MATH 305 Notes Part 1

Term: 2024W2

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W1C1 Lecture 1 (Jan 6)

Complex Numbers

📌 **Def.** A complex number $z \in \mathbb{C}$ is an expression of the form $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$.

Rules

- Addition: $z = x + iy, w = u + iv \implies z + w = (x + y) + i(y + v)$
- Multiplication: $z \cdot w = (xu - yv) + i(xv + yu)$

Inverses

- Additive inverse: $-x - iy$
- Multiplicative inverse: if $z \neq 0$, the multiplicative inverse is $z^{-1} = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

Representation

- For $z = x + iy$, we we write $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$, where $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$
- Complex plane: $z \in \mathbb{C}$ is identified with a point on the plane $\mathbb{R}^2$, with a real and imaginary axis
- Geometrically, complex numbers can be added using the parallelogram law

W1C2 Lecture 2 (Jan 8)

Modulus and Conjugate

- Modulus: $|z| = \sqrt{x^2 + y^2}$
  - We have $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$
- Complex conjugate: $\overline{z} = x - iy$

Identities

- $|z|^2 = z\overline{z}$
- $z^{-1} = \frac{\overline{z}}{|z|^2}$
- $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$

- $\text{Im}(z) = \frac{1}{2i}(z - \overline{z})$

Polar Representation

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r^2 = x^2 + y^2$
- $\tan\theta = \frac{y}{x}$ (θ is the argument of z)
    - Convention: $\arg(z) \in [0, 2\pi)$
- Complex exponential: $e^{ix} = \cos(x) + i\sin(x)$
    - $e^{i\pi} = -1$
- $z = |z|e^{i\arg(z)} = re^{i\theta}$
    - Periodic: $e^{i(\theta+2\pi n)} = e^{i\theta}$ for all $n \in \mathbb{Z}$

Identities

- $\cos(x) = \text{Re}(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix})$
- $\sin(x) = \text{Im}(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})$
- $zw = |z|e^{i\theta} \cdot |w|e^{i\phi} = |z||w|e^{i(\theta+\phi)}$

W1C3 Lecture 3 (Jan 10)

- Another convention: $\text{Arg}(z) \in (-\pi, \pi]$
    - Arg and $\arg$ are not defined for $z = 0$
- The complex exponential is helpful to derive trig identities:

💡 **Ex 1.** Derive the trig identity for $\cos(x - y)$.

$$\begin{aligned}\cos(x-y) &= \text{Re}(e^{i(x-y)}) \\ &= \text{Re}(e^{ix}e^{-iy}) \\ &= \text{Re}(\cos x + i\sin x)(\cos y - i\sin y) \\ &= \cos x\cos y + \sin x\sin y\end{aligned}$$

Roots of Unity

- $(e^{ix})^n = e^{inx} \quad n \in \mathbb{N}$, so $z^n = |z|^n e^{in\arg(z)}$
- For any $k \in \mathbb{N}$, $\left(e^{\frac{2\pi ik}{n}}\right)^n = e^{2\pi ik} = 1$

📌 **Def.** The nth roots of unity are given by $z = e^{\frac{2\pi ik}{n}}, \quad k = 1, 2, \ldots, n$ and solve $z^n = 1$.

📌 **Def.** For $z \in \mathbb{C}, z = x + iy$, the complex exponential is $e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$.

- The complex exponential is periodic in the imaginary direction: $\arg(e^z) = \operatorname{Im}(z) \pmod{2\pi}$

Complex Functions

- A subset $\Omega \subset \mathbb{C}$ is identified with the corresponding subset of $\mathbb{R}^2$.
- $\Omega \subset \mathbb{C}$ is bounded if there exists $r > 0$ such that $|z| \leq r$ for all $z \in \Omega$.
- $\Omega \subset \mathbb{C}$ is open if there is a disc in Ω around any point $z \in \mathbb{C}$
  - Ω does not contain its boundary

💡 **Ex 2.**

a. Open disc: $B_r(z_0) = \{z \in \mathbb{C} \text{ s.t. } |z - z_0| < r\}$

b. Pointed open disc: $\dot{B}_r(z_0) = B_r(z_0) \setminus \{z_0\}$

c. Closed ball: $\{z \in \mathbb{C} \text{ s.t. } |z - z_0| \leq r\}$ is not open

- An open set is (path) connected if there is a continuous path inside Ω between any two points of Ω
  - Ω can have holes, but it cannot be separated in two parts
- A domain is an open and connected subset of $\mathbb{C}$.

W2C1 Lecture 4 (Jan 13)

📌 **Def.** A complex function maps $f : \mathbb{C} \to \mathbb{C}$. Notation: $f(z) = u(z) + iv(z)$.

💡 **Ex 3.** Find the $u(z)$ and $v(z)$ for $f(z) = e^z$.

We have $u(z) = e^x \cos y, \quad v(z) = e^x \sin y$.

We are interested in how subsets of $\mathbb{C}$ map to other subsets of $\mathbb{C}$ through complex functions.

Transformations

- Translation by $\vec{w}$: pick $w \in \mathbb{C}$, and consider $f(z) = z + w$
- Rotation by φ CCW around origin: pick $\varphi \in \mathbb{R}$, and consider $f(z) = e^{i\varphi}z$
- Scaling by λ: pick $\lambda \in (0, \infty)$, and consider $f(z) = \lambda z$

💡 **Ex 4.** We examine the inverse $f(z) = z^{-1}$.

a. Consider $\Omega = \dot{B}_1(0)$

   We try to find all $\zeta \in \mathbb{C}$ of the form $\zeta = \frac{1}{z}$ with $z \in \dot{B}_1(0)$.
   We have
   $z \in \Omega \iff |z| < 1 \iff \frac{1}{|z|} > 1 \iff |\zeta| > 1$ (outside of the unit disc).

b. Consider $\tilde{\Omega} = B_1(1)$

   $z \in \tilde{\Omega} \iff |z - 1| < 1 \iff |\frac{1}{\zeta} - 1| < 1.$
   Write
   $\zeta = u + iv$ so $\frac{1}{\zeta} = \frac{u-iv}{u^2+v^2}$.

$$
\begin{aligned}
|\tfrac{1}{\zeta} - 1|^2 &= |\tfrac{u-iv}{u^2+v^2} - 1|^2 \\
&= |(\tfrac{u}{u^2+v^2} - 1) - i(\tfrac{v}{u^2+v^2})|^2 \\
&= \tfrac{1}{(u^2+v^2)^2}((u - (u^2+v^2))^2 + v^2) \\
&= \tfrac{1}{(u^2+v^2)^2}((u^2+v^2)^2 - 2u(u^2+v^2) + u^2 + v^2) \\
&= \tfrac{1}{u^2+v^2}(u^2 + v^2 - 2u + 1)
\end{aligned}
$$

   Our condition becomes
   $u^2 + v^2 > u^2 + v^2 - 2u + 1 \iff 2u - 1 > 0 \iff u > \frac{1}{2}$. The inverse maps $B_1(1)$ to the right half-plane at $u = \frac{1}{2}$.

💡 **Ex 5.** The Joukowsky map is $f(z) = z + \frac{1}{z}$.
Write $\zeta = u + iv = f(z) = (x + iy) + \frac{x-iy}{x^2+y^2}$.
Matching gives
$u = x + \frac{x}{x^2+y^2}$ and $v = y - \frac{y}{x^2+y^2}$.
Take
$\Omega = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$, then $f(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos\theta$.
Unit circle is mapped to horizontal line from
-2 to 2.
Generally, the circles
$\{z_0 + re^{i\theta} \mid \theta \in [0, 2\pi)\}$ are mapped to Joukowsky airfoils

W2C2 Lecture 5 (Jan 15)

💡 **Ex 6.** Consider $f(z) = \frac{1}{-iz+\frac{1}{2}}$ and $\Omega = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$. What is $f(\Omega)$?

This is a combination $z \xrightarrow{f_1} -iz \xrightarrow{f_2} -iz + \frac{1}{2} \xrightarrow{f_3} \frac{1}{-iz+\frac{1}{2}}$.

Upper half plane → right half plane (
f_1) [rotation]
→
$u > \frac{1}{2}$ half plane (f_2) [translation]
→
$B_1(1)$ disc (f_3) - [see **Ex 4. (b)**]

Remark: there exists a complex function mapping any complex region to any other region (Riemann mapping theorem)

Limits, Continuity, and Differentiability

📌 **Def.** Consider $f : \Omega \to \mathbb{C}$. For $z_0 \in \Omega$, we write the limit as $\lim\limits_{z \to z_0} f(z) = L$, if $f(z)$ is arbitrarily to L provided z is sufficiently close to z_0.

More precisely, for all $\varepsilon > 0$, there exists a radius $\delta > 0$ such that $|f(z) - L| < \varepsilon$ for all z such that $|z - z_0| < \delta$.

Remark: in the complex plane, z can "tend to z_0" via many different paths (same idea as multivariate limit). The value of the limit is independent of how $z \to z_0$.

💡 **Ex 7.** Notice that $\lim\limits_{z \to i} \arg(z) = \frac{\pi}{2}$. However, we claim that $\lim\limits_{z \to 1} \arg(z)$ does not exist.

From the upper half plane, the limit tends to 0, but from the lower half plane, the limit tends to 2π. For a spiral, the limit does not exist.

📌 **Def.** f is continuous at z_0 if $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

Examples of continuity

- $\arg(z)$ is continuous on $\mathbb{C} \setminus [0, \infty)$
- e^z and $|z|^2$ are continuous on $\mathbb{C}$
- $\frac{1}{z-w}$ is continuous on $\mathbb{C} \setminus \{w\}$

💡 **Ex 8.** Consider $y(z) = \begin{cases} \frac{z}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$. Show that $\lim_{z \to 0} y(z)$ does not exist.

Observe that $\frac{z}{|z|} = e^{i\theta}$, so the value of the limit depends on the direction of approach.

📌 **Def.** A function f is differentiable at z_0 if $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

W2C3 Lecture 6 (Jan 17)

- $f(z) = z^n$ for $n \in \mathbb{N}$ is differentiable everywhere, and $f'(z) = nz^{n-1}$

💡 **Ex 9.** Show that $f(z) = \bar{z}$ is not differentiable anywhere.

Let $z_0 = x + iy$ and $z = (x + h) + i(y + k)$. Then

$R = \frac{f(z) - f(z_0)}{z - z_0} = \frac{h - ik}{h + ik}$. We let $(h, k) \to (0, 0)$ via two different ways.
1)
$k = 0, h \to 0$ implies that $\lim_{(h,k) \to (0,0)} R = \lim_{h \to 0} \frac{h}{h} = 1$

2)
$k \to 0, h = 0$ implies that $\lim_{(h,k) \to (0,0)} R = \lim_{k \to 0} \frac{-ik}{ik} = -1$
Since the two limits are different, the limit does not exist.

Remark: the function $z \mapsto \bar{z}$, when seen in $\mathbb{R}^2$, is given by $(x, y) \mapsto (x, -y)$, and it is perfectly $\mathbb{R}$-differentiable

📌 **Def.** If f is $\mathbb{C}$-differentiable in a domain Ω, f is holomorphic and we write $f \in H(\Omega)$.

If $f \in H(\mathbb{C})$, f is entire.

Let $f = u + iv$ be differentiable at $z_0 = x + iy$. What does that mean for u, v?

Let $z = (x + a) + i(y + b)$.

- Horizontal limit: $b = 0, a \to 0$

  $f(z) - f(z_0) = (u(x + a, y) + iv(x + a, y)) - (u(x, y) + iv(x, y))$
  $\frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{a}\left[(u(x + a, y) + iv(x + a, y)) - (u(x, y) + iv(x, y))\right]$
  $\lim_{a \to 0} \frac{f(z) - f(z_0)}{z - z_0} = \partial_x u(x, y) + i\partial_x v(x, y)$

- Vertical limit:

$\lim_{b \to 0} \frac{f(z)-f(z_0)}{z-z_0} = -i\partial_y u(x,y) + \partial_y v(x,y)$

If f is differentiable, that means $\partial_x u(x,y) = \partial_y v(x,y)$ and $\partial_y u(x,y) = -\partial_x v(x,y)$. These are the Cauchy-Riemann equations.

📖 **Thm.** If f is a differentiable function at $x+iy$, then u, v satisfy the Cauchy-Riemann equations at (x,y).

📖 **Thm.** $f \in H(\Omega)$ if and only if the partial derivatives of u, v exist and are continuous and satisfy the Cauchy-Riemann equations.

💡 **Ex 10.** Consider $f(z) = |z|^2$ so $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$.

Differentiable nowhere except at $(0,0)$.

💡 **Ex 11.** Consider $f(z) = e^z = e^x(\cos y + i \sin y)$ so $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$.

$$\partial_x u = e^x \cos y = \partial_y v$$

$$\partial_y u = -e^x \sin y = -\partial_x v$$

Hence

f is complex differentiable.

W3C1 Lecture 7 (Jan 20)

Recall that f is differentiable if and only if partials of u, v are continuous and $\partial_x u = \partial_y v, \partial_y u = -\partial_x v$.

Differentiation Properties

- $(f+g)'(z) = f'(z) + g'(z)$
- $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
- $(f \circ g)(z) = f'(g(z))g'(z)$

Consequences: polynomials in z are entire, rational functions $\frac{p(z)}{q(z)}$ are holomorphic on $\{z \in \mathbb{C} : q(z) \neq 0\}$.

Remark: We can write $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$. Then

$\frac{\partial f}{\partial \bar{z}} = \partial_x f \frac{\partial x}{\partial \bar{z}} + \partial_y f \frac{\partial y}{\partial \bar{z}} = \frac{1}{2}(\partial_x f - \frac{1}{i}\partial_y f) = \frac{1}{2}(\partial_x u + i\partial_x v - \frac{1}{i}\partial_y u - \partial_y v) = 0$

if f is holomorphic (i.e. C-R equations apply). We conclude that holomorphic functions can only depend on z, not $\bar{z}$.

💡 **Ex 12.** Let $u(x, y) = x^3 - 3xy^2 + y$. Find $v(x, y)$ such that $f = u + v$ is entire.

Use Cauchy-Riemann equations.

$\partial_x v = -\partial_y u = -(-6xy + 1) = 6xy - 1$

$v(x, y) = \int (6xy - 1)\, dx = 3x^2 y - x + C(y)$
Other equation:

$\partial_y v = \partial_x u \implies 3x^2 + C'(y) = 3x^2 - 3y^2 \implies C'(y) = -3y^2 \implies C(y) = -y^3 + C$
The solution is
$v(x, y) = 3x^2 y - x - y^3 + C$
Overall,
$f(z) = (x^3 - 3xy^2 + y) + i(3x^2 y - x - y^2 + c) = z^3 - i(z - c)$

💡 **Ex 13.** Can any differentiable function be the real part of a holomorphic function?

No. If $f \in H(\Omega)$ then $\partial_x(\partial_x u) = \partial_x(\partial_y v) = \partial_y(\partial_x v) = -\partial_y(\partial_y u)$. Namely, the function must satisfy $\partial_x^2 u + \partial_y^2 u = 0$, same for v.

📖 **Thm.** If $f \in H(\Omega)$ then $\Delta u(x, y) = 0$ and $\Delta v(x, y) = 0$, where $\Delta = \nabla^2 = (\partial_x^2 + \partial_y^2)$ is the Laplacian.
Functions that satisfy these conditions are called
harmonics.

W3C2 Lecture 8 (Jan 22)

Elementary Functions

Exponentials

- e^z is entire and $(e^z)' = e^z$
- Power series representation: $e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is convergent $\forall z \in \mathbb{C}$.

Trig Functions

- $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$
- $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$
- They are entire, and standard trig derivatives/identities hold for all $z \in \mathbb{C}$

Hyperbolic Functions

- $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$
- $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$

- They are "rotated" versions of the trig functions: $\cosh(z) = \cos(iz)$ and $\sinh(z) = -i\sin(iz)$

💡 **Ex 14.** Solve $\cos(z) = 2$, for $z \in \mathbb{C}$.

$$\begin{aligned} \frac{1}{2}(e^{iz} + e^{-iz}) &= 2 \\ e^{iz} + e^{-iz} - 4 &= 0 \\ e^{-iz}(e^{2iz} + 1 - 4e^{iz}) &= 0 \\ (e^{iz})^2 - 4(e^{iz}) + 1 &= 0 \end{aligned}$$

So
$e^{iz} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$.
Let
$z = x + iy$, so $e^{ix}e^{-y} = 2 \pm \sqrt{3}$.
Since
$2 \pm \sqrt{3}$ is real and positive, we need $e^{ix} = 1$ and $e^{-y} = 2 \pm \sqrt{3}$.
Hence
$x = 2\pi n, n \in \mathbb{Z}$ and $y = -\ln(2 \pm \sqrt{3})$.
The solutions are
$z = \{2\pi n - i\ln(2 \pm \sqrt{3}), n \in \mathbb{Z}\}$.

Logarithm

- "Taking the log" is richer in $\mathbb{C}$ than in $\mathbb{R}$
- In $\mathbb{C}$, e^z has range $\mathbb{C} \setminus \{0\}$ and is periodic, namely many to one, i.e. two complex numbers z, w such that $z = w + 2\pi ni, n \in \mathbb{Z}$ satisfy $e^z = e^w$
- To make e^z a one-to-one function, we take a strip of width 2π and consider only z in that strip.
- The principal strip is defined as $\Omega_p = \{z \in \mathbb{C}, -\pi < \mathrm{Im}(z) \leq \pi\}$, which makes e^z one-to-one from Ω_p to $\mathbb{C} \setminus \{0\}$.

W3C3 Lecture 9 (Jan 24)

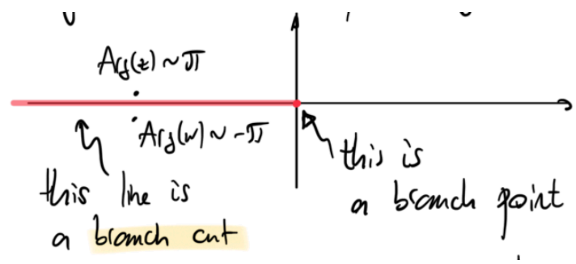
- Principal branch of the logarithm: maps a function $\mathrm{Log}(z) : \mathbb{C} \setminus \{0\} \mapsto \Omega_p$, given by $\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$, where $\mathrm{Arg}(z)$ ranges from $(-\pi, \pi]$.
  1. Range satisfies $\mathrm{Log}(z) \in \Omega_p$, for any $z \in \mathbb{C}$.
  2. $e^{\mathrm{Log}(z)} = e^{\ln|z| + i\mathrm{Arg}(z)} = e^{\ln|z|}e^{i\mathrm{Arg}(z)} = |z|e^{i\mathrm{Arg}(z)} = z$
  3. $\mathrm{Log}(e^z) = \ln e^{\mathrm{Re}(z)} + i\mathrm{Arg}(e^z) = \mathrm{Re}(z) + i(\mathrm{Im}(z) + 2\pi n)$, where n is chosen so that $\mathrm{Im}(z) + 2\pi n \in (-\pi, \pi]$. If $z \in \Omega_p$, then $\mathrm{Im}(z) \in (-\pi, \pi]$ so $n = 0$ and $\mathrm{Log}(e^z) = z$.

💡 **Ex 15.** Let $x > 0$, and consider $\text{Log}(-x)$.

We have $-x = e^{i\pi}x$, so $\text{Log}(-x) = \ln|-x| + i\text{Arg}(-x) = \ln x + i\pi$
For example,
$\text{Log}(-1) = i\pi$.

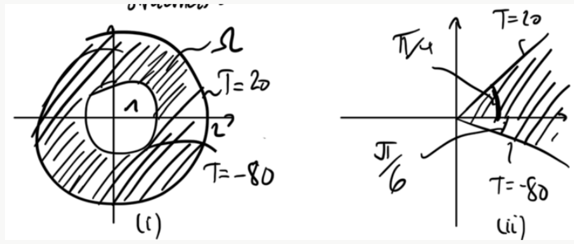
Remarks:

1. $\text{Log}(zw) = \text{Log}(z) + \text{Log}(w) + 2\pi i n$, where n is chosen so that the imaginary part lies in $(-\pi, \pi]$.
2. Log is discontinuous where Arg is discontinuous, namely $(-\infty, 0]$



3. Away from the cut, Log is holomorphic with $\text{Log'}(z) = \frac{1}{z}$
4. Another choice is to define $\log$ with $\arg(z) \in [0, 2\pi)$: $\log(z) = \ln|z| + i\arg(z)$
5. Pick any curve γ starting at 0 and extending to ∞ without self intersections; then there is a branch $\log_\gamma(z)$ that is holomorphic in $\mathbb{C} \setminus \gamma$.
6. $\log$ provides two harmonic functions: $u(x, y) = \ln|z|$ and $v(x, y) = \text{Arg}(z)$ or $\arg(z)$

💡 **Ex 16.** Find the steady state temperature distribution on the following domains:



i. Since $u(x, y) = \ln|z|$ is constant along circles, we try $\phi(x, y) = A\ln|z| + B$. Boundary conditions give $B = -80$ and $A\ln 2 - 80 = 20 \implies A = \frac{100}{\ln 2}$.
   The solution is
   $\phi(x, y) = \frac{100}{\ln 2}\ln(r) - 80.$
ii. Because of the wedge shape, we try $\phi(x, y) = C\mathrm{Arg}(z) + D$ (cannot have branch cut outside the wedge). Solving with the boundary conditions gives
   $\phi(x, y) = \frac{240}{\pi}\theta - 40$
   (This is harmonic in the interior of the wedge, but discontinuous at its tip)

W4C1 Lecture 10 (Jan 27)

- Roots: for any $\alpha \in \mathbb{C}$, we define $z^\alpha = e^{\alpha \mathrm{Log}(z)}$ for $z \in \mathbb{C} \setminus \{0\}$ and we call it the principal branch of z^α.

💡 **Ex 17.**

a. $1^{\frac{1}{2}} = e^{\frac{1}{2}\mathrm{Log}1} = e^{\frac{1}{2}(\ln 1 + i\cdot 0)} = e^0 = 1$
b. $(-1)^{\frac{1}{2}} = e^{\frac{1}{2}(\ln 1 + i\pi)} = e^{i\frac{\pi}{2}} = i$
c. $i^i = e^{i\mathrm{Log}(i)} = e^{i(\ln 1 + i\frac{\pi}{2})} = e^{-\frac{\pi}{2}}$

One could pick another branch of the logarithm to define another branch of z^α.

💡 **Ex 18.**

a. The branch cuts of Log are inherited for these derived functions: for example, $z^\alpha = e^{\alpha \text{Log}(z)}$ has a branch cut on $(-\infty, 0]$.

b. $\text{Log}(1-z^2)$ has a branch cut on $\{z \in \mathbb{C} : 1 - z^2 \in (-\infty, 0]\}$, namely:

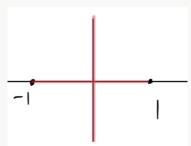
$$\begin{cases} 0 = \text{Im}(1-z^2) = -2xy \\ 0 \geq \text{Re}(1-z^2) = 1 + y^2 - x^2 \end{cases}$$

If $x = 0$, then $1 + y^2 \leq 0$ is a contradiction.

Hence, $y = 0$, so $1 - x^2 \leq 0 \implies x^2 \geq 1 \implies (-\infty, 1] \cup [1, \infty)$ is the branch cut

c. Find a branch of $(z^2-1)^{\frac{1}{2}}$ which is holomorphic in $\{|z| > 1\}$.

The branch $e^{\frac{1}{2}\text{Log}(z^2-1)}$ has cuts



Writing $(z^2-1)^{\frac{1}{2}}$ as $z(1-\frac{1}{z^2})^{\frac{1}{2}}$, we pick the branch $ze^{\frac{1}{2}\text{Log}(1-\frac{1}{z^2})}$

Branch cut on $\{z \in \mathbb{C} : 1 - \frac{1}{z^2} \in (-\infty, 0]\}$, namely:

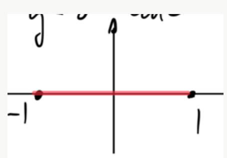
$$\begin{cases} \text{Im}(1-\frac{1}{z^2}) = 0 \\ \text{Re}(1-\frac{1}{z^2}) \leq 0 \end{cases}$$

So $\frac{1}{z^2}$ is real and ≥ 1. We have

$$\frac{1}{z^2} = \frac{1}{x^2-y^2+2ixy} = \frac{x^2-y^2-2ixy}{x^4+y^4-2x^2y^2+4x^2y^2} = \frac{x^2-y^2-2ixy}{(x^2+y^2)^2} = \frac{x^2-y^2-2ixy}{|z|^4}$$

Hence, we need $\begin{cases} xy = 0 \\ \frac{x^2-y^2}{|z|^4} \geq 1 \end{cases}$

$x = 0$ does not work so we have $y = 0$ and $\frac{x^2}{x^4} \geq 1 \implies x^2 \leq 1$



W4C2 Lecture 11 (Jan 29)

Integration

📌 **Def.** A smooth parameterized curve is a function $\alpha : [a, b] \to \mathbb{C}$ such that

i. α is differentiable with continuous derivative

ii. α'$(t) \neq 0$ for all $t \in [a, b]$

- α is closed if $\alpha(a) = \alpha(b)$
- α is simple if $\alpha(t) \neq \alpha(s)$ for all $a < t < s < b$

Remarks:

- Such curves are oriented from $\alpha(a)$ to $\alpha(b)$
- There are many different parameterizations of the same geometric curve

💡 **Ex 19.**

a. Consider the horizontal line segment from -1 to 2.

   One parameterization is $\alpha(t) = t, t \in [-1, 2]$
   Another one is
   $\alpha(t) = 3t - 1, t \in [0, 1]$

b. Consider the vertical line segment from $1 - i$ to $1 - 3i$

   One parameterization is $\alpha(t) = 1 - it, t \in [1, 3]$

c. Circle of radius r centered at $z_0 \in \mathbb{C}$

   $\alpha(t) = z_0 + re^{it}, t \in [0, 2\pi]$

📌 **Def.** Consider a domain Ω, with f defined on Ω, and a curve $\alpha \in \Omega$.

The integral of f along α is $\int_\alpha f(z)\, dz = \int_a^b f(\alpha(t))\alpha'(t)\, dt$

💡 **Ex 20.**

a. Integrate $f(z) = \bar{z}$ along the line from 1 to $2+i$.

Parameterize the curve as $\alpha(t) = 1 + (1+i)t, t \in [0,1]$

$$\begin{aligned}\int_\alpha \bar{z}\,dz &= \int_0^1 \overline{\alpha(t)}\alpha'(t)\,dt \\ &= \int_0^1 (1+(1-i)t)(1+i)\,dt \\ &= (1+i)\left(t + \frac{1}{2}(1-i)t^2\right)\Bigg|_{t=0}^{t=1} \\ &= 2+i\end{aligned}$$

b. Let $z_0 \in \mathbb{C}, n \in \mathbb{Z}, \Omega = \mathbb{C} \setminus \{z_0\}$. Then $f(z) = (z-z_0)^n$ is holomorphic in Ω.

Let $\alpha(t) = z_0 + re^{it}, t \in [0, 2\pi]$. Then the integral

$$\begin{aligned}\oint_\alpha (z-z_0)^n\,dz &= \int_0^{2\pi} (\alpha(t)-z_0)^n ire^{it}\,dt \\ &= \int_0^{2\pi} r^n e^{int} ire^{it}\,dt \\ &= ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt\end{aligned}$$

If $n \neq -1$, then $n+1 \neq 0$ and so $\int_0^{2\pi} e^{i(n+1)t}\,dt = \frac{1}{i(n+1)}e^{i(n+1)t}\Big|_{t=0}^{t=2\pi} = 0$

If $n = -1$, $\oint_\alpha \frac{1}{z-z_0}\,dz = i\int_0^{2\pi}\,dt = 2\pi i$ (independent of radius)

Remark: for $n \neq -1$, $(z-z_0)^n$ has an antiderivative along α, but not so for $n=-1$ because of the branch cut of the log.

W4C3 Lecture 12 (Jan 31)

Piecewise Smooth Curves

- If $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, then $\oint_\alpha f(z)\,dz = \oint_{\alpha_1} f(z)\,dz + \oint_{\alpha_2} f(z)\,dz + \oint_{\alpha_3} f(z)\,dz$

📌 **Def.** The length of a smooth parameterized curve $\alpha : [a,b] \to \mathbb{C}$ is given by

$\ell(\alpha) = \int_a^b |\alpha'(t)|\,dt$ (complex modulus of derivative)

A useful bound for contour integrals is:

$|\int_\alpha f(z)\,dz| \leq M(f)\ell(\alpha)$, where $M(f) = \max\{|f(z)| : z \in \alpha\}$

💡 **Ex 21.** Let $\alpha(t) = Re^{it}, t \in [0, \pi]$. Consider the integral $\int_\alpha \frac{z^{\frac{1}{2}}}{1+z^2}\, dz$.

The length is $\ell(\alpha) = \int_0^\pi |Rie^{it}|\, dt = \pi R$, as we would expect from geometry.

We have $|f(z)| = \left|\frac{z^{\frac{1}{2}}}{1+z^2}\right| = \frac{|e^{\frac{1}{2}(\ln|z| + i\text{Arg}(z))}|}{|1+z^2|}$

Along the curve, we have $|z^{\frac{1}{2}}| = R^{\frac{1}{2}}$, and $|1+z^2| \geq |z^2| - 1 = R^2 - 1$

Hence, $|f(z)| = \left|\frac{z^{\frac{1}{2}}}{1+z^2}\right| \leq \frac{R^{\frac{1}{2}}}{R^2-1} \to 0$ as $R \to \infty$.

Remark: the integral is well-defined since the curve α does not cross the branch cut.

📌 **Def.** A holomorphic function F is an antiderivative of f in a domain Ω if $f(z) = F'(z)$ for all $z \in \Omega$.

📖 **Thm.** Let α be a curve in Ω, and suppose $\alpha(a) = z_i$ and $\alpha(b) = z_j$.

If f has an antiderivative F in a neighbourhood of α, then

$\int_\alpha f(z)\, dz = F(z_j) - F(z_i)$

📖 **Corollary.** If α is a closed curve, then $\oint_\alpha f(z)\, dz = 0$, assuming that f has an antiderivative.

Another consequence is that if f has an antiderivative in Ω, then $\int_\alpha f(z)\, dz$ is independent of α, provided the endpoints z_i and z_j are fixed (the specific path between them does not matter).

📖 **Thm.** (Cauchy's Theorem). Let f be holomorphic in a disc $B_r(z_0)$. Then f has an antiderivative in $B_r(z_0)$. In particular: $\oint_\alpha f(z)\, dz = 0$ for any closed curve α in $B_r(z_0)$.

Remark:

i. The antiderivative is unique up to a constant (for any two antiderivatives F_1 and F_2, $F_1 - F_2$ is constant)

ii. Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in H(\Omega)$. Let α be a simple closed curve in Ω such that its interior is completely in Ω. Then $\oint_\alpha f(z)\, dz = 0$.