A QUICK TEST FOR SERIAL CORRELATION SUITABLE FOR USE WITH NON-STATIONARY TIME SERIES

C. W. J. GRANGER
University of Nottingham

Goodman's simplified runs test is examined as a test for serial correlation and is found to be both much quicker to apply than other available tests and to be appropriate for use with non-stationary data provided that the series has no trend in mean. Approximations to the power of the test are discussed and it is found that the power is acceptable for samples of size 100 or more. Particular consideration is given to the application of the test to economic data.

1. INTRODUCTION

TESTING for serial correlation is of importance in various aspects of time series analysis and is particularly important when econometric models, for instance, are being constructed. It frequently occurs that the data involved are both large in quantity and of dubious stationarity. In this note Goodman's [7] simplified runs test is discussed and it is suggested that it has desirable properties of quickness and ease of use, and of robustness against an assumption of stationarity.

David [3] proposed a test for serial correlation which, for our present purposes, can be stated as follows: Given a sample x_t, $t=1, 2, \cdots, n$ from a stationary stochastic process, transform the sample into a sequence of plus and minus signs by replacing x_t by $+$ if $x_t - M \geq 0$ and by $-$ if $x_t - \mathrm{M} < 0$, M being the sample median. A run is defined as a complete sequence, possibly of length one, consisting only of one symbol, the sequence being bordered at each end by the other symbol. We test the null hypothesis

$$H_0 \equiv E[(x_r - m)(x_{r-j} - m)] = 0 \qquad \text{for all } t \text{ and } j \neq 0.$$

where:

$$m = E[x_r]$$

by noting the observed total number of runs of either symbol (T), forming the statistic $S = (2T/n) - 1$ and then enquiring if the observed S appears to have come from a normal distribution with mean zero and variance $1/n$.

Although this test is extremely simple, it does not qualify as a quick test as the sample median M cannot quickly be found for a large sample. If one is prepared to assume the distribution of the stochastic process to be symmetric, the sample median M can be replaced by the sample mean $\bar{x}$, making the test a reasonably quick one to apply. The David runs test with the sample median replaced by the sample mean will be called the symmetric runs test.

For this test, David showed that if $n = 2r$, r an integer

$$\text{Prob. } [T = 2k] = \frac{2[{}_{r-1}C_{k-1}]^2}{{}_nC_r}$$

and:

$$\text{Prob. } [T = 2k + 1] = \frac{\text{Prob. } [T = 2k]}{k(k - r + 1)}$$

where:

$${}_nC_r = \frac{n!}{r!(n - r)!}$$

and further:

$$E[T] = r + 1$$

$$D^2[T] = \frac{r(r - 1)}{2r - 1} \cdot$$

The distribution of T becomes approximately normal for n large and so:

$$\frac{T}{n} \sim N\left(\frac{n + 2}{2n}, \left(\frac{n - 2}{4n(n - 1)}\right)^{1/2}\right)$$

which, for n sufficiently large, can be well approximated by $N(\frac{1}{2}, 1/2\sqrt{n})$. The reason for the statistical behaviour of T in this case being so similar to that of a binomial variate where the sample size is n and the true probability parameter is $p = \frac{1}{2}$ can be easily illustrated.

If $y_t = x_t - \bar{x}$, let X be the number of times that $y_t < 0$ is followed by $y_{t+1} \geq 0$, for $t = 1, 2, \cdots, n-1$. If the number of times that $y_t < 0$ is a then the large sample distribution of X, under the null hypothesis, is similar to that of a binomial variate where the sample size is a and the probability parameter is $p = \frac{1}{2}$. Let Y be the number of times that $y_t \geq 0$ is followed by $y_{t+1} < 0$. If the number of times that $y_t \geq 0$ is b then the large sample distribution of Y is similar, under the null hypothesis to that of a binomial variate where the sample size is b and the parameter is $p = \frac{1}{2}$. Goodman [7] has shown that X and Y are asymptotically independent and thus the large sample distribution of $X + Y$ which, for large n, is virtually indistinguishable from T is similar to that of a binomial variate with sample size $a + b = n$ and probability parameter $p = \frac{1}{2}$.

The assumption of symmetry, although common in econometrics, is not generally acceptable. However, Goodman [7] has suggested an alternative test, called by him the simplified runs test, which does not require such an assumption.

Given the sample x_t, $t = 1, 2, \cdots, n$ from a stationary stochastic process and some constant k, form a new series $z_t = x_t - k$. Transforming z_t into a sequence of plus and minus signs corresponding to $z_t \geq 0$ and $z_t < 0$ respectively, let T be the number of runs, a the number of minus signs, and b the number of plus signs, so the $a + b = n$. If we denote $\tau = E[a]/E[b]$, which will depend upon the choice of k, Goodman has shown that, under the null hypothesis, the statistics $S(\tau) = (nT/2ab) - 1$ is asymptotically distributed normally with mean zero and variance $1/n$.

The simplified runs test has various advantages over the symmetric runs test (which corresponds to the case $\tau=1$) as not only is it based on less restrictive assumptions but will usually be easier to apply as k can be chosen to be any convenient constant. Thus, k could be the integer part of $\bar{x}$ or, for very large amounts of data containing no trend in mean k could be the sample mean of the first 100 pieces of data. The choice of k affects the power of the test but, as is shown in the next section, the selection of k is by no means critical provided that $\frac{1}{2}\leq\tau\leq 2$. Thus, provided both a and b are $\geq n/3$, any value of k may be used without detriment to the power of the test.

2. POWER OF THE TESTS

We now consider the power of the tests against the alternative hypothesis (H_A) that the series is a Markov process. Transforming the series $z_t=x_t-k$ into a sequence of plus and minus signs produces, under this alternative hypothesis, a Markov chain with two states, $E_1\equiv z_t<0$, $E_2\equiv z_t\geq 0$. Let the transition probability matrix be:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where:

$$p_{ij} = \text{Prob}\left[z_{t+1} \text{ in } E_j \mid z_t \text{ in } E_i\right]$$

and, assuming the process to be stationary, let

$$P_i = \text{Prob}\left[z_t \text{ in } E_i\right], \qquad i = 1, 2.$$

For the symmetric runs test, with $k=\bar{x}$, $P_1=P_2=\frac{1}{2}$, but we first consider the general case where $P_1/P_2=\tau$, $\tau\geq 0$. Denoting $p_{11}-p_{12}=\phi=p_{22}-p_{12}$, it follows from Goodman's results that, under H_A, the statistic $S(\tau)$ is asymptotically distributed normally with mean ϕ and variance given by:

$$D^2[S(\tau)] = \frac{(1-\phi)[\tau+\phi(1-\tau+\tau^2)]}{n\tau}. \tag{2.1}$$

For the symmetric case $\tau=1$, this becomes

$$D^2[S(\tau)] = \frac{(1-\phi^2)}{n} \tag{2.2}$$

ϕ is a measure of the dependence of z_{t+1} on z_t and τ is a measure of the symmetry of z_t about zero and is thus dependent upon the choice of k. Virtually all economic series are such that adjacent terms are positively correlated and thus the case $\phi\geq 0$ is of most interest. For given ϕ and τ we denote the corresponding alternative hypothesis by $H_A(\phi, \tau)$.

Using these results, the approximate power of the simplified runs test is easily determined. If, under H_0,

$$\text{Prob}\left[\,|S(\tau)| \geq \frac{k_\alpha}{\sqrt{n}}\right] = \alpha$$

then the power of the test when using the $100\alpha\leq$ per cent confidence level, considered as a two-tailed test, can be defined as:

$$M(\phi, \tau, n) = 1 - \text{Prob}\left[\left| S(\tau) \right| \leq \frac{k_\alpha}{\sqrt{n}} \mid H_A(\phi, \tau) \right] \tag{23}$$

i.e. the probability that, under $H_A(\phi, \tau)$ we get an observed value of $S(\tau)$ which would be considered acceptable under H_0.

The extent to which the power depends upon τ can be indicated by considering, for fixed ϕ and n, the ratio of the variance of $S(\tau)$ to that of $S(1)$, under the alternative hypothesis. From (2.1) and (2.2) it is seen that this ratio, which is asymptotically independent of n, is

$$k(\phi, \tau) = \frac{[\tau + \phi(1 - \tau + \tau^2)]}{\tau(1 + \phi)}$$
$$= \frac{1 + \phi(1 + \tau)^2}{\tau(1 + \phi)}.$$

For $\phi \geq 0$

$$1 \leq k(\phi, \tau) \leq \frac{1 + \tau^2}{2\tau}$$
$$k(\phi, \tau) = k(\phi, \tau^{-1})$$

and

$$k(\phi, \tau) \text{ increases as } \phi \text{ increases.}$$

As $1 \leq k(\phi, \tau) \leq 1.25$ when $\frac{1}{2} \leq \tau \leq 2$ it is seen that the power of the test is not likely to be critically affected by the choice of a constant k such that τ does lie between $\frac{1}{2}$ and 2. This result is further illustrated in Table 1, which shows the approximate power of the simplified test against ϕ and for various values of τ in the case $n=100$ and using 95 per cent confidence levels.

The table is based on the asymptotic distribution of $S(\tau)$ and on the assumption that the true distribution for $n=100$ does not greatly differ from this asymptotic distribution. The numbers are thus only the approximate powers of the test.

TABLE 1. APPROXIMATE POWER OF SIMPLIFIED RUNS TEST FOR VARIOUS τ. (WITH $n=100$, USING TWO-TAILED TEST AND 95% CONFIDENCE BAND)

τ \ ϕ	0	0.05	0.1	0.15	0.2	0.25	0.3	0.4	0.5
τ	.050	.079	.170	.319	.516	.712	.862	.987	.999
$\frac{2}{3}$ or 3/2	.050	.079	.171	.323	.516	.709	.858	.985	.999
$\frac{1}{2}$ or 2	.050	.081	.174	.326	.516	.709	.849	.981	.999
$\frac{1}{3}$ or 3	.050	.084	.185	.334	.515	.695	.841	.971	.998
$\frac{1}{4}$ or 4	.050	.092	.214	.341	.514	.679	.813	.959	.997

The curves are seen to be almost identical for τ in the range $\frac{1}{2}$ to 2.

The implication of this result is that, for the size of sample most likely to be used with a quick test (say $50 \leq n \leq 400$) we may discuss the power of the symmetric runs test ($\tau = 1$) and carry over any results thus obtained to the simplified runs test with τ in range the $\frac{1}{2}$ to 2.

Before further discussing the power of the test, it is desirable to show how the results can be expressed in terms of ρ, the first serial correlation coefficient, rather than in terms of ϕ. Although there is no general formula relating ρ and ϕ, a formula does exist when the data is normally distributed. Thus, if $\tau = 1$, $P_{11} = \frac{1}{2}(1+\phi)$ and, for normally distributed data, using the well-known result by Sheppard [9],

$$\text{Prob}\left[x_r - \bar{x} \geq 0, x_{r-1} - \bar{x} \geq 0\right] = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\rho\right)$$

it follows that

$$P_{11} = \frac{1}{2}\left(1 + \frac{2}{\pi}\sin^{-1}\rho\right)$$

and thus

$$\phi = \frac{2}{\pi}\sin^{-1}\rho. \tag{2.4}$$

Approximate power-curves for the symmetric runs test with various values of n for both 95 per cent and 99 per cent confidence bands are shown in Figures 1 and 2. If the data is normally distributed, the power of given ρ may read off the curves directly and the same curves given the power for the test with complete generality by using the ϕ-scale.

The curves shown are actually the approximate power curves for the statistic $S'(1) = (2T/n) - 1$ which, for $n \geq 25$, Barton and Davie [2] have indicated has a frequency function well approximated by the normal curve with mean

$$\frac{2}{n} + \frac{\delta(n-2)}{4(n+1)} \quad \text{and variance} \quad \frac{n-2}{4n(n-1)},$$

where,

$$\delta = \log\left(\frac{1-\phi}{1+\phi}\right).$$

These curves have been compared with the approximate power curves for $S(1)$ using the asymptotic distribution of this statistic given above, for both $n = 100$ and $n = 200$ and are found to be virtually indistinguishable.

Thus the curves are only the approximate power curves for $S(1)$ and the degree of approximation is unknown. However, they should be sufficiently accurate to at least indicate the size of n required before the power of the test becomes reasonable.

The indication is that the power is poor for $n \leq 50$ but becomes satisfactory for n large enough to warrant the use of a quick test. For $n = 100$, for instance,

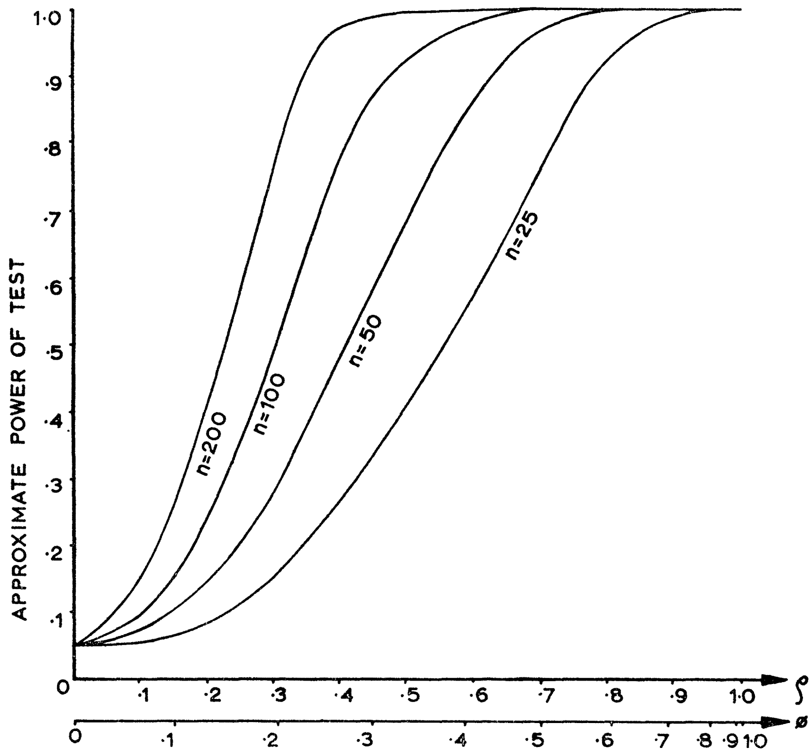

FIGURE 1. Approximate power curve for 5% level.

the approximate power function of the test at the 5 per cent level is .932 at $\rho=0.5$ and for n as large as 500 the power is 0.813 at $\rho=0.2$ and .993 at $\rho=0.3$.

Consideration of the power of the test against the alternative hypothesis of a Markov process should indicate the test's usefulness against more complicated stochastic processes. For economic time series, in particular, the first serial correlation coefficient is invariably the most important and frequently the spectra of economic time series suggest that a Markov process would provide a reasonably good approximation. (See references [1], [5], [6]). The simplified runs test is, of course, likely to be less powerful than the tests suggested by Durbin and Watson [4], [12], under the perfect conditions of stationarity and normality. The graphical test proposed by Quenouille [8] is closely related to the runs test but as it extracts slightly more information from the data it is probably a little more powerful.

3. POWER OF THE TEST WITH NON-STATIONARY DATA

Three broad classes of non-stationary series may be distinguished;

(i) the sum of a stationary series and a function of time giving a series with trend in mean only.

(ii) a series having a mean that does not change with time but with variance and possibly also autocovariances changing with time.

(iii) the sum of a series such as in (ii) and a function of time.

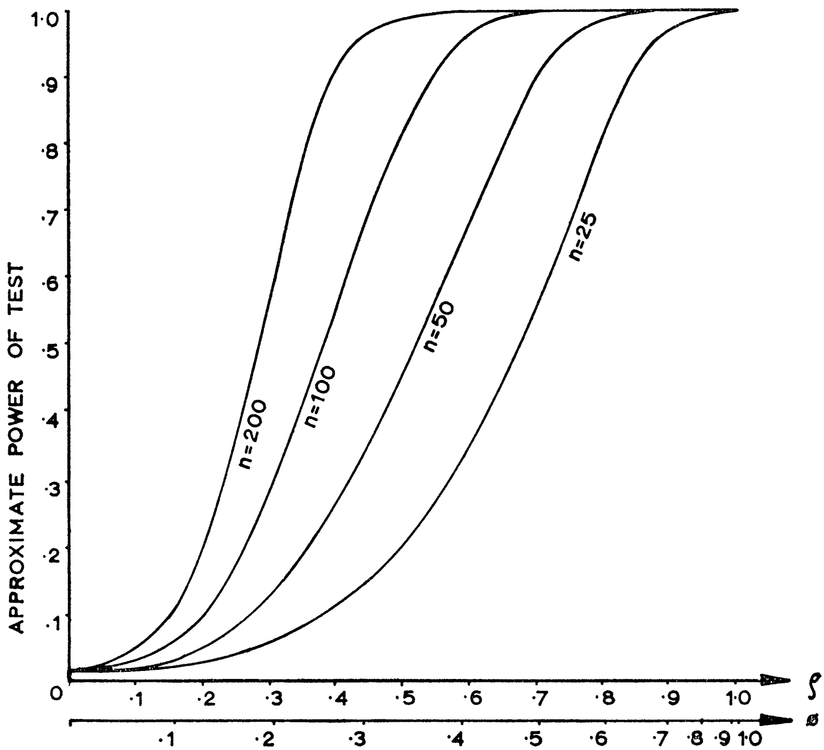

FIGURE 2. Approximate power curve for 1% level.

Clearly the effect of a series containing a trend in mean will be to induce serial correlation that may not be present had the mean remained constant. There are, however, efficient methods of removing such trends in mean when sufficient data is available (such as by filtering or by polynomial regression) which may be used not only with series in class (i) but also with those in class (iii) provided that the variance of the series is changing slowly with time (see Chapters 8 and 9 of reference [6]). If the trend in mean is not removed the runs test will reject H_0 simply because of the induced serial correlation and thus we suppose that the test will only be used on data from which such trends have been removed, giving a zero sample mean.

Of more practical importance in economics is the problem of how to deal with series of class (ii) in which the variance and (or) the serial correlations are changing with time. As a runs test effectively measures the "smoothness" of the data, such non-stationarity need not affect it to any great extent. Thus, if x_t, $t=1, \cdots, n$ is a sample from a stationary process, with $\bar{x}=0$ then the sample $a(t)x_t$, $t=1, \cdots, n$ where $a(t)$ is any positive function of time, will provide the same numbers a, b, and T, as the original sample. The power of the simplified runs test is thus entirely unaffected by a trend in variance (when $k=\bar{x}$), providing that there is no trend in mean. This cannot be said of the alternative tests available, such as those proposed by Durbin and Watson [4], [12], Theil and Nagar [10] and by von Neumann [11]. If $a(t)$ is montonically increasing with $a(n) = 4a(0)$, $n = 100$ say, all the alternative tests will lose considerable power as the latter terms will become increasingly important compared to the earlier terms, thus effectively shortening the amount of data available. This suggests that, when dealing with non-stationary data, with zero mean for all t, a runs test not only provides a quick test for serial correlation but may have power comparable to that of the alternative tests available.

If

$$E[x_t] = 0, \qquad E[x_t^2] = \sigma^2, \qquad E[x_t x_{t-1}] = \sigma^2 \rho(t)$$

i.e. the mean and variance are not functions of time but the first serial correlation is, then if $\rho(t) > 0$ all t, say, the data will still be "smoother" than a sample drawn from white noise and so the runs test will still be applicable, with power approximately the same as with

$$\bar{\rho} = \frac{1}{n} \sum_{t=1}^{n} \rho(t)$$

provided that $\rho(t)$ does not change too quickly with time. The other tests will also probably discover the serial correlation in such a model but as a more likely situation in practice is one in which σ^2 and ρ are functions of time, the runs tests again become at least equally suitable.

4. CONCLUSION

The suggested quick test for serial correlation based on Goodman's simplified runs test and for use with data, either stationary or non-stationary, either having no obvious trend in mean or having had such a trend removed is as follows:

Given data x_t, $t = 1, \cdots, n$, form a new series $z_t = x_t - k$, where k is some appropriate constant. Provided that the number of positive values of z_t lie between $n/3$ and $2n/3$, count the number of runs of positive and negative values (T), the number of positive values (a) and the number of negative values (b). Form the statistic

$$S = \frac{nT}{2ab} - 1$$

and, if the null hypothesis is true that there is no serial correlation, S will be distributed normally with mean zero and variance n^{-1}.

If the number of positive values of z_t does not lie between $n/3$ and $2n/3$, a new value of the constant k should be chosen so that this condition is obeyed. If the mean or the trend in mean has been removed from the series x_t, so that $E[x_t] = 0$, k can be taken to be zero provided the above condition is satisfied. Apart from this condition, the value of k is not critical with regard to the power of the test.

Being a quick test, it is not efficient and it is suggested that it is not used on series with less than a hundred terms.

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