Quantitative Methods: Assignment 4

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Problem 1 (35 points):

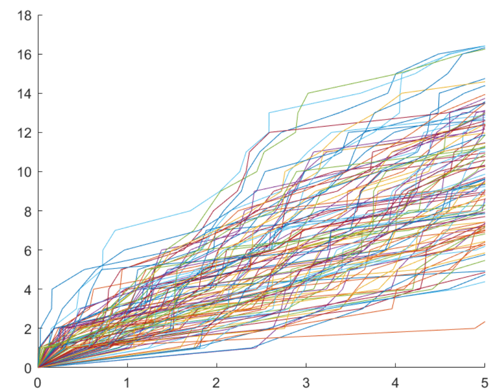
1. Write a code in Matlab to simulate a Poisson process $N(t)$ with parameter $\lambda > 0$ over the time interval $[0, T]$ where $T > 0$.

Solution:

```
lambda = 2;
T = 5;
N = 100;
hold on
for i = 1:N
    t = 0;
    x = [0];
    len = 0;
    while t < T
        r = exprnd(1/lambda);
        t = t + r;
        x = [x t];
        len = len + 1;
    end
    y = 0:len;
    x(1) = 0;
    plot(x,y);
    xlim([0 T]);
end
hold off
```

2. Take $T = 5$ year, $\lambda = 2$ per year. Plot 100 sample paths.

Solution:



Problem 2 (25 points):

Consider the following process $\{X(t), t \geq 0\}$ defined by

$$X(t) = N(t) - \lambda t$$

where N is a Poisson process with rate $\lambda > 0$

1. What is the probability mass of $X(t)$?

Solution: We note that $N(t) = N(t) - 0 = N(t) - N(0) \sim Pois(\lambda t)$. It then follows that the pmf of $X(t)$ is $\mathbb{P}[X(t) = n] = \mathbb{P}[N(t) - \lambda t = n] = \mathbb{P}[N(t) = n + \lambda t]$. We can then intuitively observe that this is a Poisson process that has mean zero at each arrival time.
Alternatively, we can describe the process by its increments after observing that $X(0) = N(0) - 0 = 0$: $\forall t, s \geq 0, X(s+t) - X(t) = N(s+t) - \lambda(s+t) - N(t) + \lambda(t) = N(s+t) - N(t) - \lambda s$. As $N(s+t) - N(t) \sim Pois(\lambda s)$, we observe that each increment of our process follows the probability distribution of a Poisson distribution with parameter λs but re-centered to have mean 0 through translation.

2. Give the expectation and the variance of $X(t)$.

Solution: The expectation is $\mathbb{E}[X(t)] = \mathbb{E}[N(t) - \lambda t] = \mathbb{E}[N(t)] - \lambda t = \lambda t - \lambda t = 0$
The variance is $Var[X(t)] = Var[N(t) - \lambda t] = Var[N(t)] = \lambda t$

3. Is X a martingale? Justify your answer

Solution: X is a martingale as $\mathbb{E}[X(t+s) \mid X(t)] = \mathbb{E}[X(t+s) - X(t) + X(t) \mid X(t)] = \mathbb{E}[X(t+s) - X(t) \mid X(t)] + \mathbb{E}[X(t) \mid X(t)] = \mathbb{E}[N(t+s) - \lambda(t+s) - N(s) + \lambda(s) \mid X(t)] + X(t) = \mathbb{E}[N(t+s) - N(s) - \lambda s \mid X(t)] + X(t) = \lambda s - \lambda s + X(t) = X(t)$

Problem 3 (40 points): We model the aggregate insurance loss, at time t, as follows:

$$S(t) = S(0) + \sum_{i=1}^{N(t)} X_i,$$

where $S(0)$ is a given real number, X_i are independent exponential variables with rate μ, and $\{N(t), t \geq 0\}$ is a Poisson process with rate λ, independent of X_i. When $N(t) = 0$, we simply set $\sum_{i=0}^{N(t)} X_i = 0$. The common mean of the exponential variables is $\frac{1}{\mu}$ and their variance is $\frac{1}{\mu^2}$.

1. Compute $\mathbb{E}[S(t)]$

Solution: $\mathbb{E}[S(t)] = \mathbb{E}[S(0) + \sum_{i=1}^{N(t)} X_i] = S(0) + \mathbb{E}[\mathbb{E}[\sum_{i=1}^{N(t)} X_i \mid N(t)]]$
$= S(0) + \sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=1}^{N(t)} X_i \mid N(t) = n] \cdot \mathbb{P}[N(t) = n]$
$= S(0) + \sum_{n=0}^{\infty} n\mathbb{E}[X_1] \cdot \mathbb{P}[N(t) = n] = S(0) + \mathbb{E}[N(t) \cdot \mathbb{E}[X_1]] = S(0) + \mathbb{E}[\frac{N(t)}{\mu}] = S(0) + \frac{1}{\mu} \cdot (\lambda t) = S(0) + \frac{\lambda t}{\mu}$

2. Compute $\mathbb{E}[S^2(t)]$

Solution: $\mathbb{E}[S^2(t)] = \mathbb{E}[S^2(0) + 2\sum_{i=1}^{N(t)} X_i S(0) + \sum_{i=1}^{N(t)} \sum_{j=1}^{N(t)} X_i X_j]$
$= S^2(0) + 2 \cdot S(0) \cdot \mathbb{E}[\sum_{i=1}^{N(t)} X_i] + \mathbb{E}[\sum_{i=1}^{N(t)} \sum_{j=1}^{N(t)} X_i X_j]$
$= S^2(0) + 2 \cdot S(0) \frac{\lambda t}{\mu} + \mathbb{E}[\sum_{i=1}^{N(t)} \sum_{j=1}^{N(t)} X_i X_j]$
[by using substitution from steps in previous problem]
$= S^2(0) + S(0) \frac{2\lambda t}{\mu} + \mathbb{E}[\mathbb{E}[\sum_{i=1}^{N(t)} \sum_{j=1}^{N(t)} X_i X_j] \mid N(t)]$
We then focus on the last term:
$\sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j] \mathbb{P}(N(t) = n)$
$= \sum_{n=0}^{\infty} \left(\sum_{i \neq j} \mathbb{E}[X_i X_j] + \sum_{i=1}^{n} \mathbb{E}[X_i^2] \right) \mathbb{P}(N(t) = n)$
$= \sum_{n=0}^{\infty} \left(\frac{n^2 - n}{\mu^2} + \sum_{i=1}^{n} (Var[X_i] + \mathbb{E}[X_i]^2) \right) \mathbb{P}(N(t) = n)$
$= \sum_{n=0}^{\infty} \left(\frac{n^2 - n}{\mu^2} + \sum_{i=1}^{n} \frac{2}{\mu^2} \right) \mathbb{P}(N(t) = n)$
$= \sum_{n=0}^{\infty} \frac{n^2 + n}{\mu^2} \mathbb{P}(N(t) = n)$
$= \frac{1}{\mu^2} \sum_{n=0}^{\infty} \left(n^2 \cdot \mathbb{P}(N(t)) + n \cdot \mathbb{P}(N(t)) \right)$
$= \frac{1}{\mu^2} \left(\mathbb{E}[N(t)] + \mathbb{E}[N(t)^2] \right)$

$= \frac{1}{\mu^2}\left(\mathbb{E}[N(t)] + Var[N(t)] + \mathbb{E}[N(t)]^2\right)$
$= \frac{1}{\mu^2}\left(\lambda t + \lambda t + \lambda^2 t^2\right) = \frac{\lambda^2 t^2 + 2\lambda t}{\mu^2}$
Thus we have:
$\mathbb{E}[S^2(t)] = S^2(0) + S(0)\frac{2\lambda t}{\mu} + \frac{\lambda^2 t^2 + 2\lambda t}{\mu^2}$