2-Neighbor Bootstrap Percolation in Graphs

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Lafayette College
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https://raymaths.github.io/Lafayette.pdf

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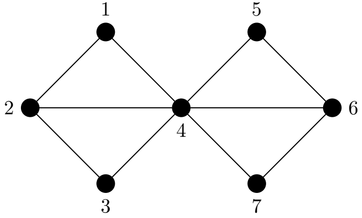
Assumptions

Graphs

The graphs we are working with today are

- Simple. No loops or multi-edges.
- Connected.



$V(G) = \{1, 2, 3, 4, 5, 6, 7\}$
$E(G) =$
$\{(1, 2), (1, 4), (2, 3), (2, 4), (3, 4), (4, 5), (4, 6), (4, 7), (5, 6), (6, 7)\}$

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r-Neighbor Bootstrap Percolation

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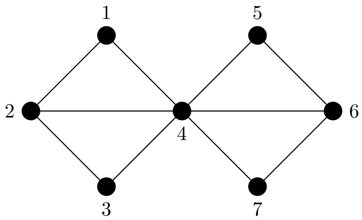
Cellular automaton – Conway's Game of Life

Example 1 ($r = 2$)



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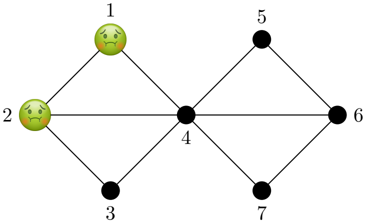

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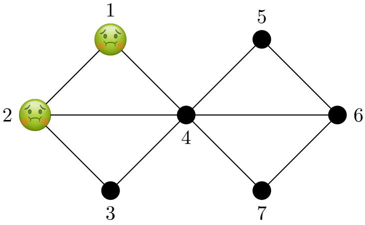

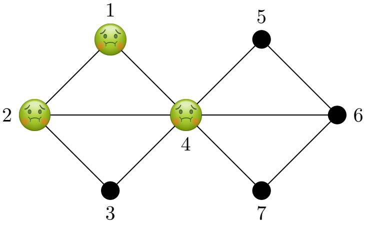

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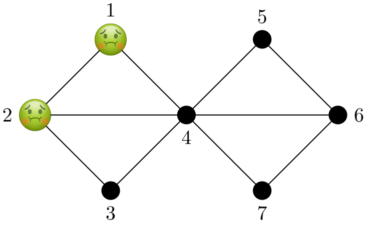
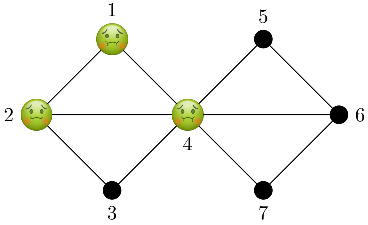
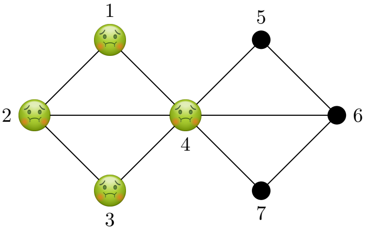


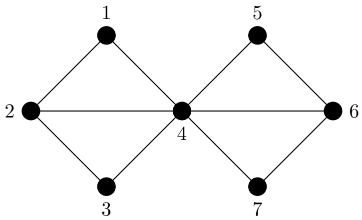




$$A_0 = \{1, 2\}$$

$$\langle A_0 \rangle = \{1, 2, 3, 4\}$$

Example 2 ($r = 2$)



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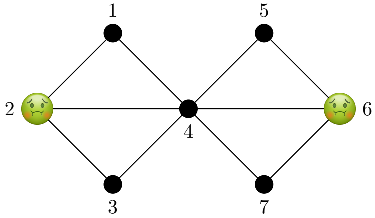

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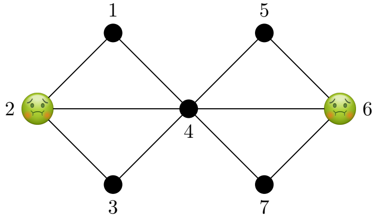

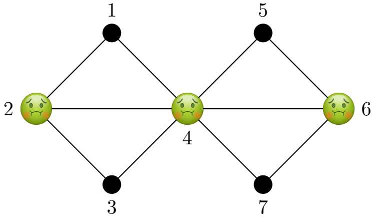

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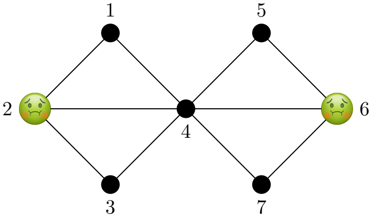

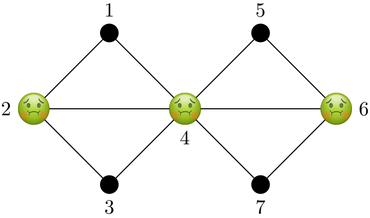

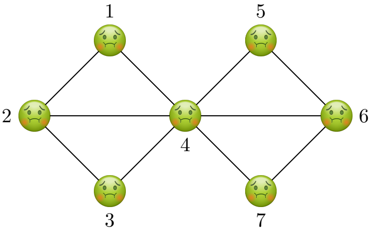

$$A_0 = \{2, 6\}$$

$$\langle A_0 \rangle = V(G)$$

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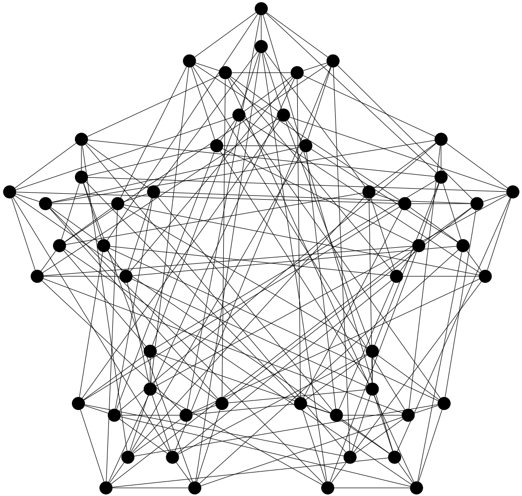

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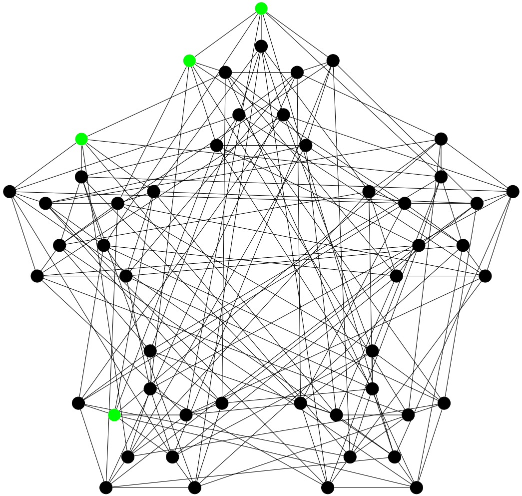

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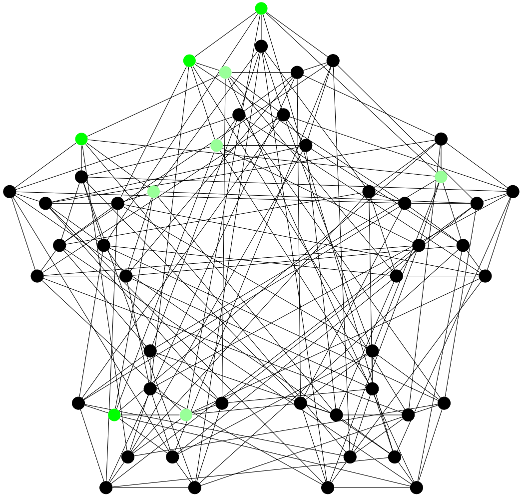

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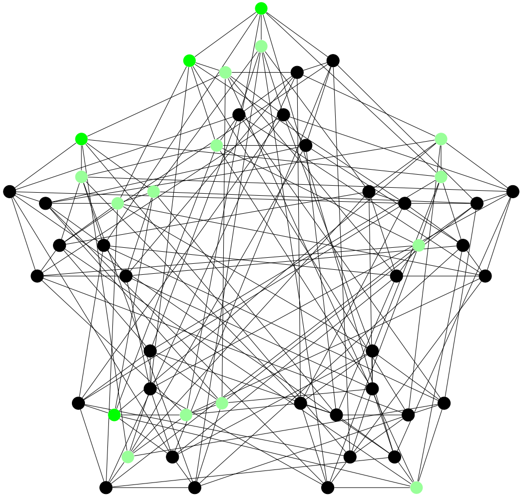

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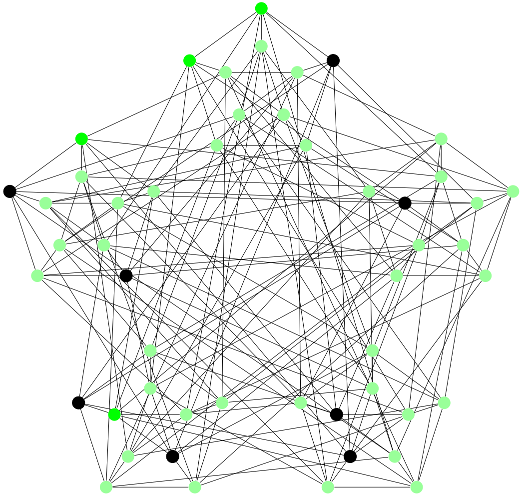

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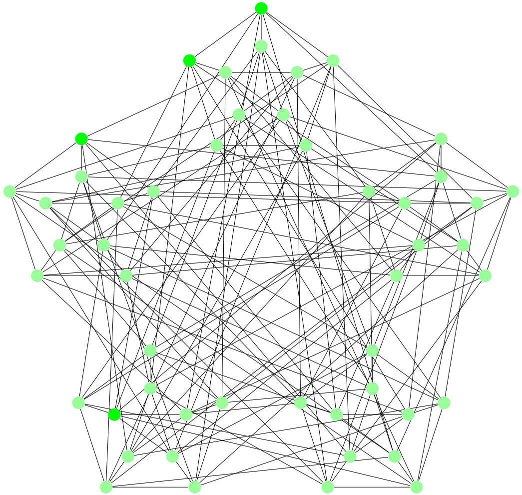

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Choosing A_0

Early models incorporate randomness; initial infected vertices are selected with probability p.

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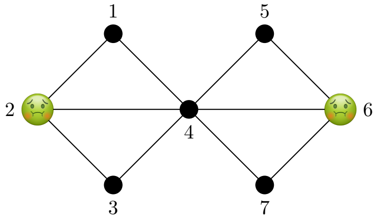

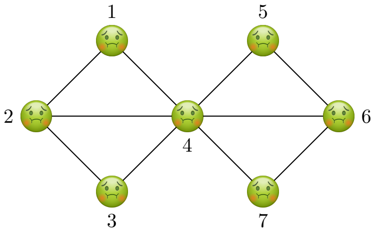

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- What are the minimum and maximum number of rounds to percolate?
  - Looking at all percolating sets of a fixed size (minimum), which set takes the most rounds to percolate? The fewest?

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Definition (Block)

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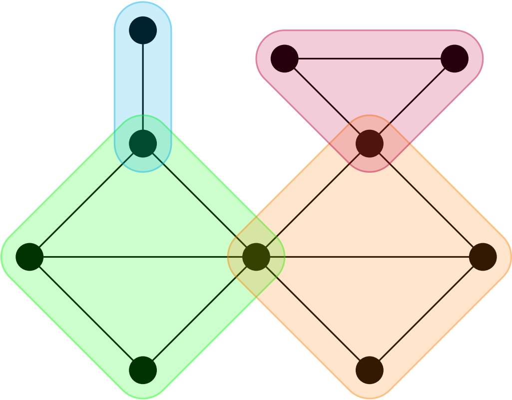

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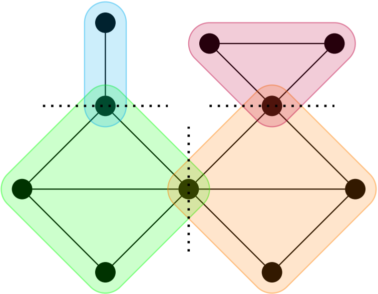

Observations

- Blocks intersect in a cut-vertex.
- Blocks are 2-connected, or K_2.

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Theorem (Bushaw et al. '23)

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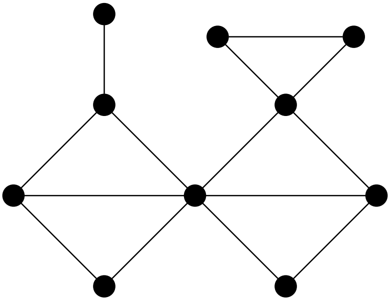

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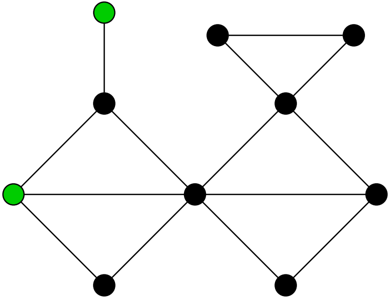

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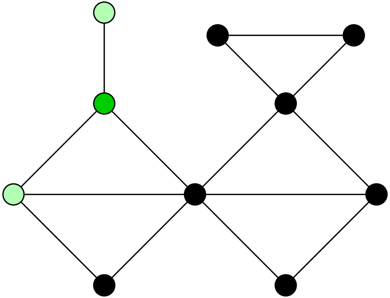

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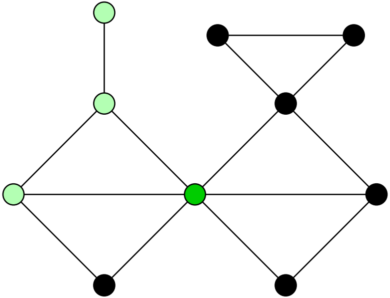

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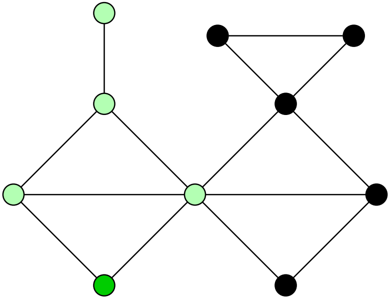

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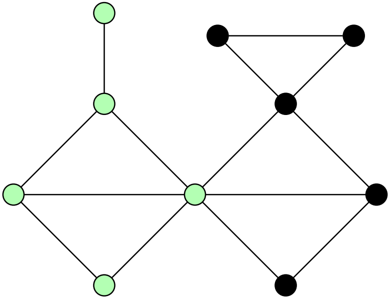

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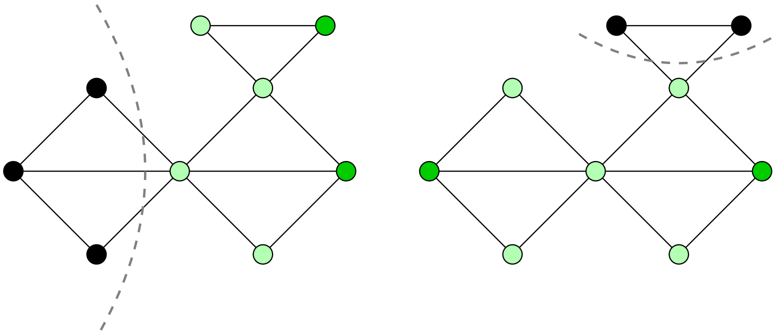

Our Graph Class - Diameter

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The *diameter* of a graph G is $\max_{u,v \in V(G)} d(u,v)$, where $d(u,v)$ is the length of the shortest path from u to v.

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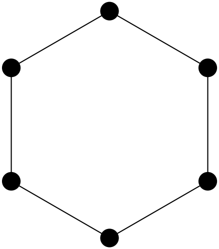
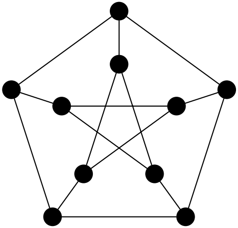


$\text{diam}(C_6) = 3$



$\text{diam}(\text{Pete}) = 2$

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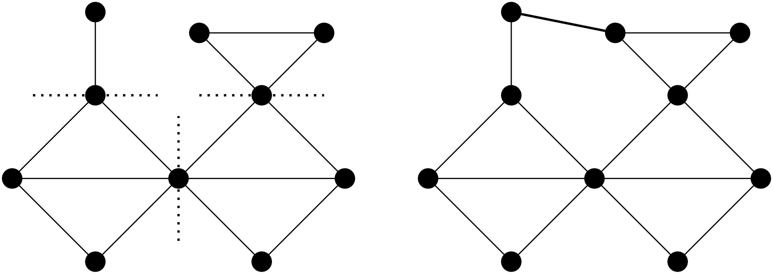

Our Graph Class - Diameter 2 and 2-connected

Graphs of interest: Diameter 2 and 2-connected

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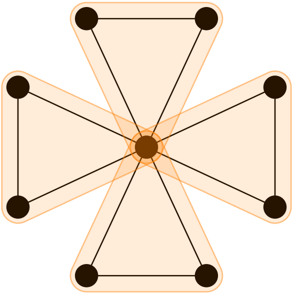

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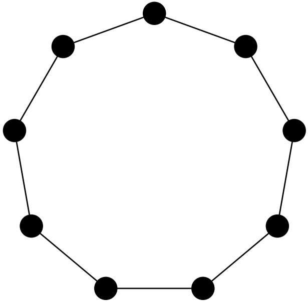

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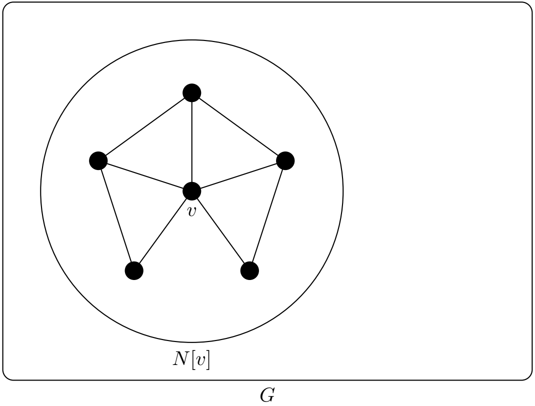

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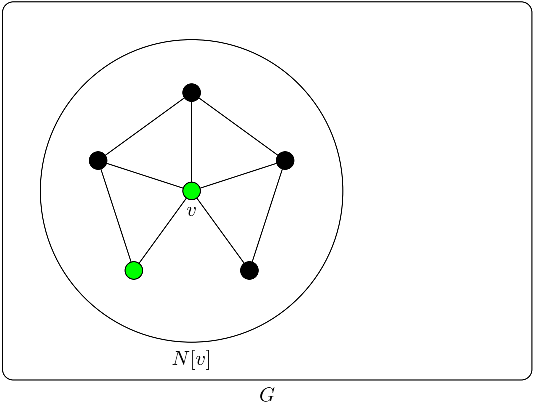

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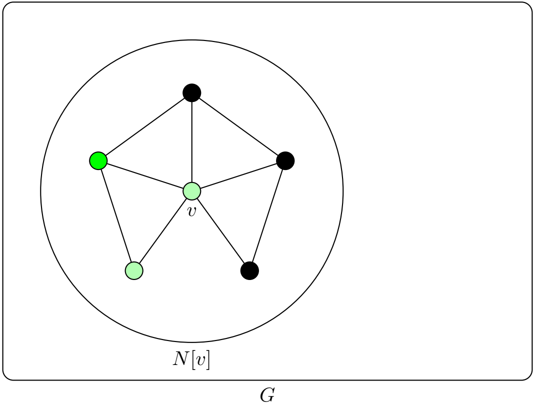

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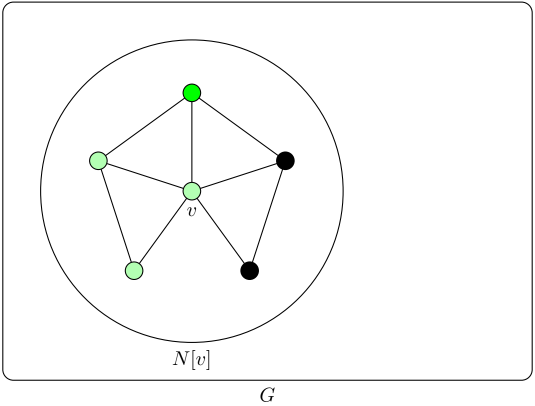

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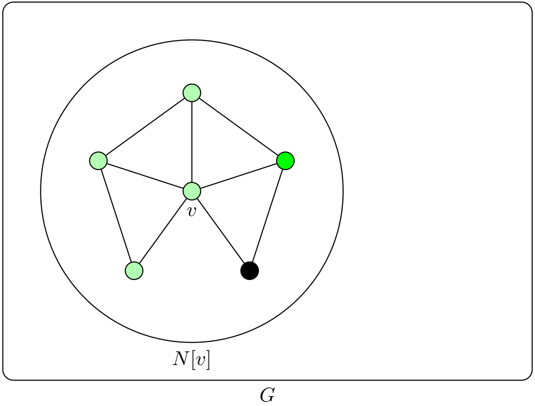

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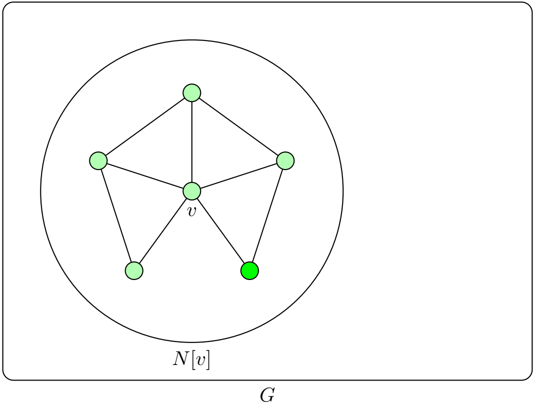

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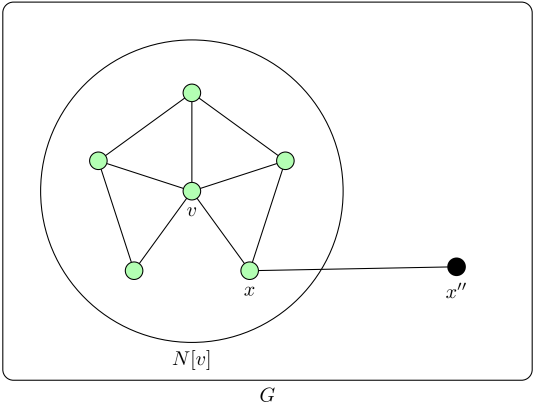

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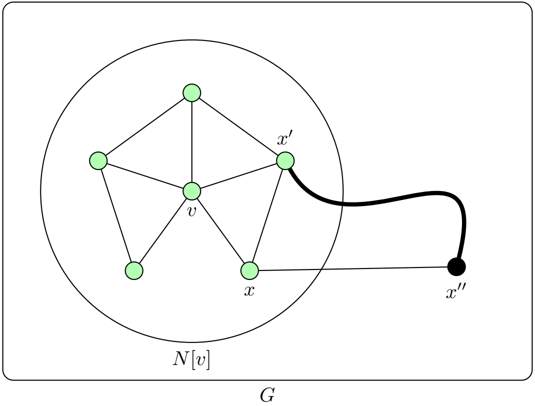

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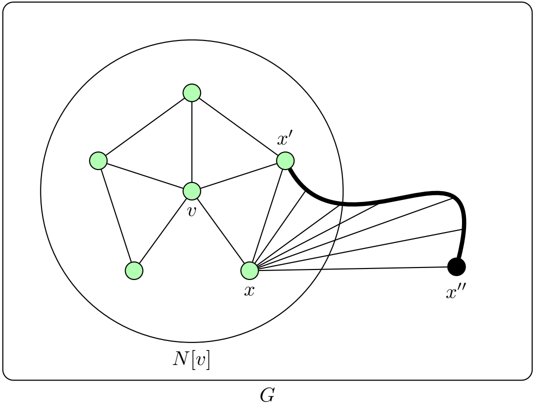

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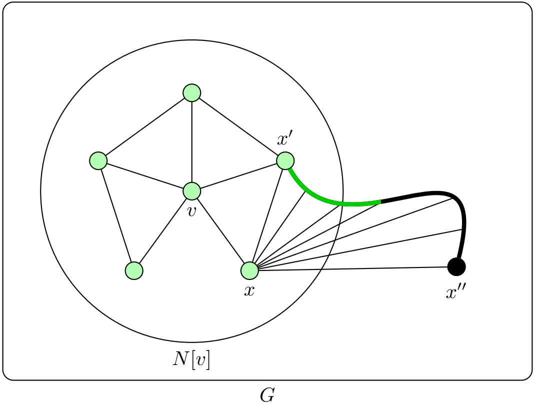

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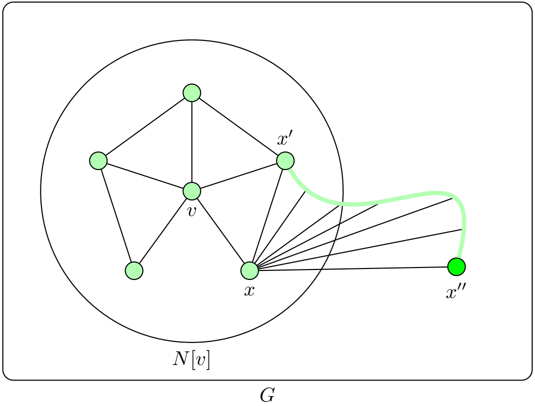

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If G is locally connected, then G is 2-BG. In particular ***any pair of adjacent vertices percolates.***

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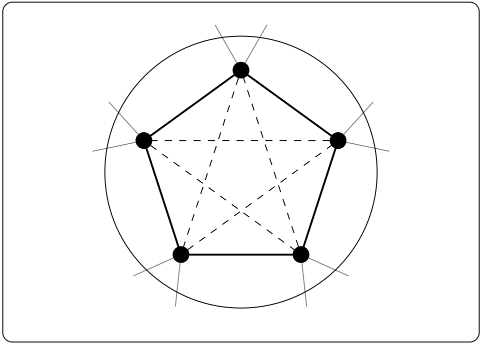

G

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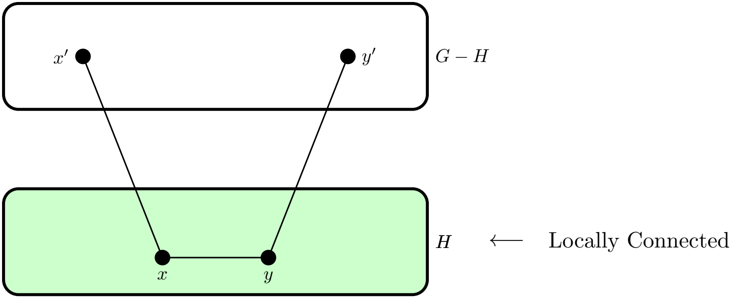

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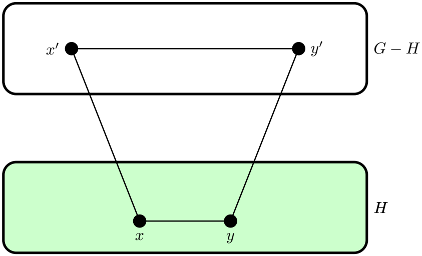

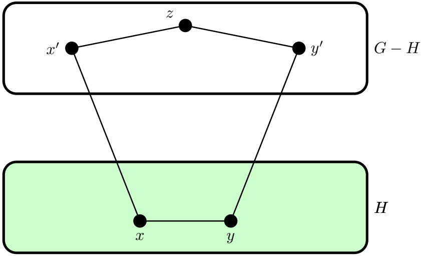

Future Directions

- What is $m(G, 2)$ when G has other properties, e.g. just diameter 2?

- A characterization of 2-BG graphs? $m(G, 2) = 2 \iff$???

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Thank you!

https://doi.org/10.26493/2590-9770.1694.1cf

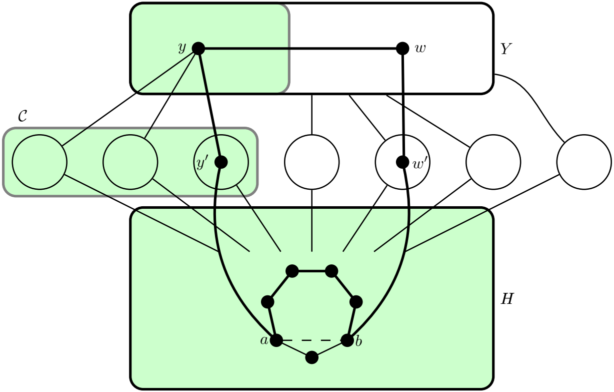
Extra

The next few slides are extra slides...

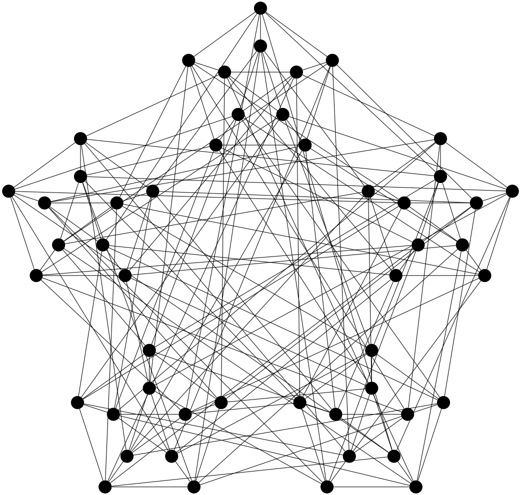
A 2-BG Theorem (Generalization)

Theorem (Ibrahim '24)

Let G be a 2-connected graph with diameter 2. If G is C_k-free, then $m(G,2) \le \lceil (k-3)/2 \rceil$.



Extra Extra



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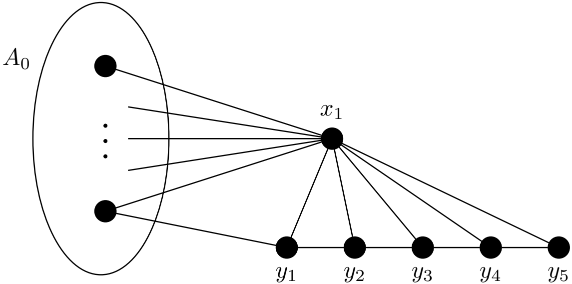

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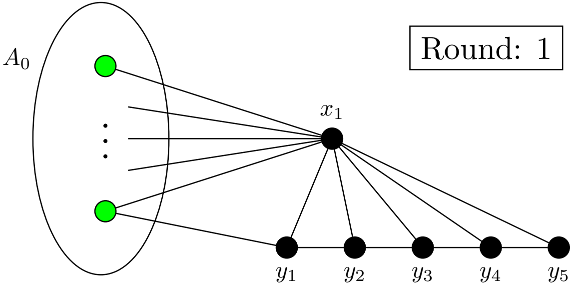

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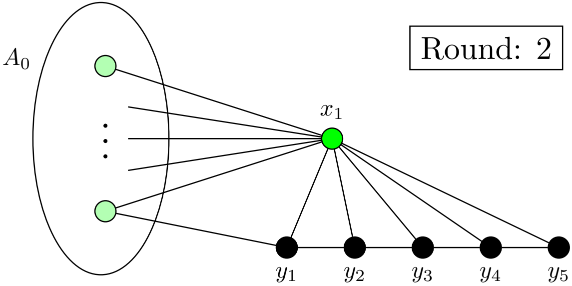

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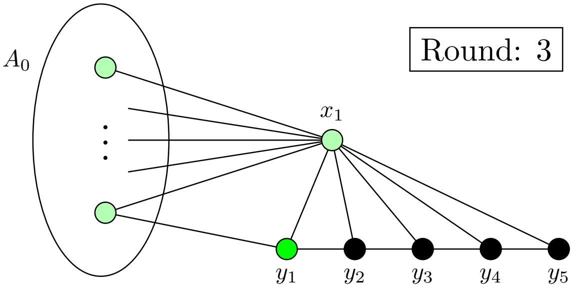

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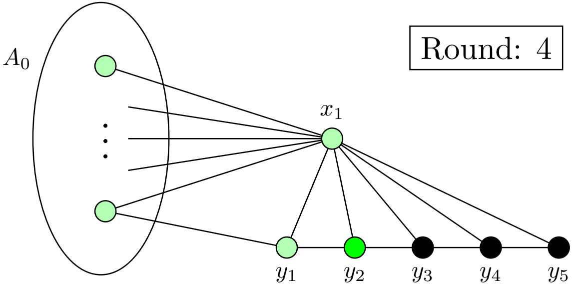

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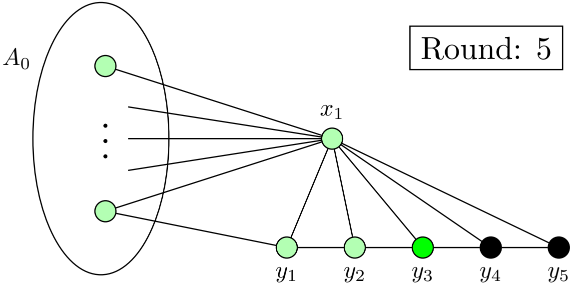

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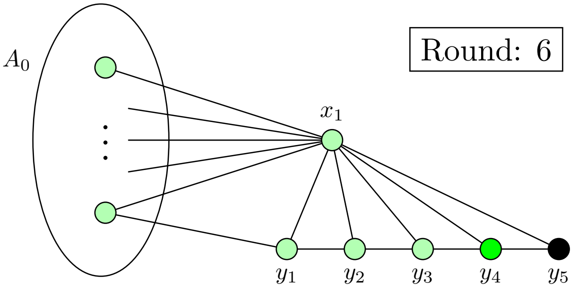

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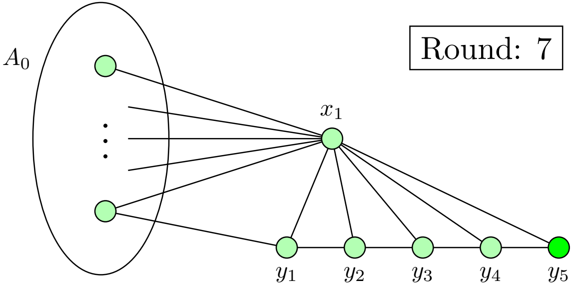

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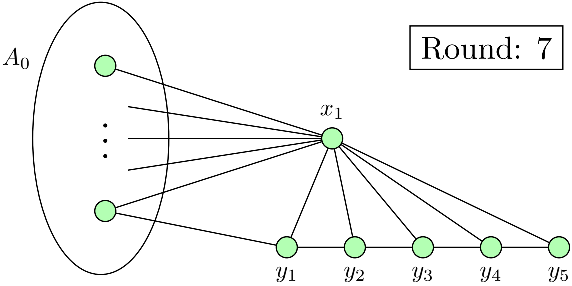
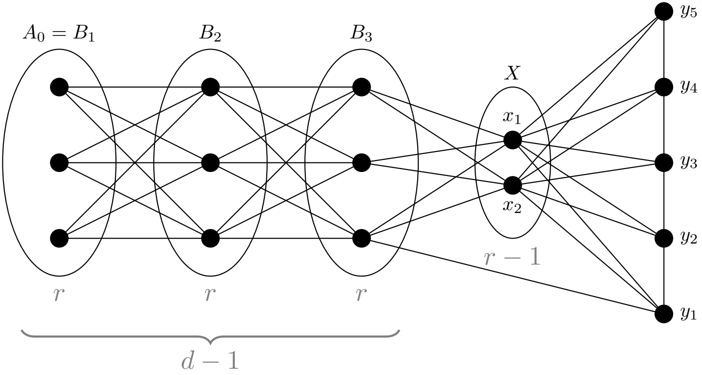


Number of rounds until infection: $2 + 5 = \mathrm{diam}(G) + |Y|$

Number of Rounds

Question: What is the maximum number of rounds until percolation?
For arbitrary r and diameter d. (Example: $r = 3$, $d = 4$.)



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