PY511 PS7

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Problem 1

(a)

$$\langle 0| e^{ik\hat{x}} |0\rangle = \langle 0| e^{ik\hat{x}} \cdot \mathbb{I} |0\rangle = \int_{-\infty}^{\infty} e^{ikx'} \langle 0|x'\rangle \langle x'|0\rangle \, dx' \tag{1}$$

Notice that

$$\langle 0|x'\rangle = \langle x'|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \tag{2}$$

is just the ground state harmonic oscillator wave function. Thus, the above equation becomes:

$$\langle 0| e^{ik\hat{x}} |0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{\pi\hbar}{m\omega}\right)^{1/2} e^{-\hbar k^2/4m\omega} \tag{3}$$

Also notice that

$$\langle 0| \hat{x}^2 |0\rangle = \Delta_x^2 [\langle 0| \hat{a^\dagger}\hat{a^\dagger} + \hat{a^\dagger}\hat{a} + \hat{a}\hat{a^\dagger} + \hat{a}\hat{a} |0\rangle] = \Delta_x^2 = \frac{\hbar}{2m\omega} \tag{4}$$

so

$$e^{-\hbar k^2/4m\omega} = e^{-\frac{k^2}{2}\langle 0|\hat{x}^2|0\rangle} \tag{5}$$

Therefore,

$$\langle 0| e^{ik\hat{x}} |0\rangle = \exp\left(-\frac{1}{2}k^2 \langle 0| \hat{x}^2 |0\rangle\right) \tag{6}$$

(b)

Similar to above, we have

$$\begin{aligned} \langle 0|k'\rangle &= \langle k'|0\rangle = \frac{1}{sqrt2\pi}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \int_{-\infty}^{\infty} e^{-ikx} e^{-m\omega x^2/2\hbar} dx \\ &= \left(\frac{\hbar}{m\omega}\right)^{1/4} e^{-\hbar k^2/2m\omega} \end{aligned} \tag{7}$$

where

$$\langle a| \hat{p}^2 |0\rangle = -\Delta_p^2 \langle 0| \hat{a^\dagger}\hat{a^\dagger} - \hat{a^\dagger}\hat{a} - \hat{a}\hat{a^\dagger} + \hat{a}\hat{a} |0\rangle = \Delta_p^2 = \frac{\hbar m\omega}{2} \tag{8}$$

$$\begin{aligned} \langle 0| e^{ix\hat{k}} |0\rangle &= \int_{-\infty}^{\infty} e^{ixk'} \langle 0|k'\rangle \langle k'|0\rangle \, dk' = \left(\frac{\hbar}{m\omega}\right)^{1/2} \int_{-\infty}^{\infty} e^{ixk'} e^{-hk'^2/m\omega} dk' \\ &= \left(\frac{\hbar}{m\omega}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right)^{1/2} e^{-m\omega x^2/2\hbar} = e^{-x^2/\hbar^2 \langle 0|\hat{p}^2|0\rangle} \end{aligned} \tag{9}$$

Therefore,

$$\langle 0| e^{i\hat{k}x} |0\rangle = \exp\left(-\frac{x^2}{\hbar^2} \langle 0| \hat{p}^2 |0\rangle\right) \tag{10}$$

Problem 2

(a)

The time independent Schrodinger's equation for delta potential is given by:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \lambda\delta(x)\psi = E\psi \tag{11}$$

and the associated bound state solution in position space is given by:

$$\psi_b(x) = \frac{\sqrt{m\lambda}}{\hbar} e^{-m\lambda|x|/\hbar^2} \tag{12}$$

Let's perform Fourier transform to Eq. 12.

$$\begin{aligned}
\tilde{\psi_b}(k) &= \mathcal{F}[\psi_b(x)] = \frac{1}{\sqrt{2\pi\hbar}}\frac{\sqrt{m\lambda}}{\hbar}\int_{-\infty}^{\infty} e^{\frac{-ikx}{\hbar}} e^{\frac{-m\lambda|x|}{\hbar^2}}dx \qquad (13)\\
&= \frac{1}{\sqrt{2\pi\hbar}}\frac{\sqrt{m\lambda}}{\hbar}\left[\int_{-\infty}^{0} e^{\frac{-ikx}{\hbar}} e^{\frac{m\lambda|x|}{\hbar^2}}dx + \int_{0}^{\infty} e^{\frac{-ikx}{\hbar}} e^{-\frac{m\lambda|x|}{\hbar^2}}dx\right]\\
&= \frac{1}{\sqrt{2\pi\hbar}}\frac{\sqrt{m\lambda}}{\hbar}\left[\frac{\hbar^2}{\lambda m + i\hbar k} + \frac{\hbar^2}{\lambda m - i\hbar k}\right]\\
&= \boxed{\sqrt{\frac{2m^3\lambda^3\hbar}{\pi}}\frac{1}{\lambda^2 m^2 + \hbar^2 k^2} = \sqrt{\frac{m\lambda^3}{2\pi\hbar^3}}\frac{1}{k^2/2m - E}\text{ , where } E = -\frac{m\lambda^2}{2\hbar^2}}
\end{aligned}$$

(b)

For an infinite square well, the nth excited state has a wavefunction of the following form in the position space:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \tag{14}$$

Let's Fourier transform the wavefunction of second excited state.

$$\tilde{\psi_2}(k) = \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\int_{-\infty}^{\infty} e^{-\frac{ikx}{\hbar}}\sin\left(\frac{2\pi x}{a}\right)dx = \frac{1}{\sqrt{\pi a\hbar}}\frac{2a\hbar^2}{a^2k^2 - 4\hbar^2}\left[e^{-iak/\hbar} - 1\right] \tag{15}$$

The probability of the particle having momentum between p and $p + dp$ is given by taking the modulus square:

$$|\tilde{\psi_2}(k)|^2 dp = \frac{8\pi^2 a^2\hbar^4}{(a^2k^2 - 4\pi^2\hbar^2)^2}\left[1 - \cos\left(\frac{ak}{\hbar}\right)\right] \tag{16}$$

Problem 3

We start from considering two observers, Alice and Bob, observing two entangled particles moving in opposite directions. They measure the spin along three mutually non-orthogonal directions $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$. Let's denote the positive measurement result of direction $\hat{\mathbf{a}}$ to be $\hat{\mathbf{a}}+$, and similar for negative result and for other directions. The total spin of the entangled pair should be 0 in every direction, so the observation result of a particle's spin in one direction should be the opposite of the result of the other particle's spin in the same direction. For example, if the measurement result of the 1st particle is $(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$, then the result of the other particle must be $(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$, each component of the spin adding up to 0.

Supposing the number of entangled pairs N is large, we can statistically say that there are equal number of entangled pairs that yield each combination of the spin direction. This is tabulated in Table 1, where $\sum_{i=1}^{8} N_i = N$ is large.

Now, let's suppose that Alice measures $\hat{\mathbf{a}}+$ and Bob measures $\hat{\mathbf{b}}+$. Then, referring to the table above, we can conclude that the pair is in either N_3 or N_4. Since $N_i \in \mathbb{N}$, it should hold true that, say,

$$N_3 + N_4 \leq (N_2 + N_4) + (N_3 + N_7) \tag{17}$$

Meanwhile, the probability of measuring $(\mathbf{S_1}\cdot\hat{\mathbf{a}}, \mathbf{S_2}\cdot\hat{\mathbf{b}}) = (\hat{\mathbf{a}}+, \hat{\mathbf{b}}+)$ is given by:

$$P(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) = \frac{N_3 + N_4}{N} \tag{18}$$

Number of Such Pairs	Outcome of Particle 1	Outcome of Particle 2
N_1	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$
N_2	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$
N_3	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$
N_4	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$
N_5	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$
N_6	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$
N_7	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$
N_8	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$

Table 1: All possible combination of measurement outcomes

, and similarly,

$$P(\hat{\mathbf{a}}+, \hat{\mathbf{c}}+) = \frac{N_2 + N_4}{N} \tag{19}$$

$$P(\hat{\mathbf{c}}+, \hat{\mathbf{b}}+) = \frac{N_3 + N_7}{N} \tag{20}$$

Thus, Eq. 17 becomes

$$\boxed{P(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+) \leq P(\hat{\mathbf{a}}+, \hat{\mathbf{c}}+) + P(\hat{\mathbf{c}}+, \hat{\mathbf{b}}+)} \tag{21}$$

Now we have retrieved Bell's inequality. The significance of this inequality is that, had there been a hidden variable that ensures the correlation between the entangled particles, the outcomes of such experiments that measure the spins of entangled particles should obey the inequality. Recent Nobel prize laureates, namely, Aspect, Clauser, and Zeilinger, has experimentally shown that such quantum measurements do not obey the Bell's inequality, thus proving that the hidden variables are not present, and the principle of nonlocality holds for quantum mechanics.

Problem 4

Done below.

Problem 5

Done below.

Problem HW7.4 Position operator in the Heisenberg picture

Prior to the Midterm, you learned about the Schrödinger picture and the Heisenberg picture. The operator $\hat{x}$ discussed in lecture is in the Schrödinger picture (the subscript "S" is omitted in common practice).

The time-dependent position operator in the Heisenberg picture is

$$\hat{x}_H(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{x}e^{-\frac{i}{\hbar}\hat{H}t}$$

Remember that at time $t = 0$ the operators in the two pictures are the same, $x_H(t = 0) = \hat{x}$

(a) Find the two time-dependent *quantum autocorrelation functions* in the energy eigenstate $|n\rangle$ defined as

$$g_x(t, 0) \equiv \langle n|\hat{x}_H(t)\hat{x}_H(0)|n\rangle$$

$$g_x(0, t) \equiv \langle n|\hat{x}_H(0)\hat{x}_H(t)|n\rangle$$

Note the order in which the two operators appear. Would you expect the two autocorrelation functions to be equal?

(b) Find the symmetric correlation functions

$$g_{x+} = \frac{1}{2}(g_x(t, 0) + g_x(0, t))$$

(a)

$$\hat{x}_H(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{x}e^{-\frac{i}{\hbar}\hat{H}t}, \quad \hat{x}_H(0) = \Delta_x(\hat{a} + \hat{a}^\dagger)$$

$$\langle n|\hat{x}_H(t)\hat{x}_H(0)|n\rangle = \langle n|e^{\frac{i}{\hbar}\hat{H}t}\hat{x}e^{-\frac{i}{\hbar}\hat{H}t}\hat{x}|n\rangle = \Delta_x^2\left[\langle n|e^{\frac{i}{\hbar}\hat{H}t}\hat{x}e^{-\frac{i}{\hbar}\hat{H}t}\hat{a}|n\rangle + \langle n|e^{\frac{i}{\hbar}\hat{H}t}\hat{x}e^{-\frac{i}{\hbar}\hat{H}t}\hat{a}^\dagger|n\rangle\right]$$

$$= \Delta_x^2\left[\langle n|e^{\frac{i}{\hbar}\hat{H}t}\hat{x}e^{-iw(n-\frac{1}{2})t}\sqrt{n}|n-1\rangle + \langle n|e^{\frac{i}{\hbar}\hat{H}t}\hat{x}e^{-iw(n+\frac{3}{2})t}\sqrt{n+1}|n+1\rangle\right]$$

$$= \Delta_x^2\left[e^{-iw(n-\frac{1}{2})t}\sqrt{n}\left(\langle n|e^{\frac{i}{\hbar}\hat{H}t}a^\dagger|n-1\rangle + \cancelto{0}{\langle n|e^{\frac{i}{\hbar}\hat{H}t}a|n-1\rangle}\right) + e^{-iw(n+\frac{3}{2})t}\sqrt{n+1}\left(\cancelto{0}{\langle n|e^{\frac{i}{\hbar}\hat{H}t}a^\dagger|n+1\rangle} + \langle n|e^{\frac{i}{\hbar}\hat{H}t}\hat{a}|n+1\rangle\right)\right]$$

$$= \Delta_x^2\left[e^{-iw(n-\frac{1}{2})t}\sqrt{n}\langle n|e^{iw(n+\frac{1}{2})t}\sqrt{n}|n\rangle + e^{-iw(n+\frac{3}{2})t}\sqrt{n+1}\langle n|e^{iw(n+\frac{1}{2})t}\sqrt{n+1}|n\rangle\right]$$

$$= \Delta_x^2\left[n\cdot e^{iwt} + (n+1)e^{-iwt}\right] \qquad \therefore \boxed{g_x(t,0) = \langle n|\hat{x}_H(t)\hat{x}_H(0)|n\rangle = \Delta_x^2\left[ne^{iwt} + (n+1)e^{-iwt}\right]}$$

Similar calculation shows that $g_x(0,t) = g_x^\dagger(t,0)$

(b) $$g_{x+} = \frac{1}{2}\left[g_x(t,0) + g_x(0,t)\right] = \frac{1}{2}\Delta_x^2\left[ne^{iwt} + (n+1)e^{-iwt} + ne^{-iwt} + (n+1)e^{iwt}\right] = \Delta_x^2\left[(2n+1)\cos(wt)\right]$$

$$\therefore \boxed{g_{x+} = \Delta_x^2(2n+1)\cos(wt)}$$

Problem HW7.5 Momentum operator in the Heisenberg picture

Again start with the momentum operator $\hat{p}$ discussed in lecture in the Schrödinger picture (with the subscript "S" omitted).

The time-dependent momentum operator in the Heisenberg picture is

$$\hat{p}_H(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{p}e^{-\frac{i}{\hbar}\hat{H}t}$$

Remember that at time $t = 0$ the operators in the two pictures are the same, $\hat{p}_H(t = 0) = \hat{p}$

(a) Find the two time-dependent autocorrelation functions for state $|n\rangle$ defined as

$$g_p(t, 0) \equiv \langle n|\hat{p}_H(t)\hat{p}_H(0)|n\rangle$$

$$g_p(0, t) \equiv \langle n|\hat{p}_H(0)\hat{p}_H(t)|n\rangle$$

(b) Find the symmetric correlation function

$$g_{+p} = \frac{1}{2}\left(g_p(t, 0) + g_p(0, t)\right)$$

(a) $g_p(t,0) = \langle n|\, p_H(t)\, p_H(0)\,|n\rangle = \langle n|\, e^{\frac{i\hat{H}}{\hbar}t}\,\hat{p}\, e^{-\frac{i\hat{H}}{\hbar}t}\,\hat{p}\,|n\rangle = -\Delta_p^2 \langle n|\, e^{\frac{i\hat{H}}{\hbar}t}\,\hat{p}\, e^{-\frac{i\hat{H}}{\hbar}t}\left(\hat{a}^\dagger|n\rangle - \hat{a}|n\rangle\right)$

$= -\Delta_p^2 \langle n|\, e^{\frac{i\hat{H}}{\hbar}t}\,\hat{p}\left[e^{-i(n+\frac{3}{2})\omega t}\sqrt{n+1}\,|n+1\rangle - e^{-i(n-\frac{1}{2})\omega t}\sqrt{n}\,|n-1\rangle\right]$

$= -\Delta_p^2\left[e^{-i(n+\frac{3}{2})\omega t}\sqrt{n+1}\,\langle n|\, e^{\frac{i\hat{H}}{\hbar}t}\left(\hat{a}^\dagger|n+1\rangle - \hat{a}|n+1\rangle\right) - e^{-i(n-\frac{1}{2})\omega t}\sqrt{n}\,\langle n|\, e^{\frac{i\hat{H}}{\hbar}t}\left(\hat{a}^\dagger|n-1\rangle - \hat{a}|n-1\rangle\right)\right]$

$= \Delta_p^2\left[(n+1)e^{-i\omega t} + ne^{i\omega t}\right]$ $\quad\therefore\quad \boxed{g_p(t,0) = \Delta_p^2\left[(n+1)e^{-i\omega t} + ne^{i\omega t}\right]}$

Similarly, $g_p(t,0) = g_p^*(0,t)$

(b) $g_{pt} = \frac{1}{2}\left[g_p(t,0) + g_p(0,t)\right] = \Delta_p^2(2n+1)\cos(\omega t) = \frac{1}{2}\Delta_p^2\left[(n+1)e^{-i\omega t} + ne^{i\omega t} + (n+1)e^{i\omega t} + ne^{-i\omega t}\right]$

$\therefore\quad \boxed{g_{pt} = \Delta_p^2(2n+1)\cos(\omega t)}$