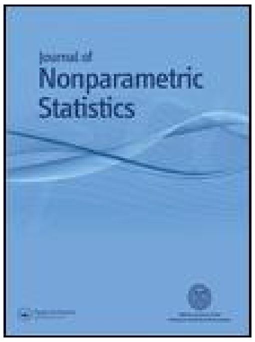

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GENERALIZED RUNS TESTS FOR HETEROSCEDASTIC TIME SERIES

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The problem of testing for nonhomogeneous white noise (i.e., independently but possibly nonidentically distributed observations, with a common, specified or unspecified, median) against alternatives of serial dependence is considered. This problem includes as a particular case the important problem of testing for heteroscedastic white noise. When the value of the common median is specified, invariance arguments suggest basing this test on a generalized version of classical *runs*: the *generalized runs* statistics. These statistics yield a run-based correlogram concept with exact (under the hypothesis of nonhomogeneous white noise) p-values. A run-based *portmanteau test* is also provided. The local powers and *asymptotic relative efficiencies* (AREs) of run-based correlograms and the corresponding run-based tests with respect to their traditional parametric counterparts (based on classical correlograms) are investigated and explicitly computed. In practice, however, the value of the exact median of the observations is seldom specified. For such situations, we propose two different solutions. The first solution is based on the classical idea of replacing the unknown median by its empirical counterpart, yielding *aligned* runs statistics. The asymptotic equivalence between *exact*

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and *aligned* runs statistics is established under extremely mild assumptions. These assumptions do not require that the empirical median consistently estimates the exact one, so that the continuity properties usually invoked in this context are totally helpless. The proofs we are giving are of a combinatorial nature, and related to the so-called *Banach match box problem*. The second solution is a finite-sample, nonasymptotic one, yielding (for fixed n) strictly conservative testing procedures, irrespectively of the underlying densities. Instead of the empirical median, a nonparametric confidence interval for the unknown median is considered. Run-based correlograms can be expected to play the same role in the statistical analysis of time series with nonhomogeneous innovation process as classical correlograms in the traditional context of second-order stationary *ARMA* series.

Keywords: Runs; time series; heteroscedasticity; test of randomness

AMS 1980 Subject Classification: 62G10; 62M10

1. INTRODUCTION

A huge literature exists on the problem of testing for *homogeneous* white noise (exchangeable or *i.i.d.* observations) against various alternatives—probably one of the most fundamental problems in statistical inference. Much less is known about testing for *nonhomogeneous* white noise (independently but possibly non identically distributed observations), a hypothesis of which *heteroscedastic* white noise constitutes an important particular case. This paper deals with the problem of testing for nonhomogeneous white noise against alternatives of serial dependence. More precisely, denoting by $H^{(n)}(\xi_{1/2})$ the hypothesis under which $X^{(n)} = (X_1^{(n)}, \dots, X_n^{(n)})$ is a n-tuple of independently distributed observations, with unspecified, possibly distinct, densities, f_t, $t = 1, \dots, n$, but with common median $\xi_{1/2}$, we are interested in null hypotheses of the form $H^{(n)}(\xi_{1/2})$ (specified median) and $H^{(n)} = \cup_{\xi_{1/2} \in \mathbb{R}} H^{(n)}(\xi_{1/2})$ (unspecified median).

Testing $H^{(n)}(\xi_{1/2})$ against location alternatives is traditionally achieved by means of a *sign test*, which is optimal in this context: see, e.g., Lehmann (1986), Section 4.9 (the proof there is easily adapted, for $p = 1/2$, to the nonhomogeneous case). Testing $H^{(n)}(\xi_{1/2})$ or $H^{(n)}$ against serial dependence, besides being an important problem in its own right, also would provide an important step towards a general methodology for the analysis of time series with nonhomogeneous (or simply heteroscedastic) innovation processes.

Heterogeneous or, at least, heteroscedastic noise, is to be feared in a variety of applications, and in general very seriously affects the validity of classical inference methods (here, correlogram-based analysis), which take *i.i.d.* noise for granted (not to mention those requiring Gaussian densities). This problem is not new, and a variety of methods have been proposed, relying mainly (i) on two-stage sampling, (ii) on variance-stabilizing transformations, or (iii) on nonparametric estimates of the densities f_t. These methods however either do not apply in the present context (case (i)), perform very poorly unless something is known about the densities f_t at hand (case (ii)), or typically require regularity assumptions (on f_t, with respect to t) and extremely large samples (case (iii)).

The methods we are proposing here are of a totally different nature. They rely on a generalized concept of *runs* which naturally follows from invariance arguments, in the same spirit as the classical *sign tests*. The main advantage of runs statistics is that they are distributions-free under extremely general conditions, and extremely robust. This advantage has been recognized long ago, and *exact* tests (i.e., using *exact* null distribution) based on runs can be traced back to the very first developments of nonparametric inference: Fisher (1926); Kermack and McKendrick (1937); Mood (1940); Olmstead (1940); Wallis and Moore (1941); see also the extensive bibliography by Dufour, Lepage and Zeidan (1982).

The popularity of runs tests has been increasing until the late fifties, when it seems to reach a peak: 24 references between 1955 and 1959 in Dufour *et al.* (1982). Runs tests indeed offer a number of theoretical and practical advantages which make them extremely attractive in a variety of applications, among which economic and econometric ones: see Karsten (1927); Cowles and Jones (1937); Mann (1945, 1950), Wright (1950) for early applications in these domains. Runs tests have been adapted, quite successfully, to a very wide range of statistical situations [for example, 2- and k-sample problems (Barton and David, 1957), contingency tables (Moore, 1949), regression (Stuart, 1956), quality control (Wolfowitz, 1943; Weiler, 1953, 1954)], but time series analysis always has been one of their favorite and most natural domains of application. Runs and the allied tests based on signs of first-order differences have been used in testing against trend [Kuznets (1929); Cochran (1936); Wallis and Moore (1941); Moore and Wallis (1943);

Mann (1945a, b, 1950); Stuart (1956)] and first-order serial dependence [David (1947); Goodman (1958); Granger (1963); Denny and Yakowitz (1978)], but also against higher-order Markov dependence [Denny and Yakowitz (1978)], or in the study of comovements between two series [Goodman and Grundfeld (1961); Yang and Schreckengost (1981)].

The attractive features of classical runs unfortunately are seriously hampered by the fact that they are designed against first-order serial dependence problems. This severely restricts their role as a tool for time series analysis. Our *generalized runs* are lagged versions of this classical concept, and are flexible enough to handle higher-order serial dependence problems. Being invariant (under a group of transformations generating the hypothesis $H^{(n)}(\xi_{1/2})$), our *generalized runs* statistics are distribution-free under $H^{(n)}(\xi_{1/2})$, and their null distributions are given under explicit form. Moreover, they are fairly robust, and can be interpreted as providing a sign-based correlogram, with the same interpretation as usual correlograms, but exact p-values instead of asymptotic or approximate ones. This correlogram interpretation is an important advantage, since much of the expertise of time-series analysts is related with correlogram inspection.

Though invariance, distribution-freeness, robustness, exact explicit distributions, etc. are extremely desirable features, the price to be paid for such properties in general is a severe loss of efficiency. Granger (1963), in the particular context of first-order Markovian dependence, suggests that this loss, for traditional runs tests, is not dramatic at all. His conclusions, however, are of a qualitative nature, and depend very much on the particular alternative considered. Granger's findings here are confirmed and extended to the much broader context of ARMA processes, by the computation of the asymptotic relative efficiencies (AREs) of generalized runs procedures with respect to their classical correlogram-based counterparts. Such comparisons, of course, are inherently unfair to runs methods, since they have to be carried out under restricted conditions of *i.i.d.* observations which, unlike their competitors, generalized runs tests do not require and do not exploit. Note that choosing between traditional correlogram methods and the proposed ones based on runs is not just a matter of going for the most powerful method. Actually, depending on the assumptions made, there is no real choice. Either the noise process underlying the observations can be assumed to be *i.i.d.* and the correlogram (or rank-based correlogram) methods are the obvious option, or the noise cannot be assumed to be *i.i.d.*, and the methods based on runs are the only valid ones. Still, the ARE figures, obtained under *i.i.d.* assumptions (and shown in Tab. 3.1), are surprisingly good: 0.4053 under Gaussian innovations, 0.5000 under double exponential ones.

Generalized runs statistics are computed from the signs of the differences $X_t - \xi_{1/2}$, and thus allow for testing $H^{(n)}$ $(\xi_{1/2})$, with specified $\xi_{1/2}$. Now, in practice, $\xi_{1/2}$ is seldom specified, and a much more realistic problem is that of testing for $H^{(n)}$. To deal with this nuisance parameter problem, we propose two different solutions. The first solution, of an asymptotic nature, is based on the classical idea of replacing the unknown median $\xi_{1/2}$ by its empirical counterpart $X^{(n)}_{1/2}$. The second one is entirely nonasymptotic, and relies on constructing a preliminary confidence interval for $\xi_{1/2}$.

In the first approach, considering the signs of the differences $X_t - X^{(n)}_{1/2}$ yields (generalized) *aligned* runs statistics. But the alignment device destroys the invariance and distribution-freeness properties of runs. Fortunately, it can be shown that the effect of substituting $X^{(n)}_{1/2}$ for $\xi_{1/2}$ is asymptotically nill, so that aligned runs tests are at least asymptotically invariant, hence asymptotically distribution-free. Such a result, however, is by no mean obvious. There is no guarantee, indeed, that $X^{(n)}_{1/2}$ under $H^{(n)}(\xi_{1/2})$ even converges to $\xi_{1/2}$, so that all methods (generally relying on some stochastic continuity property) which are traditionally used to study the impact of substituting an estimate for an unknown nuisance parameter here are totally helpless. The proof we are giving is of a combinatorial nature, and related to the so-called *Banach match box problem.*

The second approach can be viewed as an extension to a nonparametric context of a procedure proposed by Dufour (1990) in a totally different, parametric, setting. Instead of the point estimate $X^{(n)}_{1/2}$, a nonparametric confidence interval, based on a standard sign test, is considered for $\xi_{1/2}$, and the run-based test statistic, as a function of $\xi_{1/2}$, is minimized over this interval. We show that conservative critical values easily can be obtained for this modified statistic. The resulting test is strictly conservative (i.e., the probability of type I error is uniformly less than the nominal level, which is likely to induce some bias), for fixed n, irrespectively of the underlying densities, whereas the aligned runs procedure is only asymptotically correct.

One may argue that this second approach is likely to result in a significant loss of power. Such an objection again is methodologically unfair, since this method we are proposing is the only valid one so far: pointing out a loss of power with respect to a competitor which does not meet the same requirements on Type-I error actually does not make too much sense. Numerical simulations (such simulations were conducted in a closely related problem by Campbell and Dufour (1997)) nevertheless suggest that the power of the conservative tests is by no means negligible. As a rule, one may like to consider the conservative procedure if validity under the null hypothesis is to be emphasized, and the aligned runs procedure if the sample size is big enough, and more emphasis is to be put on the power.

The paper is organized as follows. Section 2 briefly presents the problem and the invariance arguments leading to runs tests. Section 3 introduces runs and generalized runs, and provides an exact distribution theory for them, as well as some asymptotic theory. Section 4 is devoted to the alignment problem, Section 5 to the finite-sample conservative procedure. A simulation study of the size and power of (aligned) runs tests under various heteroscedasticity patterns is presented in Section 6.

2. NONHOMOGENEOUS WHITE NOISE: UNBIASEDNESS AND INVARIANCE

For mathematical convenience, we assume in this section that, under $H^{(n)}(\xi_{1/2})$, each observation $X_t^{(n)}$ is of the form $X_t^{(n)} = \xi_{1/2} + Z_t^{(n)}$, where $Z_1^{(n)}, \ldots, Z_n^{(n)}$ are independent, with nowhere vanishing densities f_t, $t = 1, \ldots, n$ such that $\int_{-\infty}^{0} f_t(z)\,dz = 1/2$. This assumption later on will be relaxed.

The unspecified densities f_t accordingly play the role of nuisance parameters which—unless very restrictive additional assumptions are made — cannot be estimated, and therefore somehow should be eliminated. The classical theory of hypothesis testing in such situations suggests two main approaches: *unbiasedness* arguments (*Neyman structure*: see Lehmann (1986, Chapter 4)) and *invariance* arguments (same reference, Chapter 6).

The unbiasedness argument leads to conditional tests, resulting from conditioning upon a sufficient, boundedly complete statistic. The observed vector $\mathbf{X}^{(n)}$ in the present case unfortunately is minimal sufficient for $H^{(n)}(\xi_{1/2})$ (moreover, it is not boundedly complete, since the signs sgn $(Z_t^{(n)} - \xi_{1/2})$ obviously are *ancillary*). As a consequence, the unbiasedness approach in this context is totally helpless.

The invariance argument is more successful. A generating group for $H^{(n)}(\xi_{1/2})$ indeed is the group $\mathcal{G}^{(n)}, \circ$ of transformations $g^{(n)} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$g^{(n)}(x_1, \ldots, x_n) = (\xi_{1/2} + g_1(x_1 - \xi_{1/2}), \ldots, \xi_{1/2} + g_n(x_n - \eta_{1/2})),$$

where g_t, $t = 1, \ldots, n$ belongs to the set $\mathcal{G}$ of all continuous, (strictly) monotonically increasing functions $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{z \to \pm\infty} g(z) = \pm \infty \text{ and } g(0) = 0.$$

It is easily checked that a maximal invariant for this generating group is the vector of signs $s^{(n)} = (s_1^{(n)}, \ldots, s_n^{(n)})$, with

$$s_t^{(n)} = \text{sgn}\,(X_t^{(n)} - \xi_{1/2}), \quad t = 1, \ldots, n. \tag{2.1}$$

The invariance principle then implies that, in the problem of testing for $H^{(n)}(\xi_{1/2})$, only those tests which are measurable with respect to the vector of signs $s^{(n)}$ should be considered. Being invariant, $s^{(n)}$ under $H^{(n)}(\xi_{1/2})$ is distribution-free. The signs $s_t^{(n)}$ indeed clearly are *i.i.d. symmetric Bernoulli variables*:

$$P(s_t^{(n)} = 1) = \frac{1}{2} = P(s_t^{(n)} = -1) \tag{2.2}$$

(this terminology will be used throughout the paper).

The continuity assumption in $H^{(n)}(\xi_{1/2})$ is not necessary for (2.2) to hold, but it considerably simplifies the group invariance argument. Later on (Section 3.3), it will be relaxed (and (2.2) as well).

3. RUNS AND GENERALIZED RUNS

3.1. Exact Distributions

All runs in this section are computed from the signs $s_t^{(n)}$ defined in (2.1), and all distributions are considered under $H^{(n)}(\xi_{1/2})$. With probability one, all the observations $X_t^{(n)}$ then are distinct from $\xi_{1/2}$ (this is tacitly assumed throughout Sections 3.1 and 3.2): then, the number of runs, in $\mathbf{X}^{(n)}$, or in some given subseries thereof, is the number of successive sequences of 1 or -1 values in the corresponding series of signs $\mathbf{s}^{(n)}$.

Denoting by $[x]$ the largest integer smaller than or equal to x, let $n(j,k) = j + k[(n-j)/k]$, $1 \le j, k < n \in \mathbf{N}$, and divide $\mathbf{X}^{(n)}$ into the $l(k) = \min(k, n-k)$ subseries

$$\begin{array}{lllll} (X_1^{(n)}, & X_{1+k}^{(n)}, & X_{1+2k}^{(n)}, & \ldots, & X_{n(1,k)}^{(n)}), \\ (X_2^{(n)}, & X_{2+k}^{(n)}, & X_{2+2k}^{(n)}, & \ldots, & X_{n(2,k)}^{(n)}), \\ \vdots & & & & \vdots \\ (X_{l(k)}^{(n)}, & X_{l(k)+k}^{(n)}, & X_{l(k)+2k}^{(n)}, & \ldots, & X_{n(l(k),k)}^{(n)}). \end{array} \tag{3.1}$$

Let

$$S_{+,k}^{(n)} = (n-k)^{-1} \sum_{t=k+1}^{n} I[(X_t^{(n)} - \xi_{1/2})(X_{t-k}^{(n)} - \xi_{1/2}) < 0], \qquad k = 1, \ldots, n-1; \tag{3.2}$$

$(n-k) S_{+;k}^{(n)}$ clearly represents the cumulated number of runs (in the sense described at the beginning of this section) in the $l(k)$ subseries (3.1): call them *generalized runs* (taken with respect to $\xi_{1/2}$), and call (3.3) a *generalized lag k runs statistic.* It is easily checked that (still, with probability one)

$$S_{+;k}^{(n)} = \frac{1}{2}\left[1 - \frac{1}{n-k} \sum_{t=k+1}^{n} s_t^{(n)} s_{t-k}^{(n)}\right], \tag{3.3}$$

so that generalized runs statistics are closely related to the autocovariance coefficients of the series of signs $s_1^{(n)}, \ldots, s_n^{(n)}$ (see

Section 3.2). Since the products $s_t^{(n)} s_{t-k}^{(n)}$, $t = k+1, \ldots, n$ under $H^{(n)}(\xi_{1/2})$ are also independent symmetric Bernoulli variables (see Dufour, 1981), $(n-k)\, S_{+;k}^{(n)}$ is binomial Bin $(n-k;\ 1/2)$:

$$P[S_{+;k}^{(n)} = m/(n-k)] = \left(\frac{1}{2}\right)^{n-k} \frac{(n-k)!}{m!(n-k-m)!}, \quad m = 0, \ldots, n-k. \tag{3.4}$$

It follows that, still under $H^{(n)}(\xi_{1/2})$,

$$E(S_{+;k}^{(n)}) = \frac{1}{2} \quad \text{and} \quad \operatorname{var}(S_{+;k}^{(n)}) = \frac{1}{4(n-k)}, \qquad k = 1, \ldots, n-1. \tag{3.5}$$

For $k = 1$, $S_{+;1}^{(n)}$ coincides with the traditional runs statistic, and one retrieves well-known classical results (see, e.g., Gibbons and Chakraborti 1992, Chapter 3).

Exact joint distributions also can be obtained, at least numerically. Consider, for instance, $S_{+;k'}^{(n)}$ and $S_{+;k''}^{(n)}$, $k' < k''$. Then,

$$\begin{pmatrix} (n-k') & S_{+;k'}^{(n)} \\ (n-k'') & S_{+;k''}^{(n)} \end{pmatrix} \stackrel{\mathcal{D}}{=} \begin{pmatrix} \xi_1 + \xi_2 \\ \xi_1 + \xi_3 \end{pmatrix} + \begin{pmatrix} \xi_5 \\ 0 \end{pmatrix},$$

where

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \sum_{t=k''+1}^{n} \begin{pmatrix} I[s_t^{(n)} s_{t-k'}^{(n)} < 0] I[s_t^{(n)} s_{t-k''}^{(n)} < 0] \\ I[s_t^{(n)} s_{t-k'}^{(n)} < 0] I[s_t^{(n)} s_{t-k''}^{(n)} > 0] \\ I[s_t^{(n)} s_{t-k'}^{(n)} > 0] I[s_t^{(n)} s_{t-k''}^{(n)} < 0] \\ I[s_t^{(n)} s_{t-k'}^{(n)} > 0] I[s_t^{(n)} s_{t-k''}^{(n)} > 0] \end{pmatrix}$$

is multinomial Mult $(n-k'';\ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$,

$$\xi_5 = \sum_{t=k'+1}^{k''} I[s_t^{(n)} s_{t-k'}^{(n)} < 0]$$

is Bin $(k''-k';\ 1/2)$, and ξ and ξ_5 are independent. Note furthermore that $S_{+;k'}^{(n)}$ and $S_{+;k''}^{(n)}$ are uncorrelated (for $k' \neq k''$), still under $H^{(n)}(\xi_{1/2})$, since $s_t^{(n)} s_{t-k'}^{(n)}$ and $s_t^{(n)} s_{t-k''}^{(n)}$ are.

3.2. An Exact Correlogram Based on Generalized Runs

Classical correlograms consist in a plot of autocorrelation coefficients $r_k^{(n)}$, along with p-values resulting from the asymptotic normal approximation $(n-k)^{1/2} r_k^{(n)} \approx N(0,1)$. More sophisticated standardizations are possible (see Moran (1948); Dufour (1985); Dufour and Roy (1985)), but still only yield approximate distributions. If we define

$$r_{+;k}^{(n)} = \frac{1}{n-k} \sum_{t=k+1}^{n} s_t^{(n)} s_{t-k}^{(n)} = 1 - 2S_{+;k}^{(n)}, \quad k = 1, \ldots, n-1, \quad (3.6)$$

$r_{+;k}^{(n)}$ can be interpreted as the empirical autocorrelation (at lag k) of the series of signs $s_t^{(n)}$. Moreover, $(n-k)^{1/2} r_{+;k}^{(n)}$ is *exactly* standardized (under $H^{(n)}(\xi_{1/2})$), and *exact* p-values can be computed from (3.4) using binomial tables. More specifically, let $p_{+;k}^{(n)}(z) = P[|r_{+;k}^{(n)}| \geq z]$, and denote by F_ν the distribution function of a Bin $(\nu;\ 1/2)$ variable. Then, the appropriate two-sided p-value associated with $r_{+;k}^{(n)}$ under $H^{(n)}(\xi_{1/2})$ is

$$p_{+;k}^{(n)}(|r_{+;k}^{(n)}|) = 2\left[1 - F_{n-k}\left(\frac{(n-k)}{2}(1 + |r_{+;k}^{(n)}|)\right)\right]. \quad (3.7)$$

The resulting correlogram admits the same interpretation as traditional correlograms, but under considerably weaker validity assumptions.

3.3. Relaxing the Assumptions; Treatment of Observed Zeroes

The probability under $H^{(n)}(\xi_{1/2})$ that some $X_t^{(n)}$ coincide with $\xi_{1/2}$ is zero, and the possibility of such an event accordingly has been ignored in the above sections. Now, in practice, $X_t^{(n)} = \xi_{1/2}$ (hence, $s_t^{(n)} = 0$) does occur, even under the continuity assumption. Moreover, one may be willing to relax this continuity assumption, in such a way that $s_1^{(n)}, \ldots, s_n^{(n)}$ under $H^{(n)}(\xi_{1/2})$ remain mutually independent, but

$$P[s_t^{(n)} = 0] = p_t, \quad P[s_t^{(n)} = -1] = \frac{1}{2}(1 - p_t) = P[s_t^{(n)} = 1], \quad (3.8)$$

with unspecified values of $p_t \in [0, 1]$, $t = 1, \ldots, n$.

Denote by $H_0^{(n)}(\xi_{1/2})$ this new hypothesis: it contains, among many others, the case of observations $X_t^{(n)}$ of the form $\xi_{1/2} + Z_t^{(n)}$, where the $Z_t^{(n)}$'s are independent, have median zero, and possibly discontinuous distribution functions F_t satisfying $F_t(-0) = 1 - F_t(0)$. A common practice, in traditional runs tests, would consist in removing from the observed series all observations which are equal to $\xi_{1/2}$. Such procedure, in the present time-series context, is highly inadequate, since it quite artificially alters the serial structure of the observations: if, e.g., X_{t_0} is removed, the lag between X_{t_0-k} and X_{t_0+l} drops from $k+l$ to $k+l-1$, which may seriously affect the analysis.

We therefore rather suggest the following conditional approach, which is valid under $H_0^{(n)}(\xi_{1/2})$, hence also under $H^{(n)}(\xi_{1/2})$. For all vector $\mathbf{s} \in \{-1, 0, 1\}^n$, denote by $n_0(\mathbf{s})$ and $\mathcal{T}_0(\mathbf{s}) = \{t_1(\mathbf{s}), \ldots, t_{n_0(\mathbf{s})}(\mathbf{s})\}$ the number of zeroes and their positions, respectively.

Let $n_k^* = \sum_{t=k+1}^n I[s_t s_{t-k} \neq 0] : n_k^*$ and n_0 are measurable with respect to $\mathcal{T}_0$. The probability, under $H_0^{(n)}(\xi_{1/2})$, that $\mathbf{s}^{(n)}$ takes value $\mathbf{s}$ is

$$P[\mathbf{s}^{(n)} = \mathbf{s}] = \left(\prod_{t \in \mathcal{T}_0(\mathbf{s})} p_t\right)\left(\prod_{t \in \mathcal{T}_0(\mathbf{s})} (1-p_t)\right)(1/2)^{n-n_0(\mathbf{s})},$$

so that the conditional distribution of the non-null components of $\mathbf{s}^{(n)}$, given $\mathcal{T}_0(\mathbf{s}^{(n)})$, again is that of a $(n-n_0)$-tuple of independent symmetric Bernoulli variables. It follows that $(n-k)\,S_{+;k}^{(n)}$, with $n^* - k$ non-null summands, is conditionally Bin $(n_k^* - k; 1/2)$. Since this conditional distribution depends only on the $\mathcal{T}_0$-measurable variable n_k^*, it also holds if we condition with respect to n_k^* itself. Summing up, in case zeroes occur in $\mathbf{s}^{(n)}$, we propose the following conditional correlogram (same notation as in Section 3.2):

$$r_{*;k}^{(n)} = \frac{1}{n_k^* - k} \sum_{t=k+1}^{n} s_t^{(n)} s_{t-k}^{(n)} = \frac{n-k}{n_k^* - k}[1 - 2S_{+;k}^{(n)}],$$

where $n_k^* = \sum_{t=k+1}^n I[s_t^{(n)} s_{t-k}^{(n)} \neq 0]$, with the (conditional) p-values

$$p_{*;k}^{(n)}(|r_{*;k}^{(n)}|) = 2\left[1 - F_{n_k^*-k}\left(\frac{n_k^* - k}{2}(1 + |r_{*;k}^{(n)}|)\right)\right].$$

3.4. Asymptotic Distributions and Asymptotic Relative Efficiencies (AREs)

The main advantage of generalized runs of course lies in the finite-sample exactness of their null distributions. However, for the sake of efficiency comparisons with traditional procedures, we also briefly consider their asymptotic behaviour, under $H^{(n)}(\xi_{1/2})$ as well as under $H_0^{(n)}(\xi_{1/2})$ and contiguous alternatives of serial dependence.

The asymptotic normality, with mean zero and variance one, of $(n-k)^{1/2}\, r^{(n)}_{+;k}$ readily follows from the asymptotic behaviour of the binomial distribution. Similarly, under $H_0^{(n)}(\xi_{1/2}), (n-k)^{1/2}\, r^{(n)}_{+;k}$ is (unconditionally) asymptotically standard normal, provided that there exists some $\delta > 0$ such that

$$P\left[\limsup_{n\to\infty} \frac{n_0(\mathbf{s}^{(n)})}{n} < 1-\delta\right] = 1. \tag{3.9}$$

The asymptotic joint distribution of K-tuples of the form

$$n^{1/2}\,\mathbf{r}^{(n)}_{+;K} = ((n-1)^{1/2}\, r^{(n)}_{+;1}, \ldots, (n-K)^{1/2}\, r^{(n)}_{+;K}),$$

with mean $\mathbf{0}$ and identity covariance matrix, confirms the "correlogram behaviour" of our run-based correlation coefficients.

PROPOSITION 3.1 *For all K, $n^{1/2}\,\mathbf{r}^{(n)}_{+;K}$ is asymptotically multinormal, under $H^{(n)}(\xi_{1/2})$, as $n\to\infty$, with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}$.*

Proof Since $S^{(n)}_{+;k}$, $k = 1,\ldots,n-1$ are distribution-free under $H^{(n)}(\xi_{1/2})$, there is no loss of generality in proving the proposition under the particular assumption of an *i.i.d.* sequence $X_t^{(n)} = Z_t^{(n)} + \xi_{1/2}$, where the distribution function F of $Z_t^{(n)}$ is symmetric with respect to the origin. Denoting by $R^{(n)}_{+;t}$ the rank of $|Z_t^{(n)}|$ among $|Z_1^{(n)}|, \ldots, |Z_n^{(n)}|$, $S^{(n)}_{+;k}$ then is a particular case of the *linear serial signed rank statistics*, of the form

$$S^{(n)}_{+;k} = (n-k)^{-1} \sum_{t=k+1}^{n} a_+^{(n)}(s_t R^{(n)}_{+;t}, \ldots, s_{t-k} R^{(n)}_{+;t-k}),$$

considered in Hallin and Puri (1991). The asymptotic results of this latter paper apply, with a score-generating function $J_+(v_1,\ldots,v_{k+1})$

$=I[v_1\ v_{k+1} < 0]$, so that (Proposition 2.1, same reference), for any $\alpha_1, \ldots, \alpha_K \in \mathbb{R}^K$,

$$\sum_{k=1}^{K} \alpha_k\, S^{(n)}_{+;k} = \sum_{k=1}^{K} \alpha_k\, \mathcal{S}^{(n)}_{+;k} + o_P(n^{-1/2}),$$

where

$$\mathcal{S}^{(n)}_{+;k} = (n-k)^{-1} \sum_{t=k+1}^{n} I[(2F(Z^{(n)}_t) - 1)\,(2F(Z^{(n)}_{t-k}) - 1) < 0].$$

Due to the fact that the expectations $E(S^{(n)}_{+;k}) = E(\mathcal{S}^{(n)}_{+;k}) = \frac{1}{2}$ are constant, the U-statistic approach used in Hallin and Puri (1991) is not required here, and it readily follows from standard results on K-dependent processes (see, e.g., Brockwell and Davis 1991, Chapter 6) that $n^{\frac{1}{2}} \sum_{k=1}^{K} \alpha_k\,(\mathcal{S}^{(n)}_{+;k} - \frac{1}{2})$ is asymptotically normal, with mean zero and variance

$$\begin{aligned}
&Var\left[\sum_{k=1}^{K} \alpha_k I[(2F(Z_1) - 1)\,(2F(Z_{k+1} - 1) < 0]\right] \\
&\quad + 2 \sum_{l=1}^{K} Cov\left[\sum_{k=1}^{K} \alpha_K I[(2F(Z_1) - 1)(2F(Z_{k+1}) - 1) < 0],\right. \\
&\quad \left.\sum_{k=1}^{K} \alpha_k I[(2F(Z_{l+1}) - 1)\,(2F(Z_{l+k+1} - 1) < 0]\right] \\
&\quad = \frac{1}{4}\left[\sum_{k=1}^{K} \alpha^2_k + 0\right].
\end{aligned}$$

The usual Cramér-Wold argument completes the proof. ■

An immediate corollary to Proposition 3.1 is that a *portmanteau test* based on generalized runs can be constructed for testing $H^{(n)}(\xi_{1/2})$ against alternatives of serial dependence of order $k \leq K$. This test rejects $H^{(n)}(\xi_{1/2})$ whenever

$$Q^{(n)}_{+;K} = \sum_{k=1}^{K} (n-k)\,(r^{(n)}_{+;k})^2 > \chi^2_{K;1-\alpha}, \tag{3.10}$$

where $\chi^2_{K;1-\alpha}$ denotes the $(1-\alpha)$-quantile of a chi-square distribution with K degrees of freedom. Of course, this test, in contrast with those based on the p-values given in Sections 3.2 and 3.3, is only asymptotically correct. But the convergence of symmetric binomial distributions to the normal is rather fast, and the chi-square approximation in (3.10) should be pretty good, even for moderate series lengths (much better, presumably, than in the classical Box-Pierce case); moreover, the quality of the approximation is uniform over $H^{(n)}(\xi_{1/2})$. Provided that (3.9) holds, this portmanteau test remains valid when $Q^{(n)}_{+;K}$ is computed from the modified autocorrelation coefficients $r^{(n)}_{*;k}$.

Since run-based methods remain valid under much more general conditions than classical time-series ones, it may be feared that their power be considerably lower. As claimed in the introduction, this is not the case—at least, in terms of asymptotic relative efficiencies under alternatives of ARMA dependence.

In order to obtain local power comparisons, we consider the sequence of (stationary) alternatives $K^{(n)}_f$ (**a**, **b**; $\xi_{1/2}$) under which $X^{(n)}_t$ is generated by the ARMA (p, q) model

$$(X_t - \xi_{1/2}) - n^{-\frac{1}{2}} \sum_{i=1}^{p} a_i(X_{t-i} - \xi_{1/2}) = \varepsilon_t + n^{-\frac{1}{2}} \sum_{i=1}^{q} b_i \varepsilon_{t-i},$$

where $\{\varepsilon_t\}$ is *i.i.d.*, with density f symmetric with respect to 0 and satisfying the following technical assumptions (Hallin and Puri (1994); these are slightly weaker than in Hallin and Puri (1991)):

(i) $f(x) = f(-x) > 0,\ x \in \mathbb{R};\ 0 < \ x^2 f(x)\, dx = \sigma^2 < \infty$

(ii) there exists a function $\dot{f}$ such that $f(b) - f(a) = \int_a^b \dot{f}(x)\, dx$ for all $-\infty < a < b < \infty$, and, letting $\phi_f = -\dot{f}/f$, $\int \phi_f^2(x)\, dx = I(f) < \infty$.

PROPOSITION 3.2

(*i*) *The asymptotic powers against* $K^{(n)}_f(\mathbf{a}, \mathbf{b})$ *of* (*a*) *the exact level* α *test rejecting* $H^{(n)}(\xi_{1/2})$ *whenever* $r^{(n)}_{+;k}$ *is positive and* $p^{(n)}_{+;k}(|r^{(n)}_{+;k}|)$, *as given in* (3.7), *is larger than or equal to* 2α *and* (*b*) *the approximate level* α *test rejecting* $H^{(n)}(\xi_{1/2})$ *whenever* $(n-k)^{1/2}\, r^{(n)}_{+;k}$ *is larger than the* $(1-\alpha)$ *standard normal quantile* z_α, *are equal to*

$$1 - \Phi\left(z_\alpha + 2(a_k + b_k) f(0) \int |x| f(x)\, dx\right), \qquad (3.11)$$

where Φ as usual stands for the standard normal distribution function.

(*ii*) *The asymptotic power against $K_f^{(n)}(\mathbf{a}, \mathbf{b})$ of the portmantean test* (3.10) *is*

$$1 - F_K\left(\chi^2_{K;1-\alpha};\, 4\sum_{k=1}^{K} (a_k + b_k)^2 f^2(0) \left[\int |x| f(x)\, dx\right]^2\right), \qquad (3.12)$$

where $F_K(\cdot\,; \lambda^2)$ denotes the distribution function of a noncentral chi-square variable with K degrees of freedom and noncentrality parameter λ^2.

Proof Since $S^{(n)}_{+;k}$, $k = 1, \ldots, n-1$ are distribution-free under $H^{(n)}$ $(\xi_{1/2})$, there is no loss of generality in proving the proposition under the particular assumption of an *i.i.d.* sequence $X_t^{(n)} = Z_t^{(n)} + \xi_{1/2}$, where the distribution function F of $Z^{(n)}_{+;t}$ is symmetric with respect to the origin. Denoting by $R^{(n)}_{+;t}$ the rank of $|Z_t^{(n)}|$ among $|Z_t^{(n)}|, \ldots, |Z_n^{(n)}|$, $S^{(n)}_{+;k}$ is a particular case of *linear serial signed rank statistic.* The proposition then follows from Hallin and Puri (1991, Proposition 2.2), where it is shown that the asymptotic mean of $(n-k)^{1/2}$ $(S^{(n)}_{+;k} - 1/2)$ under $K_f^{(n)}(\mathbf{a}, \mathbf{b})$ is $\sum_{i=1}^{k}(a_i + b_i) C_i^+$, with

$$\begin{aligned} C_i^+ &= E\left[I[V_1 V_{k+1} < 0] \sum_{j=0}^{k-i} \phi_f(F_+^{-1}(V_{1+j}))\, F_+^{-1}(V_{1+i+j})\right] \\ &= \begin{cases} 0 & i \neq k, \\ 2\int_{[0,\frac{1}{2}]\times[\frac{1}{2},1]} \phi(F^{-1}(u_1)) F^{-1}(u_2)\, du_1 du_2 & i = k, \end{cases} \end{aligned}$$

hence

$$C_k^+ = -2\int_{-\infty}^{0} \dot{f}(x)\, dx \int_0^\infty x f(x)\, dx = -f(0) \int_{-\infty}^{\infty} |x| f(x)\, dx,$$

where $F_+ = 2F - 1$, and $V_1, \ldots V_{k+1}$ are independent and uniform over $[-1, 1]$. Changing signs and standardizing yields for the run-based autocorrelation $(n-k)^{1/2} r^{(n)}_{+;k}$ an asymptotic mean $2(a_k + b_k) f(0) \int |x|\,.$ $f(x)\, dx$; parts (i) and (ii) of the proposition readily follow. ∎

Comparing the local powers of Proposition 3.2 with those of traditional correlogram-based ones, or with those of the rank-based methods proposed in Hallin and Puri (1994) yields the following table of asymptotic relative efficiencies.

TABLE 3.1 Asymptotic relative efficiencies (AREs) of generalized runs with respect to parametric or rank-based correlogram methods, under ARMA alternatives with normal, logistic and double-exponential innovation densities, respectively

Statistics	*Innovation density*		
	Normal	*Logistic*	*Laplace*
Parametric (Pearson) autocorrelations	.405	.438	.500
Signed and unsigned rank autocorrelations			
van der Waerden	.405	.418	.408
Wilcoxon	.428	.399	.337
Laplace	.661	.538	.250
Spearman	.444	.438	.395

The AREs reported in Table 3.1 are obtained, classically, as ratios of noncentrality parameters under contiguous alternatives of the form $K_f^{(n)}(\mathbf{a}, \mathbf{b})$; the influence of $\mathbf{a}$ and $\mathbf{b}$ cancels out, provided that $\|\mathbf{a}+\mathbf{b}\| \neq 0$; see, e.g., Hallin and Puri (1994) for details.

4. ALIGNED RUNS

4.1. Nonhomogeneous White Noise with Unspecified Median

The considerable advantage of runs with respect to $\xi_{1/2}$, in the problem of testing for $H^{(n)}(\xi_{1/2})$, lies in their invariance, hence distribution-freeness property. The trouble is that computing these runs is possible only when $\xi_{1/2}$ is specified, which is seldom the case in practice. A simple and intuitively attractive idea consists in substituting the empirical median $X_{1/2}^{(n)}$ for the unknown one $\xi_{1/2}$. This is, apparently, the way practitioners, on heuristic grounds, use to overcome the problem: see, for instance, the study of the Lake Michigan-Huron data reported in Kruskal and Tanur (1968). Unfortunately, the *aligned runs* resulting from considering the empirical median $X_{1/2}^{(n)}$ are no longer invariant; and they are no longer distribution-free under $H^{(n)}(\xi_{1/2})$ (they are distribution-free under the more restrictive assumption of identically distributed X_t's). This precludes using exact runs tests in testing for nonhomogeneous white noise with unspecified location, and at first sight considerably reduces their attractiveness.

Now, this attractiveness would be partly restored if approximate distributional results could palliate the lack of exact ones. The prospects for approximate distributional results however do not look very bright. Indeed, all traditional approaches to the alignment problem at least require the consistency, with appropriate rate, of the estimate—here, the convergence, in probability under $H^{(n)}(\xi_{1/2})$, of $X^{(n)}_{1/2}$ to $\xi_{1/2}$.

Such a convergence is by no means guaranteed under heterogeneous or heteroscedastic distributions. This consistency problem is investigated in Mizera and Wellner (1996) and Hallin and Mizera (1996, 1997), where necessary and sufficient convergence conditions are given. Roughly speaking, in the heteroscedastic case, where the noise is of the form $X_t = c_t U_t$, with *i.i.d.* random variables U_t and scaling constants c_t, consistency does not hold unless the expanding rate of the scaling constants is less than $t^{1/2}$. This, typically, is an assumption we would not like to make here. As a consequence, all traditional approaches to the *alignment* problem, which require consistent estimates, are inapplicable in the present context.

Surprisingly enough, though, combinatorial techniques are successful in obtaining the desired result under far more general conditions. More precisely, it is possible to proof (under mild regularity conditions: see Theorem 4.6) that the effect on $S^{(n)}_{+;k}$ of substituting $X^{(n)}_{1/2}$ for $\xi_{1/2}$ is asymptotically $o_P(n^{-1/2})$, as desired, *even when* $X^{(n)}_{1/2}$ *does not converge to* $\xi_{1/2}$.

This section is devoted to a proof of this result. The proof relies on a series of adequate conditionings, in a probabilistic context related to the celebrated *Banach's match boxes* problem (Feller, 1968). Section 4.2 explores the behaviour of *misclassified* observations—those for which the actual and aligned signs do not coincide—and shows how the rank of $|X^{(n)}_{1/2}|$ among $|X^{(n)}_1|, \ldots, |X^{(n)}_n|$ relates to the *Banach's match boxes* distribution. Section 4.3 studies the impact of alignment on the number of runs, and derives an explicit form, relying on the concept of *contact function* (Wolfowitz, 1942) for the resulting *alignment bias*. Finally, Section 4.4 establishes the main consistency results (with rates): Theorem 4.12 for homogeneous noise, Theorem 4.13 for the general case of heterogeneous noise.

The reader interested only in a statement of the main results (Theorems 4.10–4.13) can proceed directly to Section 4.4 after reading the few notational conventions below, and the definition of a *contact function* at the end of Section 4.3.

Throughout this section, we assume, without any loss of generality, that $\xi_{1/2} = 0$ (hypothesis $H^{(n)}(0)$ holds). Also, we are proving the required asymptotic equivalence result for runs of lag one; the general lag k case follows exactly along the same lines. Rather than $S^{(n)}_{+;k}$ defined in (3.3), we here consider the number of runs itself, i.e., $T^{(n)}_{+;k} = (n-k)\, S^{(n)}_{+;k}$. Let $\sigma(z_1, z_2) = I\,[z_1\, z_2 < 0]$: for simplicity, we write $T^{(n)}(\theta)$ for

$$T^{(n)}_{+;1}(\mathbf{X}^{(n)};\theta) = \sum_{t=2}^{n} \sigma(X^{(n)}_t - \theta, X^{(n)}_{t-1} - \theta),$$

$T^{(n)}$ for $T^{(n)}(0)$, and $\hat{T}^{(n)}$ for $T^{(n)}(X^{(n)}_{1/2})$. The objective of this section is to clarify the asymptotic relation, under $H^{(n)}(0)$, between $T^{(n)}$ and $\hat{T}^{(n)}$.

Denote by $\mathbf{s}^{(n)} = (s^{(n)}_1, \ldots, s^{(n)}_n)$ the signs of $X^{(n)}_1, \ldots, X^{(n)}_n$, by

$$|\mathbf{X}|^{(n)}_{(\cdot)} = \left(|X|^{(n)}_{(1)}, \ldots, |X|^{(n)}_{(n)}\right)$$

the order statistic computed from $|\mathbf{X}^{(n)}| = (|X^{(n)}_1|, \ldots, |X^{(n)}_n|)$, by $R^{(n)}_{+;t}$ the rank of $|X^{(n)}_t|$ among $|X^{(n)}_1|, \ldots, |X^{(n)}_n|$, and by $s^{(n)}_{(i)}$ the sign of the observation X_t for which $R^{(n)}_{+;t} = i$. Under $H^{(n)}(0)$, all these quantities are correctly defined with probability one.

Finally, all proofs are for $n = 2m + 1$, odd; for n even, they are entirely similar. The superscript $^{(n)}$ is omitted when no confusion is possible.

4.2. Critical and Misclassified Points

The sample median $X^{(n)}_{1/2}$ takes its values on a lattice consisting of all quantities of the form $\pm|X^{(n)}_t|$, $t = 1, \ldots, n$. Denote by $M^{(n)}$ the random variable characterizing $X^{(n)}_{1/2}$'s position on this lattice:

$$M^{(n)} = \begin{cases} k & \text{if } X^{(n)}_{1/2} = |X^{(n)}|_{(k)} \\ -k & \text{if } X^{(n)}_{1/2} = -|X^{(n)}|_{(k)} \end{cases}, \quad k = 1, 2, \ldots, m+1.$$

In other terms, $|M^{(n)}|$ is the rank of $|X^{(n)}_{1/2}|$ among $|X^{(n)}_1|, \ldots, |X^{(n)}_n|$, whereas $sgn\,(M^{(n)}) = sgn\,(X^{(n)}_{1/2})$.

PROPOSITION 4.1

(*i*) *Conditionally upon* $M^{(n)}$ *and* $s_{(|M^{(n)}|)} = sgn\,(X^{(n)}_{1/2})$, *the* $(n - |M^{(n)}|)$ $-$*tuple*$(s_{(|M^{(n)}|+1)}, s_{(|M^{(n)}|+2)}, \ldots, s_{(n)})$ *is uniformly distributed over the* $\binom{2m+1-|M^{(n)}|}{m}$ *possible arrangements of m signs equal to* $s_{(|M^{(n)}|)}$ *and* $m - |M^{(n)}| + 1$ *signs equal to* $-s_{(|M^{(n)}|)}$.

(*ii*) *Conditionally upon* $M^{(n)}$, *the* $|M^{(n)}| - 1$ *signs* $(s_{(1)}, s_{(2)}, \ldots, s_{(|M^{(n)}|-1)})$ *are i.i.d. symmetric Bernoulli variables.*

(*iii*) *The distribution of* $M^{(n)}$ *is given by*

$$P[M^{(n)} = k] = v_k/2 = \frac{1}{2}\binom{n-|k|}{m}\left(\frac{1}{2}\right)^{n-|k|},$$
$$k = \pm 1, \ldots, \pm(m+1).$$

(*iv*) $E[M^{(n)}] = 0$, *and* $E[|M^{(n)}|] = (2m+1)\binom{2m}{m}\left(\frac{1}{2}\right)^{2m}$.

(*v*) $\lim_{n\to\infty} n^{-1/2} E[|M^{(n)}|] = \sqrt{\frac{2}{\pi}}$, *hence* $\lim_{n\to\infty} n^{-1} E[|M^{(n)}|] = 0$.

Proof The signs $s_{(n)}, s_{(n-1)}, \ldots, s_{(1)}$ are *i.i.d.* symmetric Bernoulli. The value of $M^{(n)}$ can be thought of as arising from the corresponding successive Bernoulli trials (in this order, i.e., starting with the sign $s_{(n)}$ associated with the highest absolute value) as follows: as soon as $(m+1)$ signs of the same kind have occurred, this sign— call it the *dominant sign*, since it automatically coincides with that of the majority of the observations—is the sign $s_{(|M^{(n)}|)}$ of the median. Parts (i) and (ii) of the Proposition follow.

This sign allocation process is closely related to the so-called *Banach's match boxes problem* (Feller 1968 I, pp 166–167): imagine a mathematician carrying one match box containing m positive signs in his right pocket, and one match box containing m negative signs in his left pocket. The values of $s_{(n)}, s_{(n-1)}, \ldots$ are determined (in this order) by selecting one pocket at random, then picking a sign out of the corresponding match box. The process stops as soon as, choosing a value for $s_{(|M^{(n)}|)}$, the mathematician discovers that one of the boxes is empty: $n - (m - |M^{(n)}|) = |M^{(n)}| - 1$ signs then are left in the other box. The distribution (iii) of $M^{(n)}$ follows from formula (8.5) in Feller (1968 I, VI. 8); note that, given $|M^{(n)}|$, and due to symmetry, the sign of $M^{(n)}$ is itself symmetric Bernoulli. Symmetry also yields the first part of (iv); for the second part, we refer to Problem 11 in Feller (1968 I, IX.9). The same source also indicates how to use Stirling's formula in order to obtain the first part of (v), of which the second part is an immediate consequence. ∎

The observations with rank $R_{+;t}^{(n)}$ larger than $|M^{(n)}|$— those involved in part (*i*) of Proposition 4.1—will be called *stable*; observations with rank $R_{+;t}^{(n)}$ less than $|M^{(n)}|$—involved in part (ii) of Proposition 4.2—will be called *critical*. Also, let $\mathcal{S}^{(n)} = \{t \mid X_t \text{ is stable}\}$, $\mathcal{C}^{(n)} = \{t \mid X_t \text{ is critical}\}$: clearly, $\#\mathcal{C}^{(n)} = |M^{(n)}| - 1$, and $\#\mathcal{S}^{(n)} = n - |M^{(n)}|$. Denote by $\hat{s}_t^{(n)}$ the sign of $X_t^{(n)} - X_{1/2}^{(n)}$: the observations for which $\hat{s}_t^{(n)} = -s_t^{(n)}$ will be called *misclassified*. Put $\mathcal{W}^{(n)} = \{t | \hat{s}^{(n)} = s_t^{(n)}\}$ and $\mathcal{U}^{(n)} = \{t \mid \hat{s}_t^{(n)} = -s_t^{(n)}\}$.

PROPOSITION 4.2

(*i*) *Stable observations are well-classified*: $\mathcal{S}^{(n)} \subseteq \mathcal{W}^{(n)}$, *and* $\mathcal{U}^{(n)} \subseteq \mathcal{C}^{(n)}$.

(*ii*) *For all* $t \in \mathcal{C}^{(n)}$, $\hat{s}_t^{(n)} = -sgn(M^{(n)})$.

(*iii*) *For all* $t \in \mathcal{U}^{(n)}$, $\hat{s}_t^{(n)} = -sgn(M^{(n)})$ *and* $s_t^{(n)} = sgn(M^{(n)})$.

Proof The fist part of (i) follows from noting that, if $t \in \mathcal{S}^{(n)}$, $s_t^{(n)} = 1$ implies

$$X_t^{(n)} = |X_t^{(n)}| \geq |X_{1/2}^{(n)}| \geq X_{1/2}^{(n)};$$

on the other hand, if $s_t^{(n)} = -1$, then

$$X_t^{(n)} = -|X_t^{(n)}| \leq -|X_{1/2}^{(n)}| \leq X_{1/2}^{(n)}.$$

The second part of (i) is a consequence of the first one. Let $X_t^{(n)}$ be critical. If $sgn(M^{(n)}) = 1$, then

$$X_t^{(n)} \leq |X_t^{(n)}| \leq |X_{1/2}^{(n)}| = X_{1/2}^{(n)},$$

and $\hat{s}_t^{(n)} = -1$. Analogously, if $sgn\,(M_n) = -1$, then

$$X_t^{(n)} \geq -|X_t^{(n)}| \geq -|X_{1/2}^{(n)}| = X_{1/2}^{(n)},$$

and $\hat{s}_t^{(n)} = 1$, which entails (ii). Finally, from (i), all misclassified observations are critical, so that they have *aligned* signs $\hat{s}_t^{(n)} = -sgn\,(M^{(n)})$; since they are misclassified $s_t^{(n)} = -\hat{s}_t^{(n)}$ and (iii) holds. ∎

Let $U^{(n)} = \#\mathcal{U}^{(n)}$ and $V^{(n)} = \#\{t \,|\, s_t^{(n)} = 1\}$.

PROPOSITION 4.3 *Conditionally on $M^{(n)}$, each critical point is, independently of other critical points, misclassified with probability 1/2. As a consequence, still conditionally on $M^{(n)}$, $U^{(n)}$ has binomial Bin $(|M^{(n)}| - 1;\, 1/2)$ distribution.*

Proof The statement directly follows from Proposition 4.2 (i) and Proposition 4.1 (ii). ∎

PROPOSITION 4.4 *$U^{(n)} = |V^{(n)} - \frac{n}{2}| - \frac{1}{2}$ and $sgn\,(V^{(n)} - \frac{n}{2}) = sgn\,(M^{(n)})$.*

Proof If $sgn\,(X_{1/2}^{(n)}) = sgn\,(M^{(n)}) = 1$, then m observations are lying above the median, which are well-classified, and m observations are below the median, of which $n - V^{(n)}$ are well-classified. The number of misclassified observations is thus

$$m - (n - V^{(n)}) = V^{(n)} - (m+1) = V^{(n)} - \frac{1}{2}\,(n+1).$$

The case when $sgn\,(M^{(n)}) = -1$ is treated similarly. ∎

PROPOSITION 4.5 *Under $H^{(n)}$ (0), as $n \to \infty$,*

(i) $2V^{(n)} - n = M^{(n)} + o_P\,(n^{1/2})$;
(ii) $n^{-1/2}\, M^{(n)} \xrightarrow{\mathcal{D}} N(0,\, 1)$;
(iii) $E(M^{(n)})^2 = O(n)$, $E|M^{(n)}|^3 = O(n^{3/2})$, $E|M^{(n)}|^4 = O(n^2)$;
(iv) if $n^{1/2}\, c^{(n)} \to \infty$, then the truncated expectation $\sum_{k=c^{(n)}}^{m+1} k v_k$, where v_k is given in Proposition 4.1 (iii), is $o(n^{1/2})$.

Proof By Proposition 4.4,

$$\begin{aligned} Z^{(n)} &= 2V^{(n)} - n = 2\left|V^{(n)} - \frac{1}{2}n\right| sgn\left(V^{(n)} - \frac{1}{2}n\right) \\ &= (2U^{(n)} + 1)\, sgn\,(M^{(n)}), \end{aligned}$$

hence

$$\begin{aligned} Z^{(n)} - M^{(n)} &= (2U^{(n)} + 1 - |M^{(n)}|)\, sgn\,(M^{(n)}) \\ &= 2\left(U^{(n)} - \frac{1}{2}\,(|M^{(n)}| - 1)\right) sgn\,(M^{(n)}). \end{aligned} \tag{4.1}$$

Proposition 4.3 then implies that

$$\begin{aligned} E\,[(Z^{(n)} - M^{(n)})^2] &= E\left[4E\left[\left(U^{(n)} - \frac{1}{2}\left(|M^{(n)}| - 1\right)\right)^2 |M^{(n)}\right]\right] \\ &= 4E\,[Var\,(U^{(n)}\,|M^{(n)})] \\ &= 4E\left[\frac{1}{4}\,(|M^{(n)}| - 1)\right] = E\left[|M^{(n)}| - 1\right]. \end{aligned}$$

Hence, from Proposition 4.1 (v),

$$E\,[(Z^{(n)} - M^{(n)})^2] = E\,[|M^{(n)}| - 1] = o(n);$$

part (i) of the Proposition then follows from applying the Markov inequality.

To establish (ii), note that, as a consequence of the de Moivre-Laplace theorem, and since $V^{(n)}$ is binomial, with $E(V^{(n)}) = \frac{n}{2}$ and $\operatorname{Var}(V^{(n)}) = \frac{n}{4}$,

$$n^{-1/2}\,(Z^{(n)}) = \left(\frac{n}{4}\right)^{-1/2}\left(V^{(n)} - \frac{n}{2}\right) \xrightarrow{\mathcal{D}} N\,(0, 1); \tag{4.2}$$

(ii) accordingly follows from (i) and the traditional Cramér-Slutzky argument.

As for (iii), it is sufficient to prove the uniform integrability of $(n^{-1/2}\,|M^{(n)}|)^i$ for $i = 2, 3, 4$, since then part (ii) of this proposition and

Theorem 1.4. A of Serfling (1980) yield the desired result that

$$\lim_{n\to\infty} E|n^{-1/2}M^{(n)}|^i = E[|\eta|^i], \text{ where } \eta \sim N(0,1). \tag{4.3}$$

From Minkowski's inequality,

$$\begin{aligned} E[|n^{-1/2}M^{(n)}|^6] \le \Big\{ & (E[|n^{-1/2}M^{(n)}|^6])^{\frac{1}{6}} \\ & + (E[|n^{-1/2}(M^{(n)} - Z^{(n)})|^6])^{1/6}\Big\}^6. \end{aligned} \tag{4.4}$$

The centered moment of order six of the symmetric binomial Bin $(n; \frac{1}{2})$ variable is

$$\mu_6(n) = \frac{15}{64}n^3 - \frac{15}{34}n^2 + \frac{1}{4}n. \tag{4.5}$$

This moment, which seems hardly available in the literature, was obtained with the help of MAPLE (Release 2), by computing the sixth derivative of the centered binomial characteristic function. By (4.5), (4.4), (4.2), (4.1) and Proposition 4.3, we have

$$\begin{aligned} \limsup_{n\to\infty} E[|n^{-1/2}M^{(n)}|^6] &\le \Bigg\{\left(\limsup_{n\to\infty}\frac{64}{n^3}\mu_6(n)\right)^{\frac{1}{6}} \\ &\quad + \left(\limsup_{n\to\infty}\frac{1}{n^3}E[(M^{(n)} - Z^{(n)})^6]\right)^{\frac{1}{6}}\Bigg\}^6 \\ &\le \Bigg\{1 + \Bigg(\limsup_{n\to\infty}\frac{64}{n^3} \\ &\quad E\left[\left(U^{(n)} - \frac{1}{2}(|M^{(n)}|-1)\right)^6\right]\Bigg)^{\frac{1}{6}}\Bigg\}^6 \\ &\le \left\{1 + \left(\limsup_{n\to\infty}\frac{64}{n^3}\mu_6(|M^{(n)}|-1)^{\frac{1}{6}}\right)^6\right\} < \infty, \end{aligned}$$

since $|M^{(n)}| - 1 < n$. This and Lemma 1.4. A (ii) of Serfling (1980) entails the required uniform integrability of (4.3)−as well as that of $n^{-1/2}|M^{(n)}|$.

As for part (iv) of the proposition, let $n^{1/2}c^{(n)} \to \infty$. For any $c > 0$, there exists $n_c \in \mathbb{N}$ such that $n \geq n_c$ implies $n^{1/2}c^{(n)} \geq c$. Let $L_n = M^{(n)} I[M^{(n)} \neq n^{1/2}c]$ and $L'_n = M^{(n)} I[M^{(n)} \neq c_n]$. For $n \geq n_c$, we have $L_n \geq L'_n$; hence, for ν_k given in Proposition 4.1 (iii),

$$\sum_{k=c_n}^{m+1} k\nu_k = E[L'_n] \leq E[L^{(n)}]. \tag{4.6}$$

Let $f_c : \mathbb{R} \to \mathbb{R}$ be such that $f_c(u) = |u| I[|u| \geq c]$. By (ii) and the continuous mapping theorem,

$$n^{-1/2} E(L_n) = f_c(n^{-1/2} M^{(n)}) \xrightarrow{D} f_c(\eta), \tag{4.7}$$

where $\eta \sim N(0, 1)$. Since $|L_n| \leq |M^{(n)}|$, the uniform integrability of $n^{-1/2}|M^{(n)}|$ implies that of $n^{-1/2} L_n$. Combining (4.6) and (4.7),

$$n^{-1/2} E[L'_n] \leq n^{-1/2} E[L_n] \to E[f_c(\eta)]$$

for $n \geq n_c$; (iv) follows, since $c > 0$ is arbitrary and $E[f_c(\eta)]$ decreases to 0 as c increases to ∞. ■

4.3. "Aligned" Versus "Exact" Runs

In view of Proposition 4.2, we have a decomposition of $T^{(n)}$, the number of runs with respect to $\xi_{1/2} = 0$ or "exact" runs, into

$$\begin{aligned} T^{(n)} = &\sum_{\substack{t-1 \in S_n \\ t \in S_n}} \sigma(s_t, s_{t-1}) + \sum_{\substack{t-1 \in S_n \\ t \in C_n}} \sigma(s_t, s_{t-1}) \\ &+ \sum_{\substack{t-1 \in C_n \\ t \in S_n}} \sigma(s_t, s_{t-1}) + \sum_{\substack{t-1 \in C_n \\ t \in C_n}} \sigma(s_t, s_{t-1}) + t^{(n)}, \end{aligned} \tag{4.8}$$

where $t^{(n)}$ corresponds to those (two, at most) sign changes in which $X^{(n)}_{1/2}$ might be involved— this term of course is $O_P(1)$. An analogous decomposition takes place for the number $\hat{T}^{(n)}$ of runs with respect to $X^{(n)}_{1/2}$, where s_t is replaced with $\hat{s}_t$.

PROPOSITION 4.6

$$\sum_{\substack{t-1\in\mathcal{S}_n\\ t\in\mathcal{S}_n}} [\sigma(s_t, s_{t-1}) - \sigma(\hat{s}_t, \hat{s}_{t-1})] = 0.$$

Proof: From Proposition 4.2 (i), all stable points are well-classified.

PROPOSITION 4.7

(*i*) $$\sum_{\substack{t-1\in\mathcal{C}_n\\ t\in\mathcal{S}_n}} [\sigma(s_t, s_{t-1}) - \sigma(\hat{s}_t, \hat{s}_{t-1})] = O_P(n^{1/4});$$

(*ii*) $$\sum_{\substack{t-1\in\mathcal{S}_n\\ t\in\mathcal{C}_n}} [\sigma(s_t, s_{t-1}) - \sigma(\hat{s}_t, \hat{s}_{t-1})] = O_P(n^{1/4}).$$

Proof: The proof is given for (i); for (ii), it is completely similar. Put

$$\zeta^{(n)} = \sum_{\substack{t-1\in\mathcal{C}_n\\ t\in\mathcal{S}_n}} [\sigma(s_t, s_{t-1}) - \sigma(\hat{s}_t, \hat{s}_{t-1})] = \sum_{\substack{t-1\in\mathcal{C}_n\\ t\in\mathcal{S}_n}} I_t, \text{ say.} \qquad (4.9)$$

Since (Proposition 4.2 (i)) $s_t = \hat{s}_t$, I_t is a random variable taking values -1, 0, or 1 according to the following scheme:

(a) either $t-1 \in \mathcal{C}^{(n)} \cap \mathcal{W}^{(n)}$; this happens (Proposition 4.2 (ii)) iff $s_{t-1} = -\operatorname{sgn}(M^{(n)})$: then $I_t = 0$;
(b) or $t-1 \in \mathcal{C}^{(n)} \cap \mathcal{U}^{(n)}$, and $s_t = \operatorname{sgn}(M^{(n)})$; this implies (Proposition 4.2 (iii)) $s_{t-1} = \operatorname{sgn}(M^{(n)})$, thus $I_t = -1$;
(c) or $t-1 \in \mathcal{C}^{(n)} \cap \mathcal{U}^{(n)}$, and $s_t = -\operatorname{sgn}(M^{(n)})$; again, from Proposition 4.2 (iii), $s_{t-1} = \operatorname{sgn}(M^{(n)})$, and thus $I_t = 1$.

Conditionally on $M^{(n)}$, we thus have, from Proposition 4.1 (ii) and 4.1 (i), respectively,

$$P[I_t = 0] = \frac{1}{2}, \quad P[I_t = -1] = \frac{m}{2(n - |M^{(n)}|)},$$

and

$$P[I_t = 1] = \frac{m - |M^{(n)}| + 1}{2(n - |M_n|)},$$

so that

$$E[I_t|M^{(n)}] = \frac{1 - |M^{(n)}|}{2(n - |M^{(n)}|)}.$$

Note that, still conditionally on $M^{(n)}$, the number of terms in (4.9) is fixed, and bounded from above by $|M^{(n)}| - 1$; hence, using the fact that $2(n - |M^{(n)}|) \geq n - 1$,

$$\begin{aligned} |E[\zeta^{(n)}]| &= |E[E[\zeta^{(n)}|M^{(n)}]]| \leq E[(|M^{(n)}| - 1)|E[I_t|M^{(n)}]|] \\ &= E\left[\frac{(|M^{(n)}| - 1)^2}{2(n - |M^{(n)}|)}\right] \leq E\frac{(|M^{(n)}| - 1)^2}{(n-1)}, \end{aligned} \tag{4.10}$$

a quantity which, from Proposition 4.1 (v) and 4.5 (iii), is O(1) as $n \to \infty$. For $(\zeta^{(n)})^2$, we have

$$(\zeta^{(n)})^2 = \sum_t I_t^2 + \sum_{t' \neq} \sum_{t''} I_{t'} I_{t''}.$$

Similar reasoning as above yields $P[I_t^2 = 1] = P[I_t^2 = 0] = 1/2$, conditionally on $M^{(n)}$ as well as unconditionally; hence,

$$E[\sum_t I_t^2] = E\left[\sum_t E[I_t^2|M^{(n)}]\right] \leq E\left[\frac{1}{2}(|M^{(n)}| - 1)\right], \tag{4.11}$$

which, according to Proposition 4.1 (v), is $O(n^{1/2})$. For the cross-products (note that, since $t \in \mathcal{S}^{(n)}$ and $t - 1 \in \mathcal{C}^{(n)}$, the sets of summands in $I_{t'}$ and $I_{t''}$ are mutually disjoint), we have, still from Proposition 4.1 (i) and (ii), conditionally on $M^{(n)}$,

$$\begin{aligned} P[I_{t'}I_{t''} = 1] &= \frac{(m - |M^{(n)}| + 1)(m - |M^{(n)}|) + m(m-1)}{4(n - |M^{(n)}|)(n - |M^{(n)}| - 1)}, \\ P[I_{t'}I_{t''} = -1] &= \frac{2m(m - |M^{(n)}| + 1)}{4(n - |M^{(n)}|)(n - |M^{(n)}| - 1)}, \\ P[I_{t'}I_{t''} = 0] &= \frac{3}{4}. \end{aligned}$$

Hence,

$$\begin{aligned}\left|E\Big(\sum_{t'\neq}\sum_{t''} I_{t'}I_{t''}\Big)\right| &\leq E\Big(\sum_{t'\neq}\sum_{t''} |E[I_{t'}I_{t''}|M^{(n)}]|\Big)\\ &\leq E\Bigg[\frac{1}{2}(|M^{(n)}|-1)(|M^{(n)}|-2)\\ &\qquad \left|\frac{(M^{(n)})^2-|M^{(n)}|-2m}{4(n-|M^{(n)}|)(n-|M^{(n)}|-1)}\right|\Bigg]\\ &\leq \frac{1}{2}E\frac{(M^{(n)})^4+|M^{(n)}|^3+n(M^{(n)})^2}{(n-3)^2}\end{aligned} \tag{4.12}$$

which, in view of Proposition 4.5 (iii) is O(1). Piecing together (4.10), (4.11) and (4.12) yields the desired result.

We now turn to the last term in (4.8), where $t-1$ and t both are running over $\mathcal{C}^{(n)}$. This term can be decomposed further into

$$\begin{aligned}\sum_{\substack{t-1\in\mathcal{C}_n\\ t\in\mathcal{C}_n}}\sigma(s_t,s_{t-1}) &= \sum_{\substack{t-1\in\mathcal{W}_n\cap\mathcal{C}_n\\ t\in\mathcal{W}_n\cap\mathcal{C}_n}}\sigma(s_t,s_{t-1}) + \sum_{\substack{t-1\in\mathcal{W}_n\cap\mathcal{C}_n\\ t\in\mathcal{U}_n}}\sigma(s_t,s_{t-1})\\ &+ \sum_{\substack{t-1\in\mathcal{U}_n\\ t\in\mathcal{W}_n\cap\mathcal{C}_n}}\sigma(s_t,s_{t-1}) + \sum_{\substack{t-1\in\mathcal{U}_n\\ t\in\mathcal{U}_n}}\sigma(s_t,s_{t-1});\end{aligned} \tag{4.13}$$

$\hat{T}^{(n)}$ decomposes similarly.

PROPOSITION 4.8

(i) $\displaystyle\sum_{\substack{t-1\in\mathcal{C}_n\\ t\in\mathcal{C}_n}}\sigma(\hat{s}_t,\hat{s}_{t-1}) = 0;$

(ii) $\displaystyle\sum_{\substack{t-1\in\mathcal{U}_n\\ t\in\mathcal{U}_n}}\sigma(s_t,s_{t-1}) = 0 = \sum_{\substack{t-1\in\mathcal{U}_n\\ t\in\mathcal{U}_n}}\sigma(\hat{s}_t,\hat{s}_{t-1});$

(iii) $\displaystyle\sum_{\substack{t-1\in\mathcal{W}_n\cap\mathcal{C}_n\\ t\in\mathcal{W}_n\cap\mathcal{C}_n}}\sigma(s_t,s_{t-1}) = 0 = \sum_{\substack{t-1\in\mathcal{W}_n\cap\mathcal{C}_n\\ t\in\mathcal{W}_n\cap\mathcal{C}_n}}\sigma(\hat{s}_t,\hat{s}_{t-1}).$

Proof For (i), note that all values of t and $t-1$ involved in the sum correspond to critical observations; hence, by Proposition 4.2 (ii), $\hat{s}_t = -\text{sign}(M^{(n)}) = \hat{s}_{t-1}$. Similarly, in (ii), $-\hat{s}_t = s_t = \text{sgm}(M^{(n)}) = s_{t-1} = -\hat{s}_{t-1}$ (Proposition 4.2 (iii)) whereas in (iii), $s_t = \hat{s}_t = -\text{sign}(M^{(n)}) = \hat{s}_{t-1} = s_{t-1}$ (Proposition 4.2 (ii)).

If, however, s_t is substituted for $\hat{s}_t$ in part (i) of Proposition 4.8 the result can be extremely different from zero. Let $\mathcal{P}: \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ be a permutation. A *contact function* $K_{\mathcal{P}}$ of $\mathcal{P}$ [the name is taken from Wolfowitz (1942) who apparently first investigated this aspect of permutations] is a function whose value at $k \in \{1,2,\ldots,n\}$ indicates the number of *contacts*:

$$K_{\mathcal{P}}(k) = \#\{t | \mathcal{P}(t) \leq k \quad \text{and} \quad \mathcal{P}(t+1) \leq k\}.$$

It is easily seen that $K_{\mathcal{P}}(k) \leq k-1$, where the bound is attained, for instance, at the identity permutation. Also note that

$$K_{\mathcal{P}}(k+1) - 2 \leq K_{\mathcal{P}}(k) \leq K_{\mathcal{P}}(k+1). \tag{4.14}$$

The contact function $K_{\mathbf{R}_+^{(n)}}$ of the absolute ranking permutation $\mathbf{R}_+^{(n)}$ is crucial in the evaluation of the value of the *alignment bias*

$$\begin{aligned} T^{(n)} - \hat{T}^{(n)} &= \sum_{\substack{t-1 \in C_n \\ t \in C_n}} [\sigma(s_t, s_{t-1}) - \sigma(\hat{s}_t, \hat{s}_{t-1})] \\ &= \sum_{\substack{t-1 \in C_n \\ t \in C_n}} \sigma(s_t, s_{t-1}). \end{aligned}$$

PROPOSITION 4.9 *Conditionally on* $M^{(n)}$, *the alignment bias is Bin* $(K_{\mathbf{R}_+^{(n)}}(|M^{(n)}| - 1), 1/2)$.

Proof By Proposition 4.3, $T^{(n)} - \hat{T}^{(n)}$ is a sum of independent Bin (1; 1/2) variables I_t; the number of terms in this sum depends on $|M^{(n)}|$, and is precisely $K_{\mathbf{R}_+^{(n)}}(|M^{(n)}| - 1)$. ■

4.4. Asymptotic Equivalence of Aligned and "Exact" Runs Statistics

We are now ready to characterize the behaviour of the difference $T^{(n)} - \hat{T}^{(n)}$ between the "exact" and aligned runs statistics. The following theorem summarizes the results of sections 4.1 through 4.3.

THEOREM 4.10 *Conditionally on* $|X_1^{(n)}|, |X_2^{(n)}|, \ldots, |X_n^{(n)}|$,

$$T^{(n)} - \hat{T}^{(n)} = B_n + O_P(n^{1/4}),$$

under $H^{(n)} = \bigcap_{\xi_{1/2} \in \mathbf{R}} H^{(n)}(\xi_{1/2})$, *as* $n \to \infty$, *where* B_n *is a variable with nonnegative integer values, whose distribution is a mixture of binomials. More precisely,* $B_n \sim Bin(K_{\mathbf{R}_+^{(n)}}(C_n), 1/2)$, *where* $K_{\mathbf{R}_+^{(n)}}$ *is the contact function of* $\mathbf{R}_+^{(n)}$,

$$P(C_n = k) = \begin{cases} \binom{n-k}{[\frac{n}{2}]} \left(\frac{1}{2}\right)^{n-k} & k = 1, 2, \ldots, n - [n/2], \\ 0 & elsewhere \end{cases}$$

and $\mathbf{R}_+^{(n)}$ *and* C_n *are independent (conditionally on* $|\mathbf{X}^{(n)}|$ *as well as unconditionally).*

Proof This statement is a consequence of Propositions 4.6–4.9, and the decomposition (4.8). Note that, in view of (4.14), the effect of substituting C_n for $|M^{(n)}| - 1$ is $O_P(n^{1/4})$. ■

Now, the asymptotic equivalence result we are hoping for of course is that $T^{(n)} - \hat{T}^{(n)}$ is *unconditionally* $O_P(n^{1/2})$. Does Theorem 4.10 allow for such a result ? As shown by the following corollary, the answer, disappointingly, is negative.

COROLLARY 4.11 *Still under* $H^{(n)}$, *as* $n \to \infty$,

$$T^{(n)} - \hat{T}^{(n)} = O_P(n^{1/2}), \tag{4.15}$$

and this result is sharp.

Proof The first part of the statement actually is trivial, since

$$|T^{(n)} - \hat{T}^{(n)}| \leq 2U^{(n)}, \tag{4.16}$$

where $U^{(n)}$ denotes the number of misclassified observations. By Proposition 4.4, $2U^{(n)} = |V^{(n)} - \frac{n}{2}| - \frac{1}{2}$, where (conditionally on $|M^{(n)}|$), $V^{(n)} \sim$ Bin $(|M^{(n)}| - 1;\ 1/2)$. The uniform integrability of $U^{(n)}$ can be proved along the same lines as in the proof of Proposition 4.5 (iii), which via (4.16) implies the uniform integrability of $T^{(n)} - \hat{T}^{(n)}$ and (4.15). It thus follows from Theorem 4.10 and Proposition 4.1 (v) that, if $K_{\mathbf{R}_+^{(n)}}(k) = k - 1$ with probability one, then

$$\lim_{n\to\infty} n^{-1/2}E[T^{(n)} - \hat{T}^{(n)}] = \lim_{n\to\infty} n^{-1/2}E[E[B_n|C_n]]$$
$$= \lim_{n\to\infty} n^{-1/2}E\left[\frac{1}{2}(C_n - 1)\right] = (2\pi)^{-1/2}, \tag{4.17}$$

which establishes that (4.15), under such unfavorable contact function, cannot be improved upon. Now, consider the case when $|X_t^{(n)}| = t$ with probability one, whereas the signs $s_t^{(n)} = \text{sgn}(X_t^{(n)})$ are *i.i.d.* symmetric Bernoulli variables: this yields the contact function $K_{\mathbf{R}_+^{(n)}}(k) = k - 1$, thus the unpleasant asymptotic behavior (4.17). Of couse, this discontinuous case, involving Dirac measures at t, does not occur under $H^{(n)}$; but replacing the Dirac measure with a continuous distribution concentrated on a short interval centered at t can be achieved in such a way that $\mathbf{R}_+^{(n)}$, hence its contact function, remains almost surely unaffected. Note that the fact that t is unbounded as $n \to \infty$ has no influence here: replacing t with $g\ (t)$, where g is an arbitrary continuous, order-preserving, possibly bounded function does not affect the argument. ■

Asymptotic equivalence (up to $o_P\ (n^{1/2})$ terms for $T^{(n)} - \hat{T}^{(n)}$, up to $o_P(n^{-1/2})$ terms for $S^{(n)} - \hat{S}^{(n)}$) thus does not hold under $H^{(n)}$. However, in order to attain the bound in (4.15), a very special "heteroscedasticity pattern" had to be assumed. One may argue that such a pattern is highly artifical, and does not enter the intuitive scope of heterogeneous white noise. If we consider, for instance, the particular case of homogeneous white noise ($X_t^{(n)}$ *i.i.d.*, with unspecified median $\xi_{1/2}$) and unspecified density f, we have the following result.

THEOREM 4.12 *Under homogeneous white noise, as $n \to \infty$,*

$$T^{(n)} - \hat{T}^{(n)} = O_P(n^{1/4}).$$

Proof Conditional on C_n,

$$E[B_n|C_n| = \sum_{t:R_{+;t}^{(n)} \le C_n} E[I_t|C_n], \tag{4.18}$$

where $I_t = I[R_{+;t+1}^{(n)} \le C_n]$. Since $\mathbf{R}_+^{(n)}$ now is uniformly distributed over the $n!$ permutations of $\{1,2,\ldots,n\}$, for all values of t involved in (4.18),

$$E[I_t|C_n] = P[I_t = 1|C_n] \leq \frac{C_n - 1}{n - 1}$$

(inequality is due to the possibility of t being equal to n). Hence, by Proposition 4.5 (iii) and 4.1 (iv), as $n \to \infty$,

$$E(B_n) = E[E[B_n|C_n]] = \frac{C_n^2 - C_n}{n - 1} = O(1).$$

The Markov inequality thus implies that $B_n = O_P(1)$, and the result directly follows from Theorem 4.10.

In most realistic situations, it is plausible that observations do not follow such contrived patterns as in the proof of Corollary 4.11. Though we still would like to allow for very general heteroscedasticity patterns, it may be reasonable to rule out "too organized" *inliers*, occurring in patches. Formalizing this feature seems quite delicate, but intuitively acceptable sufficient conditions (assumptions C1 and C2 below) can be stated. A careful potential user can decide about the acceptability of such conditions, possibly by investigating the patchy behaviour of inlying observations.

Condition C1 relaxes the quadratic $O(n)$ growth condition on $K_{\mathbf{R}_+^{(n)}}(k)$ in Theorem 4.12 into a condition which asymptotically involves $K_{\mathbf{R}_+^{(n)}}(n/2), K_{\mathbf{R}_+^{(n)}}(n/3), \ldots$ only. This condition basically requires that the probability of two successive observations having "nearby" small absolute ranks should be "small".

(C1) For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \sup_{t<n} P\left[R_{+;t+1}^{(n)} \leq n\delta \middle| R_{+;t}^{(n)} \leq n\delta\right] < \varepsilon.$$

Conditional probabilities are considered rather than unconditional ones, since the behavior of *outlying* values of $R_{+;t}^{(n)}$ is quite irrelevant. Note that in the case of homogeneous symmetric white noise, δ can be taken approximatively equal to ε.

Condition C1 also can be interpreted as a form of nondegeneracy in the vicinity of zero of the joint distribution of two successive observations. Of somewhat simpler appearance is

(**C2**) There exists $\varepsilon > 0$ such that

$$\sup_{t<n}\ \sup_{\gamma,\rho<n\varepsilon} P\left[R^{(n)}_{+;t+1} = \rho | R^{(n)}_{+;t} = r\right] = o(n^{-1/2}).$$

The interpretation is similar to that of C1; for homogeneous symmetric white noise, C2 is satisfied with rate $O(n)$. It can be checked that none of the conditions C1 or C2 implies, or is implied, by the other one (proof is omitted).

Denote by $H^{(n)}_1$ and $H^{(n)}_2$ the intersections of $H^{(n)}$ with assumptions C1 and C2, respectively. Under $H^{(n)}_1$ and $H^{(n)}_2$, we have the following asymptotic equivalence result, which implies that the exact and aligned versions of $S^{(n)}_{+;k}$ differ only by $o_P\,(n^{-1/2})$ terms.

THEOREM 4.13 *Both under* $H^{(n)}_1$ *and* $H^{(n)}_2$, *as* $n \to \infty$,

$$T^{(n)} - \hat{T}^{(n)} = o_P(n^{1/2}).$$

Proof In view of Theorem 4.10 and Proposition 4.9, it is sufficient to show that C1 and C2 both imply that $E(B_n)$ is $o(n^{1/2})$; the theorem then follows on applying the Markov inequality. Now,

$$E(B_n) = E[E[B_n|C_n]] = \frac{1}{2}E\left[K_{\mathbf{R}^{(n)}_+}(C_n)\right]. \tag{4.19}$$

For fixed k, we have

$$E(K_{\mathbf{R}^{(n)}_+}(k)) = \sum_{t:R^{(n)}_{+;t}\leq k} E[I_t],$$

where $I_t = I[R^{(n)}_{+;t+1} \leq k]$.

Let C1 hold. For given $\varepsilon > 0$ and $k < n\delta$,

$$E[K_{\mathbf{R}^{(n)}_+}(k)] \leq \sum_{t:R^{(n)}_{+;t}\leq k} \varepsilon \leq k\varepsilon, \tag{4.20}$$

whereas, for $k \geq n\delta$, trivially,

$$E[K_{\mathbf{R}^{(n)}_+}(k)] \leq (k-1) < k. \tag{4.21}$$

Combining (4.19), (4.20) and (4.21) yields

$$\begin{aligned} E(B_n) &\leq \frac{1}{2}\left(\varepsilon \sum_{k<n\delta} k\nu_k + \sum_{k\geq n\delta} k\nu_k\right) \\ &\leq \frac{1}{2}\varepsilon E(C_n) + \frac{1}{2}\sum_{k\geq n\delta} k\nu_k. \end{aligned} \tag{4.22}$$

By Proposition 4.5 (iv), the last sum in (4.22) is $o(n^{1/2})$, since $n^{-1/2}n\delta = n^{1/2}\,\delta \to \infty$. Hence, by Proposition 4.1 (v),

$$\limsup_{n\to\infty} n^{-1/2}E(B_n) \leq \limsup_{n\to\infty} \frac{1}{2} n^{-1/2}\varepsilon E(C_n) \leq \varepsilon(2\pi)^{-1/2}.$$

Since $\varepsilon > 0$ is arbitrarily small, the result follows.

Let C2 hold. Instead of (4.20), we have, for all $k < n\varepsilon$,

$$E[K_{\mathbf{R}_+^{(n)}}(k)] \leq \sum_{t:R_{+;t}\leq k} n^{-1/2}\gamma_n k = n^{-1/2}\gamma_n k^2, \tag{4.23}$$

where, according to C2, $\gamma_n \to 0$ as $n \to \infty$. For $k > n\varepsilon$, (4.21) still holds. Hence by Proposition 4.5 (iii) and (iv),

$$\begin{aligned} \lim_{n\to\infty} n^{-1/2}E(B_n) &= \lim_{n\to\infty} n^{-1}\gamma_n \sum_{k<n\varepsilon} k^2\nu_k + n^{-1/2}\sum_{k\geq n\varepsilon} k\nu_k \\ &\leq \lim_{n\to\infty} n^{-1}\gamma_n E(C_n^2) = 0. \end{aligned}$$

The proof is thus complete. ■

As a consequence, the run-based correlograms (along with their p-values) and tests described in Sections 3.2 and 3.3 still remain approximately valid when aligned runs are substituted for the exact ones.

5. A CONSERVATIVE FINITE-SAMPLE PROCEDURE

The alignment device described in Section 4 is of an asymptotic nature, and the resulting runs tests are only asymptotically valid: for fixed n, there is no strict guarantee that the probability of type-I error does not exceed the nominal level α. The procedure we are describing in this section provides strictly conservative tests for fixed n. It is based on an idea used, in a totally different context, in Dufour (1990) and Campbell and Dufour (1997), and mainly consists in considering a confidence interval for $\xi_{1/2}$ instead of the point estimate $X^{(n)}_{1/2}$.

Denote by

$$T^{(n)}(\xi^{0}_{1/2}) = \sum_{i=1}^{n} I[X^{(n)}_t - \xi^{0}_{1/2} \geq 0]$$

the traditional *sign test* statistic for testing $H^{(n)}\ (\xi^{0}_{1/2})$ against location alternatives. Clearly, $T^{(n)}(\xi^{0}_{1/2})$ is Bin $(n;\ 1/2)$ under $H^{(n)}\ (\xi^{0}_{1/2})$. It follows that the set

$$C(\alpha_1) = \left\{\xi | n - b^{+}_n\left(\frac{\alpha_1}{2}\right) \leq T^{(n)}(\xi) \leq b^{+}_n\left(\frac{\alpha_1}{2}\right)\right\}, \tag{5.1}$$

where

$$b^{+}_n(\alpha) = \inf\{m \in \mathbb{N} | P[Bin(n; 1/2) \leq m] \geq 1 - \alpha\},$$

is a confidence region for $\xi_{1/2}$, at confidence level $1 - \alpha_1$, under $H^{(n)}$. Letting $X^{(n)}_{(r)}$ denote the rth order statistic computed from $\mathbf{X}^{(n)}$, the same confidence region with probability one also takes the form $[X^{(n)}_{(m*+1)}; X^{(n)}_{(n-m*)}]$, with

$$m^{*} = n - b^{+}_n\left(\frac{\alpha_1}{2}\right) - 1 = \sup\left\{m \ \in \ \mathbb{N} | P[Bin(n; 1/2) \leq m] \leq \frac{\alpha_1}{2}\right\};$$

see David (1981, Chapter 2), or Hettmansperger (1984, pp. 12–15).

Letting

$$S^{(n)}_{+;k}(\xi) = (n-k)^{-1} \sum_{t=k+1}^{n} I\big[(X^{(n)}_t - \xi)(X^{(n)}_{t-k} - \xi) < 0\big],$$
$$\xi \ \in \ \mathbb{R}, k = 1, \ldots, n-1,$$

consider the two statistics

$$m^{(n)}_k(C(\alpha_1)) = \inf\left\{\left|S^{(n)}_{+;k}(\xi) - \frac{1}{2}\right| \ \middle|\ \xi \in C(\alpha_1)\right\} \tag{5.2}$$

and

$$M_k^{(n)}(\mathcal{C}(\alpha_1)) = \sup\left\{ \left| S_{+;k}^{(n)}(\xi) - \frac{1}{2} \right| \,\middle|\, \xi \in \mathcal{C}(\alpha_1) \right\} \tag{5.3}$$

for testing the null hypothesis $H^{(n)}$ against two-sided alternatives of serial dependence at lag k. Then, defining

$$c_k^{(n)}(\alpha) = (n-k)^{-1} b_{n-k}^{+}(\alpha) - \frac{1}{2},$$

for all α_1 and α_2 in (0, 1),

$$\begin{aligned}
&P\left[m_k^{(n)}(\mathcal{C}(\alpha_1)) > c_k^{(n)}\left(\frac{\alpha_2}{2}\right)\right] \\
&\quad \le P\left[m_k^{(n)}(\mathcal{C}(\alpha_1)) > c_k^{(n)}\left(\frac{\alpha_2}{2}\right) \text{ and } \xi_{1/2} \in \mathcal{C}(\alpha_1)\right] \\
&\quad + P\left[m_k^{(n)}(\mathcal{C}(\alpha_1)) > c_k^{(n)}\left(\frac{\alpha_2}{2}\right) \text{ and } \xi_{1/2} \notin \mathcal{C}(\alpha_1)\right] \\
&\quad \le P\left[\left| S_{+;k}^{(n)}(\xi) - \frac{1}{2} \right| > c_k^{(n)}\left(\frac{\alpha_2}{2}\right)\right] + P[\xi_{1/2} \notin \mathcal{C}\,(\alpha_1)] \\
&\quad \le \alpha_2 + \alpha_1.
\end{aligned}$$

If we thus select α_1 and α_2 such that $\alpha_1 + \alpha_2 = \alpha$ (e.g., $\alpha_1 = \alpha_2 = \alpha/2$), the test rejecting the null hypothesis $H^{(n)}$ whenever

$$\left| S_{+;k}^{(n)}(\xi) - \frac{1}{2} \right| > c_k^{(n)}\left(\frac{\alpha_2}{2}\right) \text{ for all } \xi \in \mathcal{C}(\alpha_1) \tag{5.5}$$

is conservative at probability level α.

Similarly, for all α_1 and α_2' in (0, 1),

$$\begin{aligned}
&P\left[M_k^{(n)}(\mathcal{C}(\alpha_1)) \le c_k^{(n)}\left(\frac{\alpha_2'}{2}\right)\right] \\
&\quad \le P\left[\left| S_{+;k}^{(n)}(\xi) - \frac{1}{2} \right| \le c_k^{(n)}\left(\frac{\alpha_2'}{2}\right)\right] + P\left[\xi_{1/2} \notin \mathcal{C}(\alpha_1)\right] \\
&\quad \le 1 - (\alpha_2' - \alpha_1).
\end{aligned} \tag{5.6}$$

Thus, if we select α_1 and α_2' such that $\alpha = \alpha_2' - \alpha_1$, the test rejecting the null hypothesis $H^{(n)}$ whenever

$$\left| S_{+;k}^{(n)}(\xi) - \frac{1}{2} \right| > c_k^{(n)}\left(\frac{\alpha'_2}{2}\right) \text{ for some } \xi \in \mathcal{C}(\alpha_1) \tag{5.7}$$

is *liberal* at probability level α, i.e., its probability of *not* rejecting $H^{(n)}$ is never greater than $1 - \alpha$ under $H^{(n)}$.

This suggest the following generalized bounds test. Select α_1, α_2 and α_2' such that $\alpha = \alpha_1 + \alpha_2$ and $\alpha = \alpha_2' - \alpha_1$ (e.g., let $\alpha = 0.05$, $\alpha_1 = \alpha_2 = 0.025$ and $\alpha_2' = 0.075$, and

$$\text{reject } H^{(n)} \text{ whenever } m_k^{(n)}(\mathcal{C}(\alpha_1)) > c_k^{(n)}\left(\frac{\alpha_2}{2}\right)$$
$$\textit{accept} H^{(n)} \text{whenever} M_k^{(n)}(\mathcal{C}(\alpha_1)) \leq c_k^{(n)}\left(\frac{\alpha_2'}{2}\right).$$

Unilateral versions of the above testing procedures are easily obtained. The same ideas also adapt to the portmanteau tests described in Section 3.4. We refer to Dufour (1989, 1990) for further discussion of generalized bounds tests.

6. SIMULATIONS

Asymptotic results, as those derived in the previous sections, are only a substitute for unavailable approximation results. In order to assess the validity of aligned runs methods in finite sample situations, an extensive simulation study was carried out on a variety of heteroscedastic series. Due to space limitations, we only report part of them here; additional figures, as well as the XLISPSTAT code used in the simultations, can be obtained from the WWW location[1] http: // www. dcs. fmph. uniba.sk/~mizera/dhm.html.

This section mainly consists of two parts. Part one deals with the behaviour of runs, aligned runs and their traditional autocorrelation competitors under various null hypotheses of homogeneous and

[1]Thus, we hope to conform the rules on the reproducibility of numerical investigations, as postulated by Claerbout, Buckheit, Donoho, and others.

heterogeneous white noise, hence with their finite-sample validity as test statistics for the null hypothesis of independent, possibly nonidentically distributed observations. Part two investigates the finite-sample power of the corresponding tests under alternatives of first-order serial dependence.

Similar simulation experiments have been carried out at higher lags, but, as expected, did not bring anything new (when compared with their higher lag parametric and rank-based counterparts) except of course for the fact that, unlike classical runs, they efficiently detect higher lag serial dependence. We also explored the possibility of using the sample mean as an alternative to the empirical median in the alignment process. The rejection frequencies under heteroscedastic null hypotheses were absolutely disastrous (as could be expected): we strongly recommend not considering this device. We do not report these results here: interested readers are referred to the mentioned web pages.

6.1. Finite-Sample Behaviour Under the Null

$N = 10\,000$ replications of series of independent observations, of lengths $n = 30$, 60 and 120, respectively, were obtained under five heteroscedasticity patterns (homoscedasticity, playing the role of a control, two *stable* heteroscedasticity patterns and three *expanding* ones):

(i) *i.i.d.* (*homoscedastic*): $X_t = \eta_t$, where $\eta_1,\ldots,\eta_n$ are *i.i.d.*, with centered Gaussian and Cauchy densities, respectively (note that all procedures considered here are invariant under scale transformations of η_t);

(ii) *Heteroscedastic* (*change-point type*): $X_t = \eta_t = c_t U_t$, where $U_1,\ldots,U_n$ are *i.i.d.*, with centered Gaussian and Cauchy densities as above, and $c_t = I\,[t \leq n/2] + 4I\,[t > n/2]$;

(iii) *Heteroscedastic* (*alternate type*): $X_t = \eta_t = c_t U_t$, where $U_1,\ldots,U_n$ are *i.i.d.*, with centered Gaussian and Cauchy densities as above, $c_t = I\,[t \text{ odd}] + 4I\,[t \text{ even}]$;

(iv) *Expanding heteroscedastic* ($\sqrt{t}$ *rate*): $X_t = \eta_t = c_t U_t$, where $U_1,\ldots U_n$ are *i.i.d.*, with centered Gaussian and Cauchy densities as above, $c_t = \sqrt{t}$;

(v) *Expanding heteroscedastic* (*t rate*): $X_t = \eta_t = c_t U_t$, where $U_1, \ldots, U_n$ are *i.i.d.*, with centered Gaussian and Cauchy densities as above, $c_t = t$;

(vi) *Expanding heteroscedastic* (t^2 *rate*): $X_t = \eta_t = c_t U_t$, where $U_1, \ldots U_n$ are *i.i.d.*, with centered Gaussian and Cauchy densities as above, $c_t = t^2$.

An exponenially expanding pattern $X_t = \eta_t = c_t U_t$, with $c_t = e^t$ also has been considered, for the sake of comparison with previous simulation result by Dufour (1981). The behaviour of aligned runs was quite the same as under the t^2 expanding case (vi), the deterioration of parametric and rank-based methods even more pronounced. Such a pattern looking quite extreme and unrealistic, we decided to omit it; once again, we refer to the mentioned web pages.

Each of the series thus obtained was subjected to the following testing procedures:

(a) *Classical*, exact, first-order *runs test* (based on the observed signs): exact binomial critical values have been considered, which explains the *natural* levels of $\alpha = 0.06143$ for $n = 30$, $\alpha = 0.06744$ for $n = 60$ and $\alpha = 0.06629$ for $n = 120$; these levels have been selected as the smallest ones beyond 5% allowing for exact nonrandomized two-sided binomial tests;

(b) *Aligned runs* (based on the signs of the differences $X_t - X^{(n)}_{1/2}$): same critical values as in (a);

(c) *Traditional first-order autocorrelations*: critical values based on the asymptotic standard normal distribution of $n^{\frac{1}{2}} r_1^{(n)}$, at the same probability levels as in (a) and (b);

(d) *Spearman first-order autocorrelations*: same critical values as in (c).

Only the results obtained for standard normal and Cauchy densities are reported here. Student t with 3 degrees of freedom, truncated Cauchy and centered lognormal distributions also were investigated, but it turned out that the impact on rejection frequencies of the density type is relatively small compared with that of the heteroscedasticity pattern. Therefore, we decided to concentrate on normal and Cauchy densities, the behaviors of which in some sense appear to be extremal: the observed rejection rates for the other density types typically are intermediate between the normal and the Cauchy ones, and we could not spot any particular effect of asymmetry. As already mentioned, higher-order procedures also have been studied, yielding exactly the same conclusions as the first-order ones. Still for the sake of saving space, we do not report them here.

The resulting rejection frequencies are reported in Table 6.1. A rapid inspection reveals that they essentially support the idea that the asymptotic results derived in the previous sections for aligned runs statistics provide reasonable approximations to their finite-sample behaviours (note that $n = 30$ and $n = 60$ still are very short series lengths). Significant (in a two-sided test at probability level 1%) deviations from the nominal sizes α are identified with $^{-}$ (for underrejections) and $^{+}$ (for overrejections) subscripts, respectively. The very severe deviations (allowing for rejecting the null hypothesis that the nominal level is correct at probability level 0.001%) are marked as $^{-*}$ and $^{+*}$, respectively.

The exact runs column is somewhat particular. The binomial distributions on which it is based indeed is an exact distribution, and only one out of the 36 rejection percentages shown in the table differs significantly from the nominal size. The three rows corresponding to *i.i.d.* observations also bring relatively little surprise: still, the asymptotics of aligned runs seem much closer to their finite-sample behaviour, even under normal densities, than for traditional and rank-based correlograms (note the significant underrejection phenomenon for traditional first-order autocorrelation, indicating strictly conservative, hence also significantly biased tests). Under Cauchy densities, whereas the size of the correlogram-based test collapses down to about half its nominal level (revealing a severe bias), aligned runs successfully maintain their nominal rejection rates, thus exhibiting remarkable robustness against heavy-tailed observations. As expected, Spearman autocorrelations also behave quite well, under both Cauchy and normal densities.

It should be insisted that a test obtaining a $^{+*}$ mark simply and definitely should be ruled out in the nonparametric context considered here. Such a test just does not qualify, since it size fails to meet, even very roughly, the nominal α-level requirement. Table 6.1 reveals that this disqualification holds for the classical Pearson autocorrelation test under *all* series lengths and heteroscedasticity patterns but the *alternated* one (under which, on the other hand, a very systematic and

TABLE 6.1 Observed rejection (type I error) frequencies in $N = 10\,000$ replications of simulated white noise series of length $n = 30$, 60, 120, under various heteroscedasticity patterns, with Normal and Cauchy parent distributions, respectively. Critical values correspond to the *natural* significance levels $\alpha = 0.06143$ for $n = 30$, $\alpha = 0.06744$ for $n = 60$ and $\alpha = 0.06629$ for $n = 120$. Significant under- and overrejection cases (allowing for rejecting the nominal size α in a two-sided 1% test) are marked as $^{+}$ and $^{-}$, respectively; *very* significant under- and overrejection cases (allowing for rejecting the nominal size α in a two-sided 0.001% test) are marked as $^{+*}$ and $^{-*}$, respectively

Heteroscedasticity patterns		*Procedure* *Runs*		*Aligned runs*		*Pearson*		*Spearman*	
	n	*Normal*	*Cauchy*	*Normal*	*Cauchy*	*Normal*	*Cauchy*	*Normal*	*Cauchy*
i.i.d.	30	.0638	.0643	.0649	.0636	.0519^{-}	.0241^{-*}	.0575	.0542^{-}
(*homoscedastic*)	60	.0685	.0682	.0716	.0732	.0626	.0336^{-*}	.0646	.0667
	120	.0661	.0653	.0705	.0685	.0681	.0305^{-*}	.0682	.0658
heteroscedastic	30	.0612	.0593	.0671	.0676	.1120^{+*}	.0439^{-*}	.0868^{+*}	.0706^{+}
(*change point*)	60	.0695	.0681	.0722	.0724	.1498^{+*}	.0484^{-*}	.1051^{+*}	.0829^{+*}
	120	.0673	.0658	.0683	.0700	.1529^{+*}	.0491^{-*}	.1068^{+*}	.0899^{+*}
heteroscedastic	30	.0642	.0627	.0722^{+*}	.0699^{+}	.0039^{-*}	.0151^{-*}	.0318^{-*}	.0423^{-*}
(*alternated*)	60	.0672	.0658	.0744^{+}	.0747^{+}	.0017^{-*}	.0219^{-*}	.0304^{-*}	.0474^{-*}
	120	.0681	.0659	.0708	.0729^{+}	.0001^{-*}	.0194^{-*}	.0281^{-*}	.0500^{-*}
expanding	30	.0614	.0612	.0634	.0681^{+}	.0787^{+*}	.0305^{-*}	.0705^{+}	.0619
($\sqrt{t}$ *rate*)	60	.0708	.0700	.0691	.0747^{+}	.0966^{+*}	.0371^{-*}	.0818^{+*}	.0718
	120	.0631	.0675	.0661	.0674	.1065^{+*}	.0386^{-*}	.0863^{+*}	.0657
expanding	30	.0589	.0607	.0702^{+}	.0662	.1073^{+*}	.0364^{-*}	.0835^{+*}	.0682^{+}
(t *rate*)	60	.0702	.0652	.0741^{+}	.0739^{+}	.1453^{+*}	.0475^{-*}	.0990^{+*}	.0863^{+*}
	120	.0670	.0729^{+}	.0700	.0743^{+}	.1519^{+*}	.0427^{-*}	.1013^{+*}	.0862^{+*}
expanding	30	.0564	.0634	.0782^{+*}	.0768^{+*}	.1713^{+*}	.0535^{-}	.1064^{+*}	.0902^{+*}
(t^2 *rate*)	60	.0640	.0649	.0878^{+*}	.0808^{+*}	.2179^{+*}	.0717	.1161^{+*}	.1074^{+*}
	120	.0660	.0659	.1020^{+*}	.0874^{+*}	.2455^{+*}	.0629	.1222^{+*}	.1067^{+*}

severe bias is observed). The Spearman autocorrelation test does not perform any better, except, perhaps, under the mild root-t expansion rate, for short series length ($n = 30$). None of the proposed procedures survives the t^2 expanding rate (except, of course, the exact runs test, since it remains totally insensitive to heteroscedasticity). Aligned runs resist all other heteroscedasticity patterns, except for the short length ($n = 30$) *alternated* one, with a significant (not dramatic, though) 1% overrejection frequency. These facts should be kept in mind in the finite-sample power discussions of Section 6.2: all rejection frequencies marked as $^{+*}$ in Tables 6.2 and 6.3 are meaningless in terms of power comparisons.

Now, the main motivation for considering runs instead of classical or rank-based autocorrelations is the possible presence of heteroscedasticity. It is plain from Table 6.1 that all correlogram-based methods behave quite miserably in that case, with rejection rates collapsing as low as 0.01% under alternated heteroscedasticity patterns (under normal density, for $n = 120$), and shooting up to 24.5% in the t^2 expanding case (still under normal density, for $n = 120$).

The size distortions (due to overlooking the fact that the empirical median was substituted for the true median) for aligned runs, quite on the contrary, appear to be rather moderate: comparing the *exact runs column* and the *aligned runs* one, it appears that the differences between rejection frequencies are generally less than 1%, and seem to disappear about $n = 120$ under *stable* heteroscedasticity patterns (*change-point and alternated* types); none of them is significant at a probability level of 1%. In the *expanding* case, these differences also remain small; they decrease with the series length, and increase with the *expanding rate*: quite understandably, expanding heteroscedasticity has a tendency to slow down the pace of asymptotics.

6.2. Finite-Sample Power

The same types of homo- and heteroscedastic white noises X_t as in the previous section were used as innovations in the simulation of MA (1) series generated from the models $X_t = \eta_t + \theta\eta_{t-1}$, with $\theta = 0.5$ and $\theta = 0.9$, successively. The same series lengths ($n = 30$, 60 and 120), the same number of replications ($N = 10\,000$) and the same four testing

TABLE 6.2 Observed rejection frequencies in $N = 10\,000$ replications of the MA (1) series of length $n = 30, 60, 120$ generated from the model $X_t = \eta_t + 0.5\eta_{t-1}$, with the same heteroscedastic white noises $\eta_t = c_t U_t$, and under the same four testing procedures as in Table 6.1. The figures associated with *very* significant underrejection rates (as shown in Table 6.1) are marked as $^{-*}$; those associated with *very* significant overrejection rates (as shown in Table 6.1) are marked as $^{+*}$, and should not be taken into account in power comparison discussions

Heteroscedasticity patterns		Procedure							
		Runs		*Aligned runs*		*Pearson*		*Spearman*	
	n	*Normal*	*Cauchy*	*Normal*	*Cauchy*	*Normal*	*Cauchy*	*Normal*	*Cauchy*
i.i.d.	30	.3199	.3886	.2197	.2872	.4892	$.5607^{-*}$	.4452	.5224
(*homoscedastic*)	60	.5903	.7063	.5148	.6265	.8972	$.9355^{-*}$	.8606	.9215
	120	.8742	.9507	.8476	.9363	.9976	$.9891^{-*}$	.9951	.9991
heteroscedastic	30	.3072	.3853	.2468	.3132	$.4690^{+*}$	$.5585^{-*}$	$.4134^{+*}$	.5070
(*change point*)	60	.5795	.7039	.5227	.6522	$.8196^{+*}$	$.9204^{-*}$	$.7776^{+*}$	$.8902^{+*}$
	120	.8707	.9476	.8497	.9372	$.9824^{+*}$	$.9843^{-*}$	$.9742^{+*}$	$.9968^{+*}$
heteroscedastic	30	.5873	.5360	$.4306^{+*}$	.3931	$.5531^{-*}$	$.6167^{-*}$	$.6071^{-*}$	$.6132^{-*}$
(*alternated*)	60	.8885	.8463	.8194	.7737	$.9675^{-*}$	$.9627^{-*}$	$.9734^{-*}$	$.9587^{-*}$
	120	.9957	.9901	.9915	.9816	$.9999^{-*}$	$.9945^{-*}$	$.9999^{-*}$	$.9999^{-*}$
expanding	30	.3031	.3848	.2235	.2974	$.4714^{+*}$	$.5706^{-*}$	.4307	.5268
($\sqrt{t}$ *rate*)	60	.5702	.7028	.5081	.6384	$.8615^{+*}$	$.9298^{-*}$	$.8278^{+*}$	.9117
	120	.8683	.9477	.8454	.9321	$.9929^{+*}$	$.9864^{-*}$	$.9894^{+*}$	.9987
expanding	30	.2779	.3742	.2399	.3109	$.4502^{+*}$	$.5495^{-*}$	$.4048^{+*}$	.5079
(t *rate*)	60	.5576	.6860	.5462	.6541	$.8200^{+*}$	$.9185^{-*}$	$.7837^{+*}$	$.8946^{+*}$
	120	.8590	.9454	.8650	.9388	$.9842^{+*}$	$.9842^{-*}$	$.9796^{+*}$	$.9971^{+*}$
expanding	30	.2589	.3651	$.3048^{+*}$	$.3646^{+*}$	$.4244^{+*}$	.5350	$.3765^{+*}$	$.4747^{+*}$
(t^2 *rate*)	60	.5137	.6701	$.5890^{+*}$	$.6972^{+*}$	$.7427^{+*}$	.8853	$.7029^{+*}$	$.8456^{+*}$
	120	.8412	.9380	$.8873^{+*}$	$.9526^{+*}$	$.9562^{+*}$	.9754	$.9493^{+*}$	$.9906^{+*}$

TABLE 6.3 Observed rejection frequencies in $N = 10\,000$ replications of the MA (1) series of length $n = 30, 60, 120$ generated from the model $X_t = \eta_t + 0.9\eta_{t-1}$, with the same heteroscedastic white noises $\eta_t = c_t U_t$, and under the same four testing procedures as in Table 6.1. The figures associated with *very* significant underrejection rates (as shown in Table 6.1) are marked as $^{-*}$; those associated with *very* significant overrejection rates (as shown in Table 6.1) are marked as $^{+*}$, and should not be taken into account in power comparison discussions

Heteroscedasticity patterns		*Procedure*							
		Runs		*Aligned runs*		*Pearson*		*Spearman*	
	n	*Normal*	*Cauchy*	*Normal*	*Cauchy*	*Normal*	*Cauchy*	*Normal*	*Cauchy*
i.i.d.	30	.4686	.4691	.3579	.3537	.7507	.8385^{-*}	.6895	.6504
(*homoscedastic*)	60	.8030	.8092	.7396	.7425	.9897	.9732^{-*}	.9803	.9730
	120	.9822	.9829	.9758	.9766	.9999	.9927^{-*}	1.0000	1.0000
heteroscedastic	30	.4632	.4698	.3714	.3904	.6972^{+*}	.8156^{-*}	.6261^{+*}	.6311
(*change point*)	60	.7920	.8074	.7427	.7560	.9575^{+*}	.9698^{-*}	.9420^{+*}	.9572^{+*}
	120	.9815	.9820	.9722	.9798	.9996^{+*}	.9906^{-*}	.9996^{+*}	.9995^{+*}
heteroscedastic	30	.6977	.5963	.5563^{+*}	.4734	.8380^{-*}	.8824^{-*}	.7900^{-*}	.7251^{-*}
(*alternated*)	60	.9549	.9030	.9228	.8504	.9987^{-*}	.9887^{-*}	.9962^{-*}	.9853^{-*}
	120	.9994	.9961	.9987	.9943	1.0000^{-*}	.9970^{-*}	1.0000^{-*}	1.0000^{-*}
expanding	30	.4727	.4811	.3738	.3753	.7253^{+*}	.8371^{-*}	.6776	.6536
($\sqrt{t}$ *rate*)	60	.7976	.8011	.7417	.7464	.9791^{+*}	.9771^{-*}	.9681^{+*}	.9641
	120	.9795	.9807	.9739	.9771	1.0000^{+*}	.9935^{-*}	1.0000^{+*}	1.0000
expanding	30	.4631	.4755	.4023	.3999	.6934^{+*}	.8116^{-*}	.6422^{+*}	.6382
(t *rate*)	60	.7976	.8039	.7784	.7794	.9607^{+*}	.9733^{-*}	.9519^{+*}	.9573^{+*}
	120	.9812	.9817	.9794	.9807	.9996^{+*}	.9941^{-*}	.9993^{+*}	.9995^{+*}
expanding	30	.4384	.4567	.4609^{+*}	.4547^{+*}	.6200^{+*}	.7773	.5781^{+*}	.5897^{+*}
(t^2 *rate*)	60	.7843	.8046	.8101^{+*}	.8167^{+*}	.9143^{+*}	.9533	.9118^{+*}	.9267^{+*}
	120	.9804	.9813	.9861^{+*}	.9874^{+*}	.9958^{+*}	.9902	.9973^{+*}	.9991^{+*}

procedures as in Table 6.1 were considered for each combination of a heteroscedasticity pattern and a density type (normal or Cauchy) for η_t. Rejection frequencies are shown in Table 6.2 for $\theta = 0.5$, in Table 6.3 for $\theta = 0.9$. These rejection rates (their interpretation as estimated power values) however are not quite meaningful when they correspond to strongly biased, or strongly overrejecting testing procedures. Therefore, the *very* significant over- and underrejection rates cases are still identified with the $^{+*}$ and $^{-*}$ superscripts obtained from Table 6.1.

Except for the i.i.d. case, under which all procedures considered are asymptotically correct, one should not pay too much attention to a comparison between the rejection frequencies resulting from the runs tests and those resulting from the correlogram-based methods: as already mentioned, the latter correspond either to severely over-rejecting, or severely biased testing procedures (see Table 6.1). The main findings in Tables 6.2 and 6.3 thus are to be found in the *i.i.d. row*, and in the comparison between the two *runs columns*.

The chosen *MA* parameter values hardly can be considered as *local*; nor can $n = 30$ or 60 be considered as a very large sample size. The situation described in these tables thus appears quite remote from the asymptotic results derived in Section 4 under the null hypothesis of independence. Nevertheless, the rejection frequencies reported in the *i.i.d. row* of Tables 6.2 and 6.3 suggest that these asymptotics still provide a fair picture of the actual situation. In the *i.i.d.* case, Table 3.1 corroborates the theoretical ARE values (for runs, with respect to the classical correlogram test, under Gaussian densities) of about 40%. This means that the power attained by the runs tests for n observations (n *large*) should be about the same as that of the classical correlogram test for $4n/10$ observations. The observed frequencies in Table 6.2 and 6.3 roughly support this asymptotic evaluation, with a slight advantage in favour of runs. Similarly, Theorems 4.12 and 4.13 establish the asymptotic equivalence, still under the null, of the exact and aligned runs tests. The comparison between their respective rejection frequencies in Tables 6.2 and 6.3 suggests that this asymptotic equivalence still provides a reasonable picture of the finite-sample situation under MA dependence, even under the t^2 expanding heteroscedasticity pattern (certainly so at $n = 120$).

7. CONCLUSION

Towards a General Methodology for the Analysis of Time Series with Nonhomogeneous Innovations?

Generalized runs apparently provide an appropriate tool in the problem of testing for non-homogeneous white noise. Due to their invariance properties and their interpretation as a generalized residual correlogram, they also probably constitute the most adequate tool for the analysis of time series with independent but nonhomogeneous innovation process. Testing the hypothesis under which the coefficients $\boldsymbol{\theta}$ of a given model of this type (e.g., an ARMA model with heteroscedastic innovation process) take on some preassigned value $\boldsymbol{\theta}_0$ obviously reduces to the problem of testing for nonhomogeneous white noise treated in this paper. Most problems of practical relevance, however, involve hypotheses under which the model coefficients $\boldsymbol{\theta}$ are only partially specified. The residuals corresponding to such null hypotheses cannot be computed exactly, and typically are replaced with estimated residuals resulting from some constrained estimates $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. In the spirit of the alignment process considered in Section 4 (alignment with respect to the empirical median), one would like to be able to consider the aligned run-based correlograms associated with these estimates. Numerical experience (Campbell and Dufour, 1995 and 1997; Campbell and Galbraith, 1994) suggests that the behaviour of such aligned run-based correlograms is essentially similar to that of the "exact" ones. Extending the asymptotic equivalence results of Section 4 to the more general situation of, e.g., estimated ARMA residuals, however is by no means a trivial problem. In particular, the choice of the possibly nonconsistent estimate $\hat{\boldsymbol{\theta}}$ (replacing the empirical median) to be used is not clear.

Though much work clearly remains to be done until run-based techniques allow for a complete and flexible methodology in the analysis of heteroscedastic time series, so that the question mark in the title of this conclusion section can be removed, the numerical results just mentioned nevertheless allow for some optimism.

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