FRE-GY 6233: Assignment 1

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Problem 1: Finish the problem from slide 14 we started in the class.

1. List all the sets in σ-algebra in $\mathcal{F}$

> **Solution:** $\mathcal{F} = \{\emptyset, \Omega, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\},$
> $\{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}, \{a,c,d\}, \{a,b,d\}, \{a,b,c\}\}$

2. List all the sets $\sigma(X)$

> **Solution:** $\sigma(X) = \{\emptyset, \Omega, \{a,b\}, \{c,d\}\}$

3. Define the r.v. $\mathbb{E}[X|\sigma(Y))]$, so calculate it for a, b, c, d.

> **Solution:** We note that $\mathbb{P}(a) = \frac{3}{8}, \mathbb{P}(b) = \frac{1}{8}, \mathbb{P}(c) = \frac{1}{6}, \mathbb{P}(d) = \frac{1}{3}$.
> As $\mathbb{E}[Y|X] = \Sigma_{i\in\{-1,1\}}\mathbb{E}[Y|X = i]\mathbb{1}_i(x)$, we consider the following:
> If $X = 1$, $\mathbb{P}(X^{-1}(1) = a) = \frac{\mathbb{P}(a)}{\mathbb{P}(a)+\mathbb{P}(b)} = \frac{3}{4}, \mathbb{P}(X^{-1}(1) = b) = \frac{1}{4}$
> So, $\mathbb{E}[Y|X = 1] = (-1 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4}) = -\frac{1}{2}$
> If $X = -1$, $\mathbb{P}(X^{-1}(-1) = c) = \frac{\mathbb{P}(c)}{\mathbb{P}(c)+\mathbb{P}(d)} = \frac{1}{3}, \mathbb{P}(X^{-1}(1) = d) = \frac{2}{3}$
> So, $\mathbb{E}[Y|X = -1] = (1 \cdot \frac{1}{3} - 1 \cdot \frac{2}{3}) = -\frac{1}{3}$

Problem 2:

1. Suppose that $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space where $\Omega = \{a, b, c, d, e, f\}$, $\mathcal{F}$ is σ-algebra, and $\mathcal{P}$ is uniform (so $\mathcal{P}(a) = \mathcal{P}(b) = \cdots = \frac{1}{6}$).

2. Let X, Y, Z be r.v. given by

$$X(a) = 1, X(b) = X(c) = 3, X(d) = X(e) = 5, X(f) = 7$$
$$Y(a) = Y(b) = 2, Y(c) = Y(d) = 1, Y(e) = Y(f) = 7$$
$$Z(a) = Z(b) = Z(c) = Z(d) = 3, Z(e) = Z(f) = 2$$

1. Write down $\sigma(X), \sigma(X), \sigma(Z)$. Are there any relationships between them?

Solution:

$$\sigma(X) = \{\emptyset, \Omega, \{a\}, \{b,c\}, \{d,e\}, \{f\}, \{a,b,c\}, \{a,d,e\}, \{a,f\},$$
$$\{b,c,d,e\}, \{b,c,f\}, \{d,e,f\}, \{a,b,c,d,e\}, \{b,c,d,e,f\}, \{a,d,e,f\}, \{a,b,c,f\}\}$$
$$\sigma(Y) = \{\emptyset, \Omega, \{a,b\}, \{c,d\}, \{e,f\}, \{a,b,c,d\}, \{a,b,e,f\}, \{c,d,e,f\}\}$$
$$\sigma(Z) = \{\emptyset, \Omega, \{a,b,c,d\}, \{e,f\}\}$$

2. Define a r.. $\mathbb{E}[X \mid \sigma(Y)]$

Solution: As $\mathbb{E}[X|Y] = \Sigma_{i\in\{1,2,7\}}\mathbb{E}[X|Y=i]\mathbb{1}_i(y)$, we consider the following:
If $Y=1$, $\mathbb{P}(Y^{-1}(1)=c) = \frac{\mathbb{P}(c)}{\mathbb{P}(c)+\mathbb{P}(d)} = \frac{1}{2}, \mathbb{P}(Y^{-1}(1)=d) = \frac{1}{2}$
So, $\mathbb{E}[X|Y=1] = (X(c)\cdot\frac{1}{2} + X(d)\cdot\frac{1}{2}) = 3\cdot\frac{1}{2} + 5\cdot\frac{1}{2} = 4$
If $Y=2$, $\mathbb{P}(X^{-1}(2)=a) = \frac{\mathbb{P}(a)}{\mathbb{P}(a)+\mathbb{P}(b)} = \frac{1}{2}, \mathbb{P}(X^{-1}(2)=b) = \frac{1}{2}$
So, $\mathbb{E}[X|Y=2] = (X(a)\cdot\frac{1}{2} + X(b)\cdot\frac{1}{2}) = 2$
If $Y=7$, $\mathbb{P}(Y^{-1}(7)=e) = \frac{\mathbb{P}(e)}{\mathbb{P}(e)+\mathbb{P}(f)} = \frac{1}{2}, \mathbb{P}(Y^{-1}(7)=f) = \frac{1}{2}$
So, $\mathbb{E}[X|Y=7] = (X(e)\cdot\frac{1}{2} + X(f)\cdot\frac{1}{2}) = 6$
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3. Check directly the averaging property

$$\mathbb{E}[\mathbb{E}[X \mid \sigma(Y)]] = \mathbb{E}[X]$$

Solution: $\mathbb{E}[\mathbb{E}[X \mid \sigma(Y)]] = \Sigma_{i\in\{1,2,7\}}\mathbb{E}[X|Y=i]\cdot\mathbb{P}[Y=i]$. As $\mathbb{P}[Y=i] = \frac{1}{3}$ for $i \in \{1,2,7\}$, we have that $\mathbb{E}[\mathbb{E}[X \mid \sigma(Y)]] = \frac{1}{3}\cdot(4+2+6) = 4$.
We then note that:
$\mathbb{E}[X] = \Sigma_{i\in\{a,b,c,d,e,f\}}\mathbb{P}[i]\cdot X(i) = \frac{1}{6}(1+3+3+5+5+7) = \frac{24}{6} = 4$.
From this we have checked the averaging property.