PY511 PS3

Raymond Kil

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Problem 1

(a)

i)

The angular frequency is given by:

$$\omega = 2\pi f = 2\pi \frac{1}{0.2} = \boxed{10\ \frac{\text{rad}}{s}} \tag{1}$$

ii)

The wavelength is $\boxed{\lambda = 14\text{ m}}$.

iii)

Speed of the wave is given by:

$$v = f \cdot \lambda = (5\text{s}^{-1})(14\text{m}) = \boxed{70\text{ ms}^{-1}} \tag{2}$$

iv)

Comparing two graphs, we can see that the $\boxed{\text{wave is traveling to the left}}$.

(b)

i)

Looking at the graph, the de Broglie wavelength is $\boxed{\lambda = 14\text{ nm}}$.

ii)

We have the following relation: $E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$, where $k = \frac{2\pi}{\lambda}$ is the angular wavenumber. Thus,

$$\boxed{\frac{E}{m_e c^2} = \frac{\hbar^2 k^2}{2m_e^2 c^2}} \tag{3}$$

iii)

We have $\omega = \frac{E}{\hbar}$ and $\frac{1}{k} = \frac{\lambda}{2\pi}$. Thus, we have:

$$v_1 = \frac{\omega}{k} = \frac{E}{\hbar}\frac{\lambda}{2\pi} = \frac{1.22 \times 10^{-21}\text{J}}{1.05 \times 10^{-34}\text{Js}} \cdot \frac{14 \times 10^{-9}\text{m}}{2\pi} \approx \boxed{25900\text{ m/s}} \tag{4}$$

Compared to speed of light,

$$\frac{v_1}{c} = \frac{25900\text{m/s}}{3 \times 10^8\text{m/s}} \approx \boxed{8.6 \times 10^{-5}} \tag{5}$$

iv)

As a corollary to above, we have the relation $p = \frac{h}{\lambda}$. Thus,

$$v_2 = \frac{p}{m_e} = \frac{h}{\lambda m_e} = \frac{6.62 \times 10^{-34}\text{Js}}{(14 \times 10^{-9}\text{m})(9.11 \times 10^{-31}kg)} \approx \boxed{51900 \text{ m/s}} \tag{6}$$

Again, compared to the speed of light,

$$\frac{v_2}{c} = \frac{51900\text{m/s}}{3 \times 10^8\text{m/s}} \approx \boxed{1.73 \times 10^{-4}} \tag{7}$$

v_1 and v_2 are different. v_2 is about twice v_1. We can see from the above result that the speed of the electron is less than the speed of light by a factor of $\approx 10^{-4}$. Therefore, a non-relativistic description is justified for this electron.

v)

Let's look at the probability density.

$$J = -\frac{ih}{2m}\left[\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x} - \frac{\partial\psi^*(x,t)}{\partial x}\psi(x,t)\right] = \frac{hA^2k}{m} > 0 \tag{8}$$

The probability current density is positive, which means that $\boxed{\text{de Broglie wave is traveling to the left}}$.

(c)

We can see that both classical and de Briglie waves have wavelength and speed. However, de Broglie wave has different phase and group velocities.

Problem 2

(a)

Let's Taylor expand $e^A B e^{-A}$.

$$\begin{aligned} e^A B e^{-A} &= \left[\mathbb{1} + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\right] B \left[\mathbb{1} - A + \frac{1}{2!}A^2 - \frac{1}{3!}A^3 + \cdots\right] \\ &= \mathbb{1}B\mathbb{1} - \mathbb{1}BA + \frac{1}{2!}\mathbb{1}BA^2 - \frac{1}{3!}\mathbb{1}BA^3 + \cdots + AB\mathbb{1} - ABA + \frac{1}{2!}ABA^2 - \frac{1}{3!}ABA^3 + \cdots \\ &\quad + \frac{1}{2!}A^2B\mathbb{1} - \frac{1}{2!}A^2BA + \frac{1}{2!}\frac{1}{2!}A^2BA^2 - \frac{1}{3!}\frac{1}{2!}A^2BA^3 + \cdots \\ &\quad + \frac{1}{3!}A^3B\mathbb{1} - \frac{1}{3!}A^3BA + \frac{1}{2!}\frac{1}{3!}A^3BA^2 - \frac{1}{3!}\frac{1}{3!}A^3BA^3 + \cdots \\ &= B + (AB - BA) + \frac{1}{2!}\left[A^2B - 2ABA + BAA\right] + \frac{1}{3!}\left[-BA^3 + 3ABA^2 - 3A^2BA + A^3B\right] + \cdots \\ &= B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots \text{ Q.E.D.} \end{aligned} \tag{9}$$

(b)

To prove that $[\hat{A}, f(\hat{B})] = [\hat{A}, \hat{B}]f'(\hat{B})$, given that $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, let's first note some of the properties. First, a function of an operator can be Taylor expanded as:

$$f(\hat{B}) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\hat{B}^n. \tag{10}$$

Second, a derivative of $f(\hat{B})$ can also be Taylor expanded:

$$f'(\hat{B}) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n+1)}(0)\hat{B}^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} f^{(n)}(0)\hat{B}^{n-1}. \tag{11}$$

Third, suppose $[\hat{A}, \hat{B}^n] = n[\hat{A}, \hat{B}]\hat{B}^{n-1}$ is true for some $n \in \mathbb{Z}^+$. Then the following relation holds if the commutator commutes.

$$\begin{aligned}
[\hat{A}, \hat{B}^{n+1}] &= [\hat{A}, \hat{B} \cdot \hat{B}^n] = \hat{B}[\hat{A}, \hat{B}^n] + [\hat{A}, \hat{B}^n]\hat{B} = \hat{B}n[\hat{A}, \hat{B}]\hat{B}^{n-1} + [\hat{A}, \hat{B}]^n \qquad (12)\\
&= [\hat{A}, \hat{B}](n\hat{B}\hat{B}^{n-1} + \hat{B}^n) = (n+1)[\hat{A}, \hat{B}]\hat{B}^n
\end{aligned}$$

Thus, by induction,

$$[\hat{A}, \hat{B}^n] = n[\hat{A}, \hat{B}]\hat{B}^{n-1} \qquad (13)$$

is true. Now let's prove that $[\hat{A}, f(\hat{B})] = [\hat{A}, \hat{B}]f'(\hat{B})$.

$$\begin{aligned}
[\hat{A}, f(\hat{B})] &= \hat{A}f(\hat{B}) - f(\hat{B})\hat{A} = f(\hat{B}) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\hat{A}\hat{B}^n - \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\hat{B}^n\hat{A} \qquad (14)\\
&= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)[\hat{A}, \hat{B}^n] = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)n[\hat{A}, \hat{B}]\hat{B}^{n-1} \qquad (15)\\
&= \sum_{n=0}^{\infty} \frac{1}{(n-1)!} f^{(n)}(0)[\hat{A}, \hat{B}]\hat{B}^{n-1} = [\hat{A}, \hat{B}]f'(\hat{B}) \qquad (16)
\end{aligned}$$

where in equality 14 we used Eq. 10, in equality 15 we used Eq. 13, and in equality 16 we used Eq. 11. Q. E. D. In case of $[g(\hat{A}), \hat{B}]$, the order of the operators will be reversed throughout the proof, which will end up in

$$[g(\hat{A}), \hat{B}] = g'(\hat{A})[\hat{A}, \hat{B}] \qquad (17)$$

Problem 3

(a)

$$\det \begin{pmatrix} \varepsilon_0 - \lambda & \gamma e^{-i\chi} \\ \gamma e^{i\chi} & \varepsilon_1 - \lambda \end{pmatrix} = (\varepsilon_0 - \lambda)(\varepsilon_1 - \lambda) - \gamma^2 = 0 \Rightarrow \boxed{\lambda_\pm = \frac{\varepsilon_0 + \varepsilon_1 \pm \sqrt{(\varepsilon_0 + \varepsilon_1)^2 - 4(\varepsilon_0\varepsilon_1 - \gamma^2)}}{2}} \qquad (18)$$

(b)

The eigenvectors are given by:

$$\begin{aligned}
|\psi_g\rangle &= \cos(\theta)e^{-\frac{i\chi}{2}} |0\rangle + \sin(\theta)e^{\frac{i\chi}{2}} |1\rangle \qquad (19)\\
|\psi_e\rangle &= -\sin(\theta)e^{-\frac{i\chi}{2}} |0\rangle + \cos(\theta)e^{\frac{i\chi}{2}} |1\rangle \qquad (20)
\end{aligned}$$

(c)

Time evolution operator is given by:

$$\begin{aligned}
U(t, t_0) &= \begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta)e^{-i\chi} \\ \cos(\theta)\sin(\theta)e^{i\chi} & \sin^2(\theta) \end{pmatrix} e^{-\frac{i}{\hbar}\varepsilon_g}(t - t_0) \qquad (21)\\
&+ \begin{pmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta)e^{-i\chi} \\ -\sin(\theta)\cos(\theta)e^{i\chi} & \cos^2(\theta) \end{pmatrix} e^{-\frac{i}{\hbar}\varepsilon_e}(t - t_0)
\end{aligned}$$

(d)

Time-dependent phase angle will result in time-dependent Hamiltonian. This will result in having a time dependent eigenstates. There will be a time factor in the answers for parts (a) and (b).

Problem 4

(a)

The following expression holds for any $\lambda \in \mathbb{C}$:

$$(\langle\alpha| + \lambda^* \langle\beta|) \cdot (|\alpha\rangle + \lambda |\beta\rangle) \geq 0 \qquad (22)$$

Since this is true, we can plug in $\lambda = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$. Then, Eq. 22 becomes:

$$\begin{aligned}
(\langle\alpha| + \lambda^* \langle\beta|) \cdot (|\alpha\rangle + \lambda |\beta\rangle) &= \langle\alpha|\alpha\rangle + \lambda \langle\alpha|\beta\rangle + \lambda^* \langle\beta|\alpha\rangle + \lambda^*\lambda \langle\beta|\beta\rangle \\
&= \langle\alpha|\alpha\rangle + -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} \langle\alpha|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle} \langle\beta|\alpha\rangle + \frac{\langle\alpha|\beta\rangle \langle\beta|\alpha\rangle}{(\langle\beta|\beta\rangle)^2} \langle\beta|\beta\rangle \\
&= \langle\alpha|\alpha\rangle \langle\beta|\beta\rangle - |\langle\alpha|\beta\rangle|^2 \geq 0
\end{aligned} \tag{23}$$

Therefore, we have proven the Schwarz inequality:

$$\boxed{\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2} \text{ Q.E.D.} \tag{24}$$

(b)

Let $\hat{X}$ be a general operator, and $|\psi\rangle$ and $|\varphi\rangle$ be general state kets. Then, the following relation holds:

$$\langle\psi| \hat{X} |\varphi\rangle = \langle\varphi| \hat{X}^\dagger |\psi\rangle^* \tag{25}$$

Now, if $\hat{X}$ is Hermitian, we have:

$$\langle\psi| \hat{X} |\varphi\rangle = \langle\varphi| \hat{X} |\psi\rangle^* \tag{26}$$

A number that is equal to its complex conjugate is real. Therefore, the expectation value of a Hermitian operator is real.

(c)

Similarly, for an anti-Hermitian operator, Eq. 25 becomes

$$-\langle\psi| \hat{X} |\varphi\rangle = \langle\varphi| \hat{X} |\psi\rangle^* \tag{27}$$

A number whose complex conjugate is the negative of itself is a purely imaginary number. Therefore, the expectation value of an anti-Hermitian operator is purely imaginary.

(d)

Now let A and B be observables, $\Delta\hat{A} \equiv \hat{A} - \langle\hat{A}\rangle$ and $\Delta\hat{B} \equiv \hat{B} - \langle\hat{B}\rangle$ be operators, and $|\cdot\rangle$ to be any ket. Substituting $|\alpha\rangle \equiv \Delta\hat{A} |\cdot\rangle$ and $|\beta\rangle \equiv \Delta\hat{B} |\cdot\rangle$ to Eq. 24 gives:

$$\langle\cdot| \Delta\hat{A}^*\Delta\hat{A} |\cdot\rangle \langle\cdot| \Delta\hat{B}^*\Delta\hat{B} |\cdot\rangle = \langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq |\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2 \tag{28}$$

Let's look at the RHS. We can divide $\Delta\hat{A}\Delta\hat{B}$ into commutator and anti-commutator parts.

$$\Delta\hat{A}\Delta\hat{B} = \frac{1}{2}[\Delta\hat{A}, \Delta\hat{B}] + \frac{1}{2}\{\Delta\hat{A}, \Delta\hat{B}\} \tag{29}$$

The commutator part of Eq. 29 can be reduced as following:

$$[\Delta\hat{A}, \Delta\hat{B}] = [\hat{A} - \langle\hat{A}\rangle, \hat{B} - \langle\hat{B}\rangle] = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A}, \hat{B}] \tag{30}$$

The commutator is anti-Hermitian and the anti-commutator is Hermitian:

$$\begin{aligned}
[\hat{A}, \hat{B}]^\dagger &= \hat{B}\hat{A} - \hat{A}\hat{B} = -[\hat{A}, \hat{B}] \\
\{\Delta\hat{A}\Delta\hat{B}\}^\dagger &= \hat{B}\hat{A} + \hat{A}\hat{B} = \{\Delta\hat{A}\Delta\hat{B}\}
\end{aligned} \tag{31}$$

Taking the expectation value of Eq. 29 gives:

$$\langle\Delta\hat{A}\Delta\hat{B}\rangle = \frac{1}{2}\langle[\hat{A}, \hat{B}]\rangle + \frac{1}{2}\langle\{\hat{A}, \hat{B}\}\rangle \tag{32}$$

where the first term is purely imaginary and the second term is purely real (as shown in part (b) and part (c)). Taking the modulus square of Eq. 32 gives:

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq |\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2 = \frac{1}{4}|\langle[\hat{A}, \hat{B}]\rangle|^2 + \frac{1}{4}|\langle\{\hat{A}, \hat{B}\}\rangle|^2 \tag{33}$$

Omitting the anti-commutator term which is always positive preservs the inequality.

$$\boxed{\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}|\langle[\hat{A}, \hat{B}]\rangle|^2} \tag{34}$$

Problem 5

(a)

To normalize the wave function, we divide coefficients by the norm. The norm is given by:

$$\sqrt{7^2 + 24^2} = \sqrt{625} = 25 \tag{35}$$

Thus, the normalized wave function will be:

$$\boxed{\langle x|\Psi\rangle_{t=0} = \langle x| \left(\frac{7}{25} |0\rangle + \frac{24}{25} |1\rangle \right)} \tag{36}$$

(b)

The expectation value is given by the sum of probabilities of each energy eigenvalue.

$$\langle \hat{H} \rangle = \left(\frac{7}{25} \right)^2 \left(\frac{1}{2} \hbar\omega_0 \right) + \left(\frac{24}{25} \right)^2 \left(\frac{3}{2} \hbar\omega_0 \right) = \boxed{\frac{1777}{1250} \hbar\omega_0} \tag{37}$$

(c)

Energy of nth state of harmonic oscillator is given by $E_n = \left(\frac{1}{2} + n\right) \hbar\omega_0$. Since we have a superposition of the ground state ($n = 0$) and the first excited state ($n = 1$), the possible result of measurement are $\frac{1}{2}\hbar\omega_0$ and $\frac{3}{2}\hbar\omega_0$.

(d)

Of the two possible results above, the bigger one is $\frac{3}{2}\hbar\omega_0$.

(e)

The wave function collapses into the measured state, i.e., $|\psi\rangle = |1\rangle$.

(f)

The expectation value of $\hat{x}$ is given by:

$$\langle 1| \hat{x} |1\rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \langle 1| (a_- + a_+) |1\rangle = \sqrt{\frac{\hbar}{2m\omega_0}} (\langle 1|0\rangle + \langle 1|2\rangle) = 0 \tag{38}$$

(g)

The possible measurement result are $\frac{1}{2}\hbar\omega_0$, $\frac{3}{2}\hbar\omega_0$, and $\frac{5}{2}\hbar\omega_0$. This is because measuring the position results in a superposition of energy eigenstates.

(h)

What I newly learned here is that measuring the position completely destructs the knowledge on the energy of the system. This was explained with an analogy of "color" and "shape" attributes of a particle in the first lecture. This is the case where we apply color sorter, shape sorter, and then the color sorter again, where sorting according to shape destructs the color nature of particles.