FRE 6233 Stochastic Calculus and Option pricing
Week 3 The Ito integral and their properties

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Most of the material for this Lecture can be studied from the textbook by Steven Shreve *Stochastic Calculus for Finance, II*, chapters 2 and 4. I follow the same logic with some modifications (and different notations).

Outline of the lecture:

- The notion of martingale
- The ordinary Riemann Integral
- Motivating example
- The Itô Stochastic Integral for Simple Process
- The General Itô Stochastic Integral
- Properties of the General Itô Stochastic Integral
- The Wiener integral

Stochastic Processes

- A stochastic process on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is a family of r.v. X_t parameterized by $t \in \mathcal{T}$, $\mathcal{T} \subset \mathbb{R}$. If $\mathcal{T}$ is an interval we say that X_t is a stochastic process in *continuous* time. If $\mathcal{T} = 1, 2, ...$ we say that X_t is a stochastic process in *discrete* time.
- The later describes a sequence of r.v. X_t converges in the strong sense to X as $t \to \infty$ if

$$\mathbb{P}\left(\omega : \lim_{t\to\infty} X_t(\omega)\right) = 1$$

- The evolution in time for a given state $\omega \in \Omega$ given by function $t \mapsto X_t(\omega)$ is called a *path* or *realization* of X_t.
- Consider all the information accumulated until time t contained in the σ-field $\mathcal{F}_t$. The information is growing in time:

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$$

for any $s, t \in \mathcal{T}$, $s \leq t$. The family $\mathcal{F}_t$ is called a *filtration*

Notion of a *martingale*

A stochastic process X_t is called *adapted* to the filtration $\mathcal{F}_t$ if X_t is $\mathcal{F}_t$ measurable.

- If X is a r.v., consider the conditional expectation

$$X_t = \mathbb{E}[X|\mathcal{F}_t]$$

  It follows that the r.v. X_t is $\mathcal{F}_t$ measurable and can be regarded as the measurement of X using the information $\mathcal{F}_t$. If the accumulated knowledge $\mathcal{F}_t$ increases and eventually equals the σ-field $\mathcal{F}$ then $X = \mathbb{E}[X|\mathcal{F}]$, the entire r.v.
- *A process X_t , $t \in \mathcal{T}$ is called a martingale w.r.t the filtration $\mathcal{F}_t$ if*
  1. *X_t is integrable for each $t \in \mathcal{T}$*
  2. *X_t is adapted to the filtration $\mathcal{F}_t$*
  3. *$X_s = \mathbb{E}[X_t|\mathcal{F}_s], \forall s < t.$*

The Classical Riemann Integral

Suppose that f is real-valued function defined on $[0, 1]$. Consider a partition of $[0, 1]$ and intermediate points:

$$\tau_n : 0 = t_0 < t_1 < ...t_{n-1} < t_n = 1$$
$$\Delta_i = t_i - t_{i-1}$$
$$y_i : t_{i-1} \leq y_i \leq t_i, i = 1, ..., n$$

We can define the Riemann sum:

$$S_n(\tau_n, y_n) = \sum_{i=1}^{n} f(y_i)(t_i - t_{i-1}) = \sum_{i=1}^{n} f(y_i)\Delta_i$$

Definition of Rieman Integral

Definition

If the limit

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(y_i)\Delta_i$$

exists as $||\tau_n|| = \max_i \Delta_i \to 0$ *and* S *is independent of the choice of the partition* τ_n *and the choice of* y_i*, then* S *is called the Riemann integral of* f *on* $[0,1]$*.*

The Riemann integral is taken as a model for the definition of any kind of integral, in particular, it should share as many properties in common with the Riemann integral as possible

Properties of the Riemann Integral

For the Riemann integrable functions f, f_1, and f_2 on $[0,1]$ the following properties hold:

(i) The Riemann integral is *linear*, i.e. for any constants c_1 and c_2

$$\int_0^1 [c_1 f_1(t) + c_2 f_2(t)]dt = c_1 \int_0^1 f_1(t)dt + c_2 \int_0^1 f_2(t)dt$$

(ii) The Riemann integral is linear on adjacent intervals:

$$\int_0^1 f(t)dt = c_1 \int_0^a f(t)dt + \int_a^1 f(t)dt, 0 \leq a \leq 1$$

The Riemann Stieltjes Integral

In probability theory it is usual to denote the expectation of a random variable X by

$$\mathbb{E}X = \int_{-\infty}^{\infty} t dF_X(t)$$

where F_X denotes the distribution function of X. Roughly speaking

$$\int_{-\infty}^{\infty} t dF_X(t) \approx \sum_i y_i [F_X(t_i) - F_X(t_{i-1})]$$

for a partition (t_i) of $\mathbb{R}$ and choice of points y_i. The integral $\int_0^1 f(t) dg(t)$ can be defined as $\int_0^1 f(t) g'(t) dt$, provided the derivative $g' t)$ exists.

Constructing the Riemann Stieltjes Integral

As before we consider a partition of the interval and intermediate points:

$$\tau_n : 0 = t_0 < t_1 < ...t_{n-1} < t_n = 1$$
$$y_i : t_{i-1} \leq y_i \leq t_i, i = 1, ..., n$$

Let f and g are real valued functions on $[0, 1]$ and define

$$\Delta_i g = g(t_i) - g(t_{i-1})$$

The Riemann-Stieltjes sum corresponding to τ_n and y_n is given by

$$S_n = S_n(\tau_n, y_n) = \sum_{i=1}^{n} f(y_i)\Delta_i g$$

Defintion of R-S integral

Definition

If the limit

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(y_i)\Delta_i g$$

exists as $||\tau_n|| = \max_i \Delta_i \to 0$ *and* S *is independent of the choice of the partition* τ_n *and the choice of* y_i*, then* S *is called the Riemann-Stieltjes integral of* f *with respect to* g *on* $[0,1]$.

Question:

When does the Riemann-Stieltjes integral $\int_0^1 f(t)dg(t)$ *exist and is it possible to take* $g = W$ *for Brownian motion on* $[0,1]$*?*

A Motivating Example

Let's consider the following integral

$$I_\omega(W(t)) = \int_0^t W_\omega(s) dW_\omega(s)$$

the integral does not exist in the R-S sense. Therefore, we will try to define this integral in probabilistic average.
We consider the R-S sums:

$$S_n = \sum_{i=1}^{n} W(t_{i-1}) \left(W(t_i) - W(t_{i-1}) \right)$$

where $\tau_n : 0 = t_0 < t_1 < .. < t_{n-1} < t_n = t$ is a partition of $[0, 1]$
Thus, this is the R-S sum S_n with the choice of y_i being the left points of the intervals: $y_i = t_{i-1}$. This is the choice for the *Itô integral*.

Integration of the example

Let's present the factor $W(t_{i-1})$ in the previous sum as follows

$$W(t_{i-1}) = \frac{1}{2}(W(t_i) + W(t_{i-1})) - \frac{1}{2}(W(t_i) - W(t_{i-1}))$$

Then the R-S sum can written as

$$S_n = \frac{1}{2}\sum_{i=1}^{n}\left(W^2(t_i) - W^2(t_{i-1})\right) - \frac{1}{2}\sum_{i=1}^{n}(W(t_i) - W(t_{i-1}))^2$$

Since $W(0) = 0$, the first part in the sum is $\frac{1}{2}W^2(t_n) = \frac{1}{2}W^2(t)$. The second part Q_n in the sum is nothing else but the quadratic variation, considered in the previous lecture. We showed that the expectation of Q_n converges to t, and its variance converges to 0, for $n \to \infty$ (or mesh size of the partition goes to zero).

The Itô integral of the example

We define the Itô integral of the example

$$I(t) = \int_0^t W(t)dW(t) = \frac{1}{2}\left(W^2(t) - t\right)$$

as the limit of $S_n, n \to \infty$ in a mean square sense.
The additional term $-t/2$ is a consequence of the "roughness" , no-differentiability of the Brownian motion. It it was smooth, that additional term would be zero.

The beauty of the Itô integral is that it is a martingale. It was defined for that purpose.

The Itô Stochastic Integral for Simple Processes

We will start with process whose paths assume only a finite number of values. Let

$$\mathcal{F}(t) = \sigma(W(s), s \leq t), t \geq 0$$

is the corresponding natural filtration. Recall that a stochastic process $X = (X(t), t \geq 0)$ is adapted to Brownian motion if X is adapted to $(\mathcal{F}(t), t \geq 0)$, This means that for every t, $X(t)$ is a function of the past and present Brownian motion (but "can not see in the future").

Simple Process

Definition

The stochastic process $C = (C(t), t \in [0, T])$ *is said to be simple is it satisfies the following properties:*

(i) *There exists a partition:*

$$\tau_n : 0 = t_0 < t_1 < ...t_{n-1} < t_n = T$$

and there is a sequence $(Z_i, i = 1, ...n)$ *of random variables such that*

$$C(t) = \begin{cases} Z_n & \text{if } t = T. \\ Z_i, & \text{if } t_{i-1} \le t \le t_i, i = 1, ...n. \end{cases}$$

(iii) *The sequence* Z_i *is adapted to* $\mathcal{F}(t_{i-1}), i = 1, ...n)$, *i.e.* Z_i *is a function of Brownian motion up to time* t_{i-1} *and satisfies* $\mathbb{E}(Z_i^2) < \infty$ *for all* i.

Examples of simple processes

1. The deterministic function

$$f_n(t) = \begin{cases} \frac{n-1}{n} & \text{if } t = T. \\ \frac{i-1}{n}, & \text{if } t_{i-1} \leq t \leq t_i, i = 1, \ldots n. \end{cases}$$

2. Next, define the process we used in the previous ("motivating") example:

$$C(t) = \begin{cases} Z_n = W(t_{n-1}) & \text{if } t = T. \\ Z_i = W(t_{i-1}), & \text{if } t_{i-1} \leq t \leq t_i, i = 1, \ldots n. \end{cases}$$

   for a given partition τ_n of $[0, T]$. It is a simple process: the paths are piecewise constant and, and C_t is a function of Brownian motion until time t.

Definition of the Itô integral for a simple process

Definition

The Itô stochastic integral of a simple process C on $[0, T]$ is given by

$$\int_0^T C_s dW(t) := \sum_{i=1}^{n} C(t_{i-1})\left(W(t_i) - W(t_{i-1})\right) = \sum_{i=1}^{n} Z_i \Delta_i W$$

Thus the value of the Itô integral is the R-S sum of the path C evaluated at the left points of the intervals $y_i = t_{i-1}$ with respect to Brownian motion.

The Itô izometry

The Itô integral satisfies the *isometry* property:

$$\mathbb{E}\left(\int_0^t C(s)dW(t)\right)^2 = \mathbb{E}[\int_0^T C(s)^2 ds], t \in [0, T]$$

Sketch of the proof: Let the random variables $Y_i = Z_i \Delta_i W$. Assume that $t = t_k$. Then

$$\mathbb{E}[I_t(C)]^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} \mathbb{E}(Y_i Y_j)$$

The random variables Y_i and Y_j are uncorrelated for $i \neq j$. (prove it). Therefore the only non-zero terms would be for $i = j$, so the property follows.

Linearity and Continuity of the Itô integral

(i) For constants c_1 and c_2 and simple processes $C^{(1)}$ and $C^{(2)}$ on $[0, T]$

$$\int_0^T [c_1 C^{(1)}(s) + c_2 C^{(2)}(s)] dW(s) = \\ c_1 \int_0^T C^{(1)}(s) dW(s) + c_2 \int_0^T C^{(2)}(s) dW(s)$$

(ii) The Itô integral is linear on adjacent intervals:

$$\int_0^T C(s) dW(s) = \int_0^t C(s) dW(s) + \int_t^T C(s) dW(s)$$

(iii) The process $I(C)$ has continuous sample paths. This follows from the definition of $I(C)$ and continuity of sample path of Brownian motion:
$I_t(C) = I_{t_{i-1}}(C) + Z_i(W(t) - W(t_{i-1})), t_{i-1} \leq t \leq t_i$

Martingale Property of the Itô property

The stochastic process $I_t(C) = \int_0^t C(s)dW(s), t \in [0, T]$ is a martingale with respect to the natural Brownian filtration $(\mathcal{F}(t), t \in [0, T])$.

We check the following properties of a martingale:

1. $I(C)$ is adapted to $\mathcal{F}(t)$
2. $\mathbb{E}(I_t(C)|\mathcal{F}(s)) = I_s(C), s < t$

The first property follows from the fact that the random variable $Z_1, ..Z_k$ and $\Delta_1 W, ..\Delta_{k-1} W$ are functions of Brownian motion up time t.

The Martingale property for the Itô integral

We assume $s < t$ and $t, s \in [t_{k-1}, t_k]$. Other cases are treated similarly. Then we have

$$I_t(C) = I_{t_{k-1}}(C) + Z_k(W(s) - W(t_{k-1})) + Z_k(W(t) - W(s)) =$$
$$= I_s(C) + Z_k(W(t) - W(s))$$

where $I_s(C)$ and Z_k are functions of Brownian motion up to time s, and $W(t) - W(s)$ is independent of $\mathcal{F}(s)$. Hence

$$\mathbb{E}\left(I_t(C)|\mathcal{F}(s)\right) = I_s(C) + Z_k\mathbb{E}\left((W(t) - W(s))|\mathcal{F}(s)\right) = I_s(C)$$

Itô integral for general integrands

So far we introduced the Itô integral for simple processes C, i.e. for stochastic processes whose sample paths are step functions. Consider $\sigma(t), t \in [0, T]$ be a process, which serves as the integrand of the Itô stochastic integral. We assume the following conditions:

Assumptions on the Integrand Process σ:

- $\sigma(t)$ is adapted to the filtration $\mathcal{F}(t)$
- The integral

$$\mathbb{E} \int_0^T \sigma(u)^2 dt < \infty.$$

The conditions are satisfied for a simple process. Another class of admissible integrands consists of the deterministic functions $c(t)$ on $[0, T]$ with $\int_0^T c^2(t)dt < \infty$. It includes the continuous functions on $[0, T]$.

Approximating General Integrands with Simple Processes

Let σ be a process satisfying the Assumptions. Then one can find a sequence $C^{(n)}$ of simple processes such that

$$\int_0^T \mathbb{E}[\sigma(s) - C^{(n)}(s)]^2 ds \to 0, n \to \infty$$

The next step is show that the sequence of $I(C^{(n)})$ of Itô stochastic integrals converges in a certain mean square sense to a unique limit process.

The mean square limit $I(C)$ is called the *Itô stochastic integral of* $\sigma(t)$ It is denoted by

$$I_t(\sigma) = \int_0^t \sigma(s)dW(s), t \in [0, T]$$

The "meaning" of the Itô integral

The Itô stochastic integrals $I_t(\sigma) = \int_0^t \sigma(s)dW(s)$, $t \in [0, T]$, constitute a stochastic process. For a given partition

$$\tau_n : 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$$

and $t \in [t_{k-1}, t_k]$, the random variables $I_t(\sigma)$ is "close" to the R-S sum

$$\sum_{i=1}^{k-1} \sigma(t_{i-1}) \left(W(t_i) - W(t_{i-1})\right) + \sigma(t_{k-1}) \left(W(t) - W(t_{k-1})\right)$$

and this approximation becomes closer (in the mean square sense) to the value $I_t(\sigma)$ for more dense partition τ_n.
All properties of the Itô integrals , listed before for simple processes, as linearity, continuity, isometry and the martingale property hold.

The Wiener Integral

An Itô integral in the case of a deterministic function $f(t)$ is called a *Wiener* integral $\int_a^b f(t)dW(t)$. It is the m.s. limit of

$$S_n = \sum_{i=1}^{n} f(t_{i-1})\left(W(t_i) - W(t_{i-1})\right)$$

1. All properties of Itô integrals hold. The Wiener integral is a r.v. with zero mean

   $$\mathbb{E}[\int_a^b f(t)dW(t)] = 0$$

   and variance

   $$\mathbb{E}[\left(\int_a^b f(t)dW(t)\right)^2] = \int_a^b f^2(t)dt \qquad (1)$$

2. Moreover, the Wiener integral $\int_a^b f(t)dW(t)$ is a normal variable with mean 0 and variance given by 1. This is can be seen from the fact that increments $W(t_i) - W(t_{i-1})$ are normally distributed with zero mean and variance $t_i - t_{i-1}$.