

An Infection Process in Graphs

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<https://raymaths.github.io/Hamilton.pdf>

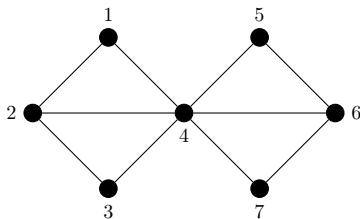
* Acknowledgement: Supported in part by The Thomas F. and Kate Miller Jeffress Memorial Trust and by NSF DMS-2204148.

Assumptions

Graphs

The graphs we are working with today are

- Simple. No loops or multi-edges.
- Connected.



$$V(G) = \{1, 2, 3, 4, 5, 6, 7\}$$

$$E(G) =$$

$$\{(1, 2), (1, 4), (2, 3), (2, 4), (3, 4), (4, 5), (4, 6), (4, 7), (5, 6), (6, 7)\}$$

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r -Neighbor Bootstrap Percolation

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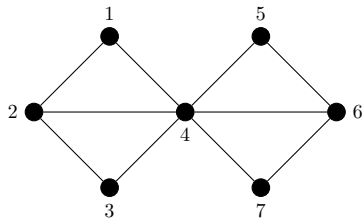
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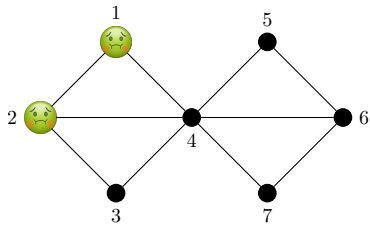
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Cellular automaton – Conway’s Game of Life

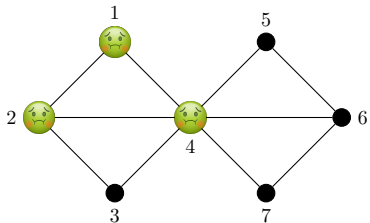
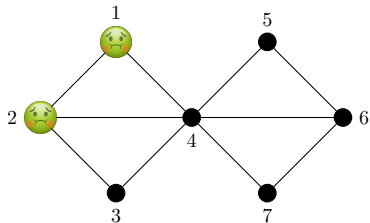
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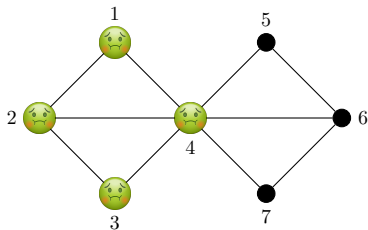
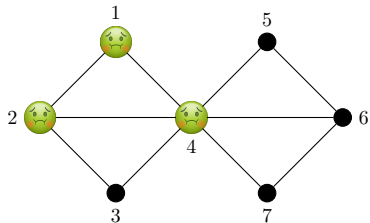
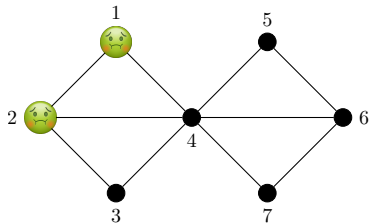
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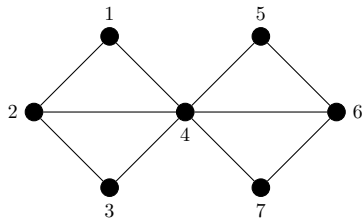
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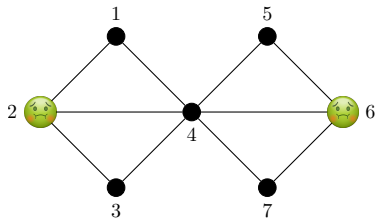
$$A_0 = \{1, 2\}$$

$$\langle A_0 \rangle = \{1, 2, 3, 4\}$$

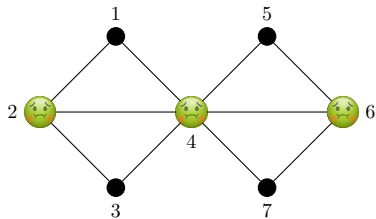
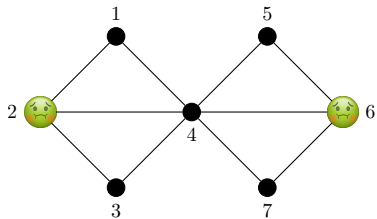
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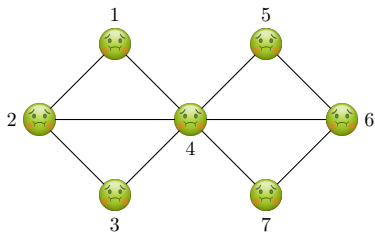
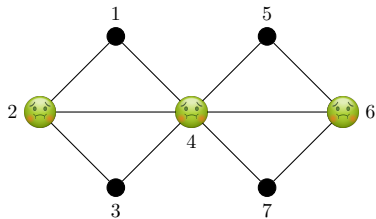
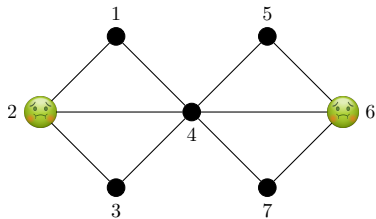
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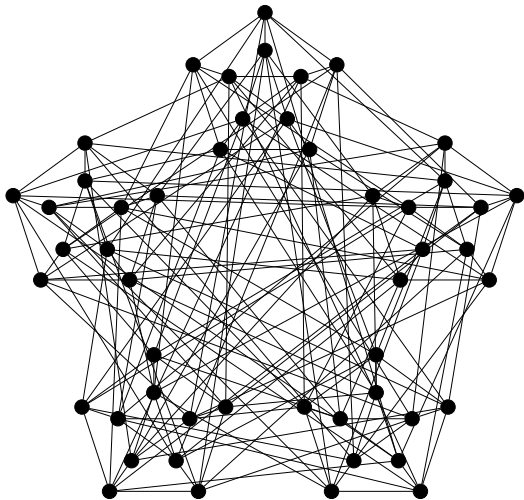
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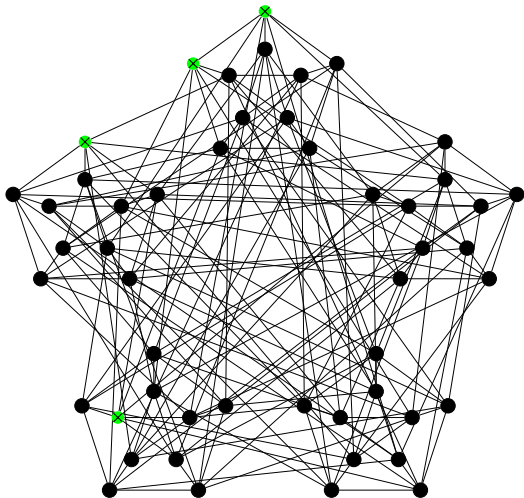
$$A_0 = \{2, 6\}$$

$$\langle A_0 \rangle = V(G)$$

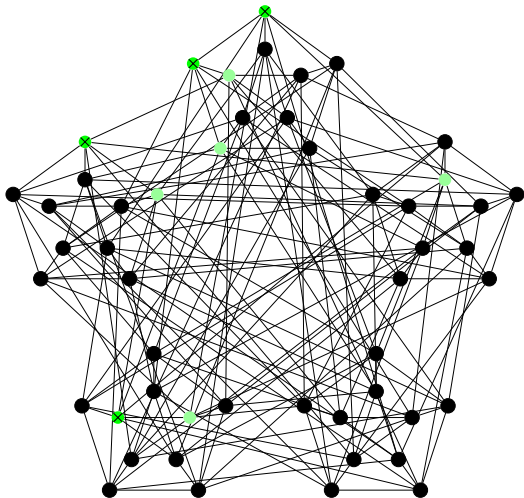
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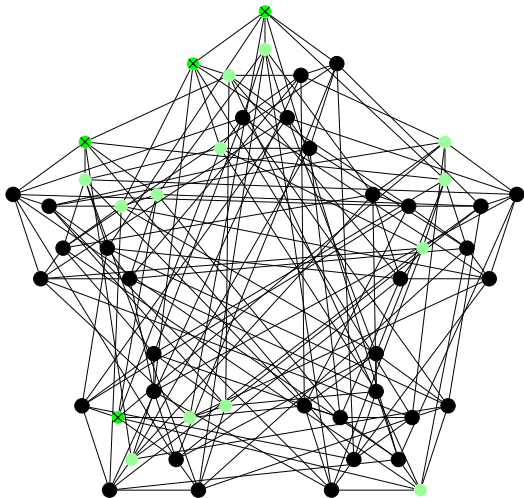
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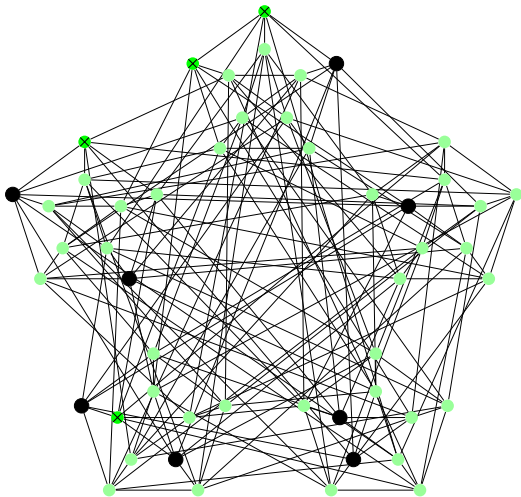
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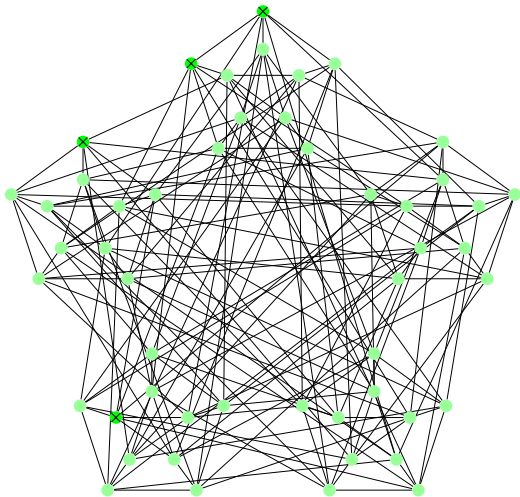
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Choosing A_0

Early models incorporate randomness; initial infected vertices are selected with probability p .

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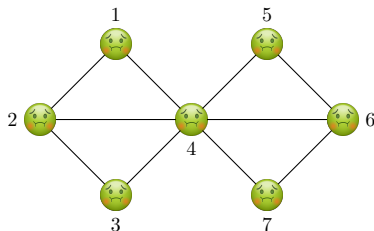
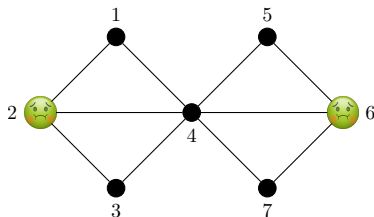
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- What are the **minimum** and **maximum** number of rounds to percolate?
 - Looking at all percolating sets of a fixed size (minimum), which set takes the most rounds to percolate? The fewest?

A Necessary Condition Involving Blocks

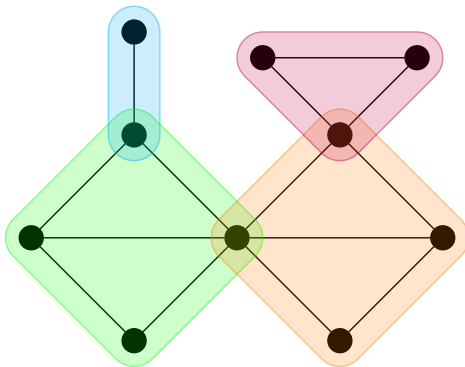
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A *block* in a graph G is a maximal connected subgraph with no cut-vertex.

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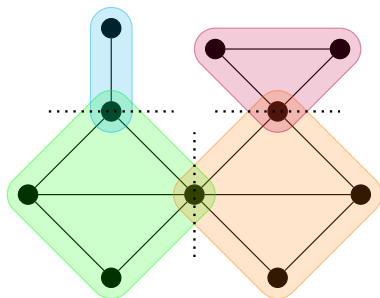
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Observations

- Blocks intersect in a cut-vertex.
- Blocks are 2-connected, or K_2 .

A Necessary Condition Involving Blocks (2-BG)

Theorem (Bushaw et al. '23)

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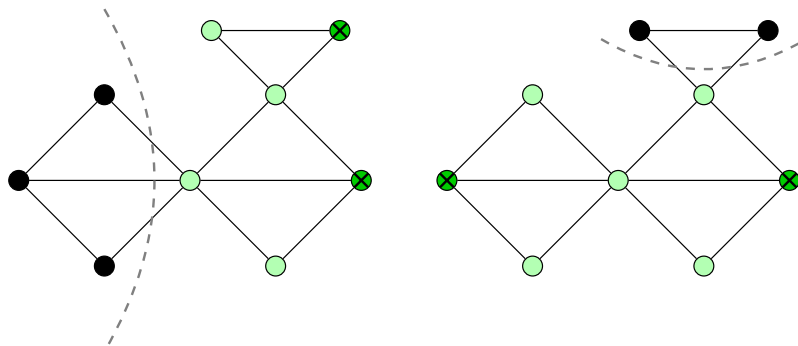
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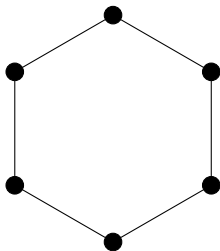
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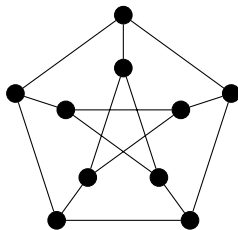
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$$\text{diam}(C_6) = 3$$



$$\text{diam}(\text{Pete}) = 2$$

Our Graph Class - Connectivity

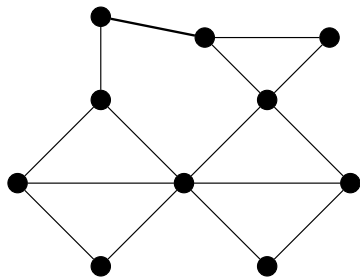
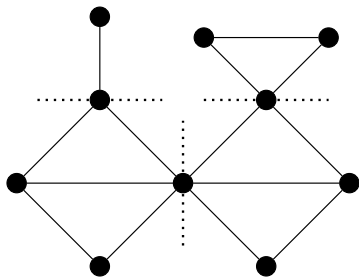
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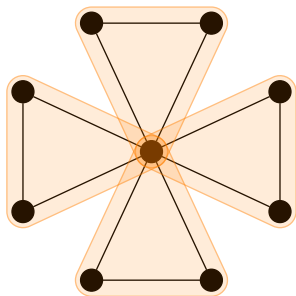


Our Graph Class - Diameter 2 and 2-connected

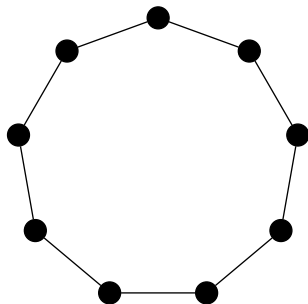
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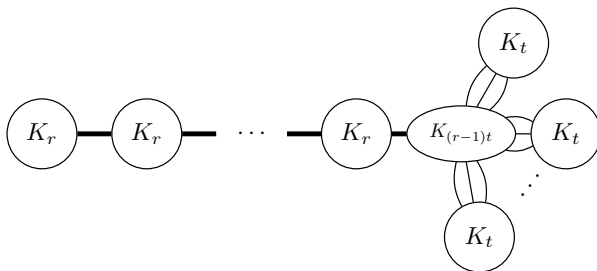
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Basis of the Construction

r -Forbidden Subgraph

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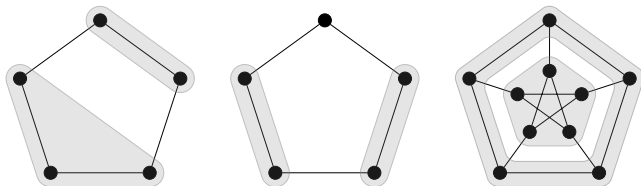
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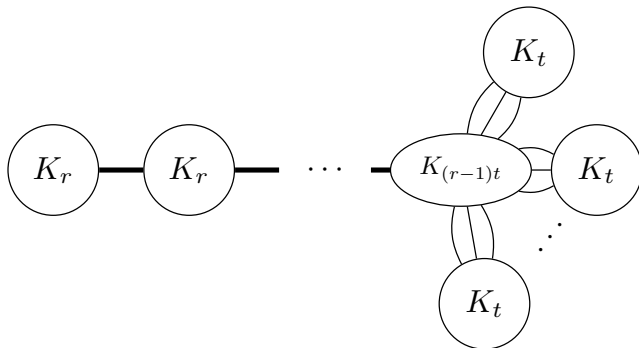
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Theorem (Ibrahim '24+)

If G is 2-connected with diameter 2, and contains exactly 2 vertex-disjoint 2-forbidden subgraphs, then $m(G, 2) \leq 3$.

A Cool Upper Bound

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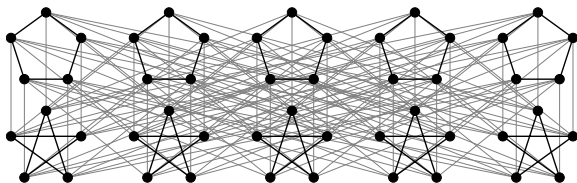
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$$\Delta(G) = 7 \implies m(G, 2) \leq 4$$

Chasing an Upper Bound

Theorem (Ibrahim, LaFayette, McCall '24)

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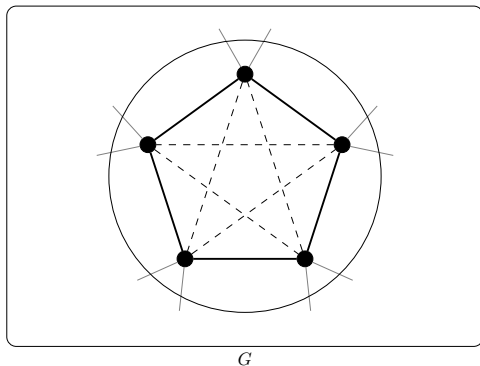
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Continuing the Pattern (H -free)

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Theorem (Ibrahim '24+)

Let G be 2-connected and diameter 2. If G is C_6 -free or C_7 -free then $m(G, 2) \leq 3$.

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k	5	6	7	8	9	10	11	12	...
$\lceil (k-3)/2 \rceil + 1$	2	3	3	4	4	5	5	6	...

Bonus Results

Theorem (Ibrahim '24+)

If G is 2-connected, diameter two, and $2K_2$ -free then $m(G, 2) \leq 3$.

Theorem (Ibrahim '24+)

If G is a connected graph such that \overline{G} is connected, then $m(G\overline{G}, 2) \leq 3$ where $G\overline{G}$ is the complementary prism of G .

Theorem (Ibrahim et al. '25)

Let MMG be the missing Moore graph. Then $3 \leq m(\text{MMG}, 2) \leq 4$.

My Wish List

- What is $m(G, 2)$ when we forbid other interesting subgraphs?
- Open Problem: Is there a constant k such that for all 2-connected graphs G with diameter 2, we have $m(G, 2) \leq k$?
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