Precalculus

Course Notes

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This work would not have been possible without the hard work and dedication of many individuals to make many technical things possible. This document was produced in LaTeX, making use of KOMA-script, as well as TikZ and PGF.

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π **Preface**

This document is a result of course notes that I have written throughout my teaching as a graduate student at Virginia Commonwealth University, mainly over three summers teaching Precalculus. The purpose of these notes is two-fold:

1. Supplemental material to a lecture or self-learning environment, where a student can refresh their memory or work through example.

2. Material for teaching assistants and tutors; to provide a comprehensive document with easy to find examples and explanations that may enlighten a student.

I would like to extend a special thank you to my students from VCU. I would also like to thank Aimee Ellington, Andy Bristow, and Mason Hester, with who I've had conversations about teaching with, or have learned from their teaching materials.

0 Prerequisites

0.1 Exponent Rules

Exponents are a convenient notation in mathematics. They are used to represent the multiplication of the same number some number of times. For example, instead of writing $5 \times 5 \times 5$, we may write 5^3 to communicate to the reader: "multiply 5 by itself 3 times". Not only does this make reading mathematics simpler, but it also allows us to more easily manipulate algebraic expressions. For example, we use these concepts to expand expressions like $(x + 5)^2$ or simplify expressions like $\frac{(x+5)^2(x-2)}{(x+7)(x+5)^{10}}$.

Here a, b, p, q are any real number. Verify each property using your own numbers.

Name	Form	Example
Zero Rule	$a^0 = 1$	$2^0 = 1$
Product Rule	$a^p \cdot a^q = a^{p+q}$	$2^10 \cdot 2^4 = 2^{14}$
Quotient Rule	$a^p / a^q = a^{p-q}$	$2^{10} / 2^4 = 2^6$
Power of a Product	$(a \cdot b)^p = a^p b^p$	$(2 \cdot 5)^2 = 2^2 \cdot 5^2$
Power of a Quotient	$(a/b)^p = a^p / b^p$	$(5/4)^3 = 5^3 / 4^3$
Power of a Power	$(a^p)^q = a^{pq}$	$(2^3)^2 = 2^6$
Negative Exponent	$a^{-p} = 1/a^p$	$2^{-1} = 1/2$
Fractional Exponent	$a^{p/q} = (\sqrt[q]{a})^p = (\sqrt[q]{a^p})$	$5^{4/3} = \sqrt[3]{5^4}$

Most often we will use a combination of these rules to simplify algebraic expressions. **Note**: when simplifying an expression, there may be more than one way to apply a sequence of rules. In other words, there is no set order, so long as the rule is implemented properly from one expression to another.

Example 0.1: Simplifying Expressions With Exponents.

Simplify.
$$\left(\frac{15a^3b^{-5}c^{-2}}{5a^{-7}b^8}\right)^{-2}$$

$$\left(\frac{15a^3b^{-5}c^{-2}}{5a^{-7}b^8}\right)^{-2} = \left(\frac{3a^3b^{-5}c^{-2}}{a^{-7}b^8}\right)^{-2} = \frac{3^{-2}a^{-6}b^{10}c^4}{a^{14}b^{-16}} = \frac{b^{10}b^{16}c^4}{3^2a^6a^{14}} = \frac{b^{26}c^4}{9a^{20}}$$

Steps:

1. We simplify the $\frac{15}{5}$ to 3 in the numerator.

2. Using the **Power of a quotient** and **Power of a product** rules, we "distribute" the exponent -2 to each term on the inside.

3. Using the **Negative Exponent Rule**, we take variables with negative exponents from the top and move them to the bottom while making their exponents positive, and from the bottom to the top while making their exponents positive.

4. Using the **Product Rule**, we can combine identical variables and their exponents. E.g. $a^3a^7 = a^{3+7} = a^{10}$.

0.2 Factoring

Similar to how one uses exponent rules, factoring is a technique we can use to simplify expressions. For example, the fraction $\frac{36}{8}$. One may deduce that 36 and 8 have common factors, and so if we take the *greatest common factor*, in this case 4, we can simplify:

$$\frac{36}{8} = \frac{4 \cdot 9}{4 \cdot 2} = \frac{9}{2}.$$

Similarly for expressions with variables, we may do the same:

$$\frac{4x^2 + 2x}{2x^4 + 4x^2} = \frac{2x(2x+1)}{2x^2(x^2+2)} = \frac{2x+1}{x(x^2+2)}.$$

Factoring allows us to do many things; by writing a quadratic function (and later, polynomial functions in general) as a product of its factors, we can more easily find things like the x-intercept. For example

$$f(x) = x^2 + 5x + 6 = (x+2)(x+3).$$

Multiply the factors out and check this for yourself. In factored form we see that plugging in -2 or -3 results in an evaluation of 0, i.e. $f(-2) = f(-3) = 0$.

0.2.1 Quadratics

We will explore the ways to factor a quadratic expression $ax^2 + bx + c$ with integer coefficients. **If you simply need a quick refresher, look over the examples**.

If $a = 1$, then we have $x^2 + bx + c$. We want to write

$$x^2 + bx + c = (x + p)(x + q).$$

If we multiply out the right hand side:

$$\begin{aligned}(x + p)(x + q) &= x^2 + qx + px + pq \\ &= x^2 + (p + q)x + pq \\ &= x^2 + bx + c.\end{aligned}$$

So, if we equate the coefficients we must find two numbers p and q so that

$$p + q = b, \text{and}$$
$$pq = c.$$

To do this, we construct a list of factors of c, and check the sums. Let's do an example.

Example 0.2: Factoring $x^2 + bx + c$.

Factor $x^2 + 3x - 18$ completely.
Solution. We want two numbers that sum to 3 and multiply out to -18. We can factor 18 as: 1×18, 2×9, or 3×6.
Since what we actually need to factor is -18, **exactly one** of the factors in the pair we choose must be negative, and one must be positive. The choice in this case is $(-3, 6)$, since $-3 + 6 = 3$ and $(-3)(6) = -18$. We get $(x - 3)(x + 6)$ as our answer.

If $a \neq 1$, then we have $ax^2 + bx + c$. Now in this case if we wish to factor, the leading terms of the factors must have a coefficient which is not 1.

We can use the "ac-method" here, and *grouping*. First, we find the factors of ac. Then we find a pair of factors of ac which sum to b. Then, we split the linear term bx into two terms and group.

Example 0.3: Factoring $ax^2 + bx + c$ using the ac-method.

Factor completely:
$$3x^2 + 13x + 10$$

Solution. Using the ac-method: We have $a = 3, c = 10$, and so $ac = 30$. We want two numbers that add up to $b = 13$ and multiply to $ac = 30$. The number 30 can be factored as:

$$
\begin{array}{cc}
 & 30 \\
1 & 30 \\
2 & 15 \\
3 & 10 \\
5 & 6
\end{array}
$$

We pick 3 and 10:

$$3x^2 + 13x + 10 = 3x^2 + \overbrace{3x}^{3x} + \overbrace{10x}^{10} + 10$$
$$= 3x(x + 1) + 10(x + 1)$$
$$= (x + 1)(3x + 10)$$

Finally, one can use the quadratic formula to find the roots of a quadratic, and then write the quadratic in standard form. The quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quadratic formula will give two roots α_1 and α_2 (sometimes they are the same, this happens when the original quadratic is a perfect square.) Then we write $f(x)$ as $(x - \alpha_1)$ and $(x - \alpha_2)$.

Example 0.4: Using the quadratic formula.

Let's look at our previous two quadratic expressions.
For $x^2 + 3x - 18$ we have $a = 1, b = 3$, and $c = -18$. Plugging into the quadratic

formula we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-18)}}{2(1)}$$
$$= \frac{-3 \pm \sqrt{81}}{2}$$
$$= \frac{-3 \pm 9}{2}.$$

So one root $\alpha_1 = \frac{-3+9}{2} = 3$ and the other root $\alpha_2 = \frac{-3-9}{2} = -6$. Finally to write our quadratic in factored form, we have

$$x^2 + 3x - 18 = (x - \alpha_1)(x - \alpha_2) = (x - 3)(x - (-6)).$$

One can also find each factor by solving two equations.

$$x = 3 \text{ and } x = -6$$
$$x - 3 = 0 \text{ and } x + 6 = 0$$

For $3x^2 + 13x + 10$ we have $a = 3$, $b = 13$, and $c = 10$. Plugging in we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(13) \pm \sqrt{(13)^2 - 4(3)(10)}}{2(3)}$$
$$= \frac{-13 \pm \sqrt{49}}{6}$$
$$= \frac{-13 \pm 7}{6}$$

So one root $\alpha_1 = \frac{-13+7}{6} = -1$ and the other root $\alpha_2 = \frac{-13-7}{6} = \frac{-10}{3}$.
So the roots are $x = -1$ and $x = \frac{-10}{3}$. Solving each, we get factors $(x + 1)$ and $(3x + 10)$.

0.3 Functions

0.3.1 What is a function?

Recall what a function is:

> **Definition 0.5: Function**
>
> A *function* is an assignment of the elements of one set to another; a function is an object that takes inputs and produces exactly one output. For example, $f(x) = x + 5$. The f is the *name* of the function, the input is x, and $f(x)$ is read "f of x". The *output* $f(x)$ is $x + 5$, or 5 more than the input. Notice, any input for $f(x)$ receives one, and only one, output. The set of inputs can be all real numbers (and consequently the outputs are real numbers.)

Functions can be represented using formulas, or as a picture on a grid.
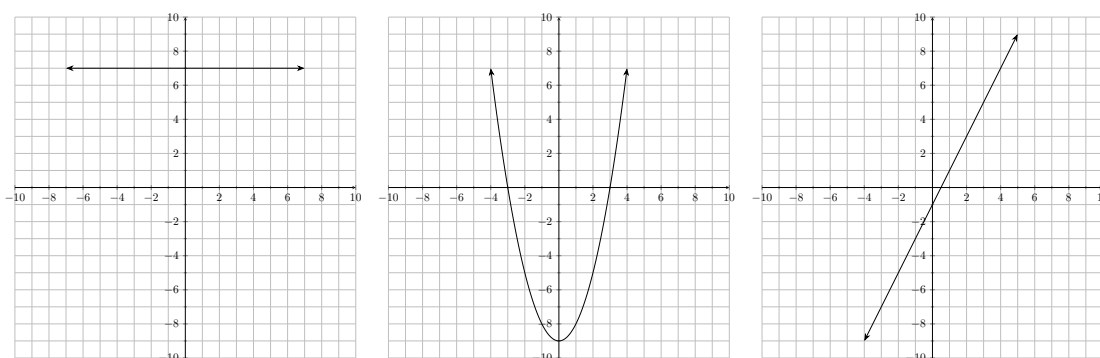
Figure 1: Three functions, from left to right: $f(x) = 7$ (a constant function), $g(x) = x^2 - 9$ (a quadratic function), and $h(x) = 2x - 1$ (a linear function).
.

Note how each of the functions in Figure 1 passes the *vertical line test*; any vertical line drawn on the grid intersects the function in exactly one point. This tests the requirement that a function has exactly one output for each of its inputs (you plug one number in, you get one number out.)

> **Definition 0.6: Domain and Range**
>
> The set of allowable inputs of a function is its *domain*, and the set of possible outputs given those inputs is its *range.* The domain and range can be expressed

using *interval notation*, a real number line, or in words.

Example 0.7: Functions – Domain and Range.

Taking a look at Figure 1, we can make the following table

Function	Domain	Range
$f(x)$	$(-\infty, \infty)$	$\{7\}$
$g(x)$	$(-\infty, \infty)$	$[-9, \infty)$
$h(x)$	$(-\infty, \infty)$	$(-\infty, \infty)$

The range of f only contains 7, since it is the constant function $f(x) = 7$. The range of g is $[-9, \infty)$; g is a quadratic function with a positive leading coefficient, so it has a *minimum*, in this case -9. Finally, $h(x) = 2x - 1$ is a line, with unrestricted domain and range. Notice that none of these examples has a restricted domain, but there is a function we all know of that can serve as an example. Consider $p(x) = \frac{1}{x}$. Since $\frac{1}{0}$ is undefined, the function p has no output for $x = 0$, that is, 0 is not in the domain of p. This is the only *excluded value* of p, and so the domain of p can be written as $(-\infty, 0) \cup (0, \infty)$. We will discuss functions like p when we talk about rational functions.

We will learn many different classes of functions in addition to the ones we know already (lines, quadratic.) For each family of functions, it is important to know how the domain and range behave.

You should be familiar with *lines* and *quadratic functions*. Recall that x-intercepts are where a function crosses the x axis (when $f(x) = 0$), and the y-intercept is where a function crosses the y-axis (the point $(0, f(0))$).

0.3.2 Lines and Quadratic Functions

Definition 0.8: Line

A *line* is a function of the form $f(x) = mx + b$, where x is the input, m is the *slope*, and b is the y-intercept.
Given two points, one is able to define the unique line passing through the two points. To calculate the slope between two points $(x_1, y_1), (x_2, y_2)$ we use the formula:
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Slope is often referred to as *rise over run*, and this can be seen in the formula. The
"rise" is the difference between the y coordinates of the points, $y_2 - y_1$, and the
"run" is the difference between the x coordinates, $x_2 - x_1$. Notice that one can
compute $y_2 - y_1$ or $y_1 - y_2$, so long as one remains consistent. In other words,
take care to not to compute something like $\frac{y_2 - y_1}{x_1 - x_2}$.
The *domain* of any line is $(-\infty, \infty)$. The *range* of a line with slope $m \neq 0$ is
$(-\infty, \infty)$. If the slope $m = 0$, then the line is *horizontal* with equation $f(x) = b$,
and thus has range $\{b\}$.

Let's do an example.

Example 0.9: Equation of a Line Given Two Points

Find the equation of the line passing through the points $(-2, 1)$ and $(1, 7)$.

Solution. A line has equation $f(x) = mx + b$. We must find m and b. Let
$(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (1, 7)$.
To find m we calculate the rise over run:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 1}{1 - (-2)} = \frac{6}{3} = 2.$$

Now there are two ways we can obtain the equation. First, we can solve $f(x) = mx + b$ using one of the given points. For example using $(1, 7)$:

$$7 = (2)(1) + b \implies b = 5.$$

One can also use **point-slope form.** This is essentially the same. Let's use the
other point here.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - (1) &= 2(x - (-2)) \\
y - 1 &= 2(x + 2) \\
y - 1 &= 2x + 4 \\
y &= 2x + 5
\end{aligned}$$

Given pieces of information of a line, e.g. two points, or one point and the slope,
one should be able to find the equation of the line.

Definition 0.10: Quadratic Function

A *quadratic* function is of the form $f(x) = ax^2 + bx + c$ where a, b, and c are *coefficients*, with c referred to specially as the *constant* term. The *vertex* (h, k) of a quadratic function is the minimum or maximum point of the function. If a is positive, then the vertex is a minimum ($\smile$). If a is negative, then the vertex is a maximum ($\frown$).

Every quadratic function can be written in what is called *vertex form* $f(x) = a(x - h)^2 + k$. The former is *standard form.*

The domain of any quadratic function is $(-\infty, \infty)$. The range of a quadratic function depends on whether a is negative or positive. For a vertex (h, k), the range of a quadratic function is

1. $[k, \infty)$ if $a > 0$ (positive)

2. $(-\infty, k]$ if $a < 0$ (negative)

Example 0.11: A Quadratic Function

Let's take a look at the following two quadratic functions.
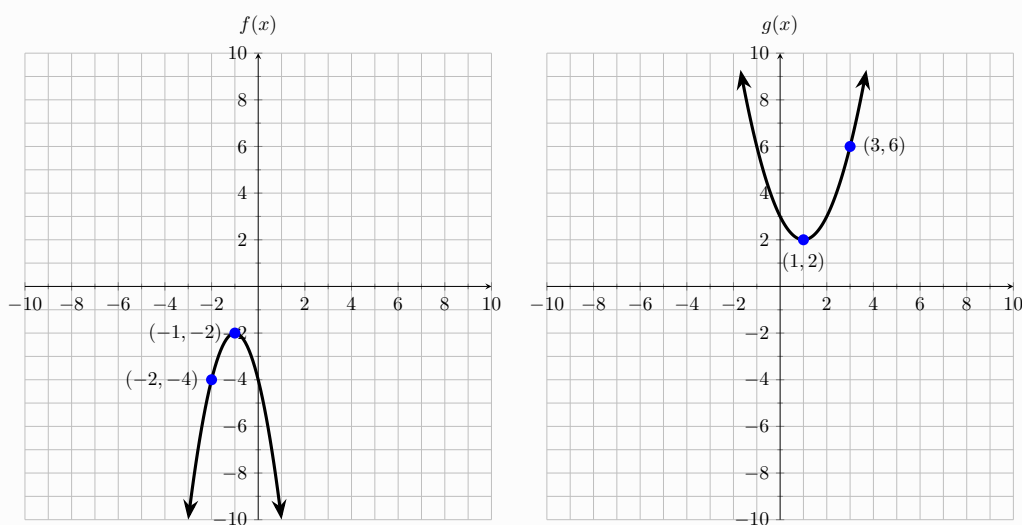
On the left we have a quadratic function $f(x)$ with vertex $(h, k) = (-1, -2)$. To find the equation of f, we can use the vertex form. So $f(x) = a(x - h)^2 + k$. Plugging in we get $f(x) = a(x - (-1))^2 + (-2)$, or $f(x) = a(x + 1)^2 - 2$. To find a, we can plug in a point *other than the vertex* to solve for a. If we plug in

the vertex, then $a(x+1)^2$ will become 0, leaving you unable to solve for 0. Let's use the point $(-2, -4)$ given to us instead. Then

$$f(-2) = a(-2+1)^2 - 2$$
$$= a - 2$$

and since we know $f(-2) = -4$, we can solve $a - 2 = -4$ to get that $a = -2$.

For a quick mental check, notice how f opens up downward ($\frown$); this matches with the fact that the a we found is negative. So, $f(x) = -2(x+1)^2 - 2$. The domain of f is $(-\infty, \infty)$ and the range of f is $(-\infty, -2]$.
Try and follow the same procedure to find that $g(x) = (x-1)^2 + 2$. The domain of g is $(-\infty, \infty)$ and the range is $[2, \infty)$.

If one is only given a quadratic function in standard form, $f(x) = ax^2 + bx + c$, then one can still find information, such as the vertex, in different ways. If the vertex is (h, k), then an equation for h is

$$h = \frac{-b}{2a}.$$

Once h is calculated, one can plug h into f to get k. The a in the standard form is the same as the a in the vertex form.

Given the equation of a quadratic function that is not in standard form, we can take steps to rewrite it in vertex form. One way is by completing the square.

Example 0.12: Completing the Square.

Let $f(x) = 3x^2 + 12x + 6$. First, factor the GCF so that $f(x) = 3(x^2 + 4x + 2)$. We complete the square with the inside function by first computing $(b/2)^2$. In this case $(b/2)^2 = (4/2)^2 = 4$. Then we add and subtract $(b/2)^2$. This is like adding 0, but it provides us with a convenience!

$$f(x) = 3(x^2 + 4x + 4 + 2 - 4)$$
$$= 3(x^2 + 4x + 4 - 2)$$
$$= 3(x^2 + 4x + 4) - 6 \qquad \text{(Move } -2 \text{ out.)}$$
$$= 3(x + 2)^2 - 6$$

So we see that f is a quadratic function with vertex $(-2, -6)$.

Completing the square can be useful for solving algebra equations involving radicals, which we will see later.

0.3.3 Function Translations

Recall that functions can be stretched, compressed, shifted left to right, and shifted up and down. These concepts are crucial when understanding families of functions, and how to get from a standard example (for example, $f(x) = x^2$, to something like $f(x) = (x + 5)^2 + 2$.

Let f be a function, and let a be any **positive** number. Then we have the following rules.

- $f(x - a)$ shifts a function a units to the right.

- $f(x + a)$ shifts a function a units to the left.

- $f(x) + a$ shifts a function a units up.

- $f(x) - a$ shifts a function a units down.

- If $a > 1$ then $af(x)$ is $f(x)$ but compressed horizontally. If a is a proper fraction (between 0 and 1) then $af(x)$ is $f(x)$ stretched horizontally. This rule applies the same for when a is negative, except the negativity of a reflects $f(x)$ across the x-axis.

To see the shifting in action, go back to the examples of quadratic functions. Notice that vertex form of a quadratic function shows you exactly what shifting is going on. For example, $g(x) = (x - 1)^2 + 2$ is the function $f(x) = x^2$ shifted to the right 1 unit, and up 2 units. In other words, $g(x) = f(x - 1) + 2$.

You can play with the following Desmos activity to see how the shifting occurs: https://www.desmos.com/calculator/mmkk5legqy.

> ### Summary
> ### Chapter 0
>
> List of things you need to know.
>
> - Exponent rules.
>
> - What are the different factoring techniques.
>   - Factor $ax^2 + bx + c$ when $a = 1$ and when $a \neq 1$.
>
> - What is a function? How to work with functions (their names, inputs, outputs, etc.) What is the vertical line test?
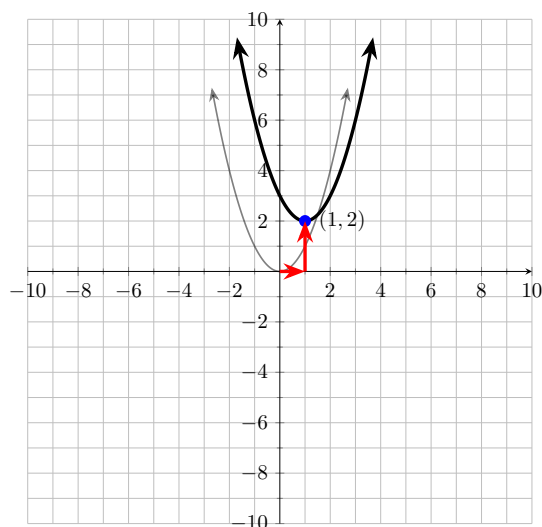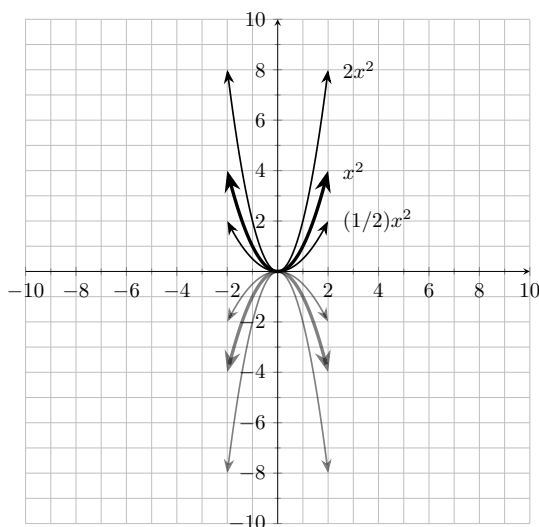
- What is the domain and range of a line? A quadratic function?

- How do you find the equation of a line given some information?

- How do you find the equation of a quadratic function given some information?

  - How do you find the vertex ($h = \frac{-b}{2a}$).

  - How do you convert a quadratic function written in standard form to vertex form?

    * Completing the square.

    * OR finding the vertex, then find another point of the function and use that to solve for a in the vertex form.

- Understand function translations.

1 Quadratic and Radical Equations

1.1 Quadratic Equations

The equation we are interested in solving is of the form

$$ax^2 + bx + c = d.$$

How do we solve something like

$$x^2 = 64 \tag{1.1}$$

for x. When we solve an equation for a variable, previously we would isolate the variable using operations like addition, subtraction, multiplication, and division. Here however, we introduce a new operation, raisin both sides to the same power. For example, if we raise both sides of Equation 4.1 to the $1/2$ power, we get

$$(x^2)^{1/2} = (64)^{1/2}$$
$$x = 8.$$

Recall the exponent rules; $(x^2)^{1/2} = x^{2 \cdot 1/2} = x^1$. Now we have to be a bit careful. While $x = 8$ is indeed a solution to $x^2 = 64$, it is not the only solution. Consider $x = -8$. If we plug in -8, we get $(-8)^2 = 64$. If we raise both sides of an equation to the $1/2$ power, or we take the *square root*, we must remember to add a $\pm$.

So

$$\sqrt{x^2} = \sqrt{64}$$
$$x = \pm 8.$$

Example 1.1: Simple Quadratic Equations.

Some examples of quadratic equations:

1. $-3x^2 + 202 = 10$

$$
\begin{aligned}
-3x^2 + 202 &= 10 \\
-3x^2 &= -192 && \text{(Subtract 202)} \\
x^2 &= 64 && \text{(Divide by } -3) \\
x &= \pm 8 && (\sqrt{} \text{ each side})
\end{aligned}
$$

2. $(x - 6)^2 = 25$

$$
\begin{aligned}
(x - 6)^2 &= 25 \\
x - 6 &= \pm\sqrt{25} && (\sqrt{} \text{ each side}) \\
x - 6 &= \pm 5 \\
x &= \pm 5 + 6
\end{aligned}
$$

So $x = 5 + 6 = 11$ or $x = -5 + 6 = 1$.

Some times we require some more complicated techniques to solve quadratic equations.

Example 1.2: Harder Quadratic Equations

1. Find all solutions to $x^2 - 12x + 11 = 0$.

    Solution. We can factor the left hand side of the equation as $(x - 11)(x - 1)$. So we have
    $$(x - 11)(x - 1) = 0.$$
    We have a product of two things equal to 0, meaning either $x - 11 = 0$ or $x - 1 = 0$. So either $x = 11$ or $x = 1$.

2. Solve $x^2 + 4x - 45 = 0$ by **completing the square**.

    Solution. First we move -45 to the right hand side to get
    $$x^2 + 4x = 45.$$
    Then we add $(b/2)^2$ to both sides. Here $b = 4$.
    $$x^2 + 4x + 4 = 49.$$

By doing so, we can now factor the left hand side as $(x + b/2)^2$.

$$(x + 2)^2 = 49$$
$$x + 2 = \pm 7 \qquad (\sqrt{\,} \text{ each side})$$
$$x = \pm 7 - 2$$

So $x = 7 - 2 = 5$ or $x = -7 - 2 = -9$. One can also check these solutions by factoring.

3. Solve $x^2 - 8x + 3 = 11$.

   Solution. Move 3 to the right hand side, and add $(b/2)^2$ to both sides, then factor and solve.

   $$x^2 - 8x = 8$$
   $$x^2 - 8x + 16 = 24$$
   $$(x - 4)^2 = 24$$
   $$x - 4 = \pm\sqrt{24}$$
   $$x = 4 \pm \sqrt{24}$$

   Notice here that $\sqrt{24}$ can be simplified as $\sqrt{4 \cdot 6} = 2\sqrt{6}$. So we have $x = 4 + 2\sqrt{6}$ and $x = 4 - 2\sqrt{6}$.

1.2 Radical Equations

Now we'd like to solve equations containing radicals. With quadratic equations, we had to make sure that we did not leave out possible solutions. That is, we had to keep track of a $\pm$ every time we raised both sides of an equation to the $1/2$ power (square root).

We experience an opposite occurrence with equations which have a square root. For example, consider

$$\sqrt{-2x + 8} = x.$$

Here, we would like to isolate the variable x, however there is an x under a radical. Similar to how we solved quadratic equations by square rooting both sides, to undo a radical we square both sides of an equation.

$$\sqrt{-2x + 8} = x$$
$$-2x + 8 = x^2$$
$$0 = x^2 + 2x - 8$$

Now we have a quadratic equation. Using techniques from the previous section we find that $x = 2$ or $x = -4$. However, what happens when we check our work?

$$\text{Check } x = 2 \qquad\qquad \text{Check } x = -4$$

$$\sqrt{-2(2) + 8} = 2 \qquad\qquad \sqrt{-2(-4) + 8} = -4$$

$$\sqrt{-4 + 8} = 2 \qquad\qquad \sqrt{8 + 8} = -4 \qquad \text{So the}$$

$$\sqrt{4} = 2 \qquad\qquad \sqrt{16} = -4$$

$$2 = 2 \; \checkmark \qquad\qquad 4 = -4 \; \text{\ding{55}}$$

only solution is $x = 2$, while $x = -4$ is not a solution to the original equation.

When we square two sides of an equation, we may introduce what we call *extraneous solutions*. These are extra solutions that are not solutions for the original equation. Therefore, it is **necessary** that one **checks their work** when solving radical equations.

Example 1.3: Radical Equation With Two Radicals

Solve the following equation for x.

$$\sqrt{-x + 4} + \sqrt{5x + 1} = 3$$

For equations with two terms as radicals, solving requires a bit of extra effort. First we isolate one of the radical terms. Here let's isolate $\sqrt{-x + 4}$. Then we square both sides and simplify.

$$\sqrt{-x + 4} = 3 - \sqrt{5x + 1}$$
$$-x + 4 = (3 - \sqrt{5x + 1})^2$$
$$-x + 4 = (3 - \sqrt{5x + 1})(3 - \sqrt{5x + 1})$$
$$-x + 4 = 9 - 3\sqrt{5x + 1} - 3\sqrt{5x + 1} + \sqrt{5x + 1} \cdot \sqrt{5x + 1}$$
$$-x + 4 = 9 - 6\sqrt{5x + 1} + 5x + 1$$
$$-x + 4 = 10 + 5x - 6\sqrt{5x + 1}$$

Notice now we have an equation with a single radical, and linear terms. Again, we isolate the radical to get a new equation,

$$-6x - 6 = -6\sqrt{5x + 1}.$$

Let's divide both sides by -6, to make it a bit nicer,

$$x + 1 = \sqrt{5x + 1}.$$

Now we square both sides, and solve.

$$(x+1)^2 = 5x+1$$
$$x^2 + 2x + 1 = 5x + 1$$
$$x^2 - 3x = 0$$
$$x(x-3) = 0$$

So either $x = 0$ or $x = 3$. Remember, we need to **check the solutions**, as we may have introduced extraneous solutions.

Check $x = 0$

$$\sqrt{-(0)+4} = 3 - \sqrt{5(0)+1}$$
$$\sqrt{4} = 3 - \sqrt{+1}$$
$$2 = 2 \quad \checkmark$$

Check $x = 3$

$$\sqrt{-(3)+4} = 3 - \sqrt{5(3)+1}$$
$$\sqrt{1} = 3 - \sqrt{16}$$
$$1 = 3 - 4$$
$$1 = -1 \quad ✗$$

The most difficult part of a radical equation with two radicals is multiplying out side of the form $\blacksquare + \sqrt{\square}$. For good measure, let's see it abstractly, then let's show a quick example with variables to round out this section.

$$(\blacksquare + \sqrt{\square})^2 = (\blacksquare + \sqrt{\square})(\blacksquare + \sqrt{\square})$$
$$= \blacksquare \cdot \blacksquare + \blacksquare\sqrt{\square} + \blacksquare\sqrt{\square} + \sqrt{\square} \cdot \sqrt{\square}$$
$$= \blacksquare^2 + 2\blacksquare\sqrt{\square} + \square$$

If you know how to multiply out $(x+a)^2$, then this is nothing new. Remember to keep track of signs. Also note, the last term turns from a product of radicals into whatever was inside ($\sqrt{\square}\sqrt{\square} = \square$); the radical disappears so long as what is inside is the same.

$$(3 + \sqrt{2x+5})(3 + \sqrt{2x+5}) = 9 + 3\sqrt{2x+5} + 3\sqrt{2x+5} + \sqrt{2x+5}\sqrt{2x+5}$$
$$= 9 + 6\sqrt{2x+5} + (2x+5)$$
$$= 14 + 2x + 6\sqrt{2x+5}.$$

Summary
Chapter 1

List of things you need to know.

- When we take a square root, we must remember $\pm$.

- How to solve a quadratic equation.
  - Complete the square.
  - Factoring (move everything over so you get $ax^2 + bx + c = 0$, then factor and set each factor equal to 0.)

- When solving radical equations, you must **check your solutions.**

2 Absolute Value and Piecewise Functions

2.1 The Absolute Value Function

Definition 2.1: Absolute Value

The *absolute value* of a number x, denoted $|x|$, is defined as follows:

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

In words, if the input x is negative, then the absolute value function multiplies x by -1 to make it positive. If the input x is positive, then the absolute value leaves it alone. For example, $|-5| = -(-5) = 5$ and $|300| = 300$.
Like quadratic functions, the minimum (or maximum) point of an absolute value function is called the *vertex*.

Graphically, the absolute value function says "for all negative values of x, I am the line $y = -x$, and for all positive values of x, I am the line x." Let's try to understand this through a picture.
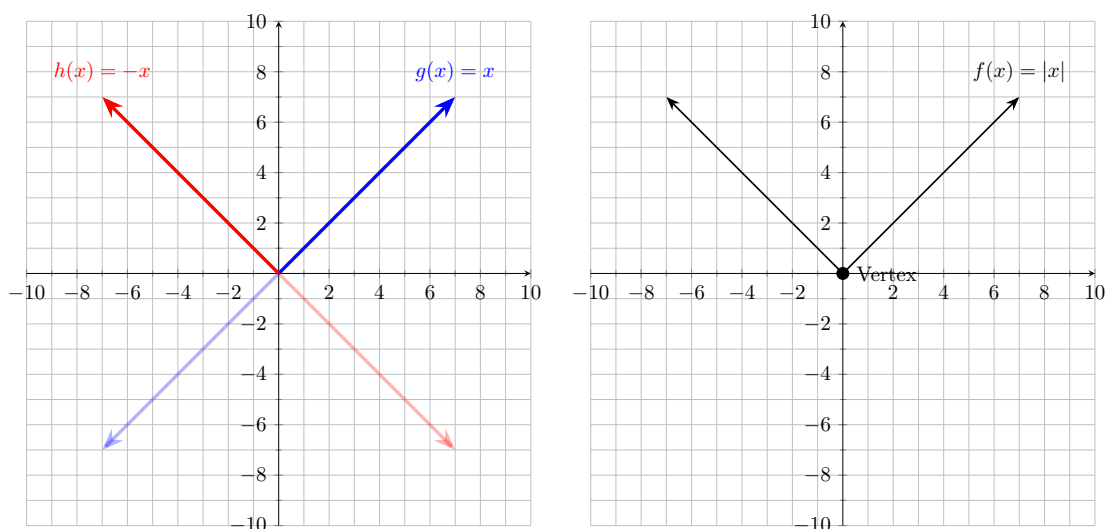
Figure 2.1: On the left we see the lines $g(x) = x$ and $h(x) = -x$, and on the positive x-axis we take $g(x) = x$, while on the negative x-axis we take $h(x) = -x$. On the right we have a graph of the function $f(x) = |x|$.

Absolute value functions can come in many different flavors. For example, we may have

1. $f(x) = 3|x|$

2. $f(x) = |x + 2|$

3. $f(x) = |x| + 1$

4. $f(x) = 2|x + 5| - 1$

We can apply what we know about function transformations to graph an absolute value function. In the general form

$$f(x) = a|x - h| + k,$$

we have that (h, k) is the vertex. As usual, a will vertically stretch or compress the function, and reflect it if negative. For example, $f(x) = 2|x + 5| - 1$ is $|x|$ vertically stretched by a factor of 2, shifted to the left 5 units, and shifted down 1 unit.

Given an absolute value function, one first locates the vertex (h, k). Then, find an additional point to the left or right of the vertex. Since we know the absolute value function is **symmetric**, if we have a point to the right of the vertex, we can also quickly find a point to the left. Let's do a quick example.

Example 2.2: Absolute Value Function

Let $f(x) = -2|x+3|+4$. We will graph this function. First we vertically stretch by a factor of 2, and reflect since a is negative (-2). Then we locate the vertex; here the vertex is $(-3, 4)$. Then we can find an extra point by plugging in either -2 or -4. Doing so will yield $(-2, 2)$ and $(-4, 2)$.
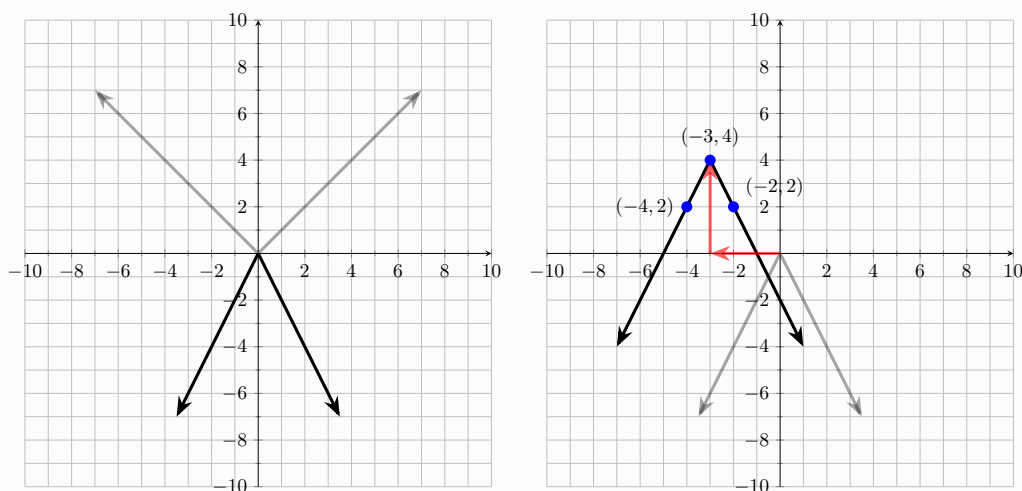
Play with the following Desmos link to understand how transforming an absolute value function works: https://www.desmos.com/calculator/cu0rctptin.

2.2 Piecewise Functions

Review; functions, linear and quadratics, domains and ranges.

Definition 2.3: Piecewise Function

A function is a *piecewise function* (some times referred to as a *piecewise defined* function) is a function defined by multiple sub functions. The sub functions are defined on segments of the domain.

Let's understand what a piecewise function is through a basic example.

Example 2.4: Piecewise Functions – Easy

Suppose that the function f is defined, for all real numbers, as follows.

$$f(x) = \begin{cases} 2 & \text{if } x \neq 0 \\ -5 & \text{if } x = 0 \end{cases}$$

1. Evaluate the function f for $x = -1$, $x = 0$, and $x = 1$.

2. Graph the function f.

Solution. To evaluate a piecewise function for a given input, we must first determine where that input is in the domain. We have $f(-1) = 2$, since $-1 \neq 0$. Continuing, $f(0) = -5$, since $0 = 0$. Lastly, $f(1) = 2$, since $1 \neq 0$. Now we can move on to graphing f.
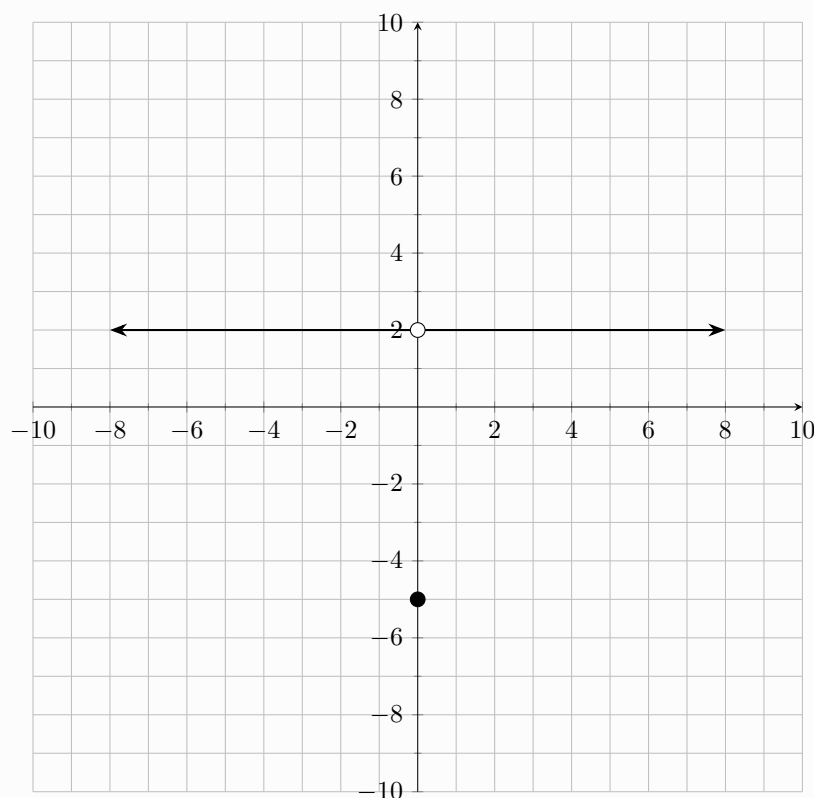
Notice that the function f outputs the value 2 for any input, except for $x = 0$, for which it outputs 5. Graphically, we draw the line $y = 2$, with an empty circle at $(0, 2)$, and a filled in circle at $(0, -5)$. This indicates the behavior of the function at $x = 0$, as we described; the function does not take on the value 2 at $x = 0$, but rather it takes on the value -5 at $x = 0$.

In order to understand and graph more complicated piecewise functions, the ability to evaluate them at given values is important. In your homework assignments, you may see some piecewise functions like the previous example, or ones where each piece of the function is a horizontal line. Let's do a more complicated example where there are three pieces (horizontal line, sloped line, and quadratic.)

Example 2.5: Piecewise Functions – Hard

Suppose the function q is defined, for all real numbers, as follows.

$$q(x) = \begin{cases} 7 & \text{if } x < -4 \\ x^2 - 9 & \text{if } -4 \leq x \leq 4 \\ 2x - 1 & \text{if } x > 4 \end{cases}$$

Graph the function q. What is the domain and range of q?

Solution. Notice that our function q is three pieces, a constant function from $-\infty$ to -4, a quadratic function between -4 and 4, and a sloped line from 4 to ∞. The constant function is straightforward, we draw the horizontal line $y = 7$ from the left up until -4, where we put place open circle, since $q(x) = 7$ only when $x < 4$.
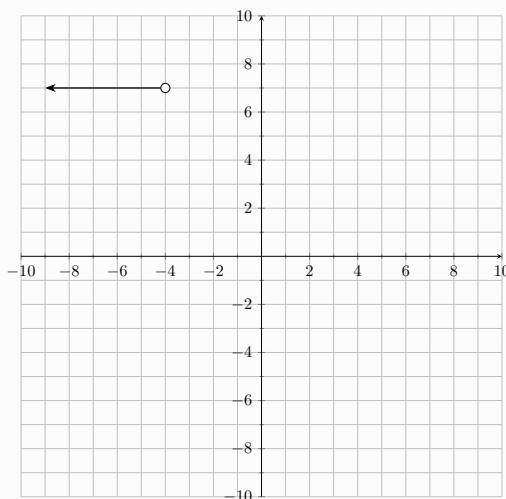
Next, we will handle the quadratic part: $y = x^2 - 9$. We know that this quadratic has vertex $(0, -9)$. We can tell this in three different ways:

1. $y = x^2 - 9$ is x^2 shifted down by 9.

2. The x coordinate of the vertex (h, k) of any quadratic function is $h = \frac{-b}{2a}$. In this case $b = 0$ and $a = 1$. So $h = 0$, and plugging this in we get $(h, k) = (0, -9)$.

3. Recall the vertex standard form: $a(x - h)^2 + k$, where the vertex is (h, k). Here we have $y = 1(x - 0)^2 + (-9)$. So the vertex $(h, k) = (0, -9)$.

Now that we have the vertex, we need to evaluate the quadratic on the edge points (namely, $x = -4$ and $x = 4$). Doing so gives 7 for each point: $(4)^2 - 9 = (-4)^2 - 9 = 7$. Since $q(x) = x^2 - 9$ for $-4 \leq x \leq 4$, we put filled circles on $(-4, 7)$ and $(4, 7)$, and using our vertex we are now able to draw in the partial parabola.
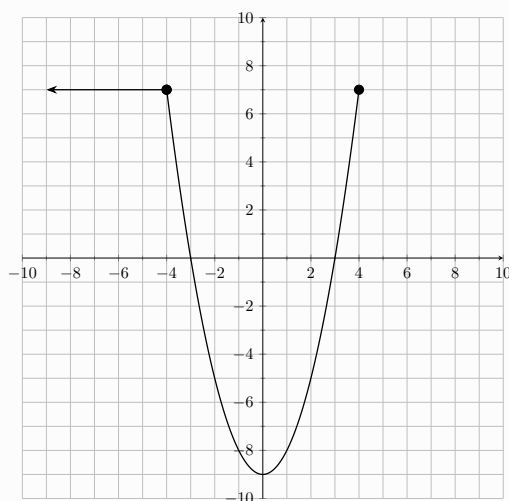
Notice that the open circle we had previously is now filled in! This is because $x^2 - 9 = 7$ at $x = -4$, that is, the horizontal piece and the quadratic piece both share a point.

To finish off, we have $q(x) = 2x - 1$ for $x > 4$. We need at least two points to plot a line, one of them having $x = 4$. Remember, if we want to calculate $q(4)$, we use the quadratic piece; plugging $x = 4$ into $2x - 1$ is only done to find where the open circle is located. Plugging in $x = 4$ to $2x - 1$ we obtain $2(4) - 1 = 7$. So we would place an open circle at $(4, 7)$, but since there is already a filled in circle, we leave it as is ($x^2 - 9$ accounts for $(4, 7)$). Using $x = 5$ for our second point, we get $2(5) - 1 = 9$.
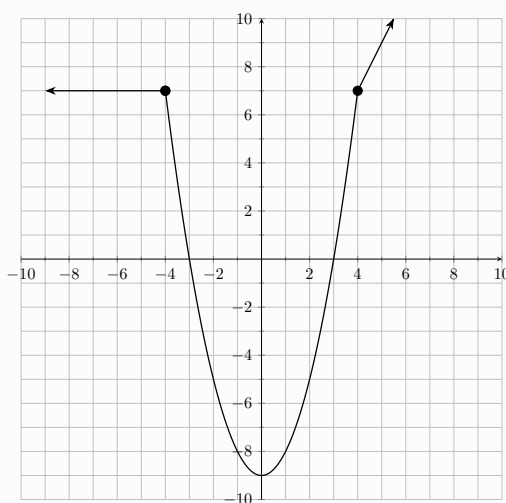
We have graphed the piecewise function in its entirety. Now what about the **domain and range**?

Notice that we can plug in any real number into q. Conceptually, in interval notation, we can see that for $(-\infty, -4)$, $f(x) = 7$ and for $(4, \infty)$, $q(x) = 2x - 1$. The quadratic closes the gap by taking on the values $[-4, 4]$. So, in total the domain is $(-\infty, -4) \cup [-4, 4] \cup (4, \infty) = (-\infty, \infty)$. We can figure out the range quickly. The quadratic portion has a minimum value of -9, and increases until 7. The sloped line portion starts at $(4, 7)$, and increases to infinity. So we have the range of q as $[-9, \infty)$.

Notice that the function we obtained in the previous example is *continuous.*

Definition 2.6: Continuity

A function is *continuous* if you can draw its graph without lifting your pen off paper. Some scenarios where a function is not continuous are if it has a hole, or a jump.

We will finish off with a modification of the previous example, that will have a more complicated domain and range.


Example 2.7: Piecewise Functions – Hard

Suppose the function f is defined, for all real numbers, as follows.

$$f(x) = \begin{cases} x + 10 & \text{if } x < -5 \\ x^2 - 4 & \text{if } -3 < x \le 3 \\ \frac{1}{2}x + 5 & \text{if } x > 4 \end{cases}$$

Graph the function f. What is the domain and range of f?
Solution. Immediately, one should notice that the domain will be broken into pieces: we have three pieces, each living on $(-\infty, -5)$, $(-3, 3]$, and $(4, \infty)$. Using what we now know, we should be able to graph this function. (Evaluating piecewise functions, picking the proper points to evaluate, and placing open or filled circles, calculating the number of points needed to plot a piece, etc.).
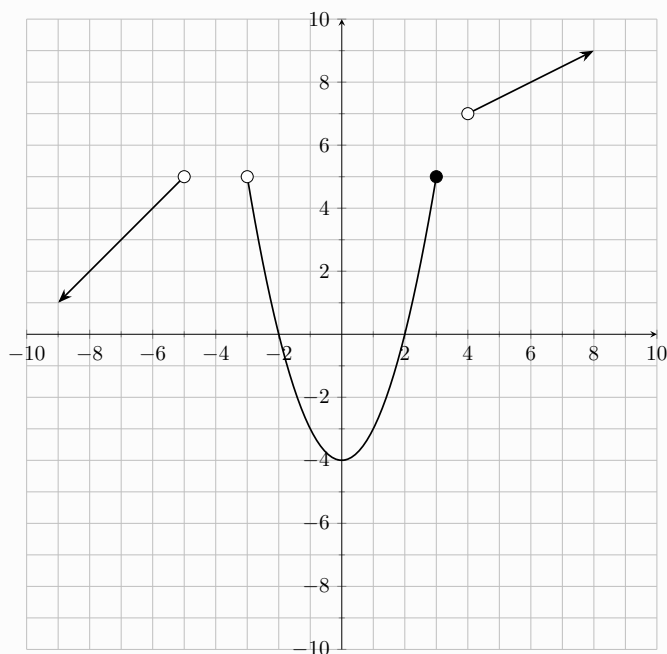
Looking at the y-axis, there are also gaps in the range. Can you see why the domain of f is $(-\infty, -5) \cup (-3, 3] \cup (4, \infty)$ and the range is $(-\infty, 5] \cup (7, \infty)$?

The function in the previous example is **not** continuous. There are two gaps, one between each piece of the piecewise function from left to right.

Note that you may encounter some differences in your homework problems. Some times plotting a quadratic piece is not as symmetric as these examples. For example,

you may need to graph the function

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 7 - 5x & \text{if } x > 1 \end{cases}$$

which looks like
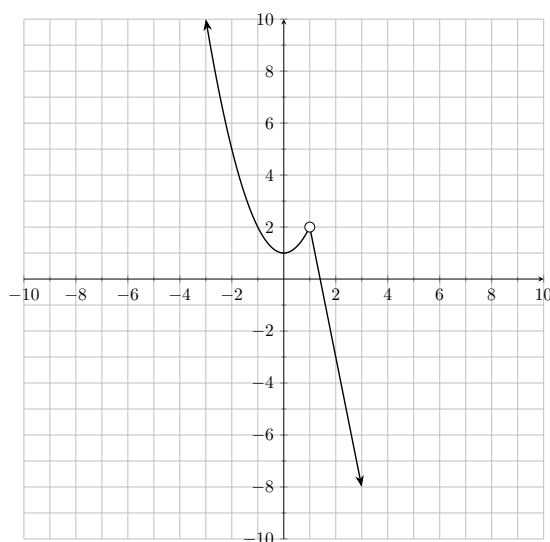
This function is not continuous, however we can fill in the hole by adding a "piece" to f. That is, we can define $f(1) = 2$, and $f(x) = 2$ only when $x = 1$. Then the graph would look like:
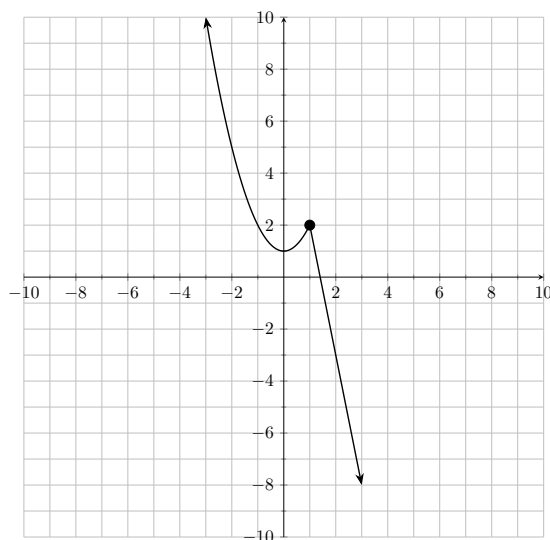
**Summary
Chapter 2**

List of things you need to know.

- What is a function? How to work with functions (their names, inputs, outputs, etc.) What is the vertical line test?

- What is a piecewise function?
    - How to evaluate piecewise functions.
    - How to graph piecewise functions. (This involves graphing lines and parabolas.) Finding the proper points to plot, and where to put open and filled circles.
    - What is the domain and range of a given piecewise function.
    - Is a given piecewise function continuous – A function is continuous if you can trace the function without lifting off the paper (there are no holes, jumps, or gaps.)

3 Polynomials

3.1 What are polynomials?

We begin with the definition of a polynomial in the context of this class, and some of the language used when dealing with polynomials.

Definition 3.1: Polynomial

A polynomial is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Here the *variable* is x and the *coefficients* are the a_i. The exponent of any term in a polynomial must be a nonnegative integer. (We could say positive integer, but 0 can be an exponent.)

The *degree* of a polynomial is n, the largest exponent. The *leading coefficient*, some times abbreviated L.C., is a_n, the coefficient of the term with the largest exponent. The *constant* term is a_0. The *leading term* is $a_n x^n$; the leading term is the full highest degree term.

Do not let the fancy definition confuse you. Let's look at some examples.

Example 3.2: A Quadratic Polynomial: $a_2 x^2 + a_1 x + a_0$.

Let's look at a familiar example: quadratic polynomials. A quadratic polynomial is a polynomial with degree 2. You may have seen the general form written as $ax^2 + bx + c$, this is fine.

Here are some examples of quadratic polynomials.

- $2x^2 + 4x + 1$. The L.C. is 2, the degree is 2, the constant term is 1 and the leading term is $2x^2$.

- $x^2 - 1$. The L.C. is 1, the degree is 2. Notice, one can say that the coefficient of x is 0, and you can instead write this polynomial as $x^2 + 0x - 1$. This line

of thinking will come in handy when we discuss polynomial long division. The constant term is -1 and the leading term is x^2.

Other examples of polynomials:

- $x^3 + 2x + 1$

- $45x^{100} + x^{23} - 98x^2 - 786$

- $-x + 1$

- 5

Notice that lines are polynomial functions, and further, a number itself is a polynomial. For example, the number 5 can be written as $5x^0$, that is, any number by itself is a polynomial of degree 0.
We must be careful, however. Here are some examples of expressions which are **not** polynomials.

- $\dfrac{1}{x-1}$

- $x^2 + x^{\frac{1}{2}} + 1$

- $x^{-3} + 3x^2 - 6x + 8$

- $x^2 - x^{0.3} + 5$

A polynomial can also be written in *factored form.* We will see this a bit later when we talk about zeros, or roots. For now, let us see some examples of polynomials in factored form, and how we can get information from them.

Example 3.3: Polynomial in Factored Form.

Consider the polynomial

$$-2x \left(\frac{1}{2}x + 1 \right) (x - 2)(x + 3).$$

Given a polynomial in this form, we want to be able to know certain properties. Here we will focus on the **leading term.** In order to find the leading term of a polynomial in factored form, we multiply the highest term in each factor. Do not fall for the trap; there is a factor $-2x$, which you could write as $(-2x + 0)$

to not confuse you. So the leading term is:

$$-2x \cdot \frac{1}{2}x \cdot x \cdot x = -x^4.$$

By finding the leading term like this, we will easily be able to find the leading coefficient and degree of a polynomial.

It is not always so trivial, however. Suppose instead we have

$$-2x \left(\frac{1}{2}x + 1 \right)^2 (x - 2)(3x + 1)^3.$$

We must be careful in finding the leading term, as now some of our factors have exponents. However, this polynomial can be rewritten into something we know how to handle:

$$- 2x \left(\frac{1}{2}x + 1 \right)^2 (x - 2)(3x + 1)^3$$

$$- 2x \left(\frac{1}{2}x + 1 \right) \left(\frac{1}{2}x + 1 \right)(x - 2)(3x + 1)(3x + 1)(3x + 1)$$

The leading term of the new polynomial is then

$$-2x \cdot \frac{1}{2}x \cdot \frac{1}{2}x \cdot x \cdot 3x \cdot 3x \cdot 3x = -2x \cdot \left(\frac{1}{2} \right)^2 x^2 \cdot x \cdot 3^3 x^3$$

$$= -2x \cdot \frac{1}{4}x^2 \cdot x \cdot 27x^3$$

$$= -\frac{27}{2}x^7$$

So, to get the terms we need to multiply, we take the highest degree term **in each factor and raise it to the exponent of the factor. Then, multiply all the terms.** Let's look at one final example.

Consider the polynomial

$$3x^4(2x + 1)^3(x + 5)^2.$$

To find the leading term, we multiply

$$3x^2 \cdot (2x)^3 \cdot x^2 = 3x^2 \cdot 8x^3 \cdot x^2$$

$$= 24x^9$$

With that, we will move on to polynomial functions.

3.2 Polynomial Functions - Roots and End Behavior

By now we should be familiar with polynomial functions and how to graph them, such as lines (polynomials of degree 1) and quadratics (polynomials of degree 2). Further, in the case of quadratic functions, we should know how to factor them into linear factors. For example, $f(x) = x^2 + 4x + 4 = (x+2)(x+2)$, and $g(x) = 3x^2 + 13x + 10 = (3x + 10)(x + 1)$. The first can be factored by finding two numbers which add and multiply to 4, while the second requires the use heavier machinery (either the ac method, or quadratic formula, for example.)

The goal of this section is to be able to sketch a graph of a given polynomial in factored form.

Definition 3.4: Zeros or Roots

A *zero* or *root* of a polynomial is a number α so that the polynomial evaluated at α is 0. The *multiplicity* of a root is the exponent of the factor $(x - \alpha)$ in the factored form of the polynomial.

Another seemingly terrifying definition, but something we can work through with some examples. We will see later how this comes together visually. For example, the roots or zeroes are the x-intercepts of polynomial functions.

Example 3.5: Roots and Multiplicities

Let $f(x) = x^2 + 4x + 4$. We mentioned earlier that this polynomial factors into $(x + 2)(x + 2)$, or $(x + 2)^2$. So we can rewrite $f(x) = (x + 2)^2$. Note that $f(x) = (x + 2)^2 = 0$ if and only if $x = -2$. That is, -2 is a root of f. The multiplicity of -2 is 2, since the exponent of $(x - (-2)) = (x + 2)$ in the factored form of f is 2.

Let $g(x) = x^3 + 6x^2 + 8x$. This is a polynomial of degree 3, but it is still one we can factor.

$$\begin{aligned} g(x) &= x^3 + 6x^2 + 8x \\ &= x(x^2 + 6x + 8) \\ &= x(x + 2)(x + 4) \end{aligned}$$

So $g(x) = x(x + 2)(x + 4)$. To find the roots, we set $x(x + 2)(x + 4) = 0$, and

---

the left hand side can only be 0 if and only if one of the three factors is 0. We can make the table:

Factor	Root	Multiplicity
x	0	1
$x + 2$	-2	1
$x + 4$	-4	1

Here x is a factor (can be written as $(x - 0)$), and each root has multiplicity 1. Some times a polynomial comes in factored form. For example, consider the following polynomial:

$$h(x) = -2x^4(x + 1)^2(x - 2)(x + 5)^3.$$

One may ask what the degree of this polynomial is. To find the degree, we simply add the exponent of each factor (or the multiplicities of all the roots.)

Factor	Root	Multiplicity
$(x - 0)^4$	0	4
$(x + 1)^2$	-1	2
$x - 2$	2	1
$(x + 5)^3$	-5	3

The degree of $h(x)$ is $4 + 2 + 1 + 3 = 10$.

A harder example is the following: Find all x and y intercepts of the function $f(x) = x^3 + 5x^2 - 4x - 20$. This requires some creativity and experience. To find the x-intercepts, we must factor:

$$\begin{aligned}
x^3 + 5x^2 - 4x - 20 &= (x^3 + 5x^2) + (-4x - 20) \\
&= x^2(x + 5) - 4(x + 5) \\
&= (x^2 - 4)(x + 5) \\
&= (x - 2)(x + 2)(x + 5)
\end{aligned}$$

Since the degree of this polynomial is 3, we cannot rely on our methods for factoring quadratics (ac-method, quadratic formula, etc.) Instead, we notice that we can conveniently group two portions of the polynomial, and factoring from those portions leads to a nice result. The quadratic factor $x^2 - 4$ can be factored by methods we already know. We can now find the roots by setting each factor equal to 0, giving us $x = -2, 2, -5$, each with multiplicity 1. For the y-intercept, we evaluate $f(0)$ (the y-axis has $x = 0$.) So $f(0) = (0)^3 + 5(0)^2 - 4(0) - 20 = -20$. So the y-intercept is $(0, -20)$.

The multiplicity of a root α tells us how the function behaves around the $x - axis$, at $x = \alpha$. Simply put,

1. If a root α has even multiplicity, the function bounces on the x-axis (does not cross.)

2. If a root α has odd multiplicity, the function crosses the x-axis.

Example 3.6: Multiplicity

Let $f(x) = -(x+1)^2(x-2)(x+2)$. The degree of f is 4, and the roots are

Factor	Root	Multiplicity	Behavior
$(x+1)^2$	-1	2	Bounce
$x+2$	-2	1	Cross
$x-2$	2	1	Cross

Sure enough, we can graph the function and examine the roots. We see that indeed, the function crosses the x-axis at $x = -2$, bounces at $x = -1$, then crosses again at $x = 2$.
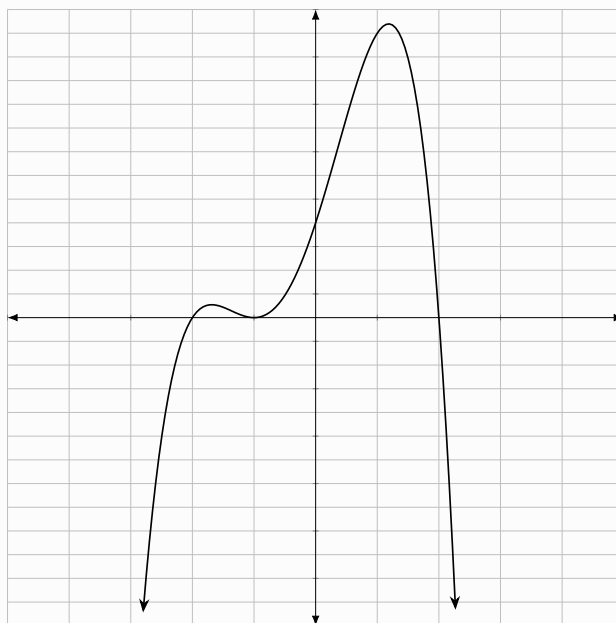
Go back to the previous example, and plot the functions (using Desmos, for example) and see how the function behaves around the roots!

Roots of polynomials are only one piece of the puzzle. We are often interested in other characteristics, such as a polynomial's *end behavior*. How does the polynomial behave when x goes toward infinity (gets very large), or negative infinity (very small.) We will try to understand this concept graphically. Consider the two basic polynomials, $f(x) = x^2$ and $g(x) = x^3$. For $f(x) = x^2$, as $x \to -\infty$ (moves far left), f goes up
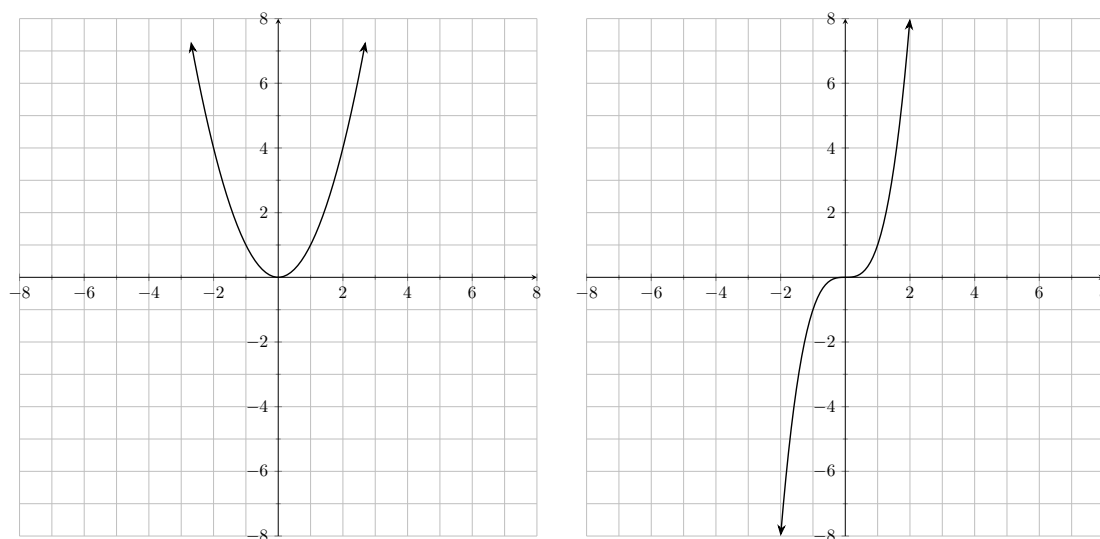
Figure 3.1: The functions $f(x) = x^2$ (left) and $g(x) = x^3$ (right).

to infinity, and similarly when $x \to \infty$, $f(x)$ goes up to infinity as well. Here we say f **rises to the left and rises to the right.**

On the other hand, $g(x) = x^3$ displays different end behavior; as x becomes really negative, so does $g(x)$, and when x becomes really positive, so does $g(x)$. We say g **falls to the left and rises to the right.** Take a look at Figure 3.1.

One last thing to note is the L.C. Recall from your previous knowledge of function transformations, what happens when the L.C. is negative (reflection about the x-axis.)

We can generalize this behavior; given any polynomial function, we will be able to tell how the output behaves as the input becomes really negative ($x \to -\infty$) or really positive ($x \to \infty$). This will depend on two things: if the degree is odd or even, and if the L.C. is positive or negative. We have the following rules:

Positive L.C.　　　　　Negative L.C.

Even Degree

Odd Degree
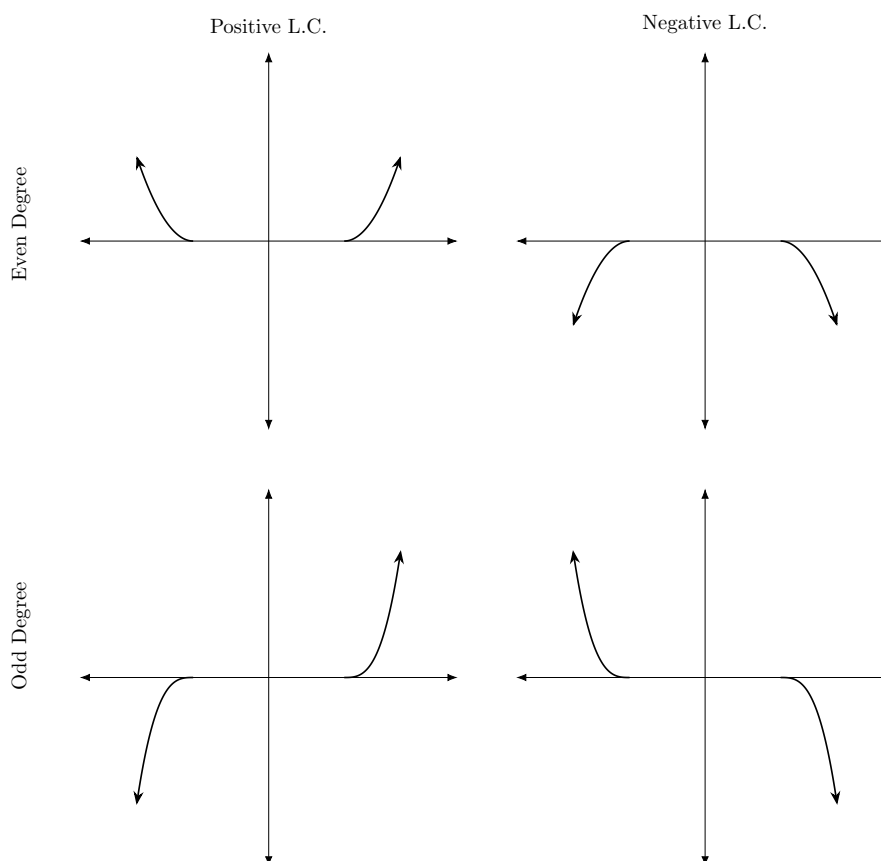
Now try going through all of the previous examples and examine their graphs to understand root behavior and end behavior. Can you fill in the following statement for each of the previous examples?: the graph _____ to the left and _____ to the right.

3.3 Graphing a Polynomial Function

At this point, we now have all the knowledge we need to *sketch* the graph of a polynomial. Do the following example. Let $f(x) = -(x + 5)^2(x - 1)(x + 1)^3(x - 4)^2$. Sketch a graph of f. Be sure to include all x and y intercepts. **Remember: to find the y-intercept we plug in $x = 0$.**
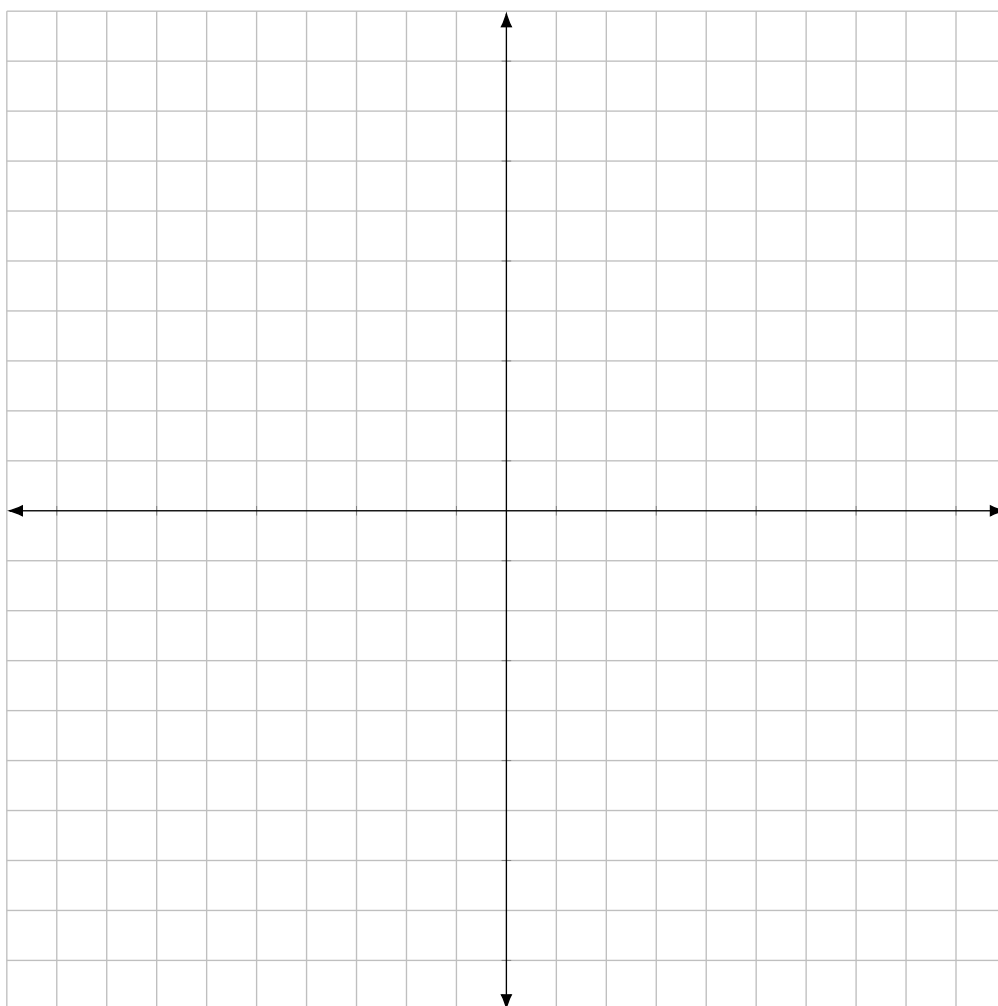
3.4 Polynomial Long Division

So far we have been only concerned with multiplying polynomials. What happens when we divide polynomials? Or, *how* do we divide polynomials? We've divided polynomials before. For example, we know that

$$\frac{x+1}{x+1} = 1.$$

A step further, we know how to simplify something like $(x^2+6x+8) \div (x+4)$, because we can write

$$\frac{x^2 + 6x + 8}{x + 4} = \frac{(x + 2)(x + 4)}{x + 4} = x + 2.$$

Here $x^2 + 6x + 8$ is the **dividend** and $x + 4$ is the **divisor**.

We can learn a procedure to compute a division of polynomials.

Example 3.7: Polynomial Long Division

Divide $24x^3 + 2x^2 - 20x - 6$ by $4x + 3$.

Solution. . We first start by setting up the division. We write the dividend and divisor like so:

$$4x + 3 \overline{\big)\quad 24x^3 \;\; + 2x^2 - 20x - 6}$$

The divisor is written on the outside, and the dividend is written on the inside. Now we look at the highest term in the dividend and divide it by the highest term of the divisor. Here, the highest term of $24x^3 + 2x^2 - 20x - 6$ is $24x^3$ and the highest term of $4x + 3$ is $4x$. Dividing them we obtain $6x^2$. So we write $6x^2$ on top. In order to stay neat, we can keep it in our minds that there is a column for each power of x. So, we write the $6x^2$ term above the $2x^2$ term like so.

$$\begin{array}{r} 6x^2 \qquad\qquad\qquad \\ 4x + 3 \overline{\big)\quad 24x^3 \;\; + 2x^2 - 20x - 6} \end{array}$$

Now we multiply the divisor, $4x + 3$, by $6x^2$ and subtract it from the dividend. So $(4x + 3) \cdot 6x^2 = 24x^3 + 18x^2$. Notice the highest term here is $24x^3$, the same highest term as the dividend. This is not a coincidence. To subtract the result, we write $-(24x^3 + 18x^2)$ underneath the dividend.

$$\begin{array}{r} 6x^2 \qquad\qquad\qquad \\ 4x + 3 \overline{\big)\quad 24x^3 \;\; + 2x^2 - 20x - 7} \\ \underline{-(24x^3 + 18x^2)\qquad\qquad} \end{array}$$

Notice the $24x^3$ term cancels, and $2x^2 - 18x^2 = -16x^2$. We bring down the other terms.

$$\begin{array}{r} 6x^2 \qquad\qquad\qquad \\ 4x + 3 \overline{\big)\quad 24x^3 \;\; + 2x^2 - 20x - 7} \\ \underline{-(24x^3 + 18x^2)\qquad\qquad} \\ -16x^2 - 20x - 7 \end{array}$$

Now we repeat the process, with a new dividend $-16x^2 - 20x - 7$. **We keep repeating the process until the degree of the dividend is lower than the degree of the divisor.** Since the divisor $4x + 3$ has degree 1, this means we stop when the degree of the dividend reaches 0. Continuing on, we compute $\frac{-16x^2}{4x} = -4x$ and repeat.

$$
\begin{array}{r}
6x^2 - 4x \\
4x + 3 \overline{)\; 24x^3 + 2x^2 - 20x - 7} \\
\underline{-(24x^3 + 18x^2)} \\
-16x^2 - 20x - 7 \\
\underline{-(-16x^2 - 12x)} \\
-8x - 7
\end{array}
$$

Since the degree of $-8x - 7$ is not less than the degree of $4x + 3$, we continue.

$$
\begin{array}{r}
6x^2 - 4x - 2 \\
4x + 3 \overline{)\; 24x^3 + 2x^2 - 20x - 7} \\
\underline{-(24x^3 + 18x^2)} \\
-16x^2 - 20x - 7 \\
\underline{-(-16x^2 - 12x)} \\
-8x - 7 \\
\underline{-(-8x - 6)} \\
-1
\end{array}
$$

Now we stop, since the degree of -1 is 0, which is less than $4x + 3$. Here, $6x^2 - 4x - 2$ is the **quotient** and -1 is the **remainder**.

Here is an alternate way to write your work, where the negative sign is distributed at the subtraction steps and terms are added instead. If this does not resonate with you, you may disregard it.

$$
\begin{array}{r}
6x^2 \quad - 4x - 2 \\
4x + 3 \overline{\smash{)}\ 24x^3\ + 2x^2 - 20x - 7} \\
\underline{-\ 24x^3 - 18x^2} \\
-16x^2 - 20x \\
\underline{16x^2 + 12x} \\
-8x - 7 \\
\underline{8x + 6} \\
-1
\end{array}
$$

The answer is then written in the form

$$
\text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.
$$

In this case,

$$
\frac{24x^3 + 2x^2 - 20x - 7}{4x + 3} = 6x^2 - 4x - 2 + \frac{-1}{4x + 3}.
$$

The remainder is not always a polynomial of degree 0. For example, try to work out $(2x^3 + x + 1) \div (x^2 + 1)$. You should get a quotient of $2x$ and a remainder of $-x + 1$.

$$
x^2 + 1 \overline{\smash{)}\ \ 2x^3\ + x + 1}
$$

So the steps to long division are:

1. Write the dividend underneath and the divisor outside.

2. Divide the highest degree term of the inside by the highest degree term of the outside.

3. Multiply the result of (2) by the divisor and subtract it from the dividend.

4. If the degree of the dividend is **less than or equal to** the degree of the divisor, then continue. Once the degree of the dividend is strictly less than the degree of the divisor, we stop and have a remainder (could be 0, could be a polynomial.)

> **Summary**
> **Chapter 3**
>
> List of things you need to know.
>
> - A polynomial is an expression involving a variable, like x, with exponents that are non-negative integers. For example, $x^4 + x + 1$ is a polynomial.
>
> - The degree of a polynomial is the highest exponent. For example, the degree of $x^4 + x + 1$ is 4.
>
> - The leading coefficient (L.C.) is the coefficient of the term with the highest exponent. For example, the L.C. of $x^5 + 2x^3 + 1$ is 1.
>
> - The leading term is the term with the highest degree. For example, the leading term of $2x^3 + x + 1$ is $2x^3$.
>   - To find the leading term of a polynomial in factored form, multiply the highest degree term of each factor. For example, the leading term of $4x^2(x-1)^4(2x+3)^2$ is $4x^2 \cdot x^4 \cdot (2x)^2 = 16x^8$.
>
> - The constant term is the coefficient of the term x^0, usually written as just a number. For example, the constant term of $x^2 + 6x + 12$ is 12.
>
> - Roots, or zeros of a polynomial are numbers that, when plugged into a polynomial, give 0. They are the x-intercepts in the graph of a polynomial.
>   - The multiplicity of a root α is the exponent of the factor $(x - \alpha)$ in the factored form of the polynomial. If the multiplicity is even, the function bounces on the x-axis at the root. If the multiplicity is odd, it crosses.

- The end behavior of a polynomial is dictated by the degree and L.C. If the degree is **even**, then

    - Positive L.C. $\searcharrow\nearrow$, the graph **rises** to the left and **rises** to the right.

    - Negative L.C. $\nearrow\searrow$, the graph **falls** to the left and **falls** to the right.

  If the degree is **odd**, then

    - Positive L.C. $\nearrow\nearrow$, the graph **falls** to the left and **rises** to the right.

    - Negative L.C. $\nwarrow\searrow$, , the graph **rises** to the left and **falls** to the right.

- The domain of any polynomial function is $(-\infty, \infty)$, or all real numbers.

- Polynomial long division algorithm.

4 Rational

In this chapter we will go over how to solve rational equations, rational inequalities, and how to understand the behavior of rational functions.

4.1 Rational Equations

Consider the following equation:

$$\frac{1}{x} = 0. \tag{4.1}$$

Equation 4.1 is a *rational* equation. Notice that the equation has **no solution**; any value other than 0 for x will lead to a nonzero number, and $x = 0$ cannot be a solution, since $\frac{1}{0}$ is undefined. In this case we say that $x = 0$ is an *excluded value*; a value that cannot be a solution to the original equation. In a rational equation, the excluded values will come from setting the denominators equal to 0.

Of course, it is not always the case that rational equations have no solutions. However, we must be weary of excluded values, that is, we may obtain a plausible solution x through the use of some algebra techniques, however x may cause some term in the original equation to be undefined. This is very similar to equations with radicals. Let's see an example.

Example 4.1: Rational Equation – Excluded Values Matter

Solve the following equation for x:

$$-\frac{2}{x+5} = -\frac{8}{4x+20} + 3.$$

Solution. The first step is to always clear the denominators (by using the least common denominator, or LCD, of all the fractions.) Notice that we can simplify one of the denominators on the right hand side by factoring. Then we can clean

---

up a bit:

$$-\frac{2}{x+5} = -\frac{8}{4(x+5)} + 3$$

$$-\frac{2}{x+5} = -\frac{2}{x+5} + 3$$

Now that the denominators, namely $(x+5)$, are the same on both sides, we can multiply across by $(x+5)$ for some nice cancellation. It is at this point we note that $x+5 = 0 \implies x = -5$, and so -5 is an excluded value. That is, after the cancellation we can continue to solve

$$(x+5)\left(-\frac{2}{x+5}\right) = (x+5)\left(-\frac{2}{x+5} + 3\right)$$
$$-2 = -2 + 3(x+5)$$
$$0 = 3(x+5)$$
$$0 = x+5$$
$$-5 = x.$$

By our algebra, we've deduced that $x = -5$ is a possible solution. However, since $x = -5$ is an excluded value, this cannot be the case. Therefore the original equation has **no solution.** Try to plug in -5 into the original equation.

So far we've only seen rational equations with linear denominators. We can have equations with multiple solutions, and multiple excluded values. Let's look at some more difficult examples.

Example 4.2: Ration Equation – Quadratic Denominator

Solve the following equation for x:

$$\frac{1}{x^2-4} - \frac{4}{x-2} = \frac{1}{x+2}$$

Solution. Again, the first thing we want to do is clear denominators. We can factor the quadratic denominator $x^2 - 4$ to start simplifying.

$$\frac{1}{(x-2)(x+2)} - \frac{4}{x-2} = \frac{1}{x+2}$$

To clear the denominators, we must multiply by the LCD. Notice that we can multiply across by $(x + 2)$, then by $(x - 2)$, canceling out all denominators like so:

$$(x + 2)(x - 2) \left(\frac{1}{(x - 2)(x + 2)} - \frac{4}{x - 2} \right) = (x + 2)(x - 2) \left(\frac{1}{x + 2} \right)$$

$$\frac{\cancel{(x + 2)}\cancel{(x - 2)}}{\cancel{(x - 2)}\cancel{(x + 2)}} - \frac{4(x + 2)\cancel{(x - 2)}}{\cancel{x - 2}} = \frac{\cancel{(x + 2)}(x - 2)}{\cancel{x + 2}}$$

$$1 - 4(x + 2) = x - 2$$

Note our excluded values are $x = 2$ and $x = -2$. Solving this equation gives $x = -1$, and since this value is not one of the excluded values, $x = -1$ is a solution. Plug in $x = -1$ to the original equation to check.

Example 4.3: Rational Equation – All Together

Solve the following equation for x:

$$\frac{x}{x - 7} - \frac{28}{x^2 - 49} = \frac{2}{x + 7}.$$

Solution. Following all the steps highlighted earlier, we can solve for x.

$$\frac{x}{x-7} - \frac{28}{x^2-49} = \frac{2}{x+7}$$

$$\frac{x}{x-7} - \frac{28}{(x-7)(x+7)} = \frac{2}{x+7} \qquad \text{(Factor)}$$

$$(x-7)(x+7)\left(\frac{x}{x-7} - \frac{28}{(x-7)(x+7)}\right) = (x-7)(x+7)\left(\frac{2}{x+7}\right) \quad \text{(LCD)}$$

$$\frac{x(x-7)(x+7)}{x-7} - \frac{28(x-7)(x+7)}{(x-7)(x+7)} = \frac{2(x-7)(x+7)}{x+7} \qquad \text{(Distribute)}$$

$$x(x+7) - 28 = 2(x-7) \qquad \text{(Cancel)}$$

$$x^2 + 7x - 28 = 2x - 14$$

$$x^2 + 5x - 14 = 0$$

Note the excluded values that came from factors in the denominators: $(x+7)$ and $(x-7)$ give $x = 7$ and $x = -7$ as excluded values.

Solving the rational equation, we end up with a quadratic equation which we can solve by factoring. We get $(x+7)(x-2) = 0$, giving two solutions: $x = -7$ and $x = 2$. Since one of our solutions is an excluded value, the final solution to the original equation is $x = 2$.

4.2 Rational Inequalities

Now we focus our attention on rational inequalities. For example, there are infinitely many solutions to

$$\frac{1}{x} > 0.$$

The x values that make this inequality true are all positive real numbers, or $x > 0$. We can express this in interval notation as $(0, \infty)$, or on a number line:
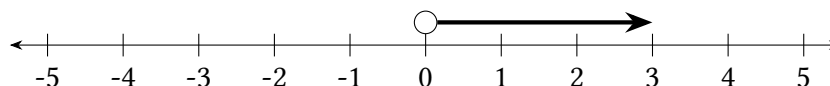
Notice that we have an open circle at 0 to indicate that x is never 0.

Like rational equations, though, these can be much more complicated.

Example 4.4: Rational Inequality – All Together

Solve the following inequality for x:

$$\frac{-2}{5-x} \geq \frac{1}{x-2}.$$

Solution. First, we move everything to one side. Let's move the term on the right to the left:

$$\frac{-2}{5-x} - \frac{1}{x-2} \geq 0.$$

Now, we want to combine both fractions. We do so by finding the least common denominator. One simple way to think of this is to look at the denominator of each term and ask "what is this term missing in the denominator that is present in the other terms." For example, the term $\frac{-2}{5-x}$ is missing a factor of $x-2$ in the denominator, and the term $-\frac{1}{x-2}$ is missing a factor of $5-x$ in the denominator. To add in the missing factors, we have to make sure we multiply both the numerator and the denominator. We have

$$\frac{-2}{5-x} \cdot \frac{(x-2)}{(x-2)} - \frac{1}{x-2} \cdot \frac{(5-x)}{(5-x)} \geq 0.$$

Note, when dealing with rational inequalities, for the most part we want to keep things in factored form. Now that the denominators are the same for each term, we can combine the terms:

$$\frac{-2(x-2) - (5-x)}{(5-x)(x-2)} \geq 0.$$

Simplify the numerator, so that we arrive at

$$\frac{-x-1}{(5-x)(x-2)} \geq 0.$$

There is a method to the madness here. We now have the inequality in the form $\frac{\blacksquare}{\blacksquare} \geq 0$. Solving this kind of inequality, in words, is solving the question "when is this fraction positive," while paying special attention to whether or not we are also equal to 0. Remember: a fraction is positive when the numerator and denominator are both positive or both negative. A fraction is negative when the numerator and denominator have differing sign.

To solve this inequality, we set the numerator and denominator equal to 0 to find **critical values**. Setting the numerator equal to 0 we have $-x - 1 = 0 \implies x = -1$. Setting the denominator equal to 0 we have $(5 - x)(x - 2) = 0$, which is 0 when $5 - x = 0$ or when $x - 2 = 0$. From this we get $x = 5$ and $x = 2$ from the denominator. Now we set up our labeled sign chart like so:
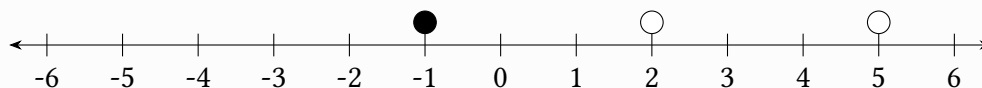
Notice, the critical values coming from the numerator have a closed circle, while the critical values coming from the denominator have an open circle. It is always the case that critical values from the denominator yield an open circle, since the denominator of a fraction can never equal 0 (undefined.) However for the critical values from the numerator, we get an open circle if the inequality sign is $<$ or $>$, and a closed circle if the inequality sign is $\geq$ or $\leq$.

These critical values on the number line represent where each factor may possibly change sign. Now all that's left to do is check some test values in each interval $(-\infty, -1)$, $(-1, 2)$, $(2, 5)$, and $(5, \infty)$.

Interval	Test Value	Plug In	T/F
$(-\infty, -1)$	-2	$\frac{-(-2)-1}{(5-(-2))((-2)-2)} = \frac{1}{(7)(-4)}$	False
$(-1, 2)$	0	$\frac{-(0)-1}{(5-(0))((0)-2)} = \frac{-1}{(5)(-2)}$	True
$(2, 5)$	3	$\frac{-(3)-1}{(5-(3))((3)-2)} = \frac{-4}{(2)(1)}$	False
$(5, \infty)$	6	$\frac{-(6)-1}{(5-(6))((6)-2)} = \frac{-7}{(-1)(4)}$	True

We find test values in each interval, then plug them and check if the equality is true. After we find which intervals satisfy the inequality, we can fill in the labeled sign chart.
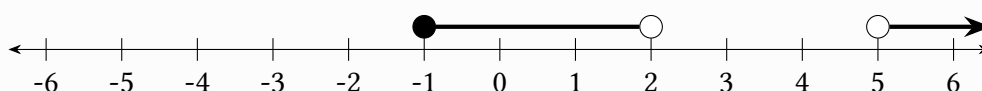
We are done!

So, the steps to solving a rational inequality are as follows:

1. Move everything to one side so that the inequality has 0 one one side and terms on the other.

2. Combine the terms by finding the least common denominator. Give each denominator the factor(s) it is missing from the other denominators.

3. We end up with an inequality like $\frac{\blacksquare}{\blacksquare} \geq 0$ or $\frac{\blacksquare}{\blacksquare} < 0$.

4. Once combined, set the numerator and denominator equal to 0 to find **critical values.**

   - Numerator critical values: If the inequality sign is $<$ or $>$, put an open circle. If the inequality sign is $\geq$ or $\leq$, put a closed circle.

   - Denominator critical values: Always put an open circle for each critical value from the denominator.

5. There will now be several intervals on the number line. Find a test value for each interval, plug the test value into the inequality, and see if the inequality is true. **Test values cannot be critical values.**

4.3 Rational Functions

We can now talk about rational functions.

Definition 4.5: Rational Function

A *rational function* is a function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0}{b_d x^d + b_{d-1} x^{d-1} + \cdots b_1 x + b_0},$$

where $N(x)$ is a polynomial of degree n and $D(x)$ is a polynomial of degree d. In other words, a rational function is a ratio of polynomial functions.

In the definition the degree of the numerator or denominator is not specified; the numerator may have higher degree than the denominator, they may be equal, and so on. In Figure 4.1 we see some examples of rational functions and their graphs.

In this section we will examine characteristics of rational functions, including their domain and range, horizontal, vertical, and slant asymptotes, and holes. We will refer back to the functions in Figure 4.1 frequently as visual aids.
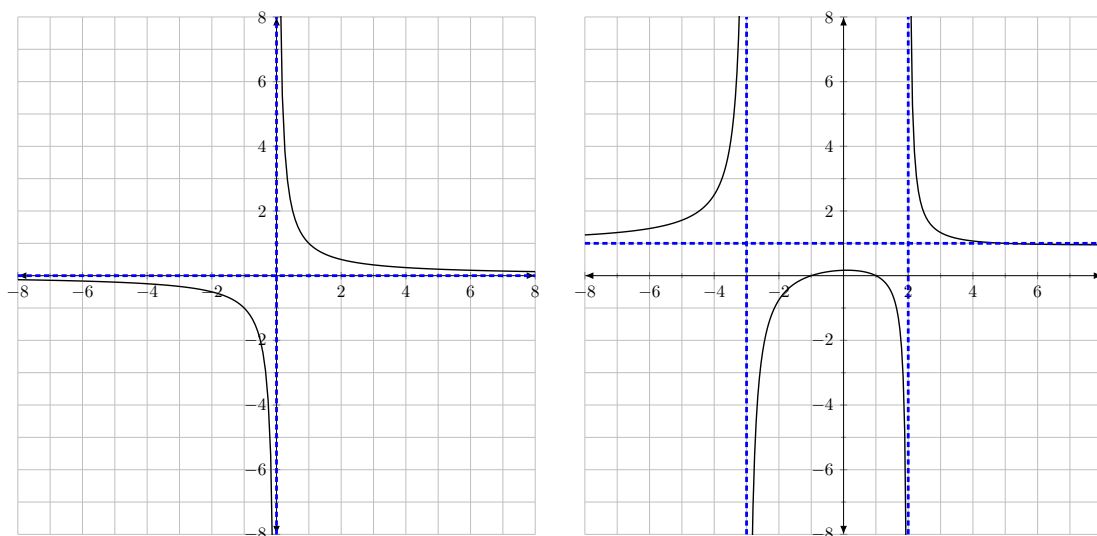
Figure 4.1: The function $f(x) = 1/x$ (left) and $g(x) = \frac{x^2-1}{x^2+x-6}$ (right). The *asymptotes* of the functions are dashed in blue.

Definition 4.6: Asymptotes

An *asymptote* is a line that acts as the limit for a curve; the curve can approach the line and get infinitely close, but they may never meet.
Let $f(x)$ be a rational function and a be a real number.

- A *vertical asymptote* of $f(x)$ is a vertical line $x = a$ where $f(a)$ is undefined.

- A *horizontal asymptote* of $f(x)$ is a horizontal line $y = a$ where $f(x) \to a$ as $x \to -\infty$ or $x \to \infty$. A horizontal asymptote may be crossed, but as soon as we observe far enough left or right, the function will get infinitely closer to $y = a$ but will not touch it.

- A *slant asymptote* (some times referred to as *oblique*) is an asymptote $y = mx + b$ with a slope.

There are three types of asymptotes, and we will learn how to find each type.

4.3.1 Vertical Asymptotes

Recall excluded values from rational equations. Excluded values were values of x which cause a term in an equation undefined. Specifically, they would cause ratios to

be undefined by yielding a 0 in the denominator. Excluded values of rational equations represent vertical asymptotes (think of the rational terms of the equations as rational functions.) **That is to say: to find vertical asymptotes of a rational function, we set the denominator equal to 0 and solve.**

Example 4.7: Vertical Asymptotes

Consider the functions in Figure 4.1.
For $f(x) = \frac{1}{x}$, we can quickly see that $x = 0$ is the only vertical asymptote. For $g(x) = \frac{x^2 - 1}{x^2 + x - 6}$, we must set the denominator equal to 0 and solve.

$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0$$

So the equations of the vertical asymptotes for $g(x)$ are $x = -3$ and $x = 2$. You may verify this by looking at Figure 4.1.

Vertical asymptotes are always written in the form of an equation $x = \alpha$. On a graph, one can visualize them as dashed vertical lines.

4.3.2 Horizontal and Slant Asymptotes

Horizontal asymptotes are more involved. Let $f(x) = \frac{N(x)}{D(x)}$ where N is a polynomial of degree n and D is a polynomial of degree d. The horizontal asymptotes will depend on the relationship between the degrees of the numerator and denominator:

- If $n > d$, there is no horizontal asymptote.

- If $n < d$, there is a horizontal asymptote at $y = 0$.

- If $n = d$, then the horizontal asymptote is a ratio of the leading coefficients of the numerator and denominator, i.e. $y = \frac{a_n}{b_d}$.

Example 4.8: Horizontal Asymptotes

Looking at $f(x) = \frac{1}{x}$, notice the degree of the numerator is 0 and the denominator is 1. Since $0 < 1$, there is a horizontal asymptote at $y = 0$.
For $g(x) = \frac{x^2 - 1}{x^2 + x - 6}$, the degree of the numerator and denominator are both 2; they are equal. Therefore there is a horizontal asymptote $y = \frac{1}{1} = 1$, since the L.C. of $N(x)$ is 1 and the L.C. of $D(x)$ is 1.

Horizontal asymptotes are a special case of a broader class of asymptotes, **slant asymptotes**. Horizontal asymptotes are lines with slope 0, but the slope can be nonzero. If $n = d + 1$, then the rational function has a slant asymptote. In words: if the degree of the numerator is exactly one more than the degree of the denominator, there will be a slant asymptote. To find the slant asymptote, we perform long division, and the equation is $y =$ quotient.

Example 4.9: Slant Asymptote

Consider the function
$$f(x) = \frac{x^2 - x + 1}{x - 1}.$$
Here $n = 2$ and $d = 1$, and since $n = d + 1$ there must be a slant asymptote. Find the equation for the slant asymptote.
Solution. To find the equation of the slant asymptote, we use polynomial long division. We obtain
$$\frac{x^2 - x + 1}{x - 1} = x + \frac{1}{x - 1}.$$
The quotient is x and the remainder is 1. The equation of the slant asymptote is $y = x$.

One doesn't have to memorize these rules if one understands what is really going on. For example, if the degree of the denominator $D(x)$ is larger than the degree of the numerator $N(x)$, then one can think of $D(x)$ as growing "faster" than $N(x)$, and as x gets really large (or really small), the gap between $N(x)$ and $D(x)$ widens. As this gap widens, the fraction $N(x)/D(x)$ shrinks, as we are dividing a numerator by a denominator which gets larger and larger. This is why if $D(x)$ has larger degree than $N(x)$, that $N(x)/D(x)$ has a horizontal asymptote at $y = 0$.

4.3.3 Holes of Rational Functions

It is important to note that there might be what we call "holes" in a rational function. These are specific points for which a function is undefined. This happens when a rational function is **defined** for us, but **not simplified.** Always remember, when you are given a rational function, make sure to factor the numerator and denominator as much as possible.

Example 4.10: Hole in Rational Function

Consider the rational function
$$f(x) = \frac{2(x^2 + 6x + 8)}{x^2 - 4}.$$
When we factor the numerator and denominator we get
$$f(x) = \frac{2(x + 2)(x + 4)}{(x + 2)(x - 2)}.$$
Notice that we have $(x + 2)$ in the numerator and the denominator. However, notice what happens when we plug in $x = -2$:
$$f(-2) = \frac{2((-2) + 2)((-2) + 4)}{((-2) + 2)((-2) - 2)} = \frac{2(0)(2)}{(0)(-4)}.$$
We cannot say that 0 cancels out 0, since $0/0$ does not make sense! Indeed, we can evaluate $0/1, 0/2$, and so on, but we cannot divide 0 into 0; it is **undefined**. So while the factors $x + 2$ in the numerator and denominators of f 'cancel' when simplifying, as f is **defined**, we must make special note of this circumstance. We set the canceling factor $x + 2 = 0$, giving a hole at $x = -2$. The simplified version of f is written
$$f(x) = \frac{2(x + 4)}{x - 2}.$$
To find the y value of the hole, we plug in $x = -2$ into the simplified version of f:
$$y = \frac{2((-2) + 4)}{(-2) - 2} = -1.$$
So we say: $f(x)$ has a hole at $(-2, -1)$, and we draw an open circle on the graph.
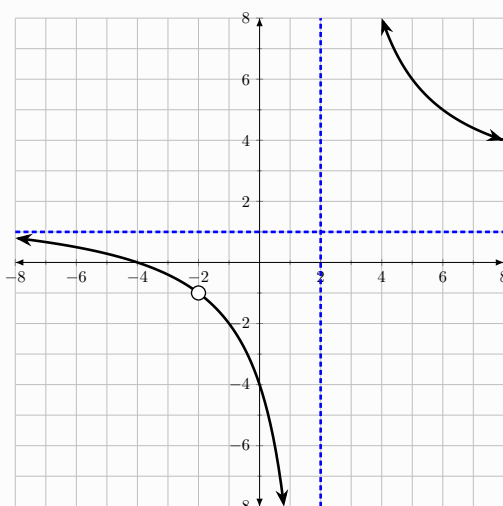
4.3.4 Finding the Domain

Now we have all the tools we need for finding the domain. We think of it in the following way: we start with the set of all real numbers, $(-\infty, \infty)$, and then we cut out x values for which the function is undefined. These x values are precisely the vertical asymptotes and holes.

Let's look at the functions from Figure 4.1.

Example 4.11: Rational Function – Domain

Let $f(x) = 1/x$ and $g(x) = \frac{x^2-1}{x^2+x-6}$. Refer back to Figure 4.1 for graphs of these functions.

For $f(x)$, we know there are no holes, but we do have a vertical asymptote at $x = 0$. So $x = 0$ is not part of the domain. So we must cut it out of $(-\infty, \infty)$. Doing so yields $(-\infty, 0) \cup (0, \infty)$ in interval notation. On a number line:
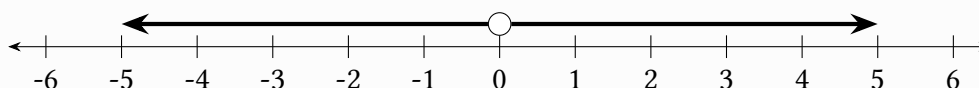
For $g(x)$, we previously factored and found that $g(x) = \frac{(x-1)(x+1)}{(x-2)(x+3)}$. Setting the denominator equal to 0 we get two vertical asymptotes: $x = 2$ and $x = -3$.
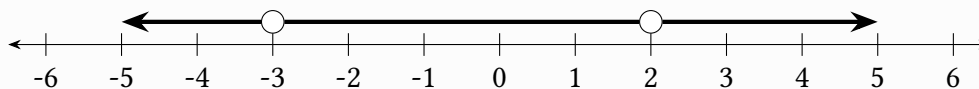
In interval notation, we get $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

One more example that we saw earlier, this time with a hole.

Example 4.12: Rational Function – Domain With A Hole

Let
$$f(x) = \frac{2(x^2 + 6x + 8)}{x^2 - 4} = \frac{2(x+2)(x+4)}{(x+2)(x-2)} = \frac{2(x+4)}{x-2}$$

As we discussed before, instead of a vertical asymptote, we get a hole at $x = -2$, and we still have a vertical asymptote at $x = 2$. So the domain is: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. On a number line:
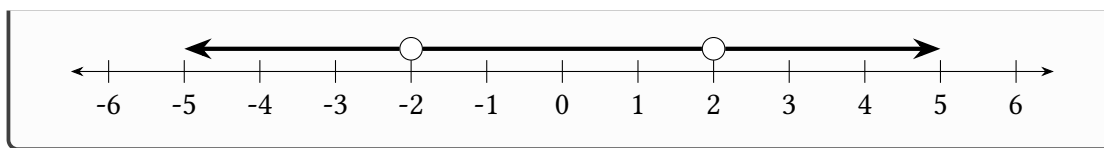
4.3.5 Intercept and Zero Behavior

Rational functions exhibit behaviors around the vertical intercepts and zeros. For example, let's take a look at the functions $f(x) = \frac{a(x+1)^2}{(x-1)^3}$ and $g(x) = \frac{a(x+1)}{(x-1)^2}$. (If you are curious, $a = 3$ here, but that knowledge is not useful for what we are trying to understand.)
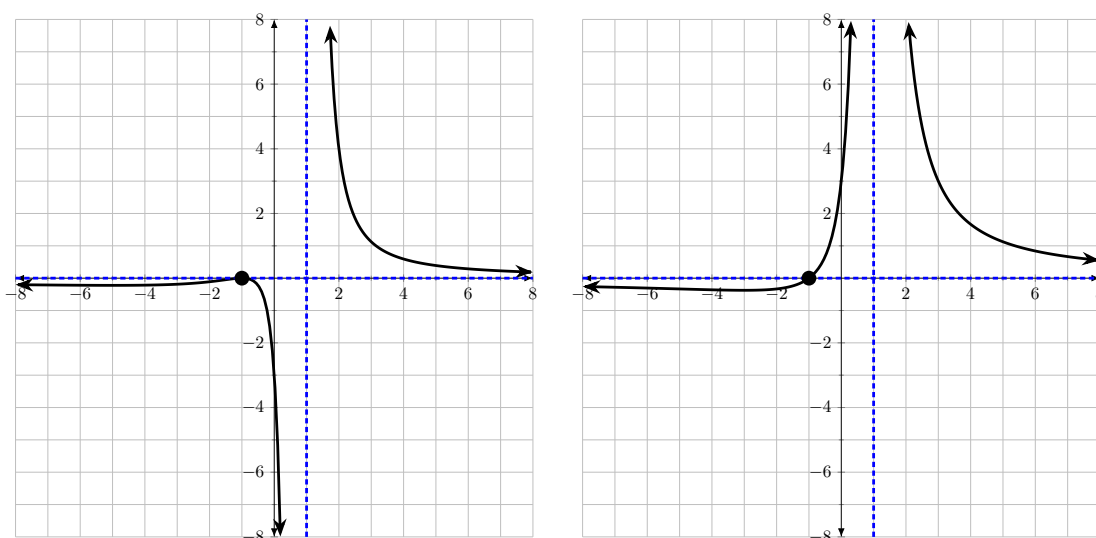
Notice that our examples both have the same x-intercept $(-1, 0)$, and the same vertical asymptote $x = 1$. However, notice the behavior of the functions. For $f(x)$ on the left of Figure **??**, the sign of $f(x)$ is **the same** on the left and right of $(-1, 0)$; the function bounces at $(-1, 0)$. The behavior of $f(x)$ (going to ∞ or $-\infty$) around the vertical asymptote $x = 1$ is **different** on each side; $f(x)$ goes to $-\infty$ as it approaches from the left of $x = -1$, and goes to ∞ as it approaches from the right of $x = -1$.

Recall: after simplifying and dealing with holes, setting the numerator equal to 0 will give us x-intercepts, and setting the denominator equal to 0 will give us vertical asymptotes. Similar to polynomials, we set each factor of the numerator or denominator equal to 0. These factors have exponents, and thus we can assign *multiplicities* to not only x-intercepts (zeros, roots, same thing) but also vertical asymptotes. Here are the rules:

- If a vertical asymptote $x = \alpha$ has even multiplicity, then the function will behave

the same on both sides of $x = \alpha$ (either going to ∞ on the left and right, or going to $-\infty$ on the left and right).

- If a vertical asymptote $x = \alpha$ has odd multiplicity, then the function will behave differently on both sides of $x = \alpha$ (either going to $-\infty$ from the left and ∞ from the right, or vice versa)

- If a zero $(\alpha, 0)$ has even multiplicity, then the function will have the same sign on both sides of $(\alpha, 0)$.

- If a zero $(\alpha, 0)$ has odd multiplicity, then the function will have different sign on both sides of $(\alpha, 0)$.

Looking at our examples and we can understand how the rules apply. This functions are purposefully similar, with only the multiplicities varying.

x-intercept from the factor $(x + 1)^2$. Even multiplicity $\rightarrow$ same sign on each side.

Vertical Asymptote from the factor $(x - 1)^3$. Odd multiplicity $\rightarrow$ different behavior on each side.

x-intercept from the factor $(x + 1)$. Odd multiplicity $\rightarrow$ different sign on each side.

Vertical Asymptote from the factor $(x-1)^2$. Even multiplicity $\rightarrow$ same behavior on each side.
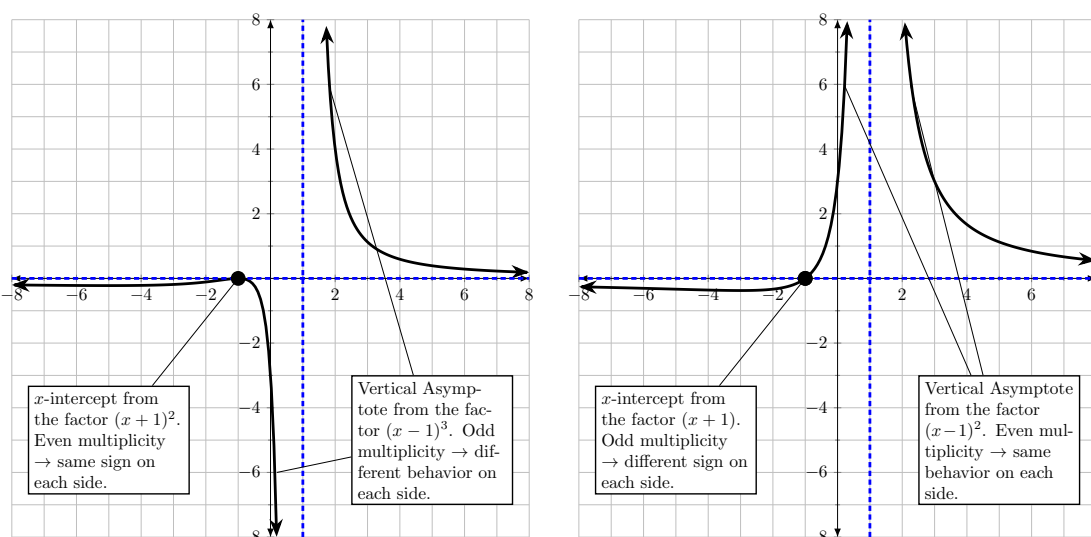
Summary
Chapter 4

- To find the vertical asymptotes of a rational function, set the denominator equal to 0 and solve.

  - Vertical asymptotes are vertical lines indicating where the function is undefined. They are written as $x = \alpha$ where α is a zero of the denominator.

  - The behavior of a rational function around a vertical asymptote is

determined by the multiplicity of the factor from which it came. Let $(x - \alpha)$ be a factor in the denominator so that $x = \alpha$ is a vertical asymptote. Recall the examples $f(x) = 1/x$ and $f(x) = 1/x^2$.

* If α has **odd** multiplicity, then the function behaves **differently** to the left and right of $x = \alpha$.

* If α has **even** multiplicity, then the function behaves **the same** to the left and right of $x = \alpha$.

- Horizontal and Slant asymptotes. Let n be the degree of the numerator and d be the degree of the denominator.

  - if $n = d + 1$ (if the numerator has degree exactly one more than the denominator), then the function has a slant asymptote. Find the equation of the slant asymptote $y = mx + b$ by polynomial long division.

  - if $n > d$, there is no horizontal asymptote.

  - if $n = d$, there is a horizontal asymptote. Divide the leading coefficient of the numerator by the leading coefficient of the denominator to get the equation $y = c$.

  - if $n < d$ there is a horizontal asymptote at $y = 0$.

- Holes - if a factor $x - \alpha$ cancels out after simplifying $f(x)$, the graph may have a hole at $x = \alpha$.

  - If, after simplifying, $x = \alpha$ ends up being a vertical asymptote, there is no need to talk about a hole.

  - Otherwise, plug in $x = \alpha$ into the **simplified** version of f to get the y value of the hole. When graphing, there will be an open circle at (α, y).

- Domain - Vertical Asymptotes and holes tell us what to exclude from the domain. For example, if a function has vertical asymptotes at $x = 1$ and $x = -2$, and a hole at $x = 5$, then the domain of f is, in interval notation, $(-\infty, -2) \cup (-2, 1) \cup (1, 5) \cup (5, \infty)$.

- x-intercepts - set the numerator equal to 0. The multiplicity of an x-intercept tells us whether the sign of the function is the same or different to the left and right of the function. Let $x = \alpha$ be an x-intercept of $f(x)$.

  - If α has **odd** multiplicity, then the sign of $f(x)$ on the left and right of α **differs** ($+/-$ or $-/+$.)

– If α has **even** multiplicity, then the sign of $f(x)$ on the left and right of α **is the same** $(+/+$ or $-/-.)$

5 Composition

In this chapter we will learn about composition of functions.

Given two functions, we may *compose* them; one function may act as an input of another. Let $f(x)$ and $g(x)$ be two functions.

We write a composition as $f(g(x))$ or $g(f(x))$, read as "f of g of x" and "g of f of x" respectively. An alternate notation of $f(g(x))$ is $(f \circ g)(x)$, which can be read as "f composed with g evaluated at x." T

To evaluate the composition $f(g(x)) = (f \circ g)(x)$, we first evaluate $g(x)$, then we plug the result in to $f(x)$. Note, it is not always the case that $f(g(x)) = g(f(x))$. In other words, **the order of composition matters**. To get the notation straight:

$$(f \circ g)(x) = f(g(x))$$
$$(g \circ f)(x) = g(f(x))$$

5.1 Evaluating Compositions

Let's do an example with some concrete functions.

Example 5.1: Evaluating Compositions

Let $f(x) = x + 5$ and $g(x) = x^2 + 2$. Then

$$f(g(x)) = (x^2 + 2) + 5 = x^2 + 7$$
$$g(f(x)) = (x + 5)^2 + 2 = x^2 + 10x + 27$$

Notice how $f(g(x)) \neq g(f(x))$.
We can compute some values for the compositions. For example, for $f(g(2))$ we compute $g(2) = (2)^2 + 2 = 6$, then plug 6 into f, so

$$f(g(2)) = f(2^2 + 2) = f(6) = (6) + 5 = 11.$$

Or, like above, we can write the composition in its general form, such as $f(g(x)) = x^2 + 7$ and then evaluate. Here are some examples, showing the many ways we can compute a composition of functions.

- $(g \circ f)(2) = ((2) + 5)^2 + 2 = (7)^2 + 2 = 51$

- $g(f(10)) = ((10) + 5)^2 + 2 = (15)^2 + 2 = 227$

- $(f \circ g)(1) = f(g(1)) = f((1)^2 + 2) = f(3) = (3) + 5 = 8$

Try to work out the following on your own. Let $p(x) = \sqrt{x+1}$ and $q(x) = x^2 + 5$. Show that

- $(p \circ q)(x) = \sqrt{x^2 + 6}$

- $(q \circ p)(x) = x + 6$

We may also evaluate compositions from a table.

Example 5.2: Compositions With Tables

Consider the following table of values for two functions r and q.

x	$r(x)$	$q(x)$
1	4	12
2	3	10
3	1	5
4	2	14

Compute $q(r(2))$ and $q(r(4))$.
To evaluate these, we first must evaluate the inside. So

- $q(r(2))$: First we evaluate $r(2)$, which from the table we know is 3.

x	$r(x)$	$q(x)$
1	4	12
2	**3**	10
3	1	5
4	2	14

So $q(r(2)) = q(3)$. Looking at the third row, we know $q(3) = 5$.

x	$r(x)$	$q(x)$
1	4	12
2	3	10
3	1	**5**
4	2	14

- $q(r(4))$: First we evaluate $r(4)$, which from the table we know is 2. So $q(r(4)) = q(2)$. Looking at the third row, we know $q(2) = 10$.

Finally we can evaluate compositions via graphs. Note, it is always the same process. For any composition $(f \circ g)(x) = f(g(x))$, we first evaluate $g(x)$ to get a number, then we plug in that number to f. On a graph this is nothing new; given an x coordinate, we find the y coordinate on a graph of g, then use that y coordinate as an x coordinate in a picture of f.

Example 5.3: Compositions With Graphs

Consider two graphs below of f and g.
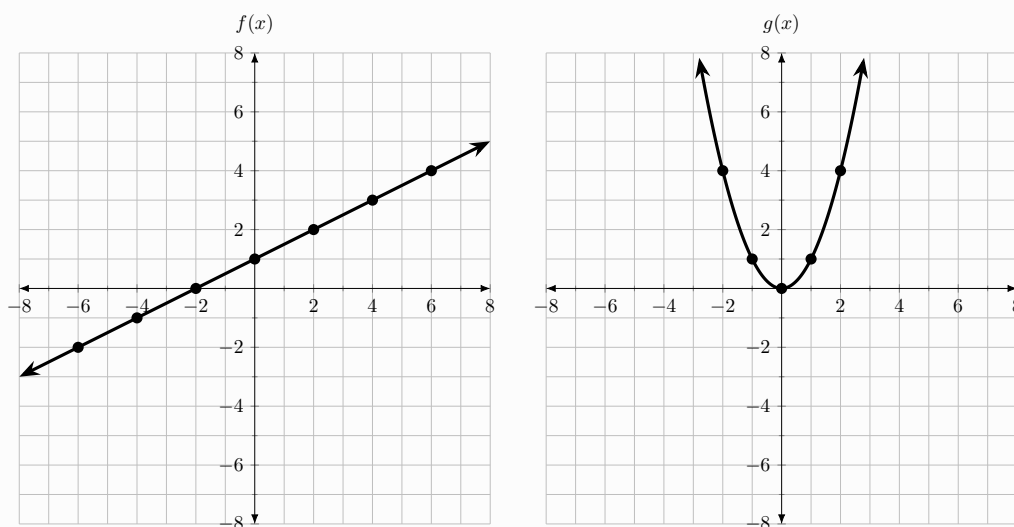
- To evaluate $f(g(2))$, we first evaluate $g(2)$. From the graph of g, we see that $g(2) = 4$. So $f(g(2)) = f(4) = 3$.

- To evaluate $g(f(2))$, we first evaluate $f(2)$. From the graph of f, we see that $f(2) = 2$. So $g(f(2)) = g(2) = 4$.

5.2 Recognizing Compositions

As we've seen, given two functions, there are two ways we can compose them. We can also reverse the process; given a single function, we can recognize and write it as a composition of functions. Any function f is a composition of itself and the *identity* function $g(x) = x$. We've been writing functions all along as *trivial* compositions; $f(x)$ is "f of x", where x is itself a function (a boring one, but still a function!) So a composition is not always unique, that is, it is possible that a function can be written as a composition in more than one way.

Example 5.4: Recognizing Compositions

The function $f(x) = (x-2)^4 + 3$ is a composition $h(g(x))$ where $g(x) = x - 2$ and $h(x) = x^4 + 3$. We can check this:

$$h(g(x)) = h(x - 2) = (x - 2)^4 + 3.$$

One will require a bit of creativity and intuition when recognizing a composition. It may be helpful to think of needing one "outside function" and one "inside function". Here, the outside function is $x^4 + 3$ and the inside function is $x - 2$. Taken together:
$$f(x) = (x - 2)^4 + 3.$$

We could have also set $h(x) = (x - 2)^4$ and $g(x) = x + 3$, and then $f(x) = g(h(x))$ (check this yourself). In colors,

$$f(x) = \left((x - 2)^4 \right) + 3$$

Suppose $H(x) = 7\sqrt{x} - 4$. Then we find $f(x) = 7x - 4$ and $g(x) = \sqrt{x}$ so that

$$f(g(x)) = f(\sqrt{x}) = 7(\sqrt{x}) - 4 = H(x)$$

In general, when given a function and asked for a composition, we want to find an "inner" function and an "outer" function, then we can check our composition. Another way to think about it is: we find an inner function, and replace it with an x. The

resulting function is the outer function. For example

$$f(x) = \sqrt{x+5} + 2.$$

We can make $x + 5$ the inner function, and replacing it with an x we get an outer function $\sqrt{x} + 2$.

5.3 Domain of Compositions

Lastly, let's look at the domain of a composition. Consider a composition $f \circ g$. The domain of $f \circ g$ is all inputs x such that x is in the domain of g and $g(x)$ is in the domain of f. That is, there are two things to check; the validity of an input x for $f \circ g$ relies on g, so that the output $g(x)$ is defined, and further, $g(x)$ must be in the domain of f so that $f(g(x))$ is defined. This all sounds a bit confusing, so let us do some examples.

Example 5.5: Domain of Compositions 1

Let $g(x) = x^2 - 1$ and $h(x) = \sqrt{x-6}$. Let's find the composition $g \circ h$ and its domain.
$$(g \circ h)(x) = g(\sqrt{x-6}) = (\sqrt{x-6})^2 - 1.$$
Notice, we may simplify further to obtain $(g \circ h)(x) = (x - 6) - 1 = x - 7$. However, notice that by simplifying, one would incorrectly assume that the domain of $(g \circ h)$ is all real numbers. However, if we write $g \circ h$ in the non simplified form $(g \circ h)(x) = (\sqrt{x-6})^2 - 1$, we may notice that we cannot have any number x less than 6 as an input, otherwise $\sqrt{x-6}$ would be undefined. That is, the domain of h, namely $[6, \infty)$ plays an important role here. Since the range of $\sqrt{x-6}$ is all real numbers, and the domain of g is all real numbers, we do not need to worry about the output $h(x)$. So the domain of $g \circ h$ is $[6, \infty)$. In total we have

$$(g \circ h)(x) = x - 7$$
$$\text{Domain: } [6, \infty)$$

Let's do a more complicated example.

Example 5.6: Domain of Compositions 2

Consider the functions $f(x) = \dfrac{x}{x+1}$ and $g(x) = \dfrac{11}{x}$. Find and simplify $(f \circ g)(x)$. Then, find the domain.

At this point, we should be able to evaluate a composition. Here it's a bit of work; the algebra can be tricky!

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\frac{11}{x}\right) \\
&= \frac{\left(\frac{11}{x}\right)}{\left(\frac{11}{x}\right) + 1} \\
&= \frac{\left(\frac{11}{x}\right)}{\frac{11}{x} + \frac{1}{1} \cdot \frac{x}{x}} && \text{(Combine terms, LCD)} \\
&= \frac{\left(\frac{11}{x}\right)}{\frac{11+x}{x}} \\
&= \left(\frac{11}{x}\right)\left(\frac{x}{11+x}\right) && \text{(Keep, Change, Flip)} \\
&= \frac{11}{11+x}
\end{aligned}
$$

Now note, we have a rational function. The domain is **not** simply obtained from finding values for which the denominator is not equal to 0. That is, the domain is **not** simply $(-\infty, -11) \cup (-11, \infty)$. This is a *part* of the domain of $(f \circ g)$. To find the domain, we go back to when we plugged $g(x)$ inside of f:

$$
f(g(x)) = \frac{\left(\frac{11}{x}\right)}{\left(\frac{11}{x}\right) + 1}.
$$

In this form, we find values of x which we must exclude from the domain. First, we have $\frac{11}{x}$ in the equation. This implies that $x \neq 0$, as then the function would be undefined. Notice that this issue does not appear in the simplified version of the composition! Then we notice that we have a rational function, so we must

check the denominator:

$$\frac{11}{x} + 1 \neq 0$$
$$\frac{11}{x} \neq -1$$
$$11 \neq -x$$
$$-11 \neq x$$

So we have two excluded values, -11 and 0. Thus the domain of the composition is $(-\infty, -11) \cup (-11, 0) \cup (0, \infty)$.

The above example is some work, but one needs to keep their head straight and obey algebra rules.

Example 5.7: Domain of Compositions 3

Let $p(x) = \sqrt{x+1}$ and $q(x) = x^2 + 5$. Find $(q \circ p)(x)$ and find the domain. We evaluate:

$$(q \circ p)(x) = q(p(x))$$
$$= q(\sqrt{x+1})$$
$$= (\sqrt{x+1})^2 + 5$$
$$= x + 1 + 5$$
$$= x + 6.$$

Again, while $(q \circ p)(x) = x + 6$, a line, the domain is **not** $(-\infty, \infty)$. We focus on when we plugged in:
$$(\sqrt{x+1})^2 + 5.$$
Here, we have to make sure that $\sqrt{x+1}$ is defined. An even root function is defined only when the inside evaluates to a number greater or equal to 0. That is in this case, $\sqrt{\blacksquare}$ is only defined when $\blacksquare \geq 0$. So we solve $x + 1 \geq 0$ which yields $x \geq -1$. So in interval notation our domain is $[-1, \infty)$.

Suppose we are given a function already composed:

Example 5.8: Domain of a Complicated Function

Let
$$f(x) = \sqrt{\frac{x-2}{(x-5)(x-3)}}$$

Now here, one needs to reason through. What do we know:

- The domain of any square root function $\sqrt{\blacksquare}$ contains all x values which satisfy $\blacksquare \geq 0$.

- There is a rational function inside the square root. We need to ensure that we find the excluded values; find the x values which make the denominator 0.

Thus, we need to solve $\frac{x-2}{(x-5)(x-3)} \geq 0$. Luckily, we've learned this before with rational inequalities. Setting the numerator equal to 0 we obtain a closed circle at $x = 2$, and setting the denominator equal to 0 we obtain open circles at $x = 5$ and $x = 3$. We find test values, and end up with (this work is left to the reader):
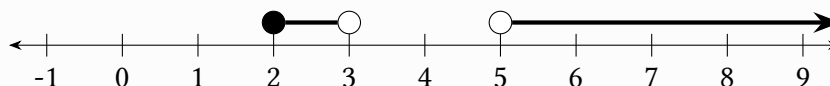
The domain in interval notation is $[2, 3) \cup (5, \infty)$.

Summary
Chapter 5

List of things you need to know.

- Compositions are written in two ways $(f \circ g)(x) = f(g(x))$.

- Evaluate compositions of functions for certain values, or give a general formula.

- To find the domain of a composition, find excluded values when you plug in a function. Do not find the domain simply based on the simplified version of a composition.

6 Inverses

Recall: A *function* is a rule which defines a relationship between an input variable, say x, and an **unique** output (often referred to as y). Functions can be given names, commonly f, g, or h. When we define a function, we may write something like

$$f(x) = x + 5$$

which is read "f of x equals x plus 5." This function, whose name is f and input is x, takes the input and adds 5 to it, giving us an output. To evaluate a function at a specified value of x, say 5, we write

$$f(5) = (5) + 5 = 10.$$

Everywhere we see an x, we substitute a 5, then we simplify. Again, the output is **unique**; when we plug in a number for x, the function will output exactly one number. To test this on a graph, we use the vertical line test. If we can draw a vertical line on top of a graph, then that signifies that there are two y values which share the same x value; the given graph is not that of a function.

Now we define what it means for a function to be one-to-one.

Definition 6.1: One-to-One Function

A function is said to be **one-to-one** if for every output there is exactly one input.

To test if a given graph is that of a one-to-one function, we use the horizontal line test; if we can draw a horizontal line on a graph where it crosses the graph twice, then there is an output value (a value on the y-axis) which comes from two different x values.

It is easy to get confused about what it means to be a function, and what it means to be a one-to-one function. A function takes an input and gives **exactly one output.** A function is one-to-one if every y-value comes from a unique x value. For example, let us look at the function $f(x) = x^2$.

Certainly $f(x) = x^2$ is a function. We plug in a number, we get exactly one number out. However, $f(x) = x^2$ is not one-to-one. The output 4 comes from two different x
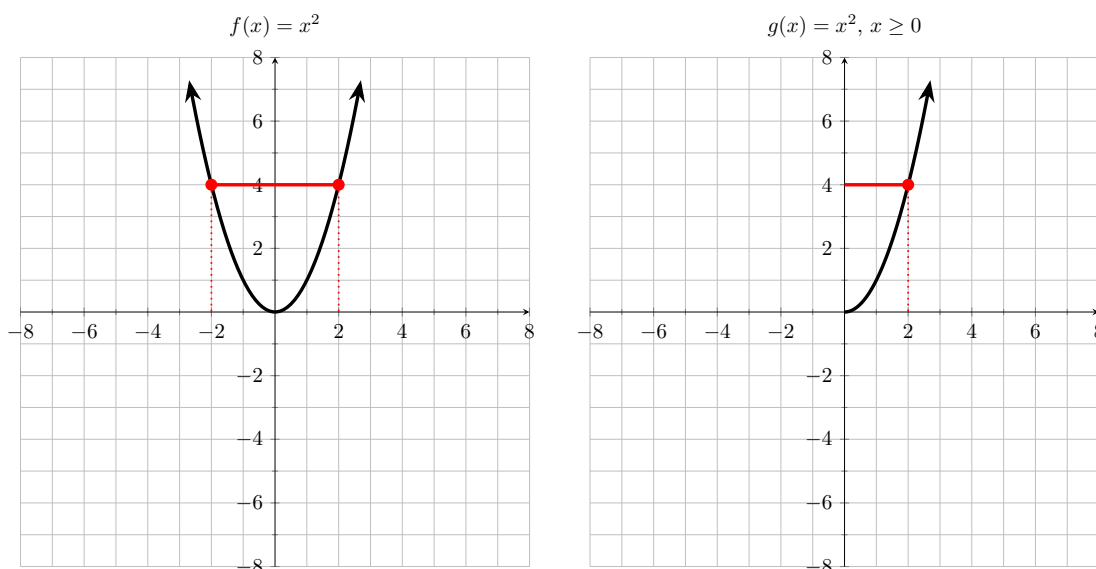
values. That is, $f(-2) = f(2) = 4$. We can see that x^2 horrifically fails the horizontal line test. However, if we are selective of the domain of our function, then we can make a one-to-one function. In the case of x^2, if we only consider the portion where $x \geq 0$, then we have a one-to-one function. In general, we can take "one half" of any quadratic function, as they are all symmetric about the vertex.

The concept of a one-to-one function comes in handy when we want to define an *inverse* for our function.

Definition 6.2: Function Inverse

Let f be a one-to-one function. Then f has an inverse function, denoted f^{-1} such that if $f(x) = y$, then $f^{-1}(y) = x$.

In words, while f takes inputs and gives outputs, f^{-1} takes outputs of f and returns the corresponding input. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

One should reason with themselves to truly understand the definition; why is it that the compositions $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Now we may understand why the concept of one-to-one is important. The function $f(x) = x^2$ is not one-to-one; it does not have an inverse. If it had an inverse f^{-1}, then $f^{-1}(4)$ would need to output both -2 and 2, and so f^{-1} would not be a function. However, if we specify that $f(x) = x^2$, $x \geq 0$, then f does indeed have an inverse
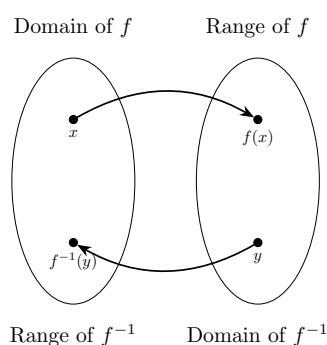
Figure 6.1: A diagram depicting a function, which assigns to elements of one set elements from another set, and an inverse.

$$f^{-1}(x) = \sqrt{x}.$$

6.1 Finding the Inverse

Again, the inverse takes y values and returns x values. That is, for every point (x, y) in the graph of f, we have the point (y, x) in the graph of f^{-1}.

Example 6.3: Finding the Inverse Given Coordinates

Let $g = \{(-8, 0), (-4, 5), (2, 3), (3, 4), (7, -4)\}$

Notice that g is a function; each x value comes with exactly one y value. We ask ourselves, is g one-to-one? We check, does each y value have its own, unique x value. Indeed, each y value appears once and only once. Now we want to evaluate, say, $g^{-1}(3)$. Two coordinates have 3 in them, $(2, 3)$ and $(3, 4)$. However, we are evaluating $g^{-1}(3)$, and the inverse takes y values as inputs. In words, $g^{-1}(3)$ asks: "what value of x, when plugged into g, gives 3?" We see then that $g^{-1}(3) = 2$, since $g(2) = 3$.
With similar reasoning, we see that $g^{-1}(-4) = 7$ or $g^{-1}(0) = -8$.

A list of coordinates is the first step. One half step and we begin to examine graphs.

Example 6.4: Finding the Inverse Given a Graph

Consider the following function:
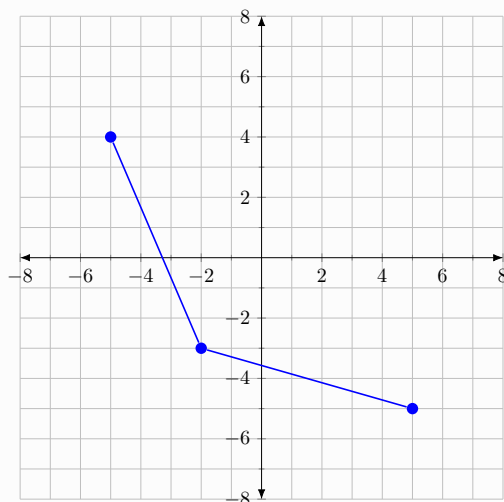
The function depicted has domain $[-5, 5]$ and range $[-5, 4]$. We make a table of the (x, y) values:

x	$f(x)$
-5	4
-2	-3
5	-5

The inverse, $f^{-1}(x)$ takes y values and gives the x values; so we switch the coordinates.

x	$f^{-1}(x)$
4	-5
-3	-2
-5	5

Now we can plot our points, and connect them:
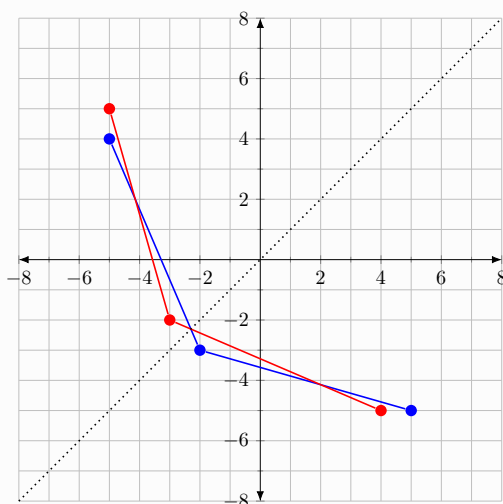
There is some geometric meaning here. The graph of a function, when reflected over the line $y = x$, yields the graph of the inverse. This is what is happening when we switch the x and y coordinates.

In general, some times we are not given such a nice graph. However, one can pick a few points on a sketch of a graph, swap the x and y coordinates, and sketch the inverse. Then, one checks their work by drawing $y = x$ and checking if the sketches are indeed reflections of each other.

Suppose instead of a set of coordinates or a graph, we are given an equation. To find the inverse of a function $f(x) = \blacksquare$ we

1. Write $y = \blacksquare$.

2. Replace every instance of x with y, and every instance of y with x. We get $x = \blacksquare$ where every x in $\blacksquare$ is replaced with a y.

3. Solve for y in terms of x; isolate y with algebra. We will get a new equation of the form $y =$.

4. Replace y with $f^{-1}(x)$.

Example 6.5: Finding the Inverse Given Equations

Let $f(x) = \frac{4x+1}{7}$. Find f^{-1}.

We are given a line, which can be written as $f(x) = \frac{4}{7}x + \frac{1}{7}$. Following our

procedure:

$$y = \frac{4x + 1}{7} \qquad\qquad (f(x) = y)$$

$$x = \frac{4y + 1}{7} \qquad\qquad (y = x)$$

$$7x = 4y + 1$$

$$7x - 1 = 4y$$

$$\frac{7x - 1}{4} = y \qquad\qquad \text{(Solve for } y\text{)}$$

$$\frac{7x - 1}{4} = f^{-1}(x)$$

We can check our answer with some specific values. Let $x = 5$. Then $f(2) = \frac{4(5)+1}{7} = \frac{21}{7} = 3$. Now when we evaluate $f^{-1}(3)$ we should get 2. Indeed:

$$f^{-1}(3) = \frac{7(3) - 1}{4}$$

$$= \frac{20}{4}$$

$$= 5.$$

The reader is encouraged to check other values of x.

Let us do another example with some more involved algebra.

Example 6.6: Finding the Inverse (Harder)

Let $f(x) = \frac{3x}{5x+1}$. Find $f^{-1}(x)$.

$$y = \frac{3x}{5x+1}$$

$$x = \frac{3y}{5y+1}$$

$$x(5y+1) = 3y \qquad \text{(Clear Denominator)}$$

$$5xy + x = 3y \qquad \text{(Distribute } x\text{)}$$

$$5xy - 3y = -x \qquad \text{(Move } 3y \text{ left and } x \text{ right.)}$$

$$y(5x-3) = -x \qquad \text{(Factor } y\text{)}$$

$$y = \frac{-x}{5x-3} \qquad \text{(Solve for } y\text{)}$$

$$f^{-1}(x) = \frac{-x}{5x-3}$$

Summary
Chapter 6

List of things you need to know.

- A one-to-one function is a function where each output has a unique input.

  - Use the horizontal line test on a graph to test if a function is one-to-one

- The inverse f^{-1} of a function f takes outputs $y = f(x)$ and produces inputs $x = f^{-1}(y)$.

- A function has an inverse if and only if it is one-to-one.

- The domain and range of an inverse is the range and domain of the function, respectively.

- To find the inverse of a function from its equation, write the equation in the form $y =$, change every x to y and every y to x, then solve for y.

7 Exponentials

In this chapter we will learn about exponential functions.

7.1 Introduction

Consider the following two functions:

$$f(x) = 2^x$$
$$g(x) = \left(\frac{1}{2}\right)^x$$

Let's make a table of some values. The reader is encouraged to calculate these values on their own.

x	$f(x)$	$g(x)$
-2	$1/4$	4
-1	$1/2$	2
0	1	1
1	2	$1/2$
2	4	$1/4$

These are *exponential functions.* There is a base b and the input x is the exponent:

$$h(x) = b^x.$$

Their graphs are given in Figure 7.1.

If b is not a proper fraction and positive, then we have exponential growth; think of setting the base to be any whole number. As $x \to \infty$, the function $f(x) \to \infty$, and as $x \to -\infty$, the function $f(x) \to 0$. Notice, if x is negative the base "becomes a fraction", for example

$$f(-2) = 2^{-2} = (2^{-1})^2 = \left(\frac{1}{2}\right)^2.$$
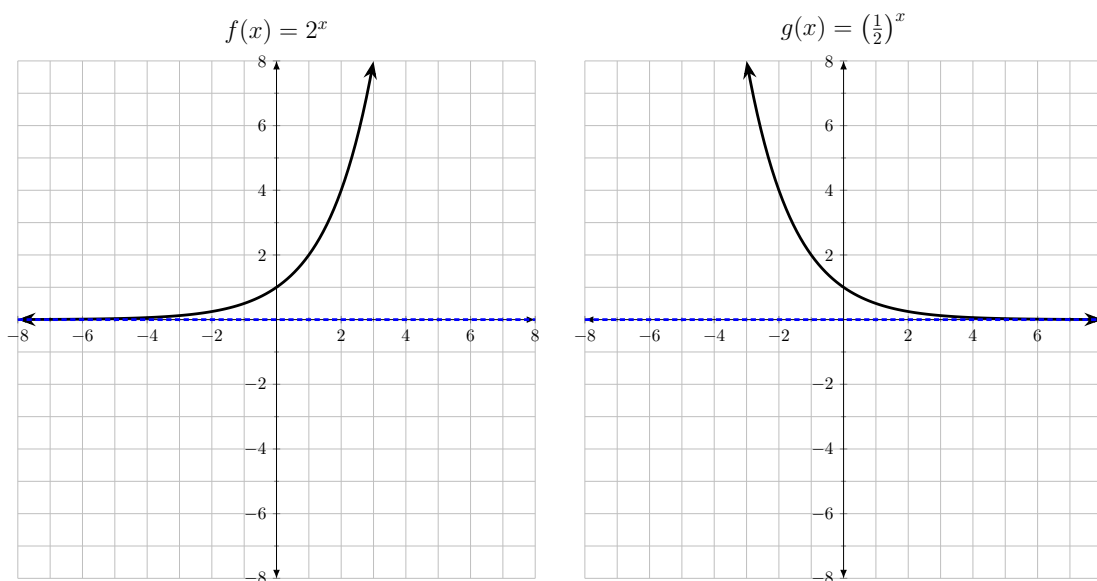
Figure 7.1: Examples of exponential functions in graph form. Notice that each func-
tion has the same y-intercept and horizontal asymptote. Also notice some
important values, at $x = 0$ and $x = 1$.

On the other hand, when b is a proper fraction, like $1/2$, as $x \to \infty$, the exponential
function goes to 0. This makes sense; in a proper fraction the denominator is larger
than the numerator, and when we raise the fraction to an exponent, the denominator
will "beat" the numerator (grow faster). This leads to exponential *decay*. Meanwhile,
as $x \to -\infty$, the function goes to ∞. Notice that if we have a proper fraction as a base,
and we plug in a negative number, the fraction "flips". For example,

$$g(-2) = \left(\frac{1}{2}\right)^{-2} = \left(\left(\frac{1}{2}\right)^{-1}\right)^2 = (2)^2.$$

So as $x \to -\infty$, the function goes to ∞.

So we have exponential decay when $0 < b < 1$ and exponential growth when $b > 1$.
Notice, we do not set b to be negative or equal to 1.

Then for any exponential function of the form $f(x) = b^x$, we have

- Domain: $(-\infty, \infty)$. We can plug any number in as an exponent.

- Range: $(0, \infty)$.

- Horizontal asymptote at $x = 0$.

- y-intercept of $(0, 1)$

Notice that these exponential functions do not have any x-intercepts. Many of these characteristics will change when we apply function transformations, but the core behaviors will be very similar.

7.2 Transformations

As we know from our prerequisite knowledge, we can tranform functions; we can shift their graphs left, right, up, or down some number of units. We can stretch and compress functions, and we can reflect. Using our knowledge, we can write a general form for exponential functions as

$$f(x) = ab^{x-h} + k.$$

In this form, $f(x)$ is b^x

- shifted h units to the left if h is negative and to the right if h is positive.

- shifted k units upward if k is positive and downward if k is negative.

- reflected about a horizontal axis (the asymptote) if a is negative.

These are the important details we focus on. Let us see some examples.

Example 7.1: Exponential Function Shifted

Let $f(x) = 2^{x-2} - 1$.
The function given to us is 2^x shifted to the right 2 units and down 1 unit. So we start with $y = 2^x$ and shift to the right 2 units.
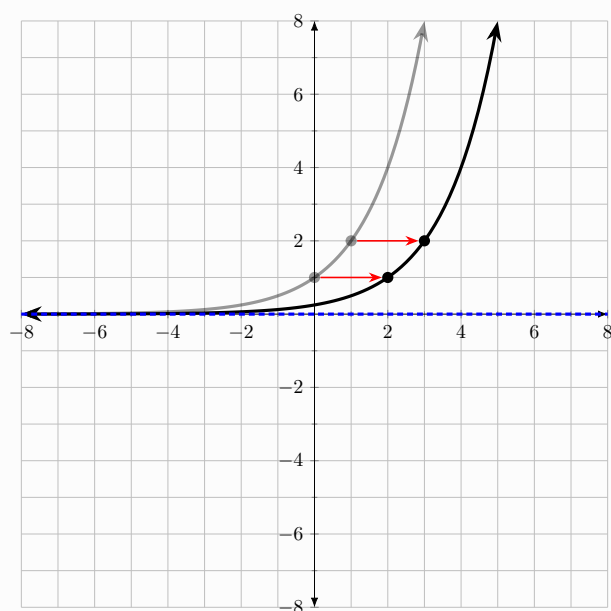
Now we shift $y = 2^{x-2}$ one unit down.
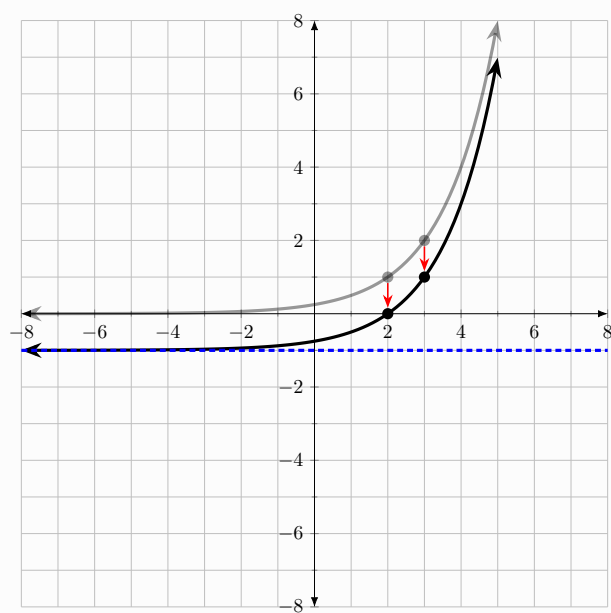
Both shifts at once:
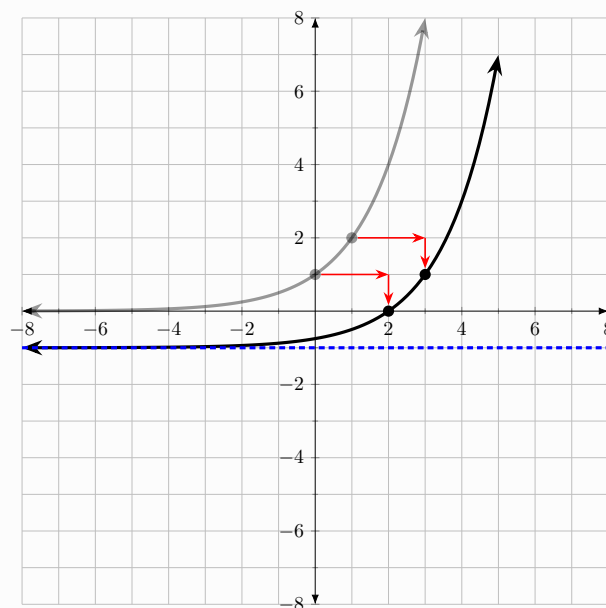
Notice that this new function has several characteristics that our basic exponential functions did not have, or are different.
By shifting downward, the horizontal asymptote goes from $y = 0$ to $y = -1$.
Additionally, there is now an x-intercept $(2, 0)$, and our y-intercept is $(0, -3/4)$.
The domain is $(-\infty, \infty)$ and the range is $(-1, \infty)$.

Let's do another example, this time with a reflection.

Example 7.2: Exponential With Reflection

Let $f(x) = -(3)^{x-1} + 2$.
Here, a is negative. So we have reflection, a shift 1 unit to the right, and a shift 2 units upward.
Let us start with $y = 3^x$ and reflect.
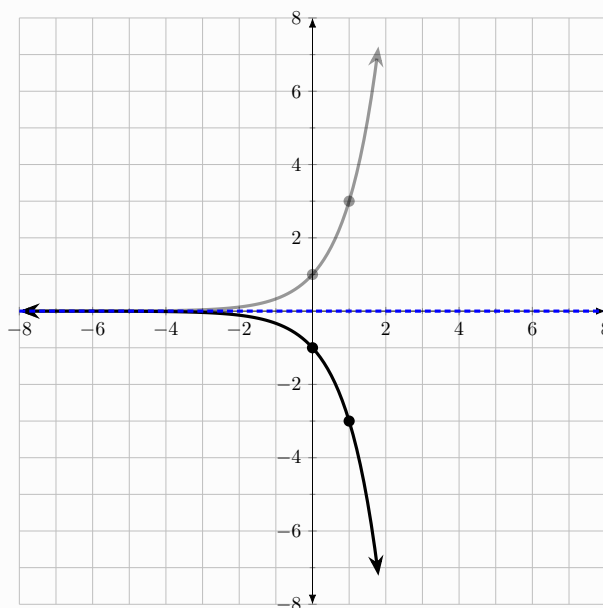
Then we shift as usual.
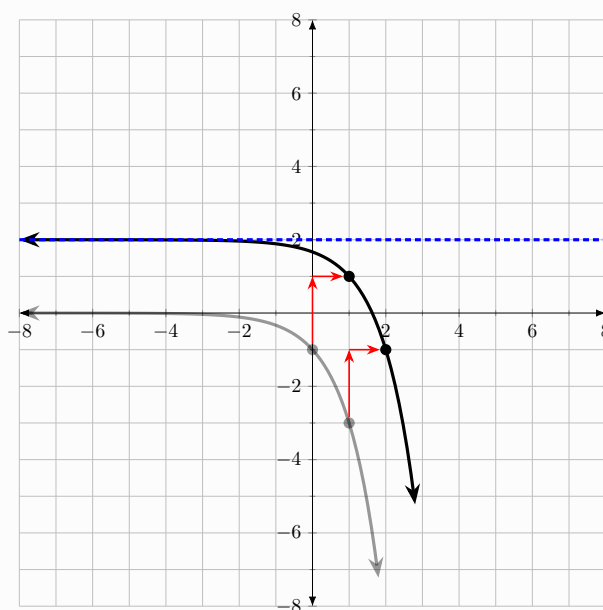
Notice here that the domain is still $(-\infty, \infty)$. The horizontal asymptote has shifted up 2 units, and is now $y = 2$. The range is $(-\infty, 2)$. There is a y-intercept, $(0, 5/3)$. There is an x-intercept, and we can find it by setting $-(3)^{x-1} + 2 = 0$. However, we will need logarithms to solve these kinds of equations; exponential equations.

So in general for an exponential function

$$f(x) = ab^{x-h} + k$$

- The domain is $(-\infty, \infty)$.

- The range is:
    - $(-\infty, k)$ if a is negative.
    - (k, ∞) if a is positive.

- $y = k$ is the horizontal asymptote.

Summary
Chapter 7

List of things you need to know.

- Translations of exponential functions and how they affect the characteristics.

8 Logarithms

In this chapter, we introduce logarithms.

8.1 Introduction

We are already familiar with exponents, and certain exponential equations. For example, $5^2 = 25$, or $3^3 = 27$. We may ask ourselves, given an exponent k, what is b^k? We know when we see 5^2, we must multiply 5 by itself twice and obtain 25. Using our exponent rules, we are able to simplify and evaluate expressions with exponents.

But now, suppose instead we are given the base and the result, but not the exponent. For example, consider the equation

$$4^x = 16.$$

How do we find x? In this case it is not too hard, as one can brute force there way to the answer $x = 2$. However, as we'll see, we need heavier machinery to eventually handle some more complicated equations.

Here is where *logarithms* come in.

$$y = \log_b(x).$$

This is read as "y equals log base b of x." This expression translates to the following: "y is the power that we must raise b to in order to get x." Or, if we are asked to evaluate something like $\log_b(x)$ we ask ourselves "to what power must we raise b to in order to get x." The keen-eyed student will notice that logarithms are the inverses of exponential functions. Here b is the base, and b is always positive. If b is not written, we assume that $b = 10$.

Let's look at a few examples:

Example 8.1: Logarithm Basics

What value of y makes the following equations true?

 1. $y = \log_2(16)$

2. $y = \log_2\left(\frac{1}{4}\right)$

3. $y = \log_5(25)$

For (1), we ask ourselves: "to what power must we raise 2 to in order to get 16." In other words, we are solving the equation $2^y = 16$ for y. The solution is 4.

For (2), we ask ourselves: "to what power must we raise 2 to in order to get $\frac{1}{4}$? Again, this is like solving the equation $2^y = \frac{1}{4}$. Here the solution is -2! Notice that we obtained a negative number. This makes sense; $\frac{1}{4}$ is a fraction, so $2^2 = 4$, however we must remember to flip (make the exponent negative to get the fraction.)

For (3), the solution is 2. So we may complete the equations by substituting the values we found for y.

1. $4 = \log_2(16)$

2. $-2 = \log_2\left(\frac{1}{4}\right)$

3. $2 = \log_5(25)$

Often one finds it helpful to convert mathematical expressions into english words when dealing with logarithms. So, in math:

$$b^y = x \iff \log_b(x) = y.$$

On the left "b raised to the y power yields x", and on the right, "y is the power that we waise b to in order to obtain x." Via this reasoning, one can convert between exponential and logarithmic equations without doing algebra. We'll see with logarithm properties that there is another way.

On a final note, notice: while $\log_b(x)$ can evaluate to a negative number like we saw with $\log_2(1/4)$, we cannot have a negative number as an input. That is, $\log_2(-2)$ is **undefined.** There is no exponent that, when given to 2, yields -2. In general, for a positive base b, there is no exponent which makes it negative. In particular, $\log_b(0)$ is undefined; there is no exponent which causes $b^y = 0$.

8.2 Properties of Logarithms

Here we cover the properties of logarithms. There are three:

Name	Property	Example
Product Rule	$\log_b(p) + \log_b(q) = \log_b(p \cdot q)$	$\log_2(4) + \log_2(4) = \log_2(16)$
Quotient Rule	$\log_b(p) - \log_b(q) = \log_b(p/q)$	$\log_2(8) - \log_2(4) = \log_2(2)$
Exponent Rule	$\log_b(p^q) = q \log_b(p)$	$\log_3(27^2) = 2 \log_3(27)$

The reader is encouraged to calculate the values in the examples given in the table to verify each rule.

8.2.1 Converting Between Exponential and Logarithmic Equations

Before we look at examples, let's get back to the point about converting exponential equations to logarithmic equations and vice versa. If $b^y = x$, then we can take the log of both sides:

$$b^y = x$$
$$\log_b(b^y) = \log_b(x)$$
$$y \log_b(b) = \log_b(x) \qquad \text{(Exponent Rule)}$$
$$y = \log_b(x) \qquad (\log_b(b) = 1)$$

All we needed was the exponent rule and the fact that $\log_b(b) = 1$ for any b; the power that we must raise b to in order to get b itself is 1. Similarly, if we want to convert $y = \log_b(x)$ into an exponential equation, we make each side an exponent of b and solve.

$$y = \log_b(x)$$
$$b^y = b^{\log_b(x)}$$
$$b^y = x \qquad (b^{\log_b x} = x)$$

Here, we relied on the fact that for any base b we have $b^{\log_b(x)} = x$. Loosely, the b and $\log_b$ cancel; however what is actually going here is $\log_b(x)$ is the exponent we raise b to in order to get x. So the expression $b^{\log_b(x)}$ is a bit circular; we obtain the exponent we raise b to in order to get x, then directly raise b to that exponent.

Let us do some examples.

Example 8.2: Converting Equations

1. If $\log_{10}(y) = 2$, what does y equal?

   Here, we can solve using algebra, or english. In algebra, we make each side an exponent of 10 since our base is 10:

   $$\log_{10}(y) = 2$$
   $$10^{\log_{10}(y)} = 10^2$$
   $$y = 10^2 \qquad (b^{\log_b(\blacksquare)} = \blacksquare)$$
   $$y = 100$$

   In english the equation reads "the power we raise 10 to in order to obtain y is 2", allowing us to write $y = 10^2$.

2. Convert $3^4 = 81$ to a logarithmic equation.

   We have a base of 3, so let's use $\log_3$.

   $$3^4 = 81$$
   $$\log_3(3^4) = \log_3(81)$$
   $$4\log_3(3) = \log_3(81) \qquad \text{(Exponent Rule)}$$
   $$4 = \log_3(81) \qquad (\log_3(3) = 1)$$

3. Convert $4^a = 12$ to a logarithmic equation.

   Here our base is 4.

   $$4^a = 12$$
   $$\log_4(4^a) = \log_4(12)$$
   $$a\log_4(4) = \log_4(12) \qquad \text{(Exponent Rule)}$$
   $$a = \log_4(12) \qquad (\log_b(b) = 1)$$

Whether translating through english, or manipulating using algebra, with enough practice one should be comfortable solving exponential and logarithmic equations of these kinds. So far we've only scratched the surface to get a better understanding of what logarithms are, and how we can use them to solve equations involving unknown variables as exponents (an x in the exponent.)

8.2.2 Expanding and Condensing Logarithmic Expressions

Similar to learning exponent rules, we must practice expanding and condensing logarithmic expressions using our rules. Just like exponent rules, there are many ways to combine and expand to get to a final result.

Example 8.3: Expanding Logarithms

1. Write the expanded form of $\log\left(\frac{3}{5}x^2\right)$.

$$
\begin{aligned}
\log\left(\frac{3}{5}x^2\right) &= \log(3/5) + \log(x^2) && \text{(Product Rule)}\\
&= \log(3) - \log(5) + \log(x^2) && \text{(Quotient Rule)}\\
&= \log(3) - \log(5) + 2\log(x) && \text{(Exponent Rule)}
\end{aligned}
$$

2. Write the expanded form of $\log\left(\frac{y^4}{\sqrt[3]{xz^2}}\right)$

   This one looks difficult, but if we know our exponent rules and keep our head straight, it is a step by step process.

$$
\begin{aligned}
\log\left(\frac{y^4}{\sqrt[3]{xz^2}}\right) &= \log(y^4) - \log(\sqrt[3]{xz^2} && \text{(Quotient Rule)}\\
&= \log(y^4) - \log((xz^2)^{1/3}) && (\sqrt[k]{\blacksquare} = \blacksquare^{1/k})\\
&= \log(y^4) - \log(x^{1/3}z^{2/3}) && \text{(Distribute Exponent)}\\
&= \log(y^4) - \log(x^{1/3}) + \log(z^{2/3}) && \text{(Product Rule)}\\
&= 4\log(y) - \frac{1}{3}\log(x) + \frac{2}{3}\log(z) && \text{(Exponent Rule)}
\end{aligned}
$$

As practice for condensing, take the end result of the previous examples and try to condense them back to get the original equations. As a practice problem, show that

$$
2\log(z) + 3(\log(y) - 2\log(x)) = \log\left(\frac{z^2 y^3}{x^6}\right).
$$

8.3 Solving Exponential and Logarithmic Equations

In this section we will see how we can use the fact that logarithms and exponentials are inverses of each other to solve special equations. Early we saw, for example, when we have a quadratic equation we can take the square root of two sides to solve the equation:

$$(x+5)^2 = 25$$
$$\sqrt{(x+5)^2} = \sqrt{25}$$
$$x + 5 = 5$$
$$\vdots$$

and similarly when solving a radical equation, we square both sides

$$\sqrt{x+4} = 3$$
$$(\sqrt{x+4})^2 = (3)^2$$
$$x + 4 = 9$$
$$\vdots$$

In the case of radical equations, recall that checking our solutions is mandatory. We will see similar behavior with exponentials and logarithms.

The idea with solving exponential equations (equations where unknown variables are in the exponent) is to use the properties of logarithms, in particular the exponent property, to isolate the unknown variable.

Example 8.4: Solving Exponential Equations Using Logarithms

1. $5e^{x+2} = 2$

    Here we have e, known as *Euler's constant*. The number e is an irrational number, approximately 2.7. Like many other special constants, we assign it a special letter. In this equation, let us use $\log_e(x) = \ln(x)$, the natural

log.

$$5e^{x+2} = 15$$
$$e^{x+2} = 3 \qquad \text{(Isolate the exponential)}$$
$$\ln(e^{x+2}) = \ln(3)$$
$$(x+2)\ln(e) = \ln(3)$$
$$x + 2 = \ln(3) \qquad \text{(}\ln(e) = 1\text{)}$$
$$x = \ln(3) - 2$$

We cannot simplify any longer, we leave our solution in **exact form.**

2. Solve for y: $5^{y+3} = 6$.

   Here we have a single exponential term with a base of 5. So let's "take the log of both sides", with the base of our logarithm as 5.

$$5^{y+3} = 6$$
$$\log_5(5^{y+3}) = \log_5(6)$$
$$(y+3)\log_5(5) = \log_5(6) \qquad \text{(Exponent Rule)}$$
$$y + 3 = \log_5(6) \qquad \text{(}\log_b(b) = 1\text{)}$$
$$y = \log_5(6) - 3 \qquad \text{(}\log_b(b) = 1\text{)}$$

   We could have used log with any base. The solution we obtained, $y = \log_5(6) - 3$, is in **exact form**. We simplified it as best we can.

3. Solve for x: $12^{4x} = 5^{-x+3}$

   Let's use the natural log, $\ln$ here, since we have two different bases and we can't be sure things will simplify nicely.

$$12^{4x} = 5^{-x+3}$$
$$\ln(12^{4x}) = \ln(5^{-x+3})$$
$$4x\ln(12) = (-x+3)\ln(5) \qquad \text{(Exponent Rule)}$$
$$4x\ln(12) = -x\ln(5) + 3\ln(5) \qquad \text{(Distribute)}$$
$$4x\ln(12) + x\ln(5) = 3\ln(5)$$
$$x(4\ln(12) + \ln(5)) = 3\ln(5) \qquad \text{(Factor } x\text{)}$$
$$x = \frac{3\ln(5)}{4\ln(12) + \ln(5)}$$

Now let's look at logarithmic equations.

Example 8.5: Solving Logarithmic Equations

1. Solve for x: $\log_5(-2x + 10) = 2$.

   In equations with logs, we can make both sides of the equation an exponent of a particular base. Here, the base of the log term is 5. This comes from the fact that if $x = y$, then $b^x = b^y$.

$$\log_5(-2x + 10) = 2$$
$$5^{\log_5(-2x+10)} = 5^2$$
$$-2x + 10 = 25 \qquad\qquad (b^{\log_b(x)} = x)$$
$$-2x = 15$$
$$x = -15/2 \text{ or } -7.5$$

   Plugging in to check our work:

$$\log_5(-2(-7.5) + 10) = 2$$
$$\log_5(15 + 10) = 2$$
$$\log_5(25) = 2$$
$$2 = 2 \checkmark$$

2. Solve for x: $\log_5(x + 4) = 1 - \log_5(x + 8)$.

   Now we have a logarithmic equation with two logs, but they each have the same base. We use our logarithm properties to solve the equation; at each step there is only one logical step forward.

$$\log_5(x + 4) = 1 - \log_5(x + 8)$$
$$\log_5(x + 4) + \log_5(x + 8) = 1$$
$$\log_5((x + 4)(x + 8)) = 1 \qquad\qquad \text{(Product Rule)}$$
$$5^{\log_5((x+4)(x+8))} = 5^1$$
$$(x + 4)(x + 8) = 5$$
$$x^2 + 12x + 32 = 5$$
$$x^2 + 12x + 27 = 0$$
$$(x + 3)(x + 9) = 0$$

Notice what we've done here. We combined the log terms into one term by using the Product Rule, which reduced the problem to something we already know how to solve (a logarithmic equation with one logarithm.) Then, we reduce the equation further to a quadratic equation. The two possible solutions are $x = -3$ and $x = -9$. Let's check our solutions.

Check $x = -3$. $\qquad\qquad$ Check $x = -9$.

$$\log_5(x + 4) = 1 - \log_5(x + 8) \quad \log_5((-9) + 4) = 1 - \log_5((-9) + 8)$$
$$\log_5((-3) + 4) = 1 - \log_5((-3) + 8) \qquad \log_5(-5) = 1 - \log_5(-1) \; \textbf{✗}$$
$$\log_5(1) = 1 - \log_5(5)$$
$$0 = 0 \; \checkmark$$

Notice that $x = -9$ is not a solution! **It is mandatory to check solutions when working with logarithmic equations.** In particular, we must check solutions when we use the product rule to combine logs, as we may introduce **extraneous solutions.**

8.4 The Logarithmic Graph

Now we consider the function $f(x) = \log_b(x)$. As we've mentioned previously, the log function $f(x) = \log_b(x)$ is the inverse function of $g(x) = b^x$; while exponentials take exponents as inputs, logarithms take numbers as inputs and produce the exponent that is needed to make the base equal to the number. Again, in math,

$$y = b^x \iff \log_b(y) = x.$$

Graphically, the inverse of a function is the reflection over the line $y = x$. Notice the base function characteristics:

- Domain: $(0, \infty)$

- Range: $(-\infty, \infty)$

- Vertical asymptote at $x = 0$.

- x-int $(1, 0)$

Knowing that the logarithmic functions are inverses of exponential functions, compare the characteristics of $f(x) = \log_b(x)$ and $f^{-1}(x) = b^x$. Notice how the domain and range are swapped, among other things.
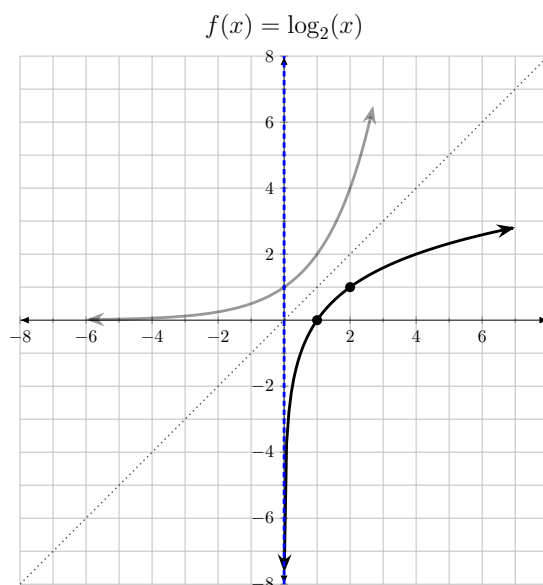
$$f(x) = \log_2(x)$$

Figure 8.1: The function $f(x) = \log_2(x)$ is depicted in bold, with the inverse $f^{-1}(x) = 2^x$ in gray.

8.4.1 Transformations

Just as we've seen with exponentials, we can shift horizontally, vertically, and reflect. We have

$$f(x) = a \log_b(x - h) + k.$$

Example 8.6: Translations of a Logarithm.

Consider the function $f(x) = -\log_4(x + 3) - 2$. Knowing our function translations, we have $a = -1$ which is negative, $h = -3$, and $k = -2$. In other words

- $a < 0$: $f(x)$ is $\log_4(x)$ reflected over a horizontal axis.

- $h = -3$: $f(x)$ is $\log_4(x)$ shifted 3 units to the left.

- $k = -2$: $f(x)$ is $\log_4(x)$ shifted 2 units downward.

So first let us apply the reflection.
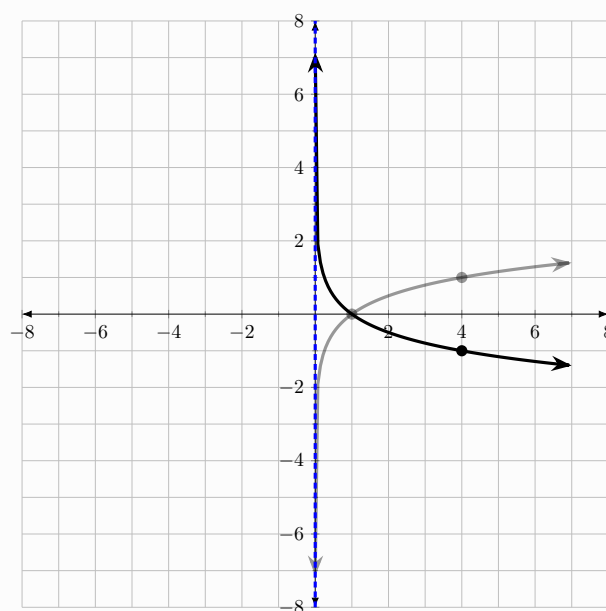
Notice the domain, range, vertical asymptote, and x-int have not changed. This will not be the case when we shift!
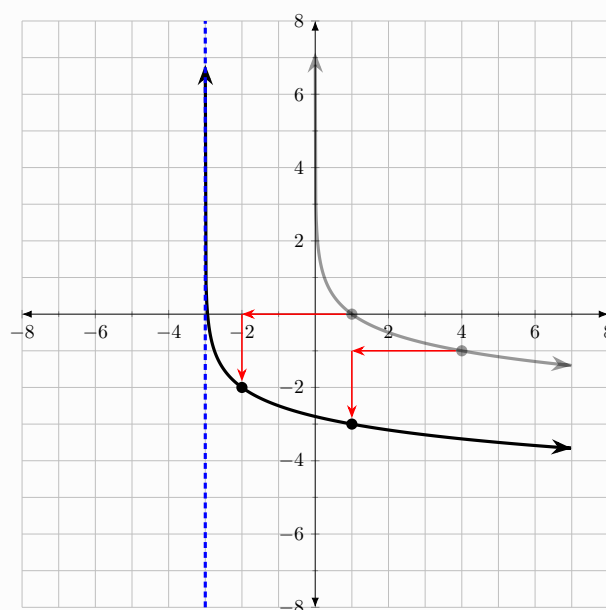
Notice now that the vertical asymptote shifted 3 units to the left. The domain as then changed from $(0, \infty)$ to $(-3, \infty)$ (just like vertical shifts change the range of the inverse exponential functions.) The range remains unchanged $(-\infty, \infty)$. We've also gained a y-intercept which we did not have before.

As we can see, function translations affect the characteristics of log functions. For $f(x) = a \log_b(x - h) + k$

- Domain: Solve for when $x - h > 0$. This gives when $x > h$. So (h, ∞).

- Range: $(-\infty, \infty)$.

- Vertical Asymptote at $x = h$.

8.4.2 Finding the Domain

We will now come back to something familiar; finding the domain of a composition of functions. We know that $\log_b(x)$ can only take positive inputs. That is, whenever we have a log term $\log_b(\blacksquare)$, the inside $\blacksquare$ must be strictly greater than 0. In math, $\blacksquare > 0$. Some times we can tell what the domain is from understanding function translations, but there is an algebraic way to solve for the domain.

Example 8.7: Finding the Domain of a Logarithmic Function

1. Let $f(x) = \log(-x + 5)$.

   The domain of $\log(\blacksquare)$ is all the values satisfying the inequality $\blacksquare > 0$. So we solve:

   $$-x + 5 > 0$$
   $$-x > -5$$
   $$x < 5 \qquad \text{(Multiply by } -1, \text{ flip sign)}$$

   So the domain of $f(x)$ is all x values such that $x < 5$. In interval notation, this is $(-\infty, 5)$. Check to see what happens when we plug numbers outside of this interval into f. What happens?

   ---

2. Let $g(x) = \log(x^2 + 6x + 8)$.

   Again, we need the inside to be positive. So we need to solve $x^2 + 6x + 8 > 0$. First we factor:

   $$x^2 + 6x + 8 > 0$$
   $$(x + 2)(x + 4) > 0$$

   Now we have a quadratic inequality, or in particular we technically have a rational inequality (with denominator 1). So we set $(x + 2)(x + 4) = 0$ to obtain critical values, then we find test values. Our critical values are $x = -2$ and $x = -4$, both having open circles on our number line.
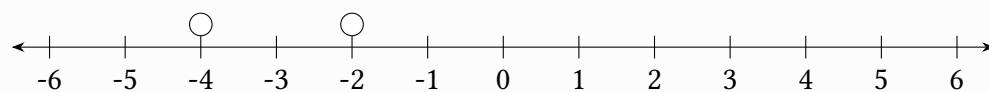
Interval	Test Value	Plug In	T/F
$(-\infty, -4)$	-5	$((-5)+2)((-5)+4) = (-3)(-1) = 3 > 0$	True
$(-4, -2)$	-3	$((-3)+2)((-3)+4) = (-1)(1) = -1 \not> 0$	False
$(-2, \infty)$	0	$((0)+2)((0)+4) = (2)(4) = 8 > 0$	True
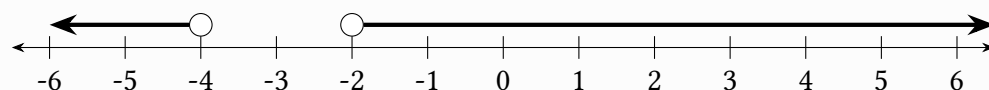
3. Let $f(x) = \log\left(\frac{(x-1)}{(x+2)(x-4)}\right)$

   Here we set the inside > 0. That is, we solve the inequality

   $$\frac{x-1}{(x+2)(x-4)} > 0.$$

   The reader is encouraged to review rational inequalities and solve for the domain. One should obtain, in interval notation, $(-2, 1) \cup (4, \infty)$.

9 Compound Interest

In this chapter we will learn about compound interest.

9.1 Introduction

Recall: when we want to compute a percentage of a number, we multiply the number by the percentage in **decimal form.** For example, 15% of 20 is $(0.15)(20) = 3$. To convert from percent to decimal, we divide by 100.

Let x be a number. If we say x has increased by 4%, then what is the new value? We convert 4% to decimal: 0.04. Then we add 4% of x to x:

$$x + 0.04x.$$

We can combine these terms, though, and write $(1 + 0.04)x = 1.04x$.

Example 9.1: Exponential Growth

1. Sandy's salary increases by 15% from $15,000. What is her current salary?

   Here our *rate* is 15%, which we convert to 0.15. Our principal, or starting value, is $15,000. To get the new value, we compute

   $$\$15,000 + (0.15)(\$15,000) = (1.15)(\$15,000) = \$17,250.$$

   Notice here we kept track of our units. Our final answer is in **dollars.**

2. Suppose Jane borrows $3,000 at an interest rate of 3% compounded yearly. Assume no payments are made on the loan.

   a) Find the amount owed after 1 year.

   Here again we have a rate of 3%, or in decimal 0.03. The interest is compounded yearly, and we are asked to find the amount owed after 1 year. Our principal is $3,000. Then $(1.03)(\$3000) = \$3090$.

   b) Find the amount owed after 2 years.

   To find the amount owed after 2 years, we must take the amount owed after 1 year and repeat the computation. Notice that we don't

---

just add $90. That is, the answer **is not** $3000 + (0.03)($3000) + (0.03)($3000).

We compute:
$$(1.03)(\$3090) = \$3182.70$$

c) Write a general formula for the amount Jane owes after t years, assuming no payments are made.

Notice, with each passing year, we take the last years value and multiply by 1.03 to compute a 3% increase. In (a) we saw after one year she owed $(1.03)(\$3000)$. After two years she owed $(1.03)(\$3090)$, or more generally, $(1.03)[(1.03)(\$3000)] = (1.03)^2(\$3000)$. In general, after t years Jane will owe $(1.03)^t(\$3000)$.

9.2 Compound Interest Formula - Discrete

To calculate the final amount in a compound interest problem where things grow discretely (the interest accrues a defined number of times per year), we use the following formula.
$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

- A is the final amount.

- P is the initial amount or initial principal.

- r is the interest rate in decimal form. For example, 15% is 0.15, or 4% is 0.04.

- n is the number of times interest is applied per time period.

- t is the number of time periods elapsed.

For our purposes, t is the **number of years** and n is the **number of times interest is applied per year.** We have special keywords for n. For example

- Compounded
    - yearly: $n = 1$
    - semiannually: $n = 2$.
    - quarterly: $n = 4$
    - monthly: $n = 12$
    - daily: $n = 365$

In compound interest word problems, one is tasked with identifying **given values** and substituting the given values into the formula properly.

Example 9.2: Compound Interest

To be added.

---

10 Trigonometry

In this chapter we will learn trigonometry, the study of angles of triangles and relations between their sides. Note that the first few pages, mainly section 11.2 on angles are **foundational.** Without a good understanding of angles, coterminal angles, reference angles, and the four quadrants, later topics will be exceedingly difficult.

10.1 Introduction

At this point in one's life, one has an understanding of what an angle is. An *angle* can be thought of as a measurement of the space between two intersecting lines, or an angle is an object formed by taking two rays (sides) which end at at common point. This measurement is commonly measured in *degrees.* As a review, we have several types of angles, depicted in Figure 10.1. We usually denote an angle measurement by the greek letter ϑ (read aloud as 'theta').

<center>Acute   Right   Obtuse</center>
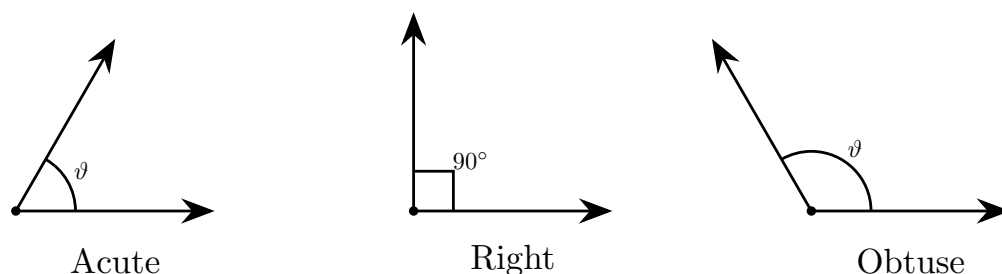
Figure 10.1: Different types of angles. In this picture from left to right, the first angle is $60°$, the second is $90°$, and the third is $120°$. Acute angles are angles that measure between $0°$ and $90°$, right angles are exactly $90°$, and obtuse angles measure between $90°$ and $180°$. There are more types of angles, such as straight angles (exactly $180°$) and reflex angles ($> 180°$). The reader is encouraged to draw pictures of straight and reflex angles on their own.

The reader is encouraged to review some basics on angles and recall knowledge of triangles from previous courses.

10.2 Angles in the xy-Plane

Instead of arbitrarily drawing lines in any orientation, we will focus on angles in the xy-plane and follow some conventions so that we can explore trigonometry in a standardized way. We will see how to sketch angles in **standard position.**

10.2.1 Sketching an Angle in Standard Position

Below we have our xy-Plane.

Like previously with functions and graphs, we have an x-axis with a positive side (right) and negative side (left). We have a y-axis with a negative side (below) and a positive si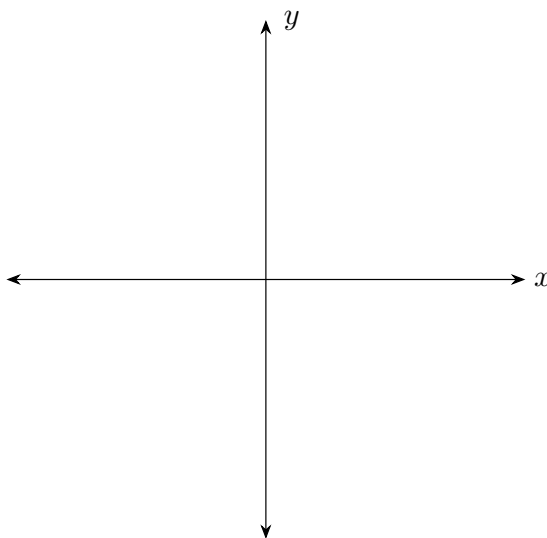de (above). Notice that the plane is divided into four sections. These sections are referred to as *quadrants.* The quadrants are labeled in order, counter clockwise starting from the positive x-axis. Later on it will be important for us to be able to tell with quadrant we are in based on the value of a given angle, or the signs of coordinates. For example, if x is positive and y is negative, then the point (x, y) is in quadrant IV (often abbreviated QIV).

Notice that the x axis is perpendicular to the y axis; that is there are four **right angles** where the x and y axis intersect. Loosely, one can think of each quadrant as being worth $90°$. Let's imagine that there is a full circle centered at the origin, so that we can better envision the rotations.
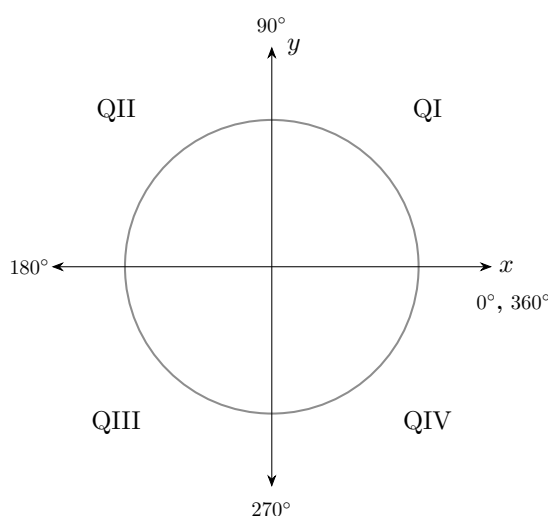
One sketches a given angle ϑ in the xy-plane in the following way. We first begin on the positive x-axis. To determine which quadrant we are in we see where our angle lands. Unless the angle is one of $0°, 90°, 180°, 270°, 360°$, or a multiple of those, we will lie inside a quadrant. The angle ϑ is in:

- QI if $0° < \vartheta < 90°$

- QII if $90° < \vartheta < 180°$

- QIII if $180° < \vartheta < 270°$

- QIV if $270° < \vartheta < 360°$.

Notice that ϑ can be negative, and these rules then would not apply. However, the quadrants always remain the same (their positions do not change.) If ϑ is positive, then we begin rotating counter clockwise (up from the x-axis). If ϑ is negative, we begin rotating clockwise (down from the x-axis). So we may redraw the above picture in the following perspective when dealing with negative angles.
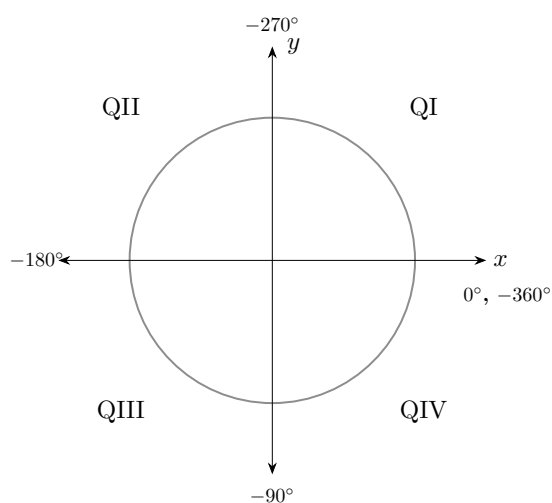
We then draw an arc with an arrow indicating the direction of rotation, and we label the arc with ϑ, whatever it may be. For example, see Figure 10.2. Some angles may be

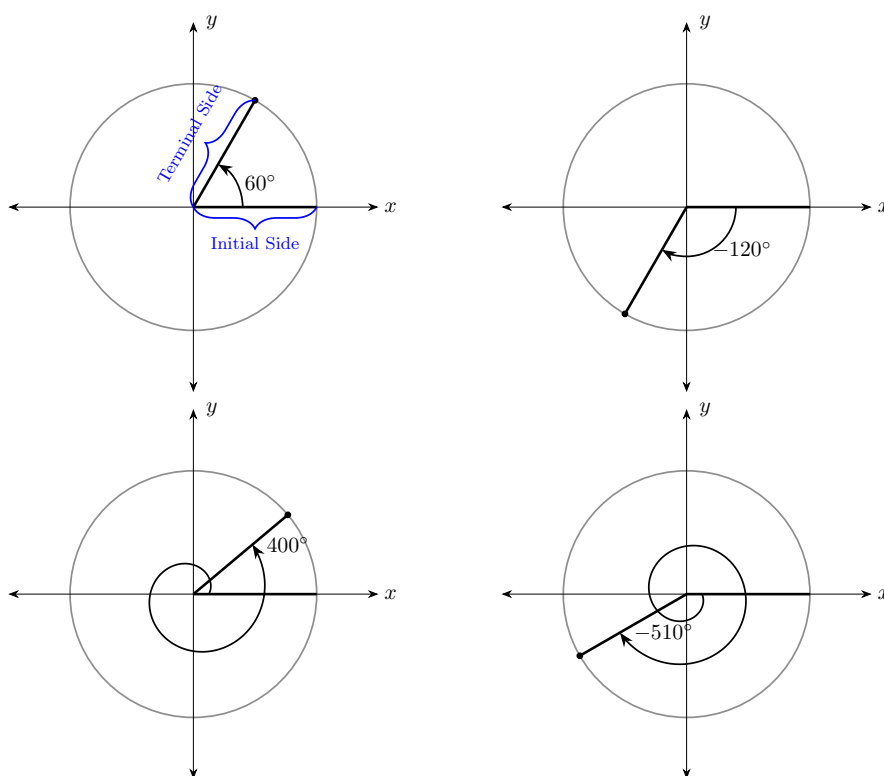Figure 10.2: Four angles sketched in standard position. Start at the positive x axis, determine what quadrant the angle is in, and then rotate in the proper direction depending on whether ϑ is positive or negative.

greater than $360°$, or less than $-360°$. That is, we are not limited to a number of full rotations; an angle can be very large or very negative, and have several full rotations before terminating. For example, the angles $400°$ and $-510°$.

To determine what quadrant a larger angle (one that is larger than $360°$) is in, it is helpful to first "subtract away" the full rotations. For example, since $400°$ is larger than $360°$, we know that there is one full rotation that lands us back on the positive x-axis. What remains is $40°$ more of rotation. Similarly for $-510°$, we rotate clockwise $-360°$, leaving us with a remaining $-150°$ starting at the positive x-axis. The concept of coterminal angles will be helpful for this, but first we will mention radians.

For some terminology, the positive x-axis is called the *initial side* and the line we draw after rotation is the *terminal side.*

10.2.2 Radians

Although understanding circles and angles in the context of a full rotation as $360°$ is useful; the number 360 has many factors:

$$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 72, 90, 120, 180, 360\}.$$

Several of those factors are "special" angles that we will care about later, but even numbers like 90 and 180 are factors, and as angles occur naturally. This allows us to cut up a circle into a whole number of pieces, which each piece represented by a whole number angle, and this can be done for each factor.

There is another intuitive way to understand angles, and it stems from the ratio between the radius and circumference of a circle. Recall that the radius r of a circle is the distance between the center point of the circle and the outer boundary. The circumference C of a circle is the distance around the circle; if one makes a circle with a piece of rope, then cuts the rope and lays it against a ruler, the length of the rope is the circumference. We then use the greek letter pi (π) to denote the following relationship:

$$\pi = \frac{C}{2r}.$$

This relationship is a nontrivial discovery; while throughout early human history there have been many estimations and hints at π, this relationship is a more formal result.

Solving the equation gives $C = 2\pi r$. Throughout this chapter we will be using the **unit circle**, or the circle of radius 1 centered at the point $(0, 0)$. The unit circle has circumference $C = 2\pi$. Then we can think of angles in the same way we think of $360°$, except in terms of 2π. For example, half a circle in degrees is half of $360°$. In radians, then, $180°$ is π radians, or half of 2π. Similarly, $90°$ is one fourth, or a quarter of a full circle. A quarter of 2π is $\frac{2\pi}{4} = \frac{\pi}{2}$, and so $90°$ and $\frac{\pi}{2}$ are equivalent. Understanding how angles are "portions" of a full circle, we see that viewing the circle in degrees or in radians is the same.
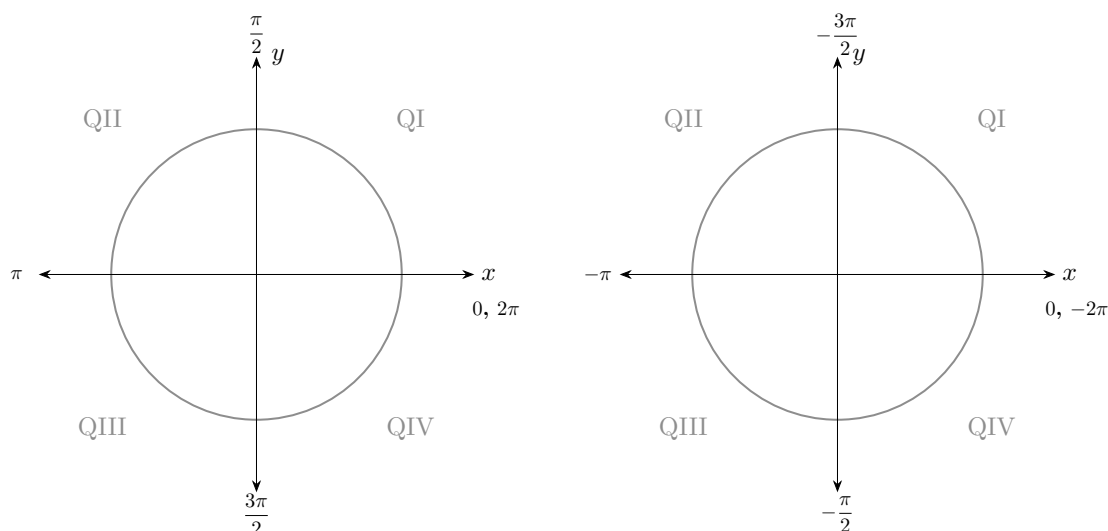
Figure 10.3: The plane labeled with radians instead of degrees.

10.2.3 Coterminal Angles

> **Definition 10.1: Definition – Coterminal**
>
> Two angles are coterminal if their terminal sides are the same when sketched in standard position.

For those already familiar with the unit circle, two angles are coterminal if they correspond to the same point (x, y) on the unit circle.

For example, consider $\theta = 150°$, which terminates in the second quadrant QII. In order to sketch $150°$, we rotate counter-clockwise until we land in QII, and then draw the terminal side. However, notice that we could have also rotated in the opposite direction; we could have rotated clockwise to QII. How many degrees would we have rotated clockwise? IF a full circle is $360°$, and we rotate counter-clockwise by $150°$, then the remaining portion of the circle is $360° - 150° = 210°$. That is, to arrive at the same terminal side as $150°$ by rotating clockwise, we rotate counter-clockwise by $210°$, or rotate by $-210°$. One could also think of this situation in the following way. We know a full revolution is $360°$. Then, one can consider the terminal side of $150°$ as an initial side. That is, if we start from the terminal side of $150°$ and rotate clockwise or counter-clockwise by full revolutions, we can find many angles which are coterminal with $150°$. By starting at the terminal side of $150°$, if we rotate clockwise, then we subtract $360°$ from $150°$, and if we rotate counter-clockwise, we add $360°$.

So, from this we can see that $-210°$ and $510°$ are coterminal with $150°$. So given
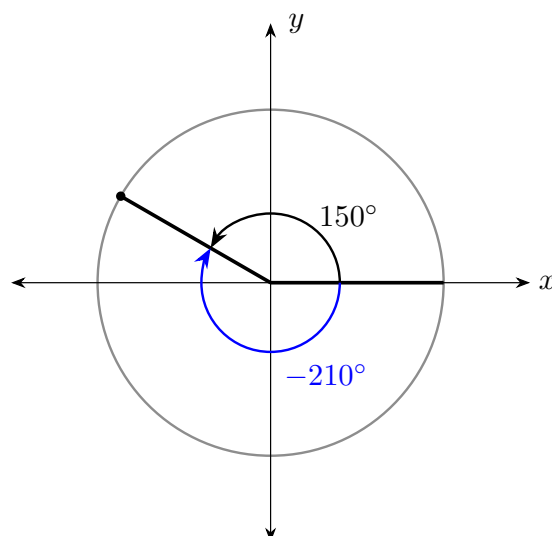
Figure 10.4: Two angles which are coterminal.

any angle ϑ, we can obtain an angle ϑ_c coterminal to ϑ by adding or subtracting $360°$. In radians, we replace $360°$ with 2π as before. In math:

$$\vartheta_c = \vartheta + 360° \quad \text{or} \quad \vartheta_c = \vartheta - 360°$$
$$\vartheta_c = \vartheta + 2\pi \quad \text{or} \quad \vartheta_c = \vartheta - 2\pi$$

Using these equations, we see that if we take the difference of any angle with an angle which is coterminal to it, in other words $\vartheta_c - \vartheta$, then the result must be either $360°$ or $-360°$ (2π or -2π in radians.) Then, **given any two angles, we can check if they are coterminal by taking their difference. If the difference is a *multiple* of a full revolution, then they are coterminal.**

Example 10.2: Coterminal Angles

1. Find one positive and one negative angle which is coterminal to $\dfrac{7\pi}{3}$.

   Here, notice that $\frac{7\pi}{3}$ is larger than 2π. That is, it is more than one full revolution. To find a positive coterminal angle then, we can subtract 2π:

   $$\frac{7\pi}{3} - 2\pi = \frac{7\pi}{3} - \frac{6\pi}{3} = \frac{\pi}{3}.$$

   Of course, we could have also added 2π as a safer option. Now to find a
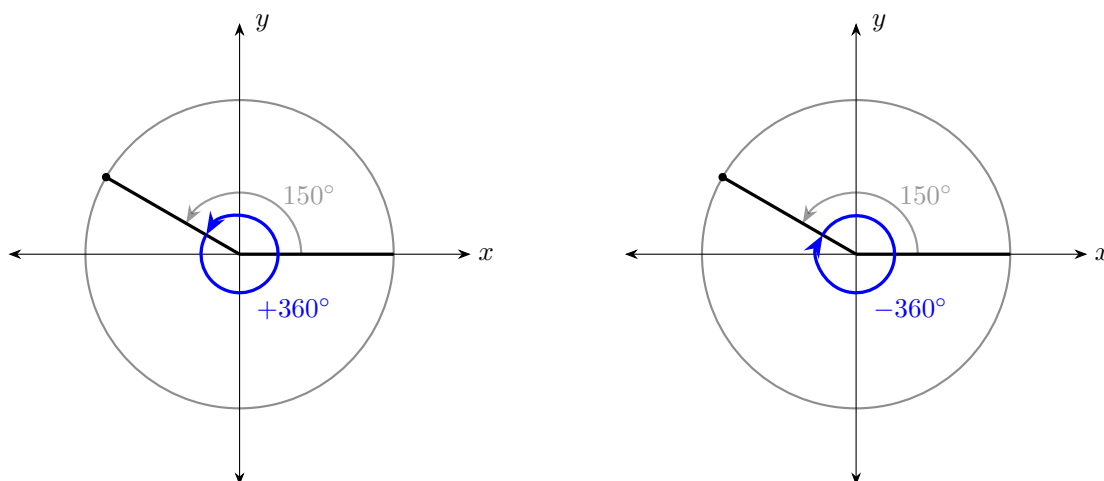
Figure 10.5: We start from $150°$, then rotate a full revolution either clockwise or counter-clockwise. If counter-clockwise, we add $360°$ to the already existing $150°$. If clockwise, we subtract $360°$.

negative coterminal angle, we will need to subtract once more.

$$\frac{\pi}{3} - 2\pi = \frac{\pi}{3} - \frac{6\pi}{3} = \frac{-5\pi}{3}.$$

The main take-away here is that it may take subtracting or adding full revolutions more than once to obtain a desired coterminal angle (in this case, one negative and one positive).

2. Let $\vartheta = 315°$. Check if the following angles are coterminal with ϑ.

    a) $-45°$

    b) $\frac{9\pi}{4}$

    c) $\frac{23\pi}{4}$

    d) $-405°$

Notice that ϑ is given to us in degrees, but we are asked to compare to other measurements in radians. We can convert ϑ to radians, or we can convert the other given angles to degrees. One should do what they are most comfortable with, though it may be more convenient to be comfortable with radians so that we only need to convert ϑ. In radians. $315°$ is $\frac{7\pi}{4}$ (check this.)

    a) $-45°$. Taking the difference, $315° - (-45°) = 315° + 45° = 360°$. ✓

b) $\frac{9\pi}{4}$. Taking the difference with our converted value of ϑ, we have $\frac{7\pi}{4} - \frac{9\pi}{4} = -\frac{2\pi}{4} = -\frac{\pi}{2}$. This is not a multiple of 2π. ✗

c) $\frac{23\pi}{4}$. Taking the difference, $\frac{7\pi}{4} - \frac{23\pi}{4} = -\frac{16\pi}{4} = -4\pi$. This is a multiple of 2π. ✓

d) $-405°$. Taking the difference, $315° - (-405°) = 720°$. This is a multiple of $360°$. ✓

10.2.4 Reference Angles

The final topic in this introduction on angles and standard position is reference angles.

Definition 10.3: Definition – Reference Angle

The reference angle of an angle ϑ is the **acute** angle between the terminal side of ϑ and the x-axis. The reference angle is always **positive**.

Four examples are highlighted in Figure 10.6. Given an angle between 0 and $360°$, one can find the reference angle by understanding what information one has access to. For example, if ϑ is in the first quadrant, then ϑ is it's own reference angle. So the reference angle of $60°$ is $60°$ (or $\pi/3$.) If ϑ is in the second quadrant, then we must think a bit more. For example, consider $150°$. We know that we rotate counter-clockwise $150°$, and we know that half of a circle is $180°$ of a rotation. Then $180° - 150° = 30°$ is our reference angle. That is, the acute space between the terminal side of $150°$ and the x-axis is half a circle, minus what was already traveled. Another way to think about it is, we first rotate $150°$, then to travel from the terminal side of $150°$ to the x-axis, we must travel $30°$ more to reach $180°$. With similar logic we can deduce that if ϑ is in QIII then the reference angle is $\vartheta - 180°$, since once again we want to measure the space between the x-axis and the terminal side.

We can make the following table. The reader is encouraged to work out the justification for QIV.

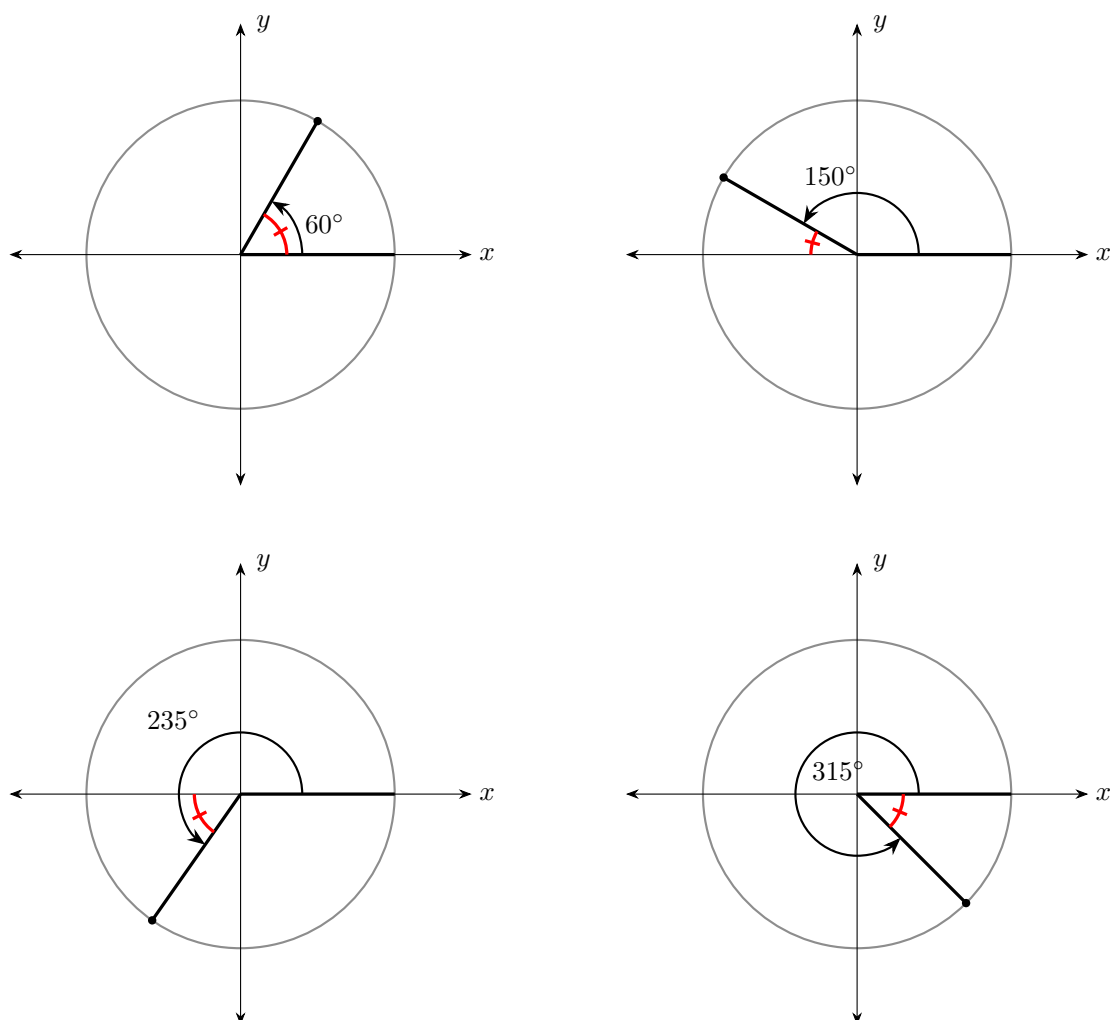| ϑ Location	Ref Angle (DEG)	Ref Angle (RAD)
QI	ϑ	ϑ
QII	$180° - \vartheta$	$\pi - \vartheta$
QIII	$\vartheta - 180°$	$\vartheta - \pi$
QIV	$360° - \vartheta$	$2\pi - \vartheta$

Figure 10.6: One angle is sketched to illustrate how reference angles work in each quadrant. In the picture, each reference angle is marked with a red arc with a dash through it, △ .

Note, these formulas only work for angles which are between 0 and $360°$, or 0 and 2π. If one is given an angle which does not fall in those ranges, then one must find the positive **coterminal** angle in the range. Coterminal angles share the same reference angle, as they have the same terminal side!

Example 10.4: Reference Angles

1. Here are some angles and their reference angles.

   a) $150°$ has a reference angle of $180° - 150° = 30°$.

   b) $\dfrac{4\pi}{3}$ has a reference angle of $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$.

   c) $\dfrac{11\pi}{6}$ has a reference angle of $2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$.

   Finding the reference angle is a process in which one identifies which quadrant the angle is in, and then applies to correct formula, or logical reasoning. One does not have to **memorize** the formulas so much as understand where they come from. If one can sketch an angle in standard position, then one can find the reference angle.

2. If an angle does not fall in the ranges $(0, 360°)$, $(0, 2\pi)$, then we first find the coterminal angle in those ranges, then we compute the reference angle.

   a) $\vartheta = -\dfrac{7\pi}{4}$ is coterminal with $\frac{\pi}{4}$. (We add 2π to ϑ). As ϑ is in the first quadrant, ϑ has a reference angle of $\frac{\pi}{4}$.

   b) $\vartheta = \dfrac{21\pi}{4}$. We find the coterminal angle in the range $(0, 2\pi)$. By subtracting 2π twice we obtain $\frac{5\pi}{4}$. Then, ϑ is in QIII, and so the reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$.

When one gets more comfortable with angles and finding reference angles, one becomes less reliant on pinpointing the correct formula to use.

10.2.5 Main Take-Away

The main take away of this section is to understand what it means to sketch an angle in standard position, what a coterminal angle is, and what a reference angle is. One should become comfortable with both degrees and radians; it can be cumbersome to constantly convert from one to another, especially when under pressure on an exam! Finally, the most important skill one must learn from this section is finding the refer-
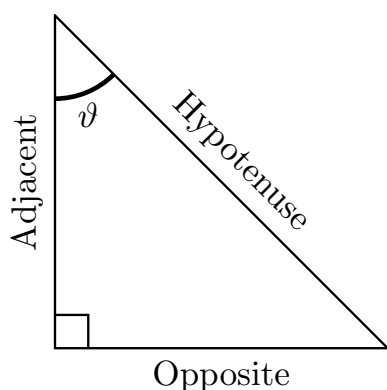
ence angle of a given angle ϑ.

1. Check if ϑ is in the range $(0, 360°)$ if degrees, and $(0, 2\pi)$ if radians.

2. If ϑ is not in the proper range, then find the coterminal angle of ϑ that is in the proper range by repeatedly adding or subtracting multiples f $360°$ or 2π.

3. Finally, identify the proper quadrant and calculate the reference angle.

10.3 Trig Functions

Triangles are fundamental in understanding the unit circle, as we will see later on. Triangles have three sides and three angles. In particular, we care about **right triangles**, triangles where one of the three angles is $90°$. Right triangles have three sides, two of which are *legs* and one of which is the *hypotenuse* (the longest side.)

Trigonometric functions are functions which define a relation between a given angle of a right triangle and ratios of the sides of triangles. We have six trig functions, namely sine, cosine, tangent, cosecant, secant, and cotangent. Given a right triangle and one of the non-right angles, we define the sides relative to the angle as 'opposite', 'adjacent' (next to), and the hypotenuse. The abbreviations used should be clear. We define the trig functions as follows.

$$\sin(\vartheta) = \frac{\text{Opp}}{\text{Hyp}} \qquad \csc(\vartheta) = \frac{1}{\sin(\vartheta)} = \frac{\text{Hyp}}{\text{Opp}}$$

$$\cos(\vartheta) = \frac{\text{Adj}}{\text{Hyp}} \qquad \sec(\vartheta) = \frac{1}{\cos(\vartheta)} = \frac{\text{Hyp}}{\text{Adj}}$$

$$\tan(\vartheta) = \frac{\text{Opp}}{\text{Adj}} \qquad \cot(\vartheta) = \frac{1}{\tan(\vartheta)} = \frac{\text{Adj}}{\text{Opp}}$$

Now one may be given a triangle and a marked angle ϑ, and then asked to compute the trig function values with ϑ as an input. Some students prefer to use the mnemonic SOHCAHTOA to remember the trig ratios; sine, opposite, hypotenuse, cosine, adjacent, hypotenuse, tangent, opposite, adjacent.

Example 10.5: The 3-4-5 Triangle

Consider the following triangle. Remember, trig ratios are calculated relative to ϑ. Here the side of length 4 is opposite ϑ, the side of length 3 is adjacent, and the side of length 5 is the hypotenuse (it is opposite the $90°$ angle.)
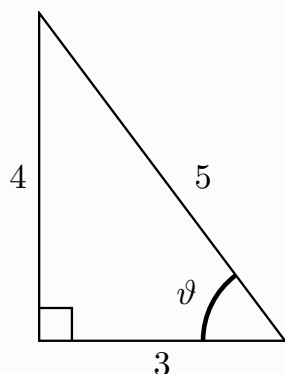
$$\sin(\vartheta) = \frac{4}{5} \qquad \csc(\vartheta) = \frac{5}{4}$$

$$\cos(\vartheta) = \frac{3}{5} \qquad \sec(\vartheta) = \frac{5}{3}$$

$$\tan(\vartheta) = \frac{4}{3} \qquad \cot(\vartheta) = \frac{3}{4}$$

Now if one is given a triangle with a missing side length, say the hypotenuse, then we would not be able to calculate $\sin(\vartheta)$ for a given angle ϑ. This is where the Pythagorean Theorem comes into play.

Theorem: Pythagorean Theorem

For any right triangle with legs a, b and hypotenuse c, the following is true:

$$a^2 + b^2 = c^2.$$

Indeed, we can verify this using the previous example of a 3-4-5 triangle. The hypotenuse is length 5, and the legs have length 3 and 4. Then

$$a^2 + b^2 = c^2$$
$$(3)^2 + (4)^2 = (5)^2$$
$$9 + 16 = 25 \quad \checkmark$$

Example 10.6: Finding $f(\vartheta)$ – Missing Hypotenuse

Consider the following triangle. Find $\csc(\vartheta)$.
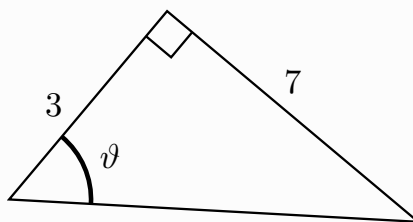
Since $\csc(\vartheta) = 1/\sin(\vartheta) = $ hyp/opp, we must find the hypotenuse. Using the Pythagorean Theorem we have

$$a^2 + b^2 = c^2$$
$$(3)^2 + (7)^2 = c^2$$
$$9 + 49 = c^2$$
$$58 = c^2$$
$$\pm\sqrt{58} = c$$

Here we pick $\sqrt{58} = c$ (the hypotenuse is positive. Soon we'll see that in the xy-plane the hypotenuse is a radius, which is always positive, however the legs may be positive or negative.) From this, we have $\csc(\vartheta) = \sqrt{58}/7$.

Example 10.7: Solving a Triangle For a Missing Leg

Consider the following triangle. Find $\sec(\vartheta)$.
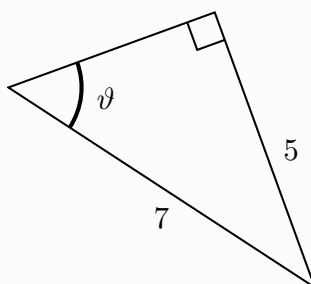
In this triangle, we are given the length of a leg and the hypotenuse. Let $a = 5$

and $c = 7$. Then using the Pythagorean Theorem, we solve for a:

$$a^2 + b^2 = c^2$$
$$(5)^2 + b^2 = (7)^2$$
$$25 + b^2 = 49$$
$$b^2 = 24$$
$$b = \pm\sqrt{24}$$
$$b = \pm 2\sqrt{6}$$

We take $b = 2\sqrt{6}$; as we are not in the xy-plane (there is no "direction" to indicate with the sign of a number), we take the positive version. So $\sec(\vartheta) = \frac{1}{\cos(\vartheta)}$, or hypotenuse divided by adjacent. We have

$$\sec(\vartheta) = \frac{7}{2\sqrt{6}} = \frac{7\sqrt{6}}{12}.$$

Note: we rationalized the denominator by multiplying the numerator and denominator by $\sqrt{6}$.

10.4 The Unit Circle

So far, we have been sketching angles in standard position. That is, we start in the xy-Plane which is divided into four quadrants which can be thought of in terms of the sign of x and y. Namely, QI: $(+, +)$, QII: $(-, +)$, QIII: $(-, -)$, and QIV: $(+, -)$. We then set up a correspondence between angles and lines in the plane; when sketching an angle ϑ the convention is to start on the positive x axis, and then rotate counter-clockwise or clockwise depending on if ϑ is positive or negative respectively. We would then draw the **terminal side** of ϑ. Earlier in these notes, we were sketching angles with an imaginary circle, but what we really should have done is sketch an angle like so:
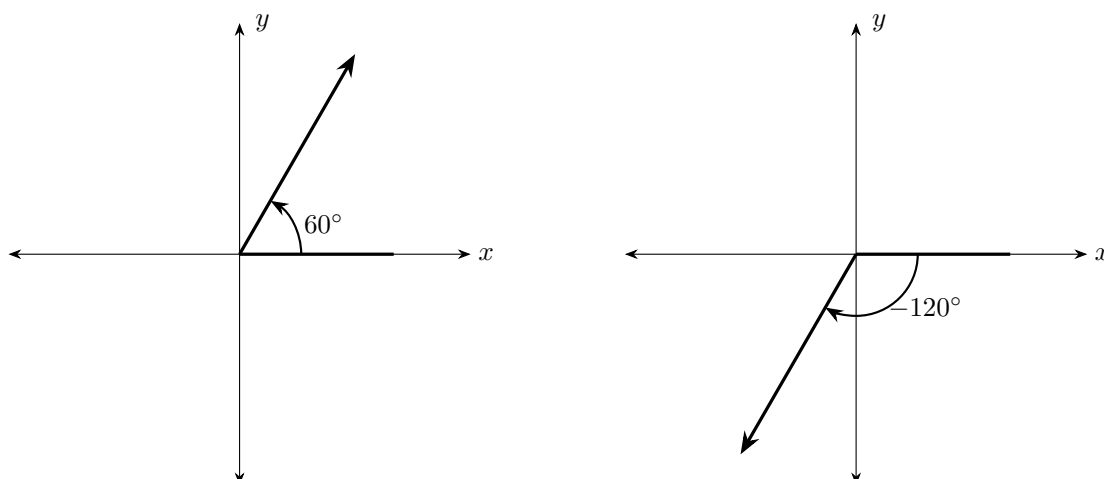
Notice, when we draw the terminal side of ϑ, we draw a *ray*, a line starting at the point $(0,0)$ with an arrow indicating that it goes on forever. Each point (x, y) on this line corresponds to the angle ϑ. This allows us to think of the xy-plane in a manner which is different than before. Previously, we'd think of the plane as two perpendicular axes, x and y, with points (x, y) representing a placement on each axis. This forms a grid. However, another way to think of the xy-plane is to think in terms of angles and a radius. Instead of thinking "to travel to the point (x, y), we move x units horizontally and y units vertically", we can instead think of points in terms of (r, ϑ): "I rotate ϑ degrees/radians, then I walk a distance of r."

We now introduce the unit circle. One can define a circle as a set of points which are all the same distance from a specified center point. The *unit circle* is the set of points which are distance 1 from the point $(0,0)$. Each point of the unit circle is some (x, y) in the xy-plane. We can think of these points in terms of the radius and an angle ϑ: we rotate ϑ degrees, then travel a distance of 1 unit.

Notice that for any angle ϑ, there is a corresponding point (x, y) on the unit circle, and one can draw a right triangle, called a *reference triangle*, with hypotenuse 1, and side lengths x and y (see Figure 10.7.) Using our knowledge of trigonometric functions, we can see that

$$\cos(\vartheta) = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x, \quad \text{and}$$

$$\sin(\vartheta) = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y.$$

So, for each point (x, y) on the unit circle, we know $x = \cos(\vartheta)$ and $y = \sin(\vartheta)$. Note that $\tan(\vartheta) = y/x$!

Before we begin to work with the unit circle, we will examine some particular values of ϑ.
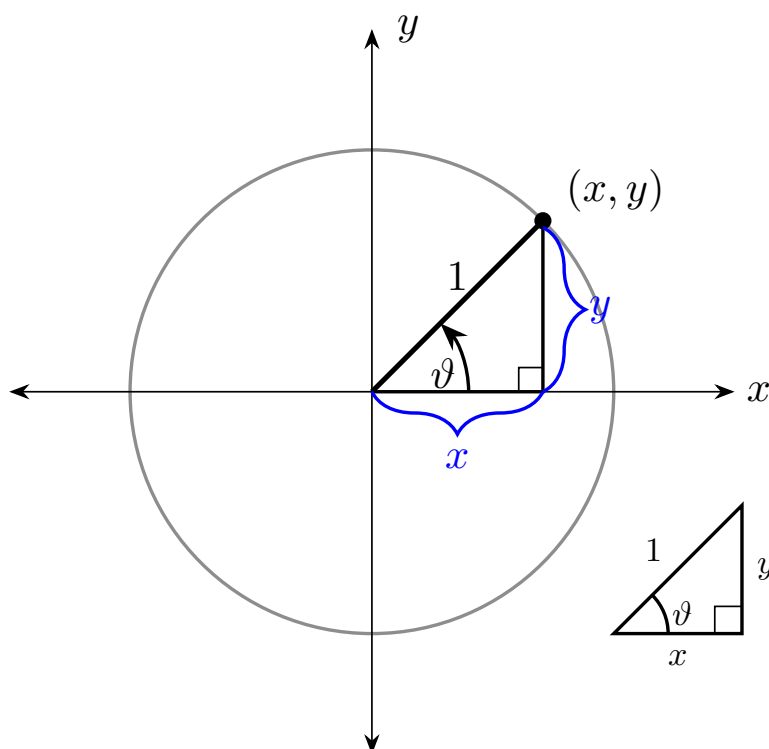
Figure 10.7: The unit circle. We rotate by an angle of ϑ, then extend out by a distance of 1 (radius 1). The angle and radius correspond to a point (x, y), and by using right triangles we can describe what x and y are in terms of trigonometric functions.

10.4.1 Special Angles and the Unit Circle

There are two special right triangles of interest to us, usually referred to by their angles. We have the 30-60-90 triangle and the 45-45-90 triangle.

In Figure 10.8 we see the special triangles. Notice the side lengths have a factor of a. We can choose a to be any number; scaling each side by the same factor does not change the angles, or particularly in this context, does not change the **ratios of the sides**. For example, from the 30-60-90 triangle we can calculate

$$\sin(30°) = \frac{a}{2a} = \frac{1}{2}.$$

If we want a hypotenuse of 1 to pretend we are on the unit circle, then we can set $a = 1/2$ for the 30-60-90 triangle and $a = 1/\sqrt{2}$ for the 45-45-90 triangle.

Figure 10.8: On the left we have the 30-60-90 triangle and on the right we have the 45-45-90 triangle. The variable a is there to show that we can scale these triangles, making each side larger or smaller by the same factor a does not change the ratios of the sides.

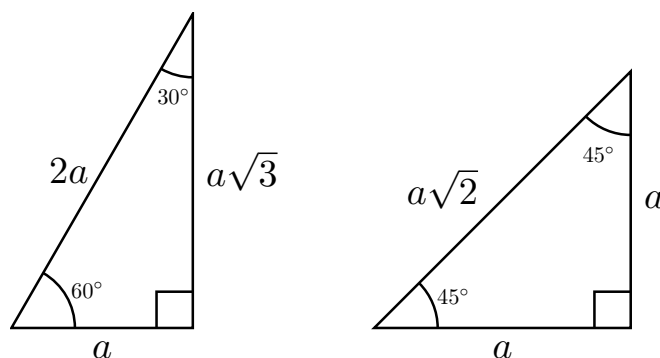| $f(\vartheta)$ $\qquad$ ϑ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$
$\sin(\vartheta)$	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$
$\cos(\vartheta)$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$
$\tan(\vartheta)$	$\dfrac{\sqrt{3}}{3}$	1	$\sqrt{3}$

This table is invaluable to us; these are all the special angles and values we will use. Tip: remembering the table is not as daunting as it seems. First, the rows are sine, cosine, and tangent, and the columns are in order by special angles 30°, 45°, and 60°. The first row, the denominators are all 2, and the numerators are $\sqrt{1} = 1$, $\sqrt{2}$, and $\sqrt{3}$ (ascending order!). Then the second row is the first row in reverse. Finally the third row need not be memorized either; the third row can be computed by computing $\sin(\vartheta)/\cos(\vartheta)$. This requires one to be comfortable with dividing fractions (keep-change-flip.) In sum, to remember the table, one needs only remember the first row!

10.4.2 Constructing the Unit Circle with Special Angles

Recall for the unit circle, $x = \cos(\vartheta)$ and $y = \sin(\vartheta)$. For our three special angles, we will have three corresponding points $(x, y) = (\cos(\vartheta), \sin(\vartheta))$ in the first quadrant. For example, the terminal side of 30° intersects the unit circle in the point
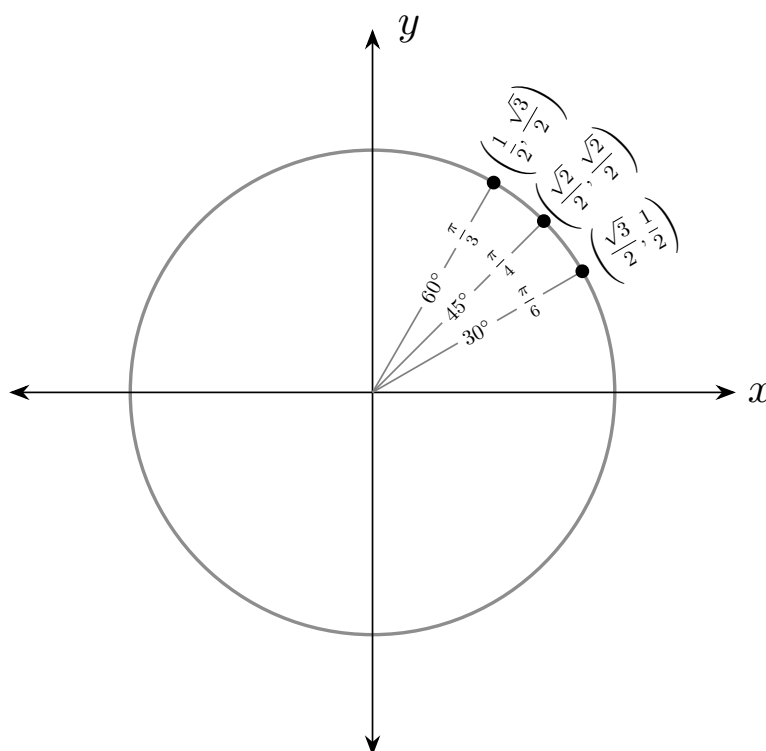
Figure 10.9: The first quadrant of the unit circle filled with our special angles.

$$(\cos(30°), \sin(30°)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Figure 10.9 shows the three special angles and their corresponding (x, y) point on the unit circle.

What about the points on the unit circle in the second, third, and fourth quadrants? First remember, every point (x, y) in the first quadrant has x and y positive. The beauty of the unit circle is that we only need two things to find the (x, y) point of any ϑ; the reference angle αp of ϑ and the corresponding point of α in the first quadrant. That is, any point (x, y) on the unit circle in the first quadrant appears in the other quadrants, up to a change in sign. By "change in sign" we mean that the x or y coordinate may become negative. Recall that the reference angle of an angle ϑ is the **positive, acute** angle between the terminal side of ϑ and the x-axis. Every angle has a corresponding reference angle (and since the reference angle is positive and acute, it lives in the first quadrant). Conversely, each angle α in QI corresponds to three other angles, one in each of QII, QIII, and QIV which have α as a reference angle. Note we are focusing on angles which are between 0 and $360°$ (or 0 and 2π.) In picture form:
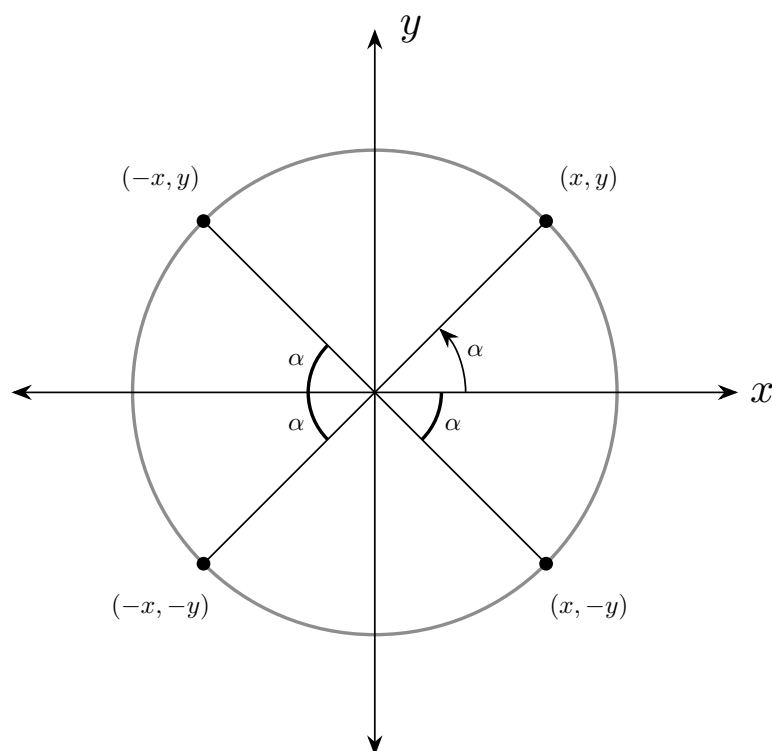
So there are angles with their terminal side in QII, QIII, and QIV with $30°$, $45°$, and $60°$ as a reference angle. Let's use $30°$ to illustrate, and come up with a strategy to remember what these angles are.

To find the angle ϑ in QII which has a reference angle of $30°$, we simply start on the negative x-axis and rotate clockwise $30°$. That is, we start at the terminal side of $180°$, then rotate 'up' $30°$, giving us $\vartheta = 180° - 30° = 150°$. By using this same logic, for QIII we start at $180°$ and rotate counter-clockwise $30°$ to get $\vartheta = 180° + 30° = 210°$. Finally, in QIV we start at $360°$ and rotate clockwise by $30°$ to get $\vartheta = 360° - 30° = 330°$. Recall previously when learning about reference angles, we were tasked with finding a reference angle given an angle. Now we are doing the reverse.

1. Given an angle ϑ.

2. Found the quadrant ϑ is in.

3. Computed the reference angle α of ϑ given the quadrant.

   a) QII: $\alpha = 180° - \vartheta$

   b) QIII: $\alpha = 180° + \vartheta$

   c) QIV: $\alpha = 360° - \vartheta$

1. Given a reference angle α.

2. Find an angle in QII, QIII, or QIV with α as a reference angle.

   a) QII: $\vartheta = 180° - \alpha$

   b) QIII: $\vartheta = 180° + \alpha$

   c) QIV: $\vartheta = 360° - \alpha$

Of course, one should be comfortable doing so with radians. Once we find all the

angles, we adjust the sign of the point in QI corresponding with 30°.

$\vartheta = 180° - 30° = 150°$
or
$\vartheta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Ref ∡: 30° or $\frac{\pi}{6}$

$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\frac{\pi}{6}$ — 150°

30° — $\frac{\pi}{6}$

$\frac{7\pi}{6}$ — 210°

330° — $\frac{11\pi}{6}$

$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$\vartheta = 180° + 30° = 210°$
or
$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

$\vartheta = 360° - 30° = 330°$
or
$\vartheta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$
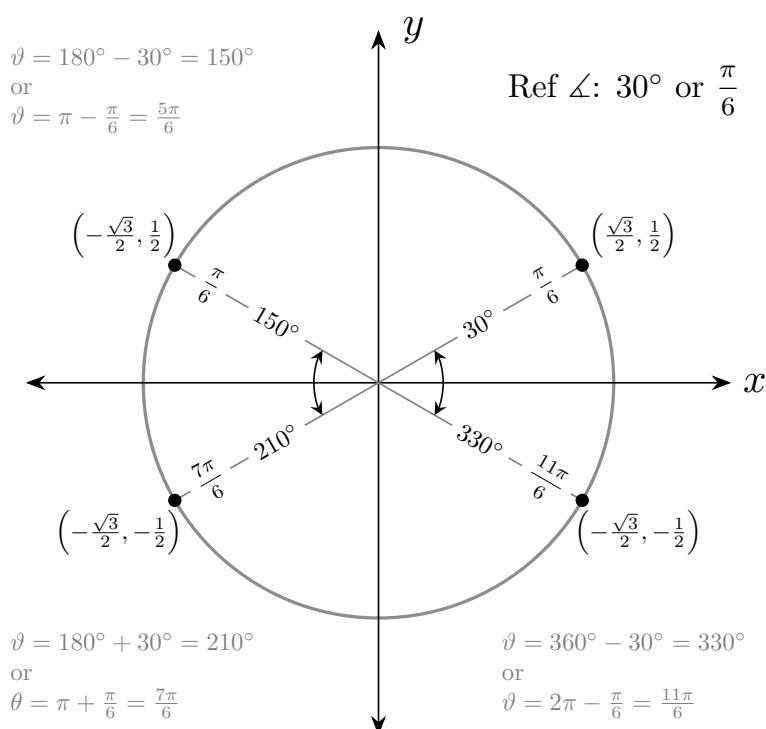
The reader is encouraged to make similar drawings for 45° and 60°. In the end we have a full unit circle in Figure 10.10. Notice that we did not point out the other special angles 90°, 180°, 270°, and 0° or 360°. Since the unit circle has radius 1, finding the coordinates for these angles should be immediate from the picture (less to memorize.)

Example 10.8: Computing Trig Functions of Special Angles

To practice using the unit circle, here are some values to double check. **In general, first one should check if an angle is in the range 0 to 2π. If not, find the coterminal angle which is in the range, and then compute the reference angle, and make sure the answer has the proper sign.**

1. $\cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

2. $\cos\left(\frac{3\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

3. $\sin\left(-\frac{5\pi}{6}\right) = \sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

4. $\tan(90°)$ is **undefined**, since $\sin(90°) = 1$ and $\cos(90°) = 0$, and $\tan(90°) = \sin(90°)/\cos(90°) = 1/0$.

   Be careful, as reciprocal trig functions may also be undefined for certain angles, since their denominators may be equal to 0.

10.5 Problems Involving The Unit Circle - Reference Triangles

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Example 10.9: Terminal Side Intersects Unit Circle

Let ϑ be an angle. What is the value of $\csc(\vartheta)$ if the terminal side of ϑ intersects the unit circle in QI at $x = \frac{13}{24}$?

We can draw a picture to represent this scenario. The terminal side is in QI, and it intersects the unit circle at a point with a specified x coordinate.
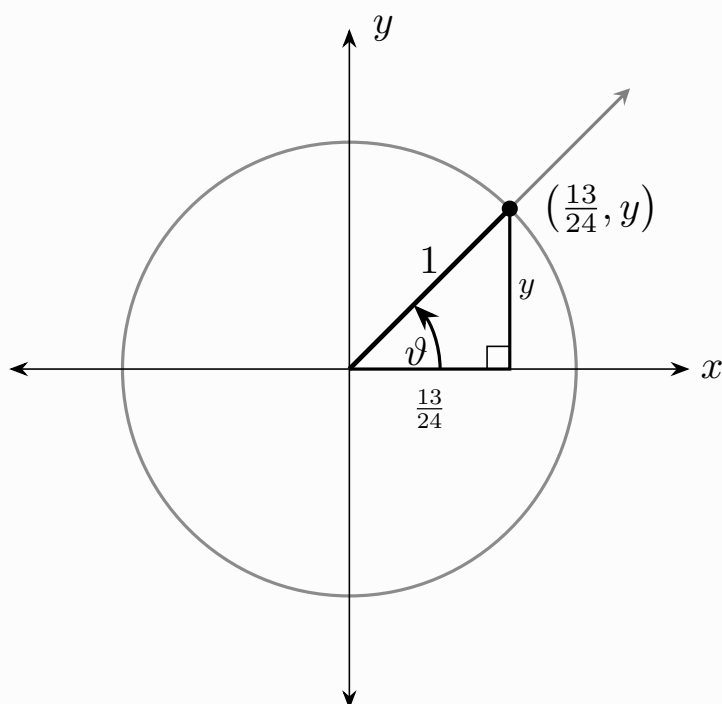
Now we have a right triangle with one missing side. Notice the hypotenuse is 1, the radius of the unit circle which is intersected by the terminal side as specified

by the problem. To find the missing side, we use the Pythagorean Theorem.

$$\left(\frac{13}{24}\right)^2 + y^2 = 1^2$$

$$\frac{13^2}{24^2} + y^2 = 1$$

$$y^2 = 1 - \frac{13^2}{24^2}$$

$$y^2 = \frac{24^2}{24^2} - \frac{13^2}{24^2}$$

$$y^2 = \frac{24^2 - 13^2}{24^2}$$

$$y^2 = \frac{407}{24^2}$$

$$y = \pm\sqrt{\frac{407}{24^2}}$$

$$y = \pm\frac{\sqrt{407}}{\sqrt{24^2}}$$

$$y = \pm\frac{\sqrt{407}}{24}$$

We take the positive solution, $y = \sqrt{407}/24$ since y is positive in QI. If we were in QIII or QIV, we would choose the negative solution.

We were asked to calculate $\csc(\vartheta)$.

$$\csc(\vartheta) = \frac{1}{\sin(\vartheta)} = \frac{1}{y} = \frac{1}{\sqrt{407}/24} = \frac{24}{\sqrt{407}} = \frac{24\sqrt{407}}{407}$$

In the previous problem, you may have also used a formula and well known identity, $\cos^2(x) + \sin^2(x) = 1$. This comes from the equation of a circle, $x^2 + y^2 = r^2$, where for the unit circle we substitute $r = 1$. Another way to see this is to look back at Figure 10.7, notice any reference triangle we draw using a point on the unit circle has sides x

and y and hypotenuse 1. Using the Pythagorean Theorem we have

$$x^2 + y^2 = 1 \quad \rightarrow \quad \cos^2(\vartheta) + \sin^2(\vartheta) = 1.$$

Example 10.10: Terminal Side Intersects Circle – Arbitrary Radius

Let ϑ be an angle with terminal side in QIV. Given that $\sin(\vartheta) = \frac{-3\sqrt{10}}{10}$, what is $\cos(\vartheta)$.

We can set up a reference triangle in two ways, both yielding the same answer. We can assume that the terminal side of ϑ intersects the unit circle, or we can instead assume we intersect a circle of larger radius. Since $\sin(\vartheta) = \frac{-3\sqrt{10}}{10}$, or opposite over hypotenuse when thinking of a reference triangle, we can take the hypotenuse to be 10, and one side to be $-3\sqrt{10}$. Or, we can assume the hypotenuse is 1 and take one side to be $\frac{-3\sqrt{10}}{10}$.
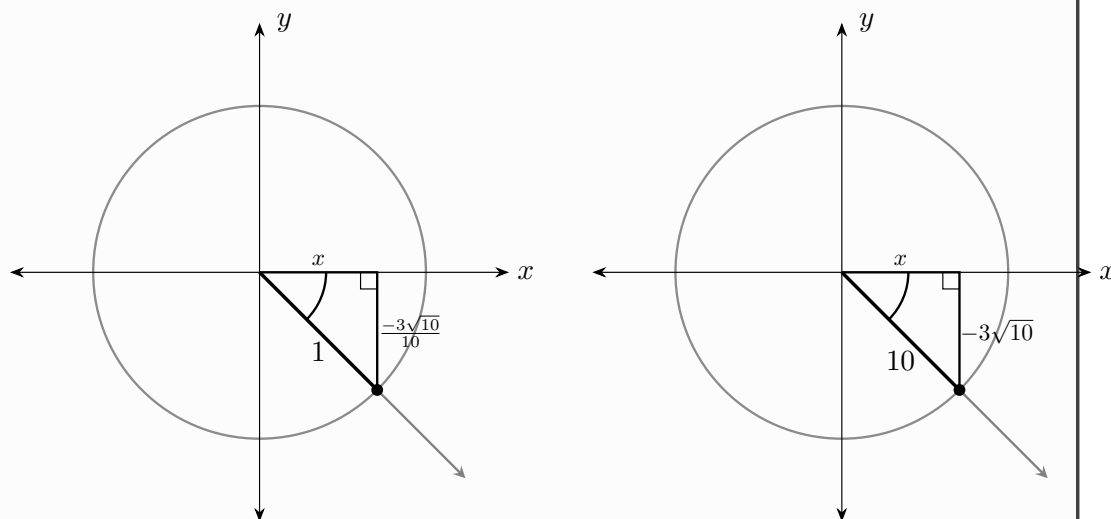
In the case of the left picture, we know on the unit circle $x = \cos(\vartheta)$, so solving for x will yield our answer. Notice the leg representing the y coordinate is negative in both pictures, since y is negative in QIV. Using the Pythagorean

Theorem

$$x^2 + \left(\frac{-3\sqrt{10}}{10}\right)^2 = 1^2$$

$$x^2 + \frac{90}{100} = 1$$

$$x^2 = 1 - \frac{90}{100}$$

$$x^2 = \frac{10}{100}$$

$$x^2 = \frac{1}{10}$$

$$x = \pm\sqrt{\frac{1}{10}}$$

$$x = \pm\frac{1}{\sqrt{10}}$$

Since we are in QIV, x is positive, and so $\cos(\vartheta) = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$.
If we used the picture on the right, then we would have to solve

$$x^2 + (-3\sqrt{10})^2 = 10^2$$

giving $x = \sqrt{10}$. Then $\cos(\vartheta)$ is adjacent over hypotenuse, or $\frac{\sqrt{10}}{10}$.

In the previous example, we saw two different set ups for approaching the same problem. We don't always have to assume we are working in the unit circle; the hypotenuse of the reference triangle, or the *radius*, does not have to be 1. We may instead interpret our trig ratios in the following way.

$$\cos(\vartheta) = \frac{x}{r}$$

$$\sin(\vartheta) = \frac{y}{r}$$

$$\tan(\vartheta) = \frac{\sin(\vartheta)}{\cos(\vartheta)} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x}$$
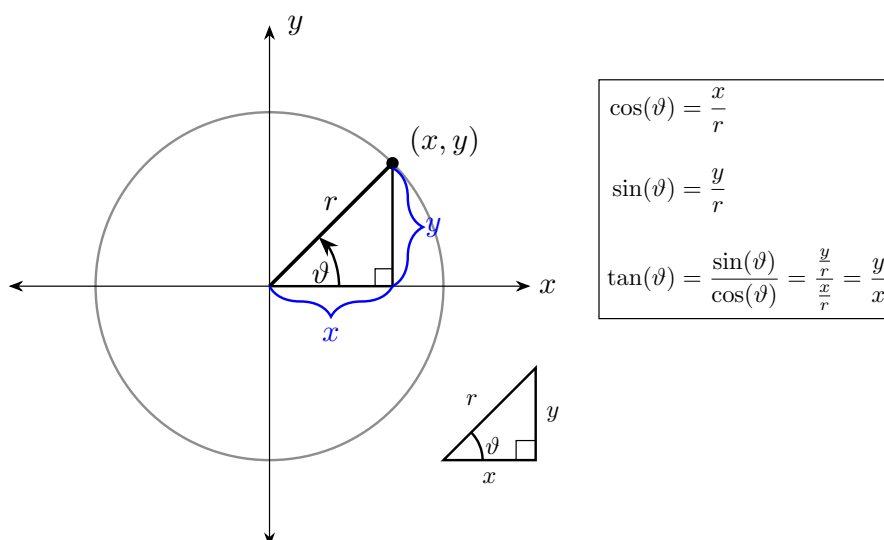
We constrain ourselves to the unit circle often, as trig functions are defined as ratios of sides of right triangles. As the radius grows bigger, the x and y coordinates change, but the ratios between the triangle sides remain the same. However, as we eluded to previously, any point in the plane corresponds to a radius and an angle. Specifically, we can rotate and then walk in the direction of the terminal side to get to any point, and this distance is not always 1 (that is, radius is not always 1) as we'll see in the next example.

Example 10.11: Given Point On Terminal Side of Angle

Let $(-3, 2)$ be a point on the terminal side of an angle ϑ. Find $\sin(\vartheta)$, $\sec(\vartheta)$, and $\tan(\vartheta)$.
The point $(-3, 2)$ is in quadrant 2, so the terminal side of ϑ is in QII. We can draw a reference triangle.
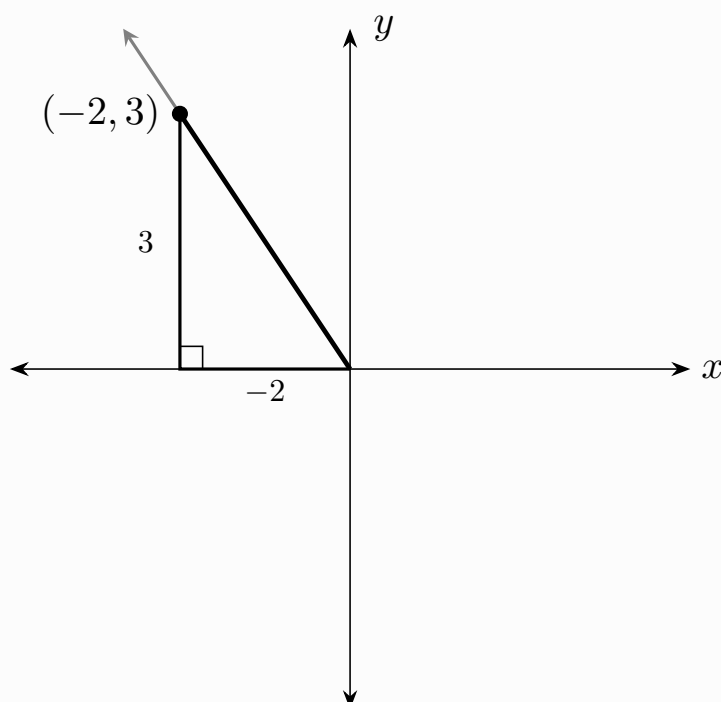
To evaluate the trig functions given, we need to solve for the hypotenuse of the reference triangle. We have

$$x^2 + y^2 = r^2$$
$$(-2)^2 + (3)^2 = r^2$$
$$4 + 9 = r^2$$
$$13 = r^2$$
$$\pm\sqrt{13} = r$$

The radius, or hypotenuse is always positive so we take $r = \sqrt{13}$. Now we have

$$\sin(\vartheta) = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\sec(\vartheta) = \frac{\sqrt{13}}{-2}$$

$$\tan(\vartheta) = \frac{3}{-2}$$

Notice that we needed to solve for the hypotenuse to calculate $\sin(\vartheta)$ and $\sec(\vartheta)$, but since $\tan(\vartheta) = y/x$ we could have evaluated that right away.
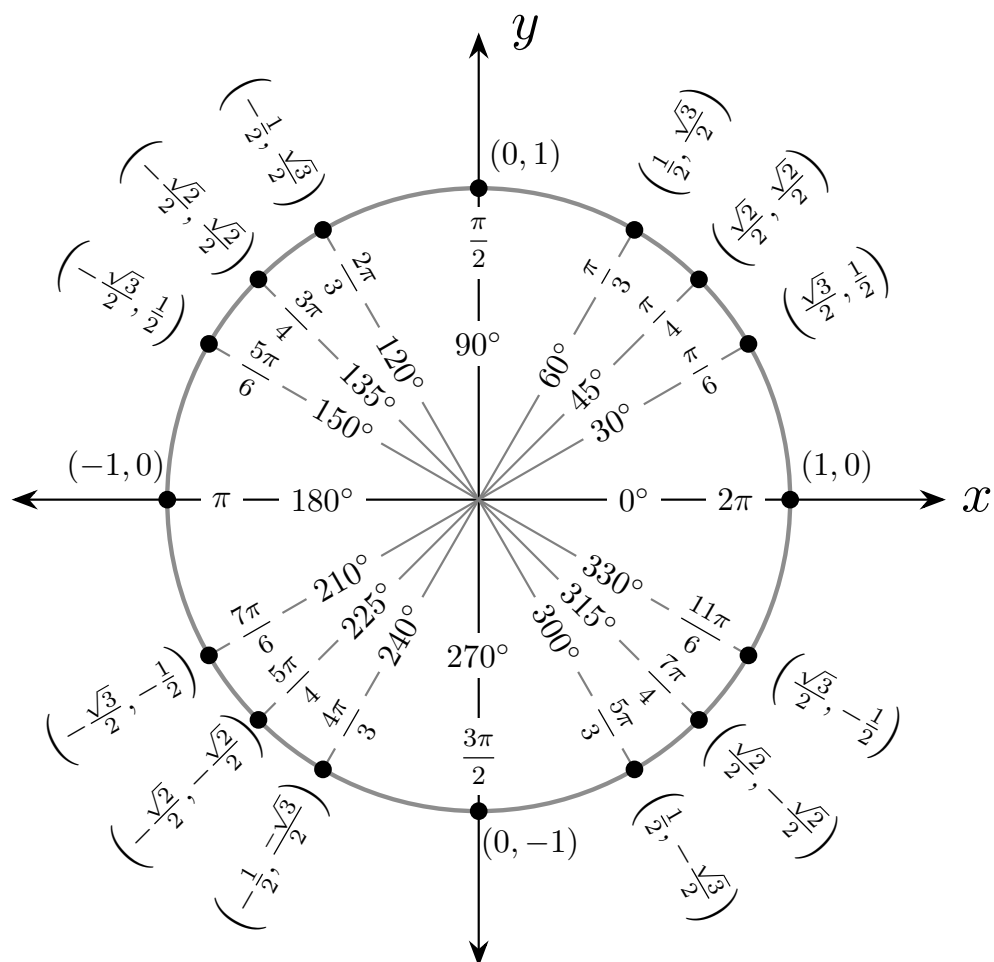
Figure 10.10: The unit circle filled out. Knowing these values, we can compute sine, cosine, tangent, cosecant, secant, or cotangent of any special angle, or any angle with a special angle as a reference angle.

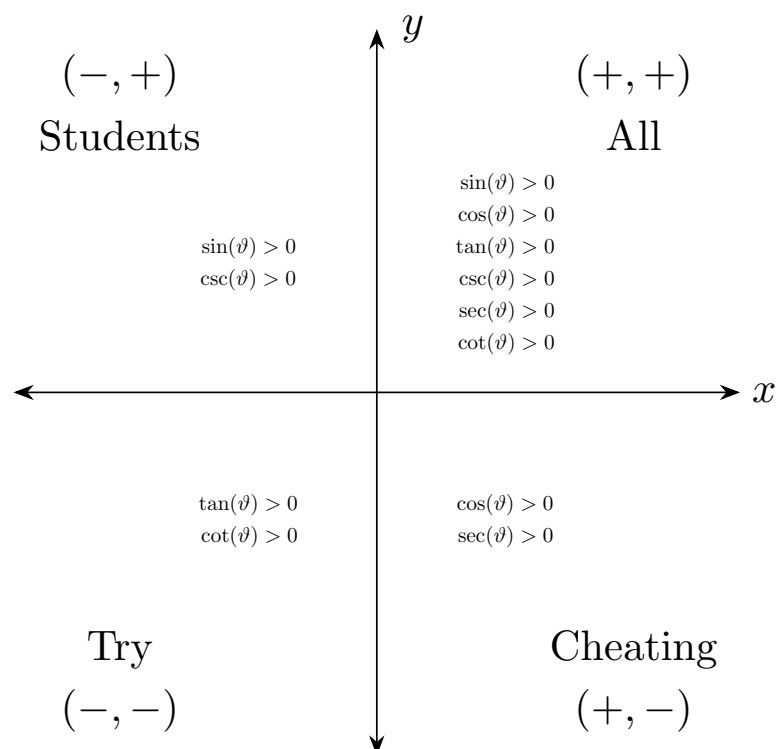Example 10.12: Find $f(\vartheta)$ Given Information

Let ϑ be an angle such that $\cos(\vartheta) = \frac{7}{8}$ and $\tan(\vartheta) > 0$. Find $\cot(\vartheta)$ and $\sin(\vartheta)$. Here we are not given the quadrant that ϑ is in, however we can use the information given to determine the quadrant. Since $\tan(\vartheta) > 0$, this implies that $y/x > 0$. In other words, either x and y are both positive, or x and y are both negative. That is, we are either in QI or QIII. Since $\cos(\vartheta) = 7/8 = x/r$, which is positive, and since r is always positive, it must be that x is positive. Since x is negative in QIII, we must be in QI.

Now we have ϑ in QI with $\cos(\vartheta) = 7/8$. Draw the reference triangle to show that $\cot(\vartheta) = 7\sqrt{15}/15$ and $\sin(\vartheta) = \sqrt{15}/8$.

In this last example we were not given a quadrant, and instead had to use context clues to figure out what quadrant the terminal side of ϑ lies in.

10.6 Identifying Quadrant From Trig Function Sign

Given information about two trig functions with regard to ϑ, we can identify which quadrant the terminal side of ϑ is in. As we saw before, $\cos(\vartheta) = x/r$, $\sin(\vartheta) = y/r$, and $\tan(\vartheta) = y/x$ (remember the reciprocals secant, cosecant, and tangent.) since r is always positive, if we know whether or not a trig function is positive or negative, we can deduce the sign of x and y. It may help some students to commit saying "**A**ll **S**tudents **T**ry **C**heating" to memory.

y

$(-,+)$

Students

$(+,+)$

All

$\sin(\vartheta) > 0$
$\cos(\vartheta) > 0$
$\tan(\vartheta) > 0$
$\csc(\vartheta) > 0$
$\sec(\vartheta) > 0$
$\cot(\vartheta) > 0$

$\sin(\vartheta) > 0$
$\csc(\vartheta) > 0$

x

$\tan(\vartheta) > 0$
$\cot(\vartheta) > 0$

$\cos(\vartheta) > 0$
$\sec(\vartheta) > 0$

Try

$(-,-)$

Cheating

$(+,-)$

Let's look at some examples.

Example 10.13: Determine the Quadrant of an Angle

Determine the quadrant in which the terminal side of ϑ lies if

1. $\cos(\vartheta) > 0$ and $\sin(\vartheta) > 0$.

   Here, $\cos(\vartheta > 0$ tells us that we are either in QI or QIV. Out of these, $\sin(\vartheta)$ is positive in QI only. Therefore ϑ terminates in QI.

2. $\cot(\vartheta) < 0$ and $\cos(\vartheta) < 0$.

   Here, $\cot(\vartheta) < 0$ tells us we are either in QII or QIV ($\cot$ follows $\tan$, the reciprocal). Then $\cos(\vartheta) < 0$ tells us ϑ terminates in QII.

Here the process is simple and step by step; find the possible quadrants for each trig function and see which quadrant they have in common.

10.7 The Graph of Sine and Cosine

- Amplitude, Frequency/Period, Phase Shift, Vertical Shift, then all together.

- Summary of what a function with all the features looks like, with example.

10.7.1 Sketching Graphs – Complete Examples

Example 10.14: Sketching sine

Sketch the graph of the function $y = 2\sin\left(3x - \frac{\pi}{4}\right) + 1$.

First, we write our function in general form:

$$y = 2\sin\left(3x - \frac{\pi}{4}\right) + 1 = 2\sin\left(3\left(x - \frac{\pi}{12}\right)\right) + 1.$$

Let's list some info:

1. $a = 2$, so we have an **amplitude** of $|2| = 2$.

2. $b = 3$, so we have a **period** of $\frac{2\pi}{3}$.

3. A **phase shift** of $\frac{\pi}{12}$ (right). One can find the phase shift by either factoring out b, or setting the inside of the sine function equal to 0 and solving.

$$3x - \frac{\pi}{4} = 0$$
$$3x = \frac{\pi}{4}$$
$$x = \frac{\pi}{12}$$

This is both the **starting point** of the primary cycle and the phase shift (in this case, $\pi/12$ units to the right.)

4. Vertical shift $d = 1$, so consequently the midline equation is $y = 1$.

We need to find the new x values for the five points of the primary cycle that we will sketch.

Finding New x Values.
Method 1 – Find Δx.

First, we solve for the first critical x value. This is the start of our cycle. We have

$$3x - \frac{\pi}{4} = 0 \rightarrow x = \frac{\pi}{12}.$$

In $\sin(x)$ function, our original five points values are $\pi/2$ apart in x.

x	y
0	0
$\pi/2$	1
π	0
$3\pi/2$	-1
2π	0

How far apart are our critical values for $2\sin\left(3x - \frac{\pi}{4}\right) + 1$? It all comes from b.
The phase shift of $\pi/12$ tells us where to start, at $x = \pi/12$.
All we do to find the new distance between critical values is calculate $\Delta x = \pi/2b$. This can be explained in two ways:

1. The original gap between the points is $\pi/2$. Divide each gap size $\pi/2$ by b, or in other words $\Delta x = \dfrac{\pi}{2b}$.

2. The length of one cycle is the **period**, $2\pi/b$. There are five original points, and so there are **four consecutive gaps** between these points. So if we divide the length of the cycle by four, we get the length of each gap. So $\dfrac{\frac{2\pi}{b}}{4} = \dfrac{\pi}{2b}$.

Some students may find it easier to remember $p/4$; calculating the period and then dividing that by 4.
Now from $\pi/12$, the value $b = 3$ tells us that the next critical value is

$$\frac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$$

away. So consecutive points are $\pi/6$ apart.

$$x = \frac{\pi}{12} \qquad\qquad (x\text{-int})$$

$$x = \frac{\pi}{12} + \frac{\pi}{6} = \frac{3\pi}{12} \qquad\qquad (\text{Max})$$

$$x = \frac{3\pi}{12} + \frac{\pi}{6} = \frac{5\pi}{12} \qquad\qquad (x\text{-int})$$

$$x = \frac{5\pi}{12} + \frac{\pi}{6} = \frac{7\pi}{12} \qquad\qquad (\text{Min})$$

$$x = \frac{7\pi}{12} + \frac{\pi}{6} = \frac{9\pi}{12} = \frac{3\pi}{4} \qquad\qquad (x\text{-int})$$

Method 2 – Five Equations

We can set up five equations to find the new x values of our five special points:

$$3x - \frac{\pi}{4} = 0 \qquad \rightarrow \qquad x = \frac{\pi}{12}$$

$$3x - \frac{\pi}{4} = \pi/2 \qquad \rightarrow \qquad x = \frac{\pi}{4}$$

$$3x - \frac{\pi}{4} = \pi \qquad \rightarrow \qquad x = \frac{5\pi}{12}$$

$$3x - \frac{\pi}{4} = 3\pi/2 \qquad \rightarrow \qquad x = \frac{7\pi}{12}$$

$$3x - \frac{\pi}{4} = 2\pi \qquad \rightarrow \qquad x = \frac{3\pi}{4}$$

To obtain the new y-values, we take our primary y values, multiply by a and add d. Then we have a complete table of new values.

x	y
0	1
$\pi/2$	3
π	1
$3\pi/2$	-1
2π	1

Sanity check: We calculated the period to be $\frac{2\pi}{3}$. This is the length of one cycle. If we take the difference between our starting and end point, we should get the period.

$$\frac{9\pi}{12} - \frac{\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}.$$

We know the amplitude is 2, so we can check for that too: $|a| = \frac{\text{max} - \text{min}}{2} = 2$.

Formulas to remember:

$$|a| = \frac{\text{max} - \text{min}}{2}$$
$$d = \frac{\text{max} + \text{min}}{2}$$
$$p = \frac{2\pi}{b}$$
$$\Delta x = p/4$$

10.8 The Graph of Secant, Cosecant, Tangent, and Cotangent.

10.9 Trig Inverses

$\sin^{-1}(x)$ and $\tan^{-1}(x)$ use the right side of the unit circle, while $\cos^{-1}(x)$ uses the upper half.

10.10 Trig Equations