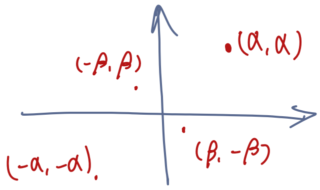
Problem 2



(a) $\mathrm{Cov}(X) = \frac{1}{N-1} \sum (X_i - \overline{X})(X_i - \overline{X})^T$

$= \frac{1}{3}\left[\begin{bmatrix}\alpha\\ \alpha\end{bmatrix}[\alpha, \alpha] + \begin{bmatrix}\beta\\ -\beta\end{bmatrix}[\beta, -\beta] + \begin{bmatrix}-\beta\\ \beta\end{bmatrix}[-\beta, \beta] + \begin{bmatrix}-\alpha\\ -\alpha\end{bmatrix}[-\alpha, -\alpha] \right]$

$= \frac{1}{3}\begin{bmatrix} \alpha^2+\beta^2+\beta^2+\alpha^2 , & \alpha^2-\beta^2-\beta^2+\alpha^2 \\ \alpha^2-\beta^2-\beta^2+\alpha^2 , & \alpha^2+\beta^2+\beta^2+\alpha^2 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 2(\alpha^2+\beta^2) , & 2(\alpha^2-\beta^2) \\ 2(\alpha^2-\beta^2) , & 2(\alpha^2+\beta^2) \end{bmatrix}$ #

(b) $\frac{1}{3}\begin{bmatrix} 2(\alpha^2+\beta^2)-\lambda , & 2(\alpha^2-\beta^2) \\ 2(\alpha^2-\beta^2) , & 2(\alpha^2+\beta^2)-\lambda \end{bmatrix}$

$\Rightarrow \frac{4}{9}(\alpha^2+\beta^2)^2 + \lambda^2 - \frac{4}{3}(\alpha^2+\beta^2) - \frac{4}{9}(\alpha^2-\beta^2)^2 = 0$

$\Rightarrow (\lambda - \frac{4}{3}\alpha^2)(\lambda - \frac{4}{3}b^2) = 0$

$\Rightarrow \lambda = \frac{4}{3}\alpha^2, \frac{4}{3}b^2$

① $\lambda = \frac{4}{3}\alpha^2$,

$\begin{bmatrix} \frac{2}{3}(\alpha^2+\beta^2)-\frac{4\alpha^2}{3}, & \frac{2}{3}(\alpha^2-\beta^2) \\ \frac{2}{3}(\alpha^2-\beta^2) , & \frac{2}{3}(\alpha^2+\beta^2)-\frac{4\alpha^2}{3} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, when $\lambda = \frac{4}{3}\alpha^2$ #

② $\lambda = \frac{4}{3}b^2$

$\begin{bmatrix} \frac{2}{3}(\alpha^2+\beta^2)-\frac{4b^2}{3}, & \frac{2}{3}(\alpha^2-\beta^2) \\ \frac{2}{3}(\alpha^2-\beta^2) , & \frac{2}{3}(\alpha^2+\beta^2)-\frac{4b^2}{3} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, when $\lambda = \frac{4}{3}b^2$ #

- eigenvector 固定方向
- eigenvalue 隨 α, β 變化 #