On The Einstein-Hilber Action with the Gibbons-Hawking Boundary Term in Causal Dynamical Triangulations

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1 Introduction

For a $(2+1)$-dimensional spacetime manifold $\mathscr{M}$ with boundary $\partial\mathscr{M}$, we must add to the Einstein-Hilbert action,

$$S_{EH}[\mathbf{g}] = \frac{1}{16\pi G}\int_{\mathscr{M}} \mathrm{d}^3x\sqrt{-g}\,(R-2\Lambda)\,, \tag{1}$$

the Gibbons-Hawking boundary term,

$$S_{GH}[\gamma] = \frac{1}{8\pi G}\int_{\partial\mathscr{M}} \mathrm{d}^2y\sqrt{|\gamma|}K. \tag{2}$$

Here, γ is the induced metric on the boundary $\partial\mathscr{M}$, and K is the trace of the extrinsic curvature of the boundary $\partial\mathscr{M}$. Regge demonstrated that, for a triangulated spacetime manifold $\mathscr{T}$, the Einstein-Hilbert action assumes the form

$$S_{EH}^{(R)}[\mathscr{T}] = \frac{1}{8\pi G}\sum_{h\in\mathscr{T}} A_h\delta_h - \frac{\Lambda}{8\pi G}\sum_{s\in\mathscr{T}} V_s. \tag{3}$$

Here, h is a 1-dimensional hinge having area A_h and deficit angle δ_h, and V_s is the spacetime volume of a 3-simplex s. Hartle and Sorkin demonstrated that, for a triangulated spacetime manifold $\mathscr{T}$ with boundary $\partial\mathscr{T}$, the Gibbons-Hawking boundary term assumes the form

$$S_{GH}^{(R)}[\partial\mathscr{T}] = \frac{1}{8\pi G}\sum_{h\in\partial\mathscr{T}} A_h\psi_h. \tag{4}$$

Here, h is a 1-dimensional hinge on the boundary $\partial\mathscr{T}$ having area A_h, and ψ_h is the angle between the two vectors normal to the two spacelike 2-simplices intersecting at the hinge h.

We wish to determine the form of the Regge-Einstein-Hilbert action supplemented by the Regge-Gibbons-Hawking boundary term in $(2+1)$-dimensional causal dynamical triangulations for two-sphere spatial topology and line interval temporal topology. Ambjørn *et al* [1] give the Regge-Einstein-Hilbert action in this case, finding that

$$\begin{aligned} S_{EH}^{(R)}[\mathscr{T}_c] &= k\left[\frac{2\pi}{i}N_1^{SL} - \frac{3}{i}k\theta_{SL}^{(3,1)}\left(N_3^{(3,1)}+N_3^{(1,3)}\right) - \frac{2}{i}\theta_{SL}^{(2,2)}N_3^{(2,2)}\right] \\ &+ k\sqrt{\alpha}\left[2\pi N_1^{TL} - 3\theta_{TL}^{(3,1)}(N_3^{(3,1)}+N_3^{(1,3)}) - 4\theta_{TL}^{(2,2)}N_3^{(2,2)}\right] \\ &- \frac{\lambda}{12}\left[(N_3^{(3,1)}+N_3^{(1,3)})\sqrt{3\alpha+1} + N_3^{(2,2)}\sqrt{4\alpha+2}\right] \end{aligned} \tag{5}$$

Here, $k=\frac{1}{8\pi G}$ and $\lambda=\Lambda k$ are coupling constants. N_1^{SL} is the number of spacelike 1-simplices, N_1^{TL} is the number of timelike 1-simplices, $N_3^{(2,2)}$ is the number of $(2,2)$ 3-simplices, $N_3^{(1,3)}$ is the number of $(1,3)$ 3-simplices, $N_3^{(3,1)}$ is the number of $(3,1)$ 3-simplices, $\theta_{SL}^{(2,2)}$ is the Lorentzian dihedral angle about a spacelike edge of a $(2,2)$ 3-simplex, $\theta_{TL}^{(2,2)}$ is the Lorentzian dihedral angle about a timelike edge of a $(2,2)$ 3-simplex, $\theta_{SL}^{(1,3)}$ is the Lorentzian dihedral angle about a spacelike edge of a $(1,3)$ 3-simplex, $\theta_{TL}^{(1,3)}$ is the Lorentzian dihedral angle about a timelike edge of a $(1,3)$ 3-simplex, $\theta_{SL}^{(3,1)}$ is the Lorentzian dihedral angle about a spacelike edge of a $(3,1)$ 3-simplex, $\theta_{TL}^{(3,1)}$ is the Lorentzian dihedral angle about a timelike edge of a $(3,1)$ 3-simplex, $V_3^{(2,2)}$ is the spacetime volume of a $(2,2)$ 3-simplex, $V_3^{(1,3)}$ is the spacetime volume of a $(1,3)$ 3-simplex, $V_3^{(3,1)}$ is the spacetime volume of a $(3,1)$ 3-simplex, and α is the ratio of the timelike to the spacelike squared edge length of a 3-simplex.The first line comes from the summation over spacelike hinges, the second line comes from the summation over timelike hinges, and the third line comes from the summation over 3-simplices. Note that we could rewrite the second term of the first line using the relation $4N_1^{SL} = 3\left(N_3^{(1,3)} + N_3^{(3,1)}\right)$.

2 The Bulk Action

If the Regge-Einstein-Hilbert action for $(2+1)$-dimensional causal dynamical triangulations (CDT) is to include a boundary term, it must be split into a bulk term and a boundary term. Hartle and Sorkin provide the framework for the Gibbons-Hawking-York boundary term [2]. However, first, we must ensure that the Regge-Einstein-Hilbert action does not take any boundary terms as input. We will rederive the Regge-Einstein-Hilbert action but making sure we don't count any part of the boundary. This is the *bulk action*.

We start with the CDT version of the Regge action [3] given by Ambjørn *et al.* [1]:

$$\begin{aligned} S_{EH}^{R}[\mathscr{T}_{bulk}] &= k \sum_{\substack{\text{space-like}\\ \text{links } l}} Vol(l)\frac{1}{i}\left(2\pi - \sum_{\substack{\text{tetrahedra}\\ \text{at } l\\ t}} \theta_D(t,l)\right) + k \sum_{\substack{\text{time-like}\\ \text{links } l}} Vol(l)\left(2\pi - \sum_{\substack{\text{tetrahedra}\\ \text{at } l\\ t}} \theta_D(t,l)\right) \\ &\quad -\lambda \sum_{\substack{(3,1) \text{ and } (1,3)\\ \text{tetrahedra}}} Vol(3,1) - \lambda \sum_{\substack{(2,2)\\ \text{tetrahedra}}} Vol(2,2). \end{aligned} \tag{6}$$

Here $Vol(l)$ is the volume of a given link l. Because we are in $(2+1)$-dimensions. We can assume that $Vol(l) = 1$ for space-like links and that $Vol(l) = \sqrt{\alpha}$ for time-like links. $\theta_D(t,l)$ is the dihedral angle of a tetrahedron t around a link l. $Vol(2,2)$ and $Vol(3,1)$ are the volumes or $(2,2)-$ and $(3,1)-$ tetrahedra respectively. These values are given in Ambjørn *et al.* [1].

If we distribute summation signs and perform obvious summations, we find that:

$$\begin{aligned} S_{EH}^{R}[\mathscr{T}_{bulk}] &= \frac{2\pi k}{i}\left[N_1^{SL}(\mathscr{T}) - N_1^{SL}(S_i^{(2)}) - N_1^{SL}(S_f'^{(2)})\right] - \frac{k}{i}\sum_{\substack{\text{space-like}\\ \text{links } l}} \sum_{\substack{\text{tetrahedra } t\\ \text{at link } l}} \theta_D(t,l) \\ &\quad + 2\pi k\sqrt{\alpha}N_1^{TL} - k\sqrt{\alpha}\sum_{\substack{\text{time-like}\\ \text{links } l}} \sum_{\substack{\text{tetrahedra } t\\ \text{at link } l}} \theta_D(t,l) \\ &\quad -\lambda\left[V_3^{(3,1)}(N_3^{(3,1)} + N_3^{(1,3)}) + V_3^{(2,2)}N_3^{(2,2)}\right], \end{aligned} \tag{7}$$

where $N_1^{SL}(S_i^{(2)})$ is the number of space-like links that lie in the initial surface at proper time $\tau = 0$. $S^{(2)}$ simply indicates that the topology of the surface is spherical. Likewise $N_1^{SL}(S_f'^{(2)}$ is the number of space-like links that lie in the final surface at proper time $\tau = \tau_{final}$. We perform the subtraction in the first term to avoid overcounting objects in the boundary multiple times. $V_3^{(3,1)}$ and $V_3^{(2,2)}$ are the 3-volumes of $(3,1)$ (and likewise $(1,3)$) tetrahedra and $(2,2)$ tetrahedra respectively.

We now need to count the number of tetrahedra connected to each link and sum over all links. To perform this operation, we first act out the inner sum over dihedral angles around an individual link:

$$\begin{aligned} S^R_{EH}[\mathscr{T}_{bulk}] &= \frac{2\pi k}{i}\left[N_1^{SL}(\mathscr{T}) - N_1^{SL}(\mathscr{S}_i^{(2)}) - N_1^{SL}(\mathscr{S}_f'^{(2)})\right] - \frac{k}{i}\sum_{\substack{\text{space-like}\\ \text{links } l\\ \text{in bulk}}}\left[N_3^{(2,2)}(l)\theta_{SL}^{(2,2)} + N_3^{(1,3)}(l)\theta_{SL}^{(3,1)} + N_3^{(3,1)}(l)\theta_{SL}^{(1,3)}\right] \\ &\quad + 2\pi k\sqrt{\alpha}N_1^{TL} - k\sqrt{\alpha}\sum_{\substack{\text{time-like}\\ \text{links } l}}\left[N_3^{(2,2)}(l)\theta_{TL}^{(2,2)} + N_3^{(1,3)}(l)\theta_{TL}^{(3,1)} + N_3^{(3,1)}(l)\theta_{TL}^{(3,1)}\right] \\ &\quad - \lambda\left[V_3^{(3,1)}(N_3^{(3,1)} + N_3^{(1,3)}) + V_3^{(2,2)}N_3^{(2,2)}\right], \end{aligned} \tag{8}$$

where $N_3^{(2,2)}(l)$, $N_3^{(3,1)}(l)$, and $N_3^{(1,3)}(l)$ are the number of $(2,2)-$, $(3,1)-$, and $(1,3)-$tetrahedra respectively around a given link. To perform the remaining summation over the entire manifold, we look at how many links that each tetrahedron connects to, and sum over tetrahedrons, rather than summing over tetrahedrons at each link and then summing over tetrahedra. We know that:

- Each $(2,2)-$simplex connects to 2 space-like links in the bulk, but only one on each boundary. Thus:

$$\sum_{\substack{\text{space-like}\\ \text{links } l\\ \text{in bulk}}} N_3^{(2,2)}(l) = 2N_3^{(2,2)}(\mathscr{T}_{bulk})$$

  and

$$\sum_{\substack{\text{space-like}\\ \text{links } l\\ \text{in boundary}}} N_3^{(2,2)}(l) = N_3^{(2,2)}(\mathscr{S}_i^{(2)}) + N_3^{(2,2)}(\mathscr{S}_f'^{(2)}),$$

  where $N_3^{(2,2)}(\mathscr{T}_{bulk})$ is the total number of $(2,2)-$simplices in the manifold that do not connect to a link in the boundary. Likewise, $N_3^{(2,2)}(\mathscr{S}_i^{(2)})$ and $N_3^{(2,2)}(\mathscr{S}_f'^{(2)})$ are the total number of simplices that have at least one link (in fact exactly one) in the initial boundary or the final boundary respectively. We will continue to use this naming convention. Thus:

$$\sum_{\substack{\text{space-like}\\ \text{links } l\\ \text{in bulk}}} N_3^{(2,2)}(l) = 2N_3^{(2,2)}(\mathscr{T}_{bulk}) = N_3^{(2,2)} - N_3^{(2,2)}(\mathscr{S}_i^{(2)}) - N_3^{(2,2)}(\mathscr{S}_f'^{(2)}), \tag{9}$$

  where $N_3^{(2,2)}$ is of course the total number of $(2,2)$-simplices in the manifold.

- Each $(2,2)-$simplex connects to 4 time-like links. Thus:

$$\sum_{\substack{\text{time-like}\\ \text{links } l}} N_3^{(2,2)}(l) = 4N_3^{(2,2)}. \tag{10}$$

  There are no time-like links in the boundary, so we don't have to worry about this distinction.

- Each $(3,1)-$ and each $(1,3)-$simplex in bulk connects to 3 spacelike links. Thus

$$\sum_{\substack{\text{space-like}\\ \text{links } l\\ \text{in bulk}}} \left(N_3^{(3,1)} + N_3^{(1,3)}\right) = 3\left(N_3^{(3,1)}(\mathscr{T}_{bulk}) + N_3^{(1,3)}(\mathscr{T}_{bulk})\right).$$

  On the initial boundary, each $(3,1)-$simplex connects to 3 links, but no $(1,3)-$simplex connects to any links at all. Similarly, on the final boundary, each $(1,3)-$simplex connects to 3 links but no $(3,1)-$simplex connacts to any. Thus:

$$\sum_{\substack{\text{space-like}\\ \text{links } l\\ \text{on-boundary}}} (N_3^{(3,1)}(l) + N_3^{(1,3)}(l)) = 3\left(N_3^{(3,1)}(\mathscr{S}_i^{(2)}) + N_3^{(1,3)}(\mathscr{S}_f'^{(2)})\right).$$

Thus:

$$\begin{aligned}\sum_{\substack{\text{space-like}\\ \text{links } l\\ \text{in bulk}}}\left(N_3^{(3,1)}+N_3^{(1,3)}\right) &= 3\left(N_3^{(3,1)}(\mathscr{T}_{bulk})+N_3^{(1,3)}(\mathscr{T}_{bulk})\right)\\ &= 3\left(N_3^{(3,1)}+N_3^{(1,3)}\right)-3\left(N_3^{(3,1)}(\mathscr{S}_i^{(2)})+N_3^{(1,3)}(\mathscr{S}_f'^{(2)})\right). \qquad (11)\end{aligned}$$

- Each $(3,1)-$ or $(1,3)-$simplex connects to 3 time-like links. Thus:

$$\sum_{\substack{\text{time-like}\\ \text{links } l}}\left(N_3^{(3,1)}(l)+N_3^{(1,3)}(l)\right)=3\left(N_3^{(3,1)}+N_3^{(1,3)}\right). \qquad (12)$$

There are no time-like links in the boundary, so we don't have to worry about this distinction.

If we take the counting relations given above into account, then we find that

$$\begin{aligned}S_{EH}^{R}[\mathscr{T}_{bulk}] &= \frac{2\pi k}{i}\left[N_1^{SL}(\mathscr{T})-N_1^{SL}(\mathscr{S}_i^{(2)})-N_1^{SL}(\mathscr{S}_f'^{(2)})\right]-\frac{k}{i}\theta_{SL}^{(2,2)}\left[\left(2N_3^{(2,2)}-N_3^{(2,2)}(\mathscr{S}_i^{(2)})-N_3^{(2,2)}(\mathscr{S}_f'^{(2)})\right)\right]\\ &\quad -\frac{3k}{i}\theta_{SL}^{(1,3)}\left[N_3^{(1,3)}+N_3^{(3,1)})-N_3^{(3,1)}(\mathscr{S}_i^{(2)})-N_3^{(1,3)}(\mathscr{S}_f'^{(2)})\right] \qquad (13)\\ &\quad +2\pi k\sqrt{\alpha}N_1^{TL}-k\sqrt{\alpha}\left[4\theta_{TL}^{(2,2)}N_3^{(2,2)}+3\theta_{TL}^{(3,1)}\left(N_3^{(3,1)}+N_3^{(1,3)}\right)\right]\\ &\quad -\lambda\left[V_3^{(3,1)}(N_3^{(3,1)}+N_3^{(1,3)})+V_3^{(2,2)}N_3^{(2,2)}\right].\end{aligned}$$

This is the bulk form of the Regge action.

3 The Gibbons-Hawking-York Term

We now supplement the Regge-Einstein-Hilbert action for $(2+1)$-dimensional causal dynamical triangulations by the appropriate Regge-Gibbons-Hawking boundary term. Given the desired spacetime topology, the boundary $\partial\mathscr{T}_c$ consists of two disconnected components: an initial or past spatial two-sphere $\mathscr{S}_i^2$ and a final or future spatial two-sphere $\mathscr{S}_f^2$. Based on the demonstration of Hartle and Sorkin [2], we propose the prescription

$$S_{GH}^{(R)}[\partial\mathscr{T}_c]=\frac{1}{8\pi G}\sum_{h\in\mathscr{S}_i^2}\frac{1}{i}\left[\pi-2\theta_{SL}^{(3,1)}-\theta_{SL}^{(2,2)}N_{3\uparrow}^{(2,2)}(h)\right]+\frac{1}{8\pi G}\sum_{h\in\mathscr{S}_f^2}\frac{1}{i}\left[\pi-2\theta_{SL}^{(1,3)}-\theta_{SL}^{(2,2)}N_{3\downarrow}^{(2,2)}(h)\right]. \qquad (14)$$

Here, $N_{3\uparrow}^{(2,2)}(h)$ is the number of future-directed $(2,2)$ 3-simplices attached to the hinge h, and $N_{3\downarrow}^{(2,2)}(h)$ is the number of past-directed $(2,2)$ 3-simplices attached to the hinge h. We justify this prescription as follows. In parallel transporting the vector normal to one component of the boundary $\partial\mathscr{T}_c$ between two spacelike 2-simplices intersecting at the hinge h, the vector rotates through the angle

$$\frac{1}{i}\left[2\theta_{SL}^{(3,1)}+\theta_{SL}^{(2,2)}N_3^{(2,2)}(h)\right]. \qquad (15)$$

When this angle is $\frac{\pi}{i}$, the extrinsic curvature vanishes locally at the hinge h; this fact dictates the deficit angle-like form of our above prescription. The absence of a relative negative sign between the contributions of the two disconnected components of the boundary $\partial\mathscr{T}_c$ to the Regge-Gibbons-Hawking boundary term stems from the fact that the future-directed orientation of the vector normal to $\mathscr{S}_i^2$ and the past-directed orientation of the vector normal to $\mathscr{S}_f^2$ are accounted for in the past-directed and future-directed orientations of the $(2,2)$ 3-simplices attached to the boundary.

Performing the summations over the hinges on the boundary $\partial\mathscr{T}_c$, we may rewrite the Regge-Gibbons-Hawking boundary term as

$$\begin{aligned} S_{GH}^{(R)}[\partial\mathscr{T}_c] \quad = \quad & \frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathscr{S}_i^2) - \frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathscr{S}_i^2) - \frac{1}{i}\theta_{SL}^{(2,2)}N_{3\uparrow}^{(2,2)}(\mathscr{S}_i^2)\right] \\ & + \frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathscr{S}_f^2) - \frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathscr{S}_f^2) - \frac{1}{i}\theta_{SL}^{(2,2)}N_{3\downarrow}^{(2,2)}(\mathscr{S}_f^2)\right]. \end{aligned} \tag{16}$$

The complete Regge action is thus

$$\begin{aligned} S^{(R)}[\mathscr{T}_c] \quad = \quad & \frac{2\pi k}{i}\left[N_1^{SL}(\mathscr{T}) - N_1^{SL}(\mathscr{S}_i^{(2)}) - N_1^{SL}(\mathscr{S}_f'^{(2)})\right] - \frac{k}{i}\theta_{SL}^{(2,2)}\left[\left(2N_3^{(2,2)} - N_3^{(2,2)}(\mathscr{S}_i^{(2)}) - N_3^{(2,2)}(\mathscr{S}_f'^{(2)})\right)\right] \\ & -\frac{3k}{i}\theta_{SL}^{(1,3)}\left[N_3^{(1,3)} + N_3^{(3,1)}) - N_3^{(3,1)}(\mathscr{S}_i^{(2)}) - N_3^{(1,3)}(\mathscr{S}_f'^{(2)})\right] \\ & +2\pi k\sqrt{\alpha}N_1^{TL} - k\sqrt{\alpha}\left[4\theta_{TL}^{(2,2)}N_3^{(2,2)} + 3\theta_{TL}^{(3,1)}\left(N_3^{(3,1)} + N_3^{(1,3)}\right)\right] \\ & -\lambda\left[V_3^{(3,1)}(N_3^{(3,1)} + N_3^{(1,3)}) + V_3^{(2,2)}N_3^{(2,2)}\right]. \\ & +\frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathscr{S}_i^2) - \frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathscr{S}_i^2) - \frac{1}{i}\theta_{SL}^{(2,2)}N_{3\uparrow}^{(2,2)}(\mathscr{S}_i^2)\right] \\ & +\frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathscr{S}_f^2) - \frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathscr{S}_f^2) - \frac{1}{i}\theta_{SL}^{(2,2)}N_{3\downarrow}^{(2,2)}(\mathscr{S}_f^2)\right]. \end{aligned} \tag{17}$$

4 Consistency Checks

We finally demonstrate that our prescription for the Regge-Gibbons-Hawking boundary term in $(2+1)$-dimensional causal dynamical triangulations is consistent with the form of the Regge-Einstein-Hilbert action determined by Ambjørn *et al*. We make such a demonstration by verifying that our prescription for the Regge-Gibbons-Hawking boundary term reproduces the Regge-Einstein-Hilbert action when we compose two spacetime regions sharing a common boundary $\mathscr{S}_c^2$. Consider two triangulated spacetime manifolds $\mathscr{T}_c$ and $\mathscr{T}_c'$ both with two-sphere spatial topology and line interval temporal topology. The boundary $\partial\mathscr{T}_c$ consists of an initial two-sphere $\mathscr{S}_i^2$ and a final two-sphere $\mathscr{S}_f^2$, and the boundary $\partial\mathscr{T}_c'$ consists of an initial two-sphere $\mathscr{S}'^2_i$ and a final two-sphere $\mathscr{S}'^2_f$. To compose the two triangulated spacetime manifolds $\mathscr{T}_c$ and $\mathscr{T}_c'$, we first take the two-spheres $\mathscr{S}_f^2$ and $\mathscr{S}'^2_i$ to have the same intrinsic geometry and then we orient the two-spheres $\mathscr{S}_f^2$ and $\mathscr{S}'^2_i$ to have coincident normal vectors. We may thus identify these two two-spheres as $\mathscr{S}_c^2$. The Regge-Gibbons-Hawking boundary term contributions of $\mathscr{S}_c^2$ from the two triangulated spacetime manifolds $\mathscr{T}_c$ and $\mathscr{T}_c'$ are

$$\begin{aligned} & \frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathscr{S}_f^2) - \frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathscr{S}_f^2) - \frac{1}{i}\theta_{SL}^{(2,2)}N_{3\downarrow}^{(2,2)}(\mathscr{S}_f^2)\right] \\ + & \frac{1}{8\pi G}\left[\frac{\pi}{i}N_1^{SL}(\mathscr{S}'^2_i) - \frac{2}{i}\theta_{SL}^{(3,1)}N_1^{SL}(\mathscr{S}'^2_i) - \frac{1}{i}\theta_{SL}^{(2,2)}N_{3\uparrow}^{(2,2)}(\mathscr{S}'^2_i)\right]. \end{aligned} \tag{18}$$

Together these two Regge-Gibbons-Hawking boundary terms combine to give the contribution to the Regge-Einstein-Hilbert action coming from the spacelike hinges on $\mathscr{S}_c^2$.

References

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