# Precalculus

## Composition of Functions

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This work would not have been possible without the hard work and dedication of many individuals to make many technical things possible. This document was produced in LaTeX, making use of KOMA-script, as well as TikZ and PGF.

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# List of Definitions

# 5 Composition of Functions

In this chapter we will learn about composition of functions.

Given two functions, we may *compose* them; one function may act as an input of another. Let $f(x)$ and $g(x)$ be two functions.

We write a composition as $f(g(x))$ or $g(f(x))$, read as "$f$ of $g$ of $x$" and "$g$ of $f$ of $x$" respectively. An alternate notation of $f(g(x))$ is $(f \circ g)(x)$, which can be read as "$f$ composed with $g$ evaluated at $x$." T

To evaluate the composition $f(g(x)) = (f \circ g)(x)$, we first evaluate $g(x)$, then we plug the result in to $f(x)$. Note, it is not always the case that $f(g(x)) = g(f(x))$. In other words, **the order of composition matters**. To get the notation straight:

$$(f \circ g)(x) = f(g(x))$$
$$(g \circ f)(x) = g(f(x))$$

## 5.1 Evaluating Compositions

Let's do an example with some concrete functions.

Example 5.1: Evaluating Compositions

Let $f(x) = x + 5$ and $g(x) = x^2 + 2$. Then

$$f(g(x)) = (x^2 + 2) + 5 = x^2 + 7$$
$$g(f(x)) = (x + 5)^2 + 2 = x^2 + 10x + 27$$

Notice how $f(g(x)) \neq g(f(x))$.
We can compute some values for the compositions. For example, for $f(g(2))$ we compute $g(2) = (2)^2 + 2 = 6$, then plug 6 into $f$, so

$$f(g(2)) = f(2^2 + 2) = f(6) = (6) + 5 = 11.$$

Or, like above, we can write the composition in its general form, such as $f(g(x)) = x^2 + 7$ and then evaluate. Here are some examples, showing the many ways we can compute a composition of functions.

- $(g \circ f)(2) = ((2) + 5)^2 + 2 = (7)^2 + 2 = 51$
- $g(f(10)) = ((10) + 5)^2 + 2 = (15)^2 + 2 = 227$
- $(f \circ g)(1) = f(g(1)) = f((1)^2 + 2) = f(3) = (3) + 5 = 8$

Try to work out the following on your own. Let $p(x) = \sqrt{x+1}$ and $q(x) = x^2 + 5$. Show that

- $(p \circ q)(x) = \sqrt{x^2 + 6}$
- $(q \circ p)(x) = x + 6$

We may also evaluate compositions from a table.

Example 5.2: Compositions With Tables

Consider the following table of values for two functions $r$ and $q$.

| $x$ | $r(x)$ | $q(x)$ |
|---|---|---|
| 1 | 4 | 12 |
| 2 | 3 | 10 |
| 3 | 1 | 5 |
| 4 | 2 | 14 |

Compute $q(r(2))$ and $q(r(4))$.
To evaluate these, we first must evaluate the inside. So

- $q(r(2))$ : First we evaluate $r(2)$, which from the table we know is 3.

| $x$ | $r(x)$ | $q(x)$ |
|---|---|---|
| 1 | 4 | 12 |
| **2** | **3** | 10 |
| 3 | 1 | 5 |
| 4 | 2 | 14 |

So $q(r(2)) = q(3)$. Looking at the third row, we know $q(3) = 5$.

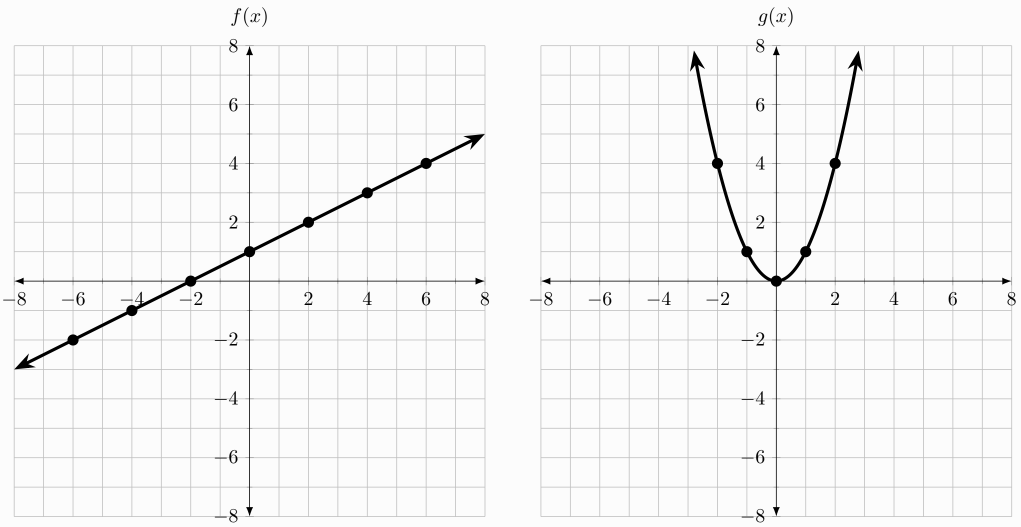
| $x$ | $r(x)$ | $q(x)$ |
|---|---|---|
| 1 | 4 | 12 |
| 2 | 3 | 10 |
| **3** | 1 | **5** |
| 4 | 2 | 14 |

- $q(r(4))$ : First we evaluate $r(4)$, which from the table we know is 2. So $q(r(4)) = q(2)$. Looking at the third row, we know $q(2) = 10$.

Finally we can evaluate compositions via graphs. Note, it is always the same process. For any composition $(f \circ g)(x) = f(g(x))$, we first evaluate $g(x)$ to get a number, then we plug in that number to $f$. On a graph this is nothing new; given an $x$ coordinate, we find the $y$ coordinate on a graph of $g$, then use that $y$ coordinate as an $x$ coordinate in a picture of $f$.

Example 5.3: Compositions With Graphs

Consider two graphs below of $f$ and $g$.




- To evaluate $f(g(2))$, we first evaluate $g(2)$. From the graph of $g$, we see that $g(2) = 4$. So $f(g(2)) = f(4) = 3$.

- To evaluate $g(f(2))$, we first evaluate $f(2)$. From the graph of $f$, we see that $f(2) = 2$. So $g(f(2)) = g(2) = 4$.

## 5.2 Recognizing Compositions

As we've seen, given two functions, there are two ways we can compose them. We can also reverse the process; given a single function, we can recognize and write it as a composition of functions. Any function $f$ is a composition of itself and the *identity* function $g(x) = x$. We've been writing functions all along as *trivial* compositions; $f(x)$ is "$f$ of $x$", where $x$ is itself a function (a boring one, but still a function!) So a composition is not always unique, that is, it is possible that a function can be written as a composition in more than one way.

Example 5.4: Recognizing Compositions

The function $f(x) = (x-2)^4 + 3$ is a composition $h(g(x))$ where $g(x) = x - 2$ and $h(x) = x^4 + 3$. We can check this:

$$h(g(x)) = h(x-2) = (x-2)^4 + 3.$$

One will require a bit of creativity and intuition when recognizing a composition. It may be helpful to think of needing one "outside function" and one "inside function". Here, the outside function is $x^4 + 3$ and the inside function is $x - 2$. Taken together:

$$f(x) = (x-2)^4 + 3.$$

We could have also set $h(x) = (x-2)^4$ and $g(x) = x + 3$, and then $f(x) = g(h(x))$ (check this yourself). In colors,

$$f(x) = \left((x-2)^4\right) + 3$$

Suppose $H(x) = 7\sqrt{x} - 4$. Then we find $f(x) = 7x - 4$ and $g(x) = \sqrt{x}$ so that

$$f(g(x)) = f(\sqrt{x}) = 7(\sqrt{x}) - 4 = H(x)$$

In general, when given a function and asked for a composition, we want to find an "inner" function and an "outer" function, then we can check our composition. Another way to think about it is: we find an inner function, and replace it with an $x$. The resulting function is the outer function. For example

$$f(x) = \sqrt{x+5} + 2.$$

We can make $x + 5$ the inner function, and replacing it with an $x$ we get an outer function $\sqrt{x} + 2$.

## 5.3 Domain of Compositions

Lastly, let's look at the domain of a composition. Consider a composition $f \circ g$. The domain of $f \circ g$ is all inputs $x$ such that $x$ is in the domain of $g$ and $g(x)$ is in the domain of $f$. That is, there are two things to check; the validity of an input $x$ for $f \circ g$ relies on $g$, so that the output $g(x)$ is defined, and further, $g(x)$ must be in the domain of $f$ so that $f(g(x))$ is defined. This all sounds a bit confusing, so let us do some examples.

**Example 5.5: Domain of Compositions 1**

Let $g(x) = x^2 - 1$ and $h(x) = \sqrt{x-6}$. Let's find the composition $g \circ h$ and its domain.

$$(g \circ h)(x) = g(\sqrt{x-6}) = (\sqrt{x-6})^2 - 1.$$

Notice, we may simplify further to obtain $(g \circ h)(x) = (x-6) - 1 = x - 7$. However, notice that by simplifying, one would incorrectly assume that the domain of $(g \circ h)$ is all real numbers. However, if we write $g \circ h$ in the non simplified form $(g \circ h)(x) = (\sqrt{x-6})^2 - 1$, we may notice that we cannot have any number $x$ less than 6 as an input, otherwise $\sqrt{x-6}$ would be undefined. That is, the domain of $h$, namely $[6, \infty)$ plays an important role here. Since the range of $\sqrt{x-6}$ is all real numbers, and the domain of $g$ is all real numbers, we do not need to worry about the output $h(x)$. So the domain of $g \circ h$ is $[6, \infty)$. In total we have

$$(g \circ h)(x) = x - 7$$
Domain: $[6, \infty)$

Let's do a more complicated example.

Example 5.6: Domain of Compositions 2

Consider the functions $f(x) = \dfrac{x}{x+1}$ and $g(x) = \dfrac{11}{x}$. Find and simplify $(f \circ g)(x)$. Then, find the domain.
At this point, we should be able to evaluate a composition. Here it's a bit of work; the algebra can be tricky!

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\frac{11}{x}\right) \\
&= \frac{\left(\frac{11}{x}\right)}{\left(\frac{11}{x}\right) + 1} \\
&= \frac{\left(\frac{11}{x}\right)}{\frac{11}{x} + \frac{1}{1} \cdot \frac{x}{x}} && \text{(Combine terms, LCD)} \\
&= \frac{\left(\frac{11}{x}\right)}{\frac{11+x}{x}} \\
&= \left(\frac{11}{x}\right)\left(\frac{x}{11+x}\right) && \text{(Keep, Change, Flip)} \\
&= \frac{11}{11+x}
\end{aligned}$$

Now note, we have a rational function. The domain is **not** simply obtained from finding values for which the denominator is not equal to $0$. That is, the domain is **not** simply $(-\infty, -11) \cup (-11, \infty)$. This is a *part* of the domain of $(f \circ g)$. To find the domain, we go back to when we plugged $g(x)$ inside of $f$:

$$f(g(x)) = \frac{\left(\frac{11}{x}\right)}{\left(\frac{11}{x}\right) + 1}.$$

In this form, we find values of $x$ which we must exclude from the domain. First, we have $\frac{11}{x}$ in the equation. This implies that $x \neq 0$, as then the function would be undefined. Notice that this issue does not appear in the simplified version of the composition! Then we notice that we have a rational function, so we must check the denominator:

$$\begin{aligned} \frac{11}{x} + 1 &\neq 0 \\ \frac{11}{x} &\neq -1 \\ 11 &\neq -x \\ -11 &\neq x \end{aligned}$$

So we have two excluded values, $-11$ and $0$. Thus the domain of the composition is $(-\infty, -11) \cup (-11, 0) \cup (0, \infty)$.

The above example is some work, but one needs to keep their head straight and obey algebra rules.

Example 5.7: Domain of Compositions 3

Let $p(x) = \sqrt{x+1}$ and $q(x) = x^2 + 5$. Find $(q \circ p)(x)$ and find the domain.
We evaluate:

$$\begin{aligned} (q \circ p)(x) &= q(p(x)) \\ &= q(\sqrt{x+1}) \\ &= (\sqrt{x+1})^2 + 5 \\ &= x + 1 + 5 \\ &= x + 6. \end{aligned}$$

Again, while $(q \circ p)(x) = x + 6$, a line, the domain is **not** $(-\infty, \infty)$. We focus on when we plugged in:

$$(\sqrt{x+1})^2 + 5.$$

Here, we have to make sure that $\sqrt{x+1}$ is defined. An even root function is defined only when the inside evaluates to a number greater or equal to $0$. That is in this case, $\sqrt{\blacksquare}$ is only defined when $\blacksquare \geq 0$. So we solve $x + 1 \geq 0$ which yields $x \geq -1$. So in interval notation our domain is $[-1, \infty)$.

Suppose we are given a function already composed:

**Example 5.8: Domain of a Complicated Function**

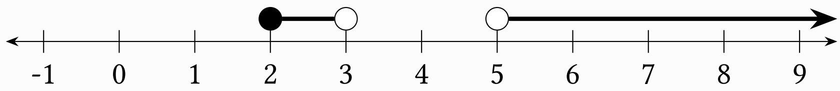
Let

$$f(x) = \sqrt{\frac{x-2}{(x-5)(x-3)}}$$

Now here, one needs to reason through. What do we know:

- The domain of any square root function $\sqrt{\blacksquare}$ contains all $x$ values which satisfy $\blacksquare \geq 0$.
- There is a rational function inside the square root. We need to ensure that we find the excluded values; find the $x$ values which make the denominator $0$.

Thus, we need to solve $\frac{x-2}{(x-5)(x-3)} \geq 0$. Luckily, we've learned this before with rational inequalities. Setting the numerator equal to $0$ we obtain a closed circle at $x = 2$, and setting the denominator equal to $0$ we obtain open circles at $x = 5$ and $x = 3$. We find test values, and end up with (this work is left to the reader):




The domain in interval notation is $[2, 3) \cup (5, \infty)$.

**Summary Chapter 5**

List of things you need to know.

- Compositions are written in two ways $(f \circ g)(x) = f(g(x))$.
- Evaluate compositions of functions for certain values, or give a general formula.
- To find the domain of a composition, find excluded values when you plug in a function. Do not find the domain simply based on the simplified version of a composition.