Sphere Generator: Techniques

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This is documentation for sphere_generator.py, a program to generate spheres of arbitrary surface area triangulated by equilateral triangles. The sphere is represented as a list of triples, each triple containing 3 vertex id numbers. Matching up the IDs yields an object homeomorphic to a sphere, and with an (ideally) uniform curvature.

This is a description of the ideas used to make the sphere generator program. The README and the programmer's guide offer more details. In this document I will describe how the program selects for standard deviation surface area of a given sphere.

1 Using Regge Calculus

Associated with each vertex is a point curvature:

$$K_p(v) = 2(2 - \theta_t N_T(v)),$$

where $\theta_t = \pi/3$ is the interior angle of an equilateral triangle and $N_t(v)$ is the total number of triangles connected to a given vertex. This description of curvature is given by Regge calculus.

The mean curvature is then

$$K = \frac{1}{N_v} \sum_{vertices,\ v} K_p(v),$$

where N_v is the total number of vertices in the sphere.

The standard deviation of curvature, σ is then given by

$$\sigma^2 = \frac{1}{N_v} \sum_{vertices,\ v} \left(K - K_p(v)\right)^2.$$

Let ϵ_a and ϵ_σ be numbers between 0 and 1. Let the target standard deviation—i.e., the standard deviation we want the sphere to have—be σ_t and the target surface area of the sphere—i.e., the surface area we want the sphere to have—be A_t. The *fitness function* for a given sphere S is

$$f(S) = e^{-\epsilon_a (A - A_t)} e^{-\epsilon_\sigma (\sigma - \sigma_t)},$$

where A is the surface area of the sphere at a given time.

2 Microscopically Optimal

One way to test whether or not a sphere is as close to "flat" as allowed is by testing for microscopically optimal. This is a global property, and doesn't care about the distribution of the curvature. Combinatorically, a sphere with N vertices is microscopically optimal if and only if the number of vertices attached to 6 triangles is

$$N - 12$$

and the number of vertices attached to 5 triangles is

$$12.$$

The number of triangles attached to a vertex is the *order* of the vertex or the *degree* of the vertex. We can test how close to microscopically optimal a sphere is by testing its deviation from these two conditions. Let $V5D$ be the number of vertices of degree 5 and $V6D$ be the number of vertices of degree 6. Then we have the following fitness functions:

$$f_1(sphere) = |(N - 12) - V6D| \texttt{ and } f_2(sphere) = |12 - V5D|.$$

let $damping_{v5D}$ and $damping_{v6D}$ be integers. Then we can construct the following convergence condition. The sphere is acceptably close to microscopically optimal if and only if

$$f_1(sphere) \leq damping_{V6D} \texttt{ and } f_2(sphere) \leq damping_{V5D}.$$

We can choose to only output sphere data when the convergence condition is met.

3 The Metropolis Algorithm

The metropolis algorithm does the following:

- Make a small change to the sphere using one of the ergodic moves. Ergodic here means that a composition of changes can bring any sphere to any other sphere.
- Test to see if the fitness function becomes larger or smaller. If the fitness function becomes larger, accept the change. Otherwise, reject it.
- Repeat.

That's it! That's how spheres are generated! For implementation details, see the programmer's guide. If you have any questions, feel free to email me.