CS 325
Due: Wed Mar 14

Homework #8

1. **Reductions**

   Let $A \leq B$ for two problems A and B mean that problem A can be solved in big $\mathbb{O}$ of the time it takes to solve problem B.

   (a) Show that **MULTIPLICATION** $\leq$ **SQUARING**.

   (b) Show that **SQUARING** $\leq$ **MULTIPLICATION**.

   (c) Show that **SQUARING** $\leq$ **RECIPROCAL**.

   (d) If $A \equiv B$ means $A \leq B$ and $B \leq A$ which of **MULTIPLICATION**, **SQUARING**, and **RECIPROCAL** are equivalent ?

   HINT: $\frac{1}{x} - \frac{1}{y} = \frac{y-x}{x\,y}$. Try $y = x + 1$.

2. **Lucas Numbers:**

   INPUT: A K bit number X.
   QUESTION: Is X a Fibonacci number?
   The Lucas numbers are defined by the recurrence

   $$L_n = L_{n-1} + L_{n-2}$$

   with the initial conditions: $L_0 = 2$, $L_1 = 1$ Show that this problem is in $\mathcal{P}$ by outlining (NO CODE, just explain what you're doing) an algorithm, **AND** showing that your algorithm runs in polynomial time in K, the number of bits.

3. **Roots:**

   Without finding the solutions, show that $x^2 - x - 1 = 0$ has:

   (a) NO positive integer solutions

   (b) NO rational solutions
   HINTS:

      i. Assume that $x = p/q$ where p and q are integers with no common factors.

      ii. $p^2 - q^2 = (p - q)(p + q)$.

      iii. Each integer is either ODD or EVEN.

4. **Average Case:**

   Do Exercise 5.5 in the NOTES on page 61.

5. **Lower Bound:**

   Exercise 6.1 in the NOTES on page 71, is about the lower bound of $\frac{3}{2}n - 2$ comparisons to find the largest and smallest elements in an array. Devise a divide-and-conquer algorithm for this problem and show that the number of comparisons used by your algorithm achieves this lower bound.

6. **Boolean Expression:**
Assume that you have an algorithm **YS( )** so that when you input a Boolean expression $\mathbf{E}(x_1, \ldots, x_n)$ into **YS( )** ,
YS(E) outputs *YES* if **E** is satisfiable, and
YS(E) outputs *NO* if **E** is not satisfiable.

   (a) Show how to use **YS( )** to construct an algorithm **FIND(** $D(x_1, \ldots, x_n)$**)** which when given a satisfiable Boolean expression $D(x_1, \ldots, x_n)$, returns an assignment $x_1 = a_1,\ x_2 = a_2,\ \ldots,\ x_n = a_n$, so that $D(a_1, \ldots, a_n)$ is *TRUE*.

   (b) Assume that **YS(** $D(x_1, \ldots, x_n)$ **)** has run time $\mathbb{O}(\, n^k\,)$ and find the run time of **FIND(** $D(x_1, \ldots, x_n)$ **)** .

7. **Platonic Hamiltonian Circiuts:**
Show that each of the *PLATONIC* solids has a Hamiltonian circuit.

8. **s-t Hamiltonian Path:**
INPUT: A graph G and two specified vertices s and t.
QUESTION: Does G have a Hamiltonian Path which starts at s and ends at t?

   (a) Assume that you know that Hamiltonian Circuit is $\mathcal{NP}-$Complete, show that s-t Hamiltonian Path is $\mathcal{NP}-$Complete.

   (b) Assume that you know that s-t Hamiltonian Path is $\mathcal{NP}-$Complete, show that Hamiltonian Circuit is $\mathcal{NP}-$Complete.

   (c) Show that the Yes/No version of TSP (Traveling Sales Person) with all edge weights in $\{1,\ 2\}$ is $\mathcal{NP}-$Complete. (You should assume that Hamiltonian Circuit is $\mathcal{NP}-$Complete.)

9. **Graph Isomorphism:**
Graph isomorphism is an example of a problem which is in $\mathcal{NP}$, but is not known to be $\mathcal{NP}$-complete, nor is it known to be in co-$\mathcal{NP}$.

INPUT: Two graphs, $G_1 = (V_1\,,\, E_1)$ and $G_2 = (V_2\,,\, E_2)$.
QUESTION: Can the vertices of G_1 be renamed so that G_1 becomes G_2? (Is there a one-to-one onto function $f : V_1 \to V_2$ so that $\forall x, y \quad (x, y) \in E_1$ iff $(\, f(x), f(y)\,) \in E_2$?

Show that GRAPH ISOMORPHISM is in $\mathcal{NP}$.

10. **Canonical Number:**

A graph with n vertices can be represented as an $n \times n$ binary matrix which has a 1 in position (i, j) if and only if there is an edge (v_i, v_j). If you "unroll" this matrix (say by rows), you will have a vector of n^2 bits and you can consider this to be a number in standard binary notation. So, there is a correspondence between n vertex graphs and n^2 bit numbers. If we re-label the vertices of the graph, we don't change the graph properties. Different re-labelings of the graph will (usually) give different numbers. Clearly among all re-labelings of the graph, there is some re-labeling which gives the smallest value for this binary number. We would like to represent a graph by the minimum number we can get by re-labeling. We'll call this minimal number the *canonical number* of the graph. It's easy to see that two graphs are isomorphic iff they have the same canonical number.

(a) The graph $v_1 — v_2 — v_3$ is isomorphic to $v_1 — v_3 — v_2$ and is also isomorphic to $v_2 — v_1 — v_3$.
Find the canonical number of $v_1 — v_2 — v_3$.

(b) Show that if finding the *canonical number* of a graph is easy, then GRAPH ISOMORPHISM is easy. (Here, *easy* means takes polynomial time.)

However, *canonical number* may be harder than GRAPH ISOMORPHISM. If I can tell that two graphs are NOT isomorphic, I know that their canonical numbers are different, but I don't know what their canonical numbers are. Further, if I know that two graphs are isomorphic, I know that their canonical numbers are identical, but again I don't know what these canonical numbers are.

(c) **Is-Canonical:**
INPUT: A graph G and an integer I.
QUESTION: Is $I <$ the canonical number of G?
EXERCISE: Show that IS-CANONICAL is in co-$\mathcal{NP}$.

11. **Tautology:**

INPUT: A Boolean Expression $E(x_1, \ldots, x_n)$.
QUESTION: Does E evaluate to **TRUE** for each and every assignment of **TRUE** and **FALSE** to the variables, the x's ?

Show that **TAUTOLOGY** is co-$\mathcal{NP}$-complete.

12. **3-SAT:**

INPUT: A Boolean Expression $E(x_1, \ldots, x_n)$ in Clause form with at most 3 literals per clause.
QUESTION: Does E evaluate to **TRUE** for some assignment of **TRUE** and **FALSE** to the variables, the x's ?

Show that if **SAT** is $\mathcal{NP}$-complete, then **3-SAT** is $\mathcal{NP}$-complete.