Quantitative Methods: Assignment 7

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Problem 1

Compute the differentials of:

1. $W^2(t)$

Solution: We do this with usage of the product rule in stochastic calculus:

$$\begin{aligned} d(W^2(t)) = d(W(t) \cdot W(t)) = dW(t)W(t) + W(t)dW(t) + dW(t)dW(t) \\ = 2W(t)dW(t) + dt \end{aligned}$$

2. $W^3(t)$

Solution: We proceed with ito's lemma for $f(x,t) = x^3$ so that:

$$\begin{aligned} d(W^3(t)) = f'(W(t))dW(t) + \frac{1}{2}f''(W(t))(dW(t))^2 \\ = 3W^2(t)dW(t) + 3W(t)dt \end{aligned}$$

3. $\exp\{-rt\}W(t)$

Solution: We proceed with ito's lemma for $f(x,t) = xe^{-rt}$ so that:

$$\begin{aligned} d(e^{-rtW(t)}) = f_t dt + f_x dW(t) + \frac{1}{2}f_{xx}dt \\ = -rW(t)e^{-rt}dt + e^{-rt}dW(t) + \frac{1}{2}0(dW(t))^2 \\ = -rW(t)e^{-rt}dt + e^{-rt}dW(t) \end{aligned}$$

4. $\exp\{t^2 - W(t)\}$

Solution: We proceed with ito's lemma for $f(x,t) = e^{t^2 - x}$ so that:

$$\begin{aligned} d(e^{t^2 - W(t)}) &= f_t dt + f_x dW(t) + \frac{1}{2} f_{xx} dt \\ &= \left(2t \cdot e^{t^2 - W(t)} + \frac{1}{2} e^{t^2 - W(t)}\right) dt - e^{t^2 - W(t)} dW(t) \\ &= (2t + \frac{1}{2}) e^{t^2 - W(t)} dt - e^{t^2 - W(t)} dW(t) \end{aligned}$$

Problem 2

1. Compute the differential of $W^4(t)$, where W is a standard Brownian motion.

Solution: We proceed with ito's lemma for $f(x,t) = x^4$ so that:

$$d(W^4(t)) = f' dW(t) + \frac{1}{2} f'' dt = 4W^3(t) dW(t) + 6W^2(t) dt$$

2. Integrate the above formula on $[0, T]$.

Solution:

$$\begin{aligned} \int_0^T d(W^4(t)) &= \int_0^T 4W^3(t) dW(t) + \int_0^T 6W^2(t) dt \\ W^4(T) - W^4(0) &= \int_0^T 4W^3(t) dW(t) + \int_0^T 6W^2(t) dt \\ W^4(T) &= \int_0^T 4W^3(t) dW(t) + \int_0^T 6W^2(t) dt \end{aligned}$$

3. Take the expectation of the left and right-hand sides and deduce $\mathbb{E}[W^4(T)]$

Solution: On the left-hand side, we have that:

$$\mathbb{E}[\int_0^T d(W^4(t))] = \mathbb{E}[W^4(T) - W^4(0)] = \mathbb{E}[W^4(T)]$$

On the right-hand side, we have that:

$$
\begin{aligned}
&\mathbb{E}[\int_0^T 4W^3(t)dW(t) + \int_0^T 6W^2(t)dt] \\
&= \mathbb{E}[\int_0^T 4W^3(t)dW(t)] + \mathbb{E}[\int_0^T 6W^2(t)dt] \\
&= 0 + \int_0^T 6\mathbb{E}[W^2(t)]dt \text{ By Fubini's theorem in Classical Calculus} \\
&= \int_0^T 6tdt = 3T^2
\end{aligned}
$$

Problem 3

We consider the following mean-reverting model for the spread $S(t)$ of two co-integrated stocks:

$$dS(t) = -\lambda S(t)dt + \sigma dW(t), S(0) = s > 0$$

where $\lambda > 0, \sigma > 0$

1. Show by using Ito-Doeblin formula that

$$S(t) = e^{-\lambda t}(s + \sigma \int_0^t e^{\lambda u} dW(u))$$

is a solution of the above Stochastic Differential Equation.

Solution:
We let $f(t,x) = e^{\lambda t} \cdot x$ and then use the Ito-Doeblin formula to find that:

$$
\begin{aligned}
d(f(t,S(t)) = d(e^{\lambda t} \cdot S(t)) &= \lambda e^{\lambda t} \cdot S(t)dt + e^{\lambda t} dS(t) + \frac{1}{2} \cdot 0(dS(t))^2 \\
&= \lambda e^{\lambda t} \cdot S(t)dt + e^{\lambda t}(-\lambda S(t)dt + \sigma dW(t)) \\
&= e^{\lambda t} \sigma dW(t)
\end{aligned}
$$

If we integrate the above expression over interval [0,t], we get:

$$
\begin{aligned}
&e^{\lambda t} \cdot S(t) - e^{\lambda 0} \cdot S(0) = \int_0^t e^{\lambda u} \sigma dW(u) \\
&e^{\lambda t} S(t) = S(0) + \int_0^t e^{\lambda u} \sigma dW(u) \\
&S(t) = e^{-\lambda t}\big(s + \sigma \int_0^t e^{\lambda u} dW(u)\big), \text{where } S(0) = s > 0
\end{aligned}
$$

2. Give the distribution of $S(t)$ and specify its mean and variance.

Solution: We have:

$$\begin{aligned}
\mathbb{E}[S(t)] &= \mathbb{E}[e^{-\lambda t}\big(s + \sigma \int_0^t e^{\lambda u} dW(u)\big)] \\
&= e^{-\lambda t}s + \mathbb{E}[e^{-\lambda t}\sigma \int_0^t e^{\lambda u} dW(u)] \\
&= e^{-\lambda t}s + e^{-\lambda t}\sigma \mathbb{E}[\int_0^t e^{\lambda u} dW(u)] \\
&= e^{-\lambda t}s + e^{-\lambda t}\sigma \cdot 0 = e^{-\lambda t}s
\end{aligned}$$

The variance follows from:

$$\begin{aligned}
Var[S(t)] &= Var[e^{-\lambda t}\big(s + \sigma \int_0^t e^{\lambda u} dW(u)\big)] \\
&= Var[e^{-\lambda t}\sigma \int_0^t e^{\lambda u} dW(u)] \\
&= e^{-2\lambda t}\sigma^2 Var[\int_0^t e^{\lambda u} dW(u)] \\
&= e^{-2\lambda t}\sigma^2 \left(\mathbb{E}[(\int_0^t e^{\lambda u} dW(u))^2] - \mathbb{E}[\int_0^t e^{\lambda u} dW(u)]^2 \right) \\
&= e^{-2\lambda t}\sigma^2 \left(\mathbb{E}[(\int_0^t e^{2\lambda u} du] - (0)^2 \right) \\
&= e^{-2\lambda t}\sigma^2 \left(\frac{1}{2\lambda} e^{2\lambda t} - \frac{1}{2\lambda} \right) \\
&= \sigma^2 \left(\frac{1}{2\lambda} - \frac{1}{2\lambda} e^{-2\lambda t} \right)
\end{aligned}$$

3. Give $\lim_{t\to+\infty} \mathbb{E}[S(t)]$ and interpret the result in plain english

Solution:

$$\lim_{t\to+\infty} \mathbb{E}[S(t)] = \lim_{t\to+\infty} e^{-\lambda t}s = 0$$

This means that our stock price will on average go to zero in the long run.

4. Does the variance of $S(t)$ increase or decrease over time?

Solution: Variance increases over time as the negative term vanishes with time.

Problem 4: Let $W(t)$ be a brownian motion, and define

$$B(t) = \int_0^t sign(W(s))dW(s)$$

where

$$sign(x) = \begin{cases} 1 & , if x \geq 0 \\ -1 & , otherwise \end{cases}$$

1. Show that $B(t)$ is a Brownian motion

Solution: We check the following:

(a) $B(0) = \int_0^0 sign(W(0))dW(s) = 0$

(b) For $\epsilon > 0$, $B(t+\epsilon) - B(t) = \int_t^{t+\epsilon} sign(W(s))dW(s)$ is independent of increments $B(m) - B(n)$ for $0 \leq n \leq m < t$.

(c) Almost-sure continuity follows from the construction of the Ito integral. As in the construction, we can find simple processes such that their integrals are continuous and converge in the mean square sense to our integral.

(d) For $\epsilon > 0$, $B(t+\epsilon) - B(t) = \int_t^{t+\epsilon} sign(W(s))dW(s)$. Note that here if we take a partition $\mathcal{P}$ of $t = t_1 < t_2 < \cdots < t_n = t + \epsilon$ where the mesh of $\mathcal{P}$ converges to zero, we can approximate the above integral as the sum $B(t+\epsilon) - B(t) = \sum_{j=1}^n sign(W(t_j))(W(t_j) - W(t_{j-1}))$ so we are considering the sum of positive and negative increments of the wiener process which are normal with mean 0 and variance $t_j - t_{j-1}$. We note that this normal distribution has mean 0 and variance $\sum_{j=1}^n t_j - t_{j-1} = t_n - t_1 = \epsilon$. So $B(t)$ has increments which are normally distributed with mean zero and variance equal to the increment length.

2. Use Ito's product rule to compute $d[B(t)W(t)]$. Integrate both sides of the resulting equation, take expectations and deduce that

$$\mathbb{E}[B(t)W(t)] = 0$$

Solution: We first note that $dB(t) = B(t+dt) - B(t) = \int_t^{t+dt} sign(W(s))dW(s) = sign(W(t))dW(t)$

$$d[B(t)W(t)] = sign(W(t))dW(t)W(t) + B(t)dW(t) + sign(W(t))dW(t)dW(t)$$

We integrate both sides from 0 to t to get:

$$\begin{aligned} B(t)W(t) - B(0)W(0) &= \int_0^t d[B(s)W(s)]ds \\ &= \int_0^t W(s)dB(s) + \int_0^t B(s)dW(s) + \int_0^t dW(s)dB(s) \end{aligned}$$

We take expectation to get:

$$\begin{aligned} \mathbb{E}[B(t)W(t)] &= \mathbb{E}[\int_0^t W(s)sign(W(s))dW(s) + \int_0^t B(s)dW(s) + \int_0^t sign(W(s))dW(s)dW(s)] \\ &= \mathbb{E}[\int_0^t W(s)sign(W(s))dW(s)] + \mathbb{E}[\int_0^t B(s)dW(s)] + \mathbb{E}[\int_0^t sign(W(s))ds] \\ &= 0 + 0 + \mathbb{E}[\int_0^t sign(W(s))ds] \\ &= \int_0^t \mathbb{E}[sign(W(s))]ds \\ &= \int_0^t \big(1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2}\big)ds = 0 \end{aligned}$$

3. Verify that

$$dW^2(t) = 2W(t)dW(t) + dt$$

Solution:

$$d(W^2(t)) = dW(t)W(t) + W(t)dW(t) + dW(t)dW(t) = 2W(t)dW(t) + dt$$

4. Use Ito product's rule to $d[B(t)W^2(t)]$. Deduce that

$$\mathbb{E}[B(t)W^2(t)] \text{ is not equal to } \mathbb{E}[B(t)]\mathbb{E}[W^2(t)]$$

Solution:

$$
\begin{aligned}
&d[B(t)W^2(t)] \\
&= dB(t)W^2(t) + B(t)d(W^2(t)) + dB(t)d(W^2(t)) \\
&= dB(t)W^2(t) + B(t)(2W(t)dW(t) + dt) + dB(t)(2W(t)dW(t) + dt) \\
&= \bigg(sign(W(t))W^2(t) + B(t)2W(t)\bigg)dW(t) + B(t)dt + sign(W(t))dW(t)dt+ \\
&+ sign(W(t))dW(t)2W(t)dW(t) \\
&= \bigg(sign(W(t))W^2(t) + B(t)2W(t)\bigg)dW(t) + B(t)dt + 0 + sign(W(t))2W(t)dt \\
&= \bigg(sign(W(t))W^2(t) + B(t)2W(t)\bigg)dW(t) + \bigg(B(t) + sign(W(t))2W(t)\bigg)dt
\end{aligned}
$$

We take integration over 0 to t and expectation to get:

$$
\begin{aligned}
&\mathbb{E}[B(t)W^2(t)] \\
&= B(0)W^2(0) + \mathbb{E}[\int_0^t \bigg(sign(W(s))W^2(s) + B(s)2W(s)\bigg)dW(s)+ \\
&+ \int_0^t \bigg(B(s) + sign(W(s))2W(s)\bigg)ds] \\
&= 0 + \mathbb{E}[\int_0^t \bigg(B(s) + sign(W(s))2W(s)\bigg)ds] \\
&= \int_0^t \mathbb{E}[B(s) + sign(W(s))2W(s)]ds \\
&= \int_0^t \mathbb{E}[sign(W(s))2W(s)]ds \\
&= 2\int_0^t \mathbb{E}[|W(s)|]ds
\end{aligned}
$$

We note that $\mathbb{E}[B(t)]\mathbb{E}[W^2(t)] = 0$ so that if $\mathbb{E}[B(t)W^2(t)] = \mathbb{E}[B(t)]\mathbb{E}[W^2(t)]$, $2\int_0^t \mathbb{E}[|W(s)|]ds = 0 \Rightarrow W(s) = 0 \forall s \in [0,t]$. But this is evidently not true. It then follows that:

$$\mathbb{E}[B(t)W^2(t)] \text{ is not equal to } \mathbb{E}[B(t)]\mathbb{E}[W^2(t)]$$

5. Conclude

Solution: We can conclude that $B(t)$ and $W^2(t)$ are not independent random processes.

Problem 5

In the Hull-white interest rate model, the interest rate is given by the stochastic differential equation

$$dR(t) = (a(t) - b(t)R(t))dt + \sigma(t)d\tilde{W}(t)$$

where $a(t), b(t), R(t), \sigma(t)$ are nonrandom positive functions and $\tilde{W}$ is a standard Brownian motion under a risk-neutral probability measure. Assume that the initial condition is given at time t by $R(t) = r$. The goal of this exercise is to solve this equation explicitly.

1. First of all, compute, by using Ito-Doeblin formula

$$d(e^{\int_0^u b(v)dv}R(u))$$

Solution: As $b(t)$ is nonrandom, we don't need the Ito-Doeblin formula and proceed with standard product rule:

$$\begin{aligned} d(e^{\int_0^u b(v)dv}R(u)) &= d(e^{\int_0^u b(v)dv})R(u) + e^{\int_0^u b(v)dv}d(R(u)) \\ &= b(u)e^{\int_0^u b(v)dv}R(u)du + e^{\int_0^u b(v)dv}((a(u) - b(u)R(u))du + \sigma(u)d\tilde{W}(u)) \\ &= e^{\int_0^u b(v)dv}a(u)du + e^{\int_0^u b(v)dv}\sigma(u)d\tilde{W}(u) \end{aligned}$$

2. Integrate both sides of the formula found above from t to T and use the initial condition $R(t) = r$ to find the formula

$$e^{\int_0^T b(v)dv}R(T) = re^{\int_0^u b(v)dv} + \int_0^T e^{\int_0^u b(v)dv}a(u)du + \int_0^T e^{\int_0^t b(v)dv}\sigma(u)d\tilde{W}$$

Solution: We integrate both sides from t to T for:

$$\begin{aligned} e^{\int_0^T b(v)dv}R(T) - e^{\int_0^t b(v)dv}R(t) &= \int_t^T e^{\int_0^u b(v)dv}a(u)du + \int_t^T e^{\int_0^u b(v)dv}\sigma(u)d\tilde{W}(u) \\ e^{\int_0^T b(v)dv}R(T) &= re^{\int_0^t b(v)dv} + \int_t^T e^{\int_0^u b(v)dv}a(u)du + \int_t^T e^{\int_0^u b(v)dv}\sigma(u)d\tilde{W}(u) \end{aligned}$$

3. Solve the above formula for $R(T)$. This yields an explicit formula for $R(T)$

Solution:

$$\begin{aligned}
R(T) &= \frac{re^{\int_0^t b(v)dv} + \int_t^T e^{\int_0^u b(v)dv} a(u)du + \int_t^T e^{\int_0^u b(v)dv} \sigma(u)d\tilde{W}(u))}{e^{\int_0^T b(v)dv}} \\
&= \frac{re^{\int_0^t b(v)dv} + \int_t^T e^{\int_0^u b(v)dv} a(u)du + \int_t^T e^{\int_0^u b(v)dv} \sigma(u)d\tilde{W}(u))}{e^{\int_0^T b(v)dv}} \\
&= re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} a(u)du + \int_t^T e^{-\int_u^T b(v)dv} \sigma(u)d\tilde{W}(u))
\end{aligned}$$

4. Give the distribution of $R(T)$

Solution: We note that as $b(t), a(t), \sigma(t)$ are nonrandom functions, $re^{-\int_t^T b(v)dv}$ and $\int_t^T e^{-\int_u^T b(v)dv} a(u)du$ are constants. We then note that the randomness of $R(T)$ comes from $\int_t^T e^{-\int_u^T b(v)dv} \sigma(u)d\tilde{W}(u) = \sum_{j=1}^n e^{-\int_{t_j}^T b(v)dv} \sigma(t_j) \cdot (W(t_j) - W(t_{j-1}))$. Where the sequence $t = t_1 < t_2 < \cdots < t_n = T$ represents a partition of $[t, T]$ with mesh converging to zero. It then follows that $R(T)$ follows a normal distribution as the sum of separately scaled independent wiener increments which are normal.

5. Give also its mean and variance.

Solution: We first observe that the mean is:

$$\begin{aligned}
\mathbb{E}[R(T)] &= \mathbb{E}[re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} a(u)du + \int_t^T e^{-\int_u^T b(v)dv} \sigma(u)d\tilde{W}(u)] \\
&= re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} a(u)du + \mathbb{E}[\int_t^T e^{-\int_u^T b(v)dv} \sigma(u)d\tilde{W}(u)] \\
&= re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} a(u)du + 0 \\
&= re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} a(u)du
\end{aligned}$$

And that the variance is:

$$Var[R(T)] = Var[re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} a(u)du + \int_t^T e^{-\int_u^T b(v)dv} \sigma(u)d\tilde{W}(u)]$$

$$
\begin{aligned}
&= Var[\int_t^T e^{-\int_u^T b(v)dv}\sigma(u)d\tilde{W}(u)] \\
&= \mathbb{E}[\big(\int_t^T e^{-\int_u^T b(v)dv}\sigma(u)d\tilde{W}(u)\big)^2] - \left(\mathbb{E}[\int_t^T e^{-\int_u^T b(v)dv}\sigma(u)d\tilde{W}(u)]\right)^2 \\
&= \mathbb{E}[\int_t^T \big(e^{-\int_u^T b(v)dv}\sigma(u)\big)^2 du] - 0 \\
&= \mathbb{E}[\int_t^T e^{-2\int_u^T b(v)dv}\sigma^2(u)du] \\
&= \int_t^T e^{-2\int_u^T b(v)dv}\sigma^2(u)du
\end{aligned}
$$