

Hypotenuse and Hyperbolic

The terms "hypotenuse" and "hyperbolic" are related only by their shared etymological root in the Greek word "hypo," meaning "under" or "beneath." However, in modern mathematics, they refer to completely different concepts. Let's break them down:

1 Introduction

* **Definition:** The hypotenuse is the **longest side** of a **right-angled triangle**, and it is always the side **opposite the right angle** (90-degree angle).

* **Context:** It's a fundamental part of Euclidean geometry and is most famously associated with the Pythagorean theorem ($a^2 + b^2 = c^2$, where c is the hypotenuse).

* **Example:** Imagine a triangle with angles of 90 degrees, 60 degrees, and 30 degrees. The side opposite the 90-degree angle is the hypotenuse.

* **Definition:** "Hyperbolic" refers to things related to **hyperbolas**, which are a type of conic section (a curve formed by the intersection of a plane and a double cone). Hyperbolic functions are analogs of trigonometric functions but defined using the hyperbola instead of the circle.

* **Context:** It's a concept in analytic geometry and calculus. It involves hyperbolic functions (\sinh , \cosh , \tanh , \coth , sech , csch), hyperbolic geometry (a non-Euclidean geometry where parallel lines diverge), and hyperbolic equations (differential equations).

* **Example:** The equation $y = \cosh(x)$ represents a hyperbolic cosine function, which has a graph shaped like a catenary curve (a hanging chain). Hyperbolic geometry is used in physics, particularly in the theory of relativity.

The Greek prefix "hypo" ("under") is present in both:

* **Hypotenuse:**

Comes from the Greek "hypoteinousa," which means "stretching under." This refers to the side that stretches under the right angle to connect the other two sides.

* **Hyperbola:** Derives from the Greek "hyperbole," meaning "overthrowing" or "exceeding." This refers to how a hyperbola "exceeds" or opens more widely than a parabola. (Although the exact connection is not immediately obvious from the shape of a hyperbola. Appolonius originally named them, as an extension of the way he named ellipses and parabolas.)

****In summary:**** While the words share a common origin, they have evolved to represent distinct and unrelated concepts in mathematics. One deals with the sides of a right-angled triangle, and the other deals with curves and functions based on a hyperbola. It's a case of etymology leading to different destinations.

Hypotenuse about Hyperbolic Cosh and Sinh Mathematics

Here's a thinking process to arrive at the explanation connecting the hypotenuse (from right triangles) to hyperbolic cosine (cosh) and hyperbolic sine (sinh):

1.1 Cosh and Sinh

1. ****Identify the Core Concepts:**** The request asks about the relationship between "hypotenuse" and "hyperbolic cosh/sinh." I know:

* ****Hypotenuse:**** Longest side of a right triangle, opposite the 90° angle. Related to the Pythagorean theorem ($a^2 + b^2 = c^2$). Connected to standard trigonometric functions (sin, cos, tan) via SOH CAH TOA, where $\sin(\vartheta) = \text{opposite/hypotenuse}$, $\cos(\vartheta) = \text{adjacent/hypotenuse}$.

* ****Hyperbolic Functions (cosh, sinh):**** Defined using the exponential function: $\sinh(x) = (e^x - e^{-x})/2$, $\cosh(x) = (e^x + e^{-x})/2$. They are **analogs** of trigonometric functions. They relate to the **unit hyperbola** ($x^2 - y^2 = 1$), not the unit circle ($x^2 + y^2 = 1$).

2. ****Initial Question:**** Is there a **direct** geometric relationship like SOH CAH TOA where cosh or sinh directly involve the hypotenuse of a **standard** right triangle?

* ***Answer:*** No, not in the standard Euclidean geometry sense. Cosh and sinh are defined algebraically using exponentials or geometrically using the **hyperbola**. The hypotenuse belongs to right triangles in Euclidean space, linked to the **circle** (via Pythagorean theorem and trig functions).

3. ****Look for Analogies:**** Since cosh and sinh are **analogs** of cos and sin, what's the core relationship for cos and sin?

* The fundamental identity: $\cos^2(\vartheta) + \sin^2(\vartheta) = 1$.

* This comes from the Pythagorean theorem applied to a right triangle inscribed in a unit circle. If the hypotenuse is 1, the adjacent side is $\cos(\vartheta)$ and the opposite side is $\sin(\vartheta)$. So, $(\text{adjacent})^2 + (\text{opposite})^2 = (\text{hypotenuse})^2$, which becomes $\cos^2(\vartheta) + \sin^2(\vartheta) = 1^2$.

4. ****What's the Analogous Identity for Hyperbolic Functions?****

* Recall or derive the hyperbolic identity: $\cosh^2(x) - \sinh^2(x) = 1$.

* Derivation:

* $\cosh^2(x) = [(e^x + e^{-x})/2]^2 = (e^{2x} + 2e^x e^{-x} + e^{-2x})/4 = (e^{2x} + 2 + e^{-2x})/4$

- * $\sinh^2(x) = [(e^x - e^{-x})/2]^2 = (e^{2x} - 2e^x e^{-x} + e^{-2x})/4 = (e^{2x} - 2 + e^{-2x})/4$
- * $\cosh^2(x) - \sinh^2(x) = [(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})] / 4 = (2 - (-2)) / 4 = 4/4 = 1.$

5. **Connect the Identities:**

- * Circular Trig: $\cos^2(\vartheta) + \sin^2(\vartheta) = 1$ (related to $x^2 + y^2 = 1$, the unit circle)
- * Hyperbolic Trig: $\cosh^2(x) - \sinh^2(x) = 1$ (related to $x^2 - y^2 = 1$, the unit hyperbola)

6. **Geometric Interpretation:**

* **Circle:** A point on the unit circle (x, y) can be parameterized as $x = \cos(\vartheta)$, $y = \sin(\vartheta)$. Here, ϑ is related to the angle and the arc length. The distance from the origin (radius/hypotenuse of an implied triangle) is 1.

* **Hyperbola:** A point on the *right branch* of the unit hyperbola (x, y) can be parameterized as $x = \cosh(a)$, $y = \sinh(a)$. Here, *'a'* is *not* a standard angle but is related to the *area* of a hyperbolic sector. The coordinates $(\cosh(a), \sinh(a))$ satisfy the hyperbola equation $x^2 - y^2 = 1$.

7. **Can we form a triangle involving cosh and sinh?**

- * Consider the point $P(\cosh(a), \sinh(a))$ on the hyperbola $x^2 - y^2 = 1$.
- * Consider the origin $O(0, 0)$.
- * Consider the point $X(\cosh(a), 0)$ on the x-axis.
- * We have a right triangle OXP.
 - * Side OX (adjacent) has length $\cosh(a)$.
 - * Side XP (opposite) has length $\sinh(a)$.
 - * What about the side OP (the distance from the origin to the point on the hyperbola)? This would be the hypotenuse of triangle OXP.
- * Using the Pythagorean theorem for *this specific triangle OXP*:
 - * $(\text{Hypotenuse OP})^2 = (\text{Adjacent OX})^2 + (\text{Opposite XP})^2$
 - * $(\text{Hypotenuse OP})^2 = \cosh^2(a) + \sinh^2(a)$
 - * **Wait!** This is *not* the fundamental identity. The fundamental identity is $\cosh^2(a) - \sinh^2(a) = 1$.

8. **Reconciling the Triangle and the Identity:**

- * The triangle OXP exists in standard Euclidean space. Its sides are indeed $\cosh(a)$, $\sinh(a)$, and $\sqrt{\cosh^2(a) + \sinh^2(a)}$.
- * The identity $\cosh^2(a) - \sinh^2(a) = 1$ comes from the definition of the *hyperbola* itself ($x^2 - y^2 = 1$) where $x = \cosh(a)$ and $y = \sinh(a)$.
- * So, while $\cosh(a)$ and $\sinh(a)$ can be the lengths of the legs of a Euclidean right triangle, the *hypotenuse* of that triangle is $\sqrt{\cosh^2(a) + \sinh^2(a)}$, which is *not* equal to 1 (unlike the circular case where the hypotenuse *is* 1 for the unit circle).

9. **Synthesize the Explanation:**

- * Start by stating the *lack* of direct relationship in the SOH CAH TOA sense.
- * Explain the *analogous* nature: \sin/\cos relate to the circle ($x^2+y^2=1$), \sinh/\cosh relate to the hyperbola ($x^2-y^2=1$).
- * Bring in the fundamental identities: $\cos^2(\vartheta)+\sin^2(\vartheta)=1$ vs. $\cosh^2(a)-\sinh^2(a)=1$.
- * Explain the unit circle parametrization: $(\cos \vartheta, \sin \vartheta)$. The hypotenuse of the triangle formed by $(0,0)$, $(\cos \vartheta, 0)$, $(\cos \vartheta, \sin \vartheta)$ is 1.
- * Explain the unit hyperbola parametrization: $(\cosh a, \sinh a)$.
- * Form the Euclidean right triangle with vertices $(0,0)$, $(\cosh a, 0)$, and $(\cosh a, \sinh a)$.
- * Identify the legs: The legs have lengths $|\cosh a|$ and $|\sinh a|$. Since $\cosh(a)$ is always ≥ 1 for real 'a', and we usually consider the right branch, the lengths are $\cosh(a)$ and $|\sinh(a)|$.
- * Calculate the hypotenuse of *this* triangle using the Pythagorean theorem: $\text{Hypotenuse}^2 = \cosh^2(a) + \sinh^2(a)$.
- * **Crucial Point:** This hypotenuse is *not* generally equal to 1. It is related to $\cosh^2(a) + \sinh^2(a)$, while the defining identity involves $\cosh^2(a) - \sinh^2(a)$.
- * Mention the connection via the identity: $\cosh^2(a) + \sinh^2(a) = \cosh(2a)$. So the hypotenuse length is $\sqrt{\cosh(2a)}$.
- * Conclude by reiterating that the connection is *analogous* and relates to the fundamental identities and parameterizations, but \cosh and \sinh don't define ratios involving the hypotenuse in the same direct way as \cos and \sin do for a *general* right triangle. The triangle involving $\cosh(a)$ and $\sinh(a)$ as legs is a specific construction related to the hyperbola's parameterization.

Okay, let's clarify the relationship (or lack thereof) and the analogies between the hypotenuse (from right triangles and standard trigonometry) and the hyperbolic functions \cosh and \sinh .

Direct Relationship? No.

- * The **hypotenuse** is a geometric concept specific to **right-angled triangles** in Euclidean geometry.
- * Standard trigonometric functions (sine, cosine) are defined using ratios involving the sides of a right triangle (SOH CAH TOA):
 - * $\sin(\vartheta) = \text{Opposite} / \text{Hypotenuse}$
 - * $\cos(\vartheta) = \text{Adjacent} / \text{Hypotenuse}$
- * **Hyperbolic functions (\sinh, \cosh)** are defined algebraically using the exponential function *e* or geometrically using the **unit hyperbola** ($x^2 - y^2 = 1$), *not* a right triangle in the same direct way.
 - * $\cosh(x) = (e^x + e^{-x}) / 2$
 - * $\sinh(x) = (e^x - e^{-x}) / 2$

The Analogous Relationship (The Key Connection):

The connection comes from the **fundamental identities** that mirror each other:

1. **Standard Trigonometry (Unit Circle: $x^2 + y^2 = 1$):**

* The fundamental identity is: $\cos^2(\vartheta) + \sin^2(\vartheta) = 1$

* Geometric interpretation: If you have a point (x, y) on the unit circle (radius 1), then $x = \cos(\vartheta)$ and $y = \sin(\vartheta)$. The distance from the origin to the point is the radius, which acts like the hypotenuse of an implied right triangle with legs x and y . The Pythagorean theorem ($x^2 + y^2 = \text{hypotenuse}^2$) becomes $\cos^2(\vartheta) + \sin^2(\vartheta) = 1^2$.

2. **Hyperbolic Functions (Unit Hyperbola: $x^2 - y^2 = 1$):**

* The fundamental identity is: $\cosh^2(a) - \sinh^2(a) = 1$

* Geometric interpretation: If you have a point (x, y) on the right branch of the unit hyperbola ($x^2 - y^2 = 1$), then $x = \cosh(a)$ and $y = \sinh(a)$. Here, 'a' is related to the *area* of a hyperbolic sector, not a simple angle like ϑ .

Forming a Triangle with cosh and sinh:

While \sinh and \cosh aren't defined *by* a hypotenuse ratio, we *can* construct a right triangle using them:

* Consider the point P on the unit hyperbola: $P = (\cosh(a), \sinh(a))$.

* Consider the origin $O = (0, 0)$.

* Consider the point X on the x-axis: $X = (\cosh(a), 0)$.

Now, look at the right-angled triangle **OX**:

* The length of the adjacent side (OX) is $\cosh(a)$.

* The length of the opposite side (XP) is $|\sinh(a)|$.

* What is the length of the *hypotenuse* (OP) of *this specific triangle*? We use the standard Pythagorean theorem:

$$\text{Hypotenuse}^2 = (\text{Adjacent})^2 + (\text{Opposite})^2$$

$$\text{Hypotenuse}^2 = \cosh^2(a) + \sinh^2(a)$$

$$\text{Hypotenuse} = \sqrt{\cosh^2(a) + \sinh^2(a)}$$

Important Distinction:

* Notice that the hypotenuse of this triangle OXP is $\sqrt{\cosh^2(a) + \sinh^2(a)}$.

* This is **NOT** equal to 1 (unlike the unit circle case).

* The identity related to the *hyperbola itself* is $\cosh^2(a) - \sinh^2(a) = 1$.

(As a side note: using another identity, $\cosh(2a) = \cosh^2(a) + \sinh^2(a)$, the hypotenuse length of triangle OXP is actually $\sqrt{\cosh(2a)}$.)

****In Summary:****

- * There's no direct definition of cosh or sinh using a hypotenuse ratio like SOH CAH TOA.
- * The connection is through the ****analogous fundamental identities****: $\cos^2(\vartheta) + \sin^2(\vartheta) = 1$ (circle) vs. $\cosh^2(a) - \sinh^2(a) = 1$ (hyperbola).
- * You *can* form a Euclidean right triangle where the legs have lengths related to $\cosh(a)$ and $\sinh(a)$ based on the coordinates of a point on the hyperbola. The hypotenuse of *that* specific triangle is $\sqrt{\cosh^2(a) + \sinh^2(a)}$, which is linked to the identity $\cosh^2(a) + \sinh^2(a) = \cosh(2a)$, not the defining identity $\cosh^2(a) - \sinh^2(a) = 1$.

Hypotenuse and Hyperbolic in Quest of Cosh and Sinh to Solver Equation

Here's a thinking process to construct the explanation about using cosh and sinh to solve equations, connecting back to the ideas of hypotenuse and hyperbolic functions:

1.1.1 Quest Cosh and Sin

1. ****Deconstruct the Request:**** The user wants to understand how "hypotenuse" and "hyperbolic" concepts, specifically related to ``cosh`` and ``sinh``, are used to *solve equations*.
2. ****Recall Previous Context:**** We've established:
 - * Hypotenuse belongs to right triangles (Euclidean geometry).
 - * Hyperbolic (\cosh , \sinh) relate to the hyperbola $x^2 - y^2 = 1$ and have the identity $\cosh^2(a) - \sinh^2(a) = 1$.
 - * There's no *direct* definition of \cosh/\sinh using a hypotenuse ratio.
 - * The connection is *analogous* identities and the geometry of the unit circle vs. the unit hyperbola.
3. ****Identify the Core Problem:**** How do ``cosh`` and ``sinh`` help *solve equations*? This usually involves substitutions that simplify the equation, leveraging their properties.
4. ****Brainstorm Equation Types:**** Where might ``cosh`` and ``sinh`` substitutions be useful?
 - * Equations involving square roots of the form $\sqrt{x^2 - 1}$ or $\sqrt{x^2 + 1}$.
 - * Integrals involving these forms.

- * Differential equations (like the catenary).
- * Systems of equations where the variables might naturally represent $\cosh(a)$ and $\sinh(a)$.
- * Equations that look structurally similar to trigonometric identities, but with a minus sign.

5. ****Focus on the Most Common Use Case (Algebra/Calculus):**** The most direct link to the "hypotenuse" analogy (even though it's imperfect) is through substitutions simplifying expressions involving sums or differences of squares, reminiscent of the Pythagorean theorem and its hyperbolic counterpart.

* ****Case 1: Expressions like $\sqrt{x^2 - 1}$ ****

* This looks like the hyperbolic identity: $\cosh^2(a) - 1 = \sinh^2(a)$.

* ***Idea:** If we let $x = \cosh(a)$, then $\sqrt{x^2 - 1}$ becomes $\sqrt{\cosh^2(a) - 1} = \sqrt{\sinh^2(a)} = |\sinh(a)|$. This removes the square root.

* ***Constraint:** For $x = \cosh(a)$ to be valid, we need $x \geq 1$ (since $\cosh(a) \geq 1$ for all real 'a').

* ****Case 2: Expressions like $\sqrt{x^2 + 1}$ ****

* This looks like rearranging the ***other*** hyperbolic identity: $\sinh^2(a) + 1 = \cosh^2(a)$.

* ***Idea:** If we let $x = \sinh(a)$, then $\sqrt{x^2 + 1}$ becomes $\sqrt{\sinh^2(a) + 1} = \sqrt{\cosh^2(a)} = \cosh(a)$ (since $\cosh(a)$ is always positive). This removes the square root.

* ***Constraint:** $x = \sinh(a)$ can represent any real number.

* ****Contrast with Trigonometric Substitution (Hypotenuse Link):****

* For $\sqrt{1 - x^2}$, we use $x = \sin(\vartheta)$ or $x = \cos(\vartheta)$, leveraging $\cos^2(\vartheta) + \sin^2(\vartheta) = 1$. This directly relates to a right triangle with hypotenuse 1.

* For $\sqrt{1 + x^2}$, we use $x = \tan(\vartheta)$, leveraging $1 + \tan^2(\vartheta) = \sec^2(\vartheta)$. This relates to a right triangle with adjacent side 1, opposite side x , and hypotenuse $\sqrt{1 + x^2}$.

* For $\sqrt{x^2 - 1}$, we use $x = \sec(\vartheta)$, leveraging $\sec^2(\vartheta) - 1 = \tan^2(\vartheta)$. This relates to a right triangle with hypotenuse x , adjacent side 1, and opposite side $\sqrt{x^2 - 1}$.

* ****Connecting Back to "Hypotenuse":**** While the ***hyperbolic*** substitution itself doesn't ***directly*** use a hypotenuse ratio in its ***definition***, the ***reason*** it works is analogous to how trigonometric substitutions work, which ***are*** derived from right triangles and the Pythagorean theorem (involving a hypotenuse). The goal is the same: simplify expressions involving sums/differences of squares using a fundamental identity. The hyperbolic identity $\cosh^2(a) - \sinh^2(a) = 1$ plays the role that $\cos^2(\vartheta) + \sin^2(\vartheta) = 1$ plays in trig substitutions.

6. ****Structure the Explanation:****

- * Start by acknowledging the user's quest – using \cosh/\sinh to solve equations.

- * Reiterate the *indirect* connection to the hypotenuse concept (via the analogous identity structure).

- * State the primary technique: **Hyperbolic Substitution**.

- * Explain the two main cases ($\sqrt{x^2 - 1}$ and $\sqrt{x^2 + 1}$), detailing the substitution, the simplification using the identity, and any constraints.

- * Provide a concrete example equation for each case.

- * Briefly mention other areas (integrals, differential equations) where these functions appear.

- * Summarize: The power lies in the algebraic identity $\cosh^2(a) - \sinh^2(a) = 1$ (and related forms), which mirrors the Pythagorean identity used in trigonometric substitutions (which *is* directly related to the hypotenuse).

7. **Refine and Clarify:**

- * Ensure the distinction between the *definition* of cosh/sinh and their *use* in substitutions is clear.

- * Emphasize that the "hypotenuse" connection is more about the *structural analogy* of the identities than a direct geometric application in the hyperbolic case.

- * Use clear formatting (bolding, bullet points) to break down the information.

- * Check that the examples are simple enough to illustrate the point. Make sure to solve *for* the original variable *in* the examples.

8. **Self-Correction/Review:** Did I directly answer the user's prompt about solving equations? Yes. Did I link it back to the "hypotenuse" and "hyperbolic" concepts as requested? Yes, by explaining the analogous identities and contrasting with trig substitution. Is the explanation clear and accurate? Seems so.

Okay, let's connect the concepts. While the **hypotenuse** itself isn't directly *used* in the definition or fundamental properties of **hyperbolic functions** (cosh, sinh), the *structure* of the core hyperbolic identity allows us to use these functions to solve certain types of equations, often those involving square roots that resemble the Pythagorean theorem (which *does* involve the hypotenuse).

The key is the fundamental identity:

$$\cosh^2(a) - \sinh^2(a) = 1$$

This identity is the hyperbolic analogue of the trigonometric (Pythagorean) identity:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
 (This one *is* directly related to a right triangle with hypotenuse 1).

How cosh and sinh Help Solve Equations (Hyperbolic Substitution):

We use \cosh and \sinh as substitutions to simplify equations, particularly those involving expressions like $\sqrt{x^2 - c^2}$ or $\sqrt{x^2 + c^2}$. This technique is analogous to trigonometric substitution used in calculus and algebra.

Scenario 1: Equations involving $\sqrt{x^2 - c^2}$

Problem: Suppose you have an equation with a term like $\sqrt{x^2 - c^2}$, where c is a constant.

Analogy: This structure resembles $\cosh^2(a) - 1 = \sinh^2(a)$. We want to make the expression under the square root a perfect square using this identity.

Substitution: Let $x = c \cosh(a)$. (We need $x \geq c$ for this to work, as $\cosh(a) \geq 1$).

Simplification:

$$x^2 - c^2 = (c \cosh(a))^2 - c^2$$

$$= c^2 \cosh^2(a) - c^2$$

$$= c^2 (\cosh^2(a) - 1)$$

$$= c^2 \sinh^2(a)$$

So, $\sqrt{x^2 - c^2} = \sqrt{c^2 \sinh^2(a)} = |c \sinh(a)|$. This often simplifies the original equation significantly by removing the square root.

Example: Solve $\sqrt{x^2 - 9} = 4$

Let $x = 3 \cosh(a)$. (Here $c = 3$)

The equation becomes $\sqrt{(3 \cosh(a))^2 - 9} = 4$

$$\sqrt{9 \cosh^2(a) - 9} = 4$$

$$\sqrt{9 (\cosh^2(a) - 1)} = 4$$

$$\sqrt{9 \sinh^2(a)} = 4$$

Assume $\sinh(a)$ is positive for simplicity here (we might need \pm depending on context): $3 \sinh(a) = 4$

$$\sinh(a) = 4/3$$

To find 'a', we use the inverse hyperbolic sine: $a = \operatorname{arsinh}(4/3)$ (or $\sinh^{-1}(4/3)$).

Now, substitute back to find x : $x = 3 \cosh(a) = 3 \cosh(\operatorname{arsinh}(4/3))$.

Using the identity $\cosh^2(a) - \sinh^2(a) = 1 \Rightarrow \cosh^2(a) = 1 + \sinh^2(a)$.

$$\cosh(a) = \sqrt{1 + \sinh^2(a)} \text{ (since } \cosh(a) \text{ is always positive)}$$

$$\cosh(a) = \sqrt{1 + (4/3)^2} = \sqrt{1 + 16/9} = \sqrt{25/9} = 5/3.$$

Therefore, $x = 3 * (5/3) = 5$.

Check: $\sqrt{5^2 - 9} = \sqrt{25 - 9} = \sqrt{16} = 4$. It works. (Note: $x = -5$ also works, which corresponds to the negative \sinh case).

Scenario 2: Equations involving $\sqrt{x^2 + c^2}$

Problem: Suppose you have an equation with a term like $\sqrt{x^2 + c^2}$.

Analogy: This structure resembles $\sinh^2(a) + 1 = \cosh^2(a)$.

Substitution: Let $x = c \sinh(a)$. ($\sinh(a)$ can be any real number, so x can be any real number).

Simplification:

$$x^2 + c^2 = (c \sinh(a))^2 + c^2$$

$$= c^2 \sinh^2(a) + c^2$$

$$= c^2 (\sinh^2(a) + 1)$$

$$= c^2 \cosh^2(a)$$

$$\text{So, } \sqrt{x^2 + c^2} = \sqrt{c^2 \cosh^2(a)} = |c \cosh(a)| = c \cosh(a) \text{ (since } c > 0 \text{ and } \cosh(a) > 0 \text{)}.$$

Example: Solve $\sqrt{x^2 + 16} = 5$

$$\text{Let } x = 4 \sinh(a). \text{ (Here } c = 4 \text{)}$$

$$\text{The equation becomes } \sqrt{(4 \sinh(a))^2 + 16} = 5$$

$$\sqrt{16 \sinh^2(a) + 16} = 5$$

$$\sqrt{16 (\sinh^2(a) + 1)} = 5$$

$$\sqrt{16 \cosh^2(a)} = 5$$

$$4 \cosh(a) = 5$$

$$\cosh(a) = 5/4$$

$a = \operatorname{arcosh}(5/4)$ (or $\cosh^{-1}(5/4)$). Note: there are two values of 'a' (positive and negative) that give the same cosh value, but they lead to opposite sinh values.

$$\text{Now, substitute back to find } x: x = 4 \sinh(a).$$

$$\text{Using } \cosh^2(a) - \sinh^2(a) = 1 \Rightarrow \sinh^2(a) = \cosh^2(a) - 1.$$

$$\sinh^2(a) = (5/4)^2 - 1 = 25/16 - 16/16 = 9/16.$$

$$\sinh(a) = \pm \sqrt{9/16} = \pm 3/4.$$

$$\text{Therefore, } x = 4 \cdot (\pm 3/4) = \pm 3.$$

$$\text{Check: } \sqrt{(\pm 3)^2 + 16} = \sqrt{9 + 16} = \sqrt{25} = 5. \text{ It works.}$$

Connecting Back to Hypotenuse:

The *reason* this substitution strategy works is the algebraic structure of the identity $\cosh^2(a) - \sinh^2(a) = 1$. This structure is the hyperbolic counterpart to the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$, which *is* fundamentally derived from the relationship between the sides (including the hypotenuse) of a right triangle inscribed in a unit circle. So, while you aren't directly using a hypotenuse *in* the hyperbolic calculation, you're using an analogous mathematical tool derived from the geometry of the hyperbola, just as trigonometric substitutions use a tool derived from the geometry of the circle (and thus, right triangles and hypotenuses).