Quantitative Methods: Finals

Raymond Luo

December 16, 2020

Question 1: Consider the rescaled random walk in discrete time $B^n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{nt} X_k$ where X_k is the increment of the random walk that goes up with probability $p = \frac{1}{2}(1 + \mu/\sqrt{n})$ and down with probability $1 - p$, where μ is a given real number.

$$X_k = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

1. Compute $\mathbb{E}[X_k]$

> **Solution:**
>
> $$\begin{aligned} \mathbb{E}[X_k] &= (+1) \cdot \mathbb{P}[X_k = 1] + (-1) \cdot \mathbb{P}[X_k = -1] \\ &= \frac{1}{2}(1 + \mu/\sqrt{n}) - (1 - \frac{1}{2}(1 + \mu/\sqrt{n})) = 1 + \mu/\sqrt{n} - 1 = \mu/\sqrt{n} \end{aligned}$$

2. Compute $var[X_k]$

> **Solution:**
>
> $$\begin{aligned} var[X_k] &= (+1)^2 \cdot \mathbb{P}[X_k = 1] + (-1)^2 \cdot \mathbb{P}[X_k = -1] \\ &= \frac{1}{2}(1 + \mu/\sqrt{n}) + (1 - \frac{1}{2}(1 + \mu/\sqrt{n})) = 1 \end{aligned}$$

3. Argue that, as $n \to +\infty$, the process $B^n(t)$ converges to a Brownian motion with drift rate μ

Solution: We first note that for fixed $n \in \mathbb{N}, t \in \mathbb{R}_{\geq 0}$, that:

$$\mathbb{E}[B^n(t)] = \mathbb{E}[\frac{1}{\sqrt{n}}\sum_{k=1}^{nt} X_k] = \frac{1}{\sqrt{n}}\sum_{k=1}^{nt}\mathbb{E}[X_k]$$
$$= \frac{1}{\sqrt{n}}\sum_{k=1}^{nt}\mu/\sqrt{n} = \mu \cdot t$$

And that:

$$Var[B^n(t)] = Var[\frac{1}{\sqrt{n}}\sum_{k=1}^{nt} X_k] = (\frac{1}{\sqrt{n}})^2 Var[X_k] = t$$

By the central limit theorem, as $nt \to \infty$ for sufficiently large n, $B^n(t)$ converges to a normal distribution with mean $\mu \cdot t$ and variance t.
Moreover note that:

- $B^n(0) = \frac{1}{\sqrt{n}}\sum_{\emptyset} X_k = 0$,
- For $t_1 > t_2 > t_3 > t_4 \geq 0$, $B^n(t_1) - B^n(t_2) = \frac{1}{\sqrt{n}}\sum_{k=nt_2}^{nt_1} X_k$ is independent of $B^n(t_3) - B^n(t_4) = \frac{1}{\sqrt{n}}\sum_{k=nt_4}^{nt_3} X_k$ following the independence of $\{X_i\}_{i\in\mathbb{N}}$.
- For $t_1 > t_2 > 0$, $B^n(t_1) - B^n(t_2) = \frac{1}{\sqrt{n}}\sum_{k=nt_2}^{nt_1} X_k = N$ and this converges in distribution by the Central Limit Theorem for sufficiently big n, $nt \to \infty$ and that $\mathbb{E}[N] = \mu \cdot (t_1 - t_2)$, $Var[N] = (t_1 - t_2)$

Question 2:

We consider a standard Brownian motion W.

1. Is the process $t \in [0, +\infty) \to W(ct^2)$, where c is a positive constant, a standard Brownian motion? Justify your answer.

Solution: No, the increments do not have variance equal to the length in time of the increment. For $s, t > 0$, we have that $a = c(t+s)^2, b = ct^2$, $W(c(t+s)^2) - W(ct^2) = W(a) - W(b) \sim N(0, a-b)$. Note that $a - b = c\Big((t+s)^2 - t^2\Big) = c(t+s+t)(t+s-t) = c(2ts + s^2) \neq s = (t+s) - t$

2. Is $t \in [0, +\infty) \to \sqrt{t}W(1)$ a standard Brownian motion? Justify your answer.

Solution: No, this process does not have independent increments. Note that for $t_1 > t_2 > 0, \sqrt{t_1}W(1) - \sqrt{t_2}W(1) = W(1)(\sqrt{t_1} - \sqrt{t_2}) \sim N(0, t-s)$ but for $s_1 > s_2 > t_1, \sqrt{s_1}W(1) - \sqrt{s_2}W(1) = \frac{\sqrt{t_1}-\sqrt{t_2}}{\sqrt{s_1}-\sqrt{s_2}}W(1)$ is clearly dependent on the previous increment.

Question 3:

Consider the Stochastic Differential Equation

$$dX(t) = \alpha(\nu - \ln X(t))X(t)dt + \sigma X(t)dW(t), X(0) = x$$

where $W(t)$ is a standard Brownian motion and α, ν, σ and x are positive real numbers.

1. Consider the process $Y(t) = \ln(X(t))$; Find the Stochastic Differential Equation satisfied by Y by using Ito-Doeblin formula.

Solution: We note that:

$$\begin{aligned}
dY(t) &= \frac{\partial}{\partial x}Y dX + \frac{1}{2}\frac{\partial^2}{\partial x^2}Y(dX)^2 \\
&= \frac{1}{X(t)}dX + \frac{1}{2}\frac{\partial}{\partial x}\frac{1}{X(t)}\left(\sigma^2 X^2(t)dt\right) \\
&= \frac{1}{X(t)}dX - \frac{\sigma^2}{2}dt \\
&= \alpha(\nu - \ln X(t))dt + \sigma dW(t) - \frac{\sigma^2}{2}dt \\
&= \left(\alpha(\nu - Y(t)) - \frac{\sigma^2}{2}\right)dt + \sigma dW(t)
\end{aligned}$$

2. Show by using Ito-Doeblin formula, that the solution of the above SDE is given by

$$Y(t) = e^{-\alpha t}\ln x + (\nu - \frac{\sigma^2}{2\alpha})(1 - e^{-\alpha t}) + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)$$

Solution: Let $f(x, t) = e^{\alpha t} \cdot x$ so that by the Ito-Doeblin formula, we have that:

$$\begin{aligned}
d(f(Y(t), t)) &= f_t dt + f_x dx + \frac{1}{2}f_{xx}(dx)^2 \\
&= e^{\alpha t}Y(t)dt + e^{\alpha t}dY + \frac{1}{2}e^{\alpha t} \cdot 0(dY)^2
\end{aligned}$$

$$= e^{\alpha t}Y(t)dt + e^{\alpha t}\left[\left(\alpha(\nu - Y(t)) - \frac{\sigma^2}{2}\right)dt + \sigma dW(t)\right]$$
$$= e^{\alpha t}\alpha\left(\nu - \frac{\sigma^2}{2\alpha}\right)dt + e^{\alpha t}\sigma dW(t)$$

We then integrate the above expression:

$$\int_0^t d(e^{\alpha s}Y(s))ds = \int_0^t e^{\alpha s}\alpha(\nu - \frac{\sigma^2}{2\alpha})ds + \int_0^t e^{\alpha s}\sigma dW(s)$$
$$e^{-\alpha t}Y(t) - e^0 Y(0) = \left[e^{\alpha s}(\nu - \frac{\sigma^2}{2\alpha})\right]_0^t + \int_0^t \sigma e^{\alpha s}dW(s)$$
$$e^{\alpha t}Y(t) = Y(0) + (e^{\alpha t} - 1)(\nu - \frac{\sigma^2}{2\alpha}) + \int_0^t \sigma e^{\alpha s}dW(s)$$
$$Y(t) = e^{-\alpha t}\ln x + (1 - e^{-\alpha t})(\nu - \frac{\sigma^2}{2\alpha}) + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)$$

3. Deduce the solution $X(t)$ of the first SDE

Solution: From $Y(t) = \ln(X(t)) \Rightarrow X(t) = e^{Y(t)}$, we have that:

$$X(t) = \exp\left\{e^{-\alpha t}\ln x + (1 - e^{-\alpha t})(\nu - \frac{\sigma^2}{2\alpha}) + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)\right\}$$

4. What is the distribution of

$$\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)$$

Give also its mean and variance.

Solution: We note that $\int_0^t e^{\alpha s}dW(s)$ has normal distribution as a sum of normal distributions over some partition of $[0, t]$ with mesh approaching 0.
$\mathbb{E}[\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)] = \sigma e^{-\alpha t}\mathbb{E}[\int_0^t e^{\alpha s}dW(s)] = \sigma e^{-\alpha t}\cdot 0 = 0$
$Var[\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)] = \sigma^2 e^{-2\alpha t}Var[\int_0^t e^{\alpha s}dW(s)]$
$= \sigma^2 e^{-2\alpha t}\left(\mathbb{E}[(\int_0^t e^{\alpha s}dW(s))^2] - \mathbb{E}[\int_0^t e^{\alpha s}dW(s)]^2\right).$
By ito's isometry, the above expression becomes:

$Var[\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)] = \sigma^2 e^{-2\alpha t}\Bigg(\mathbb{E}[\int_0^t e^{2\alpha s}ds] - \mathbb{E}[\int_0^t e^{\alpha s}dW(s)]^2\Bigg)$
$= \sigma^2 e^{-2\alpha t}\Bigg(\mathbb{E}[\int_0^t e^{2\alpha s}ds] - 0^2\Bigg) = \sigma^2 e^{-2\alpha t}\Bigg(\mathbb{E}[[\frac{1}{2\alpha}e^{2\alpha s}]_0^t]\Bigg)$
$= \frac{\sigma^2}{2\alpha}e^{-2\alpha t}\Bigg(e^{2\alpha t} - 1\Bigg) = \frac{\sigma^2}{2\alpha}\big(1 - e^{-2\alpha t}\big)$

5. Deduce

$$\mathbb{E}[\exp\{\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)\}]$$

Solution: We note that from the previous subproblem that $X' = \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s) \sim N\big(0, \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})\big)$. It then follows that $\mathbb{E}[\exp\{\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)\}] = \phi(1)$ where ϕ is the moment generating function of X'. We know that the moment generating function of a normal distribution with parameters mean μ and variance σ^2 is $\phi(t) = \exp\{\mu t + \frac{\sigma^2 t^2}{2}\}$ so it follows that:
$\mathbb{E}[\exp\{\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)\}] = \exp\{\frac{\sigma^2}{4\alpha}\big(1 - e^{-2\alpha t}\big)\}$

6. Deduce $\mathbb{E}[X(t)]$

Solution: We have that:

$$\begin{aligned}\mathbb{E}[X(t)] &= \mathbb{E}\Bigg[\exp\Bigg\{e^{-\alpha t}\ln x + (1 - e^{-\alpha t})(\nu - \frac{\sigma^2}{2\alpha}) + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)\Bigg\}\Bigg] \\ &= \exp\{e^{-\alpha t}\ln x + (1 - e^{-\alpha t})(\nu - \frac{\sigma^2}{2\alpha})\}\cdot\mathbb{E}[\exp\{\sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW(s)\}] \\ &= \exp\{e^{-\alpha t}\ln x + (1 - e^{-\alpha t})(\nu - \frac{\sigma^2}{2\alpha})\}\cdot\exp\{\frac{\sigma^2}{4\alpha}\big(1 - e^{-2\alpha t}\big)\} \\ &= \exp\{e^{-\alpha t}\ln x + (1 - e^{-\alpha t})(\nu - \frac{\sigma^2}{2\alpha})\} + \frac{\sigma^2}{4\alpha}\big(1 - e^{-2\alpha t}\big)\}\end{aligned}$$

Question 4:

The price of a share of a dividend-paying stock, $S(t)$, satisfies the Stochastic Differential Equation

$$dS(t) = (r - \delta)S(t)dt + \sigma S(t)d\tilde{W}(t)$$

where $\tilde{W}$ is a standard Brownian motion under the risk-neutral probability measure $\tilde{\mathbb{P}}$, $r > 0$ is the risk free rate, δ is the continuous-time dividend rate, and $\sigma > 0$ is the volatility coefficient. Furthermore, the dividend is deposited in the bank account that pays the rate r.

1. Give a closed formula for the stock price at time $T, S(T)$, in terms of the stock price at time $t, S(t)$.

Solution: We have that:

$$dS(s) = (r - \delta)S(s)ds + \sigma S(s)d\tilde{W}(s)$$

We have by Ito's lemma that:

$$\begin{aligned} d(\ln(S(t))) &= \frac{1}{S}dS(t) - \frac{1}{2} \cdot \frac{1}{S^2}(dS(t))^2 = \frac{dS(t)}{S} - \frac{\sigma^2 S^2 dt}{S^2} \\ d(\ln(S(t))) &= \frac{(r-\delta)S(t)dt + \sigma S(t)d\tilde{W}(t)}{S} - \sigma^2 dt \\ d(\ln(S(t))) &= (r - \delta - \frac{\sigma^2}{2})dt + \sigma d\tilde{W}(t) \\ S(t) &= S(0)\exp\{(r - \delta - \frac{\sigma^2}{2})t + \sigma\tilde{W}(t)\} \end{aligned}$$

2. Show that $e^{-(r-\delta)t}S(t)$ is a martingale under $\tilde{\mathbb{P}}$

Solution: Let $f(x,t) = e^{-(r-\delta)t}x$, by Ito's Lemma, we have that:
$d(e^{-(r-\delta)t}S(t)) = -(r-\delta)e^{-(r-\delta)t}S(t)dt + e^{-(r-\delta)t}dS + \frac{1}{2} \cdot 0(dS)^2$
$= -(r-\delta)e^{-(r-\delta)t}S(t)dt + e^{-(r-\delta)t}\big((r-\delta)S(t)dt + \sigma S(t)d\tilde{W}(t)\big)$
$= 0 \cdot dt + e^{-(r-\delta)t}\sigma S(t)d\tilde{W}(t)$
As there is no drift-term, we have that $e^{-(r-\delta)t}S(t)$ is a martingale.

3. Write the dynamics of the self-financing hedging portfolio which is invested in the bank account and the dividend paying underlying asset. We denote by $X(t)$, the value at time t of the hedging portfolio. Identify the martingale.

Solution: We choose a portfolio with $\Delta(t)$ shares of stock and $X(t) - \Delta(t)S(t)$ investments in market account. Note that at time t, we also have $\delta\Delta(t)S(t)$ dividend so that the evolution of the portfolio is now:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt + \delta\Delta S(t)dt$$

$$
\begin{aligned}
&= \Delta(t)\big((r-\delta)S(t)dt + \sigma S(t)d\tilde{W}(t)\big) + r(X(t) - \Delta(t)S(t))dt + \delta\Delta S(t)dt \\
&= rX(t)dt + \Delta(t)(r-r)S(t)dt + \Delta(t)\sigma S(t)d\tilde{W}(t) \\
&= rX(t)dt + \Delta(t)\sigma S(t)d\tilde{W}(t)
\end{aligned}
$$

If we denote $f(x,t) = e^{-r(T-t)}x$, by Ito's Lemma, we have that:
$d(e^{-rt}X(t)) = -re^{-rt}X(t)dt + e^{-rt}dX(t) + \frac{1}{2}\cdot 0(dX(t))^2$
$= -re^{-rt}X(t)dt + e^{-rt}\big(rX(t)dt + \Delta(t)\sigma S(t)d\tilde{W}(t)\big)$
$= e^{-rt}\Delta(t)\sigma S(t)d\tilde{W}(t)$.
From this, we note that $e^{-rt}X(t)$, the discounted portfolio value, is a martingale.

4. Derive an analytical formula for the price $c(t,s)$ of the European call. Show your work.

Solution: As our self-financing hedging portfolio satisfies $X(t) = c(t,s)$ for all $0 \le t \le T$, we have risk neutral pricing formula:
$T = 0$ and $0 \le t < T$,

$$
\begin{aligned}
\mathbb{E}[e^{-rT}X(T) \mid S(t) = s] &= X(t) \\
\mathbb{E}[e^{-rT}(S(T)-K)^+ \mid S(t) = s] &= X(t) \\
X(t) = \mathbb{E}[(e^{-rt}S(T) - Ke^{-rT})^+ &\mid S(t) = s]
\end{aligned}
$$

Note that we have that under the risk neutral measure $\tilde{\mathbb{P}}$, the stock price is $S(T) = S(t)\exp\{\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \delta - \frac{1}{2}\sigma^2)(T-t)\}$. If we define $Y = -\frac{\tilde{W}(T)-\tilde{W}(t)}{\sqrt{T-t}}$, we get $S(T) = S(t)\exp\{-\sigma\sqrt{T-t}Y + (r-\delta-\frac{1}{2}\sigma^2)(T-t)\}$. From this, we have that:

$$
\begin{aligned}
&X(t) = \mathbb{E}[(e^{-r(T-t)}(S(T) - Ke^{-rT})^+ \mid S(t) = s] \\
&= \tilde{\mathbb{E}}[\exp(-r(T-t)) \times (s\exp\{-\sigma\sqrt{T-t}Y + (r-\delta-\frac{1}{2}\sigma^2)(T-t)\} - K)^+] \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(-r(T-t)) \times (s\exp\{-\sigma\sqrt{T-t}Y + (r-\delta-\frac{1}{2}\sigma^2)(T-t)\} - K)^+ \exp(-\frac{1}{2}y^2)dy
\end{aligned}
$$

Note that $S(T) - K > 0 \Rightarrow s\exp\{-\sigma\sqrt{T-t}Y + (r-\delta-\frac{1}{2}\sigma^2)(T-t)\} > K$. This is equivalent to $y < \frac{1}{\sigma\sqrt{T-t}}[\ln\frac{s}{K} + (r-\delta-\frac{\sigma^2}{2})(T-t)] := d_-$. We then have:

$$
c(t,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} s\exp\{-\sigma\sqrt{T-t}Y + (r-\delta-\frac{1}{2}\sigma^2)(T-t) - \frac{1}{2}y^2\}dy
$$

$$
\begin{aligned}
&- \frac{1}{2\pi} \int_{-\infty}^{d_-} \exp(-r(T-t)) K \exp(-\frac{1}{2}y^2) \\
&= \frac{s\exp(-\delta(T-t))}{\sqrt{2\pi}} \int_{-\infty}^{d_-} \exp\{-\frac{1}{2}(y+\sigma\sqrt{T-t})^2\} \\
&- \exp(-r(T-t))KN(d_(T-t, s)) \\
&= \frac{s\exp(-\delta(T-t))}{\sqrt{2\pi}} \int_{-\infty}^{d_-+\sigma\sqrt{T-t}} \exp\{-\frac{1}{2}z^2\}dz \\
&- \exp(-r(T-t))KN(d_(T-t, s)) \\
&= s\exp(-\delta(T-t))N(d_+) - K\exp(-r(T-t))N(d_-)
\end{aligned}
$$

Where $d_- = \frac{1}{\sigma\sqrt{T-t}}[\ln\frac{s}{K} + (r-\delta-\frac{\sigma^2}{2})(T-t)], d_+ = d_- + \sigma\sqrt{T-t}$