FRE-GY 6233: Assignment 3

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Problem 1: Prove the "kurtosis" formula:

$$\sum_{j=0}^{n-1} \mathbb{E}[(W(t_{j+1}) - W(t_j))^4] = \sum_{j=0}^{n-1} 3(t_{j+1} - t_j)^2$$

Solution: It's sufficient to show that $\mathbb{E}[(W(t_{j+1})-W(t_j))^4] = 3(t_{j+1}-t_j)^2$. As $W(t_{j+1})-W(t_j) \sim N(\mu = 0, \sigma^2 = t_{j+1}-t_j$, to find $\mathbb{E}[(W(t_{j+1})-W(t_j))^4]$ is that same as finding the fourth moment of a normal distribution X with mean $\mu = 0$ and variance $\sigma^2 = t_{j+1} - t_j$. To simplify our calculations, we proceed by finding the fourth moment of the standard normal distribution Z and rescale it by σ.
We then have:

$$\begin{aligned}
\mathbb{E}[Z^4] &= \int_{\mathbb{R}} x^4 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{\int_{\mathbb{R}} x^4 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx}{1} \\
&= \frac{\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^4 e^{-x^2/2} dx}{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx} \text{ [From that integration over pdf of } Z \text{ is 1]} \\
&= \frac{\int_{\mathbb{R}} x^4 e^{-x^2/2} dx}{\int_{\mathbb{R}} e^{-x^2/2} dx} \\
&= \frac{[-x^3 e^{-x^2/2}]_{-\infty}^{\infty} + \int_{\mathbb{R}} 3x^2 e^{-x^2/2} dx}{\int_{\mathbb{R}} e^{-x^2/2} dx} \text{ [By integration by parts of numerator]} \\
&= \frac{0 + \int_{\mathbb{R}} 3x^2 e^{-x^2/2} dx}{\int_{\mathbb{R}} e^{-x^2/2} dx} \text{ [Integral over odd function is 0]} \\
&= \frac{[-3x e^{-x^2/2} dx]_{\infty}^{\infty} + \int_{\mathbb{R}} 3x e^{-x^2/2} dx}{\int_{\mathbb{R}} e^{-x^2/2} dx} \text{ [By integration by parts of numerator]} \\
&= \frac{0 + \int 3e^{-x^2/2} dx}{\int e^{-x^2/2} dx} \text{ [Integral over odd function is 0]} \\
&= 3
\end{aligned}$$

We rescaling this by a factor of σ, we have that $\mathbb{E}[X^4] = \mathbb{E}[(\sigma \cdot Z)^4] = \sigma^4 \mathbb{E}[Z^4] = 3\sigma^4 = 3(t_{j+1} - t_j)^2$.

Problem 2: Let $W(t)$ be a Brownian motion. Check if the processes defined below are Brownian motions (check *all* properties):

1. $-W(t), t \geq 0$
2. $cW(\frac{t}{c^2})$

Solution:

1. $-W(t), t \geq 0$

   (i) $-W(0) = -0 = 0$

   (ii) As the increments of $W(t)$ are independent, it is straightforward to see that $-W(t)$ are independent as a function of independent random variables.

   (iii) As the normal distribution centered at $\mu = 0$ is symmetric at the origin, we note that $-W(t+s) - (-W(t)) = -(W(t+s) - W(t)) \sim N(0, s)$

   (iv) As $f(x) = -x$ is a continuous function and the composition of continuous functions is continuous, $-W(t)$ is continuous.

   This is a Wiener Process.

2. $cW(\frac{t}{c^2})$
   Fix $c \in \mathbb{R} \setminus \{0\}$.

   (i) $cW(\frac{0}{c^2}) = cW(0) = 0$

   (ii) As the increments of $W(t)$ are independent, we note that for all $i, j, k, l \in \mathbb{R}$, $W_i - Wj$, $W_k - Wl$ are independent. Thus note that $cW(\frac{j}{c^2}) - cW(\frac{k}{c^2}) = c\big(W(\frac{j}{c^2}) - W(\frac{k}{c^2})\big) = c\big(W(\frac{j}{c^2}) - W(\frac{k}{c^2})\big)$ is independent of $cW(\frac{i}{c^2}) - cW(\frac{l}{c^2}) = c\big(W(\frac{i}{c^2}) - W(\frac{l}{c^2})\big)$.

   (iii) We have that $\forall \epsilon > 0$, $cW(\frac{t+\epsilon}{c^2}) - cW(\frac{t}{c^2}) = c(W(\frac{t+\epsilon}{c^2}) - W(\frac{t}{c^2})) \sim N(\mu = 0, \sigma^2 = \epsilon)$

   (iv) As scaling a continuous function maintains its continuous, $cW(\frac{t}{c^2})$ is still continuous.

   This is a Wiener Process.

Problem 3: Consider a sequence X_n of r.v. such that there is a constant k with $\mathbb{E}[X_n] \to k$, $n \to \infty$ and $Var(X_n) \to 0$, $n \to \infty$. Show that X_n converges to k in the mean square sense.

We use this statement in this last lecture while computing quadratic variation of Brownian motion.

Solution: We first note that:

$$\begin{aligned}\mathbb{E}[(X_n-k)^2] = \mathbb{E}[X_n^2 - 2k\cdot X_n + k^2] &= \mathbb{E}[X_n^2] - 2k\mathbb{E}[X_n] + k^2 \\ &= \mathbb{E}[X_n^2] - \left(\mathbb{E}[X_n]^2 - \mathbb{E}[X_n]^2\right) - 2k\mathbb{E}[X_n] + k^2 \\ &= Var[X_n] + \mathbb{E}[X_n]^2 - 2k\mathbb{E}[X_n] + k^2 = Var[X_n] + \left(\mathbb{E}[X_n]-k\right)^2\end{aligned}$$

Taking limit of n to infinity we have that:

$$\lim_{n\to\infty}\mathbb{E}[(X_n-k)^2] = \lim_{n\to\infty} Var[X_n] + \lim_{n\to\infty}(\mathbb{E}[X_n]-k)^2 = 0 + (k-k)^2 = 0$$

Problem 4: Show that a monotone increasing function f on interval $[a,b]$ has bounded variation.

Solution: This is straightforward from the definition of a monotone increasing function. As f is monotone increasing, for $\forall x, y \in [a,b]$, $x < y$, $f(x) < f(y)$. This means that $|f(y) - f(x)| = f(y) - f(x)$. From this we note that regardless of the partition we choose for our bounded variation:

$$V_a^b(f) = \sup_{P\in\mathcal{P}}\sum_{i=0}^{n_p-1}|f(x_{i+1}) - f(x_i)| = \sup_{P\in\mathcal{P}}\sum_{i=0}^{n_p-1} f(x_{i+1}) - f(x_i)$$

With partition set
$\mathcal{P} = \{x_0, x_1, \cdots, x_{n_P} \mid\ P$ is a partition over $[a,b]$ satisfying $x_i \leq x_{i+1}$ for $0 \leq i \leq n_p - 1.\}$
Evidently, the above sum telescopes so that we have:

$$V_a^b(f) = f(b) - f(a) < \infty$$

From this, we have that f has bounded variation on $[a,b]$.