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The Asymptotic Power of Runs Tests

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ABSTRACT. The number of runs of given length in a sequence of independent Bernoulli random variables with common probability of success is asymptotically normally distributed (Mood, 1940). It is shown that asymptotic normality also holds when the probabilities of success are not equal but vary in the course of the series. This leads to the asymptotic power of runs tests. It is shown that tests incorporating the lengths of different runs are better than the usual runs tests.

Key words: runs test, asymptotic normality, power

1. Introduction

Consider a sequence of N trials and let V_i be 1 or 0 as the ith trial is a success or a failure. A run of zeros of length k is a set of k successive zeros $V_i, V_{i+1}, \ldots, V_{i+k}$, followed and preceded by a 1. By the hypothesis of randomness we mean that the variables $V_1, \ldots, V_N$ are independent and identically distributed. When there is positive dependence between successive V's there will be less runs compared with the case of independence. Likewise the expected number of runs will decrease, when the probabilities of success $P(V_i=1)$ change in some smooth manner.

Runs tests may also arise as distribution-free tests for randomness of a series of continuously distributed random variables $Y_1, Y_2, \ldots, Y_N$. In this case the V_i are indicators for the signs of the Y_i (see Bradley, 1968).

Usually, runs tests are based on the length of the longest run or on the number of all runs. When just counting the number of runs and thus ignoring their lengths, information is lost. Mood (1940) showed that under the hypothesis of randomness the number of runs of given length is asymptotically normally distributed and derived a test statistic based on the number of runs of different lengths, which is asymptotically distributed as a χ^2 variable. In the next section we show that asymptotic normality also holds when there is some systematic pattern in the probabilities $P(V_i=1)$, which allows to study the power for this class of alternatives. Tests incorporating the length of the runs are more powerful than the usual runs test based on the number of all runs. In section 3, different runs tests are compared for some examples and the validity of asymptotic power and level is investigated by simulations.

2. Asymptotic distribution

Let $V_1, \ldots, V_N$ be a sequence of 0–1 random variables indicating success or failure. The V_i are thought to arise in some natural order, e.g. when i indicates time or location. We wish to test the hypothesis of randomness

$$H_0\colon V_1, \ldots, V_N \quad \text{i.i.d.} \quad P(V_i=1)=P=1-Q, \qquad i=1, \ldots, N, \tag{1}$$

against alternatives

$$H_1\colon V_1, \ldots, V_N \text{ independent}$$

$$P(V_i=1)=P+p_i, \quad -P\leqslant p_i \leqslant 1-P, \qquad i=1, \ldots, N. \tag{2}$$

First we will assume that P is known. Let $N_1=\Sigma V_i$ be the number of successes and $N_0=N-N_1$ the number of failures. Let $k\geqslant 1$ be fixed. For $1\leqslant l<k$ let R_{l1} be the number of success runs of

length l and R_{k1}^{+} the number of success runs of length greater or equal k. Likewise R_{l0} and R_{k0}^{+} are defined for failure runs. Let $R_l = R_{l1} + R_{l0}$ and $R_k^{+} = R_{k1}^{+} + R_{k0}^{+}$. Furthermore, the following notation is used:

$$\begin{aligned}
W_0 &= (N_1 - NP)/\sqrt{N} \\
W_{l1} &= (R_{l1} - NQ^2P^l)/\sqrt{N}, \qquad l<k \\
W_{l0} &= (R_{l0} - NP^2Q^l)/\sqrt{N}, \qquad l<k \\
W_{k1}^{+} &= (R_{k1}^{+} - NQP^k)/\sqrt{N} \\
W_{k0}^{+} &= (R_{k0}^{+} - NPQ^k)/\sqrt{N} \\
W_l &= W_{l1} + W_{l0}, \qquad l<k \\
W_k &= W_{k1}^{+} + W_{k0}^{+}.
\end{aligned} \tag{3}$$

Mood (1940) proved that under H_0 the runs statistics $W_1, \ldots, W_{(k-1)}, W_k^{+}$ are multivariate normally distributed with mean 0 and covariance matrix Σ_k and showed that the statistic

$$T_k = (W_1, \ldots, W_{(k-1)}, W_k^{+})\Sigma_k^{-1}(W_1, \ldots, W_{(k-1)}, W_k^{+})'$$

is asymptotically distributed as a χ^2 variable with k degrees of freedom. To study the distribution of the runs statistics under the alternatives (2), the probabilities $P + p_i$ are assumed to change in a smooth manner. For meaningful asymptotic theory the p_i must tend to zero.

Theorem 1

Let $V_1, V_2, \ldots, V_N$ be independent 0–1 random variables with

$$P(V_i = 1) = P + p_i, \qquad i = 1, \ldots, N,$$

where the $p_i = p_i(N)$ fulfil the assumptions:

$$\begin{aligned}
&A1\colon\ p_i = o(1) \qquad \text{uniformly in } i \\
&A2\colon\ p_i - p_{i+1} = O(N^{-1/2}) \qquad \text{uniformly in } i \\
&A3\colon\ \Sigma p_i/\sqrt{N} \to \delta, \qquad -\infty < \delta < \infty \\
&A4\colon\ \Sigma p_i^2/\sqrt{N} \to \kappa, \qquad 0 \leqslant \kappa < \infty.
\end{aligned} \tag{4}$$

Then the variables $W_0, W_{11}, W_{21}, \ldots, W_{(k-1)1}, W_{k1}^{+}, W_{10}, \ldots, W_{(k-1)0}, W_{k0}^{+}$ are asymptotically normally distributed with means

$$\begin{aligned}
\mu(W_0) &= \delta \\
\mu(W_{l1}) &= \delta P^{l-1}(lQ^2 - 2PQ) + \kappa P^{l-2}\{P^2 + l(l-1)Q^2/2 - 2lPQ\} \qquad l<k \\
\mu(W_{k1}^{+}) &= \delta P^{l-1}(lQ - P) + \kappa P^{k-2}\{k(k-1)Q/2 - kP\} \\
\mu(W_{l0}) &= \delta Q^{l-1}(2PQ - lP^2) + \kappa Q^{l-2}\{Q^2 + l(l-1)P^2/2 - 2lPQ\}, \qquad l<k \\
\mu(W_{k0}^{+}) &= \delta Q^{l-1}(Q - lP) + \kappa Q^{l-2}\{k(k-1)P/2 - kQ\}
\end{aligned} \tag{5}$$

and variances and covariances

$$
\begin{aligned}
\sigma(W_0,\,W_0)&=PQ\\
\sigma(W_{l1},\,W_{l1})&=P^lQ^2+P^{2l}[-(2l+1)Q^4+2Q^3P] \qquad l<k\\
\sigma(W_{k1}^+,\,W_{k1}^+)&=P^kQ-P^{2k}Q(2k+1)\\
\sigma(W_{l0},\,W_{l0})&=Q^lP^2+Q^{2l}[-(2l+1)P^4+2P^3Q] \qquad l<k\\
\sigma(W_{k0}^+,\,W_{k0}^+)&=PQ^k-Q^{2k}P(2k+1)\\
\sigma(W_0,\,W_{l1})&=P^l[lQ^3-2PQ^2] \qquad l<k\\
\sigma(W_0,\,W_{k1}^+)&=P^k[kQ^2-PQ]\\
\sigma(W_0,\,W_{l0})&=Q^l[-lP^3+2QP^2] \qquad l<k\\
\sigma(W_0,\,W_{k0}^+)&=Q^k[-kP^2+PQ] \qquad\qquad (6)\\
\sigma(W_{l1},\,W_{m1})&=P^{l+m}[-(l+m+1)Q^4+2PQ^3] \qquad l,\,m<k\\
\sigma(W_{l1},\,W_{k1}^+)&=P^{l+k}[-(l+k+1)Q^3+PQ^2] \qquad l<k\\
\sigma(W_{l1},\,W_{m0})&=P^lQ^m[-(l+m+3)P^2Q^2+2PQ] \qquad l,\,m<k\\
\sigma(W_{l1},\,W_{k0}^+)&=P^lQ^k[-(l+k+1)Q^2P+P^2Q+Q] \qquad l<k\\
\sigma(W_{k1}^+,\,W_{l0})&=P^kQ^l[-(l+k+1)P^2Q+Q^2P+P] \qquad l<k\\
\sigma(W_{k1}^+,\,W_{k0}^+)&=P^kQ^k[-(2k+1)PQ+1]\\
\sigma(W_{l0},\,W_{m0})&=Q^{l+m}[-(l+m+1)P^4+2QP^3] \qquad l,\,m<k\\
\sigma(W_{l0},\,W_{k0}^+)&=Q^{l+k}[-(l+k+1)P^3+P^2Q].
\end{aligned}
$$

Proof. Let

$$
X_{l1i}=\begin{cases}(1-V_{i-1})V_iV_{i+1}\cdots V_{i+l-1}(1-V_{i+l}), & i=2,\dots,N-1\\ V_i\cdots V_{i+l-1}(1-V_{i+l}), & i=1\\ (1-V_{i-1})V_i\cdots V_{i+l-1}, & i=N-l+1.\end{cases}
$$

Then $X_{l1i}=1$ if and only if V_i is the first member in a run of ones of length l, otherwise $X_{l1i}=0$. Therefore

$$R_{l1}=\sum_{1}^{N-l+1}X_{l1i}.$$

The first and last term of the sum may be neglected. Using assumptions A1 and A2 one gets for $2\leqslant i\leqslant N-l$

$$
\begin{aligned}
EX_{l1i}&=(Q-p_{i-1})(P+p_i)(P+p_{i+1})\cdots(P+p_{i+l-1})(Q-p_{i+l})\\
&=Q^2P^l+P^{l-1}Q^2(p_i+p_{i+1}+\cdots+p_{i+l-1})-(p_{i-1}+p_{i+l})QP^l\\
&\quad+(P^l+Q^2P^{l-2}l(l-1)/2-2lQP^{l-1})p_i^2+o(N^{-1/2}).
\end{aligned}
$$

Assumptions A3 and A4 lead to

$$
\begin{aligned}
EW_{l1}&=N^{-1/2}(E\Sigma X_{l1i}-Q^2P^lN)=\delta(lQ^2P^{l-1}-2QP^l)\\
&\quad+\kappa[P^l+Q^2P^{l-2}l(l-1)/2-2lQP^{l-1}]+o(1).
\end{aligned}
$$

In the same way, R_{l0}, R_{k1}^+ and R_{k0}^+ can be written as sums of indicator variables X_{l0i}, X_{k1i}^+, X_{k0i}^+ leading to (5). Furthermore, the expressions (6) for the asymptotic variances and covariances follow by straightforward calculations from assumptions A1–A4. Since the R's are sums of k-dependent bounded random variables, asymptotic multivariate normality follows (Orey, 1958).

For $\delta=\kappa=0$, i.e. $P(V_i=1)=P$ for all i, theorem 1 yields the asymptotic distribution of the number of runs under the null hypothesis. This is theorem 1 of section 8 in Mood (1940). However, the expressions (8.2), given in that paper, for the asymptotic covariances between the number of runs and the number of successes N_1 are wrong.

The variances and covariances (6) are the same under the alternative as under the hypothesis. The means do not depend on the particular pattern of the p_i, but only on δ and κ. Essentially there are only two interesting cases, either $\kappa=0$ or $\delta=0$. If the positive deviations p_i dominate the negative ones, the p_i being of order $N^{-1/2}$, then $\Sigma p_i/\sqrt{N}\to\delta\neq 0$ and $\kappa=0$. Often however, e.g. for goodness of fit in regression, the probabilities $P(V_i=1)$ differ from the basic probability P in positive and negative manner in such a way that $\Sigma p_i=0$. For such situations, we assume $\Sigma p_i^2/\sqrt{N}\to\kappa>0$. Since we are mainly interested in the latter case, we will assume henceforth $\delta=0$ and $\kappa>0$. The case $\kappa=0$, $\delta\neq 0$ may be treated in an analogous way. Furthermore, we will not distinguish between success and failure runs, i.e. we will concentrate on the statistics $W_l=W_{l0}+W_{l1}$. From theorem 1 the asymptotic normality of the statistics $W_0, W_1, \ldots, W_{k-1}$, W_k^+ follows. The asymptotic means are given by

$$\begin{aligned}\mu(W_l)=\kappa\mu_l=\kappa[&P^{l-2}\{P^2+l(l-1)Q^2/2-2lPQ\}\\&+Q^{l-2}\{Q^2+l(l-1)P^2/2-2lPQ\}],\qquad l<k\\\mu(W_k^+)=\kappa\mu_k^+=\kappa[&P^{k-2}\{Q(k-1)k/2-kQ\}+Q^{k-2}\{P(k-1)k/2-kP\}].\end{aligned}\tag{7}$$

The asymptotic variances and covariances can be obtained from (6). Let Σ_k be the asymptotic covariance matrix of $(W_1, \ldots, W_{k-1}, W_k^+)$ and let

$$T_k=(W_1, \ldots, W_{k-1}, W_k^+)\Sigma_k^{-1}(W_1, \ldots, W_{k-1}, W_k^+)'.\tag{8}$$

Corollary 1

Let the assumptions of theorem 1 hold with $\delta=0$. Then T_k is asymptotically distributed as a χ^2 variable with k degrees of freedom and non-centrality parameter $\kappa^2\lambda_\kappa$, where

$$\lambda_k=(\mu_1, \ldots, \mu_{k-1}, \mu_k^+)\Sigma_k^{-1}(\mu_1, \ldots, \mu_{k-1}, \mu_k^+)'.\tag{9}$$

The asymptotic distribution of the W's depends only on the parameter κ. As may be seen from (7), the influence of κ on the means is multiplicative. For testing $\kappa=0$ against $\kappa>0$, the asymptotically optimal test based on $W_1, \ldots, W_{k-1}, W_k^+$ therefore rejects for large values of

$$S_k=(\mu_1, \ldots, \mu_{k-1}, \mu_k^+)\Sigma_k^{-1}(W_1, \ldots, W_{k-1}, W_k^+)'/\sqrt{\lambda_k}.\tag{10}$$

S_k is therefore always at least as good as T_k. For $k=1$ the test T_1 and the two-sided test based on S_1 are equivalent. This is the usual runs test which depends only on R_1^+, the number of all runs. The asymptotic power of the tests S_k increases in k. This does not hold for T_k (see section 3). Of course, S_k can only be recommended for $\delta=0$, i.e. when the positive and negative deviations p_i cancel. From theorem 1, one gets

Corollary 2

Under the assumptions of theorem 1 with $\delta=0$, S_k is normally distributed with mean $\kappa\sqrt{\lambda_k}$ and variance 1.

When the probability P of success is unknown, one might replace P and Q by the estimates $\hat{P}=N_1/N$ and $\hat{Q}=N_0/N$ and base tests on the variables

$$U_l=(R_l-N\hat{P}^l\hat{Q}^2-N\hat{Q}^l\hat{P}^2)/\sqrt{N}, \qquad l<k$$

$$U_k^+=(R_k^+-N\hat{P}^k\hat{Q}-N\hat{Q}^kP)/\sqrt{N}, \tag{11}$$

Since $\hat{P}^j=P^j+jP^{j-1}(N_1/N-P)+o_p(N^{-1/2})$, one has:

Theorem 2

Let the assumptions of theorem 1 hold with $\delta=0$. Then $U_1,\ldots,U_{k-1}$, U_k^+ are asymptotically normally distributed with means $\kappa\mu_1,\ldots,\kappa\mu_{k-1},\kappa\mu_k^+$.

Test statistics based on the U's may be formed analogously to T_k (8) and S_k (10). Since $\hat{P}\to P$ in probability P may be replaced by $\hat{P}$ in the asymptotic covariance matrix and in the asymptotic means μ_l, μ_k^+, such that the test statistics do not depend on the unknown parameter P. Mood (1940) derived the asymptotic conditional distribution of the U's given the number N_1 of successes under the null hypothesis.

Usually runs tests are based on R_1^+, the number of all runs. As was shown above, better tests for testing randomness against trend alternatives are obtained when taking into account the lengths of the runs. This may lead to a considerable increase in power (see section 3). However, this does in general not hold for testing randomness against dependency. For a stationary Markov chain $V_1,\ldots,V_N$ the number of successes N_1 together with the number of success runs R_{11}^+ and the number of failure runs R_{10}^+ form a sufficient set of statistics (Goodman, 1958; Lehmann, 1959). In this case, no information is lost when ignoring the lengths of the runs. For higher-order Markov chains, however, $\{N_1, R_{11}^+, R_{10}^+\}$ is not sufficient and counting the number of runs of different lengths leads to better tests than the ordinary runs tests. Goodman (1958) discussed some tests for this situation. David (1947) and Barton & David (1958) derived the power function of the usual runs test for first-order Markov chains. Bateman (1948) also considers the power for second-order Markov chains.

Let us now assume a situation where

$$Y_i=g(t_i)+\varepsilon_i,\ t_i\in[0,1], \qquad i=1,\ldots,N, \text{ where } \varepsilon_1,\ldots,\varepsilon_N \text{ i.i.d.} \sim F.$$

Let F be symmetric about zero with smooth density f and let

$$V_i=\begin{Bmatrix}1 & Y_i>0\\ 0 & \text{otherwise}\end{Bmatrix}.$$

For g "small" one has

$$P[V_i=1]=1-F\{-g(t_i)\}\cong 1/2+f(0)g(t_i)$$

For g smooth, $\Sigma g(t_i)\approx 0$ and $N^{-1/2}\Sigma g(t_i)^2\to\theta$, the results of theorem 1 may be applied with $P=0.5$ and $\kappa=f(0)\theta$. The power of the distribution-free runs tests depends on g only through the parameter θ and on the distribution F only through $f(0)$.

When applying runs tests for goodness-of-fit in regression the V_i denote the signs of the residuals. Our results only hold when the regression function is known. When the regression function is estimated, the correlation of the residuals changes the variances of the runs statistics. Somewhat different tests based on runs in the situation above are the so-called runs-up tests. These tests are formed from runs of the signs of the differences $Y_{i+1}-Y_i$. Since the signs are not independent, the distribution of such tests are different. See Levene (1952) for more details.

Table 1. *Asymptotic power (in %) of runs tests*

α	P	κ	S_1	S_2	S_3	S_4	S_5	T_1	T_2	T_3	T_4	T_5
0.05	0.5	0.733	90.1	99.4	99.9	100	100	83.4	96.8	98.3	98.5	98.4
		0.580	75.0	94.9	97.8	98.6	98.9	65.1	84.4	88.3	88.6	87.8
		0.475	60.1	85.1	91.3	93.4	94.3	47.6	67.0	71.2	71.2	69.6
	0.25	0.733	82.0	89.3	98.0	99.9	100	72.5	73.6	88.8	97.5	99.3
		0.580	64.8	73.9	90.0	97.9	99.5	52.6	52.2	68.7	85.3	92.1
		0.475	50.5	58.9	77.4	91.4	96.4	38.2	36.9	49.8	66.8	76.1
0.10	0.5	0.733	95.1	99.8	100	100	100	90.1	98.5	99.2	99.3	99.3
		0.580	85.0	97.7	99.2	99.5	99.6	75.0	90.8	93.4	93.6	92.1
		0.475	73.2	92.0	95.8	96.9	97.4	60.1	77.5	81.0	81.0	79.7
	0.25	0.733	89.9	94.6	99.2	100	100	82.0	82.8	93.7	98.9	99.7
		0.580	77.1	84.2	95.0	99.2	99.8	64.8	64.6	78.9	91.4	95.8
		0.475	64.7	72.2	86.7	95.8	98.4	50.6	49.5	62.4	77.4	84.8
0.01	0.5	0.733	72.8	96.6	99.0	99.4	99.6	63.9	89.5	93.6	94.3	93.9
		0.580	50.0	83.0	91.0	93.6	94.6	39.8	65.8	72.3	72.9	71.4
		0.475	33.5	64.1	75.1	79.5	81.5	24.9	43.2	48.3	48.2	46.3
	0.25	0.733	59.2	71.3	91.5	99.0	99.9	49.3	50.9	73.2	91.5	96.9
		0.580	38.1	48.3	72.5	91.2	96.9	29.1	28.8	45.4	67.4	79.2
		0.475	25.2	32.4	52.8	75.4	86.7	17.9	17.1	27.0	43.2	54.2

3. Numerical results and simulations

The asymptotic power of the test statistics T_k (8) and S_k (10), $1 \leqslant k \leqslant 5$, was computed according to corollaries 1 and 2 for different values of α ($\alpha = 10\%$, 5%, 1%), P ($P = 0.5$, 0.25) and κ ($\kappa = 0.733$, 0.580, 0.475). The values for κ were chosen such that the one-sided test based on the number of all runs, i.e. S_1, has asymptotic power 90%, 75% and 60% for $P = 0.5$ and $\alpha = 5\%$. Table 1 shows the results. Because of the optimality of the tests S_k, the test S_5 has greatest power for all combinations of P and κ. The strong increase of power from S_1 to S_5 is however surprising (e.g. the power increases from 25.2% to 86.7% for $\alpha = 0.01$, $P = 0.25$ and $\kappa = 0..475$). The tests S_k are better than the T_k, especially for small κ. No monotone relationship holds for the power of the T_k, $1 \leqslant k \leqslant 5$, but for all examples T_5 is better than T_1. From Table 1 one may conclude that a considerable gain in power can be achieved when incorporating the lengths of the runs. In a simulation study, Steinberg & Gasser (1984) arrived at the same conclusion for the conditional tests given the number of successes.

To investigate the appropriateness of the asymptotic approximations for level and power of the tests T_k and S_k simulations were performed for sample sizes $N = 50$, 100, 500. For the alternative, we chose the following pattern for the probabilities of success:

$$P(V_i = 1) = P + \sqrt{2\kappa}\sin(2\pi i/N)N^{-1/4}, \qquad i = 1, \ldots, N,$$

considering $\kappa = 0$, 0.733 and 0.475 and $P = 0.5$. The tests were performed at the 5% level. For each situation N uniformly distributed random numbers were generated (subroutine GGUBS from IMSL) and 0–1 variables were formed with the above probabilities of success. Then either the tests $T_1, \ldots, T_5$ or the tests $S_1, \ldots, S_5$ were applied simultaneously to this pseudo-random sample with the asymptotic rejection regions given by corollary 1 or 2. The simulated values for power and level in Table 2 are based on 1000 replications. This gives $s/\alpha \leqslant 0.14$ and $s/(1-\beta) \leqslant 0.14$ for $\beta \leqslant 0.95$, where s is the standard error of the simulated level α or power β. From Table 2 one may conclude that the asymptotic approximations for the level and the power work quite well for $N \geqslant 50$, but the rate of convergence is not very fast.

Table 2. *Simulated and asymptotic level and power (in %) of runs tests ($P = 0.5$, $\alpha = 5\%$)*

κ	N	S_1	S_2	S_3	S_4	S_5	T_1	T_2	T_3	T_4	T_5
0	asym	5	5	5	5	5	5	5	5	5	5
	50	3.9	4.8	6.0	6.1	6.5	6.9	4.7	4.9	4.5	4.2
	100	3.8	4.3	4.0	4.1	5.1	6.3	5.3	6.5	5.3	5.1
	500	2.8	4.0	3.3	4.0	4.1	5.1	4.9	4.4	4.6	4.0
0.475	asym	60.1	85.1	91.3	93.4	94.3	47.6	67.0	71.2	71.2	69.6
	50	57.5	75.1	80.8	82.5	83.1	47.1	54.5	61.0	61.6	64.4
	100	53.3	75.3	78.9	80.9	81.9	46.6	56.8	61.9	63.4	64.7
	500	55.5	78.2	85.6	87.9	88.7	48.4	63.4	68.3	69.4	70.2
0.733	asym	90.1	99.4	99.9	99.9	100	83.4	96.8	98.3	98.5	98.4
	50	90.1	95.9	98.2	98.3	98.4	79.8	88.3	91.0	92.4	93.4
	100	82.7	95.5	97.0	97.4	97.5	78.8	88.4	91.3	91.1	91.8
	500	88.7	97.9	99.1	99.2	99.3	81.8	93.2	94.4	95.0	94.9

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