Precalculus

Quadratic and Radical Equations

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This work would not have been possible without the hard work and dedication of many individuals to make many technical things possible. This document was produced in LaTeX, making use of KOMA-script, as well as TikZ and PGF.

Contents

List of Examples

List of Definitions

2 Quadratic and Radical Equations

2.1 Quadratic Equations

The equation we are interested in solving is of the form

$$ax^2 + bx + c = d.$$

How do we solve something like

$$x^2 = 64 \tag{2.1}$$

for x. When we solve an equation for a variable, previously we would isolate the variable using operations like addition, subtraction, multiplication, and division. Here however, we introduce a new operation, raisin both sides to the same power. For example, if we raise both sides of Equation 2.1 to the $1/2$ power, we get

$$\begin{aligned}(x^2)^{1/2} &= (64)^{1/2}\\ x &= 8.\end{aligned}$$

Recall the exponent rules; $(x^2)^{1/2} = x^{2 \cdot 1/2} = x^1$. Now we have to be a bit careful. While $x = 8$ is indeed a solution to $x^2 = 64$, it is not the only solution. Consider $x = -8$. If we plug in -8, we get $(-8)^2 = 64$. If we raise both sides of an equation to the $1/2$ power, or we take the *square root*, we must remember to add a $\pm$.

So

$$\begin{aligned}\sqrt{x^2} &= \sqrt{64}\\ x &= \pm 8.\end{aligned}$$

Example 2.1: Simple Quadratic Equations.

Some examples of quadratic equations:

1. $-3x^2 + 202 = 10$

$$\begin{aligned} -3x^2 + 202 &= 10 \\ -3x^2 &= -192 && \text{(Subtract 202)} \\ x^2 &= 64 && \text{(Divide by } -3\text{)} \\ x &= \pm 8 && (\sqrt{\ }\text{ each side}) \end{aligned}$$

2. $(x-6)^2 = 25$

$$\begin{aligned} (x-6)^2 &= 25 \\ x - 6 &= \pm\sqrt{25} && (\sqrt{\ }\text{ each side}) \\ x - 6 &= \pm 5 \\ x &= \pm 5 + 6 \end{aligned}$$

So $x = 5 + 6 = 11$ or $x = -5 + 6 = 1$.

Some times we require some more complicated techniques to solve quadratic equations.

Example 2.2: Harder Quadratic Equations

1. Find all solutions to $x^2 - 12x + 11 = 0$.

   Solution. We can factor the left hand side of the equation as $(x-11)(x-1)$. So we have
   $$(x-11)(x-1) = 0.$$
   We have a product of two things equal to 0, meaning either $x - 11 = 0$ or $x - 1 = 0$. So either $x = 11$ or $x = 1$.

2. Solve $x^2 + 4x - 45 = 0$ by **completing the square**.

   Solution. First we move -45 to the right hand side to get
   $$x^2 + 4x = 45.$$
   Then we add $(b/2)^2$ to both sides. Here $b = 4$.
   $$x^2 + 4x + 4 = 49.$$

By doing so, we can now factor the left hand side as $(x + b/2)^2$.

$$\begin{aligned} (x+2)^2 &= 49 \\ x+2 &= \pm 7 && (\sqrt{\ } \text{ each side}) \\ x &= \pm 7 - 2 \end{aligned}$$

So $x = 7-2 = 5$ or $x = -7-2 = -9$. One can also check these solutions by factoring.

3. Solve $x^2 - 8x + 3 = 11$.

   Solution. Move 3 to the right hand side, and add $(b/2)^2$ to both sides, then factor and solve.

$$\begin{aligned} x^2 - 8x &= 8 \\ x^2 - 8x + 16 &= 24 \\ (x-4)^2 &= 24 \\ x - 4 &= \pm\sqrt{24} \\ x &= 4 \pm \sqrt{24} \end{aligned}$$

   Notice here that $\sqrt{24}$ can be simplified as $\sqrt{4 \cdot 6} = 2\sqrt{6}$. So we have $x = 4 + 2\sqrt{6}$ and $x = 4 - 2\sqrt{6}$.

2.2 Radical Equations

Now we'd like to solve equations containing radicals. With quadratic equations, we had to make sure that we did not leave out possible solutions. That is, we had to keep track of a $\pm$ every time we raised both sides of an equation to the $1/2$ power (square root).

We experience an opposite occurrence with equations which have a square root. For example, consider

$$\sqrt{-2x+8} = x.$$

Here, we would like to isolate the variable x, however there is an x under a radical. Similar to how we solved quadratic equations by square rooting both sides, to undo a radical we square both sides of an equation.

$$\begin{aligned} \sqrt{-2x+8} &= x \\ -2x + 8 &= x^2 \\ 0 &= x^2 + 2x - 8 \end{aligned}$$

Now we have a quadratic equation. Using techniques from the previous section we find that $x = 2$ or $x = -4$. However, what happens when we check our work?

Check $x = 2$

$$\begin{aligned}\sqrt{-2(2)+8} &= 2\\ \sqrt{-4+8} &= 2\\ \sqrt{4} &= 2\\ 2 &= 2 \quad ✓\end{aligned}$$

Check $x = -4$

$$\begin{aligned}\sqrt{-2(-4)+8} &= -4\\ \sqrt{8+8} &= -4\\ \sqrt{16} &= -4\\ 4 &= -4 \quad ✗\end{aligned}$$

So the only solution is $x = 2$, while $x = -4$ is not a solution to the original equation.

When we square two sides of an equation, we may introduce what we call *extraneous solutions.* These are extra solutions that are not solutions for the original equation. Therefore, it is **necessary** that one **checks their work** when solving radical equations.

Example 2.3: Radical Equation With Two Radicals

Solve the following equation for x.

$$\sqrt{-x+4} + \sqrt{5x+1} = 3$$

For equations with two terms as radicals, solving requires a bit of extra effort. First we isolate one of the radical terms. Here let's isolate $\sqrt{-x+4}$. Then we square both sides and simplify.

$$\begin{aligned}\sqrt{-x+4} &= 3 - \sqrt{5x+1}\\ -x+4 &= (3-\sqrt{5x+1})^2\\ -x+4 &= (3-\sqrt{5x+1})(3-\sqrt{5x+1})\\ -x+4 &= 9 - 3\sqrt{5x+1} - 3\sqrt{5x+1} + \sqrt{5x+1}\cdot\sqrt{5x+1}\\ -x+4 &= 9 - 6\sqrt{5x+1} + 5x + 1\\ -x+4 &= 10 + 5x - 6\sqrt{5x+1}\end{aligned}$$

Notice now we have an equation with a single radical, and linear terms. Again, we isolate the radical to get a new equation,

$$-6x - 6 = -6\sqrt{5x+1}.$$

Let's divide both sides by -6, to make it a bit nicer,

$$x + 1 = \sqrt{5x+1}.$$

Now we square both sides, and solve.

$$\begin{aligned}(x+1)^2 &= 5x+1\\ x^2+2x+1 &= 5x+1\\ x^2-3x &= 0\\ x(x-3) &= 0\end{aligned}$$

So either $x=0$ or $x=3$. Remember, we need to **check the solutions**, as we may have introduced extraneous solutions.

Check $x=0$

$$\begin{aligned}\sqrt{-(0)+4} &= 3-\sqrt{5(0)+1}\\ \sqrt{4} &= 3-\sqrt{+1}\\ 2 &= 2 \quad ✓\end{aligned}$$

Check $x=3$

$$\begin{aligned}\sqrt{-(3)+4} &= 3-\sqrt{5(3)+1}\\ \sqrt{1} &= 3-\sqrt{16}\\ 1 &= 3-4\\ 1 &= -1 \quad ✗\end{aligned}$$

The most difficult part of a radical equation with two radicals is multiplying out side of the form $\blacksquare+\sqrt{\square}$. For good measure, let's see it abstractly, then let's show a quick example with variables to round out this section.

$$\begin{aligned}(\blacksquare+\sqrt{\square})^2 &= (\blacksquare+\sqrt{\square})(\blacksquare+\sqrt{\square})\\ &= \blacksquare\cdot\blacksquare+\blacksquare\sqrt{\square}+\blacksquare\sqrt{\square}+\sqrt{\square}\cdot\sqrt{\square}\\ &= \blacksquare^2+2\blacksquare\sqrt{\square}+\square\end{aligned}$$

If you know how to multiply out $(x+a)^2$, then this is nothing new. Remember to keep track of signs. Also note, the last term turns from a product of radicals into whatever was inside ($\sqrt{\square}\sqrt{\square}=\square$); the radical disappears so long as what is inside is the same.

$$\begin{aligned}(3+\sqrt{2x+5})(3+\sqrt{2x+5}) &= 9+3\sqrt{2x+5}+3\sqrt{2x+5}+\sqrt{2x+5}\sqrt{2x+5}\\ &= 9+6\sqrt{2x+5}+(2x+5)\\ &= 14+2x+6\sqrt{2x+5}.\end{aligned}$$

Summary Chapter 2

List of things you need to know.

- When we take a square root, we must remember $\pm$.
- How to solve a quadratic equation.
  - Complete the square.
  - Factoring (move everything over so you get $ax^2 + bx + c = 0$, then factor and set each factor equal to 0.)
- When solving radical equations, you must **check your solutions.**