11 $A = [\,1\,]\,[\,5 \quad 0 \quad 0\,]V^{\mathrm{T}}$ and $A^{+} = V \begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}$; $A^{+}A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $AA^{+} = [\,1\,]$

12 The zero matrix has no pivots or singular values. Then Σ = same 2 by 3 zero matrix and the pseudoinverse is the 3 by 2 zero matrix.

13 If $\det A = 0$ then $\operatorname{rank}(A) < n$; thus $\operatorname{rank}(A^{+}) < n$ and $\det A^{+} = 0$.

14 A must be *symmetric and positive definite*, if $\Sigma = \Lambda$ and $U = V$ = eigenvector matrix Q is orthogonal.

15 (a) $A^{\mathrm{T}}A$ is singular (b) This $\boldsymbol{x}^{+}$ in the row space does give $A^{\mathrm{T}}A\boldsymbol{x}^{+} = A^{\mathrm{T}}\boldsymbol{b}$ (c) If $(1, -1)$ in the nullspace of A is added to $\boldsymbol{x}^{+}$, we get another solution to $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. But this $\widehat{\boldsymbol{x}}$ is longer than $\boldsymbol{x}^{+}$ because the added part is orthogonal to $\boldsymbol{x}^{+}$ in the row space.

16 $\boldsymbol{x}^{+}$ in the row space of A is perpendicular to $\widehat{\boldsymbol{x}} - \boldsymbol{x}^{+}$ in the nullspace of $A^{\mathrm{T}}A$ = nullspace of A. The right triangle has $c^2 = a^2 + b^2$.

17 $AA^{+}\boldsymbol{p} = \boldsymbol{p}, \ AA^{+}\boldsymbol{e} = \boldsymbol{0}, \ A^{+}A\boldsymbol{x}_r = \boldsymbol{x}_r, \ A^{+}A\boldsymbol{x}_n = \boldsymbol{0}$.

18 $A^{+} = V\Sigma^{+}U^{\mathrm{T}}$ is $\frac{1}{5}[.6 \ \ .8] = [.12 \ \ .16]$ and $A^{+}A = [\,1\,]$ and $AA^{+} = \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$ = projection.

19 L is determined by ℓ_{21}. Each eigenvector in S is determined by one number. The counts are $1 + 3$ for LU, $1 + 2 + 1$ for LDU, $1 + 3$ for QR, $1 + 2 + 1$ for $U\Sigma V^{\mathrm{T}}$, $2 + 2 + 0$ for $S\Lambda S^{-1}$.

20 LDL^{T} and $Q\Lambda Q^{\mathrm{T}}$ are determined by $\mathbf{1} + \mathbf{2} + \mathbf{0}$ numbers because A is *symmetric*.

21 Column times row multiplication gives $A = U\Sigma V^{\mathrm{T}} = \sum \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathrm{T}}$ and also $A^{+} = V\Sigma^{+}U^{\mathrm{T}} = \sum \sigma_i^{-1} \boldsymbol{v}_i \boldsymbol{u}_i^{\mathrm{T}}$. Multiplying $A^{+}A$ and using orthogonality of each $\boldsymbol{u}_i$ to all other $\boldsymbol{u}_j$ leaves the projection matrix $A^{+}A$: $A^{+}A = \sum 1\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}}$. Similarly $AA^{+} = \sum 1\boldsymbol{u}_i\boldsymbol{u}_i^{\mathrm{T}}$ from $VV^{\mathrm{T}} = I$.

22 Keep only the r by r corner Σ_r of Σ (the rest is all zero). Then $A = U\Sigma V^{\mathrm{T}}$ has the required form $A = \widehat{U} M_1 \Sigma_r M_2^{\mathrm{T}} \widehat{V}^{\mathrm{T}}$ with an invertible $M = M_1 \Sigma_r M_2^{\mathrm{T}}$ in the middle.

23 $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{v} \\ A^{\mathrm{T}}\boldsymbol{u} \end{bmatrix} = \sigma \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$. The singular values of A are *eigenvalues* of this block matrix.

Problem Set 8.1, page 418

1 Det $A_0^{\mathrm{T}} C_0 A_0 = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$ is by direct calculation. Set $c_4 = 0$ to find $\det A_1^{\mathrm{T}} C_1 A_1 = c_1 c_2 c_3$.

2 $(A_1^{\mathrm{T}} C_1 A_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1^{-1} & & \\ & c_2^{-1} & \\ & & c_3^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =$

$$\begin{bmatrix} c_1^{-1} & c_1^{-1} & c_1^{-1} \\ c_1^{-1} & c_1^{-1} + c_2^{-1} & c_1^{-1} + c_2^{-1} \\ c_1^{-1} & c_1^{-1} + c_2^{-1} & c_1^{-1} + c_2^{-1} + c_3^{-1} \end{bmatrix}.$$

3 The rows of the free-free matrix in equation (9) add to $[0\ \ 0\ \ 0]$ so the right side needs $f_1+f_2+f_3=0$. $\boldsymbol{f}=(-1,0,1)$ gives $c_2u_1-c_2u_2=-1, c_3u_2-c_3u_3=-1, 0=0$. Then $\boldsymbol{u}_{\text{particular}}=(-c_2^{-1}-c_3^{-1},-c_3^{-1},0)$. Add any multiple of $\boldsymbol{u}_{\text{nullspace}}=(1,1,1)$.

4 $\displaystyle\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)dx = -\left[c(x)\frac{du}{dx}\right]_0^1 = 0$ (bdry cond) so we need $\displaystyle\int f(x)\,dx = 0$.

5 $\displaystyle -\frac{dy}{dx}=f(x)$ gives $\displaystyle y(x)=C-\int_0^x f(t)dt$. Then $y(1)=0$ gives $\displaystyle C=\int_0^1 f(t)dt$ and $\displaystyle y(x)=\int_x^1 f(t)dt$. If the load is $f(x)=1$ then the displacement is $y(x)=1-x$.

6 Multiply $A_1^{\mathrm{T}}C_1A_1$ as columns of A_1^{T} times c's times rows of A_1. The first 3 by 3 "*element matrix*" $c_1E_1=[1\ \ 0\ \ 0]^{\mathrm{T}}c_1[1\ \ 0\ \ 0]$ has c_1 in the top left corner.

7 For 5 springs and 4 masses, the 5 by 4 A has two nonzero diagonals: all $a_{ii}=1$ and $a_{i+1,i}=-1$. With $C=\mathsf{diag}(c_1,c_2,c_3,c_4,c_5)$ we get $K=A^{\mathrm{T}}CA$, symmetric tridiagonal with diagonal entries $K_{ii}=c_i+c_{i+1}$ and off-diagonals $K_{i+1,i}=-c_{i+1}$. With $C=I$ this K is the $-1,2,-1$ matrix and $K(2,3,3,2)=(1,1,1,1)$ solves $K\boldsymbol{u}=\mathsf{ones}(4,1)$. ($K^{-1}$ will solve $K\boldsymbol{u}=\mathsf{ones}(4)$.)

8 The solution to $-u''=1$ with $u(0)=u(1)=0$ is $u(x)=\frac{1}{2}(x-x^2)$. At $x=\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$ this gives $\boldsymbol{u}=2,3,3,2$ (discrete solution in Problem 7) times $(\Delta x)^2=1/25$.

9 $-u''=mg$ has complete solution $u(x)=A+Bx-\frac{1}{2}mgx^2$. From $u(0)=0$ we get $A=0$. From $u'(1)=0$ we get $B=mg$. Then $u(x)=\frac{1}{2}mg(2x-x^2)$ at $x=\frac{1}{3},\frac{2}{3},\frac{3}{3}$ equals $mg/6, 4mg/9, mg/2$. This $u(x)$ is *not* proportional to the discrete $u=(3mg,5mg,6mg)$ at the meshpoints. This imperfection is because the discrete problem uses a 1-sided difference, less accurate at the free end. Perfect accuracy is recovered by a centered difference (discussed on page 21 of my CSE textbook).

10 (added in later printing, changing **10-11** below into **11-12**). The solution in this fixed-fixed case is (2.25, 2.50, 1.75) so the second mass moves furthest.

11 The two graphs of 100 points are "discrete parabolas" starting at $(0,0)$: symmetric around 50 in the fixed-fixed case, ending with slope zero in the fixed-free case.

12 Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB: $E=\mathsf{diag}(\mathsf{ones}(6,1),1)$; $K=64*(2*\mathsf{eye}(7)-E-E')$; $D=80*(E-\mathsf{eye}(7))$; $(K+D)\backslash\mathsf{ones}(7,1)$; % forward; $(K-D')\backslash\mathsf{ones}(7,1)$; % backward; $(K+D/2-D'/2)\backslash\mathsf{ones}(7,1)$; % centered is usually the best: more accurate

Problem Set 8.2, page 428

1 $A=\begin{bmatrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 1\end{bmatrix}$; nullspace contains $\begin{bmatrix}c\\c\\c\end{bmatrix}$; $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{0}$ for $\boldsymbol{y}=(1,-1,1)$; current along edge 1, edge 3, back on edge 2 (full loop).

3 Elimination on $b_1[A \;\; \boldsymbol{b}] = \begin{bmatrix} -1 & 1 & 0 & b_1 \\ -1 & 0 & 1 & b_2 \\ 0 & -1 & 1 & b_3 \end{bmatrix}$ leads to $[U \;\; \boldsymbol{c}] = \begin{bmatrix} -1 & 1 & 0 & b_1 \\ 0 & -1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 + b_1 \end{bmatrix}$. The nonzero rows of U come from edges 1 and 3 in a tree. The zero row comes from the loop (all 3 edges).

4 For the matrix in Problem 3, $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for $\boldsymbol{b} = (1,1,0)$ and not solvable for $\boldsymbol{b} = (1,0,0)$. For solvable $\boldsymbol{b}$ (in the column space), $\boldsymbol{b}$ must be orthogonal to $\boldsymbol{y} = (1,-1,1)$; that combination of rows is the zero row, and $b_1 - b_2 + b_3 = 0$ is the third equation after elimination.

5 Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is solvable for $\boldsymbol{f} = (1,-1,0)$ and not solvable for $\boldsymbol{f} = (1,0,0)$; $\boldsymbol{f}$ must be orthogonal to $(1,1,1)$ in the nullspace: $f_1 + f_2 + f_3 = 0$.

6 $A^{\mathrm{T}}A\boldsymbol{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \boldsymbol{f}$ produces $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{x} = 1, -1, 0$ and currents $-A\boldsymbol{x} = 2, 1, -1$; $\boldsymbol{f}$ sends 3 units from node 2 into node 1.

7 $A^{\mathrm{T}}\begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}$; $\boldsymbol{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ yields $\boldsymbol{x} = \begin{bmatrix} 5/4 \\ 1 \\ 7/8 \end{bmatrix} +$ any $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\boldsymbol{x} = \frac{5}{4}, 1, \frac{7}{8}$ and currents $-CA\boldsymbol{x} = \frac{1}{4}, \frac{3}{4}, \frac{1}{4}$.

8 $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ leads to $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ solving $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$.

9 Elimination on $A\boldsymbol{x} = \boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ in the zero rows of U and R: $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (those $\boldsymbol{y}$'s are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage* Law around the two *loops*.

10 The echelon form of A is $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ The nonzero rows of U keep edges 1, 2, 4. Other spanning trees from edges, 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5.

11 $A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$
diagonal entry = number of edges into the node
the trace is 2 times the number of nodes
off-diagonal entry = -1 if nodes are connected
$A^{\mathrm{T}}A$ is the **graph Laplacian**, $A^{\mathrm{T}}CA$ is **weighted** by C

12 (a) The nullspace and rank of $A^{\mathrm{T}}A$ and A are always the same (b) $A^{\mathrm{T}}A$ is always positive semidefinite because $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} = \|A\boldsymbol{x}\|^2 \geq 0$. Not positive definite because rank is only 3 and $(1,1,1,1)$ is in the nullspace (c) Real eigenvalues all ≥ 0 because positive semidefinite.

13 $A^{\mathrm{T}}CA\boldsymbol{x} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ gives four potentials $\boldsymbol{x} = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0)$ I grounded $x_4 = 0$ and solved for $\boldsymbol{x}$ currents $\boldsymbol{y} = -CA\boldsymbol{x} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2})$

14 $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{0}$ for $\boldsymbol{x} = c(1,1,1,1) = (c,c,c,c)$. If $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$ is solvable, then $\boldsymbol{f}$ in the column space (= row space by symmetry) must be orthogonal to $\boldsymbol{x}$ in the nullspace: $f_1 + f_2 + f_3 + f_4 = 0$.

15 The number of loops in this connected graph is $n - m + 1 = \mathbf{7 - 7 + 1 = 1}$. What answer if the graph has two separate components (no edges between)?

16 Start from (4 nodes) − (6 edges) + (3 loops) = 1. If a new node connects to 1 old node, $5 - 7 + 3 = 1$. If the new node connects to 2 old nodes, a new loop is formed: $5 - 8 + 4 = 1$.

17 (a) 8 independent columns (b) $\boldsymbol{f}$ must be orthogonal to the nullspace so f's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

18 A *complete graph* has $5 + 4 + 3 + 2 + 1 = 15$ edges. With n nodes that count is $1 + \cdots + (n-1) = n(n-1)/2$. Tree has 5 edges.

Problem Set 8.3, page 437

1 Eigenvalues $\lambda = 1$ and $.75$; $(A - I)\boldsymbol{x} = 0$ gives the steady state $\boldsymbol{x} = (.6, .4)$ with $A\boldsymbol{x} = \boldsymbol{x}$.

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; $A^{\infty} = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $\boldsymbol{x} = (1, 0)$; 1 and $-.8$, $\boldsymbol{x} = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $\boldsymbol{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

4 A^{T} always has the eigenvector $(1, 1, \ldots, 1)$ for $\lambda = 1$, because each row of A^{T} adds to 1. (Note again that many authors use row vectors multiplying Markov matrices. So they transpose our form of A.)

5 The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1)$ = everyone is dead.

6 Add the components of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to find sum $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

7 $(.5)^k \to 0$ gives $A^k \to A^{\infty}$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $\begin{matrix} a \leq 1 \\ .4 + .6a \geq 0 \end{matrix}$

8 If P = cyclic permutation and $\boldsymbol{u}_0 = (1, 0, 0, 0)$ then $\boldsymbol{u}_1 = (0, 0, 1, 0)$; $\boldsymbol{u}_2 = (0, 1, 0, 0)$; $\boldsymbol{u}_3 = (1, 0, 0, 0)$; $\boldsymbol{u}_4 = \boldsymbol{u}_0$. The eigenvalues $1, i, -1, -i$ are all *on the unit circle*. This Markov matrix contains zeros; a *positive* matrix has *one* largest eigenvalue $\lambda = 1$.

9 M^2 is still nonnegative; $[\,1 \ \cdots \ 1\,]M = [\,1 \ \cdots \ 1\,]$ so multiply on the right by M to find $[\,1 \ \cdots \ 1\,]M^2 = [\,1 \ \cdots \ 1\,] \Rightarrow$ columns of M^2 add to 1.

10 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $\boldsymbol{x}_1 = (b, 1 - a)$.

11 Last row .2, .3, .5 makes $A = A^{\mathrm{T}}$; rows also add to 1 so $(1, \ldots, 1)$ is also an eigenvector of A.

12 B has $\lambda = 0$ and $-.5$ with $\boldsymbol{x}_1 = (.3, .2)$ and $\boldsymbol{x}_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t}\boldsymbol{x}_1 = c_1\boldsymbol{x}_1$.

13 $\boldsymbol{x} = (1, 1, 1)$ is an eigenvector when the row sums are equal; $A\boldsymbol{x} = (.9, .9, .9)$

14 $(I-A)(I+A+A^2+\cdots) = (I+A+A^2+\cdots)-(A+A^2+A^3+\cdots) = \boldsymbol{I}$. This says that $I + A + A^2 + \cdots$ is $(I-A)^{-1}$. When $A = \begin{bmatrix} 0 & .5 \\ 1 & 0 \end{bmatrix}$, $A^2 = \frac{1}{2}I$, $A^3 = \frac{1}{2}A$, $A^4 = \frac{1}{4}I$ and the series adds to $\begin{bmatrix} 1+\frac{1}{2}+\cdots & \frac{1}{2}+\frac{1}{4}+\cdots \\ 1+\frac{1}{2}+\cdots & 1+\frac{1}{2}+\cdots \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = (I-A)^{-1}$.

15 The first two A's have $\lambda_{\max} < 1$; $\boldsymbol{p} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

16 $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state (.3, .3, .4) and (30, 30, 40).

17 *No*, A has an eigenvalue $\lambda = 1$ and $(I-A)^{-1}$ does not exist.

18 The Leslie matrix on page 435 has $\det(A-\lambda I) = \det \begin{bmatrix} F_1-\lambda & F_2 & F_3 \\ P_1 & -\lambda & 0 \\ 0 & P_2 & -\lambda \end{bmatrix} = -\lambda^3 + F_1\lambda^2 + F_2P_1\lambda + F_3P_1P_2$. This is negative for large λ. It is positive at $\lambda = 1$ provided that $F_1 + F_2P_1 + F_3P_1P_2 > 1$. Under this key condition, $\det(A-\lambda I)$ must be zero at some λ between 1 and ∞. That eigenvalue means that the population grows (under this condition connecting F's and P's reproduction and survival rates).

19 Λ times $S^{-1}\Delta S$ has the same diagonal as $S^{-1}\Delta S$ times Λ because Λ is diagonal.

20 If $B > A > 0$ and $A\boldsymbol{x} = \lambda_{\max}(A)\boldsymbol{x} > 0$ then $B\boldsymbol{x} > \lambda_{\max}(A)\boldsymbol{x}$ and $\lambda_{\max}(B) > \lambda_{\max}(A)$.

Problem Set 8.4, page 446

1 Feasible set = line segment $(6,0)$ to $(0,3)$; minimum cost at $(6,0)$, maximum at $(0,3)$.

2 Feasible set has corners $(0,0)$, $(6,0)$, $(2,2)$, $(0,6)$. Minimum cost $2x - y$ at $(6,0)$.

3 Only two corners $(4,0,0)$ and $(0,2,0)$; let $x_i \to -\infty$, $x_2 = 0$, and $x_3 = x_1 - 4$.

4 From $(0,0,2)$ move to $\boldsymbol{x} = (0,1,1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $\boldsymbol{x} = (1,0,1.5)$ with cost $5(1) + 8(1.5) = \$17$; $r = 1$ means more expensive.

5 Cost = 20 at start $(4,0,0)$; keeping $x_1+x_2+2x_3 = 4$ move to $(3,1,0)$ with cost 18 and $r = -2$; or move to $(2,0,1)$ with cost 17 and $r = -3$. Choose x_3 as entering variable and move to $(0,0,2)$ with cost 14. Another step will reach $(0,4,0)$ with minimum cost 12.

6 If we reduce the Ph.D. cost to \$1 or \$2 (below the student cost of \$3), the job will go to the Ph.D. with cost vector $\boldsymbol{c} = (2,3,8)$ the Ph.D. takes 4 hours ($x_1 + x_2 + 2x_3 = 4$) and charges \$8.

The teacher in the dual problem now has $y \le 2, y \le 3, 2y \le 8$ as constraints $A^{\mathrm{T}}\boldsymbol{y} \le \boldsymbol{c}$ on the charge of y per problem. So the dual has maximum at $y = 2$. The dual cost is also \$8 for 4 problems and maximum = minimum.

7 $\boldsymbol{x} = (2,2,0)$ is a corner of the feasible set with $x_1 + x_2 + 2x_3 = 4$ and the new constraint $2x_1 + x_2 + x_3 = 6$. The cost of this corner is $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = (5,3,8)\cdot(2,2,0) = 16$. Is this the minimum cost?

Compute the reduced cost r if $x_3 = 1$ enters (x_3 was previously zero). The two constraint equations now require $x_1 = 3$ and $x_2 = -1$. With $\boldsymbol{x} = (3,-1,1)$ the new cost is $3.5 - 1.3 + 1.8 = 20$. This is higher than 16, so the original $\boldsymbol{x} = (2, 2, 0)$ was optimal.

Note that $x_3 = 1$ led to $x_2 = -1$ and a negative x_2 is not allowed. If x_3 reduced the cost (it didn't) we would not have used as much as $x_3 = 1$.

8 $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{x} = (A^{\mathrm{T}}\boldsymbol{y})^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$. The first inequality needed $\boldsymbol{y} \geq 0$ and $A\boldsymbol{x} - \boldsymbol{b} \geq 0$.

Problem Set 8.5, page 451

1 $\int_0^{2\pi} \cos((j+k)x)\,dx = \left[\frac{\sin((j+k)x)}{j+k}\right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos((j-k)x)\,dx = 0$ Notice $j - k \neq 0$ in the denominator. If $j = k$ then $\int_0^{2\pi} \cos^2 jx\,dx = \pi$.

2 Three integral tests show that $1, x, x^2 - \frac{1}{3}$ are orthogonal on the interval $[-1,\ 1]$: $\int_{-1}^{1}(1)(x)\ \ dx = 0, \int_{-1}^{1}(1)(x^2 - \frac{1}{3})\ \ dx = 0, \int_{-1}^{1}(x)(x^2 - \frac{1}{3})\ \ dx = 0$. Then $2x^2 = 2(x^2 - \frac{1}{3}) + 0(x) + \frac{2}{3}(1)$. Those coefficients $2, 0, \frac{2}{3}$ can come from integrating $f(x) = 2x^2$ times the 3 basis functions and dividing by their lengths squared—in other words using $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/a^{\mathrm{T}}a$ for functions (where $\boldsymbol{b}$ is $f(x)$ and $\boldsymbol{a}$ is 1 or x or $x^2 - \frac{1}{3}$) exactly as for vectors.

3 One example orthogonal to $\boldsymbol{v} = (1, \frac{1}{2}, \ldots)$ is $\boldsymbol{w} = (2, -1, 0, 0, \ldots)$ with $\|\boldsymbol{w}\| = \sqrt{5}$.

4 $\int_{-1}^{1}(1)(x^3 - cx)\,dx = 0$ and $\int_{-1}^{1}(x^2 - \frac{1}{3})(x^3 - cx)\,dx = 0$ for all c (odd functions). Choose c so that $\int_{-1}^{1} x(x^3 - cx)\,dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^{1} = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.

5 The integrals lead to the Fourier coefficients $a_1 = 0,\ b_1 = 4/\pi,\ b_2 = 0$.

6 From eqn. (3) $a_k = 0$ and $b_k = 4/\pi k$ (*odd* k). The square wave has $\|f\|^2 = 2\pi$. Then eqn. (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots)$. That infinite series equals $\pi^2/8$.

7 The $-1, 1$ odd square wave is $f(x) = x/|x|$ for $0 < |x| < \pi$. Its Fourier series in equation (8) is $4/\pi$ times $[\sin x + (\sin 3x)/3 + (\sin 5x/5) + \cdots]$. The sum of the first N terms has an interesting shape, close to the square wave except where the wave jumps between -1 and 1. At those jumps, the Fourier sum spikes the wrong way to ± 1.09 (the *Gibbs phenomenon*) before it takes the jump with the true $f(x)$.

This happens for the Fourier sums of all functions with jumps. It makes shock waves hard to compute. You can see it clearly in a graph of the sum of 10 terms.

8 $\|\boldsymbol{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ so $\|\boldsymbol{v}\| = \sqrt{2}$; $\|\boldsymbol{v}\|^2 = 1 + a^2 + a^4 + \cdots = 1/(1 - a^2)$ so $\|\boldsymbol{v}\| = 1/\sqrt{1 - a^2}$; $\int_0^{2\pi}(1 + 2\sin x + \sin^2 x)\,dx = 2\pi + 0 + \pi$ so $\|f\| = \sqrt{3\pi}$.

9 (a) $f(x) = (1 + \text{square wave})/2$ so the a's are $\frac{1}{2}$, 0, 0, … and the b's are $2/\pi$, 0, $-2/3\pi, 0, 2/5\pi, \ldots$ (b) $a_0 = \int_0^{2\pi} x\,dx/2\pi = \pi$, all other $a_k = 0, b_k = -2/k$.

10 The integral from $-\pi$ to π or from 0 to 2π (or from any a to $a + 2\pi$) is over one complete period of the function. If $f(x)$ is periodic this changes $\int_0^{2\pi} f(x)\,dx$ to $\int_0^{\pi} f(x)\,dx + \int_{-\pi}^{0} f(x)\,dx$. If $f(x)$ is **odd**, those integrals cancel to give $\int f(x)\,dx = 0$ over one period.

11 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$; $\cos(x + \frac{\pi}{3}) = \cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$.

12 $\dfrac{d}{dx}\begin{bmatrix}1\\ \cos x\\ \sin x\\ \cos 2x\\ \sin 2x\end{bmatrix}=\begin{bmatrix}0\\ -\sin x\\ \cos x\\ -2\sin 2x\\ 2\cos 2x\end{bmatrix}=\begin{bmatrix}0&0&0&0&0\\ 0&0&-1&0&0\\ 0&1&0&0&0\\ 0&0&0&0&-2\\ 0&0&0&2&0\end{bmatrix}\begin{bmatrix}1\\ \cos x\\ \sin x\\ \cos 2x\\ \sin 2x\end{bmatrix}$. This shows the differentiation matrix.

13 The square pulse with $F(x) = 1/h$ for $-x \le h/2 \le x$ is an even function, so all sine coefficients b_k are zero. The average a_0 and the cosine coefficients a_k are

$$a_0 = \frac{1}{2\pi}\int_{-h/2}^{h/2}(1/h)dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi}\int_{-h/2}^{h/2}(1/h)\cos kx\,dx = \frac{2}{\pi kh}\left(\sin\frac{kh}{2}\right) \text{ which is } \frac{1}{\pi}\operatorname{sinc}\left(\frac{kh}{2}\right)$$

(introducing the sinc function $(\sin x)/x$). As h approaches zero, the number $x = kh/2$ approaches zero, and $(\sin x)/x$ approaches 1. So all those a_k approach $1/\pi$.

The limiting "delta function" contains an equal amount of all cosines: a very irregular function.

Problem Set 8.6, page 458

1 The diagonal matrix $C = W^{\mathrm{T}}W$ is $\Sigma^{-1} = \begin{bmatrix}1&&\\&1&\\&&1/2\end{bmatrix}$ with no covariances (independent trials). Then solve $A^{\mathrm{T}}CA\widehat{\boldsymbol{x}} = A^{\mathrm{T}}C\boldsymbol{b}$ for this weighted least squares problem (notice $Ct + D$ instead of $C + Dt$):

$$A\boldsymbol{x} = \widehat{\boldsymbol{b}} \quad\text{is}\quad \begin{matrix}0C + D = 1\\ 1C + D = 2\\ 2C + D = 4\end{matrix} \quad\text{or}\quad \begin{bmatrix}0&1\\1&1\\2&1\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}1\\2\\4\end{bmatrix}.$$

$$A^{\mathrm{T}}CA = \begin{bmatrix}3&2\\2&2.5\end{bmatrix} \qquad A^{\mathrm{T}}C\boldsymbol{b} = \begin{bmatrix}6\\5\end{bmatrix} \quad \widehat{\boldsymbol{x}} = \begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}10/7\\6/7\end{bmatrix}.$$

2 If the measurement b_3 is totally unreliable and $\sigma_3^2 = \infty$, then *the best line will not use* b_3. In this example, the system $A\boldsymbol{x} = \boldsymbol{b}$ becomes square (first two equations from Problem 1):

$$\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \text{ gives } \begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}. \text{ The line } b = \boldsymbol{t+1} \text{ fits exactly.}$$

3 If $\sigma_3 = 0$ the third equation is exact. Then the best line has $Ct + D = b_3$ which is $2C + D = 4$. The errors $Ct + D - b$ in the measurements at $t = 0$ and 1 are $D - 1$ and $C + D - 2$. Since $D = 4 - 2C$ from the exact $b_3 = 4$, those two errors are $D - 1 = 3 - 2C$ and $C + D - 2 = 2 - C$. The sum of squares $(3 - 2C)^2 + (2 - C)^2$ is a minimum at $8 = 5C$ (calculus or linear algebra in 1D). Then $C = 8/5$ and $D = 4 - 2C = 4/5$.

4 $0, 1, 2$ have probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ and $\sigma^2 = (0-1)^2\frac{1}{4} + (1-1)^2\frac{1}{2} + (2-1)^2\frac{1}{4} = \frac{1}{2}$.

5 Mean $(\frac{1}{2}, \frac{1}{2})$. Independent flips lead to $\mathbf{\Sigma} = \mathbf{diag}(\frac{1}{4}, \frac{1}{4})$. Trace $= \sigma^2_{\text{total}} = \frac{1}{2}$.

6 Mean $m = p_0$ and variance $\sigma^2 = (1-p_0)^2 p_0 + (0-p_0)^2(1-p_0) = p_0(1-p_0)$.

7 Minimize $P = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$ at $P' = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0$; $a = \sigma_2^2/(\sigma_1^2+\sigma_2^2)$ recovers equation (2) for the statistically correct choice with minimum variance.

8 Multiply $L\mathbf{\Sigma}L^{\mathrm{T}} = (A^{\mathrm{T}}\mathbf{\Sigma}^{-1}A)^{-1}A^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}A(A^{\mathrm{T}}\mathbf{\Sigma}^{-1}A)^{-1} = P = (A^{\mathrm{T}}\mathbf{\Sigma}^{-1}A)^{-1}$.

9 The new grade matrix A has row 3 $= -$ row 1 and row 4 $= -$ row 2, so the rank is 7. The nullspace of A now includes $(1, -1, -1, 1)$ as well as $(1, 1, 1, 1)$. Compare to the grade matrix in Example 6 (not Example 5). The other two singular vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ for Example 6 are still correct for this new $A(A\boldsymbol{v}_1$ is still orthogonal to $A\boldsymbol{v}_2)$:

$$A\begin{bmatrix} 2\boldsymbol{v}_1 & 2\boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 & -3 \\ -1 & 3 & -3 & 1 \\ -3 & 1 & -1 & -3 \\ 1 & -3 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -8 & -4 \\ -8 & 4 \\ 8 & 4 \end{bmatrix}.$$

Those last orthogonal columns are multiples of the orthonormal $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. This matrix A has $\sigma_1 = 8$ and $\sigma_2 = 4$ (only two singular values since the rank is 2). If you compute $A^{\mathrm{T}}A$ to find those singular vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ from scratch, notice that its trace is $\sigma_1^2 + \sigma_2^2 = 64 + 16 = 80$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 20 & -12 & -20 & 12 \\ -12 & 20 & 12 & -20 \\ -20 & 12 & 20 & -12 \\ 12 & -20 & -12 & 20 \end{bmatrix}.$$

Problem Set 8.7, page 463

1 (x, y, z) has homogeneous coordinates (cx, cy, cz, c) for $c = 1$ and all $c \neq 0$.

2 For an affine transformation we also need T (origin), because $T(\mathbf{0})$ need not be $\mathbf{0}$ for affine T. Including this translation by $T(\mathbf{0})$, $(x, y, z, 1)$ is transformed to $xT(\boldsymbol{i}) + yT(\boldsymbol{j}) + zT(\boldsymbol{k}) + T(\mathbf{0})$.

3 $TT_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 6 & 8 & 1 \end{bmatrix}$ is translation along $(1, 6, 8)$.

4 $S = \operatorname{diag}(c, c, c, 1)$; row 4 of ST and TS is $1, 4, 3, 1$ and $c, 4c, 3c, 1$; use $\boldsymbol{v}TS$!

5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square, starting from an 8.5 by 11 page.

6 $[x\ y\ z\ 1]\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} = [x\ y\ z\ 1]\begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ -2 & -2 & -4 & 1 \end{bmatrix}.$

The first matrix translates by $(-1, -1, -2)$. The second matrix rescales by 2.

7 The three parts of Q in equation (1) are $(\cos\theta)I$ and $(1-\cos\theta)\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}$ and $-\sin\theta(\boldsymbol{a}\times)$. Then $Q\boldsymbol{a}=\boldsymbol{a}$ because $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}=\boldsymbol{a}$(unit vector) and $\boldsymbol{a}\times\boldsymbol{a}=\mathbf{0}$.

8 If $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}=0$ and those three parts of Q (Problem 7) multiply $\boldsymbol{b}$, the results in $Q\boldsymbol{b}$ are $(\cos\theta)\boldsymbol{b}$ and $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}=\mathbf{0}$ and $(-\sin\theta)\boldsymbol{a}\times\boldsymbol{b}$. The component along $\boldsymbol{b}$ is $(\cos\theta)\boldsymbol{b}$.

9 $\boldsymbol{n}=\left(\dfrac{2}{3},\dfrac{2}{3},\dfrac{1}{3}\right)$ has $P=I-\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}=\dfrac{1}{9}\begin{bmatrix}5&-4&-2\\-4&5&-2\\-2&-2&8\end{bmatrix}$. Notice $\|\boldsymbol{n}\|=1$.

10 We can choose $(0,0,3)$ on the plane and multiply $T_{-}PT_{+}=\frac{1}{9}\begin{bmatrix}5&-4&-2&0\\-4&5&-2&0\\-2&-2&8&0\\6&6&3&9\end{bmatrix}$.

11 $(3,3,3)$ projects to $\frac{1}{3}(-1,-1,4)$ and $(3,3,3,1)$ projects to $(\frac{1}{3},\frac{1}{3},\frac{5}{3},1)$. Row vectors!

12 The projection of a square onto a plane is a parallelogram (or a line segment). The sides of the square are perpendicular, but their projections may not be ($\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}=0$ but $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y})=\boldsymbol{x}^{\mathrm{T}}P^{\mathrm{T}}P\boldsymbol{y}=\boldsymbol{x}^{\mathrm{T}}P\boldsymbol{y}$ may be nonzero).

13 That projection of a cube onto a plane produces a hexagon.

14 $(3,3,3)(I-2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})=\left(\dfrac{1}{3},\dfrac{1}{3},\dfrac{1}{3}\right)\begin{bmatrix}1&-8&-4\\-8&1&-4\\-4&-4&7\end{bmatrix}=\left(-\dfrac{11}{3},-\dfrac{11}{3},-\dfrac{1}{3}\right)$.

15 $(3,3,3,1)\to(3,3,0,1)\to\left(-\frac{7}{3},-\frac{7}{3},-\frac{8}{3},1\right)\to\left(-\frac{7}{3},-\frac{7}{3},\frac{1}{3},1\right)$.

16 Just subtracting vectors would give $\boldsymbol{v}=(x,y,z,0)$ ending in 0 (not 1). In homogeneous coordinates, add a **vector** to a point.

17 Space is rescaled by $1/c$ because (x,y,z,c) is the same point as $(x/c,y/c,z/c,1)$.

Problem Set 9.1, page 472

1 Without exchange, pivots .001 and 1000; with exchange, 1 and -1. When the pivot is larger than the entries below it, all $|\ell_{ij}|=|\text{entry/pivot}|\leq 1$. $A=\begin{bmatrix}1&1&1\\0&1&-1\\-1&1&1\end{bmatrix}$.

2 The exact inverse of hilb(3) is $A^{-1}=\begin{bmatrix}9&-36&30\\-36&192&-180\\30&-180&180\end{bmatrix}$.

3 $A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}11/6\\13/12\\47/60\end{bmatrix}=\begin{bmatrix}1.833\\1.083\\0.783\end{bmatrix}$ compares with $A\begin{bmatrix}0\\6\\-3.6\end{bmatrix}=\begin{bmatrix}1.80\\1.10\\0.78\end{bmatrix}$. $\|\Delta\boldsymbol{b}\|<.04$ but $\|\Delta\boldsymbol{x}\|>6$.
The difference $(1,1,1)-(0,6,-3.6)$ is in a direction $\Delta\boldsymbol{x}$ that has $A\Delta\boldsymbol{x}$ near zero.

4 The largest $\|\boldsymbol{x}\|=\|A^{-1}\boldsymbol{b}\|$ is $\|A^{-1}\|=1/\lambda_{\min}$ since $A^{\mathrm{T}}=A$; largest error $10^{-16}/\lambda_{\min}$.

5 Each row of U has at most w entries. Then w multiplications to substitute components of $\boldsymbol{x}$ (already known from below) and divide by the pivot. Total for n rows $<wn$.

6 The triangular L^{-1}, U^{-1}, R^{-1} need $\frac{1}{2}n^2$ multiplications. Q needs n^2 to multiply the right side by $Q^{-1}=Q^{\mathrm{T}}$. So $QR\boldsymbol{x}=\boldsymbol{b}$ takes 1.5 times longer than $LU\boldsymbol{x}=\boldsymbol{b}$.

7 $UU^{-1} = I$: Back substitution needs $\frac{1}{2}j^2$ multiplications on column j, using the j by j upper left block. Then $\frac{1}{2}(1^2 + 2^2 + \cdots + n^2) \approx \frac{1}{2}(\frac{1}{3}n^3) =$ total to find U^{-1}.

8 $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = U$ with $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ .5 & 1 \end{bmatrix}$;

$A \to \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$ with

$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & -.5 & 1 \end{bmatrix}$.

9 $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ has cofactors $C_{13} = C_{31} = C_{24} = C_{42} = 1$ and $C_{14} = C_{41} = -1$. A^{-1} is a full matrix!

10 With 16-digit floating point arithmetic the errors $\|\boldsymbol{x} - \boldsymbol{x}_{\text{computed}}\|$ for $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$ are of order $10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}$.

11 (a) $\cos\theta = 1/\sqrt{10}$, $\sin\theta = -3/\sqrt{10}$, $R = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$.
(b) A has eigenvalues 4 and 2. Put one of the unit eigenvectors in row 1 of Q: either $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $QAQ^{-1} = \begin{bmatrix} 2 & -4 \\ 0 & 4 \end{bmatrix}$ **or** $Q = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ and $QAQ^{-1} = \begin{bmatrix} 4 & -4 \\ 0 & 2 \end{bmatrix}$.

12 When A is multiplied by a plane rotation Q_{ij}, this changes the $2n$ (not n^2) entries in rows i and j. Then multiplying on the right by $(Q_{ij})^{-1} = (Q_{ij})^{\mathrm{T}}$ changes the $2n$ entries in columns i and j.

13 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos\theta$, the entries 1 and $\pm\tan\theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR.

14 The $(2,1)$ entry of $Q_{21}A$ is $\frac{1}{3}(-\sin\theta + 2\cos\theta)$. This is zero if $\sin\theta = 2\cos\theta$ or $\tan\theta = 2$. Then the $2, 1, \sqrt{5}$ right triangle has $\sin\theta = 2/\sqrt{5}$ and $\cos\theta = 1/\sqrt{5}$.

Every 3 by 3 rotation with $\det Q = +1$ is the product of 3 plane rotations.

15 This problem shows how elimination is more expensive (the nonzero multipliers are counted by **nnz**(L) and **nnz**(LL)) when we spoil the tridiagonal K by a random permutation.

If on the other hand we start with a poorly ordered matrix K, an improved ordering is found by the code **symamd** discussed in this section.

16 The "red-black ordering" puts rows and columns 1 to 10 in the odd-even order $1, 3, 5, 7, 9, 2, 4, 6, 8, 10$. When K is the $-1, 2, -1$ tridiagonal matrix, odd points are connected only to even points (and 2 stays on the diagonal, connecting every point to itself):

$$K = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ & & -1 & 2 \end{bmatrix} \text{ and } PKP^{\mathrm{T}} = \begin{bmatrix} 2I & D \\ D^{\mathrm{T}} & 2I \end{bmatrix} \text{ with}$$

$$D = \begin{bmatrix} -1 & & & & \\ -1 & -1 & & & \\ 0 & -1 & -1 & & \\ & & -1 & -1 & \\ & & & -1 & -1 \end{bmatrix} \begin{array}{l} 1 \text{ to } 2 \\ 3 \text{ to } 2,4 \\ 5 \text{ to } 4,6 \\ 7 \text{ to } 6,8 \\ 9 \text{ to } 8,10 \end{array}$$

17 Jeff Stuart's **Shake a Stick** activity has long sticks representing the graphs of two linear equations in the x-y plane. The matrix is nearly singular and Section 9.2 shows how to compute its condition number $c = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min} \approx 80,000$:

$$A = \begin{bmatrix} 1 & 1.0001 \\ 1 & 1.0000 \end{bmatrix} \|A\| \approx 2 \quad A^{-1} = 10000 \begin{bmatrix} -1 & 1.0001 \\ 1 & -1 \end{bmatrix} \quad \begin{array}{l} \|A^{-1}\| \approx 20000 \\ c \approx 40000. \end{array}$$

Problem Set 9.2, page 478

1 $\|A\| = 2$, $\|A^{-1}\| = 2, c = 4$; $\|A\| = 3$, $\|A^{-1}\| = 1, c = 3$; $\|A\| = 2 + \sqrt{2} = \lambda_{\max}$ for positive definite A, $\|A^{-1}\| = 1/\lambda_{\min}$, $c = (2+\sqrt{2})/(2-\sqrt{2}) = 5.83$.

2 $\|A\| = 2$, $c = 1$; $\|A\| = \sqrt{2}$, $c =$ infinite (singular matrix); $A^{\mathrm{T}}A = 2I$, $\|A\| = \sqrt{2}$, $c = 1$.

3 For the first inequality replace $\boldsymbol{x}$ by $B\boldsymbol{x}$ in $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$; the second inequality is just $\|B\boldsymbol{x}\| \le \|B\|\|\boldsymbol{x}\|$. Then $\|AB\| = \max(\|AB\boldsymbol{x}\|/\|\boldsymbol{x}\|) \le \|A\|\|B\|$.

4 $1 = \|I\| = \|AA^{-1}\| \le \|A\|\|A^{-1}\| = c(A)$.

5 If $\Lambda_{\max} = \Lambda_{\min} = 1$ then all $\Lambda_i = 1$ and $A = SIS^{-1} = I$. The only matrices with $\|A\| = \|A^{-1}\| = 1$ are *orthogonal matrices*.

6 All orthogonal matrices have norm 1, so $\|A\| \le \|Q\|\|R\| = \|R\|$ and in reverse $\|R\| \le \|Q^{-1}\|\|A\| = \|A\|$, then $\|A\| = \|R\|$. Inequality is usual in $\|A\| < \|L\|\|U\|$ when $A^{\mathrm{T}}A \ne AA^{\mathrm{T}}$. Use **norm** on a random A.

7 The triangle inequality gives $\|A\boldsymbol{x} + B\boldsymbol{x}\| \le \|A\boldsymbol{x}\| + \|B\boldsymbol{x}\|$. Divide by $\|\boldsymbol{x}\|$ and take the maximum over all nonzero vectors to find $\|A + B\| \le \|A\| + \|B\|$.

8 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\|A\boldsymbol{x}\|/\|\boldsymbol{x}\| = |\lambda|$ for that particular vector $\boldsymbol{x}$. When we maximize the ratio over all vectors we get $\|A\| \ge |\lambda|$.

9 $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\rho(A) = 0$ and $\rho(B) = 0$ but $\rho(A+B) = 1$. The triangle inequality $\|A+B\| \le \|A\| + \|B\|$ fails for $\rho(A)$. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ also has $\rho(AB) = 1$; thus $\rho(A) = \max|\lambda(A)|$ = spectral radius is not a norm.

10 (a) The condition number of A^{-1} is $\|A^{-1}\|\|(A^{-1})^{-1}\|$ which is $\|A^{-1}\|\|A\| = c(A)$. (b) Since $A^{\mathrm{T}}A$ and AA^{T} have the same nonzero eigenvalues, A and A^{T} have the same norm.

11 Use the quadratic formula for $\lambda_{\max}/\lambda_{\min}$, which is $c = \sigma_{\max}/\sigma_{\min}$ since this $A = A^{\mathrm{T}}$ is positive definite:

$$c(A) = \left(1.00005 + \sqrt{(1.00005)^2 - .0001}\right) / \left(1.00005 - \sqrt{}\right) \approx 40,000.$$

12 $\det(2A)$ is not $2\det A$; $\det(A+B)$ is not always less than $\det A + \det B$; taking $|\det A|$ does not help. The only reasonable property is $\det AB = (\det A)(\det B)$. The condition number should not change when A is multiplied by 10.

13 The residual $\boldsymbol{b} - A\boldsymbol{y} = (10^{-7}, 0)$ is much smaller than $\boldsymbol{b} - A\boldsymbol{z} = (.0013, .0016)$. But $\boldsymbol{z}$ is much closer to the solution than $\boldsymbol{y}$.

14 $\det A = 10^{-6}$ so $A^{-1} = 10^3 \begin{bmatrix} 659 & -563 \\ -913 & 780 \end{bmatrix}$: $\|A\| > 1$, $\|A^{-1}\| > 10^6$, then $c > 10^6$.

15 $\boldsymbol{x} = (1,1,1,1,1)$ has $\|\boldsymbol{x}\| = \sqrt{5}$, $\|\boldsymbol{x}\|_1 = 5$, $\|\boldsymbol{x}\|_\infty = 1$. $\boldsymbol{x} = (.1, .7, .3, .4, .5)$ has $\|\boldsymbol{x}\| = 1$, $\|\boldsymbol{x}\|_1 = 2$ (sum) $\|\boldsymbol{x}\|_\infty = .7$ (largest).

16 $x_1^2 + \cdots + x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $(|x_1| + \cdots + |x_n|)^2 = \|\boldsymbol{x}\|_1^2$. $x_1^2 + \cdots + x_n^2 \le n \max(x_i^2)$ so $\|\boldsymbol{x}\| \le \sqrt{n}\|\boldsymbol{x}\|_\infty$. Choose $y_i = \operatorname{sign} x_i = \pm 1$ to get $\|\boldsymbol{x}\|_1 = \boldsymbol{x} \cdot \boldsymbol{y} \le \|\boldsymbol{x}\|\|\boldsymbol{y}\| = \sqrt{n}\|\boldsymbol{x}\|$. $\boldsymbol{x} = (1, \ldots, 1)$ has $\|\boldsymbol{x}\|_1 = \sqrt{n}\,\|\boldsymbol{x}\|$.

17 For the ℓ^∞ norm, the largest component of $\boldsymbol{x}$ plus the largest component of $\boldsymbol{y}$ is not less than $\|\boldsymbol{x} + \boldsymbol{y}\|_\infty$ = largest component of $\boldsymbol{x} + \boldsymbol{y}$.

For the ℓ^1 norm, each component has $|x_i + y_i| \le |x_i| + |y_i|$. Sum on $i = 1$ to n: $\|\boldsymbol{x} + \boldsymbol{y}\|_1 \le \|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1$.

18 $|x_1| + 2|x_2|$ is a norm but $\min(|x_1|, |x_2|)$ is not a norm. $\|\boldsymbol{x}\| + \|\boldsymbol{x}\|_\infty$ is a norm; $\|A\boldsymbol{x}\|$ is a norm provided A is invertible (otherwise a nonzero vector has norm zero; for rectangular A we require independent columns to avoid $\|A\boldsymbol{x}\| = 0$).

19 $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = x_1y_1 + x_2y_2 + \cdots \le (\max|y_i|)(|x_1| + |x_2| + \cdots) = ||\boldsymbol{x}||_1\,||\boldsymbol{y}||_\infty$.

20 With $\lambda_j = 2 - 2\cos(j\pi/n+1)$, the largest eigenvalue is $\lambda_n \approx 2 + 2 = 4$. The smallest is $\lambda_1 = 2 - 2\cos(\pi/n+1) \approx \left(\frac{\pi}{n+1}\right)^2$, using $2\cos\theta \approx 2 - \theta^2$. So the condition number is $c = \lambda_{\max}/\lambda_{\min} \approx (4/\pi^2)\,n^2$, growing with n.

21 $A = \begin{bmatrix} 1 & 1 \\ 0 & 1.1 \end{bmatrix}$ has $A^n = \begin{bmatrix} 1 & q \\ 0 & (1.1)^n \end{bmatrix}$ with $q = 1 + 1.1 + \cdots + (1.1)^{n-1} = (1.1^n - 1)/(1.1 - 1) \approx 1.1^n/.1$. So the growing part of A^n is $1.1^n \begin{bmatrix} 0 & 10 \\ 0 & 1 \end{bmatrix}$ with $||A^n|| \approx \sqrt{101}$ times 1.1^n for larger n.

Problem Set 9.3, page 489

1 The iteration $\boldsymbol{x}_{k+1} = (I - A)\boldsymbol{x}_k + \boldsymbol{b}$ has $S = I$ and $T = I - A$ and $S^{-1}T = I - A$.

2 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $(I-A)\boldsymbol{x} = (1-\lambda)\boldsymbol{x}$. Real eigenvalues of $B = I-A$ have $|1-\lambda| < 1$ provided λ is between 0 and 2.

3 This matrix A has $I - A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ which has $|\lambda| = 2$. The iteration diverges.

4 Always $\|AB\| \le \|A\|\|B\|$. Choose $A = B$ to find $\|B^2\| \le \|B\|^2$. Then choose $A = B^2$ to find $\|B^3\| \le \|B^2\|\|B\| \le \|B\|^3$. Continue (or use induction) to find $\|B^k\| \le \|B\|^k$. Since $\|B\| \ge \max|\lambda(B)|$ it is no surprise that $\|B\| < 1$ gives convergence.

5 $A\boldsymbol{x} = \boldsymbol{0}$ gives $(S-T)\boldsymbol{x} = \boldsymbol{0}$. Then $S\boldsymbol{x} = T\boldsymbol{x}$ and $S^{-1}T\boldsymbol{x} = \boldsymbol{x}$. Then $\lambda = 1$ means that the errors do not approach zero. We can't expect convergence when A is singular and $A\boldsymbol{x} = \boldsymbol{b}$ is unsolvable!

6 Jacobi has $S^{-1}T = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{3}$. Small problem, fast convergence.

7 Gauss-Seidel has $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{9}$ which is $(|\lambda|_{\max}$ for Jacobi$)^{\mathbf{2}}$.

8 Jacobi has $S^{-1}T = \begin{bmatrix} a & \\ & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ -c/d & 0 \end{bmatrix}$ with $|\lambda| = |bc/ad|^{1/2}$.

Gauss-Seidel has $S^{-1}T = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ 0 & -bc/ad \end{bmatrix}$ with $|\lambda| = |bc/ad|$.

So Gauss-Seidel is twice as fast to converge (or to explode if $|bc| > |ad|$).

9 Set the trace $2-2\omega+\frac{1}{4}\omega^2$ equal to $(\omega-1)+(\omega-1)$ to find $\omega_{\text{opt}} = 4(2-\sqrt{3}) \approx 1.07$. The eigenvalues $\omega - 1$ are about .07, a big improvement.

10 Gauss-Seidel will converge for the $-1, 2, -1$ matrix. $|\lambda|_{\max} = \cos^2(\pi/n+1)$ is given on page 485, with the improvement from successive over relaxation.

11 If the iteration gives all $x_i^{\text{new}} = x_i^{\text{old}}$ then the quantity in parentheses is zero, which means $A\boldsymbol{x} = \boldsymbol{b}$. For Jacobi change $\boldsymbol{x}^{\text{new}}$ on the right side to x^{old}.

12 A lot of energy went into SOR in the 1950's! Now incomplete LU is simpler and preferred.

13 $\boldsymbol{u}_k/\lambda_1^k = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2(\lambda_2/\lambda_1)^k + \cdots + c_n\boldsymbol{x}_n(\lambda_n/\lambda_1)^k \to c_1\boldsymbol{x}_1$ if all ratios $|\lambda_i/\lambda_1| < 1$. The largest ratio controls the rate of convergence (when k is large). $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $|\lambda_2| = |\lambda_1|$ and no convergence.

14 The eigenvectors of A and also A^{-1} are $\boldsymbol{x}_1 = (.75, .25)$ and $\boldsymbol{x}_2 = (1, -1)$. The inverse power method converges to a multiple of $\boldsymbol{x}_2$, since $|1/\lambda_2| > |1/\lambda_1|$.

15 In the jth component of $A\boldsymbol{x}_1$, $\lambda_1 \sin\frac{j\pi}{n+1} = 2\sin\frac{j\pi}{n+1} - \sin\frac{(j-1)\pi}{n+1} - \sin\frac{(j+1)\pi}{n+1}$. The last two terms combine into $-2\sin\frac{j\pi}{n+1}\cos\frac{\pi}{n+1}$. Then $\lambda_1 = 2 - 2\cos\frac{\pi}{n+1}$.

16 $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ produces $\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, $\boldsymbol{u}_3 = \begin{bmatrix} 14 \\ -13 \end{bmatrix}$. This is converging to the eigenvector direction $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with largest eigenvalue $\lambda = 3$. Divide $\boldsymbol{u}_k$ by $\|\boldsymbol{u}_k\|$.

17 $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ gives $\boldsymbol{u}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\boldsymbol{u}_2 = \frac{1}{9}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\boldsymbol{u}_3 = \frac{1}{27}\begin{bmatrix} 14 \\ 13 \end{bmatrix} \to \boldsymbol{u}_\infty = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

18 $R = Q^{\mathrm{T}}A = \begin{bmatrix} 1 & \cos\theta\sin\theta \\ 0 & -\sin^2\theta \end{bmatrix}$ and $A_1 = RQ = \begin{bmatrix} \cos\theta(1+\sin^2\theta) & -\sin^3\theta \\ \boldsymbol{-\sin^3\theta} & -\cos\theta\sin^2\theta \end{bmatrix}$.

19 If A is orthogonal then $Q = A$ and $R = I$. Therefore $A_1 = RQ = A$ again, and the "QR method" doesn't move from A. But shift A slightly and the method goes quickly to Λ.

20 If $A - cI = QR$ then $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$. No change in eigenvalues because A_1 is similar to A.

21 Multiply $A\boldsymbol{q}_j = b_{j-1}\boldsymbol{q}_{j-1} + a_j\boldsymbol{q}_j + b_j\boldsymbol{q}_{j+1}$ by $\boldsymbol{q}_j^{\mathrm{T}}$ to find $\boldsymbol{q}_j^{\mathrm{T}}A\boldsymbol{q}_j = a_j$ (because the $\boldsymbol{q}$'s are orthonormal). The matrix form (multiplying by columns) is $AQ = QT$ where T is *tridiagonal*. The entries down the diagonals of T are the a's and b's.

22 Theoretically the $\boldsymbol{q}$'s are orthonormal. In reality this important algorithm is not very stable. We must stop every few steps to reorthogonalize—or find another more stable way to orthogonalize $\boldsymbol{q}, A\boldsymbol{q}, A^2q, \ldots$

23 If A is symmetric then $A_1 = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is also symmetric. $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$ has R and R^{-1} upper triangular, so A_1 cannot have nonzeros on a lower diagonal than A. If A is tridiagonal and symmetric then (by using symmetry for the upper part of A_1) the matrix $A_1 = RAR^{-1}$ is also tridiagonal.

24 The proof of $|\lambda| < 1$ when every absolute row sum < 1 uses $|\sum a_{ij}x_j| \le \sum |a_{ij}||x_i| < |x_i|$. (Here x_i is the largest component.) The application to the Gershgorin circle theorem (very useful) is printed after its statement in this problem.

25 For A and K, the maximum row sums give all $|\lambda| \le 1$ and all $|\lambda| \le 4$. The circles $|\lambda - .5| \le .5$ and $|\lambda - .4| \le .6$ around diagonal entries of A give tighter bounds. The circle $|\lambda - 2| \le 2$ for K contains the circle $|\lambda - 2| \le 1$ and all three eigenvalues $2 + \sqrt{2}, 2$, and $2 - \sqrt{2}$.

26 With diagonal dominance $a_{ii} > r_i$, the circles $|\lambda - a_{ii}| \le r_i$ don't include $\lambda = 0$ (*so A is invertible*!). Notice that the $-1, 2, -1$ matrix is also invertible even though its diagonals are only weakly dominant. They *equal* the off-diagonal row sums, $2 = 2$ except in the first and last rows, and more care is needed to prove invertibility.

27 From the last line of code, $\boldsymbol{q}_2$ is in the direction of $\boldsymbol{v} = A\boldsymbol{q}_1 - h_{11}\boldsymbol{q}_1 = A\boldsymbol{q}_1 - (\boldsymbol{q}_1^{\mathrm{T}}A\boldsymbol{q}_1)\boldsymbol{q}_1$. The dot product with $\boldsymbol{q}_1$ is zero. This is Gram-Schmidt with $A\boldsymbol{q}_1$ as the second input vector.

28 *Note* The five lines in Solutions to Selected Exercises prove two key properties of conjugate gradients—the residuals $\boldsymbol{r}_k = \boldsymbol{b} - A\boldsymbol{x}_k$ are orthogonal and the search directions are A-orthogonal ($\boldsymbol{p}_i^{\mathrm{T}}A\boldsymbol{p}_i = 0$). Then each new guess $\boldsymbol{x}_{k+1}$ is the **closest vector to $\boldsymbol{x}$** among all combinations of $\boldsymbol{b}$, $A\boldsymbol{b}$, $A^k\boldsymbol{b}$. Ordinary iteration $S\boldsymbol{x}_{k+1} = T\boldsymbol{x}_k + \boldsymbol{b}$ does not find this best possible combination $\boldsymbol{x}_{k+1}$.

The solution to Problem 28 in this Fourth Edition is straightforward and important. Since $H = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is symmetric if $A = A^{\mathrm{T}}$, and since H has only one lower diagonal by construction, then H has only one upper diagonal: H is tridiagonal and all the recursions in Arnoldi's method have only 3 terms (Problem 29).

29 $H = Q^{-1}AQ$ is *similar* to A, so H has the same eigenvalues as A (at the end of Arnoldi). When Arnoldi stops sooner because the matrix size is large, the eigenvalues of H_k (called *Ritz values*) are close to eigenvalues of A. This is an important way to compute approximations to λ for large matrices.

30 In principle the conjugate gradient method converges in 100 (or 99) steps to the exact solution $\boldsymbol{x}$. But it is slower than elimination and its all-important property is to give good approximations to $\boldsymbol{x}$ much sooner. (Stopping elimination part way leaves you nothing.) The problem asks how close $\boldsymbol{x}_{10}$ and $\boldsymbol{x}_{20}$ are to $\boldsymbol{x}_{100}$, which equals $\boldsymbol{x}$ except for roundoff errors.

Problem Set 10.1, page 498

1 (a)(b)(c) have sums 4, $-2+2i$, $2\cos\theta$ and products 5, $-2i$, 1. Note $(e^{i\theta})(e^{-i\theta}) = 1$.

2 In polar form these are $\sqrt{5}e^{i\theta}$, $5e^{2i\theta}$, $\frac{1}{\sqrt{5}}e^{-i\theta}$, $\sqrt{5}$.

3 The absolute values are $r = 10$, 100, $\frac{1}{10}$, and 100. The angles are θ, 2θ, $-\theta$ and -2θ.

4 $|z \times w| = 6$, $|z + w| \le 5$, $|z/w| = \frac{2}{3}$, $|z - w| \le 5$.

5 $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$; $w^{12} = 1$.

6 $1/z$ has absolute value $1/r$ and angle $-\theta$; $(1/r)e^{-i\theta}$ times $re^{i\theta}$ equals 1.

7 $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}\begin{bmatrix} ac-bd \\ bc+ad \end{bmatrix}$ $\begin{matrix}\textbf{real part} \\ \textbf{imaginary part}\end{matrix}$ $\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ is the matrix form of $(1+3i)(1-3i) = 10$.

8 $\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}$ gives complex matrix = vector multiplication $(A_1 + iA_2)(\boldsymbol{x}_1 + i\boldsymbol{x}_2) = \boldsymbol{b}_1 + i\boldsymbol{b}_2$.

9 $2+i$; $(2+i)(1+i) = 1+3i$; $e^{-i\pi/2} = -i$; $e^{-i\pi} = -1$; $\frac{1-i}{1+i} = -i$; $(-i)^{103} = i$.

10 $z + \overline{z}$ is real; $z - \overline{z}$ is pure imaginary; $z\overline{z}$ is positive; $z/\overline{z}$ has absolute value 1.

11 $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ includes aI (which just adds a to the eigenvalues and $b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. So the eigenvectors are $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$. The eigenvalues are $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$. We see $\overline{\boldsymbol{x}}_1 = \boldsymbol{x}_2$ and $\overline{\lambda}_1 = \lambda_2$ as expected for real matrices with complex eigenvalues.

12 (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the λ's can be real and different.

13 Complex λ's when $(a+d)^2 < 4(ad-bc)$; write $(a+d)^2 - 4(ad-bc)$ as $(a-d)^2 + 4bc$ which is positive when $bc > 0$.

14 $\det(P - \lambda I) = \lambda^4 - 1 = 0$ has $\lambda = 1, -1, i, -i$ with eigenvectors $(1,1,1,1)$ and $(1,-1,1,-1)$ and $(1,i,-1,-i)$ and $(1,-i,-1,i)$ = columns of Fourier matrix.

15 The 6 by 6 cyclic shift P has $\det(P_6 - \lambda I) = \lambda^6 - 1 = 0$. Then $\lambda = 1$, w, w^2, w^3, w^4, w^5 with $w = e^{2\pi i/6}$. These are the six solutions to $\lambda^b = 1$ as in Figure 10.3 (The sixth roots of 1).

16 The symmetric block matrix has real eigenvalues; so $i\lambda$ is real and λ is pure imaginary.

17 (a) $2e^{i\pi/3}, 4e^{2i\pi/3}$ (b) $e^{2i\theta}, e^{4i\theta}$ (c) $7e^{3\pi i/2}, 49e^{3\pi i}$ $(= -49)$ (d) $\sqrt{50}e^{-\pi i/4}$, $50e^{-\pi i/2}$.

18 $r = 1$, angle $\frac{\pi}{2} - \theta$; multiply by $e^{i\theta}$ to get $e^{i\pi/2} = i$.

19 $a + ib = 1, i, -1, -i, \pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$. The root $\overline{w} = w^{-1} = e^{-2\pi i/8}$ is $1/\sqrt{2} - i/\sqrt{2}$.

20 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$ are cube roots of 1. The cube roots of -1 are -1, $e^{\pi i/3}$, $e^{-\pi i/3}$. Altogether six roots of $z^6 = 1$.

21 $\cos 3\theta = \mathrm{Re}[(\cos\theta + i\sin\theta)^3] = \cos^3\theta - 3\cos\theta\sin^2\theta$; $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

22 If the conjugate $\overline{z} = 1/z$ then $|z|^2 = 1$ and z is any point $e^{i\theta}$ on the unit circle.

23 e^i is at angle $\theta = 1$ on the unit circle; $|i^e| = 1^e$; Infinitely many $i^e = e^{i(\pi/2 + 2\pi n)e}$.

24 (a) Unit circle (b) Spiral in to $e^{-2\pi}$ (c) Circle continuing around to angle $\theta = 2\pi^2$.

Problem Set 10.2, page 506

1 $\|\boldsymbol{u}\| = \sqrt{9} = 3$, $\|\boldsymbol{v}\| = \sqrt{3}$, $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = 3i + 2$, $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u} = -3i + 2$ (this is the conjugate of $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$).

2 $A^{\mathrm{H}}A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}$ and $AA^{\mathrm{H}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ are Hermitian matrices. They share the eigenvalues 4 and 2.

3 $z =$ multiple of $(1+i, 1+i, -2)$; $Az = \mathbf{0}$ gives $\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}} = \mathbf{0}^{\mathrm{H}}$ so $\boldsymbol{z}$ (not $\overline{\boldsymbol{z}}$!) is orthogonal to all columns of A^{H} (using complex inner product $\boldsymbol{z}^{\mathrm{H}}$ times columns of A^{H}).

4 The four fundamental subspaces are now $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, $\boldsymbol{C}(A^{\mathrm{H}})$, $\boldsymbol{N}(A^{\mathrm{H}})$. A^{H} ***and not*** A^{T}.

5 (a) $(A^{\mathrm{H}}A)^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{HH}} = A^{\mathrm{H}}A$ again (b) If $A^{\mathrm{H}}A\boldsymbol{z} = \mathbf{0}$ then $(\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}})(A\boldsymbol{z}) = 0$. This is $\|A\boldsymbol{z}\|^2 = 0$ so $A\boldsymbol{z} = \mathbf{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are always the ***same***.

6 (a) False (c) False $A = U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue when $A = A^{\mathrm{H}}$.

7 cA is still Hermitian *for real* c; $(iA)^{\mathrm{H}} = -iA^{\mathrm{H}} = -iA$ is skew-Hermitian.

8 This P is invertible and unitary. $P^2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $P^3 = \begin{bmatrix} -i & & \\ & -i & \\ & & -i \end{bmatrix} = -iI$. Then $P^{100} = (-i)^{33}P = -iP$. The eigenvalues of P are the roots of $\lambda^3 = -i$, which are i and $ie^{2\pi i/3}$ and $ie^{4\pi i/3}$.

9 One unit eigenvector is certainly $\boldsymbol{x}_1 = (1, 1, 1)$ with $\lambda_1 = i$. The other eigenvectors are $\boldsymbol{x}_2 = (1, w, w^2)$ and $\boldsymbol{x}_3 = (1, w^2, w^4)$ with $w = e^{2\pi i/3}$. *The eigenvector matrix is the Fourier matrix* F_3. The eigenvectors of any unitary matrix like P are orthogonal (using the correct complex form $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}$ of the inner product).

10 $(1, 1, 1)$, $(1, e^{2\pi i/3}, e^{4\pi i/3})$, $(1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because P is an orthogonal matrix—and therefore its eigenvector matrix is unitary.

11 Not included in 4^{th} edition $C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} = 2+5P+4P^2$ has $\begin{array}{l} \lambda = 2+5+4 = 11, \\ 2+5e^{2\pi i/3}+4e^{4\pi i/3}, \\ 2+5e^{4\pi i/3}+4e^{8\pi i/3}. \end{array}$

11 If $U^{\mathrm{H}}U = I$ then $U^{-1}(U^{\mathrm{H}})^{-1} = U^{-1}(U^{-1})^{\mathrm{H}} = I$ so U^{-1} is also unitary. Also $(UV)^{\mathrm{H}}(UV) = V^{\mathrm{H}}U^{\mathrm{H}}UV = V^{\mathrm{H}}V = I$ so UV is unitary.

12 Determinant = product of the eigenvalues *(all real)*. And $A = A^{\mathrm{H}}$ gives $\det A = \overline{\det A}$.

13 $(z^{\mathrm{H}}A^{\mathrm{H}})(Az) = \|Az\|^2$ is positive unless $Az = \mathbf{0}$. When A has independent columns this means $z = \mathbf{0}$; so $A^{\mathrm{H}}A$ is positive definite.

14 $A = \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$.

15 $K = (iA^{\mathrm{T}}$ in Problem 14$) = \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix}\begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix}\dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}$; λ's are imaginary.

16 $Q = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ has $|\lambda| = 1$.

17 $V = \dfrac{1}{L}\begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\dfrac{1}{L}\begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6+2\sqrt{3}$. Unitary means $|\lambda| = 1$. $V = V^{\mathrm{H}}$ gives real λ. Then trace zero gives $\lambda = 1$ and -1.

18 The $\boldsymbol{v}$'s are columns of a unitary matrix U, so U^{H} is U^{-1}. Then $z = UU^{\mathrm{H}}z =$ (multiply by columns) $= \boldsymbol{v}_1(\boldsymbol{v}_1^{\mathrm{H}}z) + \cdots + \boldsymbol{v}_n(\boldsymbol{v}_n^{\mathrm{H}}z)$: a typical orthonormal expansion.

19 Don't multiply $(e^{-ix})(e^{ix})$. Conjugate the first, then $\int_0^{2\pi} e^{2ix}\,dx = [e^{2ix}/2i]_0^{2\pi} = 0$.

20 $z = (1, i, -2)$ completes an orthogonal basis for $\mathbf{C}^3$. So does any $e^{i\theta}z$.

21 $R + iS = (R+iS)^{\mathrm{H}} = R^{\mathrm{T}} - iS^{\mathrm{T}}$; R is symmetric but S is skew-symmetric.

22 $\mathbf{C}^n$ has dimension n; the columns of any unitary matrix are a basis. For example use the columns of iI: $(i, 0, \ldots, 0), \ldots, (0, \ldots, 0, i)$

23 $[\,1\,]$ and $[-1\,]$; any $[\,e^{i\theta}\,]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\overline{z} \\ -z & e^{i\phi}\overline{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$ and any angle ϕ

24 The eigenvalues of A^{H} are *complex conjugates* of the eigenvalues of A: $\det(A - \lambda I) = 0$ gives $\det(A^{\mathrm{H}} - \overline{\lambda}I) = 0$.

25 $(I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}})^{\mathrm{H}} = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ and also $(I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}})^2 = I - 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}} + 4\boldsymbol{u}(\boldsymbol{u}^{\mathrm{H}}\boldsymbol{u})\boldsymbol{u}^{\mathrm{H}} = I$. The rank-1 matrix $\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ projects onto the line through $\boldsymbol{u}$.

26 Unitary $U^{\mathrm{H}}U = I$ means $(A^{\mathrm{T}} - iB^{\mathrm{T}})(A+iB) = (A^{\mathrm{T}}A + B^{\mathrm{T}}B) + i(A^{\mathrm{T}}B - B^{\mathrm{T}}A) = I$. $A^{\mathrm{T}}A + B^{\mathrm{T}}B = I$ and $A^{\mathrm{T}}B - B^{\mathrm{T}}A = 0$ which makes the block matrix orthogonal.

27 We are given $A + iB = (A+iB)^{\mathrm{H}} = A^{\mathrm{T}} - iB^{\mathrm{T}}$. Then $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$. So that $\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix}$ is symmetric.

28 $AA^{-1} = I$ gives $(A^{-1})^{\mathrm{H}}A^{\mathrm{H}} = I$. Therefore $(A^{-1})^{\mathrm{H}}$ is $(A^{\mathrm{H}})^{-1} = A^{-1}$ and A^{-1} is Hermitian.

29 $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\dfrac{1}{6}\begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S\Lambda S^{-1}$. Note real $\lambda = 1$ and 4.

30 If U has (complex) orthonormal columns, then $U^{\mathrm{H}}U = I$ and U is *unitary*. If those columns are eigenvectors of A, then $A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ is *normal*. The direct test for a normal matrix (which is $AA^{\mathrm{H}} = A^{\mathrm{H}}A$ because diagonals could be real!) and Λ^{H} surely commute:

$$AA^{\mathrm{H}}=(U\Lambda U^{\mathrm{H}})(U\Lambda^{\mathrm{H}}U^{\mathrm{H}})=U(\Lambda\Lambda^{\mathrm{H}})U^{\mathrm{H}}=U(\Lambda^{\mathrm{H}}\Lambda)U^{\mathrm{H}}=(U\Lambda^{\mathrm{H}}U^{\mathrm{H}})(U\Lambda U^{\mathrm{H}})=A^{\mathrm{H}}A.$$

An easy way to construct a normal matrix is $1+i$ times a symmetric matrix. Or take $A = S + iT$ where the real symmetric S and T commute (Then $A^{\mathrm{H}} = S - iT$ and $AA^{\mathrm{H}} = A^{\mathrm{H}}A$).

Problem Set 10.3, page 514

1 Equation (3) (the FFT) is correct using $i^2 = -1$ in the last two rows and three columns.

2 $$F^{-1} = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & -i & & i \end{bmatrix} = \frac{1}{4}F^{\mathrm{H}}.$$

3 $$F = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & -i & & i \end{bmatrix} \text{permutation last.}$$

4 $$D = \begin{bmatrix} 1 & & \\ & e^{2\pi i/6} & \\ & & e^{4\pi i/6} \end{bmatrix} \text{(note 6 not 3) and } F_3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}.$$

5 $F^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $F^{-1}\boldsymbol{v} = \boldsymbol{w}/4$. Delta vector $\leftrightarrow$ all-ones vector.

6 $$(F_4)^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \text{ and } (F_4)^4 = 16I. \text{ Four transforms recover the signal!}$$

7 $$\boldsymbol{c}=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\to\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\to\begin{bmatrix}2\\0\\0\\0\end{bmatrix}\to\begin{bmatrix}2\\0\\2\\0\end{bmatrix}=F\boldsymbol{c}.\text{ Also } C=\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\to\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\to\begin{bmatrix}0\\0\\2\\0\end{bmatrix}\to\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}=FC.$$
Adding $c + C$ gives $(1,1,1,1)$ to $(4,0,0,0) = 4$ (delta vector).

8 $\boldsymbol{c} \to (1,1,1,1,0,0,0,0) \to (4,0,0,0,0,0,0,0) \to (4,0,0,0,4,0,0,0) = F_8\boldsymbol{c}$.
$\boldsymbol{C} \to (0,0,0,0,1,1,1,1) \to (0,0,0,0,4,0,0,0) \to (4,0,0,0,-4,0,0,0) = F_8\boldsymbol{C}$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1: Key to FFT.

10 For every integer n, the nth roots of 1 add to zero. For even n, they cancel in pairs. For any n, use the geometric series formula $1 + w + \cdots + w^{n-1} = (w^n - 1)/(w - 1) = 0$. In particular for $n = 3$, $1 + (-1 + i\sqrt{3})/2 + (-1 - i\sqrt{3})/2 = 0$.

11 The eigenvalues of P are $1, i, i^2 = -1$, and $i^3 = -i$. Problem 11 displays the eigenvectors. And also $\det(P - \lambda I) = \lambda^4 - 1$.

12 $\Lambda = \text{diag}(1, i, i^2, i^3)$; $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and P^{T} lead to $\lambda^3 - 1 = 0$.

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$; E contains the four eigenvalues of $C = FEF^{-1}$ because F contains the eigenvectors.

14 Eigenvalues $e_1 = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2$. Just transform column 0 of C. Check trace $0 + 2 + 4 + 2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2}n \log_2 n$ multiplications each by the **FFT**. The total is much less than the ordinary n^2 for C times $\boldsymbol{x}$.

16 The row $1, \overline{w}^k, \overline{w}^{2k}, \ldots$ in $\overline{F}$ is the same as the row $1, w^{N-k}, w^{N-2k}, \ldots$ in F because $w^{N-k} = e^{(2\pi i/N)(N-k)}$ is $e^{2\pi i} e^{-(2\pi i/N)k} = 1$ times $\overline{w}^k$. So F and $\overline{F}$ have the **same rows in reversed order** (except for row 0 which is all ones).