FRE-GY 6233: Assignment 6

Raymond Luo

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Problem 1 Find all solutions of heat equation of the following type:

$$\phi(t,x) = a(t)b(x)$$

Solution: Given heat equation $\frac{\partial \phi}{\partial t} = c \cdot \nabla \phi(t,x)$ [in class we used $c = \frac{1}{2}$] and assuming solutions of the form $\phi(t,x) = a(t)b(x)$, we have that: $a'(t)b(x) = c \cdot a(t)b''(x)$. For nontrivial solutions of $\phi(t,x)$ (as in $\phi \neq \mathbf{0}$), $\frac{1}{c}\frac{a'(t)}{a(t)} = \frac{b''(x)}{b(x)} = \lambda$ for some constant λ. It is direct to observe that $a'(t) = c \cdot \lambda \cdot a(t) \Rightarrow a(t) = A_1 e^{c \cdot \lambda t}$ for some constant A_1; however, we have the following cases for $b(x)$:

1. $\lambda > 0$:
   $b''(x) - b(x)\lambda = 0 \Rightarrow b(x) = B_1 e^{\sqrt{\lambda}x} + B_2 e^{-\sqrt{\lambda}x}$ for some constants B_1, B_2

2. $\lambda = 0$:
   $b''(x) = 0 \Rightarrow b(x) = B_1 x + B_2$ for some constants B_1, B_2

3. $\lambda < 0$:
   $b''(x) - b(x)\lambda = 0 \Rightarrow b(x) = B_1 cos(\sqrt{-\lambda}x) + B_2 sin(-\sqrt{\lambda}x)$ for some constants B_1, B_2

Note that the above cases of $b(x)$ can be concisely written in considering Euler's formula and expressing it using complex form.

Problem 2 Using the heat equation method, calculate the following integral:

$$\int_0^T e^{\frac{t}{2}} cos(W(t))dW(t)$$

Solution: We note from problem 1 that $\phi(t,x) = a(t)b(x)$ where $a(t) = e^{\frac{t}{2}}$, $b(x) = sin(x)$ is a solution to a heat equation with $c = \frac{1}{2}$. We then note that as $b'(x) = cos(x)$, we can rewrite $\int_0^T e^{\frac{t}{2}} cos(W(t))dW(t) = \int_0^T f(t, W(t))dW(t)$ where $f(t,x) = \frac{\partial \phi}{\partial x}$. It then follows that $d(\phi(t, W(t))) = \partial_x \phi(t, W(t))dW(t) + (\partial_t \phi + \frac{1}{2}\partial_x^2 \phi)dt = \partial_x \phi(t, W(t))dW(t) = f(t, W(t)$ as the dt terms go to zero according to the heat equation. We then have $\int_0^T e^{\frac{t}{2}} cos(W(t))dW(t) = \int_0^T d(\phi(t, W(t))) = \phi(T, W(T)) - \phi(0, W(0)) = e^{\frac{T}{2}} sin(T) - e^0 sin(0) = e^{\frac{T}{2}} sin(T)$

Problem 3

1. Prove formula (7) for variance on slide 6 in the last lecture

Solution: We begin by observing that for $dX(t) = \alpha(t)X(t)dt + b(t)dW(t)$, we have by Ito's lemma with $f(x) = x^2$:

$$\begin{aligned}
dX^2(t) &= \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 \\
&= \frac{\partial f}{\partial x}(\alpha(t)X(t)dt + b(t)dW(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\alpha(t)X(t)dt + b(t)dW(t))^2 \\
&= 2X(t) \cdot (\alpha(t)X(t)dt + b(t)dW(t)) + \\
&\quad + 1 \cdot (\alpha^2(t)X^2(t)dt^2 + 2\alpha(t)X(t)b(t)d(t)dW(t) + b^2(t)dW^2(t)) \\
&= 2\alpha(t)X^2(t)dt + 2b(t)X(t)dW(t) + b^2(t)dt
\end{aligned}$$

the above follows from $dt^2 \approx 0, dW^2(t) = dt$

We then integrate to get:

$$X^2(t) = X^2(0) + \int_0^t 2\alpha(s)X^2(s) + b^2(s)ds + \int_0^t 2b(s)X(s)dW(s)$$

We then take expectation:

$$\mathbb{E}[X^2(t)] = X^2(0) + \int_0^t 2\alpha(t)\mathbb{E}[X^2(t)] + b^2(t)dt + 0$$

We take differentiate with respect to t to retrieve an ODE:

$$\frac{\partial \mathbb{E}[X^2(t)]}{\partial t} = 2\alpha(t)\mathbb{E}[X^2(t)] + b^2(t)$$

For the ODE we replace $\mathbb{E}[X^2(t)]$ with y to observe for $A(t) = \displaystyle\int_0^t \alpha(s)ds$

$$y' = 2\alpha(t)y + b^2(t)$$

$$e^{-2A(t)}y' - 2\alpha(t)e^{-2A(t)}y = b^2(t)e^{-2\alpha \cdot t}$$

$$\frac{d}{dt}(e^{-2A(t)}) = b^2(t)e^{-2A(t)}$$

$$
\begin{aligned}
e^{-2A(t)}y &= \int_0^t b^2(s)e^{-2A(t)}ds + X^2(0) \\
\mathbb{E}[X^2(t)] &= e^{2A(t)}\left(\int_0^t b^2(s)e^{-2A(t)}ds + X^2(0)\right) \\
&\text{Then we have variance:} \\
Var[X(t)] &= \mathbb{E}[X^2(t)] - \mathbb{E}[X(t)]^2 \\
&= e^{2A(t)}\left(\int_0^t b^2(s)e^{-2A(t)}ds + X^2(0)\right) - (e^{A(t)}X(0))^2 \\
&= e^{2A(t)}\left(\int_0^t b^2(s)e^{-2A(t)}ds\right)
\end{aligned}
$$

2. Give answers to questions 1 and 2 from Breakout Room 2 on slide 9

Solution:

(a) Find $\mathbb{E}[X(t)]$ for $X(t)$ given by

$$dX(t) = (X(t) + e^t)dt + b(t, X(t), W(t))dW(t)$$

We have that $a(t, W(t), X(t)) = \alpha(t)X(t) + \beta(t)$ with $\alpha(t) = 1, \beta(t) = e^t$ so it follows that we have ODE

$$
\begin{aligned}
&\frac{\partial \mathbb{E}[X(t)]}{\partial t} = \mathbb{E}[X(t)] + e^t, \\
&\frac{\partial \mathbb{E}[X(t)]}{\partial t}e^{-t} - \mathbb{E}[X(t)]e^{-t} = e^t \cdot e^{-t}, \\
&\frac{\partial}{\partial t}(\mathbb{E}[X(t)]e^{-t}) = 1, \\
&\mathbb{E}[X(t)]e^{-t} = t + X(0), \\
&\mathbb{E}[X(t)] = e^t(t + X(0));
\end{aligned}
$$

(b) Find mean and variance for the process:

$$dX(t) = -kX(t)dt + \sigma dW(t)$$

We have that $a(t, W(t), X(t)) = \alpha(t)X(t) + \beta(t)$ with $\alpha(t) = -k, \beta(t) = 0$ so it follows that we have ODE

$$
\begin{aligned}
&\frac{\partial \mathbb{E}[X(t)]}{\partial t} = -k\mathbb{E}[X(t)], \\
&\mathbb{E}[X(t)] = e^{-kt}X(0);
\end{aligned}
$$

We then also have that

$$Var[X(t)] = -kt$$

Problem 4

In the following problems, find the mean $\mathbb{E}[X(t)]$. a and b are some constants. (Find an ODE w.r.t for the mean and solve it)

1. $X(t) = cos(3aW(t))$

Solution: If we let $f(x) = cos(3a \cdot x)$ then we have by Ito's lemma:

$$\begin{aligned}
dX(t) &= d(f(x)) = \partial_t f(t,x)dt + \partial_x f(t,x)dx + \frac{1}{2}\partial_x^2 f(t,x)(dx)^2 \\
&= -3a \cdot sin(3aW(t))dW(t) - \frac{1}{2}9a^2 \cdot cos(3aW(t))dt \\
&\text{We then have the equivalent statement:} \\
X(t) &= X(0) - \int_0^t \frac{1}{2}9a^2 \cdot cos(3aW(s))ds - \int_0^t 3a \cdot sin(3aW(s))dW(s) \\
&\text{We take expectation to find:} \\
\mathbb{E}[X(t)] &= X(0) - \frac{1}{2}9a^2 \int_0^t \mathbb{E}[X(s)]ds \\
&\text{We differentiate with respect to } t \text{ for:} \\
\frac{\partial \mathbb{E}[X(t)]}{\partial t} &= -\frac{9}{2}a^2 \mathbb{E}[X(t)] \\
&\text{This is an ODE for } \mathbb{E}[X(t)] \text{ for which we know the solution is:} \\
\mathbb{E}[X(t)] &= e^{-\int_0^t \frac{9}{2}a^2 ds}\Big(X(0)\Big) \\
&\text{note that } X(0) = cos(3aW(0)) = cos(0) = 1 \\
&= e^{-9/2a^2 t}
\end{aligned}$$

2. $X(t) = sin(t + 2bW(t))$

Solution: If we let $f(t,x) = sin(t + 2b \cdot x)$ then we have by Ito's lemma:

$$\begin{aligned}
dX(t) &= d(f(t,x)) = \partial_t f(t,x)dt + \partial_x f(t,x)dx + \frac{1}{2}\partial_x^2 f(t,x)(dx)^2 \\
&= cos(t + 2b \cdot W(t))dt + 2bcos(t + 2bW(t))dW(t) - 2b^2 sin(t + 2bW(t))dt
\end{aligned}$$

We then have the equivalent statement:

$$X(t) = X(0) + \int_0^t \Big(cos(s + 2b \cdot W(s)) - 2b^2 sin(s + 2bW(s))\Big) ds +$$
$$+ \int_0^t 2bcos(s + 2bW(s))dW(s)$$

We take expectation to find:

$$\mathbb{E}[X(t)] = X(0) + \int_0^t \mathbb{E}\Big[cos(s + 2b \cdot W(s)) - 2b^2 sin(s + 2bW(s))\Big] ds$$

We differentiate with respect to t for:

$$\begin{aligned}\frac{\partial \mathbb{E}[X(t)]}{\partial t} &= \mathbb{E}[cos(t + 2b \cdot W(t))] - 2b^2\mathbb{E}[X(t)] \\ &= \mathbb{E}[cos(t)cos(2bW(t)) - sin(t)sin(2bW(t))] - 2b^2\mathbb{E}[X(t)] \\ &= cos(t)\mathbb{E}[cos(2bW(t))] - sin(t)\mathbb{E}[sin(2bW(t))] - 2b^2\mathbb{E}[X(t)]\end{aligned}$$

As we know from the last question what $\mathbb{E}[cos(2bW(s))]$ looks like, we explore $Y(t) = sin(2bW(s))$

$$dY(t) = 2bcos(2bW(t))dW(t) - 2b^2 sin(2bW(t))dt$$

We then have the equivalent statement:

$$Y(t) = Y(0) - \int_0^t 2b^2 sin(2bW(s))ds$$
$$+ \int_0^t 2bcos(2bW(s))dW(s)$$

We take expectation and differentiate with respect to t to find:

$$\frac{\partial \mathbb{E}[Y(t)]}{\partial t} = -2b^2\mathbb{E}[Y(t)],$$

This is an ODE for $\mathbb{E}[X(t)]$ for which we know the solution is:

$$\mathbb{E}[Y(t)] = e^{-\int_0^t 2b^2 ds}\Big(Y(0)\Big) = 0$$

as $Y(0) = sin(2bW(0)) = sin(0) = 0$

We then have:

$$\begin{aligned}\frac{\partial \mathbb{E}[X(t)]}{\partial t} &= cos(t)\mathbb{E}[cos(2bW(t))] - sin(t)\mathbb{E}[sin(2bW(t))] - 2b^2\mathbb{E}[X(t)] \\ &= cos(t)e^{-2b^2 t} - 2b^2\mathbb{E}[X(t)]\end{aligned}$$

We observe that this is once again an ODE with a simple closed solution $\Rightarrow$

$$\mathbb{E}[X(t)] = e^{-\int_0^t 2b^2 ds}(X(0) + \int_0^t e^{\int_0^t 2b^2 ds} cos(s)e^{-2b^2 s}ds)$$

$$
\begin{aligned}
&= e^{-2b^2t}(0 + \int_0^t cos(s)ds) \\
&= e^{-2b^2t}(sin(t) - sin(0)) \\
&= sin(t)e^{-2b^2t}
\end{aligned}
$$