Quantitative Methods: Midterm

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Problem 1 (22 points): We consider the Binomial tree model seen in the lectures, with 3 periods (i.e. $N = 3$) and with the parameters

$$r = 1/4, d = 1/2, u = 2$$

where r is a simple interest rate, d and u respectively represent the down and up factors. Next, consider a stock price with value $S_0 = 4$ at time 0. We consider the price V_n of a European call option, at time n, with payoff

$$V_3 = (S_3 - 5)^+$$

at time 3.

1. Specify the risk-neutral probability weights.

   Solution: We suppose that we are observing an arbitrage free market with a risky asset S and some risk-free asset (numeraire) with interest rate r so that we know by the Fundamental Theorem of Asset Pricing that we should have a risk-neutral measure Q (moreover, here it is unique) described by:

   $$S_0 = \mathbb{E}^{\mathbb{Q}}[S_1/(1+r)] = \frac{1}{1+r}\left[puS_0 + (1-p)dS_0\right]$$
   $$\text{where } \mathbb{P}[S_1^u] = p, \mathbb{P}[S_1^d] = 1 - p \text{ define probability measure } Q$$
   $$\Rightarrow p = \frac{S_0(1+r) - dS_0}{uS_0 - dS_0} = \frac{1+r-d}{u-d} = \frac{1+1/4-1/2}{3/2} = \frac{1}{2}$$

2. Compute the price V_0 of the call option at time 0.

   Solution: We then note that the European call option as a contingent claim in an arbitrage-free extended market should also satisfy:

   $$V_0 = \mathbb{E}^{\mathbb{Q}}[V_3/(1+r)^3] \text{ where } \mathbb{Q} \text{ is the same risk-neutral measure as before}$$
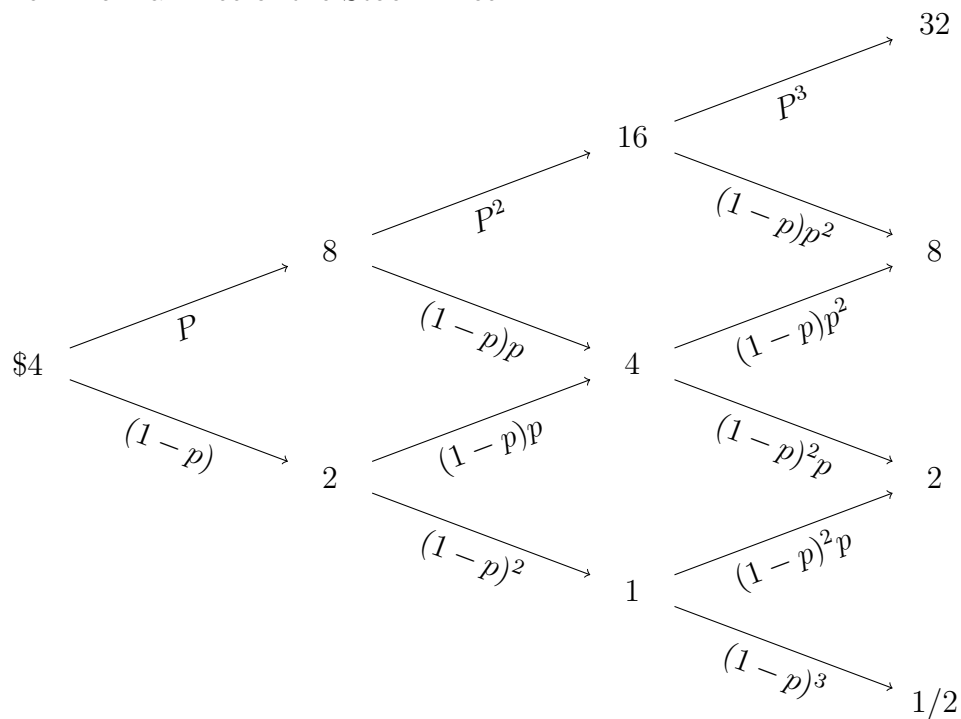
So then we have: $V_0 =$

$= \frac{1}{(1+r)^3}\left[\binom{3}{0}p^3(u^3S_0 - K) + \binom{3}{1}p^2(1-p)(u^2dS_0 - K) + 0 + 0\right]$

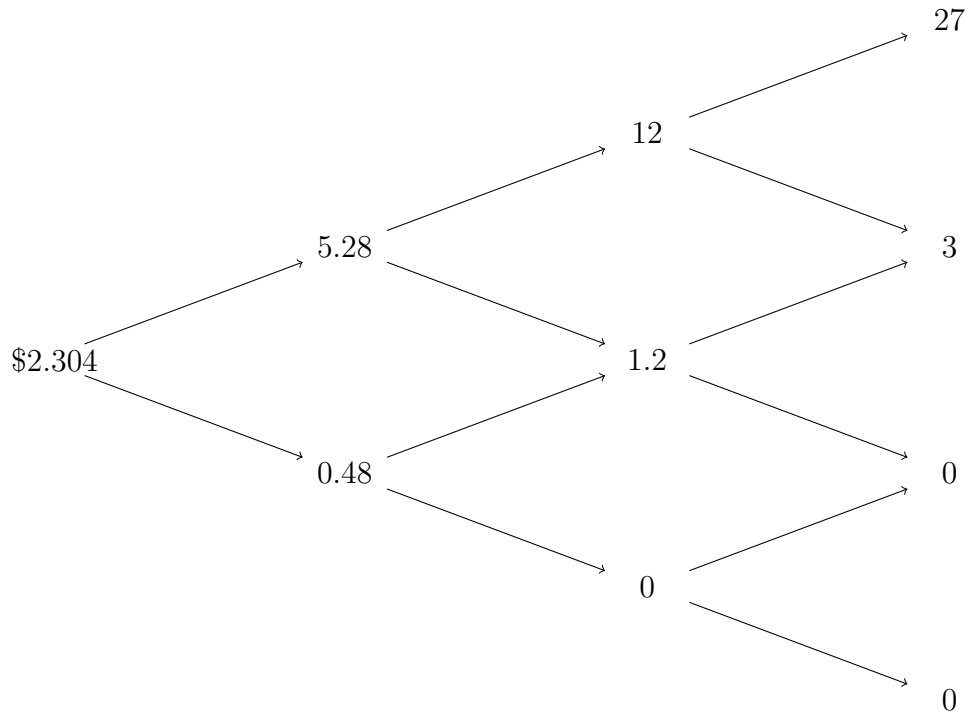$= (\frac{4}{5})^3\left[\frac{1}{8}(8 \cdot 4 - 5) + 3\frac{1}{8}(2 \cdot 4 - 5)\right] = \$2.304$

3. Give also the delta hedging strategies δ_0, δ_1, and δ_2 at the respective times $n = 0, 1, 2$.

Solution:

The Binomial Tree of the Stock Price:

The Binomial Tree of Call Option:

27

12

5.28

3

$2.304

1.2

0.48

3

0

0

0

We have the delta-hedging strategy formula by: $\delta_n(s) = \frac{v_{n+1}(2s) - v_{n+1}(\frac{1}{2}s)}{2s - \frac{1}{2}s}$ so that

$$\delta_0(4) = \frac{5.28 - 0.48}{8 - 2} = 0.8$$

$$\delta_1(8) = \frac{12 - 1.2}{16 - 4} = 0.9$$

$$\delta_1(2) = \frac{1.2 - 0}{4 - 1} = 0.4$$

$$\delta_2(16) = \frac{27 - 3}{32 - 8} = 1.0$$

$$\delta_2(4) = \frac{3 - 0}{8 - 2} = 0.5$$

$$\delta_2(1) = \frac{0 - 0}{2 - 0.5} = 0$$

4. Consider an agent who is short 100 options. How many shares of the underlying stock does the agent hold in the hedging portfolio during the first time period? What is the agent's cash position at time 0? Does the agent invest in or borrow from the money market account to finance the hedging strategy during the first time period?

Solution: We note that our delta-hedging strategy formula indicates the number

of shares of the underlying stock our agent would need to hold in order to hedge shorting one option, in the first time period. If our agent shorts 100 options, our agent holds $100\delta_0(4) = 100 \cdot 0.8 = 80$ shares of the underlying stock and invests his or her cash position of $100V_0 - 100\delta_0(4) \cdot S_0 = 100 \cdot 2.304 - 80 \cdot 4 = -\$89.6$ in a money market account. As our agent has a negative investment in a money market account, the agent borrows money from a money account to finance the hedging strategy during the first time period.

Problem 2 (15 points):

Consider the sequence of inter-arrivals times $T_1, T_2, \cdots, T_n, \cdots$ and the sequence of arrivals times $S_1, S_2, \cdots, S_n, \cdots$ of a Poisson process $\{N(t), t \geq 0\}$ with parameter $\lambda > 0$. Note that we have the relation

$$S_n = \sum_{i=1}^{n} T_i$$

We know that T_i has an exponential distribution with parameter λ, with mean $1/\lambda$ and variance $1/\lambda^2$.

Show that S_n is approximately normally distributed for n large. Give its approximate mean and variance.

Solution: We note that this follows directly from the central limit theorem by observing that as $\bar{T}_n = \frac{S_n}{n}$ and the family of $\{T_i\}_{i \in \mathbb{N}}$ are identically and independently distributed with finite variance $1/\lambda^2$ and finite mean $1/\lambda$, by the Levy Central Limit Theorem:

$$\sqrt{n}(\bar{T}_n - 1/\lambda) \xrightarrow{d} N(0, \sigma^2 = 1/\lambda^2).$$

It then follows that we can expect that for sufficiently large n,
$S_n = n\bar{T}_n \approx n \cdot N(1/\lambda, 1/(\sqrt{n}\lambda)^2)$ so that $S_n \sim N(n/\lambda, n/\lambda^2)$

Problem 3 (16 points):

Consider a sequence of independent and identically distributed random variables $X_1, X_2, \cdots, X_n, \cdots$ such that $X_i \geq 0$, with common mean $\mathbb{E}[X_i] = 1$. We define the sequence $M_n = \Pi_{i=1}^{n} X_i$

1. Compute $\mathbb{E}[M_n]$

Solution:

$$
\begin{aligned}
\mathbb{E}[M_n] &= \mathbb{E}[\Pi_{i=1}^{n} X_i] \\
&= \Pi_{i=1}^{n} \mathbb{E}[X_i] \text{ Follows from independence} \\
&= \Pi_{i=1}^{n} 1 = 1
\end{aligned}
$$

2. Consider the integers n, p such that $p < n$. Show that

$$\mathbb{E}[M_n \mid \{X_k, 1 \le k \le p\}] = M_p$$

Solution:

$$
\begin{aligned}
\mathbb{E}[M_n \mid \{X_k, 1 \le k \le p\}] &= \mathbb{E}[\Pi_{i=p+1}^n X_i \Pi_{j=1}^p X_j \mid \{X_k, 1 \le k \le p\}] \\
&= \mathbb{E}[\Pi_{i=p+1}^n X_i \mid \{X_k, 1 \le k \le p\}]\Pi_{j=1}^p X_j \\
&\quad \text{As here } \{X_k, 1 \le k \le p\} \text{ are determined} \\
&= \Pi_{i=p+1}^n \mathbb{E}[X_i \mid \{X_k, 1 \le k \le p\}] \cdot M_p \\
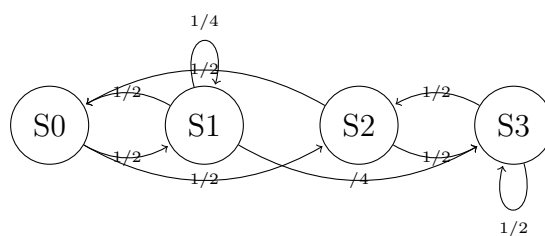&= 1 \cdot M_p = M_p
\end{aligned}
$$

3. Conclude.

Solution: Our random variable M_n is a martingale.

Problem 4 (20 points): We consider a discrete time and discrete space Markov chain X_n with state space $\{0, 1, 2, 3\}$ that has the one-step transition probability matrix.

$$
P = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 \\
1/2 & 1/4 & 0 & 1/4 \\
1/2 & 0 & 0 & 1/2 \\
0 & 0 & 1/2 & 1/2
\end{bmatrix}
$$

and assume that $X_0 = 0$

1. Plot a diagram representing the chain

Solution:

2. Is this Markov chain irreducible? Justify your answer

> **Solution:** To show that the Markov chain is irreducible, we want to show that every state in the state space communicates with all other state spaces, i.e. there is a non-zero probability to reach any other state from a given state space. We note that that state 1 communicates to all other states (0,2,3) as there is a non-zero probability to transitioning to all other state spaces in one step. It then suffices to note that any state $i, j \neq 1$ can communicate with one another through state 1 with $i \to 3 \to j$ and $j \to 1 \to i$

3. Compute $P^{(2)}$

> **Solution:**
> $$P^{(2)} = \begin{bmatrix} 1/2 \cdot 1/2 + 1/2 \cdot 1/2 & 1/2 \cdot 1/4 & 0 & 1/2 \cdot 1/4 + 1/2 \cdot 1/2 \\ 1/2 \cdot 1/4 & 1/2 \cdot 1/2 + 1/4 \cdot 1/4 & 1/2 \cdot 1/2 + 1/4 \cdot 1/4 & 1/4 \cdot 1/4 + 1/4 \cdot 1/2 \\ 0 & 1/2 \cdot 1/2 & 1/2 \cdot 1/2 + 1/2 \cdot 1/2 & 1/2 \cdot 1/2 \\ 1/2 \cdot 1/2 & 0 & 1/2 \cdot 1/2 & 1/2 \cdot 1/2 + 1/2 \cdot 1/2 \end{bmatrix}$$
> $$= \begin{bmatrix} 1/2 & 1/8 & 0 & 3/8 \\ 1/8 & 5/16 & 3/8 & 3/16 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \end{bmatrix}$$

4. Compute $\mathbb{E}[X_2 \mid X_0 = 0]$

> **Solution:** $\mathbb{E}[X_2 \mid X_0 = 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \cdot P^{(2)} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$
>
> $= \begin{bmatrix} 1/2 & 1/8 & 0 & 3/8 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \frac{5}{4}.$

5. What is the probability that the chain will move to the state 2 for the first time after exactly 4 steps?

> **Solution:** Note that in order for the chain to move to state 2 for the first time after exactly 4 steps, we must observe the probability of the event that the chain

does not move to state 2 at steps $n = 0, 1, 2, 3$ and transitions to state 2 on the 4th step. In order to do so, we consider modified transition matrix:

$$P' = \begin{bmatrix} 0 & 1/2 & \mathbf{0} & 0 \\ 1/2 & 1/4 & \mathbf{0} & 1/4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & \mathbf{0} & 1/2 \end{bmatrix}$$

This modification follows from the fact that we are observing the transition between states not involving state 2 for steps $n = 0, 1, 2, 3$ so we omit the transitions to state 2 by replacing their transition probabilities with 0 from the original matrix.

$$\mathbb{P}[\text{ first time after exactly 4 steps } | X_0 = 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \cdot (P')^3 \cdot P \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{5}{64}$$

Problem 5 (27 points):

We consider a discrete time and discrete space Markov chain X_n with state space $\{0, 1, 2, \cdots R\}$ modeling the wealth of a gambler. The gambler starts with an initial capital equal to i dollars the time $n = 0$. He or she tosses a coin at each round and either gets a head with probability p or gets a tail with probability $q = 1 - p$. Every time the gambler gets a head, he or she earns one dollar and every time he or she gets a tail, he or she loses one dollar. The gambler stops playing whenever he or she is ruined, i.e. his capital X_n goes down to 0 or when his or her capital reaches R, i.e. $X_n = R$. We also considered the variable M_i, which represents the average number of rounds that must be played until the gambler goes bankrupt or reaches a wealth equal to R, given an initial capital equal to i

1. Show that $M_0 = M_R = 0$

> **Solution:** We note that by definition, we are bankrupt when we start at capital $X_0 = 0$ and we have wealth R when we start at capital $X_0 = R$. It then follows that we don't play the game if we start at these initial positions so $M_0 = M_R = 0$.

2. Argue that for all $i = 1, \cdots, R - 1$,

$$M_i = 1 + pM_{i+1} + qM_{i-1}$$

Solution: Let us fix $i \in [1, R-1]$. Let T be a random variable that is the number of rounds played until we either go bankrupt or reach wealth R. We then have that $M_i = \mathbb{E}[T \mid X_0 = i]$ Note that $M_i = \mathbb{E}[T \mid X_0 = i]$
$= \mathbb{E}[T \mid X_0 = i \cap \text{"First coin flip is tails"}]\mathbb{P}[\text{"First coin flip is tails"}] +$
$+ \mathbb{E}[T \mid X_0 = i \cap \text{"First coin flip is heads"}]\mathbb{P}[\text{"First coin flip is heads"}]$
$= p\mathbb{E}[T \mid X_1 = i+1] + q\mathbb{E}[T \mid X_1 = i-1]$
$= p(1 + \mathbb{E}[T \mid X_0 = i+1]) + q(1 + \mathbb{E}[T \mid X_0 = i-1])$
$= p + q + pM_{i+1} + qM_{i-1}$
$= 1 + pM_{i+1} + qM_{i-1}$

3. In this question, we choose $p = 0.5$. Show that $M_i = i(R-i)$

Solution: From (2), we have that $\forall i \in [1, R-1]$,

$$M_i = 1 + pM_{i+1} + qM_{i-1} \Rightarrow$$
$$\frac{1}{2}(M_i - M_{i-1}) = 1 + \frac{1}{2}(M_{i+1} - M_i) \Rightarrow$$
$$\frac{1}{2}(M_{i+1} - M_i) = \frac{1}{2}(M_i - M_{i-1}) - 1 = \frac{1}{2}(M_1 - M_0) - i$$

We then sum the above equation:

$$2\sum_{i=1}^{j-1}\frac{1}{2}(M_{i+1} - M_i) = 2\Big(\sum_{i=1}^{j-1}\frac{1}{2}(M_1 - M_0) - i\Big)$$
$$M_j - M_1 = jM_1 - (j-1)j \Rightarrow M_j = (j-1)M_1 - j(j-1)$$
$$M_j = jM_1 - j(j-1)$$

It then remains for us to find M_1:
We have that $M_{R-1} = 1 + pM_R + qM_{R-2} = 1 + qM_{R-2} = 1 + \frac{1}{2}M_{R-2}$. Using the above result, we have that

$$M_{R-1} = (R-1)M_1 - (R-1)(R-2), M_{R-2} = (R-2)M_1 - (R-2)(R-3) \Rightarrow$$
$$2(R-1)M_1 - 2(R-1)(R-2) - 2 = (R-2)M_1 - (R-2)(R-3) \Rightarrow$$
$$R \cdot M_1 = (R-2)(R+1) + 2 \Rightarrow M_1 = \frac{(R-2)(R+1) + 2}{R}$$

We then have that $M_i = i\frac{(R-2)(R+1)+2}{R} - i(i-1)$
$= i\frac{R^2-R}{R} - i(i-1) = (R-1)i - (i-1)i = i(R-i)$

4. Now, we assume that $p \neq 0.5$. Show that

$$M_i = \frac{i}{q-p} - \frac{R}{q-p}\frac{1-(q/p)^i}{1-(q/p)^R}$$

Solution: We start by proceeding in a similar fashion as in (3) and by noting that $p + q = 1$,

$$(p+q)M_i = 1 + pM_{i+1} + qM_{i-1} \Rightarrow$$
$$q(M_i - M_{i-1}) = 1 + p(M_{i+1} - M_i) \Rightarrow$$
$$(M_{i+1} - M_i) = \frac{q}{p}(M_i - M_{i-1}) - \frac{1}{p} = \cdots$$
$$= (\frac{q}{p})^i(M_1 - M_0) - \sum_{n=0}^{i-1}\frac{1}{p}(\frac{q}{p})^n$$
$$= (\frac{q}{p})^i M_1 - \frac{1}{p}\frac{(\frac{q}{p})^i - 1}{\frac{q}{p} - 1}$$
$$= (\frac{q}{p})^i M_1 - \frac{(\frac{q}{p})^i - 1}{q - p}$$

We then sum the above equation:

$$\sum_{i=1}^{j-1}(M_{i+1} - M_i) = \sum_{i=1}^{j-1}\left((\frac{q}{p})^i M_1 - \frac{(\frac{q}{p})^i - 1}{q - p}\right)$$
$$M_j = \sum_{i=0}^{j-1}(\frac{q}{p})^i M_1 - \sum_{i=1}^{j-1}\frac{(\frac{q}{p})^i - 1}{q - p}$$
$$M_j = \frac{(\frac{q}{p})^j - 1}{q/p - 1}M_1 - \sum_{i=0}^{j-1}\frac{(\frac{q}{p})^i}{q - p} + \frac{1}{q - p} + \frac{(j-1)}{q - p}$$
$$= M_1\frac{(q/p)^j - 1}{q/p - 1} - \frac{1}{q - p}\left(\frac{(q/p)^j - 1}{q/p - 1} - j\right)$$

It then remains for us to find M_1:

We have that $M_{R-1} = 1 + pM_R + qM_{R-2} = 1 + qM_{R-2}$. Using the above result, we have that

$$M_{R-1} = M_1\frac{(q/p)^{R-1} - 1}{q/p - 1} - \frac{1}{q - p}\left(\frac{(q/p)^{R-1} - 1}{q/p - 1} - R + 1\right)$$
$$1 + pM_{R-2} = 1 + qM_1\frac{(q/p)^{R-2} - 1}{q/p - 1} - \frac{q}{q - p}\left(\frac{(q/p)^{R-2} - 1}{q/p - 1} - R + 2\right) \Rightarrow$$

$$M_1\left(\frac{(q/p)^{R-1} - 1 - q(q/p)^{R-2} + q}{q/p - 1}\right)$$

$$= \frac{(q/p - q)(q/p)^{R-2} - 1 + q + (q/p - 1)(-R + 1 + qR - 2q + q - p)}{(q/p - 1)(q - p)} \Rightarrow$$

$$M_1 = \frac{\left[(q/p - q)(q/p)^{R-2} - 1 + q\right] + \left[(q/p - 1)(-R + 1 + qR - 2q + q - p)\right]}{(q - p)((q/p - q)(q/p)^{R-2} - 1 + q)}$$

$$= \frac{1}{q - p} + \frac{R(q - 1) + 1 - q - p}{p((q/p - q)(q/p)^{R-2} - 1 + q)}$$

[Here we use the fact that $p + q = 1$ several times]

$$= \frac{1}{q - p} + \frac{-pR + 1 - 1}{p((q/p - q)(q/p)^{R-2} - p)}$$

$$= \frac{1}{q - p} + \frac{-R}{(q/p - q)(\frac{p^2}{q^2})(q/p)^R - p}$$

$$= \frac{1}{q - p} + \frac{-R}{(p/q - p^2/q)(q/p)^R - p}$$

$$= \frac{1}{q - p} + \frac{-R}{p[(q/p)^R - 1]}$$

We then have that

$$M_i = \left(\frac{1}{q - p} + \frac{-R}{p[(q/p)^R - 1]}\right)\frac{(q/p)^i - 1}{q/p - 1} - \frac{1}{q - p}\left(\frac{(q/p)^i - 1}{q/p - 1} - i\right)$$

$$= \frac{i}{q - p} + \frac{-R}{p[(q/p)^R - 1]}\frac{(q/p)^i - 1}{q/p - 1}$$

$$= \frac{i}{q - p} - \frac{R}{q - p}\frac{1 - (q/p)^i}{1 - (q/p)^R}$$