PY541 PS4

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Problem 1: Information Engines (Sethna 5.2)

(a)

This is an isothermal process, so we have $P_i V_i = P_f V_f$, where $V_f = 2V_i$ because removing the partition increases the volume by a factor of 2. The work done by the atom is given by:

$$W = \int_i^f P dV = \int_i^f \frac{P_i V_i}{V} dV = P_i V_i \log\left(\frac{V_f}{V_i}\right) = P_i V_i \log(2) \tag{1}$$

We also have the ideal gas law $PV = Nk_B T$ where $N = 1$. Rewriting Eq. 1 in terms of T, we have:

$$\boxed{W = k_B T \log(2)} \tag{2}$$

(b)

The sequence is as follows.

1. Lower the upper piston
2. Remove the partition
3. Raise the upper piston
4. Raise the lower piston
5. Insert the partition
6. Lower the lower piston

Reversing the sequence (insert $\rightarrow$ remove, raise $\rightarrow$ lower) turns bit 1 into bit 0, the original state. Thus, the sequence is reversible. There is no net work done.

(c)

If the partition is removed and the available volume doubles, the atom can be in either upper or lower part of the box, so $\Omega_f = 2$. Initially, $\Omega_i = 1$ since the location of the atom is initially known. Thus, the change in entropy is given by:

$$\Delta S_T = S_f - S_i = k_B \log(2) - 0 \Rightarrow \boxed{\Delta S_T = k_B \log(2)} \tag{3}$$

The information entropy is obtained by substituting $k_B \rightarrow k_S = \frac{1}{\log(2)}$:

$$\Delta S_I = \frac{1}{\log(2)} \log(2) = 1 \Rightarrow \boxed{\Delta S_I = 1} \tag{4}$$

(d)

After the bit has been burned, the demon is in an unknown state. Its entropy is $S = k_B \log(2)$. The energy needed to return the demon to its original state is $E = k_B T \log(2)$. The box requires work to lower its entropy, so the second law is NOT violated.

Problem 2: Does Entropy Increase? (Sethna 5.7)

We start from the chain rule.

$$\frac{\partial f(\rho)}{\partial t} = \frac{\partial f(\rho)}{\partial \rho}\frac{\partial \rho(t)}{\partial t} \tag{5}$$

Notice that $\frac{\partial \rho(t)}{\partial t}$ follows the Liouville's theorem. That is,

$$\frac{\partial \rho(t)}{\partial t} + \nabla\rho \cdot \mathbb{V} = 0 \Rightarrow \frac{\partial \rho(t)}{\partial t} = -\nabla\rho \cdot \mathbb{V} \tag{6}$$

Plugging in Eq. 6 into Eq. 5, we obtain:

$$\begin{aligned} \frac{\partial f(\rho)}{\partial t} &= \frac{\partial f(\rho)}{\partial \rho}\left[-\nabla\rho \cdot \mathbb{V}\right] = -\left(\frac{\partial \rho}{\partial \rho}\nabla f(\rho)\right) \cdot \mathbb{V} = -\nabla f \cdot \mathbb{V} \\ &= -\nabla[f\mathbb{V}] = -\sum_{\alpha} \frac{\partial}{\partial q_\alpha}(f(\rho)\dot{q}_\alpha) + \frac{\partial}{\partial p_\alpha}(f(\rho)\dot{p}_\alpha) \end{aligned} \tag{7}$$

where

$$\nabla_\alpha = \frac{\partial}{\partial q_\alpha}\hat{q_\alpha} + \frac{\partial}{\partial p_\alpha}\hat{p_\alpha}, \tag{8}$$

$$\mathbb{V}_\alpha = \dot{q}_\alpha + \dot{p}_\alpha \tag{9}$$

We can see that Eq. 8 is just a continuity equation of probability density in phase space. This follows the divergence theorem.

$$\int_\Omega \frac{\partial f(\rho)}{\partial t} d\mathbb{P}d\mathbb{Q} = \oint_\Sigma f(\rho)\mathbb{V} \cdot d\vec{\Sigma} = 0 \tag{10}$$

where Ω = phase space and Σ = surface of the phase space volume at ∞, assuming that $\rho(\mathbb{P}, \mathbb{Q}) \to 0$ as $(\mathbb{P}, \mathbb{Q}) \to \infty$. Now let's look at the change in entropy with respect to time.

$$\frac{dS}{dt} = -k_B \int \frac{\partial}{\partial t}\rho \log \rho \; d\mathbb{P}d\mathbb{Q} = 0 \tag{11}$$

where we let $f(\rho) = \rho \log \rho$. Therefore,

$$\boxed{\frac{dS}{dt} = 0} \tag{12}$$

Problem 3: Entropy Increases: Diffusion (Sethna 5.10)

Let's work out the time derivative of entropy.

$$\begin{aligned} \frac{dS}{dt} &= -k_B \int \frac{\partial}{\partial t}\rho(x,t)\log\rho(x,t)dx \\ &= -k_B \int \log\rho(x,t)\frac{\partial}{\partial t}\rho(x,t)dx - k_B \int \rho(x,t)\frac{\partial}{\partial t}\log\rho(x,t)dx \\ &= -k_B D \int \log\rho(x,t)\frac{\partial^2\rho}{\partial x^2}dx = -k_B D\left[\log\rho\frac{\partial\rho}{\partial x}\Big|_{-\infty}^{\infty} - \int \frac{1}{\rho}\frac{\partial\rho}{\partial x}\frac{\partial\rho}{\partial x}dx\right] \\ &= k_B D \int \frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2 dx \end{aligned} \tag{13}$$

We see that since probability density $\rho > 0$ and $\left(\frac{\partial\rho}{\partial x}\right)^2 > 0$, the time derivative of entropy is strictly positive.

$$\boxed{\frac{dS}{dt} > 0} \tag{14}$$