# PY541 PS3

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## Problem 1: Crooks (Sethna 4.7)

### (a)

Let's revisit the derivation of Liouville's theorem.

$$\frac{\partial \rho}{\partial t} = -\nabla(\rho \mathbf{v}) = \dot{q}\frac{\partial \rho}{\partial q} + \rho\frac{\partial \dot{q}}{\partial q} + \dot{p}\frac{\partial \rho}{\partial p} + \rho\frac{\partial \dot{p}}{\partial p}. \tag{1}$$

where, for example,

$$\frac{\partial \dot{q}}{\partial q} = \frac{\partial}{\partial q}\frac{\partial \mathcal{H}}{\partial p} \tag{2}$$

We see that the explicit time dependence of the Hamiltonian does not affect the derivation of Lioville's theorem. Thus, applies to time dependent potential energies. This is equivalent to saying that systems with time-dependent potential energies conserve phase-space volume.

### (b)

The initial energy shell is $\Omega(E)$. Due to Liouville's theorem, the probability density is conserved. Therefore, the final phase-space probability density is $\Omega(E)$. Equivalently, if the system started in energy shell $\Omega(E+W)$, the final phase-space will also be $\Omega(E+W)$.

### (c)

Let's recall the variables.

- $p^C(W)$ = probability that the evolution $U$ leaves the state in the energy shell
- $p^D(W)$ = probability that the evolution $U^{-1}$ leaves the state in the energy shell
- $\Sigma^C$ = phase-space volume of forward time evolution
- $\Sigma^D$ = phase-space volume of reversed time evolution

We have for the $p^C(W)$ and $p^D(W)$:

$$\begin{aligned} p^C(W) &= \frac{\Sigma^C}{\Omega(E)} && (3) \\ p^D(-W) &= \frac{\Sigma^D}{\Omega(E+W)} && (4) \\ & && (5) \end{aligned}$$

### (d)

The fraction of $\Omega$ is given by:

$$\frac{\Omega(E+W)}{\Omega(E)} = \frac{p^C(W)}{p^D(W)} \tag{6}$$

The entropies are given by:

$$\begin{aligned} S(E+W) &= k_B \log\left(\Omega(E+W)\right) && (7) \\ S(E) &= k_B \log\left(\Omega(E)\right) && (8) \end{aligned}$$

Now, the difference in entropy is given by:

$$S(E+W) - S(E) = k_B \left[\log\left(\frac{\Omega(E+W)}{\Omega(E)}\right)\right] \tag{9}$$

$$\frac{S(E+W) - S(E)}{k_B} = \frac{\Delta S}{k_B} = \log\left(\frac{p^C(W)}{p^D(W)}\right) \tag{10}$$

Changing the sign and exponentiating both sides of Eq. 10 gives:

$$\frac{p^D(W)}{p^C(W)} = \exp\left[\frac{-\Delta S}{k_B}\right] \tag{11}$$

# Problem 2: Jarzynski (Sethna 4.8)

## (a)

Taking the log of the equation, we have:

$$\text{avg}\left[e^{-\frac{W_c+W_e}{k_B T}}\right] = 1 \Rightarrow \text{avg}\left[-\frac{W_c+W_e}{k_B T}\right] = 0 \tag{12}$$

If $W_c + W_e > 0$ in all cases, all values of $-\frac{W_c+W_e}{k_B T}$ will be negative, which would never equal to 0. Therefore, Eq. 12 will not be true if $W_c + W_e > 0$ for all initial conditions. This implies that the system sometimes do net work on the outside world. That is, $W_c$ or $W_e$ should be negative sometimes, such that sometimes $W_c + W_e < 0$.

## (b)

In piston's moving frame of reference, the collision is seen as the gas particle moving towards the piston and then bounces off of it. I will denote the variables in piston frame with a tilde. In piston frame, the sign of the gas particle's velocity switches.

$$\tilde{V}_g^i = \frac{p}{m} - V \Rightarrow \tilde{V}_g^f = V - \frac{p}{m} \tag{13}$$

Switching to the lab frame, we add $V$ to the piston frame. In lab frame, we have:

$$V_g^i = \tilde{V}_g^i + V = \frac{p}{m} \text{ , and } V_g^f = \tilde{V}_g^f + V = 2V - \frac{p}{m} \tag{14}$$

The work done on the particle is equal to the change in kinetic energy. The final kinetic energy of the particle is given by:

$$KE_g^f = \frac{1}{2}m\left(2V - \frac{p}{m}\right)^2 = \boxed{2mV^2 - 2Vp + \frac{p^2}{2m}} \tag{15}$$

$$W_c(p) = \Delta KE = \frac{1}{2}m\left(2V - \frac{p}{m}\right)^2 - \frac{p^2}{2m} = \boxed{2mV^2 - 2Vp \text{ , where } \left|\frac{p}{m}\right| < |V|} \tag{16}$$

The opposite case has the same expression for the work, but with different condition. That is, the air particle should travel faster than the piston: $\frac{p}{m} > V$.

$$W_e(p) = \boxed{2mV^2 + \text{ , where } \left|\frac{p}{m}\right| > |V|} \tag{17}$$

Looking at Eq. 16 with the condition $\frac{p}{m} < V$, we see that $W_c(p) = 2mV^2 - 2mV\frac{p}{m} > 0$. Thus, the piston can NOT do negative work during compression. Similarly, due to the condition, $W_e(p)$ can NOT be positive.

## (c)

If the collision was to happen at the very beginning of the compression, as long as the velocity of the gas particle is smaller than that of the piston, the gas particle should be at position $x_0 = 0$ (if the initial velocity of the gas particle is greater than that of the piston, collision will not happen even if the particle is at $x_0 = 0$, because the particle will run away from the piston). This corresponds to the opposite side of the compression triangle.

If the collision was to happen at the very end of the compression, the gas particle should be at point $x = \Delta L$ at the end of compression, which corresponds to the hypotenuse of the compression triangle.

The moment when compression ends coincides with the moment when expansion starts. Thus, in order for the collision to happen at the beginning of expansion, the gas particle should have initial conditions corresponding to the same line as the hypotenuse of compression triangle. However, the magnitude of momentum should be greater than $mV$, because otherwise the piston will run away from the air particle. The vertex of expansion triangle will correspond to the situation where the piston and the air particle touches each other at the beginning of the expansion, but since the piston and particle have the same velocity, they will move together while touching, but not colliding. The upper boundary of expansion describes the initial conditions which collide at the end of expansion. As a sanity check, if we draw a horizontal line (line of same $p_0$) that passes through both upper and lower boundaries of expansion triangle ($p_0 < -mV$), the line intersects the lower boundary at lower $x_0$ and upper boundary at greater $x_0$. This means that the particle with same momentum will collide later with the piston if it started off further away.

It is straightforward to think that two collisions would never happen during one phase. This means that in order for two collisions to happen, one should happen during compression phase, and the other at expansion phase. In case of the collision in compression phase, the gas particle will end up traveling the opposite to the direction the piston is expanding. Since we are ignoring the case where the particle bounces off the wall, there will be no case where two collisions happen.

To sum up,

- beginning of compression: opposite of compression triangle
- end of compression: hypotenuse of compression triangle
- beginning of expansion: lower bound of expansion triangle
- end of expansion: upper bound of expansion triangle

### (d)

Since there will be only one collision, and the piston can only do negative work during the expansion phase, the area covered in expansion triangle will be the region where the piston does negative net work on the particle. At the limit $k_B T \ll \frac{1}{2}mV^2$, instances where the piston does negative work on the particle will be extremely rare, because the momentum of the particle will be nearly zero, whereas the piston is running away very fast.

### (e)

Attached below.

### (f)

Attached below.

## Problem 3: The Arnol'd Cat Map (Sethna 5.8)

### (a)

$$\begin{aligned}(x_n, p_n) &= \mathrm{Mod}_1\left[M\mathrm{Mod}_1\left(M^{n-1}(x_0,p_0)\right)\right] = \mathrm{Mod}_1\left[M\left(M^{n-1}(x_0,p_0) - \lfloor M^{n-1}(x_0,p_0)\rfloor\right)\right] \\ &= \mathrm{Mod}_1\left[M^n(x_0,p_0) - M\lfloor M^{n-1}(x_0,p_0)\rfloor\right] \\ &= M^n(x_0,p_0) - M\lfloor M^{n-1}(x_0,p_0)\rfloor - \left\lfloor M^n(x_0,p_0) - M\lfloor M^{n-1}(x_0,p_0)\rfloor\right\rfloor \\ &= \mathrm{Mod}_1(M^n(x_0,p_0))\end{aligned} \tag{18}$$

Therefore, the equation $\boxed{(x_n, p_n) = \mathrm{Mod}_1(M^n(x_0,p_0))}$ is true.

### (b)

Let's first find the eigenvalues of $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

$$\det(M - \lambda \mathbb{1}) = \lambda^2 - 3\lambda + 1 = 0 \tag{19}$$

$$\lambda_\pm = \frac{3 \pm \sqrt{5}}{2} \tag{20}$$

The question gives us that the eigenvectors of $M$ are $\vec{v}_+ = (\gamma, 1)$ and $\vec{v}_- = (-\frac{1}{\gamma}, 1)$, where $\gamma = \frac{1+\sqrt{5}}{2}$. Let's plug in to verify that these are indeed eigenvectors.

$$M\vec{v}_+ = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{5} \\ \frac{3+\sqrt{5}}{2} \end{pmatrix} = \frac{3+\sqrt{5}}{2} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \tag{21}$$

$$M\vec{v}_- = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{2}{1+\sqrt{5}} \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{4}{1+\sqrt{5}} \\ 1-\frac{2}{1+\sqrt{5}} \end{pmatrix} = \frac{3-\sqrt{5}}{2} \begin{pmatrix} -\frac{2}{1+\sqrt{5}} \\ 1 \end{pmatrix} \tag{22}$$

Therefore, $\vec{v}_+ = (\gamma, 1)$ and $\vec{v}_- = (-\frac{1}{\gamma}, 1)$ are the eigenvectors of $M$. We can know that the side parallel to $\vec{v_+}$ is scaled by a factor of $\frac{3+\sqrt{5}}{2} \approx 2.62$ and the side parallel to $\vec{v_-}$ is scaled by a factor of $\frac{3-\sqrt{5}}{2} \approx 0.38$. I attached a cute visualization below.

The matrix $M$ can be viewed as a Jacobian matrix that maps a set of points in $xp$ plane to itself. The Jacobian, which is equal to the determinant of the Jacobian matrix, represents the ratio between the area before the transformation and that after the transformation. Therefore, if the Jacobian of $M$ is 1, the area is preserved.

$$\det[M] = 2 - 1 = 1 \tag{23}$$

Therefore, the area is preserved.

### (c)

Towards the direction of the eigenvector $\vec{v_+}$, the side scales by a factor of $\frac{3+\sqrt{5}}{2}$. Thus, $h$ is given by the excess length after scaling that goes beyond the frame of the figure, cut and pasted so that the line with slope $\frac{2}{1+\sqrt{5}}$. Thus,

$$h = \frac{2}{1+\sqrt{5}} \times \frac{3+\sqrt{5}}{2} \approx 1.62 \tag{24}$$

Notice that $h$ is an irrational number. The multiple of an irrational number lies densely within the region $[0, 1]$. As we continue the mapping and continue drawing thin lines that lie practically anywhere within the range $[0, 1]$, the spacings between the lines will eventually get to 0. This means that the whole phase space is covered, and the process is ergodic.

### (d)

We have $x = \frac{a}{q}$ and $p = \frac{b}{q}$. Then, Eq. 5.40 (Sethna) becomes

$$(x_n, p_n) = \mathrm{Mod}_1[M^n(x_0, p_0)] = \mathrm{Mod}_1\left[M^n \frac{1}{q}(a_0, b_0)\right] = \frac{1}{q}(a_n, b_n). \tag{25}$$

Therefore, $x_n$ and $p_n$ can be written as fractions with denominator $q$. Here we have the scaling to be rational. This is the opposite case as in part (c), where irrational $h$ leads to covering up all phase space. Thus, rational scaling should eventually settle into a periodic orbit. The upper bound is a power of $q$.

The time average of such trajectory with periodic orbit is NOT ergodic, because it obviously does not cover up all the phase space. Thus, the time average of such trajectory is not equal to the microcanonical average.

### (e)

The point slightly above or below the eigenvectors will gradually be attracted to the origin, while its trajectory will have an irrational slope. An example of this can be expressed as:

$$(x_0, p_0) = (\gamma + \epsilon, 1) \tag{26}$$

, where $\epsilon$ is very small. Thus the orbit will be non periodic, and it will approach to the origin while covering the phase space between its initial phase space position and the origin. The time average will be equal to its microcanonical average, since the trajectory with such initial condition is ergodic.

## Problem 4: Invariant Measures (Sethna 4.3)

*** I didn't make it through the full question, but I gave it a try anyways.

## (a)

To find the fixed points, we set $f(x) = x$ and solve the logistic map function with respect to $x$:

$$f(x) = x = 4\mu x(1-x) \Rightarrow x^* = 1 - \frac{1}{4\mu} \tag{27}$$

Given that $0 < x < 1$, we see that fixed points other than $x = 0$ occur is the RHS of Eq. 27 is positive, i.e., when $\mu > \frac{1}{4}$. To sum up, the fixed points are:

$$\text{fixed points} \;=\begin{cases} 0 & , \mu \leq \frac{1}{4} \\ 0 \;\&\; 1-\frac{1}{4\mu} & , \mu > \frac{1}{4} \end{cases} \tag{28}$$

Now our task is to show that a nonzero, stable fixed point occurs when $\frac{1}{4} < \mu < \frac{3}{4}$. Let's look at the derivative of Eq. 27. The derivative is given by:

$$f'(x^*) = 4\mu(1-2x^*) = 4\mu - 8\mu\left(1 - \frac{1}{4\mu}\right) = -4\mu + 2 \tag{29}$$

To look at the boundaries of $\mu$ that makes the fixed points stable, we equate the last equality in Eq. 29 with $\pm 1$.

$$-4\mu + 2 = 1 \quad \Rightarrow \quad \mu = \frac{1}{4} \tag{30}$$

$$-4\mu + 2 = -1 \quad \Rightarrow \quad \mu = \frac{3}{4} \tag{31}$$

Therefore, nonzero stable fixed point $x^*(\mu)$ of the map exists for $\frac{1}{4} < \mu < \frac{3}{4}$.

Having a period two stable fixed point is for $f(f(x))$ to have a stable fixed point. We can follow the same procedure as above to find the limits for $\mu$ to have period two stable fixed points.

$$f(f(x)) = 4\mu\left[4\mu x(1-x)\right]\left(1 - \left[4\mu x(1-x)\right]\right) = x \tag{32}$$

Solving Eq. 32 using Mathematica, we get

$$x = 0 \;,\; 1 - \frac{1}{4\mu} \;,\; \frac{1 + 4\mu \pm \sqrt{16\mu^2 - 8\mu - 3}}{8\mu} \tag{33}$$

Thus we see that there are two additional fixed points. To see the limits of $\mu$, we take the derivative of Eq. 32 and set it equal to $\pm 1$. Again using Mathematica, we have:

$$\frac{d}{dx}\left[f(f(x))\right] = -256\mu^3x^3 + 384\mu^3x^2 - 128\mu^2x + 16\mu^2 \tag{34}$$

I Equated Eq. 34 to $-1$ and solving for $\mu$ using Mathematica. The result is:

$$-256\mu^3x^3 + 384\mu^3x^2 - 128\mu^2x + 16\mu^2 \;=\; -1 \tag{35}$$

$$\mu \;=\; \frac{1 \pm \sqrt{6}}{4} \tag{36}$$

What we found here is the upper bound for $\mu$ to have a stable, period two fixed point. Therefore, there is a stable, period two cycle for $\frac{3}{4} < \mu < \frac{1+\sqrt{6}}{4}$.

## (b)

Let's first derive $x_a$ and $x_b$ by deriving the probability density $\rho(x)$.

$$\rho(x) \;=\; \int \rho(x')\delta\left[f(x') - x\right] dx' \;, \text{ where} \tag{37}$$

$$\delta\left[f(x') - x\right] \;=\; \sum_i \frac{\delta(x' - x_i)}{|f'(x_i)|} \;, \text{ where} \tag{38}$$

$$x_i \;=\; \text{roots of } f(x') - x = 0 \Rightarrow x_{a,b} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-x} \tag{39}$$

We then have

$$f'(x_{a,b}) = f'\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1-x}\right) = \pm 4\sqrt{1-x} \tag{40}$$

On the other hand, we have:

$$\rho(x_{a,b}) = \frac{1}{\pi\sqrt{x_{a,b}(1-x_{a,b})}} = \frac{2}{\pi\sqrt{x}} \tag{41}$$

Plugging back to Eq. 1.14 (Sethna),

$$\rho(f(x_a)) = \frac{\rho(x_a)}{|f'(x_a)|} + \frac{\rho(x_b)}{|f'(x_b)|} = \frac{2}{\pi\sqrt{x}}\frac{1}{4\sqrt{1-x}} \times 2 = \frac{1}{\pi\sqrt{x(1-x)}} \tag{42}$$

Therefore, we have verified that, for $\mu = 1$,

$$\boxed{\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}} \tag{43}$$

Computational part is attached below.

# PY541 PS3

October 8, 2022

[1]:
```
%pylab inline
```

```
Populating the interactive namespace from numpy and matplotlib
```

## 0.1 Problem 2: Jarzynski

### 0.1.1 Part (e)

[2]:
```
# This function gives N initial conditions.
def InitCond(N,L,T,m):
    initconds = [0] * N
    stddev = sqrt(T/m)

    for i in range(N):
        x0 = uniform(0,L)
        p0 = normal(0,stddev)
        initconds[i] = [x0,p0]

    return initconds

# This function gives work done on the particle with certain initial condition.
def Work(initcond, m, V, deltaL):
    x0 = initcond[0]
    p0 = initcond[1]

    if p0 < -(V*m*x0)/(deltaL) + V*m:
        # Collision happens at compression!
        work = 2*m*V**2 - 2*V*p0
    elif p0 < -(x0*m*V)/(2*deltaL) and x0 > 2*deltaL and p0 > -(V*m*x0)/
↪(deltaL) + V*m:
        # Collision happends at expansion!
        work = 2*m*V**2 + 2*V*p0
    else:
        work = 0

    return work
```

[3]:
```
N = 100000
m = 2
T = 8
V = 2
L = 20
deltaL = 1

initconds = InitCond(N,L,T,m)
```

[4]:
```
works = zeros(N)

for i in range(len(initconds)):
    works[i] = Work(initconds[i],m,V,deltaL)

mask = works!=0
works = works[mask]
avg_work = mean(works)
print("Average work done is: {:.2f}".format(avg_work))
```

```
Average work done is: 19.91
```

At N = 100000, the average work mostly converges to ~19. Thus, N = 100000 seems to be a reasonable number for a reliable average. The average certainly obeys $\overline{W_c + W_e} \geq 0$.

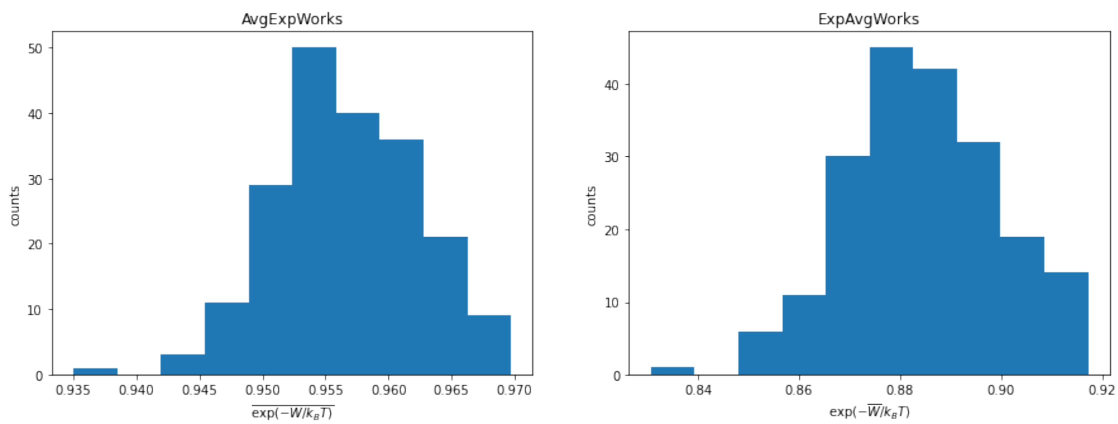
### 0.1.2 Part (f)

[5]:
```
N       = 1000
rep     = 200
works   = zeros(N)
expworks = zeros(N)

AvgExpWorks = [0]*rep
ExpAvgWorks = [0]*rep

for i in range(rep):
    initconds = InitCond(N,L,T,m)
    for j in range(N):
        works[j] = Work(initconds[j],m,V,deltaL)
        expworks[j] = exp(-works[j]/T)

    #maskWorks = works!=0
    #maskExpworks = expworks !=1

    #works = works[maskWorks]
    #expworks = expworks[maskExpworks]

    AvgExpWorks[i] = mean(expworks)
    ExpAvgWorks[i] = exp(-mean(works)/T)
```

```
    works    = zeros(N)
    expworks = zeros(N)
```

[6]:
```
fig, hist = subplots(1,2)

fig.set_figheight(5)
fig.set_figwidth(15)

hist[0].hist(AvgExpWorks)
hist[0].set_title("AvgExpWorks")
hist[0].set_xlabel("$\overline{\exp(-W/k_BT)}$")
hist[0].set_ylabel("counts")

hist[1].hist(ExpAvgWorks)
hist[1].set_title("ExpAvgWorks")
hist[1].set_xlabel("$\exp(-\overline{W}/k_BT)$")
hist[1].set_ylabel("counts")
```

[6]:
```
Text(0, 0.5, 'counts')
```




If we look at $\overline{\exp(-W/k_BT)}$, the graph is roughly centered at 1. This shows that the Jarzynki equality holds.

### 0.2 Problem 4: The Arnol'd Cat Map (Sethna 5.8)

[7]:
```
# Defining a square
x = linspace(0,1,num=50)
y = linspace(0,1,num=50)
meshx,meshy = meshgrid(x,y)

# Defining matrix M
```

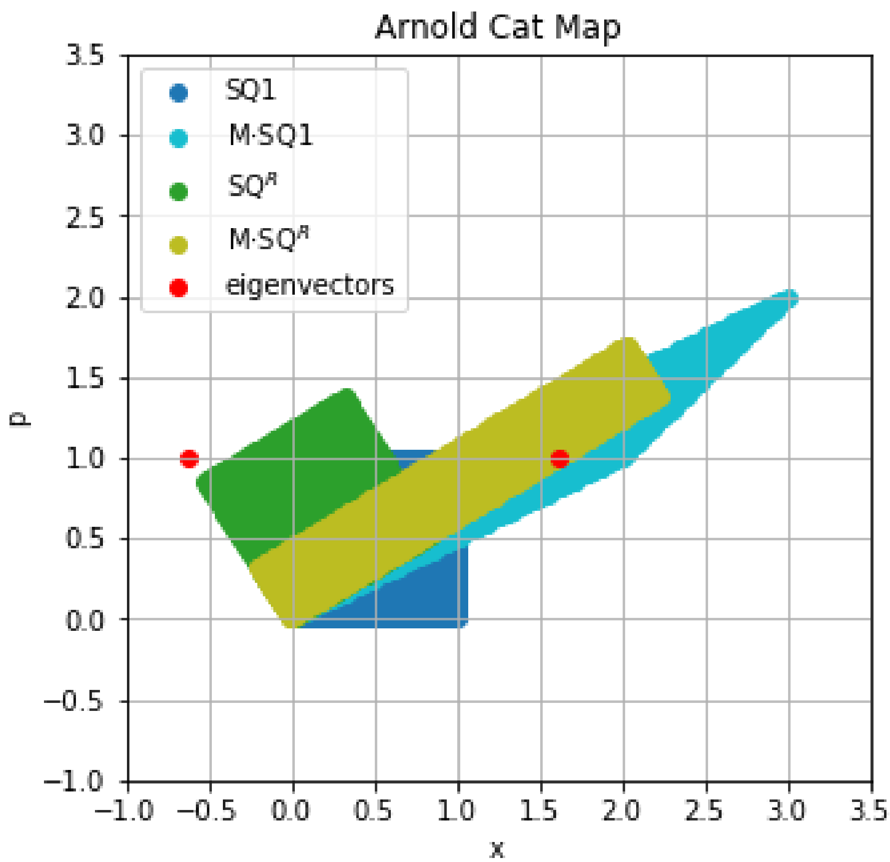
```
M = array([[2,1],[1,1]])

# Defining eigenvectors
gamma = (1+sqrt(5))/2
evec1 = [gamma,1]
evec2 = [-1/gamma,1]

# Tilting matrix to tilt the square parallel to eigenvectors
tilt = arctan(1/gamma)
rotation = [[cos(tilt),-sin(tilt)],[sin(tilt),cos(tilt)]]
```

[8]:
```
# Square in a vector form
vectors = reshape(transpose([meshx,meshy]),(len(x)*len(y),2))

# Rotating and Applying M
Mvectors = copy(vectors)
for i in range(len(vectors)):
    Mvectors[i] = M @ vectors[i]

tilted_vectors = copy(vectors)
for i in range(len(vectors)):
    tilted_vectors[i] = rotation @ vectors[i]

Mtilted_vectors = copy(tilted_vectors)
for i in range(len(tilted_vectors)):
    Mtilted_vectors[i] = M @ tilted_vectors[i]
```

[9]:
```
# Plotting

opacity = 1

figure(figsize=(5,5))
xlim(-1,3.5)
ylim(-1,3.5)

scatter(transpose(vectors)[0],transpose(vectors)[1],alpha=opacity,color="tab:
 ↪blue")
scatter(transpose(Mvectors)[0],transpose(Mvectors)[1],alpha=opacity,color="tab:
 ↪cyan")
scatter(transpose(tilted_vectors)[0],transpose(tilted_vectors)[1],alpha=opacity,color="tab:
 ↪green")
scatter(transpose(Mtilted_vectors)[0],transpose(Mtilted_vectors)[1],alpha=opacity,color="tab:
 ↪olive")

scatter(evec1[0],evec1[1],color="red")
scatter(evec2[0],evec2[1],color="red")
```

```
title("Arnold Cat Map")
xlabel("x")
ylabel("p")

grid()
legend(["SQ1","M$\cdot$SQ1","SQ$^R$","M$\cdot$SQ$^R$","eigenvectors"])
```

[9]: <matplotlib.legend.Legend at 0x7fea576908d0>




Let me first explain the legend. - SQ1 = original square with one vertex on origin - M·SQ1 = original square with M applied - $\mathrm{SQ}^R$ = tilted square to make the sides parallel to eigenvectors - M·$\mathrm{SQ}^R$ = tilted square with M applied

We can see that for the tilted square, when M is applied, the height shrinks and length increases. I have shown that the area is preserved above.

## 0.3 Problem 5: Invariant Measures (Sethna 4.3)

### 0.3.1 Part (a)

Logistic map is the following expression:

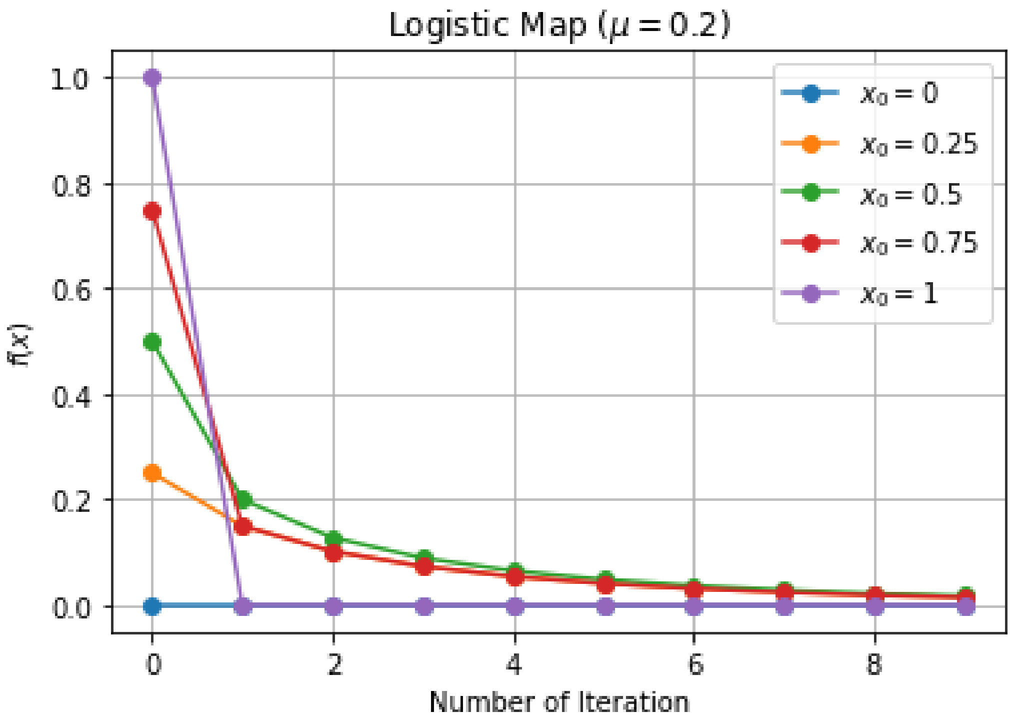
$$f(x) = 4\mu x(1-x) \tag{1}$$

[2]:
```
# Here I am defining an iterative logistic map function.
# Parameters: x = array-like initial value, c = number of iterations, mu =␣
↪logistic map parameter

def LogisticMap(x,c,mu):
    iteration = arange(c)
    snapshot = [0]*c
    count = 0
    while count < c:
        snapshot[count] = x
        x = 4 * mu * x * (1-x)
        count += 1
    return x, snapshot,iteration
```

[3]:
```
mu = 0.2
x0 = linspace(0,1,num=5)
xf,snapshot,iteration = LogisticMap(x0,10,mu)
```

[4]:
```
figure()
plot(iteration,snapshot,marker="o")
title("Logistic Map ($\mu = 0.2$)")
xlabel("Number of Iteration")
ylabel("$f(x)$")
grid()
legend(["$x_0 = 0$","$x_0 = 0.25$","$x_0 = 0.5$","$x_0 = 0.75$","$x_0 = 1$"])
```

[4]: <matplotlib.legend.Legend at 0x7f69fc00e350>



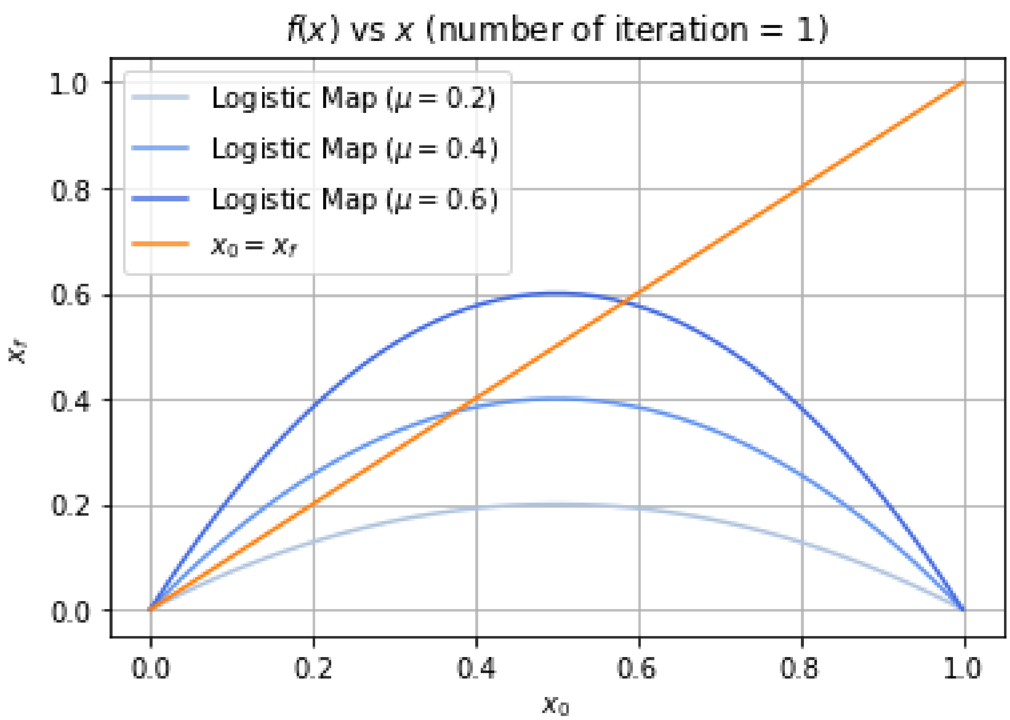

We can clearly see that all initial values converge to 0 when $\mu = 0.2$.

[5]:
```
# Now let's compute more values 0 < x0 < 1 and plot their final states after␣
↪one iteration.
x0   = linspace(0,1)
xf02 = LogisticMap(x0,1,0.2)[0]
xf04 = LogisticMap(x0,1,0.4)[0]
xf06 = LogisticMap(x0,1,0.6)[0]

# Plotting y = x together
x = linspace(0,1)
y = x
```

[6]:
```
figure()
plot(x0,xf02, color = "lightsteelblue")
plot(x0,xf04, color = "cornflowerblue")
plot(x0,xf06, color = "royalblue")
plot(x,y, color = "tab:orange")
title("$f(x)$ vs $x$ (number of iteration = 1)")
xlabel("$x_0$")
ylabel("$x_f$")
grid()
```

```
legend(["Logistic Map ($\mu = 0.2$)","Logistic Map ($\mu = 0.4$)","Logistic Map␣
↪($\mu = 0.6$)","$x_0=x_f$"])
```

[6]: <matplotlib.legend.Legend at 0x7f69dcf4bc10>




We can see from the plot above that there is a point where the logistic map graph intersects $y = x$ line for $\mu = 0.4$ and $\mu = 0.6$. This means that, for those such $\mu$s, there is a point where $f(x) = x$, i.e., a fixed point.

### 0.3.2 Part (b)

[7]:
```
mu = 1
x0 = 0.51  # A "typical" point
c  = 10000 # Number of iteration

xf,snapshot,iteration = LogisticMap(x0,c,mu)

# Plotting the theoretical curve as well
x   = linspace(0.01,0.99,100)
rho = 1/(pi*sqrt(x*(1-x)))
```

[8]:
```
figure()
hist(snapshot,bins=30,density=True)
```

```
plot(x,rho)
title("Invariant Density")
xlabel(r"Density $\rho(x)$ of Trajectory")
ylabel("Normalized counts")
legend(["Theoretical Curve",r"$\rho(x)$"])
```

[8]: <matplotlib.legend.Legend at 0x7f69dcdf4450>