A method for generating triangulated 2-spheres of arbitrary geometry

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Consider a 2-sphere $\mathcal{S}^2$ triangulated by N_2 regular 2-simplices. Let $\{|\Upsilon_\lambda^\mu\rangle\}$ for $\lambda \in \{0, \dots, l_{N_2}\}$ and $\mu \in \{-\lambda, \dots, \lambda\}$ be the set of discrete spherical harmonics required to characterize the geometry of the 2-sphere $\mathcal{S}^2$ composed of the finite number N_2 of regular 2-simplices. Denoting by $|\mathcal{G}_{\mathcal{S}^2}\rangle$ the geometry of the 2-sphere $\mathcal{S}^2$, we suggestively express the geometry as a linear combination of the discrete spherical harmonics,

$$|\mathcal{G}_{\mathcal{S}^2}\rangle = \sum_{\lambda=0}^{l_{N_2}} \sum_{\mu=-\lambda}^{\lambda} c_\mu^\lambda |\Upsilon_\lambda^\mu\rangle. \tag{1}$$

The set of coefficients $\{c_\mu^\lambda\}$ thus completely specifies the geometry $|\mathcal{G}_{\mathcal{S}^2}\rangle$. We define an inner product $A[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]$ on the space of geometries of 2-spheres as the projection of one geometry $|\mathcal{G}_{\mathcal{S}^2}\rangle$ onto another geometry $|\mathcal{G}_{\tilde{\mathcal{S}}^2}\rangle$:

$$A[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}] = \langle \mathcal{G}_{\tilde{\mathcal{S}}^2}|\mathcal{G}_{\mathcal{S}^2}\rangle. \tag{2}$$

Assuming that the set $\{|\Upsilon_\lambda^\mu\rangle\}$ of discrete spherical harmonics is orthonormal for any number N_2 of regular 2-simplices, the inner product $A[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]$ assumes the form

$$A[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}] = \sum_{\lambda=0}^{\min[l_{N_2}, l_{\tilde{N}_2}]} \sum_{\mu=-\lambda}^{\lambda} \tilde{c}_\mu^\lambda c_\mu^\lambda \tag{3}$$

Given the inner product $A[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]$, we further define a measure $\mu[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]$ of the closeness of the geometry $|\mathcal{G}_{\mathcal{S}^2}\rangle$ to the geometry $|\mathcal{G}_{\tilde{\mathcal{S}}^2}\rangle$ as

$$\mu[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}] = A[\mathcal{G}_{\tilde{\mathcal{S}}^2} - \mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]. \tag{4}$$

The measure $\mu[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]$ readily simplifies to

$$\mu[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}] = 1 - \sum_{\lambda=0}^{\min[l_{N_2}, l_{\tilde{N}_2}]} \sum_{\mu=-\lambda}^{\lambda} \tilde{c}_\mu^\lambda c_\mu^\lambda. \tag{5}$$

Conceiving of the measure $\mu[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]$ as an action functional on the space of geometries of 2-spheres, we construct a partition function

$$Z(\tilde{\mathcal{S}}^2) = \sum_{\mathcal{G}_{\mathcal{S}^2}} e^{-\beta\,\mu[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]} \tag{6}$$

for the chosen fixed 2-sphere $\tilde{\mathcal{S}}^2$ triangulated by $\tilde{N}_2$ regular 2-simplices of geometry $\mathcal{G}_{\tilde{\mathcal{S}}^2}$. β is a tunable coupling constant. The partition function $Z(\tilde{\mathcal{S}}^2)$ is related by Legendre transform to the partition function $Z_{\tilde{N}_2}(\tilde{\mathcal{S}}^2)$ for fixed number $\tilde{N}_2$ of regular 2-simplices:

$$Z(\tilde{\mathcal{S}}^2) = \sum_{N_2} Z_{\tilde{N}_2}(\tilde{\mathcal{S}}^2) e^{-N_2}. \tag{7}$$

Suppose now that we wish to generate a 2-sphere $\tilde{\mathcal{S}}^2$ triangulated by $\tilde{N}_2$ regular 2-simplices of geometry $|\mathcal{G}_{\tilde{\mathcal{S}}^2}\rangle$ specified by the set of coefficients $\{\tilde{c}_\mu^\lambda\}$. We propose the following method: design a Monte Carlo algorithm based on the partition function $Z_{\tilde{N}_2}(\tilde{\mathcal{S}}^2)$. Interpreting the weighting factor $e^{-\mu[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]}$ as the conditional probability of the geometry $|\mathcal{G}_{\mathcal{S}^2}\rangle$ given the geometry $|\mathcal{G}_{\tilde{\mathcal{S}}^2}\rangle$, those geometries $|\mathcal{G}_{\mathcal{S}^2}\rangle$ closest to the geometry $|\mathcal{G}_{\tilde{\mathcal{S}}^2}\rangle$ are most probable. Since a Monte Carlo algorithm is designed to produce an ensemble of geometries representative of the partition function $Z_{\tilde{N}_2}(\tilde{\mathcal{S}}^2)$, the algorithm should converge on geometries close to the given geometry $|\mathcal{G}_{\tilde{\mathcal{S}}^2}\rangle$.

Our Monte Carlo algorithm functions as follows. First, we choose the fixed geometry $|\mathcal{G}_{\tilde{\mathcal{S}}^2}\rangle$—in particular, the set of coefficients $\{\tilde{c}_\mu^\lambda\}$—to define the measure $\mu[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]$. Next, we initialize our simulation by generating the 2-sphere $\mathcal{S}^2_{min}$ minimally triangulated by regular 2-simplices, namely the regular tetrahedron. Then, by applying the known ergodic moves on the space of 2-surfaces triangulated by regular 2-simplices, we increase the number N_2 of regular 2-simplices of the 2-sphere $\mathcal{S}^2_{min}$ to the value $\tilde{N}_2$ of regular 2-simplices of the fixed geometry $|\mathcal{G}_{\tilde{\mathcal{S}}^2}\rangle$. Next, we run a standard Metropolis algorithm based on the partition function $Z_{\tilde{N}_2}(\tilde{\mathcal{S}}^2)$. To determine whether particular ergodic moves are accepted or rejected as the Metropolis algorithm runs, we must compute the change in the measure $\mu[\mathcal{G}_{\mathcal{S}^2}|\mathcal{G}_{\tilde{\mathcal{S}}^2}]$ upon implementation of the ergodic move. This requires that we compute the sets of coefficients $\{c_\mu^\lambda\}$ characterizing the intermediate geometries $|\mathcal{G}_{\mathcal{S}^2}\rangle$; we employ the technique of Sachs to perform these computations [1]. After the Monte Carlo algorithm has run for a sufficient computational time, the geometry $|\mathcal{G}_{\mathcal{S}^2}\rangle$ should be sufficiently close to the fixed geometry $|\mathcal{G}_{\tilde{\mathcal{S}}^2}\rangle$; we validate this closeness by employing again the technique of Sachs.

References

[1] M. K. Sachs. "Testing Lattice Quantum Gravity in $2 + 1$ Dimensions." arXiv: gr-qc/1110.6880.