Lane Formation in Pedestrian Dynamics Simulations: A Stochastic Port Hamiltonian Approach

Master's Thesis Presentation

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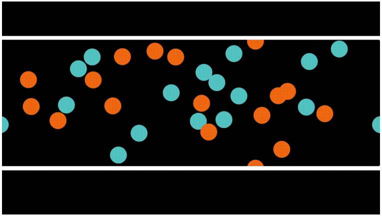
Table of Contents

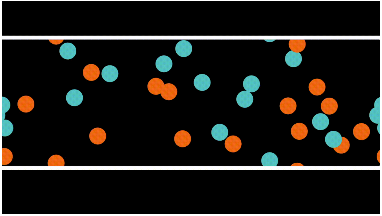
Overview

- Study of collective behavior
- from microscopic force-based pedestrian models
- using Port-Hamiltonian Framework
- with stochastic elements

Counter Flow:
Lane Formation



Cross Flow:
Stripe Formation



Microscopic Pedestrian Model

Periodic Boundaries

$$\dot{Q}_{ij}(t) = p_i(t) - p_j(t), \qquad j = 1, \ldots, N, j \neq i, \quad Q_{ij}(0) = Q_{ij}^0,$$
$$\dot{p}_i = \lambda(u_i(t) - p_i(t)) - \sum_{j \neq i} \nabla U(Q_{ij}(t)), \qquad p_i(0) = p_i^0$$

Here,

- For a pedestrian i, $\{q_i, p_i\}$ are its position and momentum respectively.
- $Q_{ij} = q_i - q_j \in \mathbb{R}^2$, distance between agents
- $\nabla U(x) = -\frac{x}{|x|} A e^{-\frac{|x|}{B}} \in \mathbb{R}_+$, distanced based interaction potential, inspired from Helbing's social force model [1]
- $\lambda \geq 0$, relaxation rate, sensitivity w.r.t to u_i
- u_i, desired velocity of an agent.

Port-Hamiltonian Systems

$$\dot{z} = (J - R)\nabla H(z(t)) + Gu(t)$$
$$y = G^T \nabla H(z(t))$$

Here, [2]

- $J \in \mathbb{R}^{n \times n}$, a skew-symmetric matrix
- $R \in \mathbb{R}^{n \times n}$, a positive semi-definite matrix
  - $R \equiv 0$ and $G \equiv 0$ results in a conservative Hamiltonian System
  - $R > 0$ results in a dissipative system
- H, Hamiltonian, total energy of the system
  - $\nabla H \in \mathbb{R}^n = \begin{bmatrix} \partial_q H & \partial_p H \end{bmatrix}^T$
- $G \in \mathbb{R}^{n \times m}$, coefficient for input u
- u, input parameter
- y, output parameter (to other interacting systems)

Port-Hamiltonian Formulation of the microscopic force-based pedestrian model

- $H(z(t)) = \frac{1}{2}||p||^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}^{N} U(Q_{ij}(t))$
- $G = \lambda \in \mathbb{R}$

Port-Hamiltonian Formulation for the pedestrian model [3]

$$\begin{bmatrix} \dot{Q} \\ \dot{p} \end{bmatrix} = \underbrace{\left(\begin{bmatrix} 0 & M \\ -M^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \lambda I \end{bmatrix} \right)}_{(J-R)} \underbrace{\begin{bmatrix} \frac{1}{2}\nabla U(Q) \\ p \end{bmatrix}}_{\nabla H} + \lambda \begin{bmatrix} 0 \\ u \end{bmatrix}$$

$$\begin{aligned} \dot{Q} &= Mp & &\rightarrow \dot{Q}_{ij} = p_i - p_j \\ \dot{p} &= -M^T \tfrac{1}{2}\nabla U(Q) - \lambda p - \lambda u & &\rightarrow \dot{p}_i = \lambda(u_i - p_i) - \textstyle\sum_{j\neq i} \nabla U(Q_{ij}) \end{aligned}$$

Example: N=3

$$M_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} M_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} M_3 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, \dot{Q} = Mp = \begin{bmatrix} p_1 - p_2 \\ p_1 - p_3 \\ p_2 - p_1 \\ p_2 - p_3 \\ p_3 - p_1 \\ p_3 - p_2 \end{bmatrix}$$

$$\begin{aligned} H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(&\nabla U_{12} + \nabla U_{13} + \\ &\nabla U_{21} + \nabla U_{23} + \\ &\nabla U_{31} + \nabla U_{32}) \end{aligned}$$

Stochastic Differential Equations (SDEs)

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t \rightarrow \text{Deterministic}$$

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}W_t \rightarrow \text{Not Deterministic [4]}$$

- μ: drift coefficient
- σ: diffusion coefficient (stochastic term)
- W_t: Wiener process (continuous time stochastic process with Gaussian increments)

Stochastic Port-Hamiltonian System

$$
\begin{aligned}
\mathrm{d}z &= [(J-R)\nabla H(z(t)) + Bu(t)]\mathrm{d}t + \sigma(z(t))\mathrm{d}W(t) \\
y &= B^T \nabla H(z(t))
\end{aligned}
$$

Stochastic Port-Hamiltonian System

- Numerical Methods of Stochastic Differential Equations (SDEs)
- Interpretation
  - Ito
  - Stratonovich
- Numerical methods for SDEs.
  - Euler-Murayama Method
- Symplectic methods for SDEs.
  - Implicit Euler-Murayama Method
  - SDE solvers in `Julia`'s DifferentialEquatons.jl
  - ...

Collective Behavior: Revisited

Finding a way to quantify macroscopic behavior

- Stability analysis [3]
- Hamiltonian as a physical order parameter [5]

[1] Dirk Helbing and Péter Molnár. "Social force model for pedestrian dynamics". In: *Physical Review E* 51.5 (May 1995), pp. 4282–4286. ISSN: 1095-3787. DOI: 10.1103/physreve.51.4282. URL: http://dx.doi.org/10.1103/PhysRevE.51.4282.

[2] Arjan Van Der Schaft. "Port-Hamiltonian systems: an introductory survey". In: *International congress of mathematicians*. European Mathematical Society Publishing House (EMS Ph). 2006, pp. 1339–1365.

[3] Antoine Tordeux and Claudia Totzeck. "Multi-scale description of pedestrian collective dynamics with port-Hamiltonian systems". In: *arXiv preprint arXiv:2211.06503* (2022).

[4] Bernt Øksendal and Bernt Øksendal. *Stochastic differential equations*. Springer, 2003.

[5] Barbara Rüdiger, Antoine Tordeux, and Baris E Ugurcan. "Stability analysis of a stochastic port-Hamiltonian car-following model". In: *Journal of Physics A: Mathematical and Theoretical* 57.29 (2024).