FRE-GY 6233: Assignment 9

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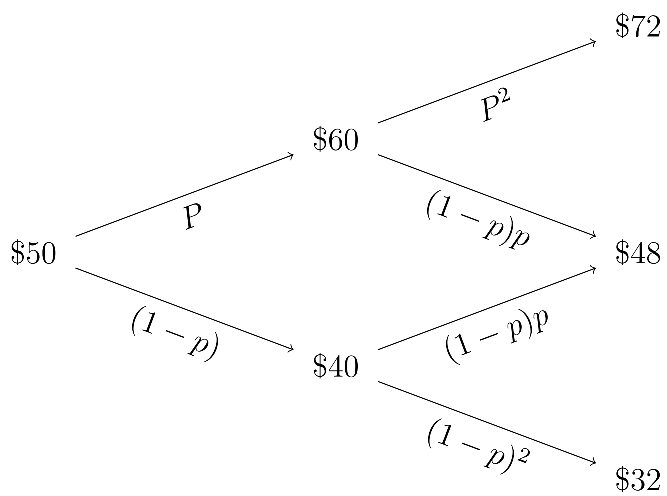
November 23, 2020

Problem 1

- We assume that over each of the next three years the stock price either moves up by 20 percent or moves down by 20 percent . The risk-free interest rate is 5 percent
- Find the value of a 2-year American put with a strike price of 49 on a stock whose current price is 50.

Solution: We have the following binomial pricing tree:



We find that the risk-neutral measure $\mathbb{P}$ for the stock is defined by $p = \mathbb{P}[S_{n+1} = 1.2 \cdot S_n] = \frac{1.05-0.8}{1.2-0.8} = \frac{5}{8}$
If we denote that the value of the option at time t, price $S(t) = s$ as $v(t, s)$, we have that:

$$
\begin{aligned}
v(2, 72) &= (49 - 72)^+ = \$0 \\
v(2, 48) &= (49 - 48)^+ = \$1 \\
v(2, 32) &= (49 - 32)^+ = \$17 \\
v(1, 60) &= \max\{\frac{1}{1+r}\big(p \cdot v(2, 72) + (1-p) \cdot v(2, 48)\big), (49 - 60)^+\} = \$0.36
\end{aligned}
$$

$$v(1,40) = \max\{\frac{1}{1+r}\big(p \cdot v(2,32) + (1-p) \cdot v(2,48)\big), (49-40)^+\} = \$9$$
$$v(0,50) = \max\{\frac{1}{1+r}\big(p \cdot v(1,40) + (1-p) \cdot v(1,60)\big), (49-50)^+\} = \$3.43$$

Problem 2

Derive the relationship (inequality) between American call C and put P on stock with price S, strike K (no dividends)

Solution: We first note that we have:

$$\begin{aligned}
&C_A - P_A \\
&= C_E - P_E + P_E - P_A \\
&= S_0 - Ke^{-r(T-t)} + P_E - P_A \\
&\leq S_0 - Ke^{-r(T-t)}
\end{aligned}$$

We can derive a left-side inequality with the following arbitrage argument:

- At time 0, we long a call option, short a put option, short the underlying stock, and borrow K cash for $-C_A + P_A + S_0 - K$.

In order to not have an arbitrage situation a maturity, we need this value to be less than 0 so that an arbitrage-free requirement is that $S(0) - K \leq C_A - P_A$ so that we get relation:

$$S(0) - K \leq C_A - P_A \leq S(0) - Ke^{-r(T-t)}$$

Problem 3

Consider notations

$$\tau_1 \vee \tau_2 = \max\{\tau_1, \tau_2\}, \tau_1 \wedge \tau_2 = \min\{\tau_1, \tau_2\}$$

Let τ_1 and τ_2 be stopping times w.r.t filtration $\mathcal{F}_t$. Show that the following are also stopping times.

1. $\tau_1 \vee \tau_2$

Solution: We have that $\{\tau_1 \leq t\}, \{\tau_2 \leq t\} \in \mathcal{F}_t$ for all $t \in T$. Note that then: $\{\tau_1 \vee \tau_2 \leq t\} = \bigcap_{i \in \{1,2\}} \{\tau_i \leq t\} \in \mathcal{F}_t$ for all $t \in T$. This naturally belongs to $\mathcal{F}_t$ as the intersection of elements of the sub-sigma-algebra.

2. $\tau_1 \wedge \tau_2$

> **Solution:** We have that $\{\tau_1 \leq t\}, \{\tau_2 \leq t\} \in \mathcal{F}_t$ for all $t \in T$. Note that then if we fix $t \in T$:
> $\{\tau_1 \wedge \tau_2 \leq t\} = \bigcup_{i\in\{1,2\}}\{\tau_i \leq t\}$. This naturally belongs to $\mathcal{F}_t$ as the union of elements of the sub-sigma-algebra.

3. $\tau_1 + \tau_2$

> **Solution:** We have that $\{\tau_1 \leq t\}, \{\tau_2 \leq t\} \in \mathcal{F}_t$ for all $t \in T$. Note that then if we fix $t \in T$:
> $\{\tau_1 + \tau_2 \leq t\} = \bigcup_{r,s\in\mathbb{Q},r+s<t}\Big(\{\tau_1 \leq s\} \cap \{\tau_2 \leq r\}\Big)$. This naturally belongs to $\mathcal{F}_t$ as the countable union of elements of the sub-sigma-algebra.

4. $f(\tau), f(t)$ is a continuous increasing function satisfying $f(t) \geq t$

> **Solution:** Note that the condition $f(t) \geq t$ for continuous increasing f is equivalent to $f^{-1}(t) \leq t$ We have that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in T$. Note that then if we fix $t \in T$:
> $\{f(\tau) \leq t\} = \{\omega \mid f(\tau(\omega)) \leq t\} = \{\omega \mid \tau(\omega) \leq f^{-1}(t)\} \in \mathcal{F}_{f^{-1}(t)}$.
> As $f^{-1}(t) \leq t$, $\mathcal{F}_{f^{-1}(t)} \subseteq \mathcal{F}_t$ so $\{f(\tau) \leq t\} \in \mathcal{F}_t$

Problem 4

An European put option on a non-dividend paying stock and expiring in one year is currently selling for 1.8. The stock price is 48 and the strike price is 51 and the risk free interest rate is 2 percent per year. Can you make a riskless profit and how?

> **Solution:** We first note that $Ke^{-rT} - S = 51 \cdot e^{-0.02} - 48 = \$1.99 > \$1.8 = p$. We have an arbitrage opportunity by the following:
>
> - At time 0, we purchase the put option for p and then exercise immediately for a payoff of $K - S = 51 - 48 = 3$. Our total cash flow is $-p + (K - S) = -1.8 + 3 = 1.2 > 0$.