FRE-GY 6233: Midterm

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Problem 1

Solution: Suppose that for purposes of contradiction that X is not constant. We note that as X is measurable with respect to the trivial σ-algebra, then $\sigma(X) = \{\emptyset, \Omega\}$ and there exists $a, b \in \Omega$ such that $X(a) \neq X(b)$.
Note that if we assume without loss of generality that $X(b) = \Omega$ then we would have $X^{-1}(X(a)) = \emptyset$. This makes no sense so we have a contradiction. But otherwise $X^{-1}(X(a)) = \Omega$ so that $X(a) = X(b)$ which is another contradiction. It follows that X must be constant.

Problem 2

Solution: $\sigma(Y)$ is defined by the sets $\omega_{y1} = (-\infty, -1]$, $\omega_{y2} = (-1, 0]$, $\omega_{y3} = (0, \infty)$. $\sigma(X)$ is defined by the sets $\omega_{x1} = (-\infty, -2]$, $\omega_{x2} = (-2, 1]$, $\omega_{y3} = (1, \infty)$.

$$\begin{aligned}
g(0) = \mathbb{E}[X \mid \omega_{y1}] &= -1 \cdot \frac{\mathbb{P}[x \leq -2]}{\mathbb{P}[x \leq -1]} + 1 \cdot \frac{\mathbb{P}[-2 < x \leq -1]}{\mathbb{P}[x \leq -1]} \\
&= -1 \frac{0.02275}{0.15866} + 1 \cdot \frac{0.15866 - 0.02275}{0.15866} = 0.713 \\
g(1) = \mathbb{E}[X \mid \omega_{y2}] &= 1 \cdot \frac{\mathbb{P}[-1 \leq x \leq 0]}{\mathbb{P}[-1 \leq x \leq 0]} = 1 \\
g(2) = \mathbb{E}[X \mid \omega_{y3}] &= 1 \cdot \frac{\mathbb{P}[-0 < x \leq 1]}{\mathbb{P}[x > 0]} + 2 \cdot \frac{\mathbb{P}[x > 1]}{\mathbb{P}[x > 0]} \\
&= \frac{0.84134 - 0.5}{0.5} + 2 \cdot \frac{1 - 0.84134}{0.5} = 1.317
\end{aligned}$$

Problem 3

Solution: We show that $V(t)$ is a Brownian motion checking the following:

1. $V(0) = 0$

2. It is straightforward to see that $V(t)$ is continuous for $t \neq 0$. We then show that it is continuous at $t = 0$ as well by showing that $\lim_{t\to 0} V(t) = \lim_{s\to\infty} V(\frac{1}{s}) = \lim_{s\to\infty} \frac{1}{s}W(s) = 0$

3. As the increments of $W(t)$ are independent, we observe that the increments of $V(t)$ are independent by observing that $\forall \epsilon > 0$, $V(t+\epsilon) - V(t) = t \cdot \left(W(\frac{1}{t+s}) - W(\frac{1}{t})\right) + \epsilon W(\frac{1}{t+\epsilon})$ represent independent increments as a summation of $W(t)$ increments which are independent.

4. We have that $\forall \epsilon > 0, (s+\epsilon)W(\frac{1}{s+\epsilon}) - sW(\frac{1}{s})$ is the difference of two scaled normal distributions so it normal. We then observe that $\mathbb{E}[(s+\epsilon)W(\frac{1}{s+\epsilon}) - sW(\frac{1}{s})] = (s+\epsilon)\cdot 0 - s\cdot 0 = 0$ and that $Var[(s+\epsilon)W(\frac{1}{s+\epsilon}) - sW(\frac{1}{s})] = Var[s\cdot(W(\frac{1}{s+\epsilon}) - W(\frac{1}{s})) + \epsilon W(\frac{1}{s+\epsilon})]$
$= s^2 \cdot Var[\cdot W(\frac{1}{s+\epsilon} - \frac{1}{s})] + \epsilon^2 Var[W(\frac{1}{s+\epsilon})]$
$= s^2(\frac{\epsilon}{s\cdot(s+\epsilon)}) + \frac{\epsilon^2}{s+\epsilon} = \epsilon.$
So it follows that $V(t+\epsilon) - V(t) \sim N(0,\epsilon)$

This is a Wiener process

Problem 4

Solution:

1. $Y(t) = W^4(t) + t^3 + W(t)$

$$\begin{aligned} dY(t) &= d(W^4(t) + t^3 + W(t)) \\ &= d(W^4(t)) + 3t^2dt + dW(t) \\ &= \frac{1}{2}\cdot 4\cdot 3W^(4-2)(t)dt + 4W^(4-1)(t)dW(t) + 3t^2dt + dW(t) \\ &= \left(6W^2(t) + 3t^2\right)dt + (4W^3(t)+1)dW(t) \end{aligned}$$

2. $Y(t) = 1 + t^4 + e^{W(t)}$

$$\begin{aligned} dY(t) &= d(1 + t^4 + e^{W(t)}) \\ &= 4t^3dt + d(e^{W(t)}) \\ &= 4t^3dt + e^{W(t)}dW(t) + \frac{1}{2}e^{W(t)}dt \\ &= (\frac{1}{2}e^{W(t)} + 4t^3)dt + e^{W(t)}dW(t) \end{aligned}$$

Problem 5

Solution:

1. $X(t) = W^3(t) - 3tW(t)$
   We observe first observe that
   $W^3(t+s) = (W(t+s) - W(t) + W(t))^3 = (W(t+s) - W(t))^3 + 3(W(t+s) - W(t))^2W(s) + 3(W(t+s) - W(t))W(t)^2 + W(t)^3$ so that we have:

   $\mathbb{E}[X(t+s) \mid \mathcal{F}_t]$
   $= \mathbb{E}[W^3(t+s) - 3(t+s)W(t+s) \mid \mathcal{F}_t]$
   $= \mathbb{E}[W(t+s) - W(t))^3 + 3(W(t+s) - W(t))^2W(t) + 3(W(t+s) - W(t))W(t)^2 +$
   $+ W(t)^3 \mid \mathcal{F}_t] - \mathbb{E}[3(t+s)W(t+s) \mid \mathcal{F}_t]$
   we note that $\mathbb{E}[(W(t+s) - W(t))^3] = 0, \mathbb{E}[(W(t+s) - W(t))^2] = s, \mathbb{E}[W(t+s) - W(t)] = 0$
   $= 3s\mathbb{E}[W(t)\mathcal{F}_t] + W^3(t) - 3(t+s)\mathbb{E}[W(t+s) \mid \mathcal{F}_t]$
   $= 3sW(t) + W^3(t) - 3(t+s)W(t) = W^3(t) - 3tW(t)$

   So it is a martingale.

2. $Y(t) = W^5(t)$ We observe that for $s > 0$:

   $$\begin{aligned}
   &\mathbb{E}[Y(t+s) - Y(s) \mid \mathbb{F}_t] \\
   &= \mathbb{E}[(W(t+s)^5 - W(t)^5 \mid \mathbb{F}_t] \\
   &= \mathbb{E}[(W(t+s) - W(t))\big(W(t+s)^4 + W(t+s)^3W(t) + \\
   &+ W(t+s)^2W(t)^2 + W(t+s)W(t)^3 + W(t)^4\big) \mid \mathbb{F}_t] \\
   &= \mathbb{E}[(W(t+s) - W(t))]\mathbb{E}[\big(W(t+s)^4 + W(t+s)^3W(t) + \\
   &+ W(t+s)^2W(t)^2 + W(t+s)W(t)^3 + W(t)^4\big) \mid \mathbb{F}_t] \\
   &= 0
   \end{aligned}$$

   It follows that $\mathbb{E}[Y(t+s) \mid \mathbb{F}_t] = Y(s)$ so that it is a martingale.

Problem 6

Solution: We let $\phi(t,x) = \frac{1}{\sqrt{2}}e^{-t} \cdot e^{\sqrt{2}x}$. Then note that:
$\phi_t(t,x) = -\frac{1}{\sqrt{2}}e^{-t} \cdot e^{\sqrt{2}x} = -\frac{1}{2} \cdot \phi_{xx}(t,x)$.
So this is a solution to the heat equation $\phi_t = c \cdot \phi_{xx}$ for $c = \frac{-1}{2}$. This implies that $\phi_t + \frac{1}{2}\phi_{xx} = 0$ so that:

$$d(\phi(t, W(t))) = \partial_x\phi(t, W(t))dW(t) + (\partial_t\phi + \frac{1}{2}\partial_x^2\phi) = \partial_x\phi(t, W(t))dW(t)$$

$$\int_0^5 \phi_x(t,x)dW(t) = \int_0^5 d(\phi(t,W(t))) = \phi(5,W(5)) - \phi(0,W(0)) = e^{-5+\sqrt(2)W(5)} - 1$$

Problem 7

Solution: We note that $\int e^{-x^2} cosx dx = \sqrt{\pi} \int cosx \cdot (\frac{1}{\sqrt{\pi}}) e^{-\frac{1}{2}(\sqrt{2}x)^2} dx$.
We note $(\frac{1}{\pi}) e^{-\frac{1}{2}(\sqrt{2}x)^2}$ is the pdf of a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = \frac{1}{2}$. The integral then becomes $\int e^{-x^2} cosx dx = \sqrt{\pi}\mathbb{E}[cos(X)]$ where $X \sim N(0, \frac{1}{2})$.
We have shown in homework that $\mathbb{E}[cos(3aW(t))] = e^{-9/2a^2 t}$. Here $t = \frac{1}{2}$, $a = 1/3$ so that:
$\int e^{-x^2} cosx dx = \sqrt{\pi}\mathbb{E}[cos(X)] = \sqrt{\pi} e^{-1/4}$

Problem 8

Solution:

1. We fix $A \in \mathcal{G}$. As $A \in \mathcal{G}$ and $\mathcal{G}$ is a σ-algebra, then $A^c \in \mathcal{G}$. It is straightfoward to see that $\chi_A^{-1}(1) = A \in \mathcal{G}$, $\chi_A^{-1}(0) = A^c \in \mathcal{G}$ so that χ_A is $\mathcal{G}$-measurable.
2. As X is independent from $\mathcal{G}$, it is measurable by some σ-algebra $\mathcal{G}'$ that is independent to $\mathcal{G}$. This means that X is by definition independent to χ_X as they are measured by independent sigma-algebra.
3. As we have established that X and χ_A are independent r.v.:

   $\mathbb{E}[X\chi_A] = \mathbb{E}[X]\mathbb{E}[\chi_A] = \mathbb{E}[X] \cdot (1 \cdot \mathbb{P}[\omega \in A]) + \mathbb{E}[X] \cdot (0 \cdot \mathbb{P}[\omega \notin A]) = \; = \mathbb{E}[X]\mathbb{P}(A)$

Problem 9

Solution:

1. By taking expectation, we have the following equation:

$$\frac{\partial \mathbb{E}[X(t)]}{dt} = \frac{t}{1+t^2}$$
$$\mathbb{E}[X(t)] = X(0) + \frac{1}{2}\ln(1+t^2)$$
$$\mathbb{E}[X(t)] = 2 + \ln(1+t^2)$$

We have the following variance:

$$Var[X(t)] = \int_0^t (s^{3/2})^2 ds = \frac{t^4}{4}$$

2. By taking expectation, we have the following equation:

$$\begin{aligned} \frac{\partial \mathbb{E}[X(t)]}{dt} &= cos(t) \\ \mathbb{E}[X(t)] &= sin(t) + X(0) \\ \mathbb{E}[X(t)] &= sin(t) + 3 \end{aligned}$$

We have the following variance:

$$\begin{aligned} Var[X(t)] &= \int_0^t (-sin(s))^2 ds = \int_0^t \frac{1}{2}(2sin^2 - 1) + \frac{1}{2} ds \\ &= \int_0^t -\frac{1}{2}cos(2s) + \frac{1}{2} ds = -\frac{1}{4}sin(2t) \end{aligned}$$