Stochastic Calculus and Option Pricing
Week 2: (Almost) All About Brownian Motion

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Most of the material for this Lecture is based on the textbook by Steven Shreve *Stochastic Calculus for Finance, II*, chapter 3.
Outline of the lecture:

- Brownian Motion as a Limit of Scaled Random Walk
- Definition and Properties of Brownian Motion
- Self Similarity of Brownian Motion
- Regularity of Brownian Motion : continuity and no-differentiability
- Quadratic Variation of Brownian motion

Key Concepts

We consider a function f on a closed interval $[a, b]$, and all partitions of the interval Π: $a = t_0 < t_1 ... < t_n = b$. The mesh size of a partition Π is the maximum of all differences $t_{j+1} - t_j$.

1. A function f has a **bounded variation**, if for all partitions there exists M so that

$$|f(t_1) - f(t_0)| + ...|f(b) - f(t_{n-1})| \leq M$$

2. A function f has a **quadratic variation** if, for all partitions, there is M so that

$$|f(t_1) - f(t_0)|^2 + ...|f(b) - f(t_{n-1})|^2 \leq M$$

3. The **total quadratic variation** of a function on the interval $[a, b]$ is defined as,

$$\sup_{\Pi} \sum_{j=1}^{n} (f(t_j) - f(t_{j-1}))^2$$

where the supremum is taken over all partitions Π with mesh size going to zero.

Back to the symmetric random walk : quadratic variation

We consider a particular path of a symmetric random walk from the first lecture: The quadratic variation up to time k is denoted by $[X,X](k)$ is

$$[X,X](k) = \sum_{j=1}^{k} (X_j - X_{j-1})^2.$$

Then,

$$[X,X](k) = \sum_{j=1}^{k} (X_j - X_{j-1})^2 = \sum_{j=1}^{k} 1 = k.$$

Note that the quadratic variation of the random walk is independent of the path considered and is equal to $Var[X_k]$. Their computations though are quite different: $Var[X_k]$ is computed by taking an average over all paths, taking into account the probabilities; while $[X,X](k)$ is computed along a single path, no probabilities involved.

Scaled symmetric random walk

We consider the same symmetric random walk, but now instead of moving by ± 1, we move by $\pm\sqrt{\delta}$, where δ is a small number. The process is the same, we toss a coin, observe head or tail.
Therefore, now

$$Y_i = \begin{cases} -\sqrt{\delta} & \text{if } \omega_i = T \\ \sqrt{\delta} & \text{if } \omega_i = H \end{cases}$$

As before $\mathbb{E}[Y_i] = 0$, and $Var[Y_i] = \delta$. Then we define the process $M_\delta(t)$ as follows:

$$M_\delta(0) = 0$$

and at time $t = n\delta$ is defined as

$$M_\delta(t) = \sum_{i=1}^{n} Y_i \text{ for } k = 1, 2, ...$$

We have: $\mathbb{E}[M_\delta(t)] = 0$, $Var[M_\delta(t)] = n\delta = t$.

Properties of Scaled Symmetric Random Walk

(i) The increments of the scaled random walk are independent: $0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m$ such that each $t_j = n_j\delta$, the increments

$$M_\delta(t_1) - M_\delta(t_0), M_\delta(t_2) - M_\delta(t_1), \ldots, M_\delta(t_m) - M_\delta(t_{m-1})$$

are independent random variables.

(ii) Expectation of an increment of the scaled random walk

$$\mathbb{E}[M_\delta(t_{i+1}) - M_\delta(t_i)] = 0$$

(iii) Variance of an increment of the scaled random walk

$$Var[M_\delta(t_i) - M_\delta(t_s)] = t_i - t_s.$$

Stationary Increments and Quadratic Variation

(iv) We say that a process $X(t)$ has *stationary increments*, if for all $0 \leq t_1 \leq t_2$ and $r > 0$ two random variables $X(t_2) - X(t_1)$ and $X(t_2 + r) - X(t_1 + r)$ have the same distributions. Then $M_\delta(t)$ has stationary increments. Let $t_1 = n_1\delta$ and $t_2 = n_2\delta$

$$M_\delta(t_2) - M_\delta(t_1) = \sum_{i=n_1+1}^{i=n_2} Y_i$$

Therefore, $\mathbb{E}[M_\delta(t_2) - M(t_1)] = 0$ and $Var[M_\delta(t_2) - M_\delta(t_1)] = t_2 - t_1$, which depend only on the length $t_2 - t_1$, not the exact location.

- Quadratic variation: We compute quadratic variation on $[t_1, t_2]$:

$$[M_\delta, M_\delta](t_1, t_2) = \sum_{i=n_1+1}^{n_2} Y_i^2 = (n_2 - n_1)\delta = t_2 - t_1$$

Central Limit Theorem

Recall the CLT:

Theorem

Central Limit Theorem Let $\{X_k\}$ be a sequence of i.i.d. (independent, identically distributed) random variables. Assume that the expectation $\mathbb{E}(X_k) = \mu$ and the variance $\sigma^2 = Var(X_k)$ exist, and let

$$S_n = X_1 + \ldots + X_n$$

Then for any β

$$\mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}}\right) \to \Phi(\beta)$$

where $\Phi(x)$ is the standard normal distribution.

Brownian Motion as a Limit a Symmetric Random Walk

In our case,

$$X_k = Y_k, S_n = M_\delta(t), \mu = 0, \sigma = \sqrt{\delta}, \sigma\sqrt{n} = \sqrt{t}$$

Therefore, as $n \to \infty$, which is equivalent to $\delta \to 0$, the distribution of

$$\frac{M_\delta(t)}{\sqrt{t}}$$

become normal, which is equivalent to the statement that $M_\delta(t)$ has a normal distribution with zero mean and variance t.
We recall pdf for a normal distribution with mean 0 and variance t:

$$f(x) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t}).$$

As t varies, we obtain a Brownian motion as the limit of the scaled random walk $M_\delta(t)$ as $\delta \to 0$ or $n \to +\infty$. We will refer to this limiting process as $W(t)$.

Another way

In Shreve' s book, the same process is defined little differently: Rescale the random walk in order to approximate a Brownian motion: we define

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} X_{nt},$$

where t is the continuous-time variable.

- How $W^{(n)}(t)$ defined in practice:
  - On one hand, if nt is a integer, X_{nt} is already defined without ambiguity.
  - On the other hand, if nt is not an integer, we define $W^{(n)}(t)$ by *linear interpolation* on the interval $[s, u]$ where s is the nearest real numbers on the left of t such that ns is an integer and u is the nearest real number to the right of t such that nu is an integer.
- Let $n \to +\infty$ in order to obtain a standard Brownian motion.

The Brownian motion

Definition (The Brownian motion)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A Brownian motion $\{W(t), t \geq 0\}$ is a continuous-time and continuous-state stochastic process such that

(i) It starts at zero: $W(0) = 0$

(ii) It has stationary, independent increments: $0 = t_0 < t_1 < t_2 < ... < t_m$

$$W(t_1) - W(t_0), W(t_2) - W(t_1), ..., W(t_m) - W(t_{m-1})$$

(iii) Each increment $W(t_{j+1}) - W(t_j)$ is normally distributed with mean 0 and variance $t_{j+1} - t_j$

$$\mathbb{E}[W(t_{j+1}) - W(t_j)] = 0, Var[W(t_{j+1}) - W(t_j)] = t_{j+1} - t_j.$$

(iv) $W(t)$ is a continuous function of t: no "jumps"

Joint distribution of the Brownian motion

- The Brownian motion is a Gaussian process.
- For all $0 = t_0 < t_1 < t_2 < ... < t_n$, the random variables $W(t_1), W(t_2), ..., W(t_n)$ are jointly normally distributed. Their joint distribution is characterized by the vector of its means and its covariance matrix.
- The mean of each $W(t_j)$ is 0

$$\mathbb{E}[W(t_j)] = 0.$$

- We want to determine its covariance matrix

$$\begin{bmatrix} \mathbb{E}[W^2(t_1)] & \mathbb{E}[W(t_1)W(t_2)] & \dots & \mathbb{E}[W(t_1)W(t_n)] \\ \mathbb{E}[W(t_2)W(t_1)] & \mathbb{E}[W^2(t_2)] & \dots & \mathbb{E}[W(t_2)W(t_n)] \\ \vdots & \vdots & \vdots & \\ \mathbb{E}[W(t_n)W(t_1)] & \dots & \dots & \mathbb{E}[W^2(t_n)] \end{bmatrix}.$$

We have

$$\mathbb{E}[W^2(t)] = t$$

and for all $s < t$,

$$\begin{aligned} & \mathbb{E}[W(s)W(t)] = \mathbb{E}[W(s)(W(t) - W(s)) + W^2(s)] \\ = & \mathbb{E}[W(s)]\mathbb{E}[W(t) - W(s)] + \mathbb{E}[W^2(s)] \\ = & 0 + s = s = \min(s, t). \end{aligned}$$

This yields the covariance matrix

$$\begin{bmatrix} t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & \dots & t_2 \\ \vdots & \vdots & \vdots & \vdots \\ t_1 & t_2 & \cdots & t_n \end{bmatrix}.$$

Transition density for the Brownian motion

For $x, y, s \leq t$, it is given by

$$p(y, x, t, s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp(-\frac{(y-x)^2}{2(t-s)}).$$

This is the probability density in the variable y for the random variable $W(t)$, conditioned by the event $W(s) = x$.

On the regularity of the Brownian motion

- Each sample path of the Brownian motion $W(t, \omega)$ is a continuous function of time. It does not have jumps!
- However the sample paths of the Brownian motion cannot be differentiated with respect to the time variable at any point! A sample path of the Brownian motion does not look smooth and curvy. It has *kinks*.
- Example of a function which is not differentiable because it does not have a continuous derivative at 0:

$$f(t) = |t|, \text{ for } t \in \mathbb{R}.$$

Self Similarity

Definition (Self Similarity)

A stochastic process $(X_t, t \in [0, \infty))$ is H-self similar for some $H > 0$ if it satisfies:

$$\left(B^H X(t_1), ...B^H X(t_n)\right) \stackrel{d}{=} (X(Bt_1)..., X(Bt_n))$$

for every $B > 0$ and any choice of $t_i > 0$, $i = 1, , , , n$.

Question: If $X(t) = W(t)$ (so Brownian motion), what is H ?
Answer: $H = 1/2$.
The self similarity of Brownian motion has a nice consequence for the simulation:
To simulate a path on $[0, T]$ is sufficient to simulate a path on $[0, 1]$, scale the time interval by factor T and then scale the sample path by the factor T^H.

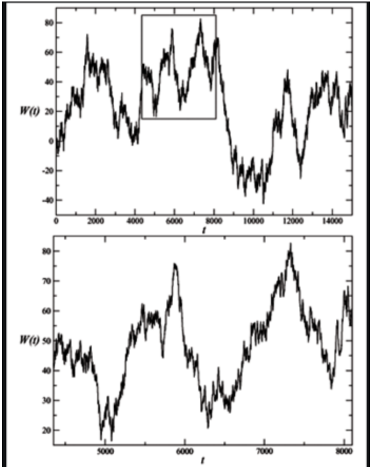

Figure 7. Self-similarity of a Brownian motion path. In (a) we plot a path of a Brownian motion with 15000 time steps. The curve in (b) is a blow-up of the region delimited by a rectangle in (a), where we have rescaled the x axis by a factor 4 and the y axis by a factor 2. Note that the graphs in (a) and (b) "look the same," statistically speaking. This process can be repeated indefinitely.

Figure: Self Similarity of Brownian Motion

Differentiability and First Order Variation

(i) A function f is differentiable at point x_0, if the limit exists and finite.

$$f^{'}(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

(ii) First Order Variation of a function on the interval $[0, T]$: Choose a partition $\Pi : 0 = t_0 < t_1 < ... < t_n = T$ and define $||\Pi|| = \max_{j=0,..,n-1}(t_{j+1} - t_j)$. Then define

$$FV_T(f) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1} - f(t_j)|$$

(iii) For a smooth differentiable function f the first order variation is finite and can be computed as (look for details in Shreve book)

$$FV_T(f) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f^{'}(t_j^*)|(t_{j+1} - t_j) = \int_0^T |f^{'}(t)|dt$$

Irregularity of Brownian motion

We fix one sample path $W_\omega(t), t \geq 0$ and consider its properties. We know it is continuous (no jumps), but very irregular. The main reason is that the increments are independent. In fact, each path is continuous but *nowhere* differentiable. Such functions were discovered by Weierstrass in 19 century.
"I turn away with horror and fear from the lamentable plague of continuous functions which do not have derivatives" ..
- Brownian motion is a process with nowhere differentiable sample paths Heuristic argument: The main reason for non-differentiability is that when scale down Δt by $B < 1$, the function values scale by $B^{1/2}$, therefore the limit to calculate the first derivative is $B^{-1/2}$ and does not exist
- Brownian sample paths *do not have bounded variation* on any finite interval $[0, T]$ We will get the intuition why it is true, after considering the quadratic variation.

Quadratic variation

Recall that quadratic variation of the scaled random walk M_δ up to time T is T. We will get the same answer for Brownian motion, taking $n \to \infty$.

Theorem (QV)

For all $T \geq 0$,

$$[W, W](T) = T \textit{almost surely}$$

- Interpretation of *almost surely*: this means that the statement is true for almost all the sample paths except for a set of sample paths. This set of sample paths for which the equality is wrong has a probability 0.
- A function which has continuous derivatives has a quadratic variation equal to 0!
- **This is a profound property of Brownian motion which makes stochastic calculus different from ordinary calculus**

Quadratic variation of a function with continuous derivatives

Assume for now that f has continuous derivatives. By the Mean Value Theorem, on each interval $[t_j, t_{j+1}]$, there is a time t_j^* and $h = ||\Pi||$,

$$
\begin{aligned}
[f,f](T) &= \lim_{h\to 0, n\to+\infty} \sum_{j=0}^{n-1} |f'(t_j^*)|^2 (t_{j+1}-t_j)^2 \\
&\leq \lim_{h\to 0, n\to+\infty} h \sum_{j=0}^{n-1} |f'(t_j^*)|^2 (t_{j+1}-t_j) \\
&= \lim_{h\to 0} h. \lim_{h\to 0, n\to+\infty} \sum_{j=0}^{n-1} |f'(t_j^*)|^2 (t_{j+1}-t_j) \\
&= \lim_{h\to 0, n\to+\infty} h. \int_0^T |f'(t)|^2 dt = 0.
\end{aligned}
$$

Quadratic variation of the Brownian motion

- We consider first the discrete quadratic variation associated with the arbitrary partition defined previously

$$Q_h = \sum_{j=0}^{n-1} [W(t_{j+1}) - W(t_j)]^2.$$

- Goal: show that its expectation is equal to T and that its variance converges to 0 as h converges to 0 (convergence in mean squares).

$$\lim_{h \to 0, n \to +\infty} \mathbb{E}[(Q_h - T)^2] = 0.$$

- First, we compute its expectation

$$\mathbb{E}[Q_h] = \sum_{j=0}^{n-1} \mathbb{E}[W(t_{j+1}) - W(t_j)]^2 = \sum_{j=0}^{n-1} (t_{j+1} - t_j) = T.$$

Next we compute its variance:

$$
\begin{aligned}
Var[Q_h] &= Var[\sum_{j=0}^{n-1}(W(t_{j+1}) - W(t_j))^2] \\
&= \sum_{j=0}^{n-1} Var[(W(t_{j+1}) - W(t_j))^2] \\
&= \sum_{j=0}^{n-1} \mathbb{E}[\{(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)\}^2] \\
&= \sum_{j=0}^{n-1} \mathbb{E}[(W(t_{j+1}) - W(t_j))^4] - 2(t_{j+1} - t_j)\mathbb{E}[(W(t_{j+1}) - W(t_j))^2] \\
&+ (t_{j+1} - t_j)^2 \\
&= \sum_{j=0}^{n-1} 3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2 = \sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2 \\
&\leq \sum_{j=0}^{n-1} 2h(t_{j+1} - t_j) = 2hT.
\end{aligned}
$$

Back to Bounded Variation

Now we can show that the first order variation for the Brownian motion is unbounded (almost surely). For a particular partition Π

$$FV[W] = \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \geq \frac{\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2}{\max_{j=0,..n-1} |W(t_{j+1} - Wj|}$$

When $||\Pi|| \to 0$, the numerator is the QV, which we showed is equal T, while the denominator goes to zero, as the Brownian motion is continuous.

Therefore, for Brownian Motion, we have:

Unbounded First Variation and Bounded Second Variation while for any smooth function f is the first one exists and bounded, and Second Variation is Zero

Cross Variation

In addition to QV, we could also calculate the following quantities:

(i)

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))(t_{j+1} - t_j) = 0$$

(ii)

$$\lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 = 0$$

This is true, as
$max_k |W(t_{k+1}) - W(t_k)| \to 0$ and $max_k (t_{k+1} - t_k) \to 0$, as
$||\Pi|| \to 0$.
We can write the above as

$$dW(t)dt = 0, dtdt = 0$$

Another "heuristic" way is to write $dW(t) = W(t + \Delta t) - W(t)$,
$dW(t) \approx \Phi(0, dt)$, $(dW(t))^2 \approx \Phi(dt, 0)$.

Filtration for a Brownian motion

- One can define rigorously a filtration for a Brownian motion. The following definition is general enough to cover a range of possibilities.
- As an example, one may simply use the filtration generated by the Brownian motion (this example was seen earlier for the random walk)
- Alternately, you may include in the filtration the information obtained by observing one or 2 other processes in addition to the Brownian motion.
- **Definition**

  A filtration for the Brownian motion is a collection of σ-algebra $\{\mathcal{F}(t), t \geq 0\}$ satisfying the 3 properties

  1. (Information accumulates) For $0 \leq s < t$, every set in $\mathcal{F}(s)$ is in $\mathcal{F}(t)$.
  2. (Adaptivity) For each $t \geq 0$, the Brownian motion $W(t)$ at time t is $\mathcal{F}(t)$-measurable.
  3. (Independence of future increments) For $0 \leq t < u$, the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$.

Filtration generated by a Brownian motion

- Important example: the simplest filtration for a Brownian motion is the one obtained by observing the Brownian motion itself up to time $t \geq 0$

$$\mathcal{F}(t) = \sigma(W(s); 0 \leq s \leq t).$$

- It is the smallest σ-algebra with respect to which $W(t)$ is measurable for every $t \in [0, T]$.

Appendix: Weierstrass' pathological function

Weierstrass concocted his example in the 1860s and presented to the Berlin Academy in July 1872. It was only published in 1875. This peculiar function is far from elementary. "Outrageous example against common sense:
If $a \geq 3$ is an odd integer and if b is a constant, $0 < b < 1$, such that

$$ab > 1 + 3\pi/2$$

then the function

$$f(x) = \sum_{k=0}^{\infty} b^k \cos(\pi a^k x)$$

is everywhere continuous and nowhere differentiable. For example, take $a = 21$ and $b = 1/3$, and plot just 3, 5, 7 .. sums for the expression.