session 9
05.01.2022

Exercise 9.1 - FTCS-method

- Develop a program that solves the 1+1 dimensional diffusion equation with the FTCS method. The solution $u(x, N\tau)$ is computed by performing N evolution steps on the initial condition $u_0(x)$ according to

$$\begin{aligned} u(x, t+\tau) &= u(x,t) \\ &+\frac{\tau}{h^2} D(x+h/2)\left[u(x+h,t)-u(x,t)\right] \\ &+\frac{\tau}{h^2} D(x-h/2)\left[u(x-h,t)-u(x,t)\right] \\ &+\tau\, S(x,t)\,. \end{aligned}$$

  Hints for the implementation:

  - Your program should allow to vary h, τ and N
  - The discretization length h must be much smaller than the typical length scales in u_0. The discretization time must satisfy $\tau \ll \frac{h^2}{2D}$ for the FTCS method to be stable.
  - N is the number of evolution steps to be done, so in the end the solution will be known for $0 \leq t \leq N\tau$.
  - Your program should be able to deal with arbitrary functions $D(x)$ and $S(x,t)$, that are given as function-handles in MATLAB.
  - To make the system finite, put it in a finite box, with Neumann boundary conditions.

    $$-L \leq x \leq +L\,, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=\pm L} = 0$$

    This can be realized numerically by replacing $D(x)$ with zero whenever $|x| > L$.
  - Store the solutions at every time-step by writing them into separate rows of a large matrix. Initialize the first row with the initial condition $u_0(x)$.

- To test your implementation choose
  - $D = 1$
  - $S(x,t) = 0$
  - $u_0(x) = \begin{cases} 1 & \text{if } |x| < 1.5 \\ 0 & \text{else} \end{cases}$
  - $L = 5$
  - $h = 0.1$
  - $\tau = 0.001$
  - $N = 2000$

  For $L = \infty$, the exact solution for this case is

  $$u(x,t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{1.5 - x}{2\sqrt{Dt}}\right) - \operatorname{erf}\left(\frac{-1.5 - x}{2\sqrt{Dt}}\right)\right],$$

  where the error-function $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int\limits_0^z e^{-y^2}dy$ appears (`erf` in MATLAB). Can you reproduce this numerically? At which time do you start seeing significant deviations? Can you explain this?

- Make a mesh-plot of your solution (hint: `mesh`)

- Now, with the same geometry and initial condition, switch on a source and non-constant $D(x)$:

  $$\begin{aligned} D(x) &= 1 - |x|/10 \\ S(x,t) &= \begin{cases} -4 & \text{if } x < 0.1 \text{ and } 0.1 < t < 0.6 \\ 0 & \text{else} \end{cases} \end{aligned}$$

  Create a mesh-plot for this case as well.