# PY511 PS5

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## Problem 1

### (a)

The ground state wave function for the harmonic oscillator is given by:

$$\psi_0(x)_{|t<0} = \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\frac{\alpha^2 x^2}{2}} \text{ , where } \alpha \equiv \sqrt{\frac{m\omega_0}{\hbar}} \tag{1}$$

The Hamiltonian when the switch is turned on is expressed as following:

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 - q\varepsilon_0 x \text{ , where } t > 0 \tag{2}$$

By letting $a = \frac{q\varepsilon_0}{m\omega_0^2}$, we can complete the square for the Hamiltonian:

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2 (x-a)^2 - \frac{1}{2}m\omega_0^2 a^2 \text{ , where } t > 0 \tag{3}$$

We can see that the Hamiltonian is the same, except the center has changed $x \to (x-a)$, and an extra constant term $-\frac{1}{2}m\omega_0^2 a^2 = -\frac{q^2\varepsilon_0^2}{2m\omega_0^2}$. Thus, with the new Hamiltonian, the ground state wave function is given in terms of the shifted center:

$$\psi_0(x-a)_{|t>0} = \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\frac{\alpha^2 (x-a)^2}{2}} \tag{4}$$

Probability amplitude of remaining in the ground state is given by:

$$c_0 = \langle \psi_0(x-a)|\psi_0(x)\rangle = \int \psi_0^*(x-a)\psi_0(x)dx = e^{-\frac{\alpha^2 a^2}{4}} \tag{5}$$

The probability is given by:

$$\boxed{P(0) = |c_0|^2 = e^{-\frac{\alpha^2 a^2}{2}}} \tag{6}$$

### (b)

If the electric field is increased very slowly, we mean that the time needed to fully increase the electric field $\tau$ is very much greater than the oscillation period of the harmonic oscillator. That is, $\tau \gg T_0 = \frac{2\pi}{\omega_0}$. Now, the electric field is given by the time evolution of the electric potential:

$$\int_{t_0}^{t} V(t')U(t', t_0)dt' = \int_{t_0}^{t} q\varepsilon_0 U(t', t_0)dt' \tag{7}$$

But in long time scale, $U$ term oscillates a lot, which averages out to zero. Thus,

$$\int_{t_0}^{t} V(t')U(t', t_0)dt' = \int_{t_0}^{t} q\varepsilon_0 U(t', t_0)dt' \approx 0 \tag{8}$$

Therefore, in adiabatic process, the time evolution operator is equal to an identity operator. Therefore, in an adiabatic process, the probability that the harmonic oscillator will be at the ground state is 1.

## Problem 2

### (a)

Let's start withe the state kets (I altered the notation just a little bit for my convenience. Please let me know if this notation is problematic)

$$|\psi(t)\rangle_I = c_0(t)\,|0\rangle + c_1(t)\,|1\rangle \tag{9}$$

Multiplying both sides of Eq. 9 by $\langle n|$ where $n = 0, 1$ gives the expression for coefficients:

$$\begin{aligned}\langle 0|\psi(t)\rangle_I &= c_0(t)\,\langle 0|0\rangle + c_1(t)\,\langle 0|1\rangle = c_0(t) &(10)\\ \langle 1|\psi(t)\rangle_I &= c_0(t)\,\langle 1|0\rangle + c_1(t)\,\langle 1|1\rangle = c_1(t) &(11)\end{aligned}$$

Meanwhile, the associated Schrödinger equation for Eq. 9 is given by:

$$i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle_I = V_I\,|\psi(t)\rangle_I \tag{12}$$

Using spectrial decomposition to the RHS and multiplying $\langle n|$ to both sides of Eq. 12 gives:

$$i\hbar\frac{\partial}{\partial t}\,\langle n|\psi(t)\rangle_I = i\hbar\frac{d}{dt}c_n(t) = \sum_m \langle n|\,V_I\,|m\rangle\,\langle m|\psi(t)\rangle_I = \sum_m V_{nm}e^{i(E_n-E_m)t/\hbar} = \sum_m V_{nm}e^{i\omega_{nm}t/\hbar} \tag{13}$$

where $n, m = 0, 1$, $\omega_{nm} = \frac{E_n-E_m}{\hbar}$, and we used $\langle n|\,V_I\,|m\rangle = \langle n|\,e^{\frac{iH_0t}{\hbar}}V_Ie^{-\frac{iH_0t}{\hbar}}\,|m\rangle = V_{nm}(t)e^{\frac{i(E_n-E_m)t}{\hbar}}$. Rewriting the second term and the last term in Eq. 13 explicitly gives:

$$\boxed{i\hbar\begin{pmatrix}\dot{c_0}\\ \dot{c_1}\end{pmatrix} = \begin{pmatrix}0 & \gamma e^{i\omega t}e^{-i\omega_{10}t}\\ \gamma e^{-i\omega t}e^{i\omega_{10}t} & 0\end{pmatrix}\begin{pmatrix}c_0\\ c_1\end{pmatrix}} \tag{14}$$

### (b)

Eq. 14 can be expressed as a system of coupled differential equations:

$$\begin{aligned}\dot{c_0} &= \frac{\gamma}{i\hbar}e^{i(\omega-\omega_L)t}c_1(t) &(15)\\ \dot{c_1} &= \frac{\gamma}{i\hbar}e^{-i(\omega-\omega_L)t}c_0(t) &(16)\end{aligned}$$

with initial conditions $c_0(0) = 1$ and $c_1(0) = 0$. Taking the time derivative of $\dot{c_1}$, we get

$$\begin{aligned}\ddot{c_1} &= \frac{\gamma}{i\hbar}\left[-i(\omega-\omega_L)e^{-i(\omega-\omega_L)t}c_0(t) + e^{-i(\omega-\omega_L)t}\dot{c_0}(t)\right] &(17)\\ &= -i(\omega-\omega_L)\dot{c_1}(t) + \frac{\gamma^2}{\hbar^2}c_1(t)\end{aligned}$$

Solving the differential equation gives us the solution of the following form:

$$c_1(t) = Ae^{-\frac{i(\omega-\omega_L)t}{2}}\sin(\Omega t) \tag{18}$$

where the normalization condition $A$ and $\theta$ are given by:

$$\begin{aligned}A &= \frac{\gamma/\hbar}{\gamma/\hbar + \frac{1}{2}(\omega-\omega_L)} &(19)\\ \Omega &= \frac{\Omega_R}{2} = \sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_L)^2}{4}} &(20)\end{aligned}$$

Thus, the full solution is given by:

$$c_1(t) = \frac{\gamma/\hbar}{\gamma/\hbar + \frac{1}{2}(\omega-\omega_L)}\sin\left[t\sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_L)^2}{4}}\right] \tag{21}$$

The probability that the particle will be in excited state is given by the modulus square:

$$\boxed{P_{0\to 1}(t) = |c_1(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega-\omega_L)^2/4}\sin^2\left[t\sqrt{\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_L)^2}{4}}\right]} \tag{22}$$

# Problem 3

## (a)

The parameters are given by:

$$\begin{aligned}
\varepsilon_0 &= \vec{\mu}_B \cdot \vec{B}_0 = \left(-9.28 \times 10^{-24} \frac{\text{J}}{\text{T}}\right)(0.35\text{T}) = -3.25 \times 10^{-24} \text{ J} && (23)\\
\varepsilon_1 &= -\varepsilon_0 = 3.25 \times 10^{-24} \text{ J} && (24)\\
\gamma &= \vec{\mu}_B \cdot \vec{B}_1 = (-9.28 \times 10^{-24} \frac{\text{J}}{\text{T}})(2.9 \times 10^{-7} \text{ T}) = 2.69 \times 10^{-30} \text{ J} && (25)
\end{aligned}$$

The paramagnetic resonance frequency is given by:

$$\omega_L = \omega_{10} = \gamma_g B_0 = \left(1.76 \times 10^{11} \frac{\text{rad}}{\text{sT}}\right)(0.35\text{T}) = 6.16 \times 10^{10} \frac{\text{rad}}{s} \tag{26}$$

The Larmor frequency is given by:

$$\nu_L = \frac{\omega_L}{2\pi \text{ rad}} = 9.80 \times 10^9 \text{ Hz} \tag{27}$$

## (b)

Attached below as a pdf version of Jupyter notebook.

## (c)

Let's calculate the value of the peak when $\omega = 9$ GHz and when $\omega = 9$ MHz. Before that, we have

$$\frac{\gamma^2}{\hbar^2} = \left(\frac{2.69 \times 10^{-30} \text{ J}}{1.05 \times 10^{-34} \text{ J} \cdot \text{s}}\right)^2 \approx 6.56 \times 10^8 \text{ s}^{-2} \tag{28}$$

Thus,

$$\begin{aligned}
\omega = 9 \text{ GHz} \quad &\rightarrow \quad |c_1(t)|^2_{\max} = \frac{6.56 \times 10^8 \text{ s}^{-2}}{6.56 \times 10^8 \text{ s}^{-2} + \frac{(9 \text{ GHz} - 9 \text{ GHz})^2}{4}} = 1 && (29)\\
\omega = 9 \text{ MHz} \quad &\rightarrow \quad |c_1(t)|^2_{\max} = \frac{6.56 \times 10^8 \text{ s}^{-2}}{6.56 \times 10^8 \text{ s}^{-2} + \frac{(9 \text{ MHz} - 9 \text{ GHz})^2}{4}} = 3.25 \times 10^{-11} && (30)
\end{aligned}$$

Thus, the signal is reduced by order of $10^{-11}$ when the RF frequency reduces from $\omega = 9$GHz to $\omega = 9$MHz.

# Problem 4

## (a)

### i)

The operator acting on the state ket is given by:

$$\begin{aligned}
e^{i\eta S_z} |\psi\rangle &= e^{\frac{i\eta}{2}\sigma_z} \left[\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right] && (31)\\
&= \begin{pmatrix} e^{\frac{i\eta}{2}} & 0 \\ 0 & e^{-\frac{i\eta}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} e^{\frac{i\eta}{2}}\cos\frac{\theta}{2} \\ e^{-\frac{i\eta}{2}} e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}
\end{aligned}$$

Multiplying by a phase factor of $e^{-\frac{i\eta}{2}}$ does not change the physics, so

$$e^{-\frac{i\eta}{2}} \begin{pmatrix} e^{\frac{i\eta}{2}}\cos\frac{\theta}{2} \\ e^{-\frac{i\eta}{2}} e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i(\phi-\eta)}\cos\frac{\theta}{2} \end{pmatrix} \tag{32}$$

Therefore, we see that $e^{\frac{i\eta}{2}}$ rotates the given ket on the surface of the Bloch sphere by the angle $\eta$.

### ii)

The rotation operator can be written as:

$$e^{i\frac{\eta}{2}\hat{n}\cdot\vec{\sigma}} = e^{i\frac{\eta}{2}n_x\sigma_x}e^{i\frac{\eta}{2}n_y\sigma_y}e^{i\frac{\eta}{2}n_z\sigma_z} \tag{33}$$

We see that the individual terms correspond to rotating by the angle $\frac{\eta}{2}n_i$ where $i = x, y, z$, with respect to $x, y, z$ axes. Therefore, the operator $e^{i\frac{\eta}{2}\hat{n}\cdot\vec{\sigma}}$ rotates the ket about $\hat{n}$ axis by the angle $\eta$. Meanwhile, the rotation operator can also be written as:

$$\begin{aligned} e^{i\theta\vec{n}\cdot\vec{\sigma}} &= \mathbb{1} + (i\theta\vec{n}\cdot\vec{\sigma}) + \frac{1}{2!}(i\theta\vec{n}\cdot\vec{\sigma})^2 + \frac{1}{3!}(i\theta\vec{n}\cdot\vec{\sigma})^3 + \frac{1}{4!}(i\theta\vec{n}\cdot\vec{\sigma})^4 + \cdots \\ &= \mathbb{1} + i\theta\vec{n}\cdot\vec{\sigma} - \frac{1}{2!}\theta^2(\vec{n}\cdot\vec{\sigma})^2 - \frac{1}{3!}i\theta^3(\vec{n}\cdot\vec{\sigma})^3 + \frac{1}{4!}\theta^4(\vec{n}\cdot\vec{\sigma})^4 + \cdots \\ &= \mathbb{1} + i\theta(\vec{n}\cdot\vec{\sigma}) - \frac{1}{2!}\theta^2\mathbb{1} - \frac{1}{3!}i\theta^3(\vec{n}\cdot\vec{\sigma}) + \frac{1}{4!}\theta^4\mathbb{1} + \cdots \\ &= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right)\mathbb{1} + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right)(\vec{n}\cdot\vec{\sigma}) \\ &= \cos(\theta)\mathbb{1} + i\sin(\theta)(\vec{n}\cdot\vec{\sigma}) \end{aligned} \tag{34}$$

which corresponds to the rotation by angle $\theta$ with respect to a vector $\vec{n}$.

## (b)

Brute force.

$$\begin{aligned} (\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) &= (A_x\sigma_x + A_y\sigma_y + A_z\sigma_z)(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z) \\ &= \begin{pmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{pmatrix}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix} \\ &= \begin{pmatrix} A_zB_z + (A_x - iA_y)(B_x + iB_y) & A_z(B_x - iB_y) - B_z(A_x - iA_y) \\ B_z(A_x + iA_y) - A_z(B_x + iB_y) & (A_x - iA_y)(B_x + iB_y) + A_zB_z \end{pmatrix} \\ &= \begin{pmatrix} A_xB_x + A_yB_y + A_zB_z & 0 \\ 0 & A_xB_x + A_yB_y + A_zB_z \end{pmatrix} \\ &\quad + i\begin{pmatrix} A_xB_y - A_yB_x & A_yB_z - A_zB_y + i(A_xB_z - A_zB_x) \\ A_yB_z - A_zB_y - i(A_xB_z - A_zB_x) & -A_xB_y + A_yB_x \end{pmatrix} \\ &= (\vec{A}\cdot\vec{B})\mathbb{1} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma} \end{aligned} \tag{35}$$

# Problem 5

## (a)

### i)

The general Dirac interaction picture state ket can be written as:

$$\begin{aligned} |\Psi(t)\rangle_I &= e^{-\frac{i}{\hbar}\hat{H}_0 t}|\Psi(t)\rangle_S \\ &= c_0(t)e^{-\frac{i\varepsilon_0}{\hbar}}|0\rangle + c_1(t)e^{-\frac{i\varepsilon_1}{\hbar}}|1\rangle \\ &= \boxed{c_0(t)|0\rangle + c_1(t)e^{i\omega_{10}t}|1\rangle} \end{aligned} \tag{36}$$

where in the last equality we multiplied a global phase factor that is unphysical (thus does not change the physics when applied) $e^{-\frac{i\varepsilon_0}{\hbar}}$ and let $\omega_{10} \equiv \frac{\varepsilon_0 - \varepsilon_1}{\hbar}$.

### ii)

The potential operator in the interaction picture is given by:

$$\begin{aligned} \hat{V}_I(t) &= e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{V}_S(t)e^{\frac{-i\hat{H}_0 t}{\hbar}} \\ &= \begin{pmatrix} \exp\left[\frac{i\varepsilon_0 t}{\hbar}\right] & 0 \\ 0 & \exp\left[\frac{i\varepsilon_1 t}{\hbar}\right] \end{pmatrix}\begin{pmatrix} 0 & 2V_0\cos\omega t \\ 2V_0\cos\omega t & 0 \end{pmatrix}\begin{pmatrix} \exp\left[\frac{-i\varepsilon_0 t}{\hbar}\right] & 0 \\ 0 & \exp\left[\frac{-i\varepsilon_1 t}{\hbar}\right] \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2V_0\exp\left[\frac{i(\varepsilon_0 - \varepsilon_1)t}{\hbar}\right]\cos\omega t \\ 2V_0\exp\left[\frac{i(\varepsilon_1 - \varepsilon_0)t}{\hbar}\right]\cos\omega t & 0 \end{pmatrix} \end{aligned} \tag{37}$$

Notice that $2\cos\omega t = e^{i\omega t} + e^{-i\omega t}$ and use $\omega_{10}$ as defined above. Then the matrix reduces to:

$$\boxed{\hat{V}_I(t) = \begin{pmatrix} 0 & V_0(e^{i\omega t} + e^{-i\omega t})e^{i\omega_{10}t} \\ V_0(e^{i\omega t} + e^{-i\omega t})e^{-i\omega_{10}t} & 0 \end{pmatrix}} \tag{38}$$

## (b)

The time evolution operator in the interaction picture is given by:

$$|\psi(t)\rangle_I = U_I(t, t_0)\,|\psi(t_0)\rangle_I \tag{39}$$

It satisfies the following equation:

$$i\hbar\frac{d}{dt}U_I(t, t_0) = V_I(t)U_I(t, t_0) \tag{40}$$

This differential equation is equivalent to the following integral equation.

$$U_I(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')U_I(t', t_0)dt' \tag{41}$$

Expanding Eq. 41 using Dyson series gives us:

$$\begin{aligned} U_I(t, t_0) &= 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\left[1 - \frac{i}{\hbar}\int_{t_0}^{t'} V_I(t'')U_I(t'', t_0)dt''\right]dt' \qquad (42) \\ &= 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t)dt' + \left(-\frac{i}{\hbar}\right)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} V_I(t')V_I(t'')dt'' + \cdots \end{aligned}$$

Taking only to the first order gives us:

$$\boxed{U_I^{(1)}(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0=0}^{t} \hat{V}_I(t')dt'} \tag{43}$$

## (c)

The coefficient $c_1(t)$ is obtained by:

$$c_1^{(1)}(t) = \langle 1|\, U_I^{(1)}(t, t_0 = 0)\, |0\rangle \tag{44}$$

Let's first work out $U_I^{(1)}$.

$$\begin{aligned} \int_0^t \hat{V}_I(t')dt' &= \begin{pmatrix} 0 & \frac{2V_0\left[-i\omega_{10}+e^{i\omega_{10}t}(i\omega_{10}\cos\omega t+\omega\sin\omega t)\right]}{\omega^2+i\omega_{10}^2} \\ \frac{2V_0\left[i\omega_{10}+e^{-i\omega_{10}t}(-i\omega_{10}\cos\omega t+\omega\sin\omega t)\right]}{\omega^2+i\omega_{10}^2} & 0 \end{pmatrix} \qquad (45) \\ U_I^{(1)}(t, t_0 = 0) &= 1 - \frac{i}{\hbar}\int_{t_0=0}^{t} \hat{V}_I(t')dt' \qquad (46) \end{aligned}$$

Thus,

$$c_1^{(1)}(t) = \langle 1|\, U_I^{(1)}(t, t_0 = 0)\, |0\rangle = \boxed{\frac{V_0}{\hbar}\left[\frac{1 - e^{i(\omega+\omega_{10})t}}{\omega_{10} + \omega} + \frac{1 - e^{-i(\omega-\omega_{10})t}}{\omega_{10} - \omega}\right]} \tag{47}$$

## (d)

The probability of finding the system in the excited state $|1\rangle$ for all time is given by:

$$\boxed{P(|1\rangle) = |c_1^{(1)}|^2 = \frac{V_0^2}{\hbar^2}\left[\frac{-1 + e^{i(\omega-\omega_{10})t}}{\omega - \omega_{10}} + \frac{1 - e^{-i(\omega+\omega_{10})t}}{\omega + \omega_{10}}\right]\left[\frac{1 - e^{-i(\omega-\omega_{10})t}}{-\omega + \omega_{10}} + \frac{1 - e^{i(\omega+\omega_{10})t}}{\omega + \omega_{10}}\right]} \tag{48}$$

## (e)

$c_1(t)$ that we obtained above represents the transition probability amplitude of the atom in two level system. In our case, the atom started off in the ground state. Thus, the only way that the state can change is to be excited. This agrees with that fact that $c_1$ is the coefficient in front of $|1\rangle$, which means that the coefficient represents the probability to find that the atom is in the excited state. In harmonic perturbation, this happens via the two-level system absorbing $\hbar\omega$ from $V$.

# PY511 HW5 Q3b

October 14, 2022

[1]:
```
%pylab inline
```

```
Populating the interactive namespace from numpy and matplotlib
```

[2]:
```
# Define variables
hbar = 1.05e-34
gamma = 2.69e-30
omegaL = 6.16e10

omega = linspace(omegaL-(1e-5)*omegaL, omegaL+(1e-5)*omegaL, 1000)
```

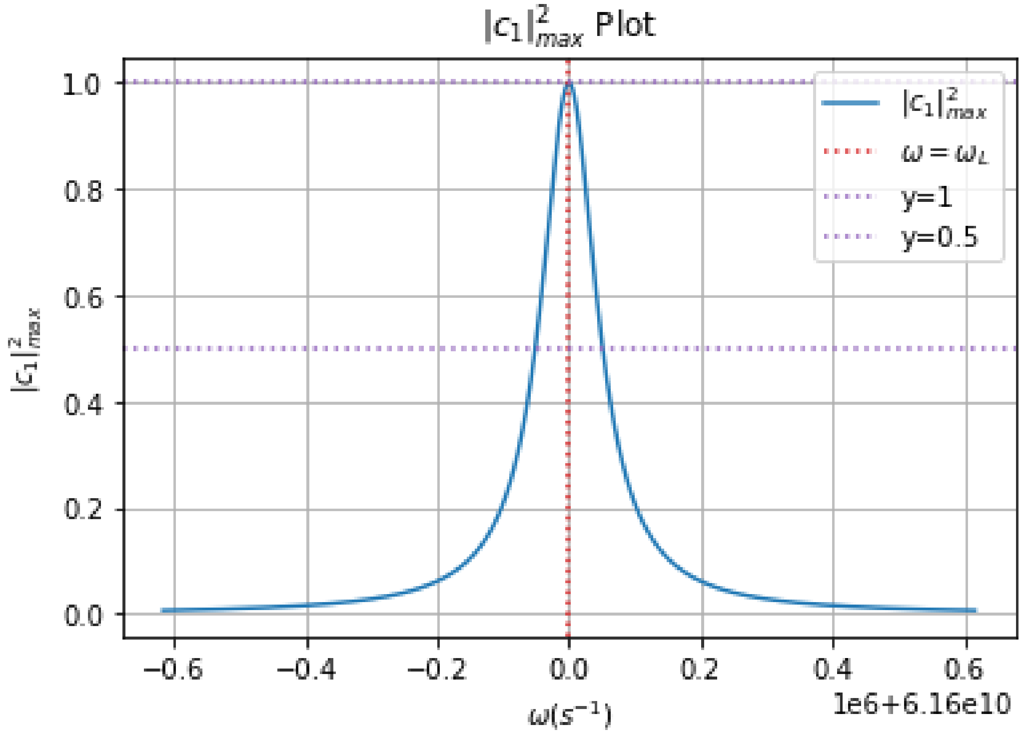
[3]:
```
cmax = (gamma**2/hbar**2)/((gamma**2/hbar**2)+((omega-omegaL)**2)/4)
```

[4]:
```
figure()
plot(omega,cmax)

title("$|c_1|^2_{max}$ Plot")
xlabel("$\omega (s^{-1})$")
ylabel("$|c_1|^2_{max}$")

axvline(x=omegaL,color="tab:red",linestyle="dotted")
axhline(y=1,color="tab:purple",linestyle="dotted")
axhline(y=0.5,color="tab:purple",linestyle="dotted")

legend(["$|c_1|^2_{max}$","$\omega=\omega_L$","y=1","y=0.5"])
grid()
```



$|c_1|^2_{max}$ Plot
1.0
0.8
0.6
0.4
0.2
0.0
$|c_1|^2_{max}$
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
1e6+6.16e10
$\omega(s^{-1})$
$|c_1|^2_{max}$
$\omega = \omega_L$
y=1
y=0.5