FRE-GY 6233: Assignment 8

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Problem 1

- Fill the gaps in getting the BS formula (17) calculating the integral (16)

Solution: We start with integral (16):

$$u(x,\tau) = \frac{1}{2\sqrt{\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} u_0(y\sqrt{2\tau}+x)e^{-y^2/2}dy$$

It's straightforward to note that for $y > -x/\sqrt{2\tau}$, and that $y\sqrt{2\tau}+x > 0 \Rightarrow u(y\sqrt{2\tau}) > 0$. This means that:

$$\begin{aligned}
&u(x,\tau) = \frac{1}{2\sqrt{\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \left(e^{\frac{1}{2}(k_1+1)(y\sqrt{2\tau}+x)} - e^{\frac{1}{2}(k_1-1)(y\sqrt{2\tau}+x)} \right) e^{-y^2/2}dy \\
&= I_1 - I_2 \\
&\text{where we have:} \\
&I_1 := \frac{1}{2\sqrt{\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k_1+1)(y\sqrt{2\tau}+x)-y^2/2}dy \\
&I_2 := \frac{1}{2\sqrt{\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k_1-1)(y\sqrt{2\tau}+x)-y^2/2}dy
\end{aligned}$$

Note that we can complete the square of the exponents (in terms of y) as follows:

$$\begin{aligned}
&\frac{1}{2}(k_1+1)(y\sqrt{2\tau}+x) - y^2/2 = \frac{1}{2}\left((k_1+1)y\sqrt{2\tau} - y^2 \right) + \frac{(k_1+1)x}{2} \\
&= \frac{1}{2}\left(-\frac{(k_1+1)^2}{2}\tau + (k_1+1)y\sqrt{2\tau} - y^2 \right) + \frac{(k_1+1)^2}{4}\tau + \frac{(k_1+1)x}{2} \\
&= \frac{-1}{2}\left(\frac{(k_1+1)}{2}\sqrt{2\tau} - y \right)^2 + \frac{(k_1+1)^2}{4}\tau + \frac{(k_1+1)x}{2}
\end{aligned}$$

Likewise, we have:

$$\frac{1}{2}(k_1-1)(y\sqrt{2\tau}+x)-y^2/2=\frac{1}{2}\bigg((k_1-1)y\sqrt{2\tau}-y^2\bigg)+\frac{(k_1-1)x}{2}$$
$$=\frac{1}{2}\bigg(-\frac{(k_1-1)^2}{2}\tau+(k_1-1)y\sqrt{2\tau}-y^2\bigg)+\frac{(k_1-1)^2}{4}\tau+\frac{(k_1-1)x}{2}$$
$$=\frac{-1}{2}\bigg(\frac{(k_1-1)}{2}\sqrt{2\tau}-y\bigg)^2+\frac{(k_1-1)^2}{4}\tau+\frac{(k_1-1)x}{2}$$

With these completed squares, we can evaluate I_1, I_2 as follows:

$$I_1=\frac{1}{2\sqrt{\pi}}\int_{-x/\sqrt{2\tau}}^{\infty}e^{\frac{1}{2}\left(\frac{(k_1+1)}{2}\sqrt{2\tau}-y\right)^2+\frac{(k_1+1)^2}{4}\tau+\frac{(k_1+1)x}{2}}dy$$
$$=e^{\frac{(k_1+1)^2}{4}\tau+\frac{(k_1+1)x}{2}}\cdot\frac{1}{2\sqrt{\pi}}\int_{-x/\sqrt{2\tau}}^{\infty}e^{-\frac{1}{2}\left(\frac{(k_1+1)}{2}\sqrt{2\tau}-y\right)^2}$$

Note that with a change of variable $y'=y-\frac{(k_1+1)}{2}\sqrt{2\tau}$

$$=e^{\frac{(k_1+1)^2}{4}\tau+\frac{(k_1+1)x}{2}}\cdot\frac{1}{2\sqrt{\pi}}\int_{-d_1}^{\infty}e^{-(y')^2/2}$$

$$I_2=\frac{1}{2\sqrt{\pi}}\int_{-x/\sqrt{2\tau}}^{\infty}e^{\frac{1}{2}\left(\frac{(k_1-1)}{2}\sqrt{2\tau}-y\right)^2+\frac{(k_1+1)^2}{4}\tau+\frac{(k_1+1)x}{2}}dy$$
$$=e^{\frac{(k_1-1)^2}{4}\tau+\frac{(k_1-1)x}{2}}\cdot\frac{1}{2\sqrt{\pi}}\int_{-x/\sqrt{2\tau}}^{\infty}e^{-\frac{1}{2}\left(\frac{(k_1-1)}{2}\sqrt{2\tau}-y\right)^2}$$

Note that with a change of variable $y'=y-\frac{(k_1-1)}{2}\sqrt{2\tau}$

$$=e^{\frac{(k_1-1)^2}{4}\tau+\frac{(k_1-1)x}{2}}\cdot\frac{1}{2\sqrt{\pi}}\int_{-d_2}^{\infty}e^{-(y')^2/2}$$

where $d_1=\frac{x}{2\tau}+\frac{k_1+1}{2}\sqrt{2\tau}$ and $d_2=\frac{x}{2\tau}+\frac{k_1-1}{2}\sqrt{2\tau}$ so that:

$$I_1=e^{\frac{(k_1+1)^2}{4}\tau+\frac{(k_1+1)x}{2}}\cdot\Phi(d_1)$$
$$I_2=e^{\frac{(k_1-1)^2}{4}\tau+\frac{(k_1-1)x}{2}}\cdot\Phi(d_2)$$

- Using (16), derive the formula for a *digital option*, which pays maturity

$$c(S,T) = \begin{cases} 1 \text{ dollar} & \text{if } S \geq K \\ 0 & \text{if } S < K \end{cases}$$

Solution: We observe that the value of the digital option follows the same PDE as the value of the call option; however, the terminal condition of the PDE is modified to be $c(S,T) = \frac{1}{S-K}\big(S-K\big)^+$. This then modifies the initial conditions of our PDE after the transformations to get (16). We now have:

$$\begin{aligned} u_0(x) &= \frac{1}{S-K}\max\left(e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, 0\right) \\ &= \frac{1}{K(e^x-1)}\max\left(e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, 0\right) \\ &= \frac{1}{K(e^x-1)}\max\left(e^{\frac{1}{2}(k_1-1)x}(e^x-1), 0\right) \end{aligned}$$

Note that for $y > -x/\sqrt{2\tau} \Rightarrow y\sqrt{2\tau} + x > 0$, we have that $e^{y\sqrt{2\tau}+x} > 0$ so that we can simplify the above equation to get $u_0(y\sqrt{2\tau}+x) = \frac{1}{K}\max\left(e^{\frac{1}{2}(k_1-1)y\sqrt{2\tau}+x}, 0\right) = \frac{1}{K}e^{\frac{1}{2}(k_1-1)y\sqrt{2\tau}+x}$. We can then substitute this into our integral (16) to get:

$$\begin{aligned} u(x,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} u_0(y\sqrt{2\tau}+x)e^{-y^2/2}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} \frac{1}{K}e^{\frac{1}{2}(k_1-1)(y\sqrt{2\tau}+x)}e^{-y^2/2}dy \\ &= \frac{1}{K}I_2 \text{ where } I_2 \text{ is the integral defined before} \end{aligned}$$

We then have the same substitutions in the normal BS derivation:

$$\begin{aligned} v &= e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(\tau,x) \\ &= e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} \cdot e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau}\frac{1}{K}\Phi(d_2) \\ &= \frac{1}{K}e^{-\frac{4k_1}{4}\tau}\Phi(d_2) = \frac{1}{K}e^{-\frac{r}{1/2\sigma^2}\tau}\Phi(d_2) \\ &= \frac{1}{K}e^{-\frac{r}{(}T-t)}\Phi(d_2) \\ &\Rightarrow \\ c &= Kv(x,\tau) = e^{-r(T-t)}\Phi(d_2) \end{aligned}$$

So that we have that the value of the digital option is

$c(t,S) = e^{-r(T-t)}\Phi(d_2)$ where $d_2 = \frac{\ln(S/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

Problem 2

Derive BS PDE for a call option on a stock with continuous dividend rate, following the same arguments as for BS PDE (set up a riskless portfolio, etc). Prove that its value is given by (19) based on the BS formula (17) and the connection between the two cases.

Solution: Suppose that our asset pays dividends at a constant rate q so that $dS = \alpha S(t)dt + \sigma S(t)dW(t) - qS(t)dt = (\alpha - q)S(t)dt + \sigma S(t)dW(t)$. We proceed similarly by choosing a portfolio with $\Delta(t)$ shares of stock and $X(t) - \Delta(t)S(t)$ investments in market account. Note that at time t, we also have $q\Delta S$ dividend so that the evolution of the portfolio is now:

$$\begin{aligned}
dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt + q\Delta S(t)dt \\
&= \Delta(t)\big((\alpha - q)S(t)dt + \sigma S(t)dW(t)\big) + r(X(t) - \Delta(t)S(t))dt + q\Delta S(t)dt \\
&= rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)
\end{aligned}$$

We assume that our portfolio can replicate the option at all times by $X(t) = c(t, S(t))$ for all $0 \le t \le T$ with $X(0) = c(0, S(0)), d(X(t) - c(t, S(t))) = 0$ so that :

$$\begin{aligned}
&c_t dt + c_s dS(t) + \frac{1}{2}c_{ss}(dS(t))^2 \\
&\qquad = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \\
&c_t dt + c_s((\alpha - q)S(t)dt + \sigma S(t)dW(t)) + \frac{1}{2}c_{ss}S^2(t)\sigma^2 dt \\
&\qquad = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \\
&c_t dt + c_s((\alpha - q)S(t)dt + \sigma S(t)dW(t)) + \frac{1}{2}c_{ss}S^2(t)\sigma^2 dt \\
&\qquad = rc(t, S(t))dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)
\end{aligned}$$

By choosing $c_x(t, S(t)) = \Delta(t)$, we have that:

$$\begin{aligned}
rc(t, S(t))dt &= c_t dt + (rc_s - qc_s)S(t)dt + \frac{1}{2}S^2(t)\sigma^2 c_{ss}dt \\
rc(t, S(t)) &= c_t + (r - q)S(t)c_s + \frac{1}{2}S^2(t)\sigma^2 c_{ss}
\end{aligned}$$

We have terminal condition $c(T, S(T)) = (S - K)^+$. We proceed with the same transformations as before ($S = Ke^x, t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, c(t, S) = Kv(x, \tau)$) but with $k_1 = \frac{r-q}{\frac{1}{2}\sigma^2}$. We then get:

$$
\begin{aligned}
rv &= \frac{-\sigma^2}{2}v_\tau + (r-q)v_x + \frac{\sigma^2}{2}\big(v_{xx} - v_x\big) \\
\frac{\partial v}{\partial \tau} &= \frac{\partial^2 v}{\partial x^2} + \left(\frac{2(r-q)}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v
\end{aligned}
$$

We then use change of variable $v(\tau, x) = exp(\alpha x + \beta\tau)u(\tau, x)$ to get:

$$
\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + \frac{\partial^2 v}{\partial x^2} + (\frac{2(r-q)}{\sigma^2} - 1)\big(\alpha u + \frac{\partial u}{\partial x}\big) - \frac{2r}{\sigma^2}u
$$

In order to get terms to vanish, we choose α, β by:

$$
\left\{ \begin{array}{l} u(\beta - \alpha^2 - (\frac{2(r-q)}{\sigma^2} - 1)\alpha + \frac{2r}{\sigma^2}) = 0 \\ \frac{\partial u}{\partial x}(2\alpha + (\frac{2(r-q)}{\sigma^2} - 1)) = 0 \end{array} \right. \Rightarrow
$$

$$
\left\{ \begin{array}{l} \alpha = \frac{1}{2} - \frac{r-q}{\sigma^2} \\ \beta = -\frac{1}{4}(\frac{2(r-q)}{\sigma^2} - 1)^2 - \frac{2r}{\sigma^2} = -\frac{1}{4}(\frac{2(r-q)}{\sigma^2} + 1)^2 - \frac{2q}{\sigma^2} \end{array} \right.
$$

We let $k_1 = \frac{2(r-q)}{\sigma^2}$ so that $v(x,\tau) = e^{\frac{-2q}{\sigma^2}\tau} \cdot exp(\frac{-1}{2}(k_1 - 1)x + \frac{-1}{4}(k_1+1)^2\tau)u(x,\tau)$. Note that at the boundary condition, our initial condition looks the same with $v(x, 0) = e^{-\alpha x}\max\left(e^x - 1, 0\right) = \max\left(e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, 0\right)$.

As our differential equation looks the same with boundary conditions of the same form (here k_1 is modified), we have the same general solution as before.

$$
\begin{aligned}
u(x,\tau) &= I_1 - I_2 \\
&\qquad \text{where } I_1, I_2 \text{ are defined the same as before but with the new } k_1 \\
v(x,\tau) &= e^{\frac{-2q}{\sigma^2}\tau} \cdot exp(\frac{-1}{2}(k_1-1)x + \frac{-1}{4}(k_1+1)^2\tau)u(x,\tau) \\
&= e^{\frac{-2q}{\sigma^2}\tau}\left(e^{\frac{(k_1+1)x}{2} + \frac{-1}{2}(k_1-1)x} \cdot \Phi(d_1) + e^{\frac{(k_1-1)^2}{4}\tau + \frac{-1}{4}(k_1+1)^2\tau} \cdot \Phi(d_2)\right) \\
&= e^{\frac{-2q}{\sigma^2}\tau}\left(e^x \cdot \Phi(d_1) + e^{-k_1\tau} \cdot \Phi(d_2)\right) \\
&= e^{\frac{-2q}{\sigma^2}\tau}\left(e^x \cdot \Phi(d_1) + e^{-\frac{2(r-q)}{\sigma^2}\tau} \cdot \Phi(d_2)\right) \\
c(s,t) &= Kv(x,\tau) \\
&= Ke^{\ln(S/K)\cdot(-q(T-t))} \cdot \Phi(d_1) + Ke^{-r(T-t)} \cdot \Phi(d_2) \\
&= Se^{-q(T-t)} \cdot \Phi(d_1) + Ke^{-r(T-t)} \cdot \Phi(d_2)
\end{aligned}
$$

Where $d_1 = \frac{x}{2\tau} + \frac{k_1+1}{2}\sqrt{2\tau} = \frac{x}{2\tau} + \frac{\frac{2(r-q)}{\sigma^2}+1}{2}\sqrt{2\tau} = \frac{\ln(S/K)+(r-q+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$
$d_2 = d_1 - \sigma^2\sqrt{T-t}$

Problem 3

Suppose a put option costs more than its valued derived from a call option with the same strike and maturity, so put-call parity:

$$p(t,S) > c(t,S) + e^{-r(T-t)}K - S$$

Show details (strategy) how one could explore the arbitrage.

Solution: We have the following arbitrage opportunity:
At time t, we short a put-option as well as one share of stock for $p(t,S)+S$ and long a call-option for $c(t,S)$. In order to finance this strategy, we invest/borrow the difference $p(t,S)+S-c(t,S)$ with interest rate r.
At time of maturity T, we receive/pay $(p(t,S)+S-c(t,S))e^{-r(T-t)}$ have the following situations:

- $S > K$, we exercise the call option to pay K to purchase a stock share to pay back our stock shorting. The put option is void in this case.
- $S < K$, so our call option is void in this case. The holder of the put-option exercises his or her option so we purchase said stock share for K to pay back our stock shorting.

In both scenarios, we have a net gain of $(p(t,S)+S-c(t,S))e^{-r(T-t)} - K > 0$ according to our inequality. As we have greater than zero net gain with probability 1 starting from 0 initial wealth, we have an arbitrage situation.

Problem 4

Consider a call option on the underlying future $F = 40$ with strike $K = 45$ and time to the option expiry $T = 1$ (year), today time $t = 0$. Assume that the volatility of the underlying future is not constant, but depends on time in the following way:

$$\sigma(t,T) = \sigma_0 e^{-B(T-t)}, 0 < t \leq T$$

where $B = 0.2$ (per year). Assume that $r = 0$. We know that the option is quoted for 3.2. *What should be the value of σ_0 so that we match the quoted price?*

Solution: We note that the implied volatility over the life time of the future is
$\overline{\sigma}(0,T) = \sqrt{\frac{1}{T}\int_0^T \sigma^2(s,T)dt} = \sqrt{\frac{1}{T}\int_0^T \sigma_0^2 e^{-2B(T-s)}dt}$.

From our quoted price, we can retrieve the implied volatility according to our underlying BS Model. We can do so using Newton-Raphson's Method to iteratively find a solution $\overline{\sigma}$ which makes $f(\sigma) - 3.2 = 0$. This process is iterative with guesses of x_n by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. The code we used to implement Newton's method is captured as follows:

Solution:

```python

from scipy.stats import norm
from math import sqrt, exp, log

def find_vol(target_value, opt_type, S, K, T, r):
    MAX_ITERATIONS = 100
    PRECISION = 1.0e-5

    sigma = 0.5
    for i in range(0, MAX_ITERATIONS):
        price = bs_price(opt_type, S, K, T, r, sigma)
        vega = bs_vega(opt_type, S, K, T, r, sigma)

        price = price
        diff = target_value - price  # our root

        print(i, sigma, diff)

        if (abs(diff) < PRECISION):
            return sigma
        sigma = sigma + diff/vega # f(x) / f'(x)

    # value wasn't found, return best guess so far
    return sigma

n = norm.pdf
N = norm.cdf

def bs_price(opt_type,S,K,T,r,v,q=0.0):
    d1 = (log(S/K)+(r+v*v/2.)*T)/(v*sqrt(T))
    d2 = d1-v*sqrt(T)
    if opt_type == 'c':
        price = S*exp(-q*T)*N(d1)-K*exp(-r*T)*N(d2)
    else:
        price = K*exp(-r*T)*N(-d2)-S*exp(-q*T)*N(-d1)
    return price

def bs_vega(opt_type,S,K,T,r,v,q=0.0):
    d1 = (log(S/K)+(r+v*v/2.)*T)/(v*sqrt(T))
    return S * sqrt(T)*n(d1)


#def main():
V_quoted = 3.2
K = 45
## Count years by market date
#T = (datetime.date(2014,10,18) - datetime.date(2014,9,8)).days / 365.
## Manual input of number of years
T = 1
S = 40
r = 0.0000
cp = 'c' # call option

implied_vol = find_vol(V_quoted, cp, S, K, T, r)

print('Implied vol: %.2f%%' % (implied_vol * 100))

print('Market price = %.2f' % V_quoted)
print('Model price = %.2f' % bs_price(cp, S, K, T, r, implied_vol))

```

We find that $\overline{\sigma} = 31.61\%$ ($\overline{\sigma}^2 = 9.991\%$). So that:

$$\overline{\sigma}^2 = \frac{1}{T}\Big[\frac{\sigma_0^2}{2B}e^{-2B(T-s)}\Big]_0^T$$

$$\sigma_0 = \sqrt{\frac{2B\overline{\sigma}^2}{1-e^{-2BT}}} = 34.82\%$$