36 $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$ give $\det \lambda_1\lambda_2 = 1$ and trace $\lambda_1 + \lambda_2 = -1$. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \dfrac{2\pi}{3}$ has this trace and det. So does every $M^{-1}AM$!

37 (a) Since the columns of A add to 1, one eigenvalue is $\lambda = 1$ and the other is $c - .6$ (to give the correct trace $c + .4$).

(b) If $c = 1.6$ then both eigenvalues are 1, and all solutions to $(A - I)\,\boldsymbol{x} = \mathbf{0}$ are multiples of $\boldsymbol{x} = (1, -1)$.

(c) If $c = .8$, the eigenvectors for $\lambda = 1$ are multiples of (1, 3). Since all powers A^n also have column sums $= 1$, A^n will approach $\dfrac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} =$ rank-1 matrix A^∞ with eigenvalues $1, 0$ and correct eigenvectors. $(1, 3)$ and $(1, -1)$.

Problem Set 6.2, page 307

1 $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

2 Put the eigenvectors in S and eigenvalues in Λ. $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

3 If $A = S\Lambda S^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still S. $A + 2I = S(\Lambda + 2I)S^{-1} = S\Lambda S^{-1} + S(2I)S^{-1} = A + 2I$.

4 (a) False: don't know λ's (b) True (c) True (d) False: need eigenvectors of S

5 With $S = I, A = S\Lambda S^{-1} = \Lambda$ is a diagonal matrix. If S is triangular, then S^{-1} is triangular, so $S\Lambda S^{-1}$ is also triangular.

6 The columns of S are nonzero multiples of (2,1) and (0,1): either order. Same for A^{-1}.

7 $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}/2 = \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}/2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ for any a and b.

8 $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$. $S\Lambda^k S^{-1} = \dfrac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \textit{2nd component} \text{ is } F_k \\ (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2) \end{bmatrix}$.

9 (a) $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}$ with $\boldsymbol{x}_1 = (1, 1),\ \boldsymbol{x}_2 = (1, -2)$

(b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

10 The rule $F_{k+2} = F_{k+1} + F_k$ produces the pattern: even, odd, odd, even, odd, odd, ...

11 (a) *True* (no zero eigenvalues) (b) *False* (repeated $\lambda = 2$ may have only one line of eigenvectors) (c) *False* (repeated λ may have a full set of eigenvectors)

12 (a) False: don't know λ (b) True: an eigenvector is missing (c) True.

13 $A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $\boldsymbol{x} = (c, -c)$.

14 The rank of $A - 3I$ is $r = 1$. Changing any entry except $a_{12} = 1$ makes A diagonalizable (A will have two different eigenvalues)

15 $A^k = S\Lambda^k S^{-1}$ approaches zero **if and only if every $|\lambda| < 1$**; $A_1^k \to A_1^\infty$, $A_2^k \to 0$.

16 $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $S\Lambda^k S^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: *steady state.*

17 $\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$, $S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $A_2^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A_2^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $A_2^{10}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ is the sum of $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

18 $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $A^k = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Multiply those last three matrices to get $A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$.

19 $B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

20 $\det A = (\det S)(\det \Lambda)(\det S^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$. This proof works when A is *diagonalizable.*

21 trace $ST = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace TS. Diagonalizable case: the trace of $S\Lambda S^{-1} =$ trace of $(\Lambda S^{-1})S = \Lambda$: *sum of the* λ's.

22 $AB - BA = I$ is impossible since trace $AB -$ trace $BA = zero \neq$ trace I. $AB - BA = C$ is possible when trace $(C) = 0$, and $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has $EE^{\mathrm{T}} - E^{\mathrm{T}}E = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

23 If $A = S\Lambda S^{-1}$ then $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix}\begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}$. So B has the additional eigenvalues $2\lambda_1, \ldots, 2\lambda_n$.

24 The A's form a subspace since cA and $A_1 + A_2$ all have the same S. When $S = I$ the A's with those eigenvectors give the subspace of diagonal matrices. Dimension 4.

25 If A has columns $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ then column by column, $A^2 = A$ means every $A\boldsymbol{x}_i = \boldsymbol{x}_i$. All vectors in the column space (combinations of those columns $\boldsymbol{x}_i$) are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$ (A might have dependent columns, so there could be less than n eigenvectors with $\lambda = 1$). Dimensions of those spaces add to n by the Fundamental Theorem, so A *is diagonalizable* (n independent eigenvectors altogether).

26 Two problems: The nullspace and column space can overlap, so $\boldsymbol{x}$ could be in both. There may not be r independent eigenvectors in the column space.

27 $R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$. $\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace is not real. Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ can have $\sqrt{-1} = i$ and $-i$, trace 0, real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

28 $A^{\mathrm{T}} = A$ gives $\boldsymbol{x}^{\mathrm{T}}AB\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}(B\boldsymbol{x}) \le \|A\boldsymbol{x}\|\|B\boldsymbol{x}\|$ by the Schwarz inequality. $B^{\mathrm{T}} = -B$ gives $-\boldsymbol{x}^{\mathrm{T}}BA\boldsymbol{x} = (B\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x}) \le \|A\boldsymbol{x}\|\|B\boldsymbol{x}\|$. Add to get Heisenberg's Uncertainty Principle when $AB - BA = I$. Position-momentum, also time-energy.

29 The factorizations of A and B into $S\Lambda S^{-1}$ are the same. So $A = B$. (This is the same as Problem 6.1.25, expressed in matrix form.)

30 $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Diagonal matrices always give $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. Then $AB = BA$ from $S\Lambda_1 S^{-1}S\Lambda_2 S^{-1} = S\boldsymbol{\Lambda_1\Lambda_2}S^{-1} = S\boldsymbol{\Lambda_2\Lambda_1}S^{-1} = S\Lambda_2 S^{-1}$ $S\Lambda_1 S^{-1} = BA$.

31 (a) $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\lambda = a$ and $\lambda = d$: $(A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 - A - I = 0$ is true, matching $\lambda^2 - \lambda - 1 = 0$ as the Cayley-Hamilton Theorem predicts.

32 When $A = S\Lambda S^{-1}$ is diagonalizable, the matrix $A - \lambda_j I = S(\Lambda - \lambda_j I)S^{-1}$ will have 0 in the j, j diagonal entry of $\Lambda - \lambda_j I$. In the product $p(A) = (A - \lambda_1 I)\cdots(A - \lambda_n I)$, each inside S^{-1} cancels S. This leaves S times (*product of diagonal matrices* $\Lambda - \lambda_j I$) times S^{-1}. That product is the zero matrix because the factors produce a zero in each diagonal position. Then $p(A) =$ zero matrix, which is the Cayley-Hamilton Theorem. (If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A.)

Comment I have also seen this reasoning but I am not convinced:

Apply the formula $AC^{\mathrm{T}} = (\det A)I$ from Section 5.3 to $A - \lambda I$ with variable λ. Its cofactor matrix C will be a polynomial in λ, since cofactors are determinants:

$$(A - \lambda I)\operatorname{cof}(A - \lambda I)^{\mathrm{T}} = \det(A - \lambda I)I = p(\lambda)I.$$

"For fixed A, this is an identity between two matrix polynomials." Set $\lambda = A$ to find the zero matrix on the left, so $p(A) =$ zero matrix on the right—which is the Cayley-Hamilton Theorem.

I am not certain about the key step of substituting a matrix for λ. If other matrices B are substituted, does the identity remain true? If $AB \ne BA$, even the order of multiplication seems unclear ...

33 $\lambda = 2, -1, 0$ are in $\boldsymbol{\Lambda}$ and the eigenvectors are in S (below). $A^k = S\boldsymbol{\Lambda}^k S^{-1}$ is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\boldsymbol{\Lambda}^k \frac{1}{6}\begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & -2 \\ 0 & 3 & -3 \end{bmatrix} = \frac{2^k}{6}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \frac{(-1)^k}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Check $k = 4$. The $(2, 2)$ entry of A^4 is $2^4/6 + (-1)^4/3 = 18/6 = 3$. The 4-step paths that begin and end at node 2 are 2 to 1 to 1 to 1 to 2, 2 to 1 to 2 to 1 to 2, and 2 to 1 to 3 to 1 to 2. Much harder to find the eleven 4-step paths that start and end at node 1.

34 If $AB = BA$, then B has the same eigenvectors $(1,0)$ and $(0,1)$ as A. So B is also diagonal $b = c = 0$. The nullspace for the following equation is 2-dimensional: $AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The coefficient matrix has rank $4 - 2 = 2$.

35 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1 and $B^4 = I$. C has $\lambda = (1 \pm \sqrt{3}i)/2$. This is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ and $C^{1024} = -C$.

36 The eigenvalues of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $\lambda = e^{i\theta}$ and $e^{-i\theta}$ (trace $2\cos\theta$ and $\det = 1$). Their eigenvectors are $(1,-i)$ and $(1,i)$:

$$A^n = S\Lambda^n S^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{in\theta} & \\ & e^{-in\theta} \end{bmatrix}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}/2i$$

$$= \begin{bmatrix} (e^{in\theta} + e^{-in\theta})/2 & \cdots \\ (e^{in\theta} - e^{-in\theta})/2i & \cdots \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Geometrically, n rotations by θ give one rotation by $n\theta$.

37 Columns of S times rows of ΛS^{-1} will give r rank-1 matrices ($r =$ rank of A).

38 Note that $\text{ones}(n) * \text{ones}(n) = n * \text{ones}(n)$. This leads to $C = 1/(n+1)$.

$$\begin{aligned} AA^{-1} &= (\text{eye}(n) + \text{ones}(n)) * (\text{eye}(n) + C * \text{ones}(n)) \\ &= \text{eye}(n) + (1 + C + Cn) * \text{ones}(n) = \text{eye}(n). \end{aligned}$$

Problem Set 6.3, page 325

1 $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_2 = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If $\boldsymbol{u}(0) = (5,-2)$, then $\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2 $z(t) = 2e^t$; then $dy/dt = 4y - 6e^t$ with $y(0) = 5$ gives $y(t) = 3e^{4t} + 2e^t$ as in Problem 1.

3 (a) If every column of A adds to zero, this means that the rows add to the zero row. So the rows are dependent, and A is singular, and $\lambda = 0$ is an eigenvalue.

(b) The eigenvalues of $A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$ are $\lambda_1 = 0$ with eigenvector $\boldsymbol{x}_1 = (3,2)$ and $\lambda_2 = -5$ (to give trace $= -5$) with $\boldsymbol{x}_2 = (1,-1)$. Then the usual 3 steps:

1. Write $u(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ as $\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \boldsymbol{x}_1 + \boldsymbol{x}_2$
2. Follow those eigenvectors by $e^{0t}\boldsymbol{x}_1$ and $e^{-5t}\boldsymbol{x}_2$
3. The solution $\boldsymbol{u}(t) = \boldsymbol{x}_1 + e^{-5t}\boldsymbol{x}_2$ has steady state $\boldsymbol{x}_1 = (3,2)$.

4 $d(v+w)/dt = (w-v) + (v-w) = 0$, so the total $v+w$ is constant. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ has $\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = -2 \end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\begin{matrix} v(1) = 20 + 10e^{-2} & v(\infty) = 20 \\ w(1) = 20 - 10e^{-2} & w(\infty) = 20 \end{matrix}$

5 $\dfrac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has $\lambda = 0$ and $+2$: $v(t) = 20 + 10e^{2t} \to \infty$ as $t \to \infty$.

6 $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ has real eigenvalues $a+1$ and $a-1$. These are both negative if $\boldsymbol{a < -1}$, and the solutions of $\boldsymbol{u}' = A\boldsymbol{u}$ approach zero. $B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$ has complex eigenvalues $b+i$ and $b-i$. These have negative real parts if $\boldsymbol{b < 0}$, and all solutions of $\boldsymbol{v}' = B\boldsymbol{v}$ approach zero.

7 A projection matrix has eigenvalues $\lambda = 1$ and $\lambda = 0$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{x}$ fill the subspace that P projects onto: here $\boldsymbol{x} = (1,1)$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{0}$ fill the perpendicular subspace: here $\boldsymbol{x} = (1,-1)$. For the solution to $\boldsymbol{u}' = -P\boldsymbol{u}$,

$$\boldsymbol{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}(t) = e^{-t}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

8 $\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ has $\lambda_1 = 5$, $\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 2$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches 20/10; e^{5t} dominates.

9 (a) $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ i \end{bmatrix} + 2\begin{bmatrix} 1 \\ -i \end{bmatrix}$. (b) Then $u(t) = 2e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + 2e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 4\cos t \\ 4\sin t \end{bmatrix}$.

10 $\dfrac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$. $A = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$ has $\det(A - \lambda I) = \lambda^2 - 5\lambda - 4 = 0$. Directly substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$ also gives $\lambda^2 = 5\lambda + 4$ and the same two values of λ. Those values are $\frac{1}{2}(5 \pm \sqrt{41})$ by the quadratic formula.

11 $e^{At} = I + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \text{ zeros } = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. Then $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$ $\begin{bmatrix} y(0) + y'(0)t \\ y'(0) \end{bmatrix}$. This $y(t) = y(0) + y'(0)t$ solves the equation.

12 $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1,3)$.

13 (a) $y(t) = \cos 3t$ and $\sin 3t$ solve $y'' = -9y$. It is $\mathbf{3}\cos 3t$ that starts with $y(0) = 3$ and $y'(0) = 0$. (b) $A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$ has det $= 9$: $\lambda = 3i$ and $-3i$ with $\boldsymbol{x} = (1, 3i)$ and $(1, -3i)$. Then $\boldsymbol{u}(t) = \frac{3}{2}e^{3it}\begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it}\begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} 3\cos 3t \\ -9\sin 3t \end{bmatrix}$.

14 When A is skew-symmetric, $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\|$ is $\|\boldsymbol{u}(0)\|$. So e^{At} is *orthogonal*.

15 $\boldsymbol{u}_p = 4$ and $\boldsymbol{u}(t) = ce^t + 4$; $\boldsymbol{u}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{u}(t) = c_1e^t\begin{bmatrix} 1 \\ t \end{bmatrix} + c_2e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

16 Substituting $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A - cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A - cI)^{-1}\boldsymbol{b}$ = particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

17 (a) $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (c) $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$. These show the unstable cases (a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$

18 $d/dt(e^{At}) = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \cdots = A(I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots)$. This is exactly Ae^{At}, the derivative we expect.

19 $e^{Bt} = I + Bt$ (short series with $B^2 = 0$) $= \begin{bmatrix}\mathbf{1} & \mathbf{-4t}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}$. Derivative $= \begin{bmatrix}0 & -4\\ 0 & 0\end{bmatrix} = B$.

20 The solution at time $t + T$ is also $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

21 $\begin{bmatrix}1 & 4\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}\mathbf{1} & 0\\ 0 & \mathbf{0}\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}$; $\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}\boldsymbol{e^t} & 0\\ 0 & \mathbf{1}\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix} = \begin{bmatrix}e^t & 4e^t - 4\\ 0 & 1\end{bmatrix}$.

22 $A^2 = A$ gives $e^{At} = I + At + \frac{1}{2}\boldsymbol{At^2} + \frac{1}{6}At^3 + \cdots = I + (e^t - 1)A = \begin{bmatrix}e^t & e^t - 1\\ 0 & 1\end{bmatrix}$.

23 $e^A = \begin{bmatrix}e & 4(e-1)\\ 0 & 1\end{bmatrix}$ from **21** and $e^B = \begin{bmatrix}1 & -4\\ 0 & 1\end{bmatrix}$ from **19**. By direct multiplication $e^Ae^B \neq e^Be^A \neq e^{A+B} = \begin{bmatrix}e & 0\\ 0 & 1\end{bmatrix}$.

24 $A = \begin{bmatrix}1 & 1\\ 0 & 3\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}\mathbf{1} & 0\\ 0 & \mathbf{3}\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix}$. Then $e^{At} = \begin{bmatrix}\boldsymbol{e^t} & \frac{1}{2}(e^{3t} - e^t)\\ 0 & \boldsymbol{e^{3t}}\end{bmatrix}$.

25 The matrix has $A^2 = \begin{bmatrix}1 & 3\\ 0 & 0\end{bmatrix}^2 = \begin{bmatrix}1 & 3\\ 0 & 0\end{bmatrix} = A$. *Then all* $A^n = A$. So $e^{At} = I + (t + t^2/2! + \cdots)A = I + (e^t - 1)A = \begin{bmatrix}e^t & 3(e^t - 1)\\ 0 & 0\end{bmatrix}$ as in Problem 22.

26 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \neq 0$. To see $e^{At}\boldsymbol{x}$, write $(I + At + \frac{1}{2}A^2t^2 + \cdots)\boldsymbol{x} = (1 + \lambda t + \frac{1}{2}\lambda^2t^2 + \cdots)\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$.

27 $(x, y) = (e^{4t}, e^{-4t})$ is a growing solution. The correct matrix for the exchanged $\boldsymbol{u} = (y, x)$ is $\begin{bmatrix}2 & -2\\ -4 & 0\end{bmatrix}$. It *does* have the same eigenvalues as the original matrix.

28 Centering produces $\boldsymbol{U}_{n+1} = \begin{bmatrix}1 & \Delta t\\ -\Delta t & 1 - (\Delta t)^2\end{bmatrix}\boldsymbol{U}_n$. At $\Delta t = 1$, $\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}$ has $\lambda = e^{i\pi/3}$ and $e^{-i\pi/3}$. Both eigenvalues have $\lambda^6 = 1$ so $A^6 = I$. Therefore $\boldsymbol{U}_6 = A^6\boldsymbol{U}_0$ comes exactly back to $\boldsymbol{U}_0$.

29 First A has $\lambda = \pm i$ and $A^4 = I$. Second A has $\lambda = -1, -1$ and $A^n = (-1)^n\begin{bmatrix}1 - 2n & -2n\\ 2n & 2n + 1\end{bmatrix}$ Linear growth.

30 With $a = \Delta t/2$ the trapezoidal step is $\boldsymbol{U}_{n+1} = \dfrac{1}{1 + a^2}\begin{bmatrix}1 - a^2 & 2a\\ -2a & 1 - a^2\end{bmatrix}\boldsymbol{U}_n$.

That matrix has orthonormal columns $\Rightarrow$ orthogonal matrix $\Rightarrow \|\boldsymbol{U}_{n+1}\| = \|\boldsymbol{U}_n\|$

31 (a) $(\cos A)\boldsymbol{x} = (\cos\lambda)\boldsymbol{x}$ (b) $\lambda(A) = 2\pi$ and 0 so $\cos\lambda = 1, 1$ and $\cos A = I$ (c) $\boldsymbol{u}(t) = 3(\cos 2\pi t)(1, 1) + 1(\cos 0t)(1, -1)$ [$\boldsymbol{u}' = A\boldsymbol{u}$ has **exp**, $\boldsymbol{u}'' = A\boldsymbol{u}$ has **cos**]

Problem Set 6.4, page 337

Note A way to complete the proof at the end of page 334, (perturbing the matrix to produce distinct eigenvalues) is now on the course website: "*Proofs of the Spectral Theorem.*" **math.mit.edu/linearalgebra**.

1 $A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \begin{matrix} = \frac{1}{2}(A + A^{\mathrm{T}}) + \frac{1}{2}(A - A^{\mathrm{T}}) \\ = \textbf{symmetric} + \textbf{skew-symmetric}. \end{matrix}$

2 $(A^{\mathrm{T}}CA)^{\mathrm{T}} = A^{\mathrm{T}}C^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}}CA$. When A is 6 by 3, C will be 6 by 6 and the triple product $A^{\mathrm{T}}CA$ is 3 by 3.

3 $\lambda = 0, 4, -2$; unit vectors $\pm(0, 1, -1)/\sqrt{2}$ and $\pm(2, 1, 1)/\sqrt{6}$ and $\pm(1, -1, -1)/\sqrt{3}$.

4 $\lambda = 10$ and -5 in $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ have to be normalized to unit vectors in $Q = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

5 $Q = \dfrac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. The columns of Q are unit eigenvectors of A
Each unit eigenvector could be multiplied by -1

6 $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ has $\lambda = 0$ and 25 so the columns of Q are the two eigenvectors: $Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$ or we can exchange columns or reverse the signs of any column.

7 (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1$ and 3 (b) The pivots have the same signs as the λ's (c) trace $= \lambda_1 + \lambda_2 = 2$, so A can't have two negative eigenvalues.

8 If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is *symmetric* then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ requires $\Lambda = 0$. The only symmetric A is $Q\,0\,Q^{\mathrm{T}} =$ zero matrix.

9 If λ is complex then $\overline{\lambda}$ is also an eigenvalue ($A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$). Always $\lambda + \overline{\lambda}$ is real. The trace is real so the third eigenvalue of a 3 by 3 real matrix must be real.

10 If $\boldsymbol{x}$ is not real then $\lambda = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is *not* always real. Can't assume real eigenvectors!

11 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0\begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25\begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

12 $[\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2\,]$ is an orthogonal matrix so $P_1 + P_2 = \boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}} = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2\,] \begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}} \\ \boldsymbol{x}_2^{\mathrm{T}} \end{bmatrix} = I$; $P_1P_2 = \boldsymbol{x}_1(x_1^{\mathrm{T}}x_2)x_2^{\mathrm{T}} = 0$. Second proof: $P_1P_2 = P_1(I - P_1) = P_1 - P_1 = 0$ since $P_1^2 = P_1$.

13 $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ has $\lambda = ib$ and $-ib$. The block matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ are also skew-symmetric with $\lambda = ib$ (twice) and $\lambda = -ib$ (twice).

14 M is skew-symmetric and orthogonal; λ's must be $i, i, -i, -i$ to have trace zero.

15 $A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$ has $\lambda = 0, 0$ and only one independent eigenvector $\boldsymbol{x} = (i, 1)$. The good property for complex matrices is not $A^{\mathrm{T}} = A$ (symmetric) but $\overline{A}^{\mathrm{T}} = A$ (Hermitian with real eigenvalues and orthogonal eigenvectors: see Problem 20 and Section 10.2).

16 (a) If $A\boldsymbol{z} = \lambda \boldsymbol{y}$ and $A^{\mathrm{T}}\boldsymbol{y} = \lambda \boldsymbol{z}$ then $B[\,\boldsymbol{y};\ -\boldsymbol{z}\,] = [\,-A\boldsymbol{z};\ A^{\mathrm{T}}\boldsymbol{y}\,] = -\lambda[\,\boldsymbol{y};\ -\boldsymbol{z}\,]$. So $-\lambda$ is also an eigenvalue of B. (b) $A^{\mathrm{T}}A\boldsymbol{z} = A^{\mathrm{T}}(\lambda\boldsymbol{y}) = \lambda^2\boldsymbol{z}$. (c) $\lambda = -1, -1, 1, 1$; $\boldsymbol{x}_1 = (1, 0, -1, 0)$, $\boldsymbol{x}_2 = (0, 1, 0, -1)$, $\boldsymbol{x}_3 = (1, 0, 1, 0)$, $\boldsymbol{x}_4 = (0, 1, 0, 1)$.

17 The eigenvalues of $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ are $0, \sqrt{2}, -\sqrt{2}$ by Problem 16 with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$, $\boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}$.

18 **1.** $\boldsymbol{y}$ is in the nullspace of A and $\boldsymbol{x}$ is in the column space $=$ row space because $A = A^{\mathrm{T}}$. Those spaces are perpendicular so $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} = 0$.
2. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $A\boldsymbol{y} = \beta\boldsymbol{y}$ then shift by β: $(A - \beta I)\boldsymbol{x} = (\lambda - \beta)\boldsymbol{x}$ and $(A - \beta I)\boldsymbol{y} = \boldsymbol{0}$ and again $\boldsymbol{x} \perp \boldsymbol{y}$.

19 A has $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B has $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$. Perpendicular for A Not perpendicular for B since $B^{\mathrm{T}} \neq B$

20 $A = \begin{bmatrix} 1 & 3 + 4i \\ 3 - 4i & 1 \end{bmatrix}$ is a *Hermitian matrix* ($\overline{A}^{\mathrm{T}} = A$). Its eigenvalues 6 and -4 are *real*. Adjust equations (1)–(2) in the text to prove that λ is always real when $\overline{A}^{\mathrm{T}} = A$:

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \text{ leads to } \overline{A}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}. \text{ Transpose to } \overline{\boldsymbol{x}}^{\mathrm{T}}A = \overline{\boldsymbol{x}}^{\mathrm{T}}\overline{\lambda} \text{ using } \overline{A}^{\mathrm{T}} = A.$$

$$\text{Then } \overline{\boldsymbol{x}}^{\mathrm{T}}A\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\lambda\boldsymbol{x} \text{ and also } \overline{\boldsymbol{x}}^{\mathrm{T}}A\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\overline{\lambda}\boldsymbol{x}. \text{ So } \lambda = \overline{\lambda} \text{ is real.}$$

21 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}}$ (c) True from $A^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

22 A and A^{T} have the same λ's but the *order* of the $\boldsymbol{x}$'s can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $\boldsymbol{x}_1 = (1, i)$ first for A but $\boldsymbol{x}_1 = (1, -i)$ first for A^{T}.

23 A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. A allows QR, $S\Lambda S^{-1}$, $Q\Lambda Q^{\mathrm{T}}$; B allows $S\Lambda S^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

24 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ if $b = 1$; repeated λ and no S if $b = -1$; singular if $b = 0$.

25 Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so $\lambda = \pm 1$. Then $A = \pm I$ or $A = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

26 Eigenvectors $(1, 0)$ and $(\boldsymbol{1}, \boldsymbol{1})$ give a 45° angle even with A^{T} very close to A.

27 The roots of $\lambda^2 + b\lambda + c = 0$ are $\frac{1}{2}(-b \pm \sqrt{b^2 - 4ac})$. Then $\lambda_1 - \lambda_2$ is $\sqrt{b^2 - 4c}$. For $\det(A + tB - \lambda I)$ we have $b = -3 - 8t$ and $c = 2 + 16t - t^2$. The minimum of $b^2 - 4c$ is $1/17$ at $t = 2/17$. Then $\lambda_2 - \lambda_1 = 1/\sqrt{17}$.

28 $A = \begin{bmatrix} 4 & 2+i \\ 2-i & 0 \end{bmatrix} = \overline{A}^{\mathrm{T}}$ has real eigenvalues $\lambda = 5$ and -1 with trace $= 4$ and $\det = -5$. The solution to **20** proves that λ is real when $\overline{A}^{\mathrm{T}} = A$ is Hermitian; I did not intend to repeat this part.

29 (a) $A = Q\Lambda\overline{Q}^{\mathrm{T}}$ times $\overline{A}^{\mathrm{T}} = Q\overline{\Lambda}^{\mathrm{T}}\overline{Q}^{\mathrm{T}}$ equals $\overline{A}^{\mathrm{T}}$ times A because $\Lambda\overline{\Lambda}^{\mathrm{T}} = \overline{\Lambda}^{\mathrm{T}}\Lambda$ (diagonal!) (b) step 2: The 1, 1 entries of $\overline{T}^{\mathrm{T}}\, T$ and $T\overline{T}^{\mathrm{T}}$ are $|a|^2$ and $|a|^2 + |b|^2$. This makes $b = 0$ and $T = \Lambda$.

30 a_{11} is $\begin{bmatrix} q_{11} \dots q_{1n} \end{bmatrix} \begin{bmatrix} \lambda_1 \overline{q}_{11} \dots \lambda_n \overline{q}_{1n} \end{bmatrix}^{\mathrm{T}} \le \lambda_{\max}\left(|q_{11}|^2 + \cdots + |q_{1n}|^2\right) = \lambda_{\max}$.

31 (a) $\boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x}) = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{x} = -\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$. (b) $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z}$ is pure imaginary, its real part is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{y} = 0 + 0$ (c) $\det A = \lambda_1 \dots \lambda_n \ge 0$: pairs of λ's $= ib, -ib$.

32 Since A is diagonalizable with eigenvalue matrix $\Lambda = 2I$, the matrix A itself has to be $S\Lambda S^{-1} = S(2I)S^{-1} = 2I$. (The unsymmetric matrix [2 1 ; 0 2] also has $\lambda = 2, 2$.)

Problem Set 6.5, page 350

1 Suppose $a > 0$ and $ac > b^2$ so that also $c > b^2/a > 0$. (i) The eigenvalues have the *same sign* because $\lambda_1\lambda_2 = \det = ac - b^2 > 0$. (ii) That sign is *positive* because $\lambda_1 + \lambda_2 > 0$ (it equals the trace $a + c > 0$).

2 Only $A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ has two positive eigenvalues. $\boldsymbol{x}^{\mathrm{T}}A_1\boldsymbol{x} = 5x_1^2 + 12x_1x_2 + 7x_2^2$ is negative for example when $x_1 = 4$ and $x_2 = -3$: A_1 is not positive definite as its determinant confirms.

3 Positive definite for $-3 < b < 3$ $\quad \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 9-b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 9-b^2 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$

Positive definite for $c > 8$ $\quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$.

4 $f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2$; $x^2 + 6xy + 9y^2 = (x + 3y)^2$.

5 $x^2 + 4xy + 3y^2 = (x + 2y)^2 - y^2 =$ *difference of squares* is negative at $x = 2$, $y = -1$, where the first square is zero.

6 $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ produces $f(x, y) = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2xy$. A has $\lambda = 1$ and -1. Then A is an *indefinite matrix* and $f(x, y) = 2xy$ has a *saddle point*.

7 $R^{\mathrm{T}}R = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$ and $R^{\mathrm{T}}R = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ are positive definite; $R^{\mathrm{T}}R = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is singular (and positive semidefinite). The first two R's have independent columns. The 2 by 3 R cannot have full column rank 3, with only 2 rows.

8 $A = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Pivots 3, 4 outside squares, ℓ_{ij} inside. $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 3(x + 2y)^2 + 4y^2$

9 $A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$ has only one pivot $= 4$, rank $A = 1$, eigenvalues are $24, 0, 0$, $\det A = 0$.

10 $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $\quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is singular; $B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

11 Corner determinants $|A_1| = 2$, $|A_2| = 6$, $|A_3| = 30$. The pivots are $2/1, 6/2, 30/6$.

12 A is positive definite for $c > 1$; determinants $c, c^2 - 1$, and $(c-1)^2(c+2) > 0$. B is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

13 $A = \begin{bmatrix} 1 & 5 \\ 5 & 10 \end{bmatrix}$ is an example with $a + c > 2b$ but $ac < b^2$, so not positive definite.

14 The eigenvalues of A^{-1} are positive because they are $1/\lambda(A)$. And the entries of A^{-1} pass the determinant tests. And $\boldsymbol{x}^{\mathrm{T}} A^{-1} \boldsymbol{x} = (A^{-1}\boldsymbol{x})^{\mathrm{T}} A (A^{-1}\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$.

15 Since $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} > 0$ and $\boldsymbol{x}^{\mathrm{T}} B \boldsymbol{x} > 0$ we have $\boldsymbol{x}^{\mathrm{T}}(A + B)\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} B \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$. Then $A + B$ is a positive definite matrix. The second proof uses the test $A = R^{\mathrm{T}} R$ (independent columns in R): If $A = R^{\mathrm{T}} R$ and $B = S^{\mathrm{T}} S$ pass this test, then $A + B = \begin{bmatrix} R & S \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R \\ S \end{bmatrix}$ also passes, and must be positive definite.

16 $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$ is zero when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal. Actually $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$ goes *negative* for $\boldsymbol{x} = (1, -10, 0)$ because the second pivot is *negative*.

17 If a_{jj} were smaller than all λ's, $A - a_{jj} I$ would have all eigenvalues > 0 (positive definite). But $A - a_{jj} I$ has a *zero* in the (j, j) position; impossible by Problem 16.

18 If $A\boldsymbol{x} = \lambda \boldsymbol{x}$ then $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} = \lambda \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}$. If A is positive definite this leads to $\lambda = \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} / \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} > 0$ (ratio of positive numbers). So positive energy $\Rightarrow$ positive eigenvalues.

19 All cross terms are $\boldsymbol{x}_i^{\mathrm{T}} \boldsymbol{x}_j = 0$ because symmetric matrices have orthogonal eigenvectors. So positive eigenvalues $\Rightarrow$ positive energy.

20 (a) The determinant is positive; all $\lambda > 0$ (b) All projection matrices except I are singular (c) The diagonal entries of D are its eigenvalues (d) $A = -I$ has $\det = +1$ when n is even.

21 A is positive definite when $s > 8$; B is positive definite when $t > 5$ by determinants.

22 $R = \dfrac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{\sqrt{2}} \begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $R = Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q^{\mathrm{T}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

23 $x^2/a^2 + y^2/b^2$ is $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$ when $A = \mathrm{diag}(1/a^2, 1/b^2)$. Then $\lambda_1 = 1/a^2$ and $\lambda_2 = 1/b^2$ so $a = 1/\sqrt{\lambda_1}$ and $b = 1/\sqrt{\lambda_2}$. The ellipse $9x^2 + 16y^2 = 1$ has axes with half-lengths $a = \frac{1}{3}$ and $b = \frac{1}{4}$. The points $(\frac{1}{3}, 0)$ and $(0, \frac{1}{4})$ are at the ends of the axes.

24 The ellipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $1/\sqrt{\lambda} = \sqrt{2}$ and $\sqrt{2/3}$.

25 $A = C^{\mathrm{T}} C = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$; $\begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

26 The Cholesky factors $C = \left(L\sqrt{D}\right)^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$ have *square roots* of the pivots from D. Note again $C^{\mathrm{T}}C = LDL^{\mathrm{T}} = A$.

27 Writing out $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}LDL^{\mathrm{T}}\boldsymbol{x}$ gives $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + \frac{ac-b^2}{a}y^2$. So the LDL^{T} from elimination is exactly the same as *completing the square*. The example $2x^2 + 8xy + 10y^2 = 2(x+2y)^2 + 2y^2$ with pivots $2, 2$ outside the squares and multiplier 2 inside.

28 $\det A = (1)(10)(1) = 10$; $\lambda = 2$ and 5; $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$, $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$; the λ's are positive. So A is positive definite.

29 $H_1 = \begin{bmatrix} 6x^2 & 2x \\ 2x & 2 \end{bmatrix}$ is semidefinite; $f_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$; $H_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite at $(0, 1)$ where 1st derivatives $= 0$. This is a saddle point of the function $f_2(x, y)$.

30 $ax^2 + 2bxy + cy^2$ has a saddle point if $ac < b^2$. The matrix is *indefinite* ($\lambda < 0$ and $\lambda > 0$) because the determinant $ac - b^2$ is *negative*.

31 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a "trough" staying at zero along the line $2x + 3y = 0$.

32 Orthogonal matrices, exponentials e^{At}, matrices with $\det = 1$ are groups. Examples of subgroups are orthogonal matrices with $\det = 1$, exponentials e^{An} for integer n. Another subgroup: lower triangular elimination matrices E with diagonal 1's.

33 A product AB of symmetric positive definite matrices comes into many applications. The "*generalized*" eigenvalue problem $K\boldsymbol{x} = \lambda M\boldsymbol{x}$ has $AB = M^{-1}K$. (often we use *eig*(K, M) without actually inverting M.) All eigenvalues λ are positive:

$$AB\boldsymbol{x} = \lambda\boldsymbol{x} \text{ gives } (B\boldsymbol{x})^{\mathrm{T}}AB\boldsymbol{x} = (B\boldsymbol{x})^{\mathrm{T}}\lambda x. \text{ Then } \lambda = \boldsymbol{x}^{\mathrm{T}}B^{\mathrm{T}}AB\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}B\boldsymbol{x} > 0.$$

34 The five eigenvalues of K are $2 - 2\cos\frac{k\pi}{6} = 2 - \sqrt{3},\ 2 - 1,\ 2,\ 2 + 1,\ 2 + \sqrt{3}$. The product of those eigenvalues is $6 = \det K$.

35 Put parentheses in $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}C(A\boldsymbol{x})$. Since C is assumed positive definite, this energy can drop to zero only when $A\boldsymbol{x} = \boldsymbol{0}$. Sine A is assumed to have independent columns, $A\boldsymbol{x} = \boldsymbol{0}$ only happens when $\boldsymbol{x} = \boldsymbol{0}$. Thus $A^{\mathrm{T}}CA$ has positive energy and is positive definite.

My textbooks *Computational Science and Engineering* and *Introduction to Applied Mathematics* start with many examples of $A^{\mathrm{T}}CA$ in a wide range of applications. I believe this is a unifying concept from linear algebra.

Problem Set 6.6, page 360

1 $B = GCG^{-1} = GF^{-1}AFG^{-1}$ so $M = FG^{-1}$. C similar to A and $B \Rightarrow A$ similar to B.

2 $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = M^{-1}AM$ with $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3 $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = M^{-1}AM$;

$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$;

$B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

4 A has no repeated λ so it can be diagonalized: $S^{-1}AS = \Lambda$ makes A similar to Λ.

5 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are similar (they all have eigenvalues 1 and 0).

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is by itself and also $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is by itself with eigenvalues 1 and -1.

6 *Eight families* of similar matrices: six matrices have $\lambda = 0, 1$ (one family); three matrices have $\lambda = 1, 1$ and three have $\lambda = 0, 0$ (two families each!); one has $\lambda = 1, -1$; one has $\lambda = 2, 0$; two matrices have $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$ (they are in one family).

7 (a) $(M^{-1}AM)(M^{-1}\boldsymbol{x}) = M^{-1}(A\boldsymbol{x}) = M^{-1}\boldsymbol{0} = \boldsymbol{0}$ (b) The nullspaces of A and of $M^{-1}AM$ have the same *dimension*. Different vectors and different bases.

8 Same Λ Same S But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues $\lambda = 0, 0$.

9 $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, every $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

10 $J^2 = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix}$ and $J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}$; $J^0 = I$ and $J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}$.

11 $\boldsymbol{u}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix}$. The equation $\dfrac{d\boldsymbol{u}}{dt} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\boldsymbol{u}$ has $\dfrac{dv}{dt} = \lambda v + w$ and $\dfrac{dw}{dt} = \lambda w$. Then $w(t) = 2e^{\lambda t}$ and $v(t)$ must include $2te^{\lambda t}$ (this comes from the repeated λ). To match $v(0) = 5$, the solution is $v(t) = 2te^{\lambda t} + 5e^{\lambda t}$.

12 If $M^{-1}JM = K$ then $JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & \boldsymbol{0} & \boldsymbol{0} & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = MK = \begin{bmatrix} \boldsymbol{0} & m_{12} & m_{13} & \boldsymbol{0} \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}$.

That means $m_{21} = m_{22} = m_{23} = m_{24} = 0$. M is not invertible, J not similar to K.

13 The five 4 by 4 Jordan forms with $\lambda = 0, 0, 0, 0$ are $J_1 =$ zero matrix and

$$J_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 12 showed that J_3 and J_4 are *not similar*, even with the same rank. Every matrix with all $\lambda = 0$ is "*nilpotent*" (its nth power is A^n = zero matrix). You see $J^4 = 0$ for these matrices. How many possible Jordan forms for $n = 5$ and all $\lambda = 0$?

14 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1} J_i M_i = M_i^{\mathrm{T}}$ in each block (2) M_0 has those diagonal blocks M_i to get $M_0^{-1} J M_0 = J^{\mathrm{T}}$. (3) $A^{\mathrm{T}} = (M^{-1})^{\mathrm{T}} J^{\mathrm{T}} M^{\mathrm{T}}$ equals $(M^{-1})^{\mathrm{T}} M_0^{-1} J M_0 M^{\mathrm{T}} = (M M_0 M^{\mathrm{T}})^{-1} A (M M_0 M^{\mathrm{T}})$, and A^{T} *is similar to* A.

15 $\det(M^{-1}AM - \lambda I) = \det(M^{-1}AM - M^{-1}\lambda I M)$. This is $\det(M^{-1}(A - \lambda I)M)$. By the product rule, the determinants of M and M^{-1} cancel to leave $\det(A - \lambda I)$.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$; $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ is similar to $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$. So two pairs of similar matrices but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: different eigenvalues!

17 (a) *False*: Diagonalize a nonsymmetric $A = S\Lambda S^{-1}$. Then Λ is symmetric and similar (b) *True*: A singular matrix has $\lambda = 0$. (c) *False*: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar (they have $\lambda = \pm 1$) (d) *True*: Adding I increases all eigenvalues by 1

18 $AB = B^{-1}(BA)B$ so AB is similar to BA. *If* $AB\boldsymbol{x} = \lambda\boldsymbol{x}$ *then* $BA(B\boldsymbol{x}) = \lambda(B\boldsymbol{x})$.

19 Diagonal blocks 6 by 6, 4 by 4; AB has the same eigenvalues as BA plus $6 - 4$ zeros.

20 (a) $A = M^{-1}BM \Rightarrow A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M$. So A^2 is similar to B^2. (b) A^2 equals $(-A)^2$ but A may not be similar to $B = -A$ (it could be!). (c) $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ is diagonalizable to $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ because $\lambda_1 \neq \lambda_2$, so these matrices are similar. (d) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ has only one eigenvector, so not diagonalizable (e) PAP^{T} is similar to A.

21 J^2 has three 1's down the *second* superdiagonal, and *two* independent eigenvectors for $\lambda = 0$. Its 5 by 5 Jordan form is $\begin{bmatrix} J_3 & \\ & J_2 \end{bmatrix}$ with $J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Note to professors: An interesting question: *Which matrices A have (complex) square roots* $R^2 = A$? If A is invertible, no problem. But any Jordan blocks for $\lambda = 0$ must have sizes $n_1 \geq n_2 \geq \ldots \geq n_k \geq n_{k+1} = 0$ that come in pairs like 3 and 2 in this example: $n_1 = (n_2 \text{ or } n_2+1)$ and $n_3 = (n_4 \text{ or } n_4+1)$ and so on.

A list of all 3 by 3 and 4 by 4 Jordan forms could be $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, $\begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$, $\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ (for any numbers a, b, c) with 3, 2, 1 eigenvectors; $\mathrm{diag}(a, b, c, d)$ and $\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & \\ & & & c \end{bmatrix}$,

$\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & 1 \\ & & & b \end{bmatrix}$, $\begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & b \end{bmatrix}$, $\begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{bmatrix}$ with 4, 3, 2, 1 eigenvectors.

22 If all roots are $\lambda = 0$, this means that $\det(A - \lambda I)$ must be just λ^n. The Cayley-Hamilton Theorem in Problem 6.2.32 immediately says that $A^n =$ zero matrix. The key example is a single n by n Jordan block (with $n-1$ ones above the diagonal): Check directly that $J^n =$ zero matrix.

23 Certainly Q_1R_1 is similar to $R_1Q_1 = Q_1^{-1}(Q_1R_1)Q_1$. Then $A_1 = Q_1R_1 - cs^2I$ is similar to $A_2 = R_1Q_1 - cs^2I$.

24 A could have eigenvalues $\lambda = 2$ and $\lambda = \frac{1}{2}$ (A could be diagonal). Then A^{-1} has the same two eigenvalues (and is similar to A).

Problem Set 6.7, page 371

1 $A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}} = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}}\begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}\dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$

2 This $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is a 2 by 2 matrix of rank 1. Its row space has basis $\boldsymbol{v}_1$, its nullspace has basis $\boldsymbol{v}_2$, its column space has basis $\boldsymbol{u}_1$, its left nullspace has basis $\boldsymbol{u}_2$:

$$\text{Row space } \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{Nullspace } \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\text{Column space } \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \boldsymbol{N}(A^{\mathrm{T}}) \ \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

3 If A has rank 1 then so does $A^{\mathrm{T}}A$. The only nonzero eigenvalue of $A^{\mathrm{T}}A$ is its trace, which is the sum of all a_{ij}^2. (Each diagonal entry of $A^{\mathrm{T}}A$ is the sum of a_{ij}^2 down one column, so the trace is the sum down all columns.) Then $\sigma_1 =$ square root of this sum, and $\sigma_1^2 =$ this sum of all a_{ij}^2.

4 $A^{\mathrm{T}}A = AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \dfrac{3+\sqrt{5}}{2}$, $\sigma_2^2 = \dfrac{3-\sqrt{5}}{2}$. But A is indefinite

$\sigma_1 = (1+\sqrt{5})/2 = \lambda_1(A)$, $\sigma_2 = (\sqrt{5}-1)/2 = -\lambda_2(A)$; $\boldsymbol{u}_1 = \boldsymbol{v}_1$ but $\boldsymbol{u}_2 = -\boldsymbol{v}_2$.

5 A proof that ***eigshow*** finds the SVD. When $\boldsymbol{V}_1 = (1, 0)$, $\boldsymbol{V}_2 = (0, 1)$ the demo finds $A\boldsymbol{V}_1$ and $A\boldsymbol{V}_2$ at some angle θ. A 90° turn by the mouse to $\boldsymbol{V}_2, -\boldsymbol{V}_1$ finds $A\boldsymbol{V}_2$ and $-A\boldsymbol{V}_1$ at the angle $\pi - \theta$. Somewhere between, the constantly orthogonal $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ must produce $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ *at angle* $\pi/2$. Those orthogonal directions give $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

6 $AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\sigma_2^2 = 1$ with $\boldsymbol{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $\sigma_2^2 = 1$ with $\boldsymbol{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$;

and $\boldsymbol{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [\,\boldsymbol{u}_1 \ \boldsymbol{u}_2\,]\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}[\,\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \boldsymbol{v}_3\,]^{\mathrm{T}}$.

7 The matrix A in Problem 6 had $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$ in Σ. The smallest change to rank 1 is **to make $\boldsymbol{\sigma}_2 = \mathbf{0}$**. In the factorization

$$A = U\Sigma V^{\mathrm{T}} = \boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\sigma_2\boldsymbol{v}_2^{\mathrm{T}}$$

this change $\sigma_2 \to 0$ will leave the closest rank–1 matrix as $\boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}}$. See Problem 14 for the general case of this problem.

8 The number $\sigma_{\max}(A^{-1})\sigma_{\max}(A)$ is the same as $\sigma_{\max}(A)/\sigma_{\min}(A)$. This is certainly ≥ 1. It equals 1 if all σ's are equal, and $A = U\Sigma V^{\mathrm{T}}$ is a multiple of an orthogonal matrix. The ratio $\sigma_{\max}/\sigma_{\min}$ is the important **condition number** of A studied in Section 9.2.

9 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

10 A rank–1 matrix with $A\boldsymbol{v} = 12\boldsymbol{u}$ would have $\boldsymbol{u}$ in its column space, so $A = \boldsymbol{u}\boldsymbol{w}^{\mathrm{T}}$ for some vector $\boldsymbol{w}$. I intended (but didn't say) that $\boldsymbol{w}$ is a multiple of the unit vector $\boldsymbol{v} = \frac{1}{2}(1, 1, 1, 1)$ in the problem. Then $A = 12\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ to get $A\boldsymbol{v} = 12\boldsymbol{u}$ when $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} = 1$.

11 If A has orthogonal columns $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ of lengths $\sigma_1, \ldots, \sigma_n$, then $A^{\mathrm{T}}A$ will be diagonal with entries $\sigma_1^2, \ldots, \sigma_n^2$. So the σ's are definitely the singular values of A (as expected). The eigenvalues of that diagonal matrix $A^{\mathrm{T}}A$ are the columns of I, so $V = I$ in the SVD. Then the $\boldsymbol{u}_i$ are $A\boldsymbol{v}_i/\sigma_i$ which is the unit vector $\boldsymbol{w}_i/\sigma_i$.

The SVD of this A with orthogonal columns is $A = U\Sigma V^{\mathrm{T}} = (A\Sigma^{-1})(\Sigma)(I)$.

12 Since $A^{\mathrm{T}} = A$ we have $\sigma_1^2 = \lambda_1^2$ and $\sigma_2^2 = \lambda_2^2$. But λ_2 is negative, so $\sigma_1 = 3$ and $\sigma_2 = 2$. The unit eigenvectors of A are the same $\boldsymbol{u}_1 = \boldsymbol{v}_1$ as for $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ and $\boldsymbol{u}_2 = -\boldsymbol{v}_2$ (notice the sign change because $\sigma_2 = -\lambda_2$, as in Problem 4).

13 Suppose the SVD of R is $R = U\Sigma V^{\mathrm{T}}$. Then multiply by Q to get $A = QR$. So the SVD of this A is $(QU)\Sigma V^{\mathrm{T}}$. (Orthogonal Q times orthogonal U = orthogonal QU.)

14 The smallest change in A is to set its smallest singular value σ_2 to zero. See # **7**.

15 The singular values of $A + I$ are not $\sigma_j + 1$. They come from eigenvalues of $(A + I)^{\mathrm{T}}(A + I)$.

16 This simulates the random walk used by *Google* on billions of sites to solve $A\boldsymbol{p} = \boldsymbol{p}$. It is like the power method of Section 9.3 except that it follows the links in one "walk" where the vector $p_k = A^k p_0$ averages over all walks.

17 $A = U\Sigma V^{\mathrm{T}} =$ [cosines including $\boldsymbol{u}_4$] $\mathbf{diag}(\mathbf{sqrt}(2-\sqrt{2}, 2, 2+\sqrt{2}))$ [sine matrix]$^{\mathrm{T}}$. $AV = U\Sigma$ says that differences of sines in V are cosines in U times σ's.

The SVD of the *derivative* on $[0, \pi]$ with $f(0) = 0$ has $\boldsymbol{u} = \sin nx$, $\sigma = n$, $\boldsymbol{v} = \cos nx$!

Problem Set 7.1, page 380

1 With $\boldsymbol{w} = \mathbf{0}$ linearity gives $T(\boldsymbol{v} + \mathbf{0}) = T(\boldsymbol{v}) + T(\mathbf{0})$. Thus $T(\mathbf{0}) = \mathbf{0}$. With $c = -1$ linearity gives $T(-\mathbf{0}) = -T(\mathbf{0})$. This is a second proof that $T(\mathbf{0}) = \mathbf{0}$.

2 Combining $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ and $T(d\boldsymbol{w}) = dT(\boldsymbol{w})$ with addition gives $T(c\boldsymbol{v} + d\boldsymbol{w}) = cT(\boldsymbol{v}) + dT(\boldsymbol{w})$. Then one more addition gives $cT(\boldsymbol{v}) + dT(\boldsymbol{w}) + eT(\boldsymbol{u})$.

3 (d) is not linear.

4 (a) $S(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $S(T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)) = S(T(\boldsymbol{v}_1)) + S(T(\boldsymbol{v}_2))$.

5 Choose $\boldsymbol{v} = (1,1)$ and $\boldsymbol{w} = (-1,0)$. Then $T(\boldsymbol{v}) + T(\boldsymbol{w}) = (\boldsymbol{v} + \boldsymbol{w})$ but $T(\boldsymbol{v} + \boldsymbol{w}) = (0,0)$.

6 (a) $T(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$ does not satisfy $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ or $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ (b) and (c) are linear (d) satisfies $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$.

7 (a) $T(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $T(T(\boldsymbol{v})) = \boldsymbol{v} + (2,2)$ (c) $T(T(\boldsymbol{v})) = -\boldsymbol{v}$ (d) $T(T(\boldsymbol{v})) = T(\boldsymbol{v})$.

8 (a) The range of $T(v_1, v_2) = (v_1 - v_2, 0)$ is the line of vectors $(c, 0)$. The nullspace is the line of vectors (c, c). (b) $T(v_1, v_2, v_3) = (v_1, v_2)$ has Range $\mathbf{R}^2$, kernel $\{(0, 0, \boldsymbol{v}_3)\}$ (c) $T(\boldsymbol{v}) = \mathbf{0}$ has Range $\{\mathbf{0}\}$, kernel $\mathbf{R}^2$ (d) $T(v_1, v_2) = (v_1, v_1)$ has Range = multiples of $(1, 1)$, kernel = multiples of $(1, -1)$.

9 If $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$ then $T(T(\boldsymbol{v})) = (\boldsymbol{v}_3, \boldsymbol{v}_1, \boldsymbol{v}_2)$; $T^3(\boldsymbol{v}) = \boldsymbol{v}$; $T^{100}(\boldsymbol{v}) = T(\boldsymbol{v})$.

10 (a) $T(1,0) = \mathbf{0}$ (b) $(0,0,1)$ is not in the range (c) $T(0,1) = \mathbf{0}$.

11 For multiplication $T(\boldsymbol{v}) = A\boldsymbol{v}$: $\boldsymbol{V} = \mathbf{R}^n$, $\boldsymbol{W} = \mathbf{R}^m$; the outputs fill the column space; $\boldsymbol{v}$ is in the kernel if $A\boldsymbol{v} = \mathbf{0}$.

12 $T(\boldsymbol{v}) = (4,4); (2,2); (2,2)$; if $\boldsymbol{v} = (a,b) = b(1,1) + \frac{a-b}{2}(2,0)$ then $T(\boldsymbol{v}) = b(2,2) + (0,0)$.

13 The *distributive law* (page 69) gives $A(M_1 + M_2) = AM_1 + AM_2$. The *distributive law* over c's gives $A(cM) = c(AM)$.

14 This A is invertible. Multiply $AM = 0$ and $AM = B$ by A^{-1} to get $M = 0$ and $M = A^{-1}B$. The kernel contains only the zero matrix $M = 0$.

15 This A is *not* invertible. $AM = I$ is impossible. $A\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The range contains only matrices AM whose columns are multiples of $(1,3)$.

16 No matrix A gives $A\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To professors: Linear transformations on matrix space come from **4** by **4** matrices. Those in Problems 13–15 were special.

17 For $T(M) = MT$ (a) $T^2 = I$ is True (b) True (c) True (d) False.

18 $T(I) = 0$ but $M = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = T(M)$; these M's fill the range. Every $M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ is in the kernel. Notice that dim (range) + dim (kernel) $= 3 + 1 =$ dim (input space of 2 by 2 M's).

19 $T(T^{-1}(M)) = M$ so $T^{-1}(M) = A^{-1}MB^{-1}$.

20 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because $T(1,0) = (a_{11}, 0)$.

21 $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ doubles the width of the house. $A = \begin{bmatrix} .7 & .7 \\ .3 & .3 \end{bmatrix}$ *projects* the house (since $A^2 = A$ from trace $= 1$ and $\lambda = 0, 1$). The projection is onto the column space of $A =$ line through $(.7, .3)$. $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ will *shear* the house horizontally: The point at (x, y) moves over to $(x + y, y)$.

22 (a) $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $d > 0$ leaves the house AH sitting straight up (b) $A = 3I$ expands the house by 3 (c) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates the house.

23 $T(\boldsymbol{v}) = -\boldsymbol{v}$ rotates the house by 180° around the origin. Then the affine transformation $T(\boldsymbol{v}) = -\boldsymbol{v} + (1,0)$ shifts the rotated house one unit to the right.

24 A code to add a chimney will be gratefully received!

25 This code needs a correction: add spaces between -10 10 -10 10

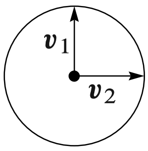
26 $\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix}$ compresses vertical distances by 10 to 1. $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ projects onto the 45° line. $\begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix}$ rotates by 45° clockwise and contracts by a factor of $\sqrt{2}$ (the columns have length $1/\sqrt{2}$). $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has determinant -1 so the house is "flipped and sheared." One way to see this is to factor the matrix as LDL^{T}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \text{(shear) (flip left-right) (shear).}$$

27 Also **30** emphasizes that circles are transformed to ellipses (see figure in Section 6.7).

28 A code that adds two eyes and a smile will be included here with public credit given!

29 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0,0)$.)

30 The circle 

transforms to the ellipse by rotating 30° and stretching the first axis by 2.

31 Linear transformations keep straight lines straight! And two parallel edges of a square (edges differing by a fixed $\boldsymbol{v}$) go to two parallel edges (edges differing by $T(\boldsymbol{v})$). So the output is a parallelogram.

Problem Set 7.2, page 395

1 For $S\boldsymbol{v} = d^2\boldsymbol{v}/dx^2$
$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 = 1, x, x^2, x^3$
$S\boldsymbol{v}_1 = S\boldsymbol{v}_2 = \boldsymbol{0}$, $S\boldsymbol{v}_3 = 2\boldsymbol{v}_1$, $S\boldsymbol{v}_4 = 6\boldsymbol{v}_2$;
The matrix for S is $B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2 $Sv = d^2v/dx^2 = 0$ for linear functions $\boldsymbol{v}(\boldsymbol{x}) = a + bx$. All $(a, b, 0, 0)$ are in the nullspace of the second derivative matrix B.

3 (Matrix $A)^2 = B$ when (transformation $T)^2 = S$ and output basis = input basis.

4 The third derivative matrix has **6** in the $(1,4)$ position; since the third derivative of x^3 is 6. This matrix also comes from AB. The fourth derivative of a cubic is zero, and B^2 is the zero matrix.

5 $T(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3) = 2\boldsymbol{w}_1+\boldsymbol{w}_2+2\boldsymbol{w}_3$; A times $(1,1,1)$ gives $(2,1,2)$.

6 $\boldsymbol{v} = c(\boldsymbol{v}_2-\boldsymbol{v}_3)$ gives $T(\boldsymbol{v}) = \mathbf{0}$; nullspace is $(0,c,-c)$; solutions $(1,0,0)+(0,c,-c)$.

7 $(1,0,0)$ is not in the column space of the matrix A, and $\boldsymbol{w}_1$ is not in the range of the linear transformation T. Key point: *Column space* of matrix matches *range* of transformation.

8 We don't know $T(\boldsymbol{w})$ unless the $\boldsymbol{w}$'s are the same as the $\boldsymbol{v}$'s. In that case the matrix is A^2.

9 Rank of $A = 2 =$ dimension of the *range* of T. The outputs $A\boldsymbol{v}$ (column space) match the outputs $T(\boldsymbol{v})$ (the range of T). The "output space" $\boldsymbol{W}$ is like $\mathbf{R}^m$: it contains all outputs but may not be filled up.

10 The matrix for T is $A = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$. For the output $\begin{bmatrix}1\\0\\0\end{bmatrix}$ choose input $\boldsymbol{v} = \begin{bmatrix}1\\-1\\0\end{bmatrix} = A^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix}$. This means: For the output $\boldsymbol{w}_1$ choose the input $\boldsymbol{v}_1-\boldsymbol{v}_2$.

11 $A^{-1} = \begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}$ so $T^{-1}(\boldsymbol{w}_1) = \boldsymbol{v}_1-\boldsymbol{v}_2, T^{-1}(\boldsymbol{w}_2) = \boldsymbol{v}_2-\boldsymbol{v}_3, T^{-1}(\boldsymbol{w}_3) = \boldsymbol{v}_3$. The columns of A^{-1} describe T^{-1} from $\boldsymbol{W}$ back to $\boldsymbol{V}$. The only solution to $T(\boldsymbol{v}) = 0$ is $\boldsymbol{v} = 0$.

12 (c) $T^{-1}(T(\boldsymbol{w}_1)) = \boldsymbol{w}_1$ is wrong because $\boldsymbol{w}_1$ is not generally in the input space.

13 (a) $T(\boldsymbol{v}_1) = \boldsymbol{v}_2, T(\boldsymbol{v}_2) = \boldsymbol{v}_1$ is its own inverse (b) $T(\boldsymbol{v}_1) = \boldsymbol{v}_1, T(\boldsymbol{v}_2) = 0$ has $T^2 = T$ (c) If $T^2 = I$ for part (a) and $T^2 = T$ for part (b), then T must be I.

14 (a) $\begin{bmatrix}2&1\\5&3\end{bmatrix}$ (b) $\begin{bmatrix}3&-1\\-5&2\end{bmatrix} =$ inverse of (a) (c) $A\begin{bmatrix}2\\6\end{bmatrix}$ must be $2A\begin{bmatrix}1\\3\end{bmatrix}$.

15 (a) $M = \begin{bmatrix}r&s\\t&u\end{bmatrix}$ transforms $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}r\\t\end{bmatrix}$ and $\begin{bmatrix}s\\u\end{bmatrix}$; this is the "easy" direction. (b) $N = \begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1}$ transforms in the inverse direction, back to the standard basis vectors. (c) $ad = bc$ will make the forward matrix singular and the inverse impossible.

16 $MW = \begin{bmatrix}1&0\\1&2\end{bmatrix}\begin{bmatrix}2&1\\5&3\end{bmatrix}^{-1} = \begin{bmatrix}3&-1\\-7&3\end{bmatrix}$.

17 Recording basis vectors is done by a *Permutation matrix*. Changing lengths is done by a *positive diagonal matrix*.

18 $(a,b) = (\cos\theta, -\sin\theta)$. Minus sign from $Q^{-1} = Q^{\mathrm{T}}$.

19 $M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$; $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ = first column of M^{-1} = coordinates of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in basis $\begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

20 $\boldsymbol{w}_2(x) = 1 - x^2$; $\boldsymbol{w}_3(x) = \frac{1}{2}(x^2 - x)$; $\boldsymbol{y} = 4\boldsymbol{w}_1 + 5\boldsymbol{w}_2 + 6\boldsymbol{w}_3$.

21 $\boldsymbol{w}$'s to $\boldsymbol{v}$'s: $\begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & -.5 \\ .5 & -1 & .5 \end{bmatrix}$. $\boldsymbol{v}$'s to $\boldsymbol{w}$'s: inverse matrix $= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$. *The key idea*: The matrix multiplies the coordinates in the $\boldsymbol{v}$ basis to give the coordinates in the $\boldsymbol{w}$ basis.

22 The 3 equations to match 4, 5, 6 at $x = a, b, c$ are $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. This Vandermonde determinant equals $(b-a)(c-a)(c-b)$. So a, b, c must be distinct to have det $\neq 0$ and one solution A, B, C.

23 The matrix M with these nine entries must be invertible.

24 Start from $A = QR$. Column 2 is $\boldsymbol{a}_2 = r_{12}\boldsymbol{q}_1 + r_{22}\boldsymbol{q}_2$. This gives $\boldsymbol{a}_2$ as a combination of the $\boldsymbol{q}$'s. So the change of basis matrix is $\boldsymbol{R}$.

25 Start from $A = LU$. Row 2 of A is ℓ_{21}(row 1 of U) + ℓ_{22} (row 2 of U). The change of basis matrix is always *invertible*, because basis goes to basis.

26 The matrix for $T(\boldsymbol{v}_i) = \lambda_i \boldsymbol{v}_i$ is $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$.

27 If T is not invertible, $T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)$ is not a basis. We couldn't choose $\boldsymbol{w}_i = T(\boldsymbol{v}_i)$.

28 (a) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ gives $T(\boldsymbol{v}_1) = \boldsymbol{0}$ and $T(\boldsymbol{v}_2) = 3\boldsymbol{v}_1$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ gives $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$ and $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \boldsymbol{v}_1$ (which combine into $T(\boldsymbol{v}_2) = \boldsymbol{0}$ by *linearity*).

29 $T(x, y) = (x, -y)$ is reflection across the x-axis. Then reflect across the y-axis to get $S(x, -y) = (-x, -y)$. Thus $ST = -I$.

30 S takes (x, y) to $(-x, y)$. $S(T(\boldsymbol{v})) = (\boldsymbol{-1}, \boldsymbol{2})$. $S(\boldsymbol{v}) = (-2, 1)$ and $T(S(\boldsymbol{v})) = (\boldsymbol{1}, \boldsymbol{-2})$.

31 Multiply the two reflections to get $\begin{bmatrix} \cos 2(\theta - \alpha) & -\sin 2(\theta - \alpha) \\ \sin 2(\theta - \alpha) & \cos 2(\theta - \alpha) \end{bmatrix}$ which is *rotation by* $2(\theta - \alpha)$. In words: $(1, 0)$ is reflected to have angle 2α, and that is reflected again to angle $2\theta - 2\alpha$.

32 False: We will not know $T(\boldsymbol{v})$ for *energy* $\boldsymbol{v}$ unless the n $\boldsymbol{v}$'s are linearly independent.

33 To find coordinates in the wavelet basis, multiply by $W^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

Then $e = \frac{1}{4}w_1 + \frac{1}{4}w_2 + \frac{1}{2}w_3$ and $v = w_3 + w_4$. Notice again: W **tells us how the bases change**, W^{-1} **tells us how the coordinates change**.

34 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, -2, -2. Therefore $c_1 = 4$ and $c_2 = 2$ and $c_3 = 1$ and $c_4 = 1$.

35 The wavelet basis is $(1,1,1,1,1,1,1,1)$ and the long wavelet and two medium wavelets $(1,1,-1,-1,0,0,0,0)$, $(0,0,0,0,1,1,-1,-1)$ and 4 wavelets with a single pair $1,-1$.

36 If $V\boldsymbol{b} = W\boldsymbol{c}$ then $\boldsymbol{b} = V^{-1}W\boldsymbol{c}$. The change of basis matrix is $V^{-1}W$.

37 Multiplying by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ gives $T(\boldsymbol{v}_1) = A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\boldsymbol{v}_1 + c\boldsymbol{v}_3$. Similarly $T(\boldsymbol{v}_2) = a\boldsymbol{v}_2 + c\boldsymbol{v}_4$ and $T(\boldsymbol{v}_3) = b\boldsymbol{v}_1 + d\boldsymbol{v}_3$ and $T(\boldsymbol{v}_4) = b\boldsymbol{v}_2 + d\boldsymbol{v}_4$. The matrix for T in this basis is $\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$.

38 The matrix for T in this basis is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Problem Set 7.3, page 406

1 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has $\lambda = 50$ and 0, $\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\sigma_1 = \sqrt{50}$.

2 Orthonormal bases: $\boldsymbol{v}_1$ for row space, $\boldsymbol{v}_2$ for nullspace, $\boldsymbol{u}_1$ for column space, $\boldsymbol{u}_2$ for $N(A^{\mathrm{T}})$. All matrices with those four subspaces are multiples cA, since the subspaces are just lines. Normally many more matrices share the same 4 subspaces. (For example, all n by n invertible matrices share $\mathbf{R}^n$.)

3 $A = QH = \frac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix} \frac{1}{\sqrt{50}}\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. H is semidefinite because A is singular.

4 $A^+ = V\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{T}} = \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; $A^+A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$, $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$.

5 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}$ has $\lambda = 18$ and 2, $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\sigma_1 = \sqrt{18}$ and $\sigma_2 = \sqrt{2}$.

6 $AA^{\mathrm{T}} = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix}$ has $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The same $\sqrt{18}$ and $\sqrt{2}$ go into Σ.

7 $\begin{bmatrix} \sigma_1\boldsymbol{u}_1 & \sigma_2\boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \end{bmatrix} = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. In general this is $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r\boldsymbol{u}_r\boldsymbol{v}_r^{\mathrm{T}}$.

8 $A = U\Sigma V^{\mathrm{T}}$ splits into QK (polar): $Q = UV^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $K = V\Sigma V^{\mathrm{T}} = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$.

9 A^+ is A^{-1} because A is invertible. Pseudoinverse equals inverse when A^{-1} exists!

10 $A^{\mathrm{T}}A = \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has $\lambda = 25, 0, 0$ and $\boldsymbol{v}_1 = \begin{bmatrix} .6 \\ .8 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} .8 \\ -.6 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Here $A = [\,3 \;\; 4 \;\; 0\,]$ has rank 1 and $AA^{\mathrm{T}} = [\,25\,]$ and $\sigma_1 = 5$ is the only singular value in $\Sigma = [\,5 \;\; 0 \;\; 0\,]$.