{bmatrix} 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 & 0.35 \\ 0.35 & 0.35 & 0.35 & 0.35 & -0.35 & -0.35 & -0.35 & -0.35 \\ 0.50 & 0.50 & -0.50 & -0.50 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.50 & 0.50 & -0.50 & -0.50 \\ 0.71 & -0.71 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.71 & -0.71 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.71 & -0.71 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & -0.71 \end{bmatrix}$$



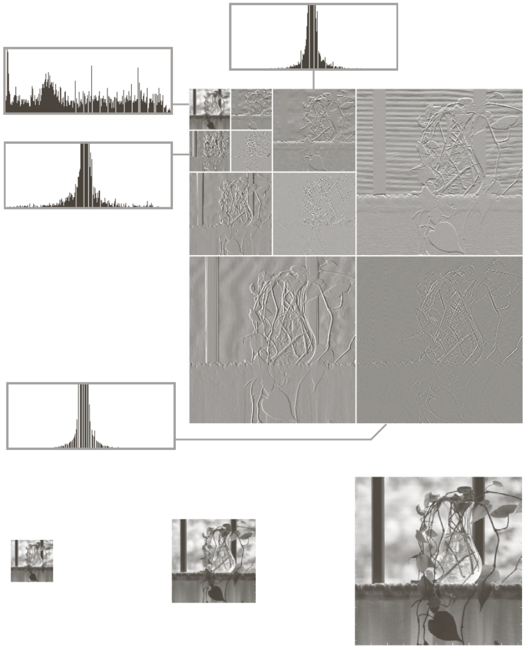
a b c

FIGURE 7.18 The transformation matrix and basis images of the discrete Haar transform for $N = 8$. (a) Graphical representation of orthogonal transformation matrix $\mathbf{A}_H$, (b) $\mathbf{A}_H$ rounded to two decimal places, and (c) basis images. For 1-D transforms, matrix $\mathbf{A}_H$ is used in conjunction with Eqs. (6-28) and (6-29); for 2-D transforms, it is used with Eqs. (6-35) and (6-36).

7.9 Haar Transform

- Example of the Haar Transform

a
b c d



(a) A discrete wavelet transform using Haar $\mathbf{H}_2$ basis functions. Its local histogram variations are also shown. (b)–(d) Several different approximations (64×64, 128×128, and 256×256) that can be obtained from (a).

7.10 Wavelet Transforms

- Scaling Functions

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k) \qquad (7\text{-}121)$$

where k determines the position of $\varphi_{j,k}(x)$ along the x-axis

j determines the width of $\varphi_{j,k}(x)$

$2^{j/2}$ determines the amplitude of the function

$$\varphi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad (7\text{-}122)$$

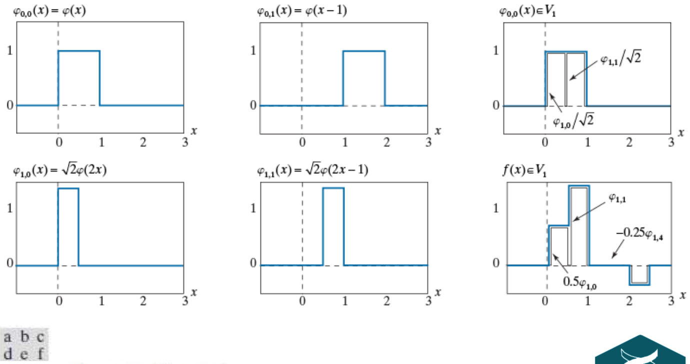
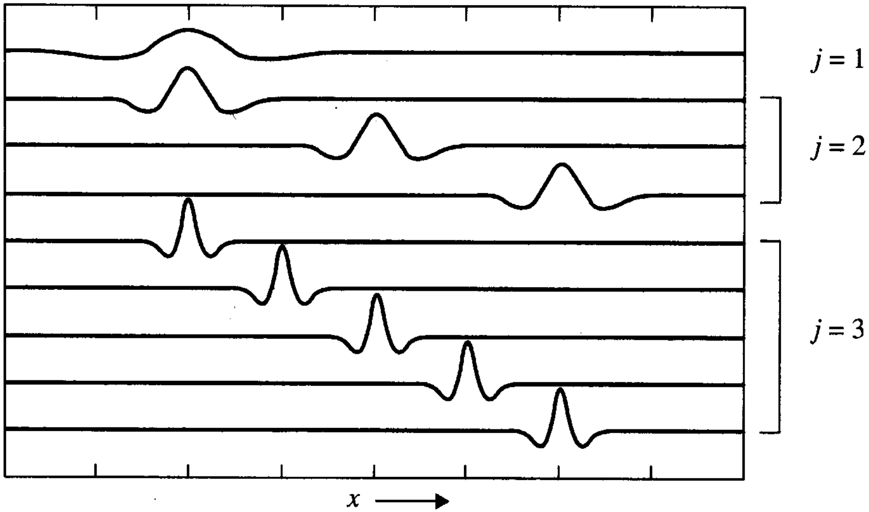


FIGURE 7.19 The Haar scaling function.

7.10 Wavelet Transforms

- Scaling Functions



Scaling and translation of a wavelet

7.10 Wavelet Transforms

Scaling Functions

Four fundamental requirements of multiresolution analysis (MRA)

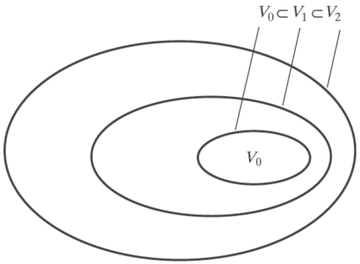
- Requirement 1:

  The scaling function is orthogonal to its integer translates

- Requirement 2:

  The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales

$$V_{-\infty} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{\infty} \qquad \text{(7-123)}$$



The nested function spaces spanned by a scaling function.

7.10 Wavelet Transforms

- Scaling Functions
- Four fundamental requirements of multiresolution analysis (MRA)
  - Requirement 3:

    The only function that is common to all V_j is $f(x) = 0$

    $$V_{-\infty} = \{0\}$$

  - Requirement 4:

    Any function can be represented with arbitrary precision

    $$V_{\infty} = \{L^2(\mathbf{R})\} \qquad (7\text{-}124)$$

    $$\varphi_{j,k}(x) = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

    $$\varphi_{j,k}(x) = \sum_n h_\varphi(n) 2^{(j+1)/2} \varphi(2^{j+1}x - n)$$

    $$\boxed{\varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2x - n) \qquad (7\text{-}125)}$$

7.10 Wavelet Transforms

- Scaling Functions
- Refinement Equation

$$\varphi(x)=\sum_{n} h_{\varphi}(n)\sqrt{2}\varphi(2x-n) \qquad (7\text{-}125)$$

- The recursive equation is called the Refinement Equation, the MRA Equation, or the Dilation Equation
- It states the recursive relationship between the scaling function and the next higher order scaling function
- $h_{\varphi}(n)$ is called Scaling Function Coefficient; h_{φ} is referred to as a Scaling Vector
- It will be used in the fast wavelet transform (FWT)

7.10 Wavelet Transforms

- Wavelet Functions

Definition of Wavelet Function

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \quad (7\text{-}127)$$

$$W_j = \overline{\underset{k}{Span}\{\psi_{j,k}(x)\}}$$

$$f(x) = \sum_k \alpha_k \psi_{j,k}(x)$$

$$V_{j+1} = V_j \oplus W_j \quad (7\text{-}128)$$

$$\langle \varphi_{j,k}(x), \psi_{j,l}(x) \rangle = 0 \quad (7\text{-}129)$$

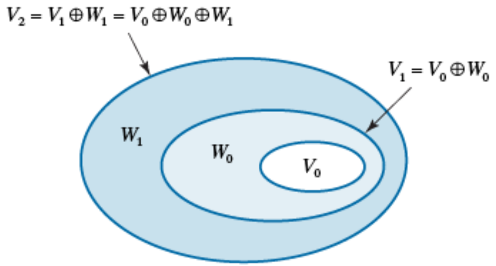


FIGURE 7.20 The relationship between scaling and wavelet function spaces.

7.10 Wavelet Transforms

- Wavelet Functions

Definition of Wavelet Function

$$L^2(\mathbf{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots$$

$$L^2(\mathbf{R}) = V_1 \oplus W_1 \oplus W_2 \oplus \cdots$$

$$L^2(\mathbf{R}) = \cdots \oplus V_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

$$L^2(\mathbf{R}) = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus \cdots$$

Any wavelet function can be expressed as a weighted sum of shifted, double-resolution scaling functions

$$\boxed{\psi(x) = \sum_n h_\psi(n)\sqrt{2}\varphi(2x-n) \qquad (7\text{-}130)}$$

Relationship between wavelet function coefficients and scaling function coefficients

$$\boxed{h_\psi(n) = (-1)^n h_\varphi(1-n) \qquad (7\text{-}131)}$$

7.10 Wavelet Transforms

- Wavelet Functions

Haar Wavelet Functions

The Haar scaling vector

$$h_\varphi(0) = h_\varphi(1) = 1/\sqrt{2}$$

Using Eq. (7-131), Wavelet vector

$$h_\psi(0) = (-1)^0 h_\varphi(1-0) = 1/\sqrt{2}$$

$$h_\psi(1) = (-1)^1 h_\varphi(1-1) = -1/\sqrt{2}$$

From Eq. (7-130),

the Wavelet function

$$\psi(x) = \varphi(2x) - \varphi(2x-1)$$

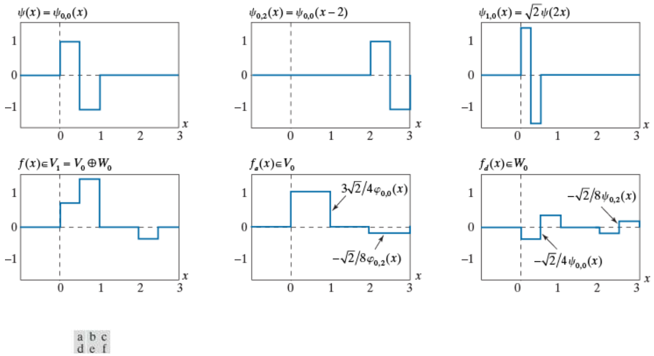


FIGURE 7.21 Haar wavelet functions.

7.10 Wavelet Transforms

- Wavelet Functions

Haar Wavelet Functions

$$\psi(x)=\begin{cases}1 & 0 \le x < 0.5\\ -1 & 0 \le x < 0.5\\ 0 & \text{elsewhere}\end{cases} \quad \text{(7-132)}$$

$$f(x)=f_a(x)+f_d(x)$$

$$f_a(x)=\frac{3\sqrt{2}}{4}\varphi_{0,0}(x)-\frac{\sqrt{2}}{8}\varphi_{0,2}(x)$$

$$f_d(x)=\frac{-\sqrt{2}}{4}\psi_{0,0}(x)-\frac{\sqrt{2}}{8}\psi_{0,2}(x)$$

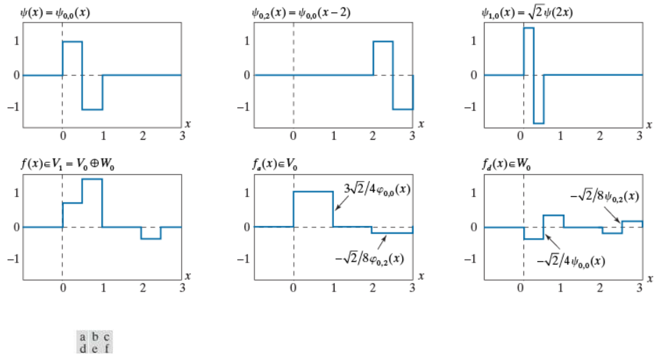


FIGURE 7.21 Haar wavelet functions.

7.10 Wavelet Transforms

- Wavelet Series Expansion

$$f(x)=\sum_k c_{j_0}(k)\varphi_{j_0,k}(x)+\sum_{j=j_0}^{\infty}\sum_k d_j(k)\psi_{j,k}(x) \qquad (7\text{-}133)$$

$c_{j_0}(k)$ are Approximate Coefficients

$$c_{j_0}(k)=\left\langle f(x),\quad \varphi_{j_0,k}(x)\right\rangle=\int f(x)\varphi_{j_0,k}(x)dx \qquad (7\text{-}134)$$

$d_j(k)$ are Wavelet Coefficients

$$d_j(k)=\left\langle f(x),\quad \psi_{j,k}(x)\right\rangle=\int f(x)\psi_{j,k}(x)dx \qquad (7\text{-}135)$$

7.10 Wavelet Transforms

- Wavelet Series Expansion
- Example

$$y = \begin{cases} x^2 & 0 \le x < 1 \\ 0 & otherwise \end{cases}$$

$$c_0(0) = \int_0^1 x^2 \varphi_{0,0}(x)dx = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$d_0(0) = \int_0^1 x^2 \psi_{0,0}(x)dx = \int_0^{0.5} x^2 dx - \int_{0.5}^1 x^2 dx = -\frac{1}{4}$$

$$d_1(0) = \int_0^1 x^2 \psi_{1,0}(x)dx = \int_0^{0.25} x^2 \sqrt{2}dx - \int_{0.25}^{0.5} x^2 \sqrt{2}dx = -\frac{\sqrt{2}}{32}$$

$$d_1(1) = \int_0^1 x^2 \psi_{1,1}(x)dx = \int_{0.5}^{0.75} x^2 \sqrt{2}dx - \int_{0.75}^1 x^2 \sqrt{2}dx = -\frac{3\sqrt{2}}{32}$$

$$y = \underbrace{\underbrace{\underbrace{\frac{1}{3}\varphi_{0,0}(x)}_{V_0} + \underbrace{\left[-\frac{1}{4}\psi_{0,0}(x)\right]}_{W_0}}_{V_1 = V_0 \oplus W_0} + \underbrace{\left[-\frac{\sqrt{2}}{32}\psi_{1,0}(x) - \frac{3\sqrt{2}}{32}\psi_{1,1}(x)\right]}_{W_1}}_{V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1} + \cdots$$

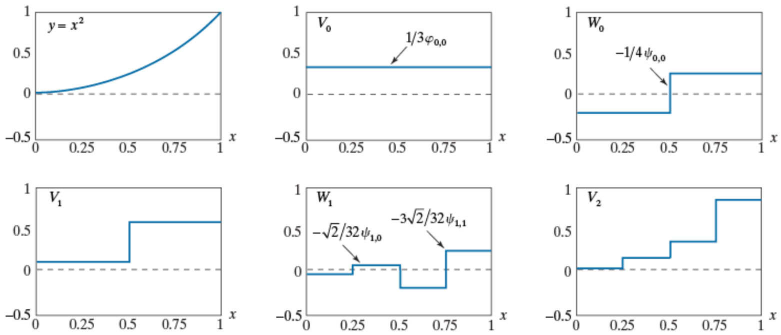
7.10 Wavelet Transforms

- Wavelet Series Expansion
- Example

$$V_1 = V_0 \oplus W_0$$

$$V_2 = V_1 \oplus W_1$$

$$= V_0 \oplus W_0 \oplus W_1$$



a b c
d e f

FIGURE 7.22 A wavelet series expansion of $y = x^2$ using Haar wavelets.

7.10 Wavelet Transforms

- The Discrete Wavelet Transform

Approximate Coefficients

$$W_{\varphi}(j_0,k)=\frac{1}{\sqrt{M}}\sum_{x} f(x)\varphi_{j_0,k}(x) \qquad (7\text{-}137)$$

Wavelet Coefficients

$$W_{\psi}(j,k)=\frac{1}{\sqrt{M}}\sum_{x} f(x)\psi_{j,k}(x) \quad \text{for } j \geq j_0 \qquad (7\text{-}138)$$

Discrete Wavelet Transform

$$f(x)=\frac{1}{\sqrt{M}}\sum_{k} W_{\varphi}(j_0,k)\varphi_{j_0,k}(x)+\frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_{k} W_{\psi}(j,k)\psi_{j,k}(x) \qquad (7\text{-}136)$$

7.10 Wavelet Transforms

- The Discrete Wavelet Transform
- Example

$$f(0)=1,\ f(1)=4,\ f(2)=\text{-}3,\ f(3)=0,$$
$$M=4,\ J=2,$$
$$W_\varphi(0,0)=\frac{1}{2}\sum_{x=0}^{3} f(x)\varphi_{0,0}(x)=\frac{1}{2}[1\cdot 1+4\cdot 1-3\cdot 1+0\cdot 1]=1$$
$$W_\psi(0,0)=\frac{1}{2}[1\cdot 1+4\cdot 1-3\cdot(-1)+0\cdot(-1)]=4$$
$$W_\psi(1,0)=\frac{1}{2}[1\cdot\sqrt{2}+4\cdot(-\sqrt{2})-3\cdot 0+0\cdot 0]=-1.5\sqrt{2}$$
$$W_\psi(1,1)=\frac{1}{2}[1\cdot 0+4\cdot 0-3\cdot\sqrt{2}+0\cdot(-\sqrt{2})]=-1.5\sqrt{2}$$
$$f(x)=\frac{1}{2}[W_\varphi(0,0)\varphi_{0,0}(x)+W_\psi(0,0)\psi_{0,0}(x)$$
$$+W_\psi(1,0)\psi_{1,0}(x)+W_\psi(1,1)\psi_{1,1}(x)]$$
$$f(0)=\frac{1}{2}[1\cdot 1+4\cdot 1-1.5\sqrt{2}\cdot(\sqrt{2})-1.5\sqrt{2}\cdot 0]=1$$

7.10 Wavelet Transforms

The Fast Wavelet Transform

From Eq. (7-125) and (7-130), we can write the coefficients of the DWT

$$W_\psi(j,k) = \sum_m h_\psi(m-2k) W_\varphi(j+1,m) \quad (7\text{-}142)$$

Similarly,

$$W_\varphi(j,k) = \sum_m h_\varphi(m-2k) W_\varphi(j+1,m) \quad (7\text{-}141)$$

Expressed by convolution operation

$$W_\psi(j,k) = h_\psi(-n) \star W_\varphi(j+1,n)\Big|_{n=2k,k\geq 0} \quad (7\text{-}144)$$

$$W_\varphi(j,k) = h_\varphi(-n) \star W_\varphi(j+1,n)\Big|_{n=2k,k\geq 0} \quad (7\text{-}143)$$

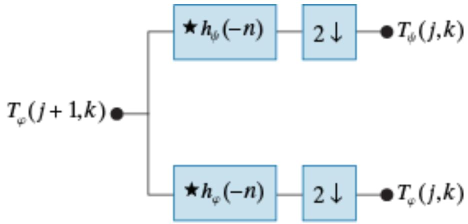
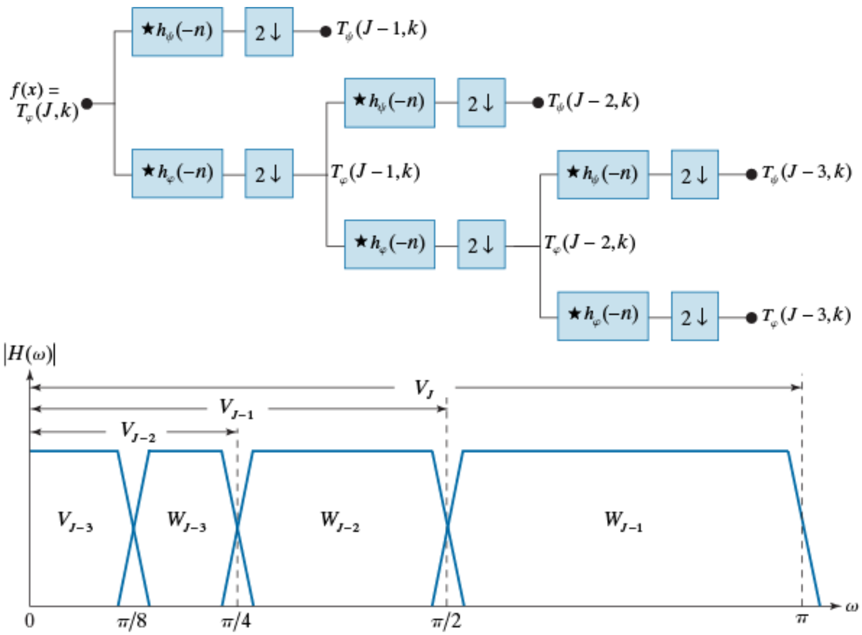


FIGURE 7.23 A FWT analysis filter bank for orthonormal filters. The $\star$ and $2\downarrow$ denote convolution and downsampling by 2, respectively.

7.10 Wavelet Transforms

- The Fast Wavelet Transform



a
b

FIGURE 7.24
(a) A three-stage or three-scale FWT analysis filter bank and (b) its frequency-splitting characteristics. Because of symmetry in the DFT of the filter's impulse response, it is common to display only the $[0, \pi]$ region.

7.10 Wavelet Transforms

- The Fast Wavelet Transform

$$h_{\varphi}(n)=\begin{cases}1/\sqrt{2} & n=0,1\\ 0 & \text{otherwise}\end{cases} \qquad (7\text{-}146)$$

$$h_{\psi}(n)=\begin{cases}1/\sqrt{2} & n=0\\ -1/\sqrt{2} & n=1\\ 0 & \text{otherwise}\end{cases} \qquad (7\text{-}147)$$

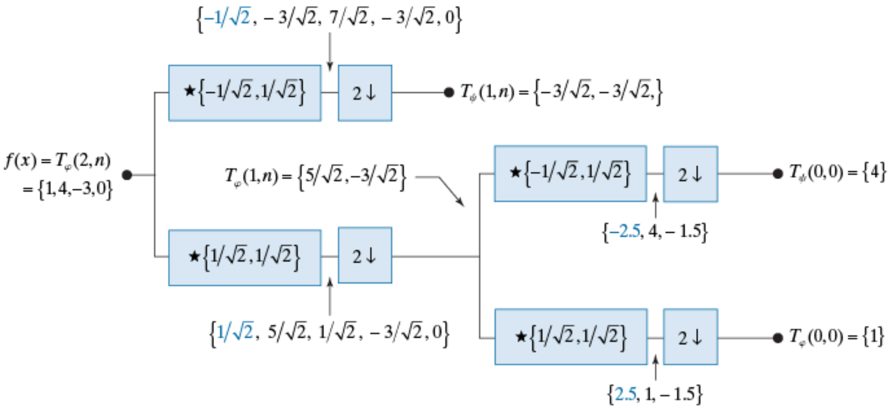


FIGURE 7.25 Computing a two-scale fast wavelet transform of sequence $\{1, 4, -3, 0\}$ using Haar scaling and wavelet coefficients.

7.10 Wavelet Transforms

- The Fast Wavelet Transform
- Sign Reversal, Order Reversal, and Modulation

Sign Reversal

$$h_2(n) = -h_1(n)$$

Order Reversal

$$h_3(n) = h_1(-n)$$

$$h_4(n) = h_1(K-1-n)$$

Modulation

$$h_5(n) = (-1)^n h_1(n)$$

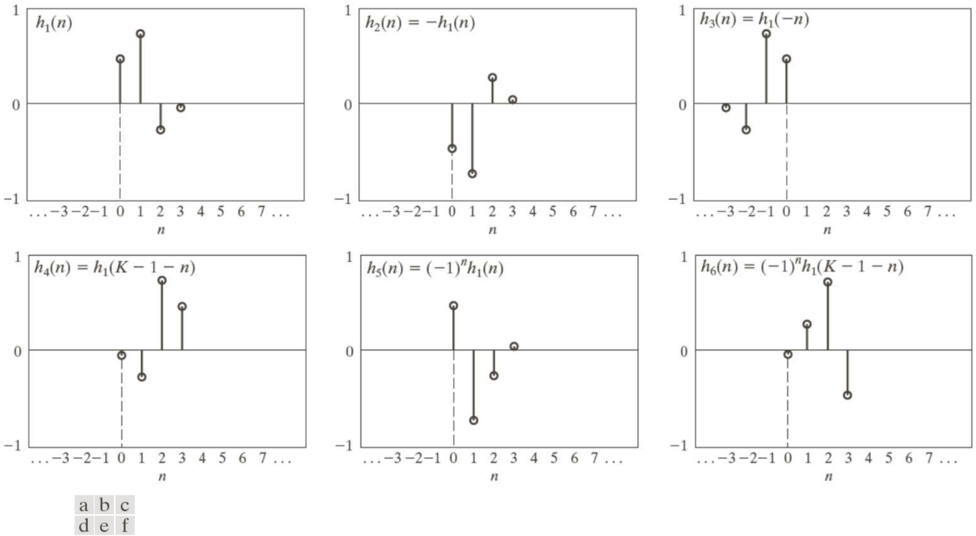
Modulation and Order Reversal

$$h_6(n) = (-1)^n h_1(K-1-n)$$

7.10 Wavelet Transforms

- The Fast Wavelet Transform

- Sign Reversal, Order Reversal, and Modulation



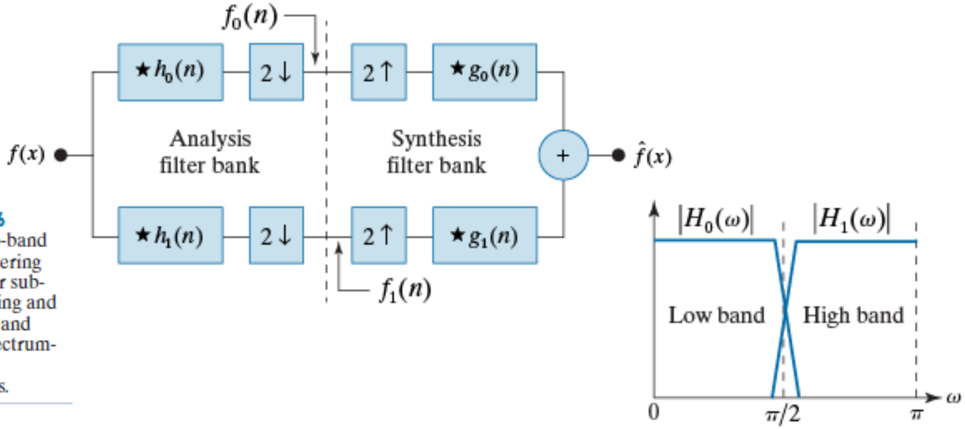
Six functionally related filter impulse responses: (a) reference response; (b) sign reversal; (c) and (d) order reversal (differing by the delay introduced); (e) modulation; and (f) order reversal and modulation.

7.10 Wavelet Transforms

- The Fast Wavelet Transform

Subband Coding

- In subband coding, an image is decomposed into a set of bandlimited components. The subbands can be reassembled to reconstruct the original image without error.
- Analysis Filter and Synthesis Filter



a b

FIGURE 6.26 (a) A two-band digital filtering system for sub-band coding and decoding and (b) its spectrum-splitting properties.

7.10 Wavelet Transforms

- The Fast Wavelet Transform
- Condition for perfect reconstruction

$$g_0(n) = (-1)^n h_1(n)$$
$$g_1(n) = (-1)^{n+1} h_0(n)$$
or
$$g_0(n) = (-1)^{n+1} h_1(n)$$
$$g_1(n) = (-1)^n h_0(n)$$

Analysis Filter and Synthesis Filter are Crossed Modulated

7.10 Wavelet Transforms

- Inverse fast wavelet transform (FWT^{-1})
  - Perfect reconstruction for two-band orthonormal filters requires $g_i(n)=h_i(-n)$ for i={0,1}. That is, the synthesis and analysis filters must be time-reversed versions of one another.

$$W_\varphi(j+1,k) = h_\varphi(k) \star W_\varphi^{2\uparrow}(j,k) + h_\psi(k) \star W_\psi^{2\uparrow}(j,k)\Big|_{k\geq 0}$$

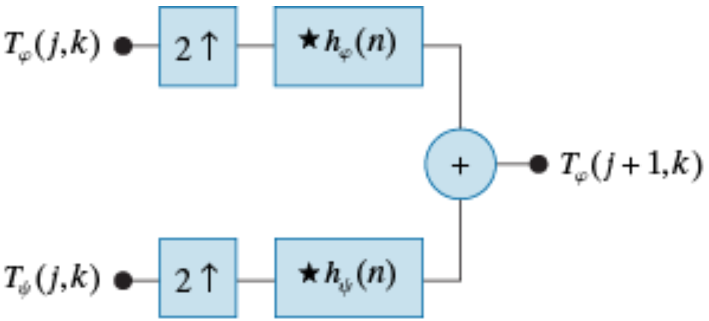
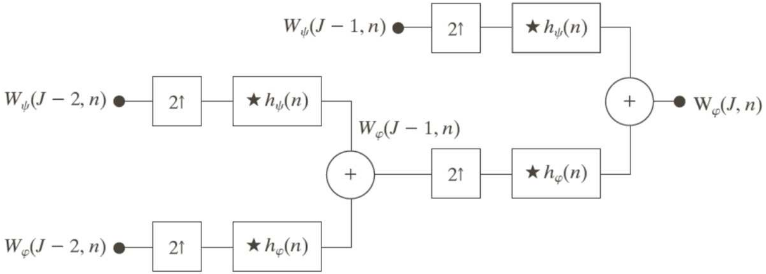


FIGURE 7.27 An inverse FWT synthesis filter bank for orthonormal filters.

7.10 Wavelet Transforms

- Inverse fast wavelet transform (FWT^{-1})

A two-stage or two-scale FWT^{-1} synthesis bank.



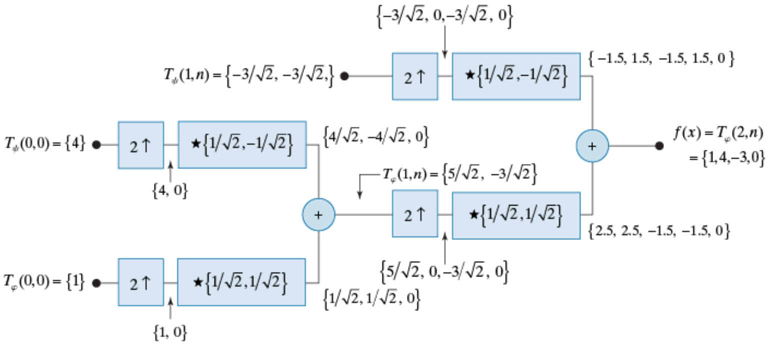
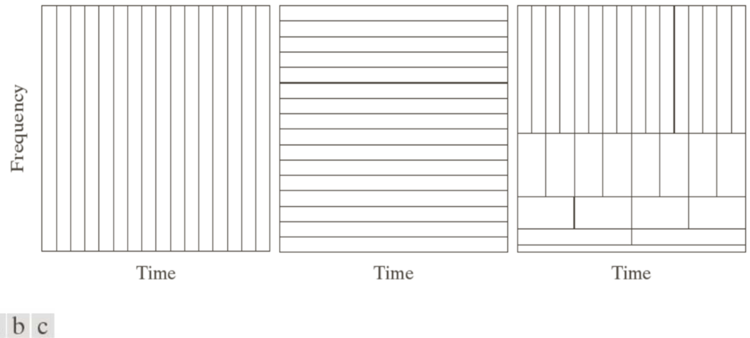


FIGURE 7.28 Computing a two-scale inverse fast wavelet transform of sequence $\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}$ with Haar scaling and wavelet functions.

7.10 Wavelet Transforms

- Comparison between FWT and FFT

1. Computational complexity: O(M) vs. O(M log M)
2. Requirement of basis: FWT > FFT
3. Comparison between spatial and frequency resolution



Time-frequency tilings for the basis functions associated with (a) sampled data, (b) the FFT, and (c) the FWT. Note that the horizontal strips of equal height rectangles in (c) represent FWT scales.

7.10 Wavelet Transforms

- Wavelet Transforms in 2-D

A 2-D wavelet transform requires a 2-D scaling function and three 2-D wavelets.

$$\varphi(x,y) = \varphi(x)\varphi(y) \tag{7-152}$$

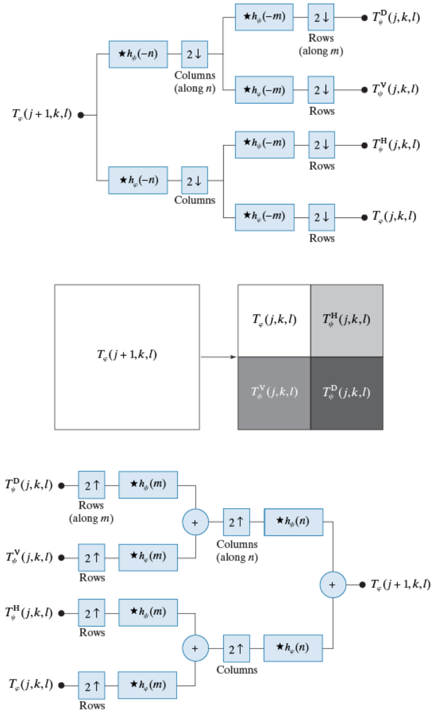
$$\psi^{H}(x,y) = \psi(x)\varphi(y) \tag{7-153}$$

$$\psi^{V}(x,y) = \varphi(x)\psi(y) \tag{7-154}$$

$$\psi^{D}(x,y) = \psi(x)\psi(y) \tag{7-155}$$

7.10 Wavelet Transforms

- Wavelet Transforms in 2-D



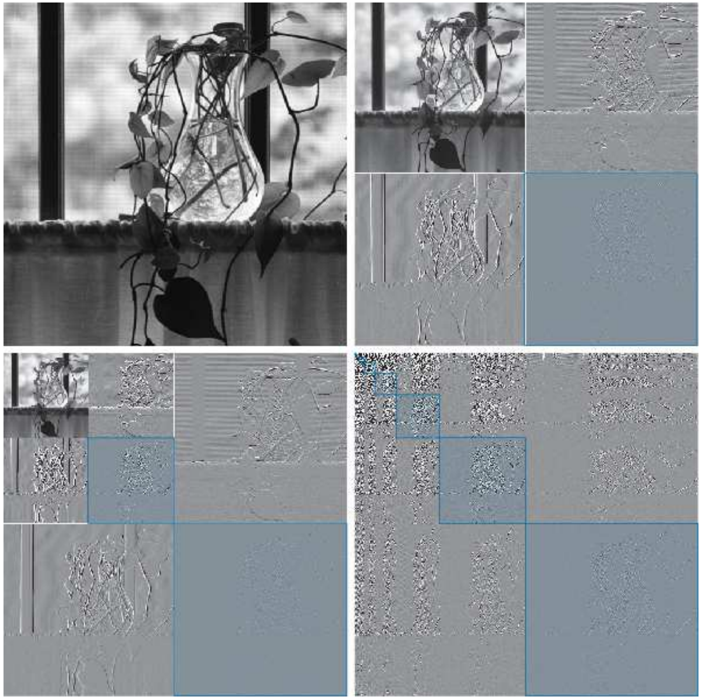
a
b
c

FIGURE 7.29
The 2-D fast wavelet transform: (a) the analysis filter bank; (b) the resulting decomposition; and (c) the synthesis filter bank.

Note m and n are dummy variables of convolution, while j, like in the 1-D case, is scale, and k and l are translations.

7.10 Wavelet Transforms

- Wavelet Transforms in 2-D



a b
c d

FIGURE 7.30
(a) A 512×512 image of a vase; (b) a one-scale FWT; (c) a two-scale FWT; and (d) the Haar transform of the original image. All transforms have been scaled to highlight their underlying structure. When corresponding areas of two transforms are shaded in blue, the correspondent pixels are identical.

7.10 Wavelet Transforms

- Wavelet Transforms in 2-D

a b

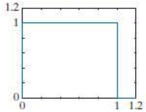
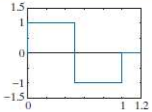
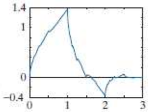
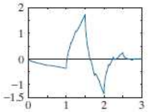
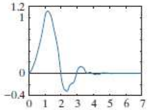
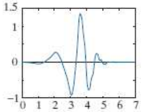
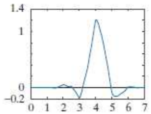
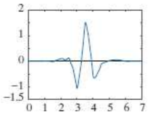
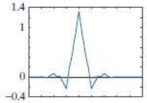
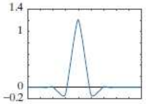
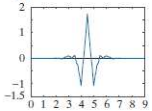
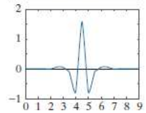
FIGURE 7.31

(a) Haar basis images of size 8×8 [from Fig. 6.18(c)] and (b) the basis images of a three-scale 8×8 discrete wavelet transform with respect to Haar basis functions.

7.10 Wavelet Transforms

Table 7.1

Some representative wavelets.

Wavelet Name or Family	Scaling Function	Wavelet Function	Filter Coefficients
Haar The oldest and simplest wavelets. Orthogonal and discontinuous.			$g_0(n) = \{1/\sqrt{2},\ 1/\sqrt{2}\}$
Daubechies family Orthogonal with the most vanishing moments for a given support. Denoted db*N*, where *N* is the number of vanishing moments; db2 and db4 shown; db1 is the Haar of the previous row.			$g_0(n) = \{0.482963,\ 0.836516, 0.224144,\ -0.129410\}$
			$g_0(n) = \{0.230372,\ 0.714847, 0.630881,\ -0.027984,\ -0.187035,\ 0.030841, 0.032883,\ -0.010597\}$
Symlet family Orthogonal with the least asymmetry and most vanishing moments for a given support (sym4 or 4th order shown).			$g_0(n) = \{0.032231,\ -0.012604,\ -0.099220,\ 0.297858, 0.803739,\ 0.497619,\ -0.029636,\ -0.075766\}$
Cohen-Daubechies-Feauveau 9/7 Biorthogonal B-spline used in the irreversable JPEG2000 compression standard (see Chapter 8).			$h_0(n) = \{0.026749,\ -0.016864,\ -0.078223,\ 0.266864, 0.602949,\ 0.266864,\ -0.078223,\ -0.016864, 0.026749\}$
			$h_1(n) = \{-0.091271,\ -0.057544, 0.591272,\ -1.115087, 0.591272,\ 0.057544,\ -0.091271, 0\}$

7.10 Wavelet Transforms

- Cohen-Daubechies-Feauveau wavelet

TABLE 14–2 DISCRETE FILTER SEQUENCES FOR THE BIORTHOGONAL WAVELETS IN FIGURE 14–33 (FROM[21] AND [22]).

Laplacian
analysis filter: $h_0 = \sqrt{2}[-.05 \quad .25 \quad .6 \quad .25 \quad -.05]^t$

Laplacian
synthesis filter: $h_0 = \sqrt{2}[-.0107 \quad -.0536 \quad .2607 \quad .6071 \quad .2607 \quad -.0536 \quad -.0107]^t$

Spline 2 filter: $h_0 = \sqrt{2}[.25 \quad .5 \quad .25]^t$

Spline 4 filter: $\tilde{h}_0 = \frac{\sqrt{2}}{128}[3 \quad -6 \quad -16 \quad 38 \quad 90 \quad 38 \quad -16 \quad -6 \quad 3]^t$

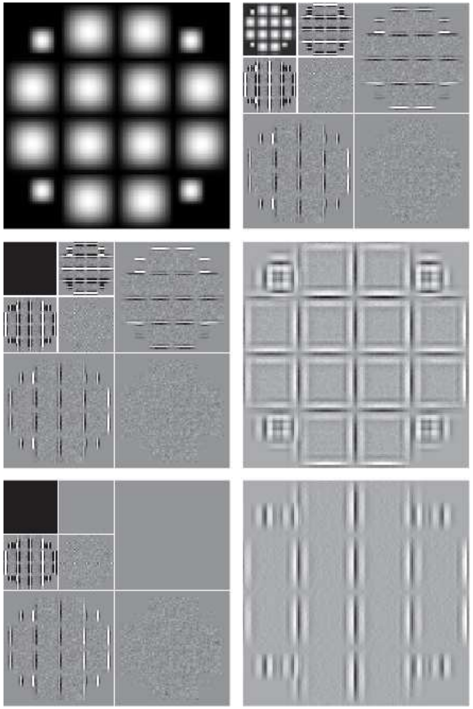
18-point
analysis filter: $h_0 = [.0012 \quad -.0007 \quad -.0118 \quad .0117 \quad .0713 \quad -.0310 \quad -.2263 \quad .0693 \quad .7318$
$.7318 \quad .0693 \quad -.2263 \quad -.0310 \quad .0713 \quad .0117 \quad -.0118 \quad -.0007 \quad .0012]^t$

18-point
synthesis filter: $h_0 = [.0012 \quad .0007 \quad -.0113 \quad -.0114 \quad .0235 \quad .0017 \quad -.0444 \quad .2044 \quad .6479$
$.6479 \quad .2044 \quad -.0444 \quad .0017 \quad .0235 \quad -.0114 \quad -.0113 \quad .0007 \quad .0012]^t$

Source: Digital Image Processing, by K. R. Cattleman

7.10 Wavelet Transforms

- Application of 2-D DWT – Edge Detection

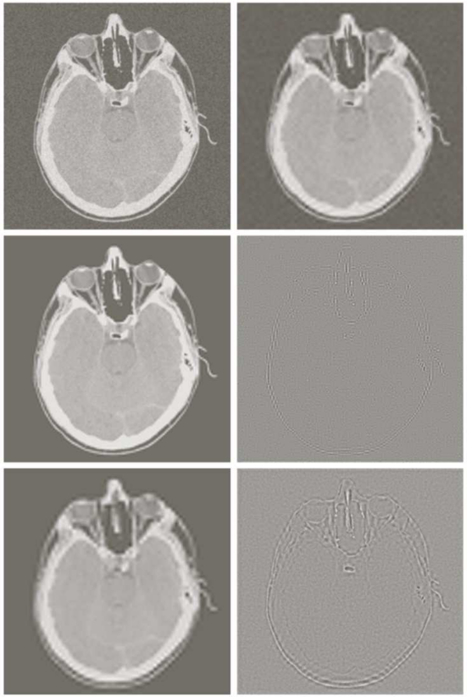


a b
c d
e f

FIGURE 7.32 Modifying a DWT for edge detection: (a) orginal image; (b) two-scale DWT with respect to 4th-order symlets; (c) modified DWT with the approximation set to zero; (d) the inverse DWT of (c); (e) modified DWT with the approximation and horizontal details set to zero; and (f) the inverse DWT of (e). (Note when the detail coefficients are zero, they are displayed as middle gray; when the approximation coefficients are zeroed, they display as black.)

7.10 Wavelet Transforms

- Application of 2-D DWT – Noise Removal



a	b
c	d
e	f

Modifying a DWT for noise removal: (a) a noisy CT of a human head; (b), (c) and (e) various reconstructions after thresholding the detail coefficients; (d) and (f) the information removed during the reconstruction of (c) and (e). (Original image courtesy Vanderbilt University Medical Center.)

7.10 Wavelet Transforms

- Application of 2-D DWT – Image Fusion

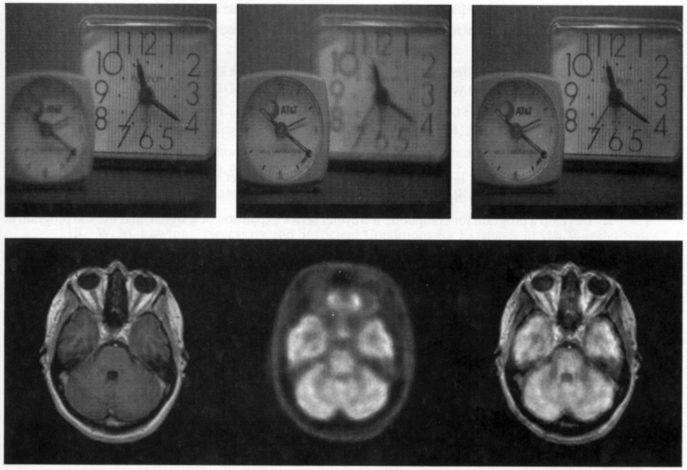
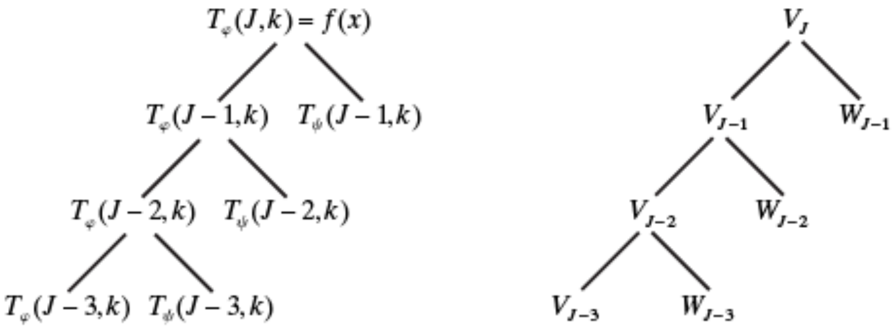


Figure 14–36 Wavelet transform image fusion: (a), (b) images taken at different focus settings; (c) fused image; (d) MRI image; (e) PET image; (f) fused image (Courtesy Henry Hui Li, reprinted by permission from [28])

After performing DWT on the two images, compute the inverse wavelet transform on the image which wavelet coefficients are bigger.

7.10 Wavelet Transforms

- Wavelet Packets

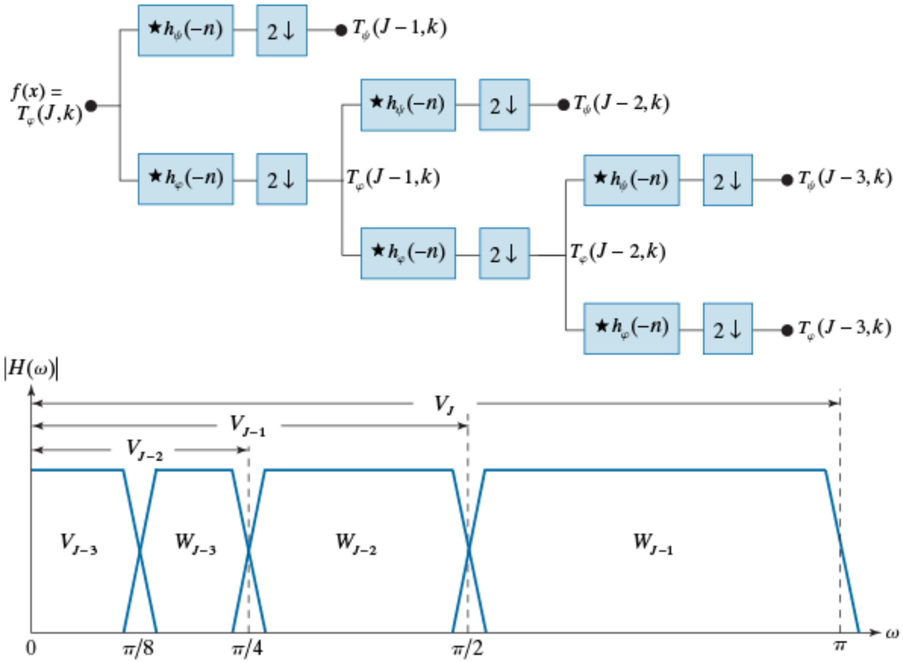


a b

FIGURE 7.33 An (a) coefficient tree and (b) analysis tree for the two-scale FWT analysis bank of Fig. 6.24.

7.10 Wavelet Transforms

- Wavelet Packets



a
b

(a) A three-stage or three-scale FWT analysis filter bank and (b) its frequency-splitting characteristics. Because of symmetry in the DFT of the filter's impulse response, it is common to display only the $[0, \pi]$ region.

7.10 Wavelet Transforms

- Wavelet Packets

$$V_j = V_{j-1} \oplus W_{j-1} \qquad (7\text{-}156)$$

$$V_j = V_{j-2} \oplus W_{j-2} \oplus W_{j-1} \qquad (7\text{-}157)$$

$$V_j = V_{j-3} \oplus W_{j-3} \oplus W_{j-2} \oplus W_{j-1} \qquad (7\text{-}158)$$

A: Approximate Filtering
D: Detail Filtering

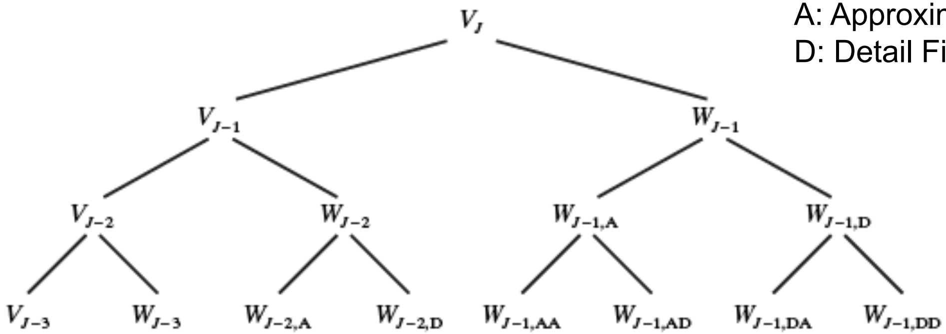
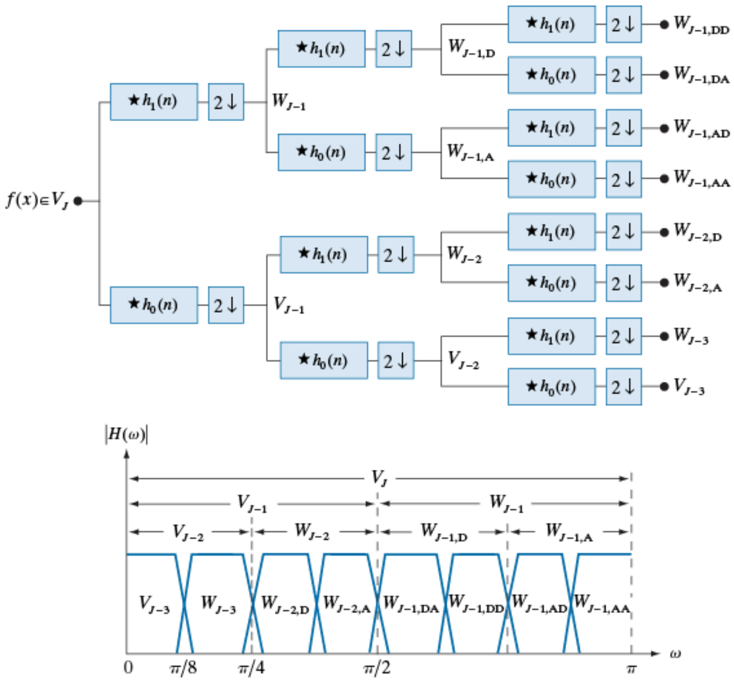


FIGURE 6.34 A three-scale wavelet packet analysis tree.

7.10 Wavelet Transforms

- Wavelet Packets

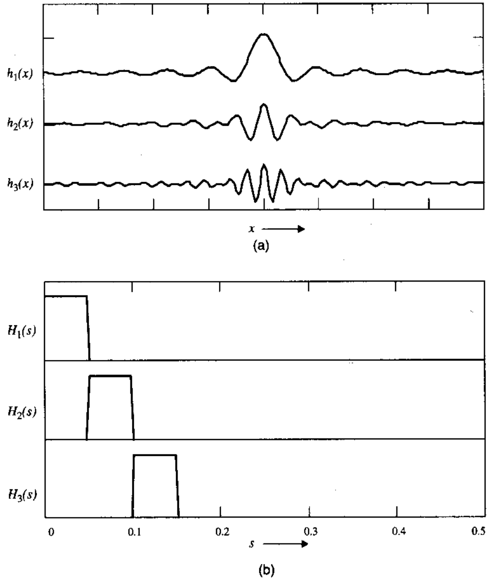


a
b

FIGURE 7.35 The (a) filter bank and (b) spectrum-splitting characteristics of a three-scale full wavelet packet analysis tree.

7.10 Wavelet Transforms

- Wavelet Packets

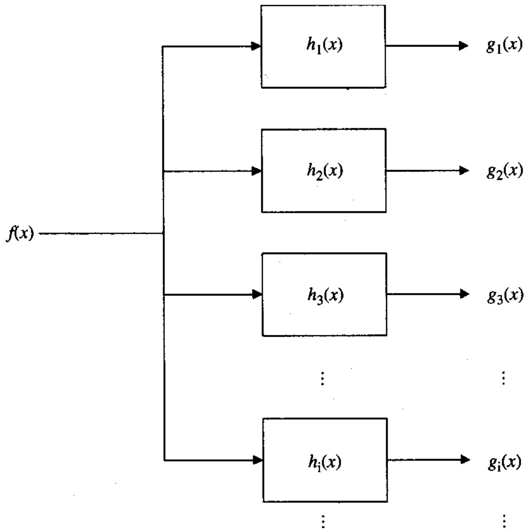


Source: Digital Image Processing, by K. R. Cattleman

Figure 14–11 Generating a series of bandpass filters by partitioning the frequency axis: (a) impulse responses; (b) transfer functions

7.10 Wavelet Transforms

- Wavelet Packets

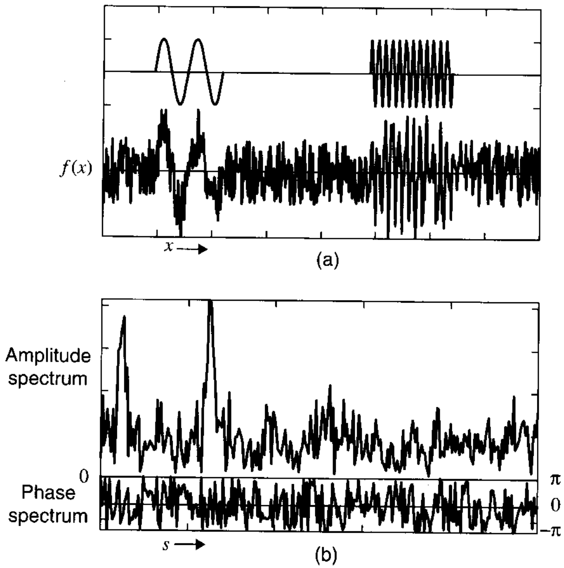


Source: Digital Image Processing, by K. R. Cattleman

Figure 14–12 Implementation of a bandpass filter bank

7.10 Wavelet Transforms

- Wavelet Packets



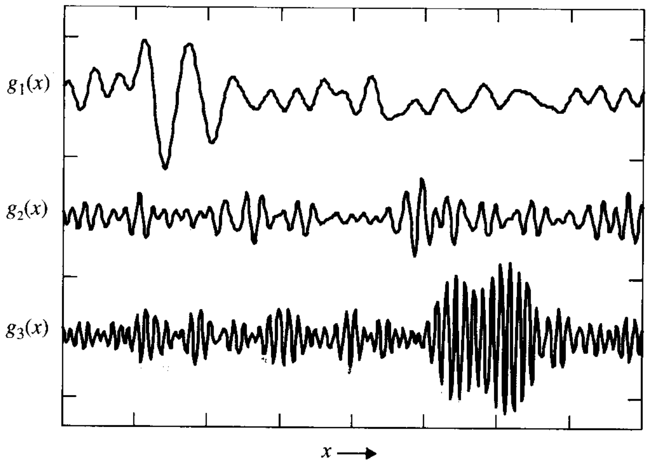
Source: Digital Image Processing, by K. R. Cattleman

Figure 14–10 Composite signal containing two tone bursts and random noise: (a) the three components; (b) amplitude and phase spectra

7.10 Wavelet Transforms

- Wavelet Packets

Concepts of Filter Bank and Bandpass Filter



Source: Digital Image Processing, by K. R. Cattleman

Figure 14–13 Bandpass filter outputs

7.10 Wavelet Transforms

Wavelet Packets Decomposition

Wavelet packets decomposition can be several ways. Number of unique decompositions of P-scale, 1-D wavelet packet transform is

$$D(P+1) = [D(P)]^2 + 1 \qquad (7\text{-}161)$$

When P=3, it supports 26 different decompositions. For instance

$$V_J = V_{J-3} \oplus W_{J-3} \oplus W_{J-2,A} \oplus W_{J-2,D} \oplus W_{J-1,AA} \oplus W_{J-1,AD} \oplus W_{J-1,DA} \oplus W_{J-1,DD} \qquad (7\text{-}159)$$

$$V_J = V_{J-1} \oplus W_{J-1,D} \oplus W_{J-1,AA} \oplus W_{J-1,AD} \qquad (7\text{-}160)$$

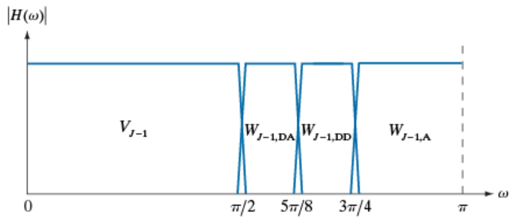


FIGURE 7.36 The spectrum of the decomposition in Eq. (6-160).

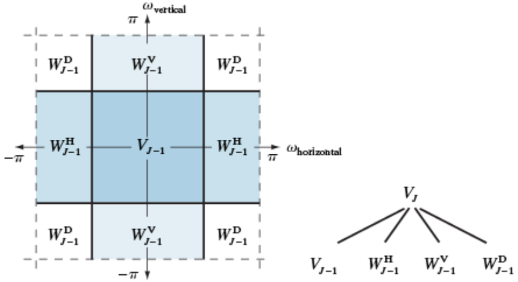
Wavelet packets transform (decomposition) provides a more flexible spectrum analysis, but it also increases the computational complexity.

7.10 Wavelet Transforms

- 2-D Wavelet Packets Transform

a b

FIGURE 7.37 The first decomposition of a 2-D FWT: (a) the spectrum and (b) the subspace analysis tree.



Number of unique decompositions of P-scale, 1-D wavelet packet transform is

$$D(P+1)=[D(P)]^4+1 \qquad (7\text{-}162)$$

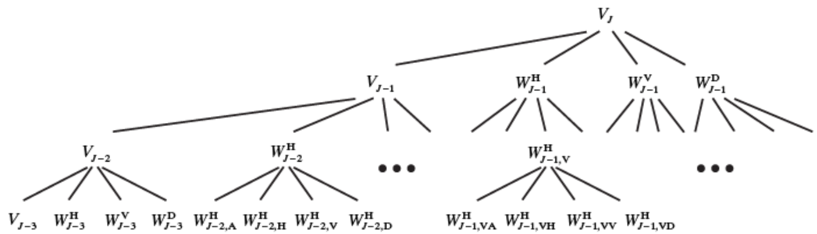
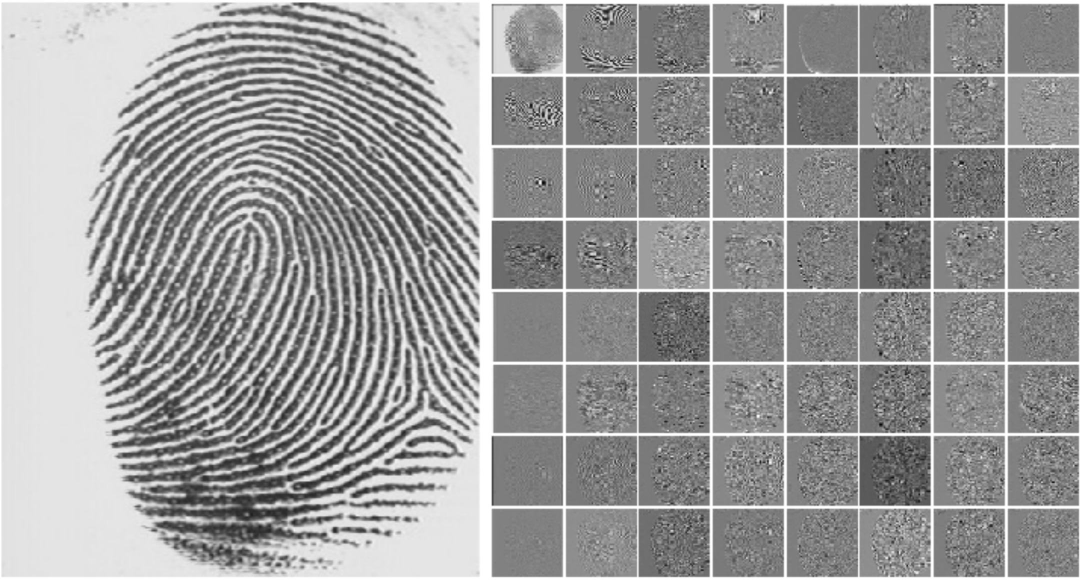


FIGURE 7.38 A three-scale, full wavelet packet decomposition tree. Only a portion of the tree is provided.

7.10 Wavelet Transforms

- 2-D Wavelet Packets Transform



a b

FIGURE 7.39 (a) A scanned fingerprint and (b) its three-scale, full wavelet packet decomposition. Although the 64 subimages of the packet decomposition appear to be square (e.g., note the approximation subimage), this is merely an aberration of the program used to produce the result. (Original image courtesy of the National Institute of Standards and Technology.)

7.10 Wavelet Transforms

- Optimize Decomposition for Image Compression
  - Optimization Criterion: Additive Cost Function

$$E(f) = \sum_{m,n} |f(m,n)| \tag{7-163}$$

  - Entropy-based cost functions are applicable
  - Optimization Algorithm

    1. Compute the entropy of the node E_P
    2. Compute the entropy of its four offspring E_A , E_H , E_V , E_D
    3. If $E_A + E_H + E_V + E_D < E_P$, include the offspring in the tree. If not, prune the offspring.
    4. For each node of the analysis tree, beginning with the root and proceeding level by level to the leaves

7.10 Wavelet Transforms

- Optimize Decomposition for Image Compression

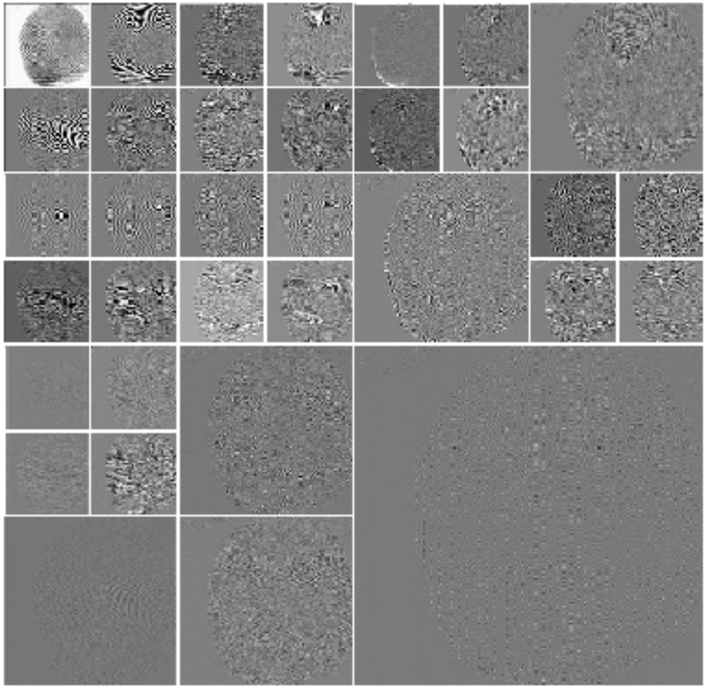
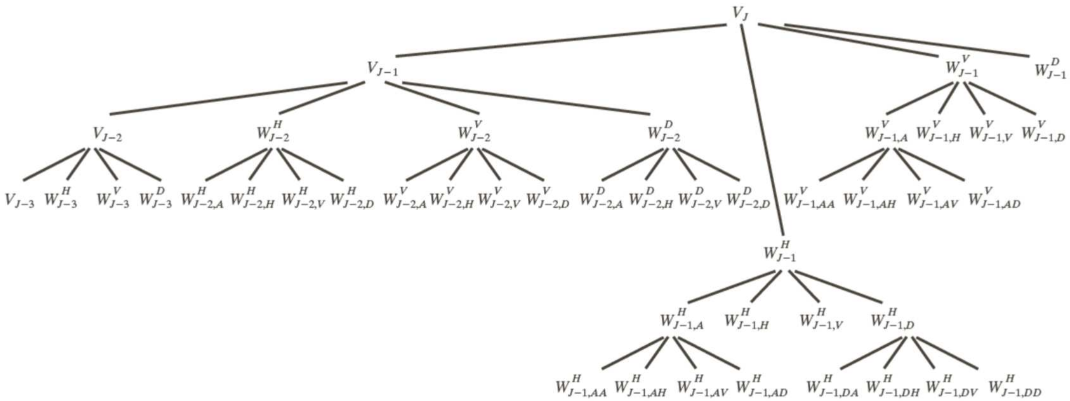


FIGURE 7.40
An optimal wavelet packet decomposition for the fingerprint of Fig. 6.39(a).

7.10 Wavelet Transforms

- Optimize Decomposition for Image Compression



The optimal wavelet packet analysis tree for the decomposition in Fig. 7.37.

7.10 Wavelet Transforms

- Cohen-Daubechies-Feauveau wavelet

Table 7.2
Biorthogonal Cohen-Daubechies-Feauveau reconstruction and decomposition filter coefficients with 6 and 8 vanishing moments, respectively. (Cohen, Daubechies, and Feauveau [1992]).

n	$h_0(n)$	$h_1(n)$	n	$h_0(n)$	$h_1(n)$
0	0	0	9	0.825923	0.417849
1	0.001909	0	10	0.420796	0.040368
2	−0.001914	0	11	−0.094059	−0.078722
3	−0.016991	0.014427	12	−0.077263	−0.014468
4	0.011935	−0.014468	13	0.049733	0.0144263
5	0.049733	−0.078722	14	0.011935	0
6	−0.077263	0.040368	15	−0.016991	0
7	−0.094059	0.417849	16	−0.0019	0
8	0.420796	−0.758908	17	0.0019	0