Quantitative Methods: Assignment 6

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Problem 1 (15 points):

Consider an investor who is holding one share of a stock whose price is evolving according to a Standard Brownian motion process, i.e.

$$S(u) = S(0) + \sigma W(u), u \geq 0$$

where $\sigma > 0$ is the volatility coefficient. This investor purchased the stock at a price $S(0) > 0$ at time 0 and decides to sell the stock if it reaches the price $S(0) + \Delta$ where $\Delta > 0$.

1. What is the cumulative distribution function of the hitting time $\tau_{S(0)+\Delta}$?

Solution: We have that from the reflection principle that $\mathbb{P}[W(t) \geq a \mid \tau_a \leq t] = \frac{1}{2}$ so that:

$$\begin{aligned}
&\mathbb{P}[W(t) \geq W(0) + a] \\
&= \mathbb{P}[W(t) \geq W(0) + a \mid \tau_{W(0)+a} \leq t]\mathbb{P}[\tau_{W(0)+a} \leq t] \\
&+ \mathbb{P}[W(t) \geq W(0) + a \mid \tau_{W(0)+a} > t]\mathbb{P}[\tau_{W(0)+a} > t] \\
&= \frac{1}{2}\mathbb{P}[\tau_{W(0)+a} \leq t] + 0 = \frac{1}{2}\mathbb{P}[\tau_{W(0)+a} \leq t]
\end{aligned}$$

So that we have that:

$$\begin{aligned}
&\mathbb{P}[\tau_{S(0)+\Delta} \geq t] = 2\mathbb{P}[S(0) + \sigma W(t) \geq S(0) + \Delta] \\
&= 2(1 - \mathbb{P}[W(t) < \frac{\Delta}{\sigma}]) = 2(1 - \Phi(\frac{\Delta}{\sigma\sqrt{t}}))
\end{aligned}$$

2. Give also the density of the distribution of the hitting time $\tau_{S(0)+\Delta}$

Solution: We differentiate our previous solution to get that:

$$f_{\tau_{S(0)+\Delta}}(t) = \Delta\Phi'(\frac{\Delta}{\sigma\sqrt{t}})t^{-3/2}$$

3. What is the distribution of the hitting time $\tau_{S(0)-\delta}$, i.e. of the first time at which the asset price falls below $S(0) - \delta$ where $\delta > 0$ is a positive constant smaller than $S(0)$?

Solution: This follows directly from the symmetry of the Wiener Process and the reflection principle. We note that the distribution of falling below $S(0) - \delta$ is the same as going above $S(0) + \delta$:

$$\begin{aligned}&\mathbb{P}[W(t) \leq W(0) - a] = \mathbb{P}[W(t) \leq -a \mid \tau_{W(0)-a} \leq t]\mathbb{P}[\tau_{W(0)-a} \leq t] \\ &+ \mathbb{P}[W(t) \leq -a \mid \tau_{W(0)-a} > t]\mathbb{P}[\tau_a > t] \\ &= \frac{1}{2}\mathbb{P}[\tau_{W(0)-a} \leq t] + 0 = \frac{1}{2}\mathbb{P}[\tau_{W(0)-a} \leq t]\end{aligned}$$

So that we have that:

$$\begin{aligned}&\mathbb{P}[\tau_{S(0)-\delta} \leq t] = 2\mathbb{P}[S(0) + \sigma W(t) \leq S(0) - \delta] \\ &= 2\mathbb{P}[W(t) \leq -\frac{\delta}{\sigma}] = 2(1 - \Phi(\frac{\delta}{\sigma\sqrt{t}}))\end{aligned}$$

Problem 2 (15 points):

Compute

$$\mathbb{E}[W(t_1)W(t_2)W(t_3)], \text{ for } t_1 < t_2 < t_3$$

where W is a standard Brownian motion.

Solution: Fix $0 < t_1 < t_2 < t_3$. We first note that $W^2(t) - t$ is a martingale by:

$$\begin{aligned}&\mathbb{E}[W^2(t_2) - t_2 \mid \mathcal{F}_{t_1}] \\ &= \mathbb{E}[(W(t_2) - W(t_1) + W(t_1))^2 - (t_2 - t_1 + t_1) \mid \mathcal{F}_{t_1}] \\ &= \mathbb{E}[(W(t_2) - W(t_1))^2 \mid \mathcal{F}_{t_1}] + W(t_1)^2 + 2\mathbb{E}[(W(t_2) - W(t_1))W(t_1) \mid \mathcal{F}_{t-1}] - t_2 + t_1 - t_1 \\ &= \mathbb{E}[(W(t_2) - W(t_1))^2 \mid \mathcal{F}_{t_1}] + W(t_1)^2 + 2W(t_1)\mathbb{E}[W(t_2) - W(t_1) \mid \mathcal{F}_{t-1}] - t_2 + t_1 - t_1 \\ &= \mathbb{E}[(W(t_2) - W(t_1))^2] + W(t_1)^2 + 2W(t_1) \cdot 0 - t_2 + t_1 - t_1 \\ &= t_2 - t_1 + W(t_1)^2 - t_2 + t_1 - t_1 \\ &= W(t_1)^2 - t_1\end{aligned}$$

We then note that $\mathbb{E}[W^3(t)] = 0$ by noting that $W(t) \sim N(0, t)$ and that $\mathbb{E}[W^3(t)]$ is then the third moment of a normal distribution centered at 0. The result follows from the fact that the odd moments of the normal distribution are all zero. We then have:

$$\begin{aligned}
&\mathbb{E}[W(t_1)W(t_2)W(t_3)] \\
&= \mathbb{E}[\mathbb{E}[W(t_1)W(t_2)W(t_3) \mid \mathcal{F}_{t_2}]] \\
&= \mathbb{E}[W(t_1)W(t_2)\mathbb{E}[W(t_3) \mid \mathcal{F}_{t_2}]] \\
&= \mathbb{E}[W(t_1)W(t_2) \cdot W(t_2)] \\
&= \mathbb{E}[\mathbb{E}[W(t_1)W^2(t_2) \mid \mathcal{F}_{t_1}]] \\
&= \mathbb{E}[W(t_1)\mathbb{E}[W^2(t_2) - t_2 + t_2 \mid \mathcal{F}_{t_1}]] \\
&= \mathbb{E}[W(t_1)\mathbb{E}[W^2(t_2) - t_2 \mid \mathcal{F}_{t_1}] + t_2 W(t_1)] \\
&= \mathbb{E}[W(t_1) \cdot (W^2(t_1) - t_1) + t_2 W(t_1)] \\
&= \mathbb{E}[W^3(t_1)] - t_1\mathbb{E}[W(t_1)] + t_2\mathbb{E}[W(t_1)] \\
&= 0
\end{aligned}$$

Problem 1 (30 points):

We consider the Geometric Brownian Motion model for a stock price:

$$d \log S(t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t)$$

We then define the log return over the interval $[t, t + \Delta]$

$$r(t, \Delta) = \log S(t + \Delta) - log(S(t))$$

Integrating the first equation over $[t, t + \Delta]$ yields

$$\log S(t + \Delta) - \log S(t) = (\mu - \frac{1}{2}\sigma^2)\Delta + \sigma(W(t + \Delta) - W(t))$$

In other words, the log return can be written as

$$r(t, \Delta) = (\mu - \frac{1}{2}\sigma^2)\Delta + \sigma(W(t + \Delta) - W(t))$$

1. What is the distribution of $r(t, \Delta)$? In particular, give its mean and variance.

Solution: We first fix $\Delta > 0$. We note that $W(t + \Delta) - W(t) \sim N(0, \Delta)$ by definition of the Wiener Process. As $(\mu - \frac{1}{2}\sigma^2)\Delta$ is deterministic, we have that:

$$r(t, \Delta) \sim ((\mu - \frac{1}{2}\sigma^2)\Delta, \sigma^2\Delta)$$

2. Suppose that we are given a set of daily data for which the above model is a good fit with $\mu = 0.1$ per year and $\sigma = 0.2$ per year. Note that $\Delta = 1$ day $= 1/252$years. We wish to estimate μ. Since the random walk model is stationary, ergodic, and has a finite variance, which allows us to apply the Central Limit Theorem, we can safely estimate μ by computing a time-average. This estimator is also the same as the Maximum Likelihood estimate for this simple model.
   The convergence rate is $\sigma/\sqrt{N}$ where N is the number of samples. Unfortunately, obtaining an accurate value for μ requires very long time series that are never available in practice. We denote by $\hat{\mu}$ an estimate of μ. If one wants to determine a 95% confidence interval of the form $[\hat{\mu} - 0.01, \hat{\mu} + 0.01]$, how many years of data do you need?

Solution: We want to find a 95% confidence interval for estimate $\hat{\mu}$ through the equation $\mathbb{P}[|\hat{\mu} - \mu| < 0.01] = 0.95$. From $\mu = 0.1, \sigma = 0.2$, we have that $\mathbb{P}[\frac{\sqrt{N}}{\sigma}|\hat{\mu} - 0.1| < \frac{\sqrt{N}}{100\sigma}] = \mathbb{P}[5\sqrt{N}|\hat{\mu} - 0.1| < \frac{\sqrt{N}}{20}] = 0.95$
As $Z = \frac{\sqrt{N}}{\sigma}(\hat{\mu} - 0.01) \sim N(0,1)$ by the Central Limit Theorem, we have that:

$$\mathbb{P}[Z < -\frac{\sqrt{N}}{20}] \geq \frac{1}{2}(1 - 0.95)$$
$$\Rightarrow -\frac{\sqrt{N}}{20} \geq \Phi^{-1}(0.025) = -1.96$$
$$N \geq (20 \cdot 1.96)^2 = 1536.4 \text{ days}$$

This means that we need $N = 1537/252 = 6.099 \approx 6$ years of data.