FRE-GY 6233: Assignment 4

Raymond Luo

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Problem 1: Generalized Breakout Question 1
Let X_t and Y_t are stochastic processes, and martingales w.r.t filtration $\mathcal{F}_t$. Show that a process Z_t defined by

$$Z_t = c_1 X_t + c_2 Y_t + c_3$$

where c_1, c_2, c_3 are some constants, is a martingale w.r.t. $\mathcal{F}$

Solution: We check the following:

(i) Z_t is integrable for each $t \in \mathcal{T}$: This follows immediately as X_t, Y_t are martingales w.r.t to $\mathcal{F}_t$. Fix $t \in \mathcal{T}$, X_t, Y_t are then integrable such that $\int |X_t| dt = a_1 < \infty, \int |Y_t| dt = a_2 < \infty$. Then $\int |Z_t| = \int |c_1 X_t + c_2 Y_t + c_3| dt \leq \int |c_1| \cdot |X_t| + |c_2| \cdot |Y_t| + |c_3| dt = |c_1| \cdot a_1 + |c_2| \cdot a_2 + |c_3| < \infty$.

(ii) Z_t is adapted to the filtration $\mathcal{F}_t$: This follows immediately as X_t, Y_t are martingales w.r.t to $\mathcal{F}_t$. We know that X_t, Y_t are $\mathcal{F}_t$-measurable and linear combinations is a measurable function, Z_t is $\mathcal{F}$-measurable as a composition of measurable functions.

(iii) $Z_s = \mathbb{E}[Z_t \mid \mathcal{F}_s] \forall s < t$: We have that $\mathbb{E}[Z_t \mid \mathcal{F}_s] = \mathbb{E}[c_1 X_t + c_2 Y_t + c_3 \mid \mathcal{F}_s] = c_1 \mathbb{E}[X_t \mid \mathcal{F}_s] + c_2 \mathbb{E}[Y_t \mid \mathcal{F}_s] + c_3 = c_1 X_s + c_2 Y_s + c_3 = Z_s$

Problem 2: Breakout Question 2
We consider a "motivating" example and make a different choice of y_i:

$$y_i = (t_i + t_{i-1})/2$$

Show details and the answer for the integral calculations in this case.

Solution: We evaluate the integral in our "motivating" example by using the Riemann-Stieltjes integral of the Wiener process with $g = W$. For the central difference choice of evaluating this integral we have that:

$$S_n = \sum_{i=1}^{n} W(y_i)[W(t_i) - W(t_{i-1})]$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot 2 \sum_{i=1}^{n} W(y_i)[W(t_i) - W(t_{i-1})] \\
&= \frac{1}{2} \big[2 \sum_{i=1}^{n} W(y_i)[W(t_i) - W(t_{i-1})] + 0 \big] * \\
&= \frac{1}{2} \big[2 \sum_{i=1}^{n} W(y_i)[W(t_i) - W(t_{i-1})] + \sum_{j=1}^{n} (W(y_j) - W(t_j))^2 - \sum_{j=1}^{n} (W(y_j) - W(t_{j-1}))^2 \big] \\
&= \frac{1}{2} \big[2 \sum_{i=1}^{n} W(y_i)[W(t_i) - W(y_i) + W(y_i) - W(t_{i-1})] + \sum_{j=1}^{n} (W(y_j) - W(t_j))^2 - \sum_{j=1}^{n} W(y_j)^2 + \\
&+ \sum_{j=1}^{n} W(y_j)^2 - \sum_{j=1}^{n} (W(y_j) - W(t_{j-1}))^2 \big] \\
&= \frac{1}{2} \big[\sum_{j=1}^{n} (W(y_j) - W(t_j))^2 - 2 \sum_{i=1}^{n} W(y_i)[W(t_2) - W(y_i)] + \sum_{j=1}^{n} W(y_j)^2 - \sum_{j=1}^{n} W(y_j)^2 + \\
&+ 2 \sum_{i=1}^{n} W(y_i)[W(y_i) - W(t_{i-1})] - \sum_{j=1}^{n} (W(y_j) - W(t_{j-1}))^2 \big] \\
&= \frac{1}{2} \big[\sum_{j=1}^{n} \big((W(t_j) - W(y_j)) + W(y_j) \big)^2 - \sum_{j=1}^{n} \big((W(t_{j-1}) - W(y_j)) + W(y_j) \big)^2 \big] \\
&= \frac{1}{2} \sum_{j=1}^{n} (W(t_j)^2 - W(t_{j-1})^2) \\
&= W(t)^2
\end{aligned}
$$

Problem 3: Isometry Property

Show details of the proof that the cross terms in the double sum are zeros

$$\mathbb{E}(Y_i Y_j) = 0, i \neq j$$

where $Y_i = Z_i \Delta_i W$

Solution: We first fix $i, j \in \mathbb{R}$ with $i < j$ so that:

$$
\begin{aligned}
&\mathbb{E}[Y_i Y_j] = \mathbb{E}[Z_i \Delta_i W \cdot Z_j \Delta_j W] = \mathbb{E}[Z_i (W_i - W_{i-1}) \cdot Z_j (W_j - W_{j-1})] \\
&= \mathbb{E}[\mathbb{E}[Z_i Z_j \cdot (W_i - W_{i-1}) \cdot (W_j - W_{j-1}) \mid \mathcal{F}_{j-1}]] \text{ [by the tower property]} \\
&= \mathbb{E}[Z_i Z_j \cdot (W_i - W_{i-1}) \big(\mathbb{E}[W_j - W_{j-1} \mid \mathcal{F}_{j-1}] \big)] \text{ [as } Z_i, Z_j, (W_i - W_{i-1}), W_{j-1} \text{ are adaptive to } \mathcal{F}_{j-1} \text{]} \\
&= \mathbb{E}[Z_i Z_j \cdot (W_i - W_{i-1}) \big(\mathbb{E}[W_j \mid \mathcal{F}_{j-1}] - W_{j-1} \big)] \\
&= \mathbb{E}[Z_i Z_j \cdot (W_i - W_{i-1}) \cdot 0] = 0
\end{aligned}
$$

Problem 4: Simple Processes
Describe a simple process obtained from a linear combination of two simple processes: For constants c_1 and c_2 and simple processes $C^{(1)}$ and $C^{(2)}$ on $[0, T]$, with (possibly) different partitions $\tau_n^{(1)}$ and $\tau_n^{(2)}$

$$C = c_1 C^{(1)} + c_2 C^{(2)}$$

Thus, you need to define a partition for C and its values. You can give an example as the illustration

Solution: Suppose we have simple processes $C^{(1)}$ and $C^{(2)}$ on $[0, T]$, with partitions $\tau_n^{(1)}$ and $\tau_n^{(2)}$. To describe simple process $C = c_1 C^{(1)} + c_2 C^{(2)}$, we need a partition $\tau = \tau_n^{(1)} \vee \tau_n^{(2)}$. Here $\tau_n^{(1)} \vee \tau_n^{(2)}$ denotes the common refinement of $\tau_n^{(1)}$ and $\tau_n^{(2)}$. More explicitly, given partitions $\tau_n^{(1)}, \tau_n^{(2)}$ respectively defined by sequences $a_0 = 0 < a_1 < a_2 < \cdots < a_n = T$ and $b_0 = 0 < b_1 < b_2 < \cdots < b_n = T$, our new partition contains all the points in the two sequences re-numbered in order. So with a simple example, if $\forall j \in [1, n-1]$, $b_{j-1} < a_j < b_j$, our resulting partition would be defined by a sequence $a_0 = b_0 < a_1 < b_1 < a_2 < \cdots < b_{n-1} < a_n = b_n$.
We then note that the definition of our new simple process follows directly as the linear combination of the values of the two contributing simple processes as defined on their respective partitions (as our common refinement is finer than both original partitions in the sense that all points of both sequences are a part of our new sequence). Following the previous example, if $C^{(1)}, C^{(2)}$ are respectively defined by a sequence of random variables X_i, Y_i over their respective partitions then:

$$C(t) = \begin{cases} c_1 X_n + c_2 Y_n, & \text{if } t = T \\ c_2 X_i + c_2 Y_j, & \text{if } b_{j-1} \leq t \leq b_j \text{ and } a_{i-1} \leq t \leq a_i \text{ for } i, j \in [1, n] \end{cases}$$

Problem 5: Wiener Integral Find a distribution of Y defined by

$$Y = \int_1^T s dW(s), T > 1$$

Solution: We first note that for a partition τ_n for some fixed $n \in \mathbb{N}$:

$$Y = \int_1^T s dW(s) = \sum_{j=1}^{n} t_j (W(t_j) - W(t_{j-1})))$$

We then observe that as $W(t_j) - W(t_{j-1})$ are independently and normally distributed with finite variance, Y is the countable sum of independent normal distributions each scaled by a corresponding factor of t_j. We conclude that this distribution should be normal as the countable sum of normal distributions so we observe the expectation and variance of the distribution.

$$
\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[TW(T) + \frac{T-1}{n}\sum_{j=1}^{n} W(t_{j-1})] \\
&= T\mathbb{E}[W(T)] + \frac{T-1}{n}\sum_{j=1}^{n} \mathbb{E}[W(t_{j-1})] = 0 \\
Var[Y] &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[Y^2] - 0 = \mathbb{E}\Big[\big(\int_1^T sdW(s)\big)^2\Big] \\
&= \mathbb{E}\Big[\big(\int_0^T sdW(s) - \int_0^1 sdW(s)\big)^2\Big] \\
&= \mathbb{E}\Big[\big(\int_0^T sdW(s)\big)^2 - 2\int_0^1 sdW(s)\int_1^T sdW(s) + \big(\int_0^1 sdW(s)\big)^2\Big] \\
&= \mathbb{E}\Big[\big(\int_0^T sdW(s)\big)^2\Big] - 2\mathbb{E}[\int_0^1 sdW(s)\int_1^T sdW(s)] + \mathbb{E}\Big[\big(\int_0^1 sdW(s)\big)^2\Big] \\
&= \mathbb{E}\Big[\big(\int_0^T sdW(s)\big)^2\Big] + \mathbb{E}\Big[\big(\int_0^1 sdW(s)\big)^2\Big] =
\end{aligned}
$$

(Here the center term evaluates to zero as the two integrals are evaluated as the sum of increments of the wiener process scaled by time; however, the increments of the wiener process are independent and have mean zero, so this term disappears)

$$
\begin{aligned}
&= \mathbb{E}[\int_0^T s^2 ds] + \mathbb{E}[\int_0^1 s^2 ds] \text{ (by Ito Isometry)} \\
&= \mathbb{E}[\frac{T^3}{3}] + \mathbb{E}[\frac{1}{3}] = \frac{T^3}{3} + \frac{1}{3}
\end{aligned}
$$