Quantitative Methods: Assignment 2

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Problem 1 (30 points):

1. Write a program to implement a symmetric random walk X_n of n steps, starting at $X_0 = 0$, using a random number generator to choose the direction of each step. Provide a printout of your code.

2. Run your program for $N = 10,000$

3. Plot X_n as a function of n, for $0 \leq n \leq N$.

4. Set

$$W_n = \frac{1}{\sqrt{n}} X_n$$

   Plot W_n as a function of n, for $0 \leq n \leq N$.

We have the following code:

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Hw2_1.m

```
N = 10000;
q = 0.5;
r = rand(1,N);
for i = 1:N
    if r(i) > q
        r(i) = 1;
    else
        r(i) = -1;
    end
end
P = cumsum(r);
x = 1:N;
w_x = sqrt(x);
W_n = P./w_x;
figure(1) ;
%hold on;
plot(x, P, '*');
xlabel('Step');
ylabel('Position');

figure(2)
plot(x, W_n, '-');
xlabel('Step');
ylabel('Position');
%hold off;

```

From which we produce the following random walk (zoomed out):

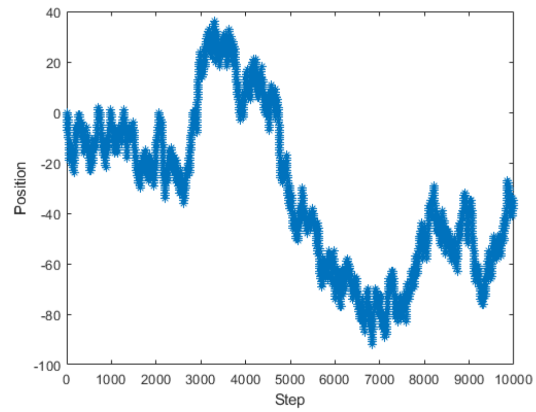

Figure 1: X_n

And we get the following W_n as a function of n:

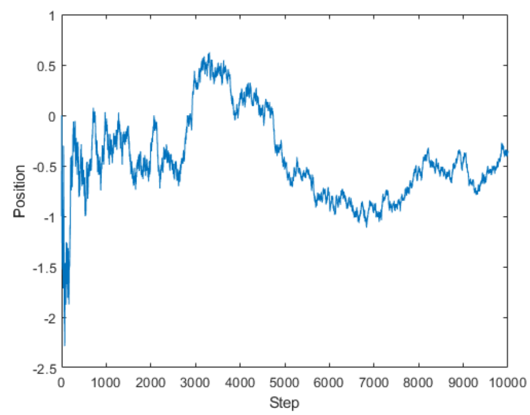

Figure 2: W_n

Problem 2 (30 points):

Consider a random walk which is not necessarily symmetric, where the probability of a head is given by $p \in (0,1)$, where as the probability of a tail is $q = 1 - p$. As you know, the random is walk is not necessarily a martingale

1. What is the expectation of the random walk for $p = 0.3$?

Solution: $\forall i \in \mathbb{N}, \mathbb{E}[X_i] = 1 \cdot p + (-1) \cdot q = 0.3 - 0.7 = -0.4$. It then follows that $\lim_{n\to\infty} \mathbb{E}[S_n]$
$= \lim_{n\to\infty} \mathbb{E}[\sum_{i=1}^{n} X_i] = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{\infty} -0.4 = -\infty$.

2. Plot a sample path of the random walk corresponding to $p = 0.3$ for $N = 10,000$ steps.
Solution:

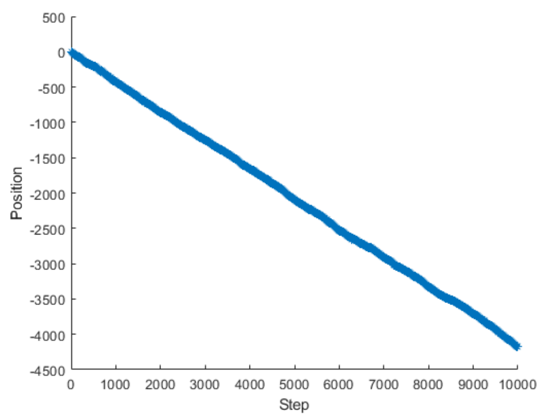

Figure 3: X_n with $p = 0.3$

3. Does this random walk have a tendency to go up for down?

Solution: This random walk has a tendency to go down.

4. Answer the previous 3 questions for a random walk corresponding to $p = 0.7$
Solution:

i) $\forall i \in \mathbb{N}, \mathbb{E}[X_i] = 1 \cdot p + (-1) \cdot q = 0.7 - 0.3 = 0.4$. It then follows that $\lim_{n\to\infty} \mathbb{E}[S_n]$ $= \lim_{n\to\infty} \mathbb{E}[\sum_{i=1}^{n} X_i] = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{\infty} 0.4 = +\infty$.
ii)
iii) This random walk has a tendency to go up.

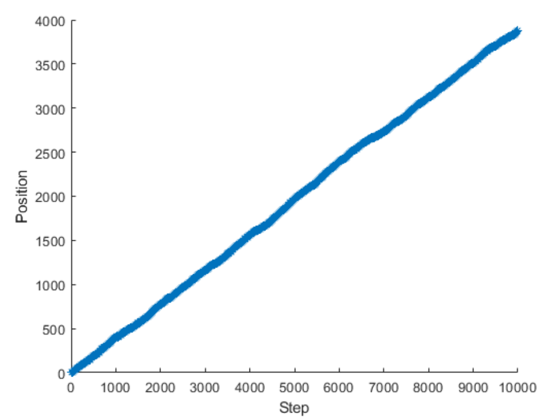

Figure 4: X_n with $p = 0.7$

Problem 3 (25 points):

Consider the integral

$$a = \int_0^1 f(x)dx$$

It can be interpreted as the following expectation

$$\mathbb{E}[f(U)]$$

where U is a random variable that is uniformly distributed on the interval (0,1). Consider next a sequence $U_1, U_2, \cdots, U_n$ of independent and uniformly distributed random variables in the interval (0,1).

1. Show that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} f(U_i) = a, \text{ almost surely .}$$

> **Solution:** This follows directly from the Strong Law of Large Numbers, as $\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} f(U_i) = \mathbb{E}[f(U)] = \int_0^1 f(x)d(U(X)) = \int_0^1 f(x)dx = a.$

2. What is the distribution of the error

$$\frac{1}{n}\sum_{i=1}^{n} f(U_i) - a?$$

> **Solution:** We note that as $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of random variables, $\{f(U_i)\}_{i\in\mathbb{N}}$ is also a sequence of random variables (as it maps the measurable codomain space of f to another measurable subspace of $\mathbb{R}$). We also observe that $\mathbb{E}[f(U)] = a$ implies that $f(U)$ has finite expectation. If we suppose that $f(U)$ has some finite variance σ^2, by the central limit theorem, we have that $\frac{1}{n}\sum_{i=1}^{n} f(U_i)$ converges in distribution to a normal distribution with mean $a = \mathbb{E}[f(U)]$ and variance σ^2/n. It then follows that the error is convergent in distribution to a normal distribution.

Problem 4 (15 points):

Let $\{X_n, n = 1, 2, \cdots\}$ be a discrete-time and discrete-state stochastic process such that, for every n, i, j and every $i_1, \cdots i_{n-1}$,

$$\mathbb{P}[X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \cdots, X_1 = i_1] = \mathbb{P}[X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}]$$

1. Is this process Markovian? Explain.

Solution: No, we are observing a Markov chain of order 2 which is not Markovian in the usual sense as it depends on the observation of two previous states as opposed to the last state.

2. Can you make it Markovian by modifying it slightly? In other words, can you define another Markov process (Y_n) which is based on X_n but whose definition differs slightly from the definition of (X_n) and which satisfies the Markov property? IF this is possible, give the definition of (Y_n) and show that (Y_n) is Markovian.

Solution: We can define $Y_n = (X_n, X_{n-1})$ to satisfy the Markov property. We note that:

$$\begin{aligned}
\mathbb{P}[Y_{n+1} = j_{n+1} \mid X_n = i, X_{n-1} = i_{n-1}, \cdots, X_1 = i_1] \\
= \mathbb{P}[Y_{n+1} = j_{n+1} \mid X_n = i, X_{n-1} = i_{n-1}] \\
= \mathbb{P}[Y_{n+1} = j_{n+1} \mid Y_n = j_n] \text{ where we define } j_n := (i, i_{n-1})
\end{aligned}$$