On the Einstein-Hilbert Action with the Gibbons-Hawking Boundary Term in Causal Dynamical Triangulations

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For a $(2+1)$-dimensional spacetime manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, we must add to the Einstein-Hilbert action,

$$S_{EH}[\mathbf{g}] = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^3x\,\sqrt{-g}\,(R-2\Lambda)\,, \tag{1}$$

the Gibbons-Hawking boundary term,

$$S_{GH}[\gamma] = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^2y\sqrt{|\gamma|}K. \tag{2}$$

Here, γ is the induced metric on the boundary $\partial\mathcal{M}$, and K is the trace of the extrinsic curvature of the boundary $\partial\mathcal{M}$. Regge demonstrated that, for a triangulated spacetime manifold $\mathcal{T}$, the Einstein-Hilbert action assumes the form

$$S_{EH}^{(R)}[\mathcal{T}] = \frac{1}{8\pi G}\sum_{h\in\mathcal{T}} A_h\delta_h - \frac{\Lambda}{8\pi G}\sum_{s\in\mathcal{T}} V_s. \tag{3}$$

Here, h is a 1-dimensional hinge having area A_h and deficit angle δ_h, and V_s is the spacetime volume of a 3-simplex s. Hartle and Sorkin demonstrated that, for a triangulated spacetime manifold $\mathcal{T}$ with boundary $\partial\mathcal{T}$, the Gibbons-Hawking boundary term assumes the form

$$S_{GH}^{(R)}[\partial\mathcal{T}] = \frac{1}{8\pi G}\sum_{h\in\partial\mathcal{T}} A_h\psi_h. \tag{4}$$

Here, h is a 1-dimensional hinge on the boundary $\partial\mathcal{T}$ having area A_h, and ψ_h is the angle between the two vectors normal to the two spacelike 2-simplices intersecting at the hinge h.

We wish to determine the form of the Regge-Einstein-Hilbert action supplemented by the Regge-Gibbons-Hawking boundary term in $(2+1)$-dimensional causal dynamical triangulations for two-sphere spatial topology and line interval temporal topology. Ambjørn *et al* derived the Regge-Einstein-Hilbert action in this case, finding that

$$\begin{aligned} S_{EH}^{(R)}[\mathcal{T}_c] \quad = \quad & \frac{1}{8\pi G}\Bigg[\frac{2\pi}{i}N_1^{SL} - \frac{2}{i}\theta_{SL}^{(2,2)}N_3^{(2,2)} - \frac{4}{i}\theta_{SL}^{(3,1)}N_1^{SL} \\ & \qquad +2\pi\sqrt{\alpha}N_1^{TL} - 4\sqrt{\alpha}\theta_{TL}^{(2,2)}N_3^{(2,2)} - 3\sqrt{\alpha}\theta_{TL}^{(1,3)}N_3^{(1,3)} - 3\sqrt{\alpha}\theta_{TL}^{(3,1)}N_3^{(3,1)}\Bigg] \\ & +\frac{\Lambda}{8\pi G}\left[V_3^{(2,2)}N_3^{(2,2)} + V_3^{(1,3)}N_3^{(1,3)} + V_3^{(3,1)}N_3^{(3,1)}\right]. \end{aligned} \tag{5}$$

Here, N_1^{SL} is the number of spacelike 1-simplices, N_1^{TL} is the number of timelike 1-simplices, $N_3^{(2,2)}$ is the number of $(2,2)$ 3-simplices, $N_3^{(1,3)}$ is the number of $(1,3)$ 3-simplices, $N_3^{(3,1)}$ is the number of $(3,1)$ 3-simplices, $\theta_{SL}^{(2,2)}$ is the Lorentzian dihedral angle about a spacelike edge of a $(2,2)$ 3-simplex, $\theta_{TL}^{(2,2)}$ is the Lorentzian dihedral angle about a timelike edge of a $(2,2)$ 3-simplex, $\theta_{SL}^{(1,3)}$ is the Lorentzian dihedral angle about a spacelike edge of a $(1,3)$ 3-simplex, $\theta_{TL}^{(1,3)}$ is the Lorentzian dihedral angle about a timelike edge of a $(1,3)$ 3-simplex, $\theta_{SL}^{(3,1)}$ is the Lorentzian dihedral angle about a spacelike edge of a $(3,1)$ 3-simplex, $\theta_{TL}^{(3,1)}$ is the Lorentzian dihedral angle about a timelike edge of a $(3,1)$ 3-simplex, $V_3^{(2,2)}$ is the spacetime volume of a $(2,2)$ 3-simplex, $V_3^{(1,3)}$ is the spacetime volume of a $(1,3)$ 3-simplex, $V_3^{(3,1)}$ is the spacetime volume of a $(3,1)$ 3-simplex, and α is the ratio of the timelike to the spacelike squared edge length of a 3-simplex.The first line comes from the summation over spacelike hinges, the second line comes from the summation over timelike hinges, and the third line comes from the summation over 3-simplices. Note that we could rewrite the third term of the first line using the relation $4N_1^{SL} = 3\left(N_3^{(1,3)} + N_3^{(3,1)}\right)$.

We now supplement the Regge-Einstein-Hilbert action for $(2+1)$-dimensional causal dynamical triangulations by the appropriate Regge-Gibbons-Hawking boundary term. Given the desired spacetime topology, the boundary $\partial\mathcal{T}_c$ consists of two disconnected components: an initial or past spatial two-sphere $\mathcal{S}_i^2$ and a final or future spatial two-sphere $\mathcal{S}_f^2$. Based on the demonstration of Hartle and Sorkin, we propose the prescription

$$S_{GH}^{(R)}[\partial\mathcal{T}_c] = \frac{1}{8\pi G}\sum_{h\in\mathcal{S}_i^2}\frac{1}{i}\left[\pi - 2\theta_{SL}^{(3,1)} - \theta_{SL}^{(2,2)}N_{3\uparrow}^{(2,2)}(h)\right] - \frac{1}{8\pi G}\sum_{h\in\mathcal{S}_f^2}\frac{1}{i}\left[\pi - 2\theta_{SL}^{(1,3)} - \theta_{SL}^{(2,2)}N_{3\downarrow}^{(2,2)}(h)\right]. \quad (6)$$

Here, $N_{3\uparrow}^{(2,2)}(h)$ is the number of future-directed $(2,2)$ 3-simplices attached to the hinge h, and $N_{3\downarrow}^{(2,2)}(h)$ is the number of past-directed $(2,2)$ 3-simplices attached to the hinge h. We justify this prescription as follows. In parallel transporting the vector normal to one component of the boundary $\partial\mathcal{T}_c$ between two spacelike 2-simplices intersecting at the hinge h, the vector rotates through the angle

$$\frac{1}{i}\left[2\theta_{SL}^{(3,1)} + \theta_{SL}^{(2,2)}N_3^{(2,2)}(h)\right]. \quad (7)$$

When this angle is $\frac{\pi}{i}$, the extrinsic curvature vanishes locally at the hinge h; this fact dictates the deficit angle-like form of our above prescription. The relative negative sign between the contributions of the two disconnected components of the boundary $\partial\mathcal{T}_c$ to the Regge-Gibbons-Hawking boundary term stems from the future-directed orientation of the vector normal to $\mathcal{S}_i^2$ and the past-directed orientation of the vector normal to $\mathcal{S}_f^2$. Performing the summations over the hinges on the boundary $\partial\mathcal{T}_c$, we may rewrite the Regge-Gibbons-Hawking boundary term as

$$\begin{aligned} S_{GH}^{(R)}[\partial\mathcal{T}_c] \;=\; & \frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathcal{S}_i^2) - \frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathcal{S}_i^2) - \frac{1}{i}\theta_{SL}^{(2,2)}N_{3\uparrow}^{(2,2)}(\mathcal{S}_i^2)\right] \\ & -\frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathcal{S}_f^2) - \frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathcal{S}_f^2) - \frac{1}{i}\theta_{SL}^{(2,2)}N_{3\downarrow}^{(2,2)}(\mathcal{S}_f^2)\right]. \end{aligned} \quad (8)$$

Before writing down the complete Regge action for $(2+1)$-dimensional causal dynamical triangulations with two-sphere spatial topology and line interval temporal topology, we must account for the presence of the boundary $\partial\mathcal{T}_c$ on the Regge-Einstein-Hilbert term. In particular, the spacelike hinges on the boundary $\partial\mathcal{T}_c$ no longer contribute to the Regge-Einstein-Hilbert term because full spacetime parallel transport about those hinges is no longer well defined. Accordingly, we modify the first line of the above prescription for the Regge-Einstein-Hilbert term to

$$\begin{aligned} \frac{1}{8\pi G}\Bigg[& \frac{2\pi}{i}\left(N_1^{SL} - N_1^{SL}(\mathcal{S}_i^2) - N_1^{SL}(\mathcal{S}_f^2)\right) - \frac{1}{i}\theta_{SL}^{(2,2)}\left(2N_3^{(2,2)} - N_{3\uparrow}^{(2,2)}(\mathcal{S}_i^2) - N_{3\downarrow}^{(2,2)}(\mathcal{S}_f^2)\right) \\ & -\frac{1}{i}\theta_{SL}^{(1,3)}\left(4N_1^{SL} - 2N_1^{SL}(\mathcal{S}_i^2) - 2N_1^{SL}(\mathcal{S}_f^2)\right)\Bigg]. \end{aligned} \quad (9)$$

The numerical factors appearing in the second and third terms of this expression require some explanation. In the second term $N_{3\uparrow}^{(2,2)}(\mathcal{S}_i^2)$ and $N_{3\downarrow}^{(2,2)}(\mathcal{S}_f^2)$ enter with the factor 1 instead of 2 because only one of the two spacelike edges of these $(2,2)$ 3-simplices attaches to the boundary $\partial\mathcal{T}_c$. In the third term $N_1^{SL}(\mathcal{S}_i^2)$ and $N_1^{SL}(\mathcal{S}_f^2)$ enter with the factor 2 instead of 4 because only two $(1,3)$ or $(3,1)$ 3-simplices attach to a hinge at the boundary $\partial\mathcal{T}_c$. The complete Regge action is thus

$$\begin{aligned}
S^{(R)}[\mathcal{T}_c] &= \frac{1}{8\pi G}\Bigg[\frac{2\pi}{i}\left(N_1^{SL}-N_1^{SL}(\mathcal{S}_i^2)-N_1^{SL}(\mathcal{S}_f^2)\right)-\frac{1}{i}\theta_{SL}^{(2,2)}\left(2N_3^{(2,2)}-N_{3\uparrow}^{(2,2)}(\mathcal{S}_i^2)-N_{3\downarrow}^{(2,2)}(\mathcal{S}_f^2)\right)\\
&\quad -\frac{1}{i}\theta_{SL}^{(1,3)}\left(4N_1^{SL}-2N_1^{SL}(\mathcal{S}_i^2)-2N_1^{SL}(\mathcal{S}_f^2)\right)+2\pi\sqrt{\alpha}N_1^{TL}-4\sqrt{\alpha}\theta_{TL}^{(2,2)}N_3^{(2,2)}\\
&\quad -3\sqrt{\alpha}\theta_{TL}^{(1,3)}N_3^{(1,3)}-3\sqrt{\alpha}\theta_{TL}^{(3,1)}N_3^{(3,1)}\Bigg]\\
&+\frac{\Lambda}{8\pi G}\left[V_3^{(2,2)}N_3^{(2,2)}+V_3^{(1,3)}N_3^{(1,3)}+V_3^{(3,1)}N_3^{(3,1)}\right]\\
&+\frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathcal{S}_i^2)-\frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathcal{S}_i^2)-\frac{1}{i}\theta_{SL}^{(2,2)}N_{3\uparrow}^{(2,2)}(\mathcal{S}_i^2)\right]\\
&-\frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathcal{S}_f^2)-\frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathcal{S}_f^2)-\frac{1}{i}\theta_{SL}^{(2,2)}N_{3\downarrow}^{(2,2)}(\mathcal{S}_f^2)\right].
\end{aligned} \tag{10}$$

We finally demonstrate that our prescription for the Regge-Gibbons-Hawking boundary term in $(2+1)$-dimensional causal dynamical triangulations is consistent with the form of the Regge-Einstein-Hilbert action determined by Ambjørn *et al.* We make such a demonstration by verifying that our prescription for the Regge-Gibbons-Hawking boundary term reproduces the Regge-Einstein-Hilbert action when we compose two spacetime regions sharing a common boundary $\mathcal{S}_c^2$. Consider two triangulated spacetime manifolds $\mathcal{T}_c$ and $\mathcal{T}_c'$ both with two-sphere spatial topology and line interval temporal topology. The boundary $\partial\mathcal{T}_c$ consists of an initial two-sphere $\mathcal{S}_i^2$ and a final two-sphere $\mathcal{S}_f^2$, and the boundary $\partial\mathcal{T}_c'$ consists of an initial two-sphere $\mathcal{S'}_i^2$ and a final two-sphere $\mathcal{S'}_f^2$. To compose the two triangulated spacetime manifolds $\mathcal{T}_c$ and $\mathcal{T}_c'$, we first take the two-spheres $\mathcal{S}_f^2$ and $\mathcal{S'}_i^2$ to have the same intrinsic geometry and then we orient the two-spheres $\mathcal{S}_f^2$ and $\mathcal{S'}_i^2$ to have coincident normal vectors. We may thus identify these two two-spheres as $\mathcal{S}_c^2$. The Regge-Gibbons-Hawking boundary term contributions of $\mathcal{S}_c^2$ from the two triangulated spacetime manifolds $\mathcal{T}_c$ and $\mathcal{T}_c'$ are

$$\begin{aligned}
&\frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathcal{S}_f^2)-\frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathcal{S}_f^2)-\frac{1}{i}\theta_{SL}^{(2,2)}N_{3\downarrow}^{(2,2)}(\mathcal{S}_f^2)\right]\\
&+\frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathcal{S'}_i^2)-\frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathcal{S'}_i^2)-\frac{1}{i}\theta_{SL}^{(2,2)}N_{3\uparrow}^{(2,2)}(\mathcal{S'}_i^2)\right],
\end{aligned} \tag{11}$$

the positive sign in the first line stemming from reorientation of the vector normal to $\mathcal{S}_f^2$. Together these two Regge-Gibbons-Hawking boundary terms combine to give the contribution to the Regge-Einstein-Hilbert action coming from the spacelike hinges on $\mathcal{S}_c^2$.