Quantitative Methods: Assignment 3

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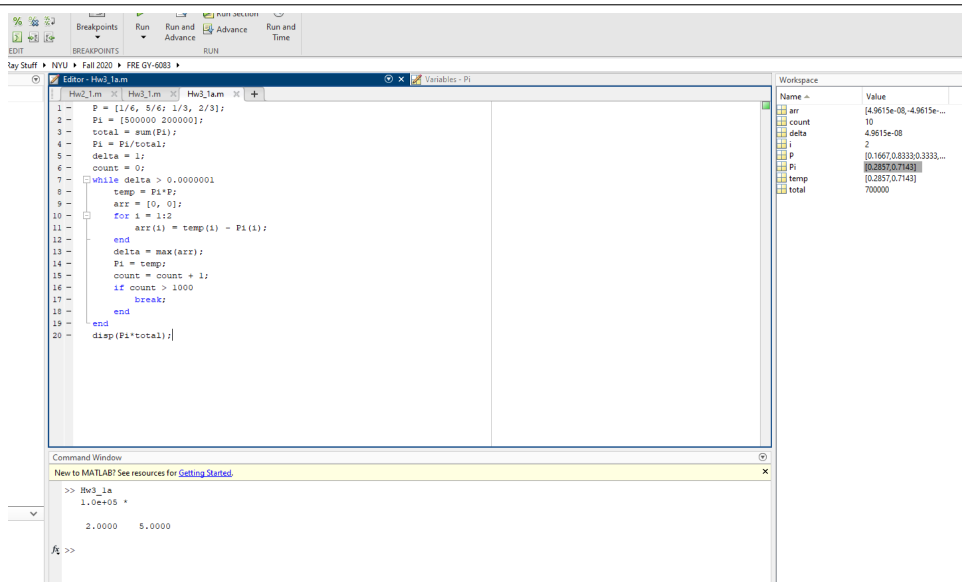
Problem 1 (25 points):

Implement a program in Matlab to compute numerically the steady state of the airline problem seen in class. Provide a printout of your code and a summary of your results

Solution:



Problem 2 (20 points):

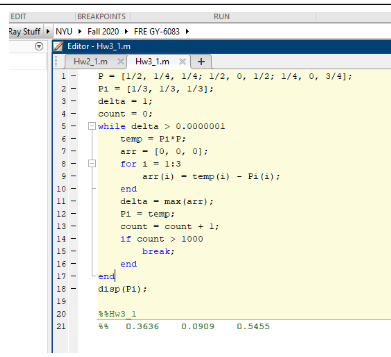
Consider a Markov chain X_n with state space $\{0, 1, 2\}$ and one-step transition probability matrix

$$P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ \alpha & 0 & 1-\alpha \\ \beta & 0 & 1-\beta \end{bmatrix}$$

1. For what values of α and β is the Markov chain irreducible?

> **Solution:** We note that as state 0 has a nonzero probability of transitioning to states 1 and 2. It is sufficient for states 1 and 2 to have a nonzero probability of transitioning to state 0 to be irreducible. As state 2 transitions to state 0 if and only if $\beta \neq 0$ and state 1 transitions to either state 0 or 2 for all values of α, we have an irreducible Markov chain if $\alpha \in [0, 1], \beta \in (0, 1]$

2. Suppose that $\alpha = 1/2$, $\beta = 1/4$. Solve the following question experimentally, using Matlab: Do the limiting probability π_j exist for $j = 0, 1, 2$?
   Solution:




```matlab
P = [1/2, 1/4, 1/4; 1/2, 0, 1/2; 1/4, 0, 3/4];
Pi = [1/3, 1/3, 1/3];
delta = 1;
count = 0;
while delta > 0.0000001
    temp = Pi*P;
    arr = [0, 0, 0];
    for i = 1:3
        arr(i) = temp(i) - Pi(i);
    end
    delta = max(arr);
    Pi = temp;
    count = count + 1;
    if count > 1000
        break;
    end
end
disp(Pi);

%%Hw3_1
%%   0.3636    0.0909    0.5455
```

3. Solve the system of linear equations

$$\pi = \pi P$$

   Compare the solutions with the limit you found in the previous question.

> **Solution:** Suppose $\pi = (x_1 \quad x_2 \quad x_3)$ so that
> $\pi P = (1/2x_1 + \alpha x_2 + \beta x_3 \qquad 1/4x_1 \qquad 1/4x_1 + (1-\alpha)x_2 + (1-\beta)x_3)$
> As $\alpha = 1/2, \beta = 1/4$, we have system of equations:
>
> $$\begin{aligned} x_1 &= 1/2x_1 + 1/2x_2 + 1/4x_3 \\ x_2 &= 1/4x_1 \\ x_3 &= 1/4x_1 + 1/2x_2 + 3/4x_3 \\ x_1 + x_2 + x_3 &= 1 \text{ condition that } \pi \cdot \mathbb{1} = 1 \end{aligned}$$
>
> So we get $\pi = (4/11 \quad 1/11 \quad 6/11)$
> The answers are approximately the same as the ones we found through matlab simulation. Truncation error can be controlled through the conditions of our for loop.s

Problem 3 (10 points)

The one-step transition probability matrix P of a Markov chain with state space $\{0, 1\}$ is given by

$$P = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Assume that $\mathbb{P}[X_0 = 0] = 1/3$. Calculate $\mathbb{E}[X_2]$

> **Solution:** We have that $\mathbb{P}[X_0 = 0] = 1/3, \mathbb{P}[X_0 = 1] = 1 - \mathbb{P}[X_0 = 0] = 2/3$. We denote this by $\pi = (1/3 \quad 1/2)$. We then note that $\mathbb{E}[X_2] = \pi \cdot P^2$.
>
> $$P^2 = \begin{bmatrix} (1/4)^2 + 3/4 \cdot 1/2 & 1/4 \cdot 3/4 + 3/4 \cdot 1/2 \\ 1/4 \cdot 1/2 + (1/2)^2 & 1/2 \cdot 3/4 + (1/2)^2 \end{bmatrix} = \begin{bmatrix} 7/16 & 9/16 \\ 6/16 & 10/16 \end{bmatrix}$$
>
> So we have that $\pi \cdot P^2 = (19/48 \quad 29/48)$

Problem 4 (15 points)

Consider the gambler's ruin problem with $k = 5$ and the initial capital $i = 2$

1. Suppose that, on any given round of the game, the gambler has a probability of 1/2 of winning one dollar, a probability of 1/4 of losing one dollar, and a probability of 1/4 of neither winning nor losing. Give the state space and the transition probability matrix P of the Markov chain representing the gambler's total wealth.

   > **Solution:** The state space $S = \{0, 1, 2, 3, 4, 5\}$ represents the wealth we can have at a time n.
   > Our transition probability matrix is:
   >
   > $$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 0 & 1/4 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
   >
   > We have initial wealth represented by matrix $\pi = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)$

2. Can you characterize the probability of ruin? Can you compute it? Hint: to some extent, you many follow the approach seen in class. However, your calculations do not need to be as general as in the lecture notes since both the initial and target capital are set to specific values in this exercise. Of course, you also need to be aware that,

unlike in the lecture notes, there is a positive probability of neither winning nor losing.

Solution: We note that the probability of ruin at any given time t is given by $\left[\pi \cdot P^t\right]_{(0)}$ and that $\mathbb{P}[\text{Ruin}] = \lim_{n\to\infty}\left[\pi P^n\right]_{(0)}$.
We proceed by the following construction: For $n \in [1,k]$, $P_n = \mathbb{P}[\text{ Ruin } \mid X_0 = i]$ where X_t denotes the amount of money we have at time t.
Note that $P_n = \mathbb{P}[\text{ Ruin } \mid X_0 = i]$
$= \mathbb{P}[\text{ Ruin } \cap \text{"First bet is a win"} \mid X_0 = i] + \mathbb{P}[\text{ Ruin } \cap \text{"First bet has no change"} \mid X_0 = i] + \mathbb{P}[\text{ Ruin } \cap \text{"First bet is a loss"} \mid X_0 = i]$
$= \mathbb{P}[\text{Win}]\mathbb{P}[\text{ Ruin } \mid X_1 = i+1] + \mathbb{P}[\text{Tie}]\mathbb{P}[\text{ Ruin } \mid X_1 = i] + \mathbb{P}[\text{Loss}]\mathbb{P}[\text{ Ruin } \mid X_1 = i-1]$
$= \mathbb{P}[\text{Win}]\mathbb{P}[\text{ Ruin } \mid X_0 = i+1] + \mathbb{P}[\text{Tie}]\mathbb{P}[\text{ Ruin } \mid X_0 = i] + \mathbb{P}[\text{Loss}]\mathbb{P}[\text{ Ruin } \mid X_0 = i-1]$
$= \frac{1}{2}P_{n+2} + \frac{1}{4}P_{n+1} + \frac{1}{4}P_n$
We then have a recurrence relation $P_{n+1} = \frac{1}{2}P_{n+2} + \frac{1}{4}P_{n+1} + \frac{1}{4}P_n \Rightarrow 3P_{n+1} = P_{n+2} + 2P_n$ with $P_0 = 1, P_k = 0$. We solve the recurrence relation by using the characteristics equation $2r^2 - 3r + 1 = (2r-1)(r-1) = 0$:

We then know that our above relation has closed form:

$$P_n = A \cdot (\frac{1}{2})^n + B \cdot (1)^n.$$
$$P_0 = A + B = 1$$
$$P_5 = A \cdot \frac{1}{32} + B = 0$$
$$\Rightarrow A = \frac{32}{31}, B = \frac{-1}{31}$$
$$P_n = \frac{32 \cdot \frac{1}{2^n} - 1}{31}$$
$$\Rightarrow \mathbb{P}[\text{Ruin} \mid i = 2] = P_2 = \frac{7}{31}$$

Problem 5 (30 points)

Consider the Markov chain with state space $\{0, 1, 2\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

1. Show that this chain is irreducible

Solution: We note that from state 0, we have nonzero probability of transitioning to state 1 and 2. We note that from state 1, we have nonzero probability of transitioning to state 2 and 3. We note that from state 2, we have nonzero probability of transitioning to state 0 and 1. It is immediate that this chain is irreducible.

2. Compute P^2 and P^3

Solution:

$$P^2 = \begin{bmatrix} 1/2 \cdot 1/3 + 1/2 \cdot 1/2 & 1/2 \cdot 1/2 & 1/2 \cdot 2/3 \\ 1/2 \cdot 2/3 & 1/2 \cdot 1/3 + 1/2 \cdot 2/3 & 1/3 \cdot 1/2 \\ 1/3 \cdot 1/2 & 1/2 \cdot 1/2 & 1/2 \cdot 1/2 + 1/2 \cdot 2/3 \end{bmatrix} = \begin{bmatrix} 5/12 & 1/4 & 1/3 \\ 1/3 & 1/2 & 1/6 \\ 1/6 & 1/4 & 7/12 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1/4 & 3/8 & 3/8 \\ 1/4 & 1/4 & 1/2 \\ 3/8 & 3/8 & 1/4 \end{bmatrix}$$

3. Deduce that the chain is aperiodic

Solution: It suffices to show that $\forall i \in \{0, 1, 2\}, \forall n \geq N$ for some $N < \infty$, that $(P^n)_{i,i} > 0$. We note that for $N = 2$, $(P^N)_{i,i} > 0$ for all $i \in \{0, 1, 2\}$. We proceed by strong induction.

4. Is the chain ergodic? Justify your answer

Solution: Ergodicity follows from the fact that the chain is both aperiodic and irreducible.

5. Determine the ergodic limit π.

Solution: We first note that ergodicity ensures the existence of limit π. Thus, we assume that $\pi = (x_1, x_2, x_3)$ such that $\pi = \pi \cdot P$. We then have the following system of equations:

$$x_1 = \frac{1}{3}x_2 + \frac{1}{2}x_3 \tag{1}$$

$$x_2 = \frac{1}{2}x_1 + \frac{1}{2}x_3 \tag{2}$$

(2) substituted into $3*(1)$ gives us:

$$3x_1 = \frac{1}{2}x_1 + \frac{1}{2}x_3 + \frac{3}{2}x_3 \Rightarrow x_1 = \frac{4}{5}x_3 \tag{3}$$

As we know that $x_1 + x_2 + x_3 = 1$, (2) substituted into this equation with (3) gives us:

$$1 - x_1 - x_3 = \frac{1}{2}x_1 + \frac{1}{2}x_3 \Rightarrow x_3 = \frac{10}{27}, x_1 = \frac{8}{27} \tag{4}$$

We then have that $\pi = (\frac{8}{27} \quad \frac{1}{3} \quad \frac{10}{27})$