FRE 6233 Stochastic Calculus and Option pricing Stochastic Differentiation and Stochastic Integration

Roza Galeeva

October 5, 2020

Basic rules of stochastic differentiation

- Most stochastic processes are not differentiable. For instance, the Brownian motion process $W(t)$ is a continuous process which is nowhere differentiable
- The only quantities allowed to be used are the infinitesimal changes of the process, in our case, $dW(t)$. We consider a process $X(t)$ and for small increments of time dt, define

$$dX(t) = X(t + dt) - X(t)$$

We list some basic rules similar to elementary differentiation rules.

1. If a, b are constants, then

$$d\left(aX(t) + bY(t)\right) = adX(t) + bdY(t)$$

2. *Product Rule*: If $X(t)$ and $Y(t)$ are stochastic processes, then

$$d(X(t), Y(t)) = dX(t)Y(t) + X(t)dY(t) + dX(t)dY(t) \quad (1)$$

Back to Quadratic Variation

- Recall definition of the discrete quadratic variation

$$Q_h = \sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)]^2.$$

- It converges in mean square sense to T

$$\lim_{h\to 0, n\to +\infty} \mathbb{E}[(Q_h - T)^2] = 0.$$

- Since $dW(t) = W(t+dt) - W(t)$,this can be written as

$$\int_0^T (dW(t))^2 = \int_0^T dt$$

- Or in the differential form

$$(dW(t))^2 = dt, dW(t) \sim N(0, dt), (dW(t))^2 \sim N(dt, 0)$$

Chain Rule

1. Let f be a differentiable function of real variable x. Let x_0 be fixed and consider $\Delta x = x - x_0$ and $\Delta f(x) = f(x) - f(x_0)$.

$$\Delta f(x) = f^{'}(x_0)\Delta x + \frac{1}{2}f^{''}(x_0)(\Delta x)^2 + \mathcal{O}(\Delta x)^3$$

2. For x infinitesimally close to x_0 we can write

$$df(x) = f^{'}(x)dx$$

3. If $x = x(t)$ is a differentiable function of t then we get the well known chain rule:

$$df(x(t)) = f^{'}(x(t))dx(t) = f^{'}(x(t))x^{'}(t)dt$$

4. In the stochastic case, the deterministic function $x(t)$ is a stochastic process $X(t)$; let $F(t) = f(X(t))$.

$$dF(t) = f^{'}(X(t))dX(t) + \frac{1}{2}f^{''}(X(t))(dX(t))^2 \qquad (2)$$

We need to take into account $(dW(t))^2 = dt$, $dW(t)dt = 0$ and $(dt)^2 = 0$

Itô formula for diffusions

1. Let the increment of $X(t)$ is given by Let $F(t) = f(X(t))$, where $f(x)$ be a twice differentiable function.

$$dX(t) = a(t, W(t))dt + b(t, W(t))dW(t) \tag{3}$$

2. Let $F(t) = f(X(t))$, $f(x)$ be a twice differentiable function.

$$(dX(t))^2 = (a(t, W(t))dt + b(t, W(t))dW(t))^2 =$$

$$= b^2(t, W(t))(dW(t))^2$$

Using 2 we get the following formula:

$$d(F(t)) = \left(a(t, W(t))f^{'}(X(t)) + \frac{b^2(t, W(t))f^{''}(X(t))}{2} \right) dt +$$

$$+ b(t, W(t))f^{'}(X(t))dW(t)$$

Particular cases

Let $X(t) = W(t)$. If $F(t) = f(W(t))$ then we have

$$dF(t) = \frac{1}{2}f^{''}(W(t))dt + f^{'}(W(t))dW(t) \tag{4}$$

Examples:

1. $f(x) = x^{\alpha}$, α is a constant, then $f^{'}(x) = \alpha x^{\alpha-1}$, $f^{''}(x) = \alpha(\alpha-1)x^{\alpha-2}$. Then 4 becomes

$$d(W^{\alpha}(t)) = \frac{1}{2}\alpha(\alpha-1)W^{\alpha-2}(t)dt + \alpha W^{\alpha-1}(t)dW(t)$$

   a. $d(W^2(t)) =$
   b. $d(W^3(t)) =$

2. $$f(x) = e^x$$

3. $$f(x) = \sin(x)$$

Case of time dependent function

In case when $f = f(t,x)$ is also time dependent, the Itô formula becomes:

$$df(t,x) = \partial_t f(t,x)dt + \partial_x f(t,x)dx + \frac{1}{2}\partial_x^2 f(t,x)(dx)^2 + \mathcal{O}(dx)^3 + \mathcal{O}(dt)^2$$

- If $X(t)$ is an Itô diffusion there is an extra term in the previous formula:

$$dF(t) = \left(\partial_t f(t,x)dt + a(W)t), t)\partial_x f(t,x) + \frac{1}{2}\partial_x^2 f(t,x)\right) dt +$$

$$+b(W(t),t)\partial_x f(t,x)dW(t)$$

- Example: calculate

$$d(tW^2(t))$$

Stochastic Integration

1. Consider a process $X(t)$ whose increments satisfy

$$dX(t) = f(t, W(t))dW(t)$$

   Integrating between t_0 and t we get

$$\int_{t_0}^{t} dX(s) = \int_{t_0}^{t} f(s, W(s))ds$$

   This integral is an Itô integral.

$$X(t) = X(t_0) + \int_{t_0}^{t} f(s, W(s))ds$$

2. Since $X(t_0)$ is a constant, for $t_0 < t$, we

$$d\left(\int_{a}^{t} f(s, W(s))dW(s)\right) = f(t, W(t))dW(t) \tag{5}$$

3. If $Y(t)$ is a stochastic process and $Y(t)dW(t) = dF(t)$, then

$$\int_{a}^{b} Y(t)dW(t) = F(b) - F(a) \tag{6}$$

Stochastic Integration by Parts

(i) Consider

$$F(t) = f(t)g(W(t))$$

where f and g are differentiable functions. Using product rule, and Itô lemma for $dg(W(t))$, we get

$$\int_a^b f(t)g^{'}(W(t))dW(t) = f(t)g(W(t))|_a^b - \int_a^b f^{'}(t)g(W(t))dt$$

$$-\frac{1}{2}\int_a^b f(t)g^{''}(W(t))dt$$

This formula is useful when integrating a product of function of t and a function of Brownian motion for which the integral is known.

Applications

1. Consider a Wiener integral
$$I_T = \int_0^T t dW(t)$$
We know its distribution (normal, variance =?). Define the integrated Brownian motion
$$Z_T = \int_0^T W(s) ds$$
Using the integration by parts, we can get the relationship between I_T and Z_T.
2. Let $g(x) = \frac{x^2}{2}$, $f(t) = 1$. Then
$$\int_a^b W(t) dW(t) = \frac{W^2(b) - W^2(a)}{2} - \frac{1}{2}(b-a)$$
3. $g(x) = \frac{x^3}{3}$, $f(t) = 1$.
4. $f(t) = e^{\alpha t}$, $g(x) = \cos(x)$.

Definition of SDE

Let $X(t)$ be a continuous stochastic process. Assume its increments can be written as a linear combination of increments in t and increments of Brownian motion $W(t)$, we may write

$$dX(t) = a(t, W(t), X(t))dt + b(t, W(t), X(t))dW(t) \tag{7}$$

and call it *stochastic differential equation*. It has the following integral meaning

$$X(t) = X(0) + \int_0^t a(s, W(s), X(s))ds + \int_0^t b(s, W(s), X(s))dW(s) \tag{8}$$

where the last integral is taken in the Itô sense. The eq 8 is taken as the definition for the stochastic differential equation 7. We will approach the SDE informally, by the analogy with the ODE, presenting the same methods of solving equations in the new stochastic environment.
The function $a(t, W(t), X(t))$ and $b(t, W(t), X(t))$ are called *drift rate* and *volatility*.

Example: Geometric Brownian Motion (GBM)

(a) In the BS world, the stock price is assumed to follow the GBM:

$$dS = \mu S dt + \sigma S dW(t) \tag{9}$$

where the expected rate on the stock μ and the volatility σ are constants. This is a version of 7 with $a = \mu S$ and $b = \sigma S$.

(b) In this case the SDE can be integrated easily:

$$\frac{dS}{S} = \mu dt + \sigma dW(t)$$

We introduce $f(S) = \ln(S)$, $f' = \frac{1}{S}$, $f'' = -\frac{1}{S^2}$. By Itô

$$df(S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t)$$

(c) Since μ and σ are constant, we will get

$$S(t) = S(0) \exp\left(\sigma W(t) + (\mu - \frac{1}{2}\sigma^2)t\right)$$