# FRE-GY 6233: Assignment 10

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**Problem 1**

Let your initial capital $x = 200$, the total capital $A = 400$, and the probability (for "you") to win a dollar $p = \frac{31}{63}$. Suppose you bet 1,2,5,10,20,40 dollars each time. For each case calculate:

- The probability of winning before you lose everything

**Solution:** We note that for $p \neq q := \mathbb{P}['Loss']$, we can deduce a closed form solution for winning based on the system of equations:

$$\begin{cases} u(x) = pu(x+1) + qu(x-1) \\ u(0) = 1, u(A) = 0 \end{cases}$$

By the difference equation $u(x) = C_1 + C_2(q/p)^x$ to get that:

$$u(x) = \frac{(q/p)^A - (q/p)^x}{(q/p)^A - 1}$$

From this, we have that:

- If we bet 1 dollars:
  This is the same as plugging directly into the formula to find that:

$$\begin{aligned} \mathbb{P}['Win' \mid X_0 = 200] &= 1 - u(200) = 1 - \frac{(q/p)^A - (q/p)^x}{(q/p)^A - 1} \\ &= \frac{(32/31)^{400} - (32/31)^{200}}{(32/31)^{400} - 1} = 0.001744 \end{aligned}$$

- If we bet 2 dollars:
  This is effectively the same setup with $A' = A/2 = 200$, $X_0' = X_0/2 = 100$ to find that:

$$\mathbb{P}['Win' \mid X_0' = 100] = 1 - u(100) = 1 - \frac{(q/p)^A - (q/p)^x}{(q/p)^A - 1}$$

$$= \frac{(32/31)^{200} - (32/31)^{100}}{(32/31)^{200} - 1} = 0.04012$$

– If we bet 5 dollars:
This is effectively the same setup with $A' = A/5 = 80$, $X_0' = X_0/5 = 40$ to find that:

$$\mathbb{P}['Win' \mid X_0' = 40] = 1 - u(40) = 1 - \frac{(q/p)^A - (q/p)^x}{(q/p)^A - 1}$$
$$= \frac{(32/31)^{80} - (32/31)^{40}}{(32/31)^{80} - 1} = 0.21927$$

– If we bet 10 dollars:
This is effectively the same setup with $A' = A/10 = 40$, $X_0' = X_0/10 = 20$ to find that:

$$\mathbb{P}['Win' \mid X_0' = 20] = 1 - u(20) = 1 - \frac{(q/p)^A - (q/p)^x}{(q/p)^A - 1}$$
$$= \frac{(32/31)^{40} - (32/31)^{20}}{(32/31)^{40} - 1} = 0.34638$$

– If we bet 20 dollars:
This is effectively the same setup with $A' = A/20 = 20$, $X_0' = X_0/20 = 10$ to find that:

$$\mathbb{P}['Win' \mid X_0' = 10] = 1 - u(10) = 1 - \frac{(q/p)^A - (q/p)^x}{(q/p)^A - 1}$$
$$= \frac{(32/31)^{20} - (32/31)^{10}}{(32/31)^{20} - 1} = 0.42129$$

– If we bet 40 dollars:
This is effectively the same setup with $A' = A/40 = 10$, $X_0' = X_0/40 = 5$ to find that:

$$\mathbb{P}['Win' \mid X_0' = 5] = u(5) = 1 - \frac{(q/p)^A - (q/p)^x}{(q/p)^A - 1}$$
$$= \frac{(32/31)^{10} - (32/31)^{5}}{(32/31)^{10} - 1} = 0.46040$$

- The expected duration of the game.

**Solution:** We note that for $p \neq q := \mathbb{P}['Loss']$, we can deduce a closed form solution for winning based on the system of equations:

$$\begin{cases} D(x) = pD(x+1) + qD(x-1) + 1 \\ D(0) = 0, D(A) = 0 \end{cases}$$

We solve inhomogeneous equation to get:

$$D(x) = \frac{x}{q-p} + \frac{A}{q-p} \frac{1-(q/p)^x}{1-(q/p)^A}$$

From this, we have that:

- If we bet 1 dollars:
  This is the same as plugging directly into the formula to find that:

$$\begin{aligned} D(200) &= \frac{x}{q-p} + \frac{A}{q-p} \frac{1-(q/p)^x}{1-(q/p)^A} \\ &= 200 \cdot 63 + 400 \cdot 63 \cdot \frac{1-(32/31)^{200}}{1-(32/31)^{400}} = 12643.95 \end{aligned}$$

- If we bet 2 dollars:
  This is effectively the same setup with $A' = A/2 = 200$, $X_0' = X_0/2 = 100$ to find that:

$$\begin{aligned} D(100) &= \frac{x}{q-p} + \frac{A}{q-p} \frac{1-(q/p)^x}{1-(q/p)^A} \\ &= 100 \cdot 63 + 200 \cdot 63 \cdot \frac{1-(32/31)^{100}}{1-(32/31)^{200}} = 6805.54 \end{aligned}$$

- If we bet 5 dollars:
  This is effectively the same setup with $A' = A/5 = 80$, $X_0' = X_0/5 = 40$ to find that:

$$\begin{aligned} D(40) &= \frac{x}{q-p} + \frac{A}{q-p} \frac{1-(q/p)^x}{1-(q/p)^A} \\ &= 40 \cdot 63 + 80 \cdot 63 \cdot \frac{1-(32/31)^{40}}{1-(32/31)^{80}} = 3625.101 \end{aligned}$$

- If we bet 10 dollars:
  This is effectively the same setup with $A' = A/10 = 40$, $X_0' = X_0/10 = 20$ to find that:

$$D(20) = \frac{x}{q-p} + \frac{A}{q-p} \frac{1-(q/p)^x}{1-(q/p)^A}$$

$$= 20 \cdot 63 + 40 \cdot 63 \cdot \frac{1-(32/31)^{20}}{1-(32/31)^{40}} = 2132.887$$

– If we bet 20 dollars:
  This is effectively the same setup with $A' = A/20 = 20$, $X_0' = X_0/20 = 10$ to find that:

$$\begin{aligned} D(10) &= \frac{x}{q-p} + \frac{A}{q-p}\frac{1-(q/p)^x}{1-(q/p)^A} \\ &= 10 \cdot 63 + 20 \cdot 63 \cdot \frac{1-(32/31)^{10}}{1-(32/31)^{20}} = 1160.82 \end{aligned}$$

– If we bet 40 dollars:
  This is effectively the same setup with $A' = A/40 = 10$, $X_0' = X_0/40 = 5$ to find that:

$$\begin{aligned} D(5) &= \frac{x}{q-p} + \frac{A}{q-p}\frac{1-(q/p)^x}{1-(q/p)^A} \\ &= 5 \cdot 63 + 10 \cdot 63 \cdot \frac{1-(32/31)^{5}}{1-(32/31)^{10}} = 605.05 \end{aligned}$$

Which strategy would you choose?

**Solution:** It appears that the best strategy is to bet with larger denominations of money for the highest chance of winning.

**Problem 2**

Consider the same problem as in class, down-and-out call with barrier $X$ below strike $K$ and $S > K$.

1. Derive the formula by solving a PDE problem with a slight modification. Instead of change of variable

$$S = Ke^x$$

use a change of variable

$$S = Xe^x$$

Show all details

**Solution:** We use change of variables by $S = Xe^x, t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, c(t, S) = Kv(x, \tau)$:

$$c(t, S) = Kv(\ln(\frac{S}{X}), \frac{\sigma^2}{2}(T - t))$$

We then have the following by differentiating:

$$\begin{aligned}
\frac{\partial c}{\partial t} = \frac{\partial \tau}{\partial t} \cdot \frac{\partial v}{\partial \tau} = \frac{-\sigma^2}{2} K \frac{\partial v}{\partial \tau} \\
\frac{\partial c}{\partial S} = \frac{\partial x}{\partial S} \cdot \frac{\partial v}{\partial x} = \frac{K}{S} \frac{\partial v}{\partial x} \\
\frac{\partial^2 c}{\partial S^2} = \frac{\partial}{\partial S} \frac{K}{S} \frac{\partial v}{\partial x} = \frac{K}{S^2} \frac{\partial^2 v}{\partial x^2} - \frac{K}{S^2} \frac{\partial v}{\partial x}
\end{aligned}$$

With $k_1 = \frac{2r}{\sigma^2}$, we get the same PDE as before:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(k_1 - 1\right) \frac{\partial v}{\partial x} - k_1 v$$

Note that we can proceed as before with $v(\tau, x) = exp(\alpha x + \beta \tau) u(\tau, x)$ to get

$$\begin{aligned}
LHS &= exp(\alpha x + \beta \tau) \left( \beta u + \frac{\partial u}{\partial \tau} \right) \\
RHS &= exp(\alpha x + \beta \tau) \left[ (\alpha^2 u + \frac{\partial^2 v}{\partial x^2}) + (k_1 - 1)\big(\alpha u + \frac{\partial u}{\partial x}\big) - k_1 u \big) \right] \\
\beta u + \frac{\partial u}{\partial \tau} &= \alpha^2 u + \frac{\partial^2 v}{\partial x^2} + (k_1 - 1)\big(\alpha u + \frac{\partial u}{\partial x}\big) - k_1 u
\end{aligned}$$

Note that we choose $\alpha = \frac{-1}{2}(k_1 - 1)$, which means that from (1), $\beta = \frac{-1}{4}(k_1 + 1)^2$. Plugging this into our change of variable equation, we have that: $v(\tau, x) = exp(\frac{-1}{2}(k_1 - 1)x + \frac{-1}{4}(k_1 + 1)^2 \tau) u(\tau, x)$ so that we have differential equation: $\frac{\partial u}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}$ with boundary condition $u(x, 0) = e^{-\alpha x} \max \left( e^x - 1, 0 \right) = \max \left( e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, 0 \right)$ with an additional condition that $u(log(X/K)) = 0$
Here we use the method of images to solve for a closed-form solution of the PDE. We reflect the initial data about $x_0 = \ln(X/K)$ to get equations:

$$\begin{aligned}
u(x, 0) &= u_0(x)(2x_0 - x) \\
u(x, 0) &= \begin{cases} \max \left( e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, 0 \right) & \text{for } x > x_0 \\ -\max \left( e^{\frac{1}{2}(k_1+1)(x_0 - \frac{1}{2}x)} - e^{\frac{1}{2}(k_1-1)(x_0 - \frac{1}{2}x)}, 0 \right) & \text{for } x < x_0 \end{cases}
\end{aligned}$$

We note that the value of call option $C(S, t) = Ke^{\alpha x + \beta \tau} u_1(x, \tau)$ and that $V(S, t) = Ke^{\alpha x + \beta \tau}(u_1(x, \tau) + u_2(x, \tau))$ where $u_2(x, \tau) = -u_1(2x_0 - x, \tau)$. We then get that $V(S, t) = C(S, t) - (\frac{S}{X})^{-(k_1 - 1)} C(X^2/S, t)$

2. Does this formula work in the case $X > K$?

**Solution:** No, our method of images would fail to preserve the Black Scholes Call option pricing for the entirety of $X > K$ so our closed form extension would change. Moreover, if $X > K$, we effectively have a call option with strike price $X$ instead of $K$ so we wouldn't need to derive the formula in this manner.

**Problem 3**

Formulate the problem for a "down-in" barrier call option: the option is activated and becomes a vanilla call option with strike $K$ only if asset price $S$ crosses a lower barrier $X$ before expiry: the payoff is $\max(S - K, 0)$ if $S \leq X$ before expiry and zero otherwise. Assume $X < S$, $X < K$.

1. Write down the formula for this option (look for connection with a "down-out" call barrier with the same barrier, strike, and expiry)

**Solution:** With the "down-out" call option value denoted $c_{out}$ and "down-in" call option denoted $c_{in}$, we have that:

$$C(s,t) = c_{out}(s,t) + c_{in}(s,t)$$

So that we have that:

$$c_{in}(s,t) = C(S,t) - c_{out}(s,t) = (\frac{S}{X})^{-(k_1-1)}C(X^2/S,t)$$

2. Calculate both "down-in" and "down-out" in the case when the call options are written on *futures*, with the initial price $F = 30, K = 40, T = 1$ (years), $\sigma = 0.5, r = 0.01, X = 20$.

**Solution:** "Down-out" Call option:

$$\begin{aligned} c_{out}(F = 30, t = 0) &= (\frac{30}{20})^{-(\frac{0.02}{0.5^2}-1)}C(20^2/30, 0) \\ &= 1.4521 * 0.0584 = \$0.0848 \end{aligned}$$

"Down-in" Call option:

$$\begin{aligned} c_{in}(F = 30, t = 0) &= C(30,0) - (\frac{30}{20})^{-(\frac{0.02}{0.5^2}-1)}C(20^2/30, 0) \\ &= 3.0728 - 0.0848 = \$2.988 \end{aligned}$$

3. When does "down-out" becomes practically a vanilla call?

> **Solution:** A "down-out" call option is practically a vanilla call as long as it does not go below a barrier $X$ at any time prior to maturity.

**Problem 4**

Using the method of images find a formula for an "up-out" put option with barrier $X$, with payoff $max(K - F, 0)$ if $F < X$ before expiry $T$ and zero otherwise. Assume $X > K, F < X$. Price it in the case option is written on futures, $F = 30, K = 15, X = 40, \sigma = 0.6, T = 0.5, r = 0.01$.

**Solution:** We begin by noting that the put option has the same PDE as our call option just with modified terminal condition.
We have differential equation:
$\frac{\partial u}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}$ with boundary condition
$u(x, 0) = e^{-\alpha x} \max\left(1 - e^x, 0\right) = \max\left(e^{\frac{1}{2}(k_1-1)x} - e^{\frac{1}{2}(k_1+1)x}, 0\right)$ with an additional condition that $u(log(X/K)) = 0$
Here we use the method of images to solve for a closed-form solution of the PDE. We reflect the initial data about $x_0 = \ln(X/K)$ to get equations:

$$u(x, 0) = u_0(x)(2x_0 - x)$$

$$u(x, 0) = \begin{cases} \max\left(e^{\frac{1}{2}(k_1-1)x} - e^{\frac{1}{2}(k_1+1)x}, 0\right) & \text{for } x > x_0 \\ -\max\left(e^{\frac{1}{2}(k_1-1)(x_0-\frac{1}{2}x)} - e^{\frac{1}{2}(k_1+1)(x_0-\frac{1}{2}x)}, 0\right) & \text{for } x < x_0 \end{cases}$$

This proceeds as before except now we observe that the value of put option $P(S, t) = Ke^{\alpha x + \beta\tau} u_1(x, \tau)$ and that $V(S, t) = Ke^{\alpha x+\beta\tau}(u_1(x, \tau)+u_2(x, \tau))$ where $u_2(x, \tau) = -u_1(2x_0 - x, \tau)$. We then get that $V(S, t) = P(S, t) - (\frac{S}{X})^{-(k_1-1)} P(X^2/S, t)$
The "up-out" put option is price at:

$$\begin{aligned} V(S, T) = V(30, 0.5) &= P(30, 0.5) - (30/40)^{1-0.02/0.6^2} P(40^2/30, 0.5) \\ &= 0.1841 - 0.0034 = 0.1807 \end{aligned}$$