Precalculus

Inverse Functions

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This work would not have been possible without the hard work and dedication of many individuals to make many technical things possible. This document was produced in LaTeX, making use of KOMA-script, as well as TikZ and PGF.

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6 Inverse Functions

Recall: A *function* is a rule which defines a relationship between an input variable, say x, and an **unique** output (often referred to as y). Functions can be given names, commonly f, g, or h. When we define a function, we may write something like

$$f(x) = x + 5$$

which is read "f of x equals x plus 5." This function, whose name is f and input is x, takes the input and adds 5 to it, giving us an output. To evaluate a function at a specified value of x, say 5, we write

$$f(5) = (5) + 5 = 10.$$

Everywhere we see an x, we substitute a 5, then we simplify. Again, the output is **unique**; when we plug in a number for x, the function will output exactly one number. To test this on a graph, we use the vertical line test. If we can draw a vertical line on top of a graph, then that signifies that there are two y values which share the same x value; the given graph is not that of a function.

Now we define what it means for a function to be one-to-one.


Definition 6.1: One-to-One Function

A function is said to be **one-to-one** if for every output there is exactly one input.

To test if a given graph is that of a one-to-one function, we use the horizontal line test; if we can draw a horizontal line on a graph where it crosses the graph twice, then there is an output value (a value on the y-axis) which comes from two different x values.

It is easy to get confused about what it means to be a function, and what it means to be a one-to-one function. A function takes an input and gives **exactly one output.** A function is one-to-one if every y-value comes from a unique x value. For example, let us look at the function $f(x) = x^2$.

Certainly $f(x) = x^2$ is a function. We plug in a number, we get exactly one number out. However, $f(x) = x^2$ is not one-to-one. The output 4 comes from two different x

$$f(x) = x^2$$

$$g(x) = x^2,\ x \geq 0$$
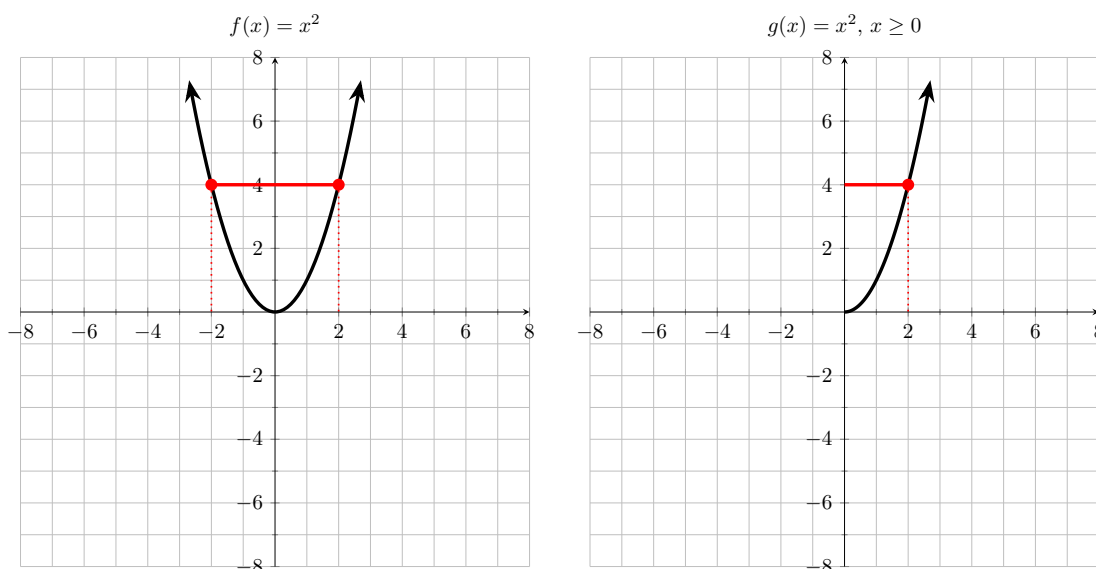
values. That is, $f(-2) = f(2) = 4$. We can see that x^2 horrifically fails the horizontal line test. However, if we are selective of the domain of our function, then we can make a one-to-one function. In the case of x^2, if we only consider the portion where $x \geq 0$, then we have a one-to-one function. In general, we can take "one half" of any quadratic function, as they are all symmetric about the vertex.

The concept of a one-to-one function comes in handy when we want to define an *inverse* for our function.

Definition 6.2: Function Inverse

Let f be a one-to-one function. Then f has an inverse function, denoted f^{-1} such that if $f(x) = y$, then $f^{-1}(y) = x$.

In words, while f takes inputs and gives outputs, f^{-1} takes outputs of f and returns the corresponding input. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

One should reason with themselves to truly understand the definition; why is it that the compositions $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Now we may understand why the concept of one-to-one is important. The function $f(x) = x^2$ is not one-to-one; it does not have an inverse. If it had an inverse f^{-1}, then $f^{-1}(4)$ would need to output both -2 and 2, and so f^{-1} would not be a function. However, if we specify that $f(x) = x^2,\ x \geq 0$, then f does indeed have an inverse
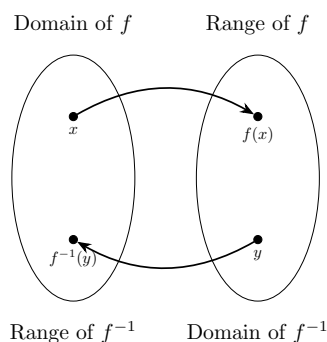
Figure 6.1: A diagram depicting a function, which assigns to elements of one set elements from another set, and an inverse.

$$f^{-1}(x) = \sqrt{x}.$$

6.1 Finding the Inverse

Again, the inverse takes y values and returns x values. That is, for every point (x, y) in the graph of f, we have the point (y, x) in the graph of f^{-1}.

Example 6.3: Finding the Inverse Given Coordinates

Let $g = \{(-8, 0), (-4, 5), (2, 3), (3, 4), (7, -4)\}$

Notice that g is a function; each x value comes with exactly one y value. We ask ourselves, is g one-to-one? We check, does each y value have its own, unique x value. Indeed, each y value appears once and only once. Now we want to evaluate, say, $g^{-1}(3)$. Two coordinates have 3 in them, $(2, 3)$ and $(3, 4)$. However, we are evaluating $g^{-1}(3)$, and the inverse takes y values as inputs. In words, $g^{-1}(3)$ asks: "what value of x, when plugged into g, gives 3?" We see then that $g^{-1}(3) = 2$, since $g(2) = 3$.
With similar reasoning, we see that $g^{-1}(-4) = 7$ or $g^{-1}(0) = -8$.

A list of coordinates is the first step. One half step and we begin to examine graphs.

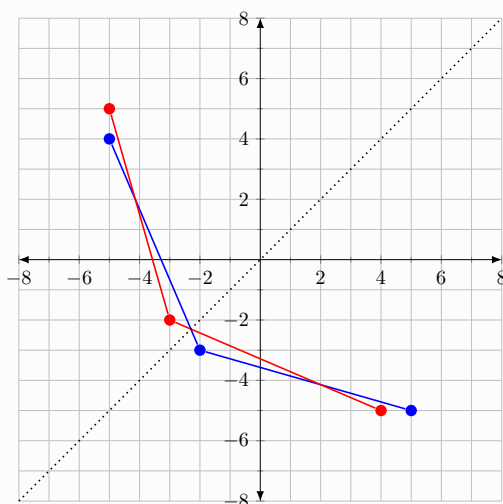**Example 6.4: Finding the Inverse Given a Graph**

Consider the following function:
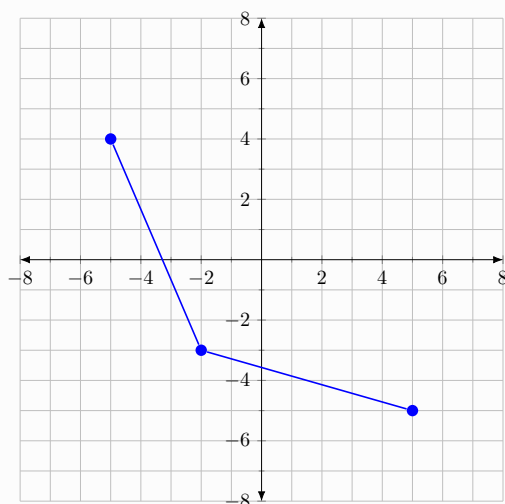
The function depicted has domain $[-5, 5]$ and range $[-5, 4]$. We make a table of the (x, y) values:

x	$f(x)$
-5	4
-2	-3
5	-5

The inverse, $f^{-1}(x)$ takes y values and gives the x values; so we switch the coordinates.

x	$f^{-1}(x)$
4	-5
-3	-2
-5	5

Now we can plot our points, and connect them:

There is some geometric meaning here. The graph of a function, when reflected over the line $y = x$, yields the graph of the inverse. This is what is happening when we switch the x and y coordinates.

In general, some times we are not given such a nice graph. However, one can pick a few points on a sketch of a graph, swap the x and y coordinates, and sketch the inverse. Then, one checks their work by drawing $y = x$ and checking if the sketches are indeed reflections of each other.

Suppose instead of a set of coordinates or a graph, we are given an equation. To find the inverse of a function $f(x) = \blacksquare$ we

1. Write $y = \blacksquare$.

2. Replace every instance of x with y, and every instance of y with x. We get $x = \blacksquare$ where every x in $\blacksquare$ is replaced with a y.

3. Solve for y in terms of x; isolate y with algebra. We will get a new equation of the form $y =$.

4. Replace y with $f^{-1}(x)$.

Example 6.5: Finding the Inverse Given Equations

Let $f(x) = \frac{4x+1}{7}$. Find f^{-1}.

We are given a line, which can be written as $f(x) = \frac{4}{7}x + \frac{1}{7}$. Following our

procedure:

$$y = \frac{4x+1}{7} \qquad\qquad (f(x) = y)$$

$$x = \frac{4y+1}{7} \qquad\qquad (y = x)$$

$$7x = 4y + 1$$

$$7x - 1 = 4y$$

$$\frac{7x-1}{4} = y \qquad\qquad (\text{Solve for } y)$$

$$\frac{7x-1}{4} = f^{-1}(x)$$

We can check our answer with some specific values. Let $x = 5$. Then $f(2) = \frac{4(5)+1}{7} = \frac{21}{7} = 3$. Now when we evaluate $f^{-1}(3)$ we should get 2. Indeed:

$$f^{-1}(3) = \frac{7(3)-1}{4}$$

$$= \frac{20}{4}$$

$$= 5.$$

The reader is encouraged to check other values of x.

Let us do another example with some more involved algebra.

Example 6.6: Finding the Inverse (Harder)

Let $f(x) = \frac{3x}{5x+1}$. Find $f^{-1}(x)$.

$$y = \frac{3x}{5x + 1}$$

$$x = \frac{3y}{5y + 1}$$

$$x(5y + 1) = 3y \qquad \text{(Clear Denominator)}$$

$$5xy + x = 3y \qquad \text{(Distribute } x\text{)}$$

$$5xy - 3y = -x \qquad \text{(Move } 3y \text{ left and } x \text{ right.)}$$

$$y(5x - 3) = -x \qquad \text{(Factor } y\text{)}$$

$$y = \frac{-x}{5x - 3} \qquad \text{(Solve for } y\text{)}$$

$$f^{-1}(x) = \frac{-x}{5x - 3}$$

Summary
Chapter 6

List of things you need to know.

- A one-to-one function is a function where each output has a unique input.

  - Use the horizontal line test on a graph to test if a function is one-to-one

- The inverse f^{-1} of a function f takes outputs $y = f(x)$ and produces inputs $x = f^{-1}(y)$.

- A function has an inverse if and only if it is one-to-one.

- The domain and range of an inverse is the range and domain of the function, respectively.

- To find the inverse of a function from its equation, write the equation in the form $y =$, change every x to y and every y to x, then solve for y.