FRE-GY 6233: Assignment 2

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Problem 1:

(i) Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space where $\Omega = \{a, b, c, d, e, f\}$, $\mathcal{F}$ is σ-algebra, and $\mathbb{P}$ is uniform (so $\mathbb{P}(a) = \mathbb{P}(b) = \cdots = \frac{1}{6}$).

(ii) Let X, Y, Z be r.v. given by

$$X(a) = 1, X(b) = X(c) = 3, X(d) = X(e) = 5, X(f) = 7$$
$$Y(a) = Y(b) = 2, Y(c) = Y(d) = 1, Y(e) = Y(f) = 7$$
$$Z(a) = Z(b) = Z(c) = Z(d) = 3, Z(e) = Z(f) = 2 \quad (1)$$

Solve the follow questions:

1. Write down $\sigma(X), \sigma(X), \sigma(Z)$. Are there any relationships between them?

Solution:

$$\sigma(X) = \{\emptyset, \Omega, \{a\}, \{b, c\}, \{d, e\}, \{f\}, \{a, b, c\}, \{a, d, e\}, \{a, f\},$$
$$\{b, c, d, e\}, \{b, c, f\}, \{d, e, f\}, \{a, b, c, d, e\}, \{b, c, d, e, f\}, \{a, d, e, f\}, \{a, b, c, f\}\}$$
$$\sigma(Y) = \{\emptyset, \Omega, \{a, b\}, \{c, d\}, \{e, f\}, \{a, b, c, d\}, \{a, b, e, f\}, \{c, d, e, f\}\}$$
$$\sigma(Z) = \{\emptyset, \Omega, \{a, b, c, d\}, \{e, f\}\}$$

We note that $\sigma(Z) \subset \sigma(Y)$.

2. Define a r.. $\mathbb{E}[X \mid \sigma(Y)]$

Solution: As $\mathbb{E}[X|Y] = \Sigma_{i \in \{1,2,7\}} \mathbb{E}[X|Y = i]\mathbb{1}_i(y)$, we consider the following:
If $Y = 1$, $\mathbb{P}(Y^{-1}(1) = c) = \frac{\mathbb{P}(c)}{\mathbb{P}(c) + \mathbb{P}(d)} = \frac{1}{2}, \mathbb{P}(Y^{-1}(1) = d) = \frac{1}{2}$
So, $\mathbb{E}[X|Y = 1] = (X(c) \cdot \frac{1}{2} + X(d) \cdot \frac{1}{2}) = 3 \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} = 4$
If $Y = 2$, $\mathbb{P}(Y^{-1}(2) = a) = \frac{\mathbb{P}(a)}{\mathbb{P}(a) + \mathbb{P}(b)} = \frac{1}{2}, \mathbb{P}(X^{-1}(2) = b) = \frac{1}{2}$

So, $\mathbb{E}[X|Y=2] = (X(a)\cdot\frac{1}{2} + X(b)\cdot\frac{1}{2}) = 2$
If $Y = 7$, $\mathbb{P}(Y^{-1}(7) = e) = \frac{\mathbb{P}(e)}{\mathbb{P}(e)+\mathbb{P}(f)} = \frac{1}{2}, \mathbb{P}(Y^{-1}(7) = f) = \frac{1}{2}$
So, $\mathbb{E}[X|Y=7] = (X(e)\cdot\frac{1}{2} + X(f)\cdot\frac{1}{2}) = 6$
.

3. Check directly the averaging property

$$\mathbb{E}[\mathbb{E}[X \mid \sigma(Y)]] = \mathbb{E}[X]$$

Solution: $\mathbb{E}[\mathbb{E}[X \mid \sigma(Y)]] = \Sigma_{i\in\{1,2,7\}}\mathbb{E}[X|Y=i]\cdot\mathbb{P}[Y=i]$. As $\mathbb{P}[Y=i] = \frac{1}{3}$ for $i \in \{1,2,7\}$, we have that $\mathbb{E}[\mathbb{E}[X \mid \sigma(Y)]] = \frac{1}{3}\cdot(4+2+6) = 4$.
We then note that:
$\mathbb{E}[X] = \Sigma_{i\in\{a,b,c,d,e,f\}}\mathbb{P}[i]\cdot X(i) = \frac{1}{6}(1+3+3+5+5+7) = \frac{24}{6} = 4$.
From this we have checked the averaging property.

4. Show directly (by calculating) that

$$\mathbb{E}[\mathbb{E}[X \mid \sigma(Y)] \mid \sigma(Z)] = \mathbb{E}[X \mid \sigma(Z)]$$

Solution: We first show the RHS:
Let us denote events $\omega_{z1} = \{a,b,c,d\}$, $\omega_{z2} = \{e,f\}$ to partition Ω to form $\sigma(Z)$.
We then denote events $\omega_{x1} = \{a\}, \omega_{x2} = \{b,c\}, \omega_{x3} = \{d,e\}, \omega_{x4} = \{f\}$ to partition Ω to form $\sigma(X)$.
We then note that $\omega_{z1}\cap\omega_{x1} = \{a\}$, $\omega_{z1}\cap\omega_{x2} = \{b,c\}$, $\omega_{z1}\cap\omega_{x3} = \{d\}$. We then have
$\mathbb{E}[X \mid \omega_{z1}] = X(\omega \in \{a\})\frac{\mathbb{P}(\{a\})}{\mathbb{P}(\omega_{z1})} + X(\omega \in \{b,c\})\frac{\mathbb{P}(\{b,c\})}{\mathbb{P}(\omega_{z1})} + X(\omega \in \{d\})\frac{\mathbb{P}(\{d\})}{\mathbb{P}(\omega_{z1})} = 1\cdot\frac{1}{4} + 3\cdot\frac{2}{4} + 5\cdot\frac{1}{4} = 3$
We also note that $\omega_{z2}\cap\omega_{x3} = \{e\}, \omega_{z2}\cap\omega_{x4} = \{f\}$ so that we have
$\mathbb{E}[X \mid \omega_{z2}] = X(\omega \in \{e\})\frac{\mathbb{P}(\{e\})}{\mathbb{P}(\omega_{z2})} + X(\omega \in \{f\})\frac{\mathbb{P}(\{f\})}{\mathbb{P}(\omega_{z2})} = 5\cdot\frac{1}{2} + 7\cdot\frac{1}{2} = 6$
We write this as:

$$\mathbb{E}[X \mid \sigma(Z)] = \begin{cases} 3, & \text{for } \omega \in \omega_{z1} = \{a,b,c,d\} \\ 6, & \text{for } \omega \in \omega_{z2} = \{e,f\} \end{cases}$$

We then look at the LHS:
In part (ii), we had subtly defined $X \mid \sigma(Y)$ over events $\omega_{y1} = \{a,b\}, \omega_{y2} = \{c,d\}, \omega_{y3} = \{e,f\}$ that partition Ω to form $\sigma(Y)$. From that we recieved the

following random variable:

$$\mathbb{E}[X \mid \sigma(Y)] = \begin{cases} 2, & \text{for } \omega \in \omega_{y1} = \{a, b\} \\ 4, & \text{for } \omega \in \omega_{y2} = \{c, d\} \\ 6, & \text{for } \omega \in \omega_{y3} = \{e, f\} \end{cases}$$

We also noted that $\sigma(Z) \subset \sigma(Y)$. It follows that as $\omega_{y1} \cup \omega_{y2} = \omega_{z1}$ and $\omega y_3 = \omega_{z3}$,

$$\begin{aligned} &\mathbb{E}[X \mid \sigma(Y)] \mid \omega_{z1}] \\ &= \left(\mathbb{E}[X \mid \omega_{y1}] \mid \omega_{z1}\right) \cdot \frac{\mathbb{P}(\omega_{y1} \cap \omega_{z1})}{\omega_{z1}} + \left(\mathbb{E}[X \mid \omega_{y2}] \mid \omega_{z1}\right) \cdot \frac{\mathbb{P}(\omega_{y2} \cap \omega_{z1})}{\omega_{z1}} \\ &= 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = 3 \end{aligned}$$

$$\begin{aligned} &\mathbb{E}[X \mid \sigma(Y)] \mid \omega_{z2}] \\ &= \left(\mathbb{E}[X \mid \omega_{y3}] \mid \omega_{z2}\right) \cdot \frac{\mathbb{P}(\omega_{y3} \cap \omega_{z2})}{\omega_{z2}} = \mathbb{E}[X \mid \omega_{z3}] = 6 \end{aligned}$$

We then have:

$$\mathbb{E}[X \mid \sigma(Y)] \mid \sigma(Z)] = \begin{cases} 3, & \text{for } \omega \in \omega_{z1} = \{a, b, c, d\} \\ 6, & \text{for } \omega \in \omega_{z2} = \{e, f\} \end{cases}$$

So it is evident that LHS = RHS.

5. Check if X and Y, or Y and Z are independent under given probability.

Solution: It is straightforward to see that X and Y are not independent; otherwise, $\mathbb{E}[X \mid \sigma(Y)] = \mathbb{E}[X]$. We have shown that $\mathbb{E}[X] = 4$, which is clearly not what we have shown to be $\mathbb{E}[X \mid \sigma(Y)]$ in the problems above.
To show that Y and Z are not independent, we proceed by using the definition of independence. We check that for some $\omega_1 \in \sigma(Y), \omega_2 \in \sigma(Z)$, that $\mathbb{P}(\omega_1 \cap \omega_2) \neq \mathbb{P}(\omega_1) \cdot \mathbb{P}(\omega_2)$. It is most evident that as $\omega_{z2} = \omega_{y3}$ we would have $\mathbb{P}(\omega_{z2} \cap \omega_{y3}) = \mathbb{P}(\omega_{z2}) = \mathbb{P}(\omega_{z2})\mathbb{P}(\omega_{y3}) = \mathbb{P}(\omega_{z2})^2$. As $\mathbb{P}(\omega_{z2}) = \frac{1}{3}$, this cannot be true. As the two sigma-algebra are not independent, random variables Y and Z are not independent.

Problem 2: Prove Markov and Tchebyshev inequalities.

Solution: Markov Inequality: For $\lambda, p > 0$ and nonnegative random variable X, we have:

$$\mathbb{P}(\omega : |X(\omega)| \geq \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}[|X|^p]$$

Proof. We first by showing the above inequality for $p = 1$. This follows directly from the definition of expectation:
If we fix $\lambda > 0$ and define set $A = \{\omega : X(\omega) \geq \lambda\}$

$$\mathbb{E}[X] = \int_{\mathbb{R}} X(\omega)dP(\omega) = \int_A X(\omega)dP(\omega) + \int_{\mathbb{R}\setminus A} X(\omega)dP(\omega)$$

$$\geq \int_A X(\omega)dP(\omega) \geq \int_A \lambda dp(\omega) \text{ [this follows from the condition that } X(\omega) \geq \lambda \text{ in set } A \text{]}$$

$$= \lambda \int_A dp(\omega) = a\mathbb{P}(\omega : |X(\omega)| \geq \lambda).$$

$$\Rightarrow$$

$$\mathbb{P}(\omega : |X(\omega)| \geq \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}[|X|^p]$$

To extend it to all other $p > 0$, we consider function $\phi(x) = |x|^p$. If ϕ is positive and non-decreasing, we have that $\mathbb{P}[X \geq \lambda] \leq \mathbb{P}[\phi(X) \geq \phi(\lambda)] \leq \mathbb{E}[\phi(X)]/\phi(\lambda)$. As ϕ is positive and non-decreasing, we are done. □

Tchebychev Inequality: If X is a r.v. with mean μ and variance σ^2, then

$$\mathbb{P}(\omega : |X(\omega) - \mu| \geq \lambda) \leq \frac{\sigma^2}{\lambda^2}$$

Proof. The above inequality follows from Markov's inequality on $(X - \mu)$ with $p = 2$. We observe that $\mathbb{P}(\omega : |X(\omega) - \mu| \geq \lambda) \leq \frac{\mathbb{E}[|X-\mu|^2]}{\lambda^2} = \frac{\mathbb{E}[X^2 - 2\mu \cdot X + \mu^2]}{\lambda^2}$
$= \frac{1}{\lambda^2} \cdot \left(\mathbb{E}[X^2] - 2\mu^2 + \mu^2\right) = \frac{1}{\lambda^2} \cdot (\mathbb{E}[X^2] - \mathbb{E}[X]^2) = \frac{\sigma^2}{\lambda^2}$ □

Problem 3: Let X be a r.v. and $\lambda > 0$. Prove that the following bound holds:

$$\mathbb{P}(X \geq \lambda) \leq \frac{\mathbb{E}[e^{tX}]}{e^{\lambda t}}, \forall t > 0$$

Use Markov inequality.

Solution: We note that the function $\phi(x) = e^{tx}$ for $t > 0$ is a positive and non-decreasing function. We first have by Markov's inequality that $\mathbb{P}(X \geq \lambda) = \mathbb{P}(\phi(x) \geq \phi(\lambda))$
$\leq \frac{1}{\phi(\lambda)}\mathbb{E}[\phi(X)] \Rightarrow \mathbb{P}(X \geq \lambda) \leq \frac{\mathbb{E}[e^{tX}]}{e^{\lambda t}}, \forall t > 0.$