7 Invertible 3 by 3 matrix A: row space basis = column space basis = $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis and left nullspace basis are *empty*. Matrix $B = \begin{bmatrix} A & A \end{bmatrix}$: row space basis $(1,0,0,1,0,0)$, $(0,1,0,0,1,0)$ and $(0,0,1,0,0,1)$; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis $(-1,0,0,1,0,0)$ and $(0,-1,0,0,1,0)$ and $(0,0,-1,0,0,1)$; left nullspace basis is empty.

8 $\begin{bmatrix} I & 0 \end{bmatrix}$ and $\begin{bmatrix} I & I; & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \end{bmatrix}$ = 3 by 2 have *row space dimensions* = 3, 3, 0 = column space dimensions; *nullspace dimensions* 2, 3, 2; left *nullspace dimensions* 0, 2, 3.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same (b) Same column space and left nullspace. Same rank (dimension of column space).

10 For **rand** (3), almost surely rank= 3, nullspace and left nullspace contain only $(0,0,0)$. For **rand** (3, 5) the rank is almost surely 3 and the dimension of the nullspace is 2.

11 (a) No solution means that $r < m$. Always $r \leq n$. Can't compare m and n here. (b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \boldsymbol{0}$ is in both $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$.

13 (a) *False*: Usually row space $\neq$ column space (same dimension!) (b) *True*: A and $-A$ have the same four subspaces (c) *False* (choose A and B same size and invertible: then they have the same four subspaces)

14 Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\boldsymbol{C}(A) = \boldsymbol{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

15 After a row exchange, the row space and nullspace stay the same; $(2,1,3,4)$ is in the new left nullspace after the row exchange.

16 If $A\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

17 Row space = yz plane; column space = xy plane; nullspace = x axis; left nullspace = z axis. For $I + A$: Row space = column space = $\mathbf{R}^3$, both nullspaces contain only the zero vector.

18 Row 3−2 row 2+ row 1 = zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

19 (a) Elimination on $A\boldsymbol{x} = \boldsymbol{0}$ leads to $0 = b_3 - b_2 - b_1$ so $(-1,-1,1)$ is in the left nullspace. (b) 4 by 3: Elimination leads to $b_3 - 2b_1 = 0$ and $b_4 + b_2 - 4b_1 = 0$, so $(-2,0,1,0)$ and $(-4,1,0,1)$ are in the left nullspace. *Why*? Those vectors multiply the matrix to give *zero rows*. Section 4.1 will show another approach: $A\boldsymbol{x} = \boldsymbol{b}$ is solvable ($\boldsymbol{b}$ is in $\boldsymbol{C}(A)$) when $\boldsymbol{b}$ is orthogonal to the left nullspace.

20 (a) Special solutions $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ are perpendicular to the rows of R (and then ER). (b) $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ has 1 independent solution = last row of E^{-1}. ($E^{-1}A = R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

22 $A = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{w} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{\mathrm{T}} & \boldsymbol{z}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}$ has column space spanned by $\boldsymbol{u}$ and $\boldsymbol{w}$, row space spanned by $\boldsymbol{v}$ and $\boldsymbol{z}$.

23 As in Problem 22: Row space basis $(3,0,3),(1,1,2)$; column space basis $(1,4,2)$, $(2,5,7)$; the rank of (3 by 2) times (2 by 3) cannot be larger than the rank of either factor, so rank ≤ 2 and the 3 by 3 product is not invertible.

24 $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y}=\boldsymbol{0}$.

25 (a) *True* (A and A^{T} have the same rank) (b) *False* $A=[1\ \ 0]$ and A^{T} have very different left nullspaces (c) *False* (A can be invertible and unsymmetric even if $\boldsymbol{C}(A)=\boldsymbol{C}(A^{\mathrm{T}})$) (d) *True* (The subspaces for A and $-A$ are always the same. If $A^{\mathrm{T}}=A$ or $A^{\mathrm{T}}=-A$ they are also the same for A^{T})

26 The rows of $C=AB$ are combinations of the rows of B. So rank $C\leq$ rank B. Also rank $C\leq$ rank A, because the columns of C are combinations of the columns of A.

27 Choose $d=bc/a$ to make $\begin{bmatrix}\mathbf{a}&\mathbf{b}\\\mathbf{c}&\mathbf{d}\end{bmatrix}$ a rank-1 matrix. Then the row space has basis (a,b) and the nullspace has basis $(-b,a)$. Those two vectors are perpendicular !

28 B and C (checkers and chess) both have rank 2 if $p\neq 0$. Row 1 and 2 are a basis for the row space of C, $B^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{0}$ has 6 special solutions with -1 and 1 separated by a zero; $N(C^{\mathrm{T}})$ has $(-1,0,0,0,0,0,0,1)$ and $(0,-1,0,0,0,0,1,0)$ and columns $3,4,5,6$ of I; $N(C)$ is a challenge.

29 $a_{11}=1,a_{12}=0,a_{13}=1,a_{22}=0,a_{32}=1,a_{31}=0,a_{23}=1,a_{33}=0,a_{21}=1$.

30 The subspaces for $A=\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B=cA$ with $c\neq 0$.

31 (a) $AX=0$ if each column of X is a multiple of $(1,1,1)$; dim(nullspace) $=3$. (b) If $AX=B$ then all columns of B add to zero; dimension of the B's $=6$. (c) $3+6=\dim(M^{3\times 3})=9$ entries in a 3 by 3 matrix.

32 The key is equal row spaces. First row of A = combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F=G$.

Problem Set 4.1, page 202

1 Both nullspace vectors are orthogonal to the row space vector in $\mathbf{R}^3$. The column space is perpendicular to the nullspace of A^{T} (two lines in $\mathbf{R}^2$ because rank $=1$).

2 The nullspace of a 3 by 2 matrix with rank 2 is $\mathbf{Z}$ (only zero vector) so $\boldsymbol{x}_n=\boldsymbol{0}$, and row space $=\mathbf{R}^2$. Column space = plane perpendicular to left nullspace = line in $\mathbf{R}^3$.

3 (a) $\begin{bmatrix}1&2&-3\\2&-3&1\\-3&5&-2\end{bmatrix}$ (b) Impossible, $\begin{bmatrix}2\\-3\\5\end{bmatrix}$ not orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ (c) $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ is impossible: not perpendicular (d) Need $A^2=0$; take $A=\begin{bmatrix}1&-1\\1&-1\end{bmatrix}$ (e) $(1,1,1)$ in the nullspace (columns add to $\mathbf{0}$) and also row space; no such matrix.

4 If $AB=0$, the columns of B are in the *nullspace* of A. The rows of A are in the *left nullspace* of B. If rank $=2$, those four subspaces would have dimension 2 which is impossible for 3 by 3.

5 (a) If $A\boldsymbol{x}=\boldsymbol{b}$ has a solution and $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{0}$, then $\boldsymbol{y}$ is perpendicular to $\boldsymbol{b}$. $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}=(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y})=0$. (b) If $A^{\mathrm{T}}\boldsymbol{y}=(1,1,1)$ has a solution, $(1,1,1)$ is in the **row space** and is orthogonal to every $\boldsymbol{x}$ in the nullspace.

6 Multiply the equations by $y_1, y_2, y_3 = 1, 1, -1$. Equations add to $0 = 1$ so no solution: $\boldsymbol{y} = (1, 1, -1)$ is in the left nullspace. $A\boldsymbol{x} = \boldsymbol{b}$ would need $0 = (\boldsymbol{y}^{\mathrm{T}}A)\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1$.

7 Multiply the 3 equations by $\boldsymbol{y} = (1, 1, -1)$. Then $x_1 - x_2 = 1$ plus $x_2 - x_3 = 1$ *minus* $x_1 - x_3 = 1$ is $0 = 1$. Key point: This $\boldsymbol{y}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ is not orthogonal to $\boldsymbol{b} = (1, 1, 1)$ so $\boldsymbol{b}$ is not in the column space and $A\boldsymbol{x} = \boldsymbol{b}$ has *no solution*.

8 $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space and $\boldsymbol{x}_n$ is in the nullspace. Then $A\boldsymbol{x}_n = \boldsymbol{0}$ and $A\boldsymbol{x} = A\boldsymbol{x}_r + A\boldsymbol{x}_n = A\boldsymbol{x}_r$. All $A\boldsymbol{x}$ are in $\boldsymbol{C}(A)$.

9 $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x}$ is also in the nullspace of A^{T}. So $A\boldsymbol{x}$ is perpendicular to itself. Conclusion: $A\boldsymbol{x} = \boldsymbol{0}$ if $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$.

10 (a) With $A^{\mathrm{T}} = A$, the column and row spaces are the same (b) $\boldsymbol{x}$ is in the nullspace and $\boldsymbol{z}$ is in the column space = row space: so these "eigenvectors" have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z} = 0$.

11 ***For A***: The nullspace is spanned by $(-2, 1)$, the row space is spanned by $(1, 2)$. The column space is the line through $(1, 3)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ is the perpendicular line through $(3, -1)$. ***For B***: The nullspace of B is spanned by $(0, 1)$, the row space is spanned by $(1, 0)$. The column space and left nullspace are the same as for A.

12 $\boldsymbol{x}$ splits into $\boldsymbol{x}_r + \boldsymbol{x}_n = (1, -1) + (1, 1) = (2, 0)$. Notice $\boldsymbol{N}(A^{\mathrm{T}})$ is a plane $(1, 0) = (1, 1)/2 + (1, -1)/2 = \boldsymbol{x}_r + \boldsymbol{x}_n$.

13 $V^{\mathrm{T}}W$ = zero makes each basis vector for $\boldsymbol{V}$ orthogonal to each basis vector for $\boldsymbol{W}$. Then every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$ (combinations of the basis vectors).

14 $A\boldsymbol{x} = B\widehat{\boldsymbol{x}}$ means that $[\,A \quad B\,]\begin{bmatrix} \boldsymbol{x} \\ -\widehat{\boldsymbol{x}} \end{bmatrix} = \boldsymbol{0}$. Three homogeneous equations in four unknowns always have a nonzero solution. Here $\boldsymbol{x} = (3, 1)$ and $\widehat{\boldsymbol{x}} = (1, 0)$ and $A\boldsymbol{x} = B\widehat{\boldsymbol{x}} = (5, 6, 5)$ is in both column spaces. Two planes in $\mathbf{R}^3$ must share a line.

15 A p-dimensional and a q-dimensional subspace of $\mathbf{R}^n$ share at least a line if $\boldsymbol{p} + \boldsymbol{q} > \boldsymbol{n}$. (The $p + q$ basis vectors of $\boldsymbol{V}$ and $\boldsymbol{W}$ cannot be independent.)

16 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ leads to $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{y} = 0$. Then $\boldsymbol{y} \perp A\boldsymbol{x}$ and $\boldsymbol{N}(A^{\mathrm{T}}) \perp \boldsymbol{C}(A)$.

17 If $\boldsymbol{S}$ is the subspace of $\mathbf{R}^3$ containing only the zero vector, then $\boldsymbol{S}^{\perp}$ is $\mathbf{R}^3$. If $\boldsymbol{S}$ is spanned by $(1, 1, 1)$, then $\boldsymbol{S}^{\perp}$ is the plane spanned by $(1, -1, 0)$ and $(1, 0, -1)$. If $\boldsymbol{S}$ is spanned by $(2, 0, 0)$ and $(0, 0, 3)$, then $\boldsymbol{S}^{\perp}$ is the line spanned by $(0, 1, 0)$.

18 $\boldsymbol{S}^{\perp}$ is the nullspace of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Therefore $\boldsymbol{S}^{\perp}$ is a *subspace* even if $\boldsymbol{S}$ is not.

19 $\boldsymbol{L}^{\perp}$ is the 2-*dimensional subspace* (*a plane*) in $\mathbf{R}^3$ perpendicular to $\boldsymbol{L}$. Then $(\boldsymbol{L}^{\perp})^{\perp}$ is a 1-*dimensional subspace* (*a line*) perpendicular to $\boldsymbol{L}^{\perp}$. In fact $(\boldsymbol{L}^{\perp})^{\perp}$ is $\boldsymbol{L}$.

20 If $\boldsymbol{V}$ is the whole space $\mathbf{R}^4$, then $\boldsymbol{V}^{\perp}$ contains only the *zero vector*. Then $(\boldsymbol{V}^{\perp})^{\perp} = \mathbf{R}^4 = \boldsymbol{V}$.

21 For example $(-5, 0, 1, 1)$ and $(0, 1, -1, 0)$ span $\boldsymbol{S}^{\perp}$ = nullspace of $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$.

22 $(1, 1, 1, 1)$ is a basis for $\boldsymbol{P}^{\perp}$. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ has $\boldsymbol{P}$ as its nullspace and $\boldsymbol{P}^{\perp}$ as row space.

23 $\boldsymbol{x}$ in $\boldsymbol{V}^{\perp}$ is perpendicular to any vector in $\boldsymbol{V}$. Since $\boldsymbol{V}$ contains all the vectors in $\boldsymbol{S}$, $\boldsymbol{x}$ is also perpendicular to any vector in $\boldsymbol{S}$. So every $\boldsymbol{x}$ in $\boldsymbol{V}^{\perp}$ is also in $\boldsymbol{S}^{\perp}$.

24 $AA^{-1} = I$: Column 1 of A^{-1} is orthogonal to the space spanned by the 2nd, 3rd, ..., nth rows of A.

25 If the columns of A are unit vectors, all mutually perpendicular, then $A^{\mathrm{T}}A = I$.

26 $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$, This example shows a matrix with perpendicular columns. $A^{\mathrm{T}}A = 9I$ is *diagonal*: $(A^{\mathrm{T}}A)_{ij} =$ (column i of A) $\cdot$ (column j of A). When the columns are *unit vectors*, then $A^{\mathrm{T}}A = I$.

27 The lines $3x + y = b_1$ and $6x + 2y = b_2$ are **parallel**. They are the same line if $b_2 = 2b_1$. In that case (b_1, b_2) is perpendicular to $(-2, 1)$. The nullspace of the 2 by 2 matrix is the line $3x + y = 0$. One particular vector in the nullspace is $(-1, 3)$.

28 (a) $(1, -1, 0)$ is in both planes. Normal vectors are perpendicular, but planes still intersect! (b) Need *three* orthogonal vectors to span the whole orthogonal complement. (c) Lines can meet at the zero vector without being orthogonal.

29 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$; A has $\boldsymbol{v} = (1, 2, 3)$ in row space and column space; B has $\boldsymbol{v}$ in its column space and nullspace. $\boldsymbol{v}$ **can not** be in the nullspace and row space, or in the left nullspace and column space. These spaces are orthogonal and $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} \neq 0$.

30 When $AB = 0$, the column space of B is contained in the nullspace of A. Therefore the dimension of $\boldsymbol{C}(B) \leq$ dimension of $\boldsymbol{N}(A)$. This means rank$(B) \leq 4 -$ rank(A).

31 null(N') produces a basis for the *row space* of A (perpendicular to $\boldsymbol{N}(A)$).

32 We need $\boldsymbol{r}^{\mathrm{T}}\boldsymbol{n} = 0$ and $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\ell} = 0$. All possible examples have the form $a\boldsymbol{c}\boldsymbol{r}^{\mathrm{T}}$ with $a \neq 0$.

33 Both $\boldsymbol{r}$'s orthogonal to both $\boldsymbol{n}$'s, both $\boldsymbol{c}$'s orthogonal to both $\boldsymbol{\ell}$'s, each pair independent. All A's with these subspaces have the form $[\boldsymbol{c}_1\ \boldsymbol{c}_2]M[\boldsymbol{r}_1\ \boldsymbol{r}_2]^{\mathrm{T}}$ for a 2 by 2 invertible M.

Problem Set 4.2, page 214

1 (a) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 5/3$; $\boldsymbol{p} = 5\boldsymbol{a}/3$; $\boldsymbol{e} = (-2, 1, 1)/3$ (b) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = -1$; $\boldsymbol{p} = \boldsymbol{a}$; $\boldsymbol{e} = \boldsymbol{0}$.

2 (a) The projection of $\boldsymbol{b} = (\cos\theta, \sin\theta)$ onto $\boldsymbol{a} = (1, 0)$ is $\boldsymbol{p} = (\cos\theta, 0)$ (b) The projection of $\boldsymbol{b} = (1, 1)$ onto $\boldsymbol{a} = (1, -1)$ is $\boldsymbol{p} = (0, 0)$ since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$.

3 $P_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P_1\boldsymbol{b} = \frac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$. $P_2 = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ and $P_2\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

4 $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. P_1 projects onto $(1, 0)$, P_2 projects onto $(1, -1)$. $P_1P_2 \neq 0$ and $P_1 + P_2$ is not a projection matrix.

5 $P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$, $P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$. P_1 and P_2 are the projection matrices onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$ $P_1P_2 =$ *zero matrix because* $\boldsymbol{a}_1 \perp \boldsymbol{a}_2$.

XXX Above solution does not fit in 3 lines.

6 $\boldsymbol{p}_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$ and $\boldsymbol{p}_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$ and $\boldsymbol{p}_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$. So $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 = \boldsymbol{b}$.

7 $P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I.$
We *can* add projections onto *orthogonal vectors*. This is important.

8 The projections of $(1, 1)$ onto $(1, 0)$ and $(1, 2)$ are $\boldsymbol{p}_1 = (1, 0)$ and $\boldsymbol{p}_2 = (0.6, 1.2)$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \neq \boldsymbol{b}$.

9 Since A is invertible, $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$: project on all of $\mathbf{R}^2$.

10 $P_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}, P_2\boldsymbol{a}_1 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, P_1P_2\boldsymbol{a}_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}.$ This is not $\boldsymbol{a}_1 = (1, 0)$ *No*, $P_1P_2 \neq (P_1P_2)^2$.

11 (a) $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (2, 3, 0)$, $\boldsymbol{e} = (0, 0, 4)$, $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$ (b) $\boldsymbol{p} = (4, 4, 6)$, $\boldsymbol{e} = \mathbf{0}$.

12 $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = projection matrix onto the column space of A (the xy plane)

$P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$ Projection matrix onto the second column space. Certainly $(P_2)^2 = P_2$.

13 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, P = square matrix $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{p} = P\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

14 The projection of this $\boldsymbol{b}$ onto the column space of A is $\boldsymbol{b}$ itself when $\boldsymbol{b}$ is in that space.
But P is not necessarily I. $P = \frac{1}{21}\begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix}$ and $\boldsymbol{b} = P\boldsymbol{b} = \boldsymbol{p} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$.

15 $2A$ has the same column space as A. $\widehat{\boldsymbol{x}}$ for $2A$ is *half* of $\widehat{\boldsymbol{x}}$ for A.

16 $\frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1) = (2, 1, 1)$. So $\boldsymbol{b}$ is in the plane. Projection shows $P\boldsymbol{b} = \boldsymbol{b}$.

17 If $P^2 = P$ then $\boldsymbol{(I - P)^2} = (I - P)(I - P) = I - PI - IP + P^2 = \boldsymbol{I - P}$. When P projects onto the column space, $I - P$ projects onto the *left nullspace*.

18 (a) $I - P$ is the projection matrix onto $(1, -1)$ in the perpendicular direction to $(1, 1)$
(b) $I - P$ projects onto the plane $x + y + z = 0$ perpendicular to $(1, 1, 1)$.

19 For any basis vectors in the plane $x - y - 2z = 0$, say $(1, 1, 0)$ and $(2, 0, 1)$, the matrix P is $\begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

20 $\boldsymbol{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $Q = \frac{\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}}{\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

21 $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. So $P^2 = P$. $P\boldsymbol{b}$ is in the column space (where P projects). Then its projection $P(P\boldsymbol{b})$ is $P\boldsymbol{b}$.

22 $P^{\mathrm{T}} = (A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}})^{\mathrm{T}} = A((A^{\mathrm{T}}A)^{-1})^{\mathrm{T}}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = P$. ($A^{\mathrm{T}}A$ is symmetric!)

23 If A is invertible then its column space is all of $\mathbf{R}^n$. So $P = I$ and $e = 0$.

24 The nullspace of A^{T} is *orthogonal* to the column space $\boldsymbol{C}(A)$. So if $A^{\mathrm{T}}\boldsymbol{b} = \mathbf{0}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{C}(A)$ should be $\boldsymbol{p} = \mathbf{0}$. Check $P\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}\mathbf{0}$.

25 **The column space of P will be S.** Then $r =$ dimension of $S = n$.

26 A^{-1} exists since the rank is $r = m$. Multiply $A^2 = A$ by A^{-1} to get $A = I$.

27 If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ is in the nullspace of A^{T}. But $A\boldsymbol{x}$ is always in the column space of A. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero. So A and $A^{\mathrm{T}}A$ have the *same nullspace*.

28 $P^2 = P = P^{\mathrm{T}}$ give $P^{\mathrm{T}}P = P$. Then the $(2,2)$ entry of P equals the $(2,2)$ entry of $P^{\mathrm{T}}P$ which is the length squared of column 2.

29 $A = B^{\mathrm{T}}$ has independent columns, so $A^{\mathrm{T}}A$ (which is BB^{T}) must be invertible.

30 (a) The column space is the line through $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \dfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$.

(b) The row space is the line through $\boldsymbol{v} = (1,2,2)$ and $P_R = \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}/\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$. Always $P_C A = A$ (columns of A project to themselves) and $AP_R = A$. Then $P_C A P_R = A$!

31 The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ must be perpendicular to all the $\boldsymbol{a}$'s.

32 Since $P_1\boldsymbol{b}$ is in $\boldsymbol{C}(A)$, $P_2(P_1\boldsymbol{b})$ equals $P_1\boldsymbol{b}$. So $P_2P_1 = P_1 = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ where $\boldsymbol{a} = (1,2,0)$.

33 If $P_1P_2 = P_2P_1$ then $\boldsymbol{S}$ is contained in $\boldsymbol{T}$ or $\boldsymbol{T}$ is contained in $\boldsymbol{S}$.

34 BB^{T} is invertible as in Problem 29. Then $(A^{\mathrm{T}}A)(BB^{\mathrm{T}}) =$ product of r by r invertible matrices, so rank r. AB can't have rank $< r$, since A^{T} and B^{T} cannot increase the rank. *Conclusion*: A (m by r of rank r) times B (r by n of rank r) produces AB of rank r.

Problem Set 4.3, page 226

1 $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ gives $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$, $E = \|\boldsymbol{e}\|^2 = \mathbf{44}$

2 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. This $A\boldsymbol{x} = \boldsymbol{b}$ is unsolvable. Change $\boldsymbol{b}$ to $\boldsymbol{p} = P\boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$; $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ exactly solves $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$.

3 In Problem 2, $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (1,5,13,17)$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = (-1,3,-5,3)$. $\boldsymbol{e}$ is perpendicular to both columns of A. This shortest distance $\|\boldsymbol{e}\|$ is $\sqrt{44}$.

4 $E = (C + \mathbf{0}D)^2 + (C + \mathbf{1}D - 8)^2 + (C + \mathbf{3}D - 8)^2 + (C + \mathbf{4}D - 20)^2$. Then $\partial E/\partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$ and $\partial E/\partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0$. These normal equations are again $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

5 $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$. $A^{\mathrm{T}} = [1 \ \ 1 \ \ 1 \ \ 1]$ and $A^{\mathrm{T}}A = [4]$. $A^{\mathrm{T}}\boldsymbol{b} = [36]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = 9 =$ best height C. Errors $\boldsymbol{e} = (-9, -1, -1, 11)$.

6 $\boldsymbol{a} = (1,1,1,1)$ and $\boldsymbol{b} = (0,8,8,20)$ give $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 9$ and the projection is $\widehat{x}\boldsymbol{a} = \boldsymbol{p} = (9,9,9,9)$. Then $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{a} = (-9,-1,-1,11)^{\mathrm{T}}(1,1,1,1) = 0$ and $\|\boldsymbol{e}\| = \sqrt{204}$.

7 $A = [0 \ \ 1 \ \ 3 \ \ 4]^{\mathrm{T}}$, $A^{\mathrm{T}}A = [26]$ and $A^{\mathrm{T}}\boldsymbol{b} = [112]$. Best $D = 112/26 = 56/13$.

8 $\widehat{\boldsymbol{x}} = 56/13$, $\boldsymbol{p} = (56/13)(0,1,3,4)$. $(C,D) = (9, 56/13)$ don't match $(C,D) = (1,4)$. Columns of A were not perpendicular so we can't project separately to find C and D.

9 Parabola Project $\boldsymbol{b}$ 4D to 3D $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$.

10 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. Then $\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$. Exact cubic so $\boldsymbol{p} = \boldsymbol{b}$, $\boldsymbol{e} = \boldsymbol{0}$. This Vandermonde matrix gives exact interpolation by a cubic at $0, 1, 3, 4$

11 (a) The best line $x = 1 + 4t$ gives the center point $\widehat{b} = 9$ when $\widehat{t} = 2$.
(b) The first equation $Cm + D\sum t_i = \sum b_i$ divided by m gives $C + D\widehat{t} = \widehat{\boldsymbol{b}}$.

12 (a) $\boldsymbol{a} = (1,\ldots,1)$ has $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = m$, $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = b_1 + \cdots + b_m$. Therefore $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/m$ is the **mean** of the b's (b) $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$ $\boldsymbol{b} = (1,2,b)$ $\|\boldsymbol{e}\|^2 = \sum_{i=1}^{m}(b_i - \widehat{x})^2 =$ **variance**
(c) $\begin{matrix} \boldsymbol{p} = (3,3,3) \\ \boldsymbol{e} = (-2,-1,3) \end{matrix}$ $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{e} = 0$. $P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

13 $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b} - A\boldsymbol{x}) = \widehat{\boldsymbol{x}} - \boldsymbol{x}$. When $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ averages to $\boldsymbol{0}$, so does $\widehat{\boldsymbol{x}} - \boldsymbol{x}$.

14 The matrix $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}}$ is $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}}A(A^{\mathrm{T}}A)^{-1}$. When the average of $(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}}$ is $\sigma^2 I$, the average of $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}}$ will be the *output covariance matrix* $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\sigma^2 A(A^{\mathrm{T}}A)^{-1}$ which simplifies to $\sigma^2(A^{\mathrm{T}}A)^{-1}$.

15 When A has 1 column of ones, Problem 14 gives the expected error $(\widehat{x} - x)^2$ as $\sigma^2(A^{\mathrm{T}}A)^{-1} = \sigma^2/m$. By taking m measurements, the variance drops from σ^2 to σ^2/m.

16 $\frac{1}{10}b_{10} + \frac{\mathbf{9}}{\mathbf{10}}\widehat{x}_9 = \frac{1}{10}(b_1 + \cdots + b_{10})$. Knowing $\widehat{x}_9$ avoids adding all b's.

17 $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

18 $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \boldsymbol{0}$.

19 If $\boldsymbol{b} =$ error $\boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection $\boldsymbol{p} = \boldsymbol{0}$.

20 If $\boldsymbol{b} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ then $\widehat{\boldsymbol{x}} = (9, 4)$ and $\boldsymbol{e} = \boldsymbol{0}$ since $\boldsymbol{b}$ is *in the column space of* A.

21 $\boldsymbol{e}$ is in $\boldsymbol{N}(A^{\mathrm{T}})$; $\boldsymbol{p}$ is in $\boldsymbol{C}(A)$; $\widehat{\boldsymbol{x}}$ is in $\boldsymbol{C}(A^{\mathrm{T}})$; $\boldsymbol{N}(A) = \{\boldsymbol{0}\} =$ zero vector only.

22 The least squares equation is $\begin{bmatrix} 5 & \mathbf{0} \\ \mathbf{0} & 10 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$. Solution: $C = 1$, $D = -1$. Line $1 - t$. Symmetric t's $\Rightarrow$ diagonal $A^{\mathrm{T}}A$

23 $\boldsymbol{e}$ is orthogonal to $\boldsymbol{p}$; then $\|\boldsymbol{e}\|^2 = e^{\mathrm{T}}(\boldsymbol{b} - \boldsymbol{p}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{p}$.

24 The derivatives of $\|A\boldsymbol{x} - b\|^2 = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} - 2\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}$ (this term is constant) are zero when $2A^{\mathrm{T}}A\boldsymbol{x} = 2A^{\mathrm{T}}\boldsymbol{b}$, or $\boldsymbol{x} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$.

25 3 points on a line: *Equal slopes* $(b_2 - b_1)/(t_2 - t_1) = (b_3 - b_2)/(t_3 - t_2)$. Linear algebra: Orthogonal to $(1, 1, 1)$ and (t_1, t_2, t_3) is $\boldsymbol{y} = (t_2 - t_3, t_3 - t_1, t_1 - t_2)$ in the left nullspace. $\boldsymbol{b}$ is in the column space. Then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ is the same equal slopes condition written as $(b_2 - b_1)(t_3 - t_2) = (b_3 - b_2)(t_2 - t_1)$.

26 $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ has $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3/2 \end{bmatrix}$. At $x, y = 0, 0$ the best plane $2 - x - \frac{3}{2}y$ has height $C = \mathbf{2} =$ average of $0, 1, 3, 4$.

27 The shortest link connecting two lines in space is *perpendicular to those lines*.

28 Only 1 plane contains $\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ unless $\boldsymbol{a}_1, \boldsymbol{a}_2$ are *dependent*. Same test for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

29 There is exactly one hyperplane containing the n points $\mathbf{0}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ *When the* $n-1$ *vectors* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ *are linearly independent*. (For $n = 3$, the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ must be independent. Then the three points $\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ determine a plane.) The equation of the plane in $\mathbf{R}^n$ will be $\boldsymbol{a}_n^{\mathrm{T}}\boldsymbol{x} = 0$. Here $\boldsymbol{a}_n$ is any nonzero vector on the line (it is only a line!) perpendicular to $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$.

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1 (a) *Independent* (b) *Independent* and *orthogonal* (c) *Independent* and *orthonormal*. For orthonormal vectors, (a) becomes $(1, 0)$, $(0, 1)$ and (b) is $(.6, .8)$, $(.8, -.6)$.

2 Divide by length 3 to get $\boldsymbol{q}_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. $\boldsymbol{q}_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. $Q^{\mathrm{T}}Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ but $QQ^{\mathrm{T}} = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}$.

3 (a) $A^{\mathrm{T}}A$ will be $16I$ (b) $A^{\mathrm{T}}A$ will be diagonal with entries 1, 4, 9.

4 (a) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $QQ^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I$. Any Q with $n < m$ has $QQ^{\mathrm{T}} \neq I$. (b) $(1, 0)$ and $(0, 0)$ are *orthogonal*, not *independent*. Nonzero orthogonal vectors *are* independent. (c) Starting from $\boldsymbol{q}_1 = (1, 1, 1)/\sqrt{3}$ my favorite is $\boldsymbol{q}_2 = (1, -1, 0)/\sqrt{2}$ and $\boldsymbol{q}_3 = (1, 1, -2)/\sqrt{b}$.

5 *Orthogonal* vectors are $(1, -1, 0)$ and $(1, 1, -1)$. *Orthonormal* are $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

6 Q_1Q_2 is orthogonal because $(Q_1Q_2)^{\mathrm T}Q_1Q_2 = Q_2^{\mathrm T}Q_1^{\mathrm T}Q_1Q_2 = Q_2^{\mathrm T}Q_2 = I$.

7 When Gram-Schmidt gives Q with orthonormal columns, $Q^{\mathrm T}Q\widehat{\boldsymbol{x}} = Q^{\mathrm T}\boldsymbol{b}$ becomes $\widehat{\boldsymbol{x}} = Q^{\mathrm T}\boldsymbol{b}$.

8 If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are *orthonormal* vectors in $\mathbf{R}^5$ then $(\boldsymbol{q}_1^{\mathrm T}\boldsymbol{b})\boldsymbol{q}_1 + (\boldsymbol{q}_2^{\mathrm T}\boldsymbol{b})\boldsymbol{q}_2$ is closest to $\boldsymbol{b}$.

9 (a) $Q = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \\ 0 & 0 \end{bmatrix}$ has $P = QQ^{\mathrm T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $(QQ^{\mathrm T})(QQ^{\mathrm T}) = Q(Q^{\mathrm T}Q)Q^{\mathrm T} = QQ^{\mathrm T}$.

10 (a) If $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ are *orthonormal* then the dot product of $\boldsymbol{q}_1$ with $c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3 = \boldsymbol{0}$ gives $c_1 = 0$. Similarly $c_2 = c_3 = 0$. *Independent* q's (b) $Q\boldsymbol{x} = \boldsymbol{0} \Rightarrow Q^{\mathrm T}Q\boldsymbol{x} = \boldsymbol{0} \Rightarrow \boldsymbol{x} = \boldsymbol{0}$.

11 (a) Two *orthonormal* vectors are $\boldsymbol{q}_1 = \frac{1}{10}(1,3,4,5,7)$ and $\boldsymbol{q}_2 = \frac{1}{10}(-7,3,4,-5,1)$
(b) Closest in the plane: *project* $QQ^{\mathrm T}(1,0,0,0,0) = (0.5, -0.18, -0.24, 0.4, 0)$.

12 (a) Orthonormal $\boldsymbol{a}$'s: $\boldsymbol{a}_1^{\mathrm T}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm T}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm T}\boldsymbol{a}_1) = x_1$
(b) Orthogonal $\boldsymbol{a}$'s: $\boldsymbol{a}_1^{\mathrm T}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm T}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm T}\boldsymbol{a}_1)$. Therefore $x_1 = \boldsymbol{a}_1^{\mathrm T}\boldsymbol{b}/\boldsymbol{a}_1^{\mathrm T}\boldsymbol{a}_1$
(c) x_1 is the first component of A^{-1} times $\boldsymbol{b}$.

13 The multiple to subtract is $\frac{\boldsymbol{a}^{\mathrm T}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm T}\boldsymbol{a}}$. Then $\boldsymbol{B} = \boldsymbol{b} - \frac{\boldsymbol{a}^{\mathrm T}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm T}\boldsymbol{a}}\boldsymbol{a} = (4,0) - 2 \cdot (1,1) = (2,-2)$.

14 $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} \|\boldsymbol{a}\| & \boldsymbol{q}_1^{\mathrm T}\boldsymbol{b} \\ 0 & \|\boldsymbol{B}\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = QR.$

15 (a) $\boldsymbol{q}_1 = \frac{1}{3}(1,2,-2)$, $\boldsymbol{q}_2 = \frac{1}{3}(2,1,2)$, $\boldsymbol{q}_3 = \frac{1}{3}(2,-2,-1)$ (b) The nullspace of $A^{\mathrm T}$ contains $\boldsymbol{q}_3$ (c) $\widehat{\boldsymbol{x}} = (A^{\mathrm T}A)^{-1}A^{\mathrm T}(1,2,7) = (1,2)$.

16 The projection $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm T}\boldsymbol{b}/\boldsymbol{a}^{\mathrm T}\boldsymbol{a})\boldsymbol{a} = 14\boldsymbol{a}/49 = 2\boldsymbol{a}/7$ is closest to $\boldsymbol{b}$; $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\| = \boldsymbol{a}/7$ is $(4,5,2,2)/7$. $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (-1,4,-4,-4)/7$ has $\|\boldsymbol{B}\| = 1$ so $\boldsymbol{q}_2 = \boldsymbol{B}$.

17 $p = (\boldsymbol{a}^{\mathrm T}\boldsymbol{b}/\boldsymbol{a}^{\mathrm T}\boldsymbol{a})\boldsymbol{a} = (3,3,3)$ and $\boldsymbol{e} = (-2,0,2)$. $\boldsymbol{q}_1 = (1,1,1)/\sqrt{3}$ and $\boldsymbol{q}_2 = (-1,0,1)/\sqrt{2}$.

18 $\boldsymbol{A} = \boldsymbol{a} = (1,-1,0,0)$; $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (\frac{1}{2}, \frac{1}{2}, -1, 0)$; $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{p}_A - \boldsymbol{p}_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Notice the pattern in those orthogonal $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. In $\mathbf{R}^5$, $\boldsymbol{D}$ would be $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$.

19 If $A = QR$ then $A^{\mathrm T}A = R^{\mathrm T}Q^{\mathrm T}QR = R^{\mathrm T}R =$ *lower* triangular times *upper* triangular (this Cholesky factorization of $A^{\mathrm T}A$ uses the same R as Gram-Schmidt!). The example has $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = QR$ and the same R appears in $A^{\mathrm T}A = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = R^{\mathrm T}R$.

20 (a) *True* (b) *True*. $Q\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2$. $\|Q\boldsymbol{x}\|^2 = x_1^2 + x_2^2$ because $\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 = 0$.

21 The orthonormal vectors are $\boldsymbol{q}_1 = (1,1,1,1)/2$ and $\boldsymbol{q}_2 = (-5,-1,1,5)/\sqrt{52}$. Then $\boldsymbol{b} = (-4,-3,3,0)$ projects to $\boldsymbol{p} = (-7,-3,-1,3)/2$. And $\boldsymbol{b} - \boldsymbol{p} = (-1,-3,7,-3)/2$ is orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

22 $A = (1,1,2)$, $B = (1,-1,0)$, $C = (-1,-1,1)$. These are not yet unit vectors.

23 You can see why $\boldsymbol{q}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \boldsymbol{q}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \boldsymbol{q}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$. $A = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&4\\0&3&6\\0&0&5\end{bmatrix} = QR$.

24 (a) One basis for the subspace $\boldsymbol{S}$ of solutions to $x_1 + x_2 + x_3 - x_4 = 0$ is $\boldsymbol{v}_1 = (1,-1,0,0)$, $\boldsymbol{v}_2 = (1,0,-1,0)$, $\boldsymbol{v}_3 = (1,0,0,1)$ (b) Since $\boldsymbol{S}$ contains solutions to $(1,1,1,-1)^{\mathrm{T}}\boldsymbol{x} = 0$, a basis for $\boldsymbol{S}^\perp$ is $(1,1,1,-1)$ (c) Split $(1,1,1,1) = \boldsymbol{b}_1 + \boldsymbol{b}_2$ by projection on $\boldsymbol{S}^\perp$ and $\boldsymbol{S}$: $\boldsymbol{b}_2 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ and $\boldsymbol{b}_1 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$.

25 This question shows 2 by 2 formulas for QR; breakdown $R_{22} = 0$ when A is singular. $\begin{bmatrix}2&1\\1&1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}2&-1\\1&2\end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix}5&3\\0&1\end{bmatrix}$. Singular $\begin{bmatrix}1&1\\1&1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}2&2\\0&\mathbf{0}\end{bmatrix}$. The Gram-Schmidt process breaks down when $ad - bc = 0$.

26 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

27 When a and b are not orthogonal, the projections onto these lines *do not add* to the projection onto the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$. We must use the orthogonal A and B (or orthonormal $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$) to be allowed to add 1D projections.

28 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

29 $\boldsymbol{q}_1 = \frac{1}{3}(2,2,-1)$, $\boldsymbol{q}_2 = \frac{1}{3}(2,-1,2)$, $\boldsymbol{q}_3 = \frac{1}{3}(1,-2,-2)$.

30 The columns of the wavelet matrix W are *orthonormal*. Then $W^{-1} = W^{\mathrm{T}}$. See Section 7.2 for more about wavelets: a useful orthonormal basis with many zeros.

31 (a) $c = \frac{1}{2}$ normalizes all the orthogonal columns to have unit length (b) The projection $(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a}$ of $\boldsymbol{b} = (1,1,1,1)$ onto the first column is $\boldsymbol{p}_1 = \frac{1}{2}(-1,1,1,1)$. (Check $\boldsymbol{e} = \mathbf{0}$.) To project onto the plane, add $\boldsymbol{p}_2 = \frac{1}{2}(1,-1,1,1)$ to get $(0,0,1,1)$.

32 $Q_1 = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$ across plane $y + z = 0$.

33 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal and zeros elsewhere.

34 (a) $Q\boldsymbol{u} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$. This is $-\boldsymbol{u}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$ equals 1 (b) $Q\boldsymbol{v} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = 0$.

35 Starting from $\boldsymbol{A} = (1,-1,0,0)$, the orthogonal (not orthonormal) vectors $\boldsymbol{B} = (1,1,-2,0)$ and $\boldsymbol{C} = (1,1,1,-3)$ and $\boldsymbol{D} = (1,1,1,1)$ are in the directions of $\boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$. The 4 by 4 and 5 by 5 matrices with *integer orthogonal columns* (not orthogonal rows, since not orthonormal Q!) are $\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{D}\end{bmatrix} = \begin{bmatrix}1&1&1&1\\-1&1&1&1\\0&-2&1&1\\0&0&-3&1\end{bmatrix}$ and

$$\begin{bmatrix}1&1&1&1&1\\-1&1&1&1&1\\0&-2&1&1&1\\0&0&-3&1&1\\0&0&0&-4&1\end{bmatrix}$$

36 $[Q, R] = \boldsymbol{q}\,\boldsymbol{r}(A)$ produces from A (m by n of rank n) a "*full-size*" square $Q = [\,Q_1 \;\; Q_2\,]$ and $\begin{bmatrix} R \\ 0 \end{bmatrix}$. The columns of Q_1 are the orthonormal basis from Gram-Schmidt of the *column space* of A. The $m - n$ columns of Q_2 are an orthonormal basis for the *left nullspace* of A. Together the columns of $Q = [\,Q_1 \;\; Q_2\,]$ are an orthonormal basis for $\mathbf{R}^m$.

37 This question describes the next $\boldsymbol{q}_{n+1}$ in Gram-Schmidt using the matrix Q with the columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ (instead of using those $\boldsymbol{q}$'s separately). Start from $\boldsymbol{a}$, *subtract its projection* $\boldsymbol{p} = Q^{\mathrm{T}}\boldsymbol{a}$ onto the earlier $\boldsymbol{q}$'s, *divide by the length* of $\boldsymbol{e} = \boldsymbol{a} - Q^{\mathrm{T}}\boldsymbol{a}$ to get $\boldsymbol{q}_{n+1} = \boldsymbol{e}/\|\boldsymbol{e}\|$.

Problem Set 5.1, page 251

1 $\det(2A) = 8$; $\det(-A) = (-1)^4 \det A = \frac{1}{2}$; $\det(A^2) = \frac{1}{4}$; $\det(A^{-1}) = 2 = \det(A^{\mathrm{T}})^{-1}$.

2 $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}$ and $\det(-A) = (-1)^3 \det A = 1$; $\det(A^2) = 1$; $\det(A^{-1}) = -1$.

3 (a) *False*: $\det(I + I)$ is not $1 + 1$ (b) *True*: The product rule extends to ABC (use it twice) (c) *False*: $\det(4A)$ is $4^n \det A$ (d) *False*: $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is invertible.

4 Exchange rows 1 and 3 to show $|J_3| = -1$. Exchange rows 1 and 4, then 2 and 3 to show $|J_4| = 1$.

5 $|J_5| = 1$, $|J_6| = -1$, $|J_7| = -1$. Determinants $1, 1, -1, -1$ repeat so $|J_{101}| = 1$.

6 To prove Rule 6, multiply the zero row by $t = 2$. The determinant is multiplied by 2 (Rule 3) but the matrix is the same. So $2\det(A) = \det(A)$ and $\det(A) = 0$.

7 $\det(Q) = 1$ for rotation and $\det(Q) = -1$ for reflection $(1 - 2\sin^2\theta - 2\cos^2\theta = -1)$.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so det can't blow up.

9 $\det A = 1$ from two row exchanges. $\det B = 2$ (subtract rows 1 and 2 from row 3, then columns 1 and 2 from column 3). $\det C = 0$ (equal rows) even though $C = A + B$!

10 If the entries in every row add to zero, then $(1, 1, \ldots, 1)$ is in the nullspace: singular A has det $= 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$ and *not* $-\det DC$. If n is even we can have an invertible CD.

12 $\det(A^{-1})$ divides twice by $ad - bc$ (once for each row). This gives $\frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc}$.

13 Pivots $1, 1, 1$ give determinant $= 1$; pivots $1, -2, -3/2$ give determinant $= 3$.

14 $\det(A) = 36$ and the 4 by 4 second difference matrix has det $= 5$.

15 The first determinant is 0, the second is $1 - 2t^2 + t^4 = (1 - t^2)^2$.

16 A singular rank one matrix has determinant $= 0$. The skew-symmetric K also $\det K = 0$ (see #**17**).

17 Any 3 by 3 skew-symmetric K has $\det(K^{\mathrm{T}}) = \det(-K) = (-1)^3\det(K)$. This is $-\det(K)$. But always $\det(K^{\mathrm{T}}) = \det(K)$. So we must have $\det(K) = 0$ for 3 by 3.

18 $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$ (to reach 2 by 2, eliminate a and a^2 in row 1 by column operations). Factor out $b-a$ and $c-a$ from the 2 by 2: $(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$.

19 For triangular matrices, just multiply the diagonal entries: $\det(U) = 6, \det(U^{-1}) = \frac{1}{6}$, and $\det(U^2) = 36$. 2 by 2 matrix: $\det(U) = ad, \det(U^2) = a^2d^2$. If $ad \neq 0$ then $\det(U^{-1}) = 1/ad$.

20 $\det\begin{bmatrix} a-Lc & b-Ld \\ c-\ell a & d-\ell b \end{bmatrix}$ reduces to $(ad-bc)(1-L\ell)$. The determinant changes if you do two row operations at once.

21 Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

22 $\det(A) = 3, \det(A^{-1}) = \frac{1}{3}, \det(A - \lambda I) = \lambda^2 - 4\lambda + 3$. The numbers $\lambda = 1$ and $\lambda = 3$ give $\det(A - \lambda I) = 0$. *Note to instructor*: If you discuss this exercise, you can explain that this is the reason determinants come before eigenvalues. Identify $\lambda = 1$ and $\lambda = 3$ as the eigenvalues of A.

23 $\det(A) = 10$, $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$, $\det(A^2) = 100$, $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$ has det $\frac{1}{10}$. $\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = \mathbf{2}$ or $\lambda = \mathbf{5}$; those are eigenvalues.

24 Here $A = LU$ with $\det(L) = 1$ and $\det(U) = -6$ product of pivots, so also $\det(A) = -6$. $\det(U^{-1}L^{-1}) = -\frac{1}{6} = 1/\det(A)$ and $\det(U^{-1}L^{-1}A)$ is $\det I = 1$.

25 When the i, j entry is ij, row 2 = 2 times row 1 so $\det A = 0$.

26 When the ij entry is $i + j$, row 3 − row 2 = row2 − row 1 so A is singular: $\det A = 0$.

27 $\det A = abc$, $\det B = -abcd$, $\det C = a(b-a)(c-b)$ by doing elimination.

28 (a) *True*: $\det(AB) = \det(A)\det(B) = 0$ (b) *False*: A row exchange gives $-\det =$ product of pivots. (c) *False*: $A = 2I$ and $B = I$ have $A - B = I$ but the determinants have $2^n - 1 \neq 1$ (d) *True*: $\det(AB) = \det(A)\det(B) = \det(BA)$.

29 A is rectangular so $\det(A^{\mathrm{T}}A) \neq (\det A^{\mathrm{T}})(\det A)$: these determinants are not defined.

30 Derivatives of $f = \ln(ad - bc)$: $\begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}$.

31 The Hilbert determinants are $1, 8\times10^{-2}, 4.6\times10^{-4}, 1.6\times10^{-7}, 3.7\times10^{-12}, 5.4\times10^{-18}, 4.8 \times 10^{-25}, 2.7 \times 10^{-33}, 9.7 \times 10^{-43}, 2.2 \times 10^{-53}$. Pivots are ratios of determinants so the 10th pivot is near 10^{-10}. The Hilbert matrix is numerically difficult (*ill-conditioned*).

32 Typical determinants of rand(n) are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. randn(n) with normal distribution gives $10^{31}, 10^{78}, 10^{186}$, Inf which means $\geq 2^{1024}$. MATLAB allows $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives Inf!

33 I now know that maximizing the determinant for $1, -1$ matrices is **Hadamard's problem** (1893): see Brenner in American Math. Monthly volume 79 (1972) 626-630. Neil Sloane's wonderful On-Line Encyclopedia of Integer Sequences (**research.att.com/~njas**) includes the solution for small n (and more references) when the problem is changed to $0, 1$ matrices. That sequence A003432 starts from $n = 0$ with 1, 1, 1, 2, 3, 5, 9. Then the $1, -1$ maximum for size n is 2^{n-1} times the $0, 1$ maximum for size $n-1$ (so $(32)(5) = \mathbf{160}$ for $n = 6$ in sequence **A003433**).

To reduce the $1, -1$ problem from 6 by 6 to the $0, 1$ problem for 5 by 5, multiply the six rows by ± 1 to put $+1$ in column 1. Then subtract row 1 from rows 2 to 6 to get a 5 by 5 submatrix S of $-2, 0$ and divide S by -2.

Here is an advanced MATLAB code and a $1, -1$ matrix with largest $\det A = 48$ for $n = 5$:

```
n = 5; p = (n − 1)^2; A0 =ones(n); maxdet= 0;
for k = 0 : 2^p − 1
Asub = rem(floor(k. ∗ 2.^(−p + 1 : 0)), 2); A = A0; A(2 : n, 2 : n) = 1 − 2∗
reshape(Asub, n − 1, n − 1);
if abs(det(A)) > maxdet, maxdet = abs(det(A)); maxA = A;
end
end
```

Output: maxA = $\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{matrix}$ maxdet = 48.

34 Reduce B by row operations to [row 3; row 2; row 1]. Then $\det B = -6$ (odd permutation).

Problem Set 5.2, page 263

1 $\det A = 1+18+12-9-4-6 = 12$, rows are independent; $\det B = 0$, row 1+row 2 = row 3; $\det C = -1$, independent rows ($\det C$ has one term, odd permutation)

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = -1$, independent.

3 All cofactors of row 1 are zero. A has rank ≤ 2. Each of the 6 terms in $\det A$ is zero. Column 2 has no pivot.

4 $a_{11}a_{23}a_{32}a_{44}$ gives -1, because $2 \leftrightarrow 3$, $a_{14}a_{23}a_{32}a_{41}$ gives $+1$, $\det A = 1 - 1 = 0$; $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$.

5 Four zeros in the same row guarantee $\det = 0$. $A = I$ has 12 zeros (maximum with $\det \neq 0$).

6 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms are sure zeros (b) 15 terms must be zero.

7 $5!/2 = 60$ permutation matrices have det $= +1$. Move row 5 of I to the top; starting from $(5, 1, 2, 3, 4)$ elimination will do four row exchanges.

8 Some term $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is not zero! Move rows $1, 2, \ldots, n$ into rows $\alpha, \beta, \ldots, \omega$. Then these nonzero a's will be on the main diagonal.

9 To get $+1$ for the even permutations, the matrix needs an *even* number of -1's. To get $+1$ for the odd P's, the matrix needs an *odd* number of -1's. So all six terms $= +1$ in the big formula and det $= 6$ are impossible: max(det) $=$ **4**.

10 The $4!/2 = 12$ even permutations are $(1,2,3,4), (2,1,4,3), (3,1,4,2), (4,3,2,1)$, and 8 P's with one number in place and even permutation of the other three numbers. $\det(I + P_{\text{even}}) = 16$ or 4 or 0 (16 comes from $I + I$).

11 $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$. Puzzle: $\det D = 441 = (-21)^2$. *Why*?

12 $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^{\mathrm{T}} = C^{\mathrm{T}}/\det A$.

13 (a) $C_1 = 0,\ C_2 = -1,\ C_3 = 0,\ C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$.

14 We must choose 1's from column 2 then column 1, column 4 then column 3,and so on. Therefore n must be even to have $\det A_n \neq 0$. The number of row exchanges is $n/2$ so $C_n = (-1)^{n/2}$.

15 The 1, 1 cofactor of the n by n matrix is E_{n-1}. The 1, 2 cofactor has a single 1 in its first column, with cofactor E_{n-2}: sign gives $-E_{n-2}$. So $E_n = E_{n-1} - E_{n-2}$. Then E_1 to E_6 is $1, 0, -1, -1, 0, 1$ and this cycle of six will repeat: $E_{100} = E_4 = -1$.

16 The 1, 1 cofactor of the n by n matrix is F_{n-1}. The 1, 2 cofactor has a 1 in column 1, with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the 1, 2 entry to find $F_n = F_{n-1} + F_{n-2}$ (so these determinants are Fibonacci numbers).

17 $|B_4| = 2\det\begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} + \det\begin{bmatrix} 1 & -1 & \\ -1 & 2 & \\ & -1 & -1 \end{bmatrix} = 2|B_3| - \det\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} =$ $2|B_3| - |B_2|$. $|B_3|$ and $-|B_2|$ are cofactors of row 4 of B_4.

18 Rule 3 (linearity in row 1) gives $|B_n| = |A_n| - |A_{n-1}| = (n + 1) - n = 1$.

19 Since x, x^2, x^3 are all in the same row, they are never multiplied in $\det V_4$. The determinant is zero at $x = a$ or b or c, so $\det V$ has factors $(x-a)(x-b)(x-c)$. Multiply by the cofactor V_3. The Vandermonde matrix $V_{ij} = (x_i)^{j-1}$ is for fitting a polynomial $p(\boldsymbol{x}) = \boldsymbol{b}$ at the points x_i. It has $\det V =$ product of all $x_k - x_m$ for $k > m$.

20 $G_2 = -1, G_3 = 2, G_4 = -3$, and $G_n = (-1)^{n-1}(n - 1) =$ (product of the λ's).

21 $S_1 = 3, S_2 = 8, S_3 = 21$. The rule looks like every second number in Fibonacci's sequence $\ldots 3, 5, 8, 13, 21, 34, 55, \ldots$ so the guess is $S_4 = 55$. Following the solution to Problem 30 with 3's instead of 2's confirms $S_4 = 81+1-9-9-9 = 55$. Problem 33 directly proves $S_n = F_{2n+2}$.

22 Changing 3 to 2 in the corner reduces the determinant F_{2n+2} by 1 times the cofactor of that corner entry. This cofactor is the determinant of S_{n-1} (one size smaller) which is F_{2n}. Therefore changing 3 to 2 changes the determinant to $F_{2n+2} - F_{2n}$ which is F_{2n+1}.

23 (a) If we choose an entry from B we must choose an entry from the zero block; result zero. This leaves entries from A times entries from D leading to $(\det A)(\det D)$
(b) and (c) Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. See #25.

24 (a) All L's have $\det = 1$; $\det U_k = \det A_k = 2, 6, -6$ for $k = 1, 2, 3$ (b) Pivots $2, \frac{3}{2}, \frac{-1}{3}$.

25 Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

26 If A is a row and B is a column then $\det M = \det AB =$ dot product of A and B. If A is a column and B is a row then AB has rank 1 and $\det M = \det AB = 0$ (unless $m = n = 1$). This block matrix is invertible when AB is invertible which certainly requires $m \le n$.

27 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. Derivative with respect to $a_{11} =$ cofactor C_{11}.

28 Row $1 - 2$ row $2 +$ row $3 = 0$ so this matrix is singular.

29 There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+\,(1,1)(2,2)(3,3)(4,4)\; +\; (1,2)(2,1)(3,4)(4,3)\; -\; (1,2)(2,1)(3,3)(4,4)\; -\; (1,1)(2,2)(3,4)(4,3)\; -\; (1,1)(2,3)(3,2)(4,4)$. Total -1.

30 The 5 products in solution 29 change to $16 + 1 - 4 - 4 - 4$ since A has 2's and -1's:

$$(2)(2)(2)(2) + (-1)(-1)(-1)(-1) - (-1)(-1)(2)(2) - (2)(2)(-1)(-1) - (2)(-1)(-1)(2).$$

31 $\det P = -1$ because the cofactor of $P_{14} = 1$ in row one has sign $(-1)^{1+4}$. The big formula for $\det P$ has only one term $(1 \cdot 1 \cdot 1 \cdot 1)$ with minus sign because three exchanges take $4, 1, 2, 3$ into $1, 2, 3, 4$; $\det(P^2) = (\det P)(\det P) = +1$ so $\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is *not right*.

32 The problem is to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule:
$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}$.

33 The difference from 20 to 19 multiplies its 3 by 3 cofactor $= 1$: then det drops by 1.

34 (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one of those choices will be zero.

35 Subtracting 1 from the n, n entry subtracts its cofactor C_{nn} from the determinant. That cofactor is $C_{nn} = 1$ (smaller Pascal matrix). Subtracting 1 from 1 leaves 0.

Problem Set 5.3, page 279

1 (a) $\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3$, $\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 6$, $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ so $x_1 = -6/3 = -2$ and $x_2 = 3/3 = 1$ (b) $|A| = 4, |B_1| = 3, |B_2| = 2, |B_3| = 1$. Therefore $x_1 = 3/4$ and $x_2 = -1/2$ and $x_3 = 1/4$.

2 (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c/(ad-bc)$ (b) $y = \det B_2/\det A = (fg-id)/D$.

3 (a) $x_1 = 3/0$ and $x_2 = -2/0$: *no solution* (b) $x_1 = x_2 = \mathbf{0/0}$: *undetermined.*

4 (a) $x_1 = \det([\,\boldsymbol{b}\ \ \boldsymbol{a}_2\ \ \boldsymbol{a}_3\,])/\det A$, if $\det A \neq 0$ (b) The determinant is linear in its first column so $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3|+x_2|\boldsymbol{a}_2\ \boldsymbol{a}_2\ \boldsymbol{a}_3|+x_3|\boldsymbol{a}_3\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$ which is $x_1 \det A$.

5 If the first column in A is also the right side b then $\det A = \det B_1$. Both B_2 and B_3 are singular since a column is repeated. Therefore $x_1 = |B_1|/|A| = 1$ and $x_2 = x_3 = 0$.

6 (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$ (b) $\dfrac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. An invertible symmetric matrix has a symmetric inverse.

7 If all cofactors $= 0$ then A^{-1} would be the zero matrix if it existed; cannot exist. (And the cofactor formula gives $\det A = 0$.) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no zero cofactors but it is not invertible.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. This is $(\det A)I$ and $\det A = 3$. The $1,3$ cofactor of A is 0. Multiplying by 4 or 100: no change.

9 If we know the cofactors and $\det A = 1$, then $C^{\mathrm{T}} = A^{-1}$ and also $\det A^{-1} = 1$. Now A is the inverse of C^{T}, so A can be found from the cofactor matrix for C.

10 Take the determinant of $AC^{\mathrm{T}} = (\det A)I$. The left side gives $\det AC^{\mathrm{T}} = (\det A)(\det C)$ while the right side gives $(\det A)^n$. Divide by $\det A$ to reach $\det C = (\det A)^{n-1}$.

11 The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1}.

12 Both $\det A$ and $\det A^{-1}$ are integers since the matrices contain only integers. But $\det A^{-1} = 1/\det A$ so $\det A$ must be 1 or -1.

13 $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ has cofactor matrix $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix}$ and $A^{-1} = \dfrac{1}{5}C^{\mathrm{T}}$.

14 (a) Lower triangular L has cofactors $C_{21} = C_{31} = C_{32} = 0$ (b) $C_{12} = C_{21}$, $C_{31} = C_{13}, C_{32} = C_{23}$ make S^{-1} symmetric. (c) Orthogonal Q has cofactor matrix $C = (\det Q)(Q^{-1})^{\mathrm{T}} = \pm Q$ *also orthogonal.* Note $\det Q = 1$ or -1.

15 For $n = 5$, C contains *25* cofactors and each 4 by 4 cofactor has *24* terms. Each term needs *3* multiplications: total 1800 multiplications vs.125 for Gauss-Jordan.

16 (a) Area $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 10$ (b) and (c) Area $10/2 = 5$, these triangles are half of the parallelogram in (a).

17 Volume$=\begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix}=20$. Area of faces $=$ length of cross product $= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} =$ $-2\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}$ length$=6\sqrt{2}$

18 (a) Area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$ (b) $5 +$ new triangle area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

19 $\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4 = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$ because the transpose has the same determinant. See #22.

20 The edges of the hypercube have length $\sqrt{1+1+1+1} = 2$. The volume $\det H$ is $2^4 = 16$. ($H/2$ has orthonormal columns. Then $\det(H/2) = 1$ leads again to $\det H = 16$.)

21 The maximum volume $L_1L_2L_3L_4$ is reached when the edges are orthogonal in $\mathbf{R}^4$. With entries 1 and -1 all lengths are $\sqrt{4} = 2$. The maximum determinant is $2^4 = 16$, achieved in Problem 20. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved by ± 1.

22 This question is still waiting for a solution! An 18.06 student showed me how to transform the parallelogram for A to the parallelogram for A^{T}, without changing its area. (Edges slide along themselves, so no change in baselength or height or area.)

23 $A^{\mathrm{T}}A = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}} \\ \boldsymbol{c}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} & 0 & 0 \\ 0 & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} & 0 \\ 0 & 0 & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} \end{bmatrix}$ has $\begin{matrix} \det A^{\mathrm{T}}A & = & (\|a\|\,\|b\|\,\|c\|)^2 \\ \det A & = & \pm\|a\|\,\|b\|\,\|c\| \end{matrix}$

24 The box has height 4 and volume $= \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = 4$. $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ and $(\boldsymbol{k} \cdot \boldsymbol{w}) = 4$.

25 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients from $(2+x)^n$ in Worked Example **2.4A**. Cube from $2I$ has volume 2^n.

26 The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$ (and $\frac{1}{n!}$ in $\mathbf{R}^n$)

27 $x = r\cos\theta$, $y = r\sin\theta$ give $J = r$. The columns are orthogonal and their lengths are 1 and r.

28 $J = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & \theta \end{vmatrix} = \rho^2\sin\varphi$. This Jacobian is needed for triple integrals inside spheres.

29 From x, y to r, θ: $\begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ (-\sin\theta)/r & (\cos\theta)/r \end{vmatrix}$
$= \dfrac{1}{r} = \dfrac{1}{\text{Jacobian in } \mathbf{27}}$.

30 The triangle with corners $(0,0), (6,0), (1,4)$ has area 24. Rotated by $\theta = 60°$ the area is *unchanged*. The determinant of the rotation matrix is $J = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} =$
$\begin{vmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{vmatrix} = 1$.

31 Base area 10, height 2, volume 20.

32 The volume of the box is $\det \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix} = 20$.

33 $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$. This is $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$.

34 $(\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v} = (\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u} = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$: *Even permutation* of $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ keeps the same determinant. Odd permutations reverse the sign.

35 $S = (2,1,-1)$, area $\|PQ \times PS\| = \|(-2,-2,-1)\| = 3$. The other four corners can be $(0,0,0)$, $(0,0,2)$, $(1,2,2)$, $(1,1,0)$. The volume of the tilted box is $|\det| = 1$.

36 If $(1,1,0)$, $(1,2,1)$, (x,y,z) are in a plane the volume is $\det\begin{bmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = x - y + z = 0$. The "box" with those edges is flattened to zero height.

37 $\det\begin{bmatrix} x & y & z \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} = 7x - 5y + z$ will be *zero* when (x,y,z) is a combination of $(2,3,1)$ and $(1,2,3)$. The plane containing those two vectors has equation $7x - 5y + z = 0$.

38 Doubling each row multiplies the volume by 2^n. Then $2\det A = \det(2A)$ only if $n = 1$.

39 $AC^{\mathrm{T}} = (\det A)I$ gives $(\det A)(\det C) = (\det A)^n$. Then $\det A = (\det C)^{1/3}$ with $n = 4$. With $\det A^{-1} = 1/\det A$, construct A^{-1} using the cofactors. *Invert to find A*.

40 The cofactor formula adds 1 by 1 determinants (which are just entries) *times* their cofactors of size $n-1$. Jacobi discovered that this formula can be generalized. For $n = 5$, Jacobi multiplied each 2 by 2 determinant from rows 1-2 (with columns $a < b$) times a 3 by 3 determinant from rows 3-5 (using the remaining columns $c < d < e$).

The key question is $+$ or $-$ sign (as for cofactors). The product is given a $+$ sign when a, b, c, d, e is an even permutation of 1, 2, 3, 4, 5. This gives the correct determinant $+1$ for that permutation matrix. More than that, all other P that permute a, b and separately c, d, e will come out with the correct sign when the 2 by 2 determinant for columns a, b multiplies the 3 by 3 determinant for columns c, d, e.

41 The Cauchy-Binet formula gives the determinant of a square matrix AB (and AA^{T} in particular) when the factors A, B are rectangular. For (2 by 3) times (3 by 2) there are 3 products of 2 by 2 determinants from A and B (printed in boldface):

$$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & c \\ \boldsymbol{d} & \boldsymbol{e} & f \end{bmatrix}\begin{bmatrix} \boldsymbol{g} & \boldsymbol{j} \\ \boldsymbol{h} & \boldsymbol{k} \\ i & \ell \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{a} & b & \boldsymbol{c} \\ \boldsymbol{d} & e & \boldsymbol{f} \end{bmatrix}\begin{bmatrix} \boldsymbol{g} & \boldsymbol{j} \\ h & k \\ \boldsymbol{i} & \boldsymbol{\ell} \end{bmatrix} \qquad \begin{bmatrix} a & \boldsymbol{b} & \boldsymbol{c} \\ d & \boldsymbol{e} & \boldsymbol{f} \end{bmatrix}\begin{bmatrix} g & j \\ \boldsymbol{h} & \boldsymbol{k} \\ \boldsymbol{i} & \boldsymbol{\ell} \end{bmatrix}$$

Check $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \qquad AB = \begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix}$

Cauchy-Binet: $(4-2)(4-2) + (7-3)(7-3) + (14-12)(14-12) = \mathbf{24}$
$(14)(66) - (30)(30) = \mathbf{24}$

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1 The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^∞. Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (the trace is now $0.2 + 0.3$). Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

2 A has $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $x_1 = (-2,1)$ and $x_2 = (1,1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to **0** and **6**. That zero eigenvalue correctly indicates that $A + I$ is singular.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1,1)$ and $\boldsymbol{x}_2 = (2,-1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

4 A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace $= -1$ and determinant $= -6$) with $x_1 = (3, -2)$ and $x_2 = (1, 1)$. A^2 has the *same eigenvectors* as A, with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.

5 A and B have eigenvalues 1 and 3. $A + B$ has $\lambda_1 = 3$, $\lambda_2 = 5$. Eigenvalues of $A + B$ *are not equal* to eigenvalues of A plus eigenvalues of B.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved in section 6.6, Problems 18-19).

7 The eigenvalues of U (on its diagonal) are the *pivots* of A. The eigenvalues of L (on its diagonal) are all 1's. The eigenvalues of A *are not* the same as the pivots.

8 (a) Multiply $A\boldsymbol{x}$ to see $\lambda\boldsymbol{x}$ which reveals λ (b) Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ to find $\boldsymbol{x}$.

9 (a) Multiply by A: $A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x} = \boldsymbol{\lambda}^2\boldsymbol{x}$ (b) Multiply by A^{-1}: $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x} = \frac{1}{\boldsymbol{\lambda}}\boldsymbol{x}$ (c) Add $I\boldsymbol{x} = \boldsymbol{x}$: $(A + I)\boldsymbol{x} = (\boldsymbol{\lambda} + \mathbf{1})\boldsymbol{x}$.

10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $\boldsymbol{x}_1 = (1, 2)$ and $\boldsymbol{x}_2 = (1, -1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 Columns of $A - \lambda_1 I$ are in the nullspace of $A - \lambda_2 I$ because $M = (A - \lambda_2 I)(A - \lambda_1 I)$ $=$ zero matrix **[**this is the *Cayley-Hamilton Theorem* in Problem 6.2.32**]**. Notice that M has *zero eigenvalues* $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_1) = 0$ and $(\lambda_2 - \lambda_2)(\lambda_2 - \lambda_1) = 0$.

12 The projection matrix P has $\lambda = 1, 0, 1$ with eigenvectors $(1, 2, 0)$, $(2, -1, 0)$, $(0, 0, 1)$. Add the first and last vectors: $(1, 2, 1)$ also has $\lambda = 1$. *Note* $P^2 = P$ leads to $\lambda^2 = \lambda$ so $\lambda = 0$ or 1.

13 (a) $P\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}) = \boldsymbol{u}$ so $\lambda = 1$ (b) $P\boldsymbol{v} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}) = \mathbf{0}$ (c) $\boldsymbol{x}_1 = (-1, 1, 0, 0)$, $\boldsymbol{x}_2 = (-3, 0, 1, 0)$, $\boldsymbol{x}_3 = (-5, 0, 0, 1)$ all have $P\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$.

14 Two eigenvectors of this rotation matrix are $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$ (more generally $c\boldsymbol{x}_1$, and $d\boldsymbol{x}_2$ with $cd \neq 0$).

15 The other two eigenvalues are $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$; the three eigenvalues are $1, 1, -1$.

16 Set $\lambda = 0$ in $\det(A - \lambda I) = (\lambda_1 - \lambda) \ldots (\lambda_n - \lambda)$ to find $\det A = (\lambda_1)(\lambda_2) \cdots (\lambda_n)$.

17 $\lambda_1 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc})$ and $\lambda_2 = \frac{1}{2}(a + d - \sqrt{})$ add to $a + d$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$.

18 These 3 matrices have $\lambda = 4$ and 5, trace 9, det 20: $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -3 & 7 \end{bmatrix}$.

19 (a) rank $= 2$ (b) $\det(B^{\mathrm{T}}B) = 0$ (d) eigenvalues of $(B^2 + I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{5}$.

20 $A = \begin{bmatrix} 0 & 1 \\ \mathbf{-28} & \mathbf{11} \end{bmatrix}$ has trace 11 and determinant 28, so $\lambda = 4$ and 7. Moving to a 3 by 3 companion matrix, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathbf{6} & \mathbf{-11} & \mathbf{6} \end{bmatrix}$ has $\det(C - \lambda I) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = (1 - \lambda)(2 - \lambda)(3 - \lambda)$. Notice the trace $6 = 1 + 2 + 3$, determinant $6 = (1)(2)(3)$, and also $\mathbf{11} = (1)(2) + (1)(3) + (2)(3)$.

21 $(A-\lambda I)$ has the same determinant as $(A-\lambda I)^{\mathrm{T}}$ because every square matrix has $\det M = \det M^{\mathrm{T}}$. $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ have *different eigenvectors*.

22 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues = trace = $\frac{1}{2}$).

23 $\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}$. Always A^2 is the zero matrix if $\lambda = 0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.32.

24 $\lambda = 0, 0, 6$ (*notice rank* 1 *and trace* 6) with $\boldsymbol{x}_1 = (0, -2, 1)$, $\boldsymbol{x}_2 = (1, -2, 0)$, $\boldsymbol{x}_3 = (1, 2, 1)$.

25 With the same n λ's and $\boldsymbol{x}$'s, $A\boldsymbol{x} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ equals $B\boldsymbol{x} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ for all vectors $\boldsymbol{x}$. So $A = B$.

26 The block matrix has $\lambda = 1, 2$ from B and $5, 7$ from D. All entries of C are multiplied by zeros in $\det(A-\lambda I)$, so C has no effect on the eigenvalues.

27 A has rank 1 with eigenvalues $0, 0, 0, 4$ (the 4 comes from the trace of A). C has rank 2 (ensuring two zero eigenvalues) and $(1, 1, 1, 1)$ is an eigenvector with $\lambda = 2$. With trace 4, the other eigenvalue is also $\lambda = 2$, and its eigenvector is $(1, -1, 1, -1)$.

28 B has $\lambda = -1, -1, -1, 3$ and C has $\lambda = 1, 1, 1, -3$. Both have $\det = -3$.

29 Triangular matrix: $\lambda(A) = 1, 4, 6$; $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$; Rank-1 matrix: $\lambda(C) = 0, 0, 6$.

30 $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}a+b\\ c+d\end{bmatrix} = (\boldsymbol{a}+\boldsymbol{b})\begin{bmatrix}1\\ 1\end{bmatrix}$; $\lambda_2 = \boldsymbol{d}-\boldsymbol{b}$ to produce the correct trace $(a+b)+(d-b) = a+d$.

31 Eigenvector $(1, 3, 4)$ for A with $\lambda = 11$ and eigenvector $(3, 1, 4)$ for PAP^{T}. Eigenvectors with $\lambda \neq 0$ must be in the column space since $A\boldsymbol{x}$ is always in the column space, and $\boldsymbol{x} = A\boldsymbol{x}/\lambda$.

32 (a) $\boldsymbol{u}$ is a basis for the nullspace, $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space
(b) $\boldsymbol{x} = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any $c\boldsymbol{u}$ from the nullspace
(c) If $A\boldsymbol{x} = \boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space: wrong dimension 3.

33 If $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$ then $A^2 = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u})\boldsymbol{v}^{\mathrm{T}}$ is the zero matrix and $\lambda^2 = 0, 0$ and $\lambda = 0, 0$ and trace $(A) = 0$. This zero trace also comes from adding the diagonal entries of $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$:

$$A = \begin{bmatrix}u_1\\ u_2\end{bmatrix}\begin{bmatrix}v_1 & v_2\end{bmatrix} = \begin{bmatrix}u_1v_1 & u_1v_2\\ u_2v_1 & u_2v_2\end{bmatrix} \quad \text{has trace } u_1v_1 + u_2v_2 = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$$

34 $\det(P-\lambda I) = 0$ gives the equation $\lambda^4 = 1$. This reflects the fact that $P^4 = I$. The solutions of $\lambda^4 = 1$ are $\lambda = 1, i, -1, -i$. The real eigenvector $\boldsymbol{x}_1 = (1, 1, 1, 1)$ is not changed by the permutation P. Three more eigenvectors are (i, i^2, i^3, i^4) and $(1, -1, 1, -1)$ and $(-i, (-i)^2, (-i)^3, (-i)^4)$.

35 3 by 3 permutation matrices: Since $P^{\mathrm{T}}P = I$ gives $(\det P)^2 = 1$, the determinant is 1 or -1. The pivots are always 1 (but there may be row exchanges). The trace of P can be 3 (for $P = I$) or 1 (for row exchange) or 0 (for double exchange). The possible eigenvalues are 1 and -1 and $e^{2\pi i/3}$ and $e^{-2\pi i/3}$.