Weiming Raymond Luo
September 21, 2016

CS 405: Algorithm Analysis II
Homework 1

Solutions.

1. For a fixed integer $k > 0$ and a fixed real $\epsilon > 0$, we can solve this using L'Hopital's Rule.

$$\lim_{n\to\infty} \frac{(\ln n)^k}{n^\epsilon} \to \frac{k(\ln n)^{k-1}\cdot\frac{1}{n}}{\epsilon\cdot n^{\epsilon-1}} \to \frac{k}{\epsilon}\cdot\frac{(\ln n)^{k-1}}{n^\epsilon} \to \frac{k}{\epsilon}\cdot\frac{(k-1)\cdot(\ln n)^{k-2}\cdot\frac{1}{n}}{\epsilon\cdot n^{\epsilon-1}} \to \frac{k(k-1)}{\epsilon^2}\cdot\frac{(\ln n)^{k-2}}{n^\epsilon} \to$$

$$\frac{k(k-1)(k-2)...(2)(1)}{\epsilon^{k-1}}\cdot\frac{\ln n}{n^\epsilon} \to \frac{k!}{\epsilon^{k-1}}\cdot\frac{\ln n}{n^\epsilon} \to \frac{k!}{\epsilon^{k-1}}\cdot\frac{\frac{1}{n}}{\epsilon\cdot n^{\epsilon-1}} \to \frac{k!}{\epsilon^k}\cdot\frac{1}{n^\epsilon} \to 0$$

2. Given that $a = 2, b = 4, f(n) = n^2$ and $g(n) = n^{log_4(2)} = n^{\frac{1}{2}}$. Since,

   $\frac{f(n)}{g(n)} \to \frac{n^2}{n^{0.5+\epsilon}} \to \infty$

   We can determine that as $n \to \infty$ for any $0 < \epsilon < 1.5$. This shows $f(n) = \omega(n^{0.5+\epsilon})$ which means $f(n) = \Omega(n^{0.5+\epsilon})$. Case (c) of the Master Theorem may apply if we can satisfy the equation, $af(n/b) \le cf(n)$ for some constant c that is $0 < c < 1$.

   We can show this from the following,

   $2(\frac{n}{4})^2 \to \frac{2n^2}{16} \to \frac{n^2}{8} \to \frac{1}{8}f(n) = cf(n)$

   Thus, $c = \frac{1}{8}$ satisfies the condition and we can conclude that $T(n) = \Theta(f(n)) = \Theta(n^2)$.

3. Given that $a = 4, b = 2, f(n) = n\lg n$ and $g(n) = n^{log_2(4)} = n^2$. Since,

   $\frac{f(n)}{g(n)} \to \frac{n\lg n}{n^{2-\epsilon}} \to 0$

   This shows that as $n \to 0$ for any $0 < \epsilon < 1$, then $f(n) = o(2-\epsilon)$ which means $f(n) = O(2-\epsilon)$. Case (a) of the Master Theorem can be applied, therefore $T(n) = \Theta(n^{log_2(4)}) = \Theta(n^2)$

4. Given that $a = 27, b = 3, f(n) = 54n^3$ and $g(n) = n^{log_3(27)} = n^3$. Since,

   $\frac{f(n)}{g(n)} \to \frac{54n^3}{n^3}$

   This implies that $f(n^3) = \Theta(n^{log_3(27)}) \to f(n^3) = \Theta(n^3)$. Therefore we can use Case (b) of the Master Theorem, concluding that $T(n) = \Theta(n^{log_3(27)}\cdot\lg n) = \Theta(n^3\lg n)$.

5. Given that $a = 2, b = 2, f(n) = \frac{n}{\lg n}$ and $g(n) = n^{log_2(2)} = n$. Since,

   $\frac{f(n)}{g(n)} \to \frac{\frac{n}{\lg n}}{n} \to \frac{1}{\lg n}$

   We determine that the Master Theorem can not be applied.

6. I. If case (a) is true, this means for some $\epsilon > 0$ then $f(n) = O(n^{log_b(a)-\epsilon})$. Suppose for some constant $A > 0$ and for some large n.

$\frac{f(n)}{g(n)} \leq \frac{An^{log_b(a)-\epsilon}}{n^{log_b(a)}} \rightarrow \frac{A \cdot \frac{n^{log_b(a)}}{n^\epsilon}}{n^{log_b(a)}} \rightarrow \frac{A}{n^\epsilon} \rightarrow 0$

This proves that for some large n, $f(n) \notin \Omega(n^{log_b(a)})$ and $f(n) \notin \Theta(n^{log_b(a)})$. Thus case (b) would fail. Similarly,

$\frac{f(n)}{g(n)} \leq \frac{An^{log_b(a)-\epsilon}}{n^{log_b(a)+\epsilon}} \rightarrow \frac{A \cdot \frac{n^{log_b(a)}}{n^\epsilon}}{n^{log_b(a)} \cdot n^\epsilon} \rightarrow \frac{A}{n^{2\epsilon}} \rightarrow 0$

This proves that for some large n and for some $\epsilon > 0$, $f(n) \notin \Omega(n^{log_b(a)+\epsilon})$. Thus case (c) would fail.

II. If case (b) is true, for some large n and $K_1 \leq \frac{f(n)}{g(n)} \leq K_2 \rightarrow K_1 g(n) \leq f(n) \leq K_2 g(n)$ then $f(n) = \Theta(n^{log_b(a)})$. Suppose for some large n.

$\frac{f(n)}{g(n)} \geq \frac{K_1 n^{log_b(a)}}{n^{log_b(a)-\epsilon}} \rightarrow \frac{K_1 n^{log_b(a)}}{\frac{n^{log_b(a)}}{n^\epsilon}} \rightarrow K_1 n^\epsilon \rightarrow \infty$

For any $\epsilon > 0$ and for some large n then $f(n) \notin O(n^{log_b(a)-\epsilon})$ for all $\epsilon > 0$. Thus case (a) would fail. Similarly,

$\frac{f(n)}{g(n)} \leq \frac{K_2 n^{log_b(a)}}{n^{log_b(a)+\epsilon}} \rightarrow \frac{K_2 n^{log_b(a)}}{n^{log_b(a)} \cdot n^\epsilon} \rightarrow \frac{K_2}{n^\epsilon} \rightarrow 0$

This shows that for any $\epsilon > 0$ and for some large n then $f(n) \notin \Omega(n^{log_b(a)+\epsilon})$ for all $\epsilon > 0$. Therefore, case (c) would fail.

III. If case (c) is true, for some large n then $f(n) = \Omega(n^{log_b(a)+\epsilon})$. Suppose for some constant $A > 0$.

$\frac{f(n)}{g(n)} \geq \frac{An^{log_b(a)+\epsilon}}{n^{log_b(a)-\epsilon}} \rightarrow \frac{A \cdot n^{log_b(a)} \cdot n^\epsilon}{\frac{n^{log_b(a)}}{n^\epsilon}} \rightarrow An^{2\epsilon} \rightarrow \infty$

For some $\epsilon > 0$ and for some large n, $f(n) \notin O(n^{log_b(a)-\epsilon})$. Thus case (a) will fail. Similarly,

$\frac{f(n)}{g(n)} \geq \frac{An^{log_b(a)+\epsilon}}{n^{log_b(a)}} \rightarrow \frac{A \cdot n^{log_b(a)} \cdot n^\epsilon}{n^{log_b(a)}} \rightarrow An^\epsilon \rightarrow \infty$

This shows that for some large n, $f(n) \notin O(n^{log_b(a)})$ and $f(n) \notin \Theta(n^{log_b(a)})$. Therefore, case (b) would fail.