3️⃣

Part 3: Green's Functions for Time-Dependent PDE

W5C2 Lecture 9 (Feb 6)

3.1 Green's Functions for Heat Equation

Let $\Omega \subset \mathbb{R}^n$, and consider

$$\begin{cases} \text{eq:} & \partial_t u - \Delta u = f \quad\quad x \in \Omega, t > 0 \\ \text{BC:} & u = g \quad\quad\quad\quad\quad x \in \partial\Omega, t > 0 \\ \text{IC:} & u(x,0) = u_0(x) \quad\quad x \in \Omega \end{cases} \textbf{(Eq. 1)}$$

Our goal is to find a solution using a Green's function $G(x,t,y,s) = G^{x,t}(y,s)$.

Let $L = \partial_t - \Delta$, which is not self-adjoint. Its adjoint operator is $L^* = -\partial_t - \Delta$.

Let $-\infty < a < b < \infty$ and $(u,v) = \int_a^b \int_\Omega u(x,t)v(x,t)\,dx\,dt$.

Similar to **(G2)**, by IBP,

$(Lu, v) = (u, L^*v) + \int_a^b \int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}) dS_y\,ds + \int_\Omega [uv]_a^b\,dy$ **(Eq. 2)**

Suppose u is a solution of **(1)** and $v = G^{x,t}(y,s)$. We would like

$$\begin{cases} L^* G^{x,t}(y,s) = \delta_x(y)\delta_y(s) & \\ G^{x,t}(y,s) = 0 & y \in \partial\Omega \\ G^{x,t}(y,s) & s > t \text{ (causality)} \end{cases} \quad \textbf{(Eq. 3)}$$

Causality means that hte solution at time t does not depend on data at time $s > t$.

Assuming **(3)**, we get from **(2)** with $a = 0$:

$u(x,t) = \int_0^t \int_\Omega G^{x,t}(y,s)f(y,s)\,dy\,ds + \int_\Omega u_0(y)G^{x,t}(y,0)\,dy + \int_0^t \int_{\partial\Omega} g\frac{\partial}{\partial n}G^{x,t}(y,s)\,dS_y\,ds$ **(Eq. 4)**

We need $G^{x,t}(y,s) = 0$ for $y \in \partial\Omega$, otherwise we cannot compute $\int_0^t \int_{\partial\Omega} G^{x,y}\frac{\partial u}{\partial n}\,dS_y\,ds$.

Because **(3)** is the backward heat equation, we change time variables to $\tau = t - s$, so $0 \le s \le t \iff 0 \le \tau \le t$ and $\partial_\tau = -\partial_s$, $\delta_t(s) = \delta_0(\tau)$. **(3)** becomes

$$\begin{cases} (\partial_\tau - \Delta_y)G = \delta_x(y)\delta_0(\tau) & \\ G = 0 & y \in \partial\Omega \\ G = 0 & \tau < 0 \end{cases} \textbf{(Eq. 5)}$$

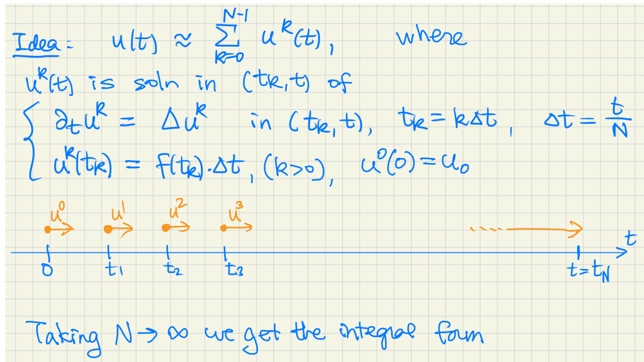
Due to translational invariance in time, we have $G(x,t,y,s) = G(x,t-a,y,s-a)$ for all a. Hence we may assume $G(x,t,y,s) = G(x,y,\underbrace{t-s}_{\tau})$.

(4) becomes

$u(x,t) = \int_0^t \int_\Omega G(x,y,t-s)f(y,s)\,dy\,ds + \int_\Omega u_0(y)G(x,y,t)\,dy + \int_0^t \int_{\partial\Omega} g(y,s)\frac{\partial G}{\partial n_y}G(x,y,t-s)\,dy\,ds$ **(Eq. 6)**

Remarks:

1. There is no translational invariance in x if $\partial\Omega \neq \varnothing$
2. **(6)** is a form of Duhamel's formula. If we denote the solution when $f = 0$ as $u(t) = g(t)u_0$ where $g(t)$ is a solution operator, then the general solution is $u(t) = g(t)u_0 + \int_0^t g(t-s)f(s)\,ds$



3.2 Heat Kernel

📌 **Def.** The heat kernel is the fundamental solution of the heat equation in $\mathbb{R}^n$, and is also the Green's function of the heat equation in $\mathbb{R}^n$.

Let $\Omega = \mathbb{R}^n$ for $n \geq 1$.

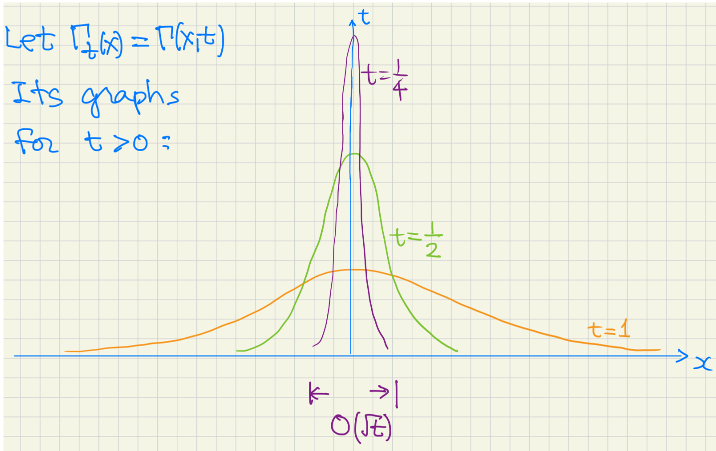
By translational invariance in time and space, we can write $G(x,t,y,s) = \Gamma(x-y, t-s)$, where the heat kernel Γ satisfies

$$\begin{cases} (\partial_t - \Delta)\Gamma(x,t) = \delta_0(x)\delta_0(t) & \\ \Gamma(x,t) = 0 & t \leq 0 \end{cases} \quad \textbf{(Eq. 6)}$$

We claim that the heat kernel is

$$\Gamma(x,t) = \begin{cases} (4\pi t)^{-n/2} e^{-x^2/4t}, x \in \mathbb{R}^n, & t > 0 \\ 0 & t \leq 0 \end{cases} \quad \textbf{(Eq. 7)} \text{ where } x^2 = |x|^2 \text{ (for higher dimensions)}$$

Let $\Gamma_t(x) = \Gamma(x,t)$, and we plot some graphs for $t > 0$:



Note that $\Gamma_1(x) = (4\pi)^{-n/2}e^{-x^2/4}$ is a Gaussian, and $\Gamma_t(x)$ is a rescaled Gaussian with support scale $\sqrt{t}$ and height scale $t^{-n/2}$.

Exponential decay as $|x| \to \infty$: for any fixed t, $\lim_{|x|\to\infty} \partial_t^m \nabla_x^k \Gamma(x,t) = 0$

Exponential decay as $t \to 0_+$ at $x^0 \neq 0$: $\lim_{(x,t)\to(x^0,0)} \partial_t^m \nabla_x^k \Gamma(x,t) = 0$

The heat kernel is continuous everywhere except at $(0,0)$.

Additionally, $\Gamma(x,t) = \Gamma^{(1)}(x_1,t) \cdot \Gamma^{(1)}(x_2,t) \cdot \ldots \cdot \Gamma^{(1)}(x_n,t)$ where $\Gamma^{(1)}$ is the 1D-heat kernel.

To show that the heat kernel satisfies **(6)**:

- $t > 0$ by direct computation
- $t < 0$ is trivial
- $t = 0, x \neq 0$: because all derivatives go to 0 exponentially as $t \to 0$
- $t = 0, x = 0$: use definition of generalized functions (skipped)

📖 **Lemma 1.**

a. $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$

b. $\int_{\mathbb{R}^n} \Gamma_t(x)\, dx = 1, \forall t > 0$

Proof of a):

Denote the integral as I, so

$I^2 = \int_{\mathbb{R}^2} e^{-x^2-y^2}\, dx\, dy = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\, dr\, d\theta = 2\pi \int_0^{\infty} e^{-u^2} \frac{du}{2} = \pi \implies I = \sqrt{\pi}$

Proof of b):

Use a change of variables $y = \frac{x}{\sqrt{4t}}$, so $dy = \frac{1}{\sqrt{4t}^n} dx$

$\int_{\mathbb{R}^n} \Gamma_t(x)\, dx = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{x^2}{4t}}\, dx = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-y^2} \sqrt{4t}^n\, dy = (\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2}\, ds)^n = 1$ by part a).

📖 **Lemma 2.** For all $R > 0$ and $v(x,t)$ continuous at $(0,0)$, we have $\lim_{t\to 0^+} \int_{B_R} \Gamma(x,t)v(x,t)\,dx = v(0,0)$

The idea is that $\{\Gamma_t\}_{t>0}$ is an approximation of δ, as in $\lim_{t\to 0^+} \Gamma_t = \delta_0(x)$.

Since Γ_t is concentrating to x near 0, the part in $\mathbb{R}^d \setminus B_R$ does not matter (proof skipped).

Now, recall the solution formula **(6)** with $G(x,y,t) = \Gamma(x-y,t)$ in $\mathbb{R}^n$. We get (no BC g):

$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-s)f(y,s)\,dy\,ds + \int_{\mathbb{R}^n} \Gamma(x-y,t)u_0(y)\,dy$ **(Eq. 8)**

📖 **Thm.** Suppose $f(x,t) \in \mathrm{BC}(\mathbb{R}^n \times [0,T])$ and $u_0(x) \in \mathrm{BC}(\mathbb{R}^n)$, where BC denotes bounded and connected. Then for $0 < t < T$, $u(x,t)$ given by **(8)** is C^2 in x, C^1 in t, and satisfies **(1)**:

$\partial_t u - \Delta u = f, \quad \lim_{t\to 0^+} u(x,t) = u_0(x)$

The PDE needs justification (skipped), and the limit to the IC is by Lemma 2.

Properties of the solution $u(x,t)$: (when assuming $f = 0$):

- Instantaneous smooth: $u(x,t) \in C^\infty_{x,t}$ for $t > 0$
- Infinite propagation speed: $u_0(x) \geq 0, \begin{cases} u_0(x) > 0 & |x| < 1 \\ u_0(x) = 0 & |x| \geq 2 \end{cases}$, then $u(x,t) > 0$ for all $x \in \mathbb{R}^n, t > 0$.

  The initial localized disturbance is propagated with infinite speed.
- $u(\cdot,t) = \Gamma_t * u_0$ (convolution), and

  $||u(\cdot,t)||_{L^q(\mathbb{R}^n)} \leq ||\Gamma_t||_{L^1} \cdot ||u_0||_{L^q} = ||u_0||_{L^q}$ (using Young's convolution inequality)

W6C1 Lecture 10 (Feb 11)

💡 **Ex.** Let $k(x) = \frac{1}{1+x^2}$, $f(x) \in C(\mathbb{R})$, $f(x) = 0$ for $|x| > 1$, and

$k * f(x) = \int_{-\infty}^{\infty} k(x-y)f(y)\,dy$

a)

Since f has compact support (zero outside of a bounded set), convolution with f is like local mixing, and $k * f$ has the same decay property as $k(x)$

$|k * f(x)| \leq \frac{C}{1+x^2}$ for $|x| > 2$

Proof:

For $|y| < 1, |x| > 2$, there exists constants such that $c_1 k(x) \leq k(x-y) \leq c_2 k(x) \implies$

$|k * f(x)| \leq \int_{-\infty}^{\infty} c_2 k(x)|f(y)|\,dy = (c_2 \int_{-\infty}^{\infty} |f(y)|\,dy)k(x)$

Remark: if $f(y) \geq 0$, we can use $c_1 k(x)$ to get a lower bound.

b)

If $\int_{-\infty}^{\infty} f(y)\,dy = 0$, we have cancellation during mixing, and we expect faster decay of $k * f(x)$ than $k(x)$:

$$
\begin{aligned}
k * f(x) &= \int_{-\infty}^{\infty} k(x-y)f(y) - k(x) \int_{-\infty}^{\infty} f(y)\,dy \\
&= \int_{-\infty}^{\infty} (k(x-y) - k(x))f(y)\,dy
\end{aligned}
$$

By MVT, $k(x-y) - k(x) = -yk'(z)$ (or $-y \cdot \nabla k(z)$ for dim ≥ 2) for some z between $x-y$ and x .

For $|y| < 1 < 2 \leq |x|$, we have $|k(x-y) - k(x)| \leq |y||k'(z)| \leq C|k'(x)|$

Hence,

$$
\begin{aligned}
|k * f(x)| &\leq \int_{-\infty}^{\infty} c|k'(x)||f(y)|\,dy \\
&= (C \int |f(y)|\,dy)|k'(x)| \leq \frac{C}{|x|^3}
\end{aligned}
$$

3.3 Green's Functions for Domains

In a domain Ω, the Green's function $G(x, y, t)$ satisfies (recall **(5)**):

$$
\begin{cases}
(\partial_t - \Delta_y)G = \delta_x(y)\delta_0(t) & \\
G = 0 & y \in \partial\Omega \\
G = 0 & t < 0
\end{cases}
$$

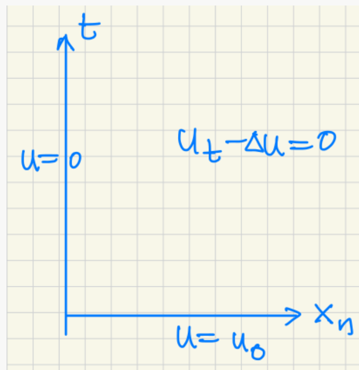
If we try $G(x, y, t) = \Gamma(x-y, t) - \phi(x, y, t)$, the correction ϕ needs to satisfy (for a fixed $x \in \Omega$):

$$
\begin{cases}
(\partial_t - \Delta_y)\phi = 0 & y \in \Omega \\
\phi(x, y, t) = \Gamma(x-y, t) & y \in \partial\Omega \\
\phi(x, y, t) = 0 & t < 0
\end{cases}
$$

We will consider method of images and eigenfunction expansion.

Ex 1. IBVP in $\mathbb{R}^n_+ \times (0,\infty)$, $n \geq 1$

$$\begin{cases} u_t - \Delta u = f \quad x \in \mathbb{R}^n_+, \quad t > 0 \\ u|_{t=0} = u_0 \\ u|_{x_n=0} = 0 \end{cases} \tag{Eq. 9}$$



Similar to the Laplace equation in $\mathbb{R}^n_+$, denote the reflection point as $\tilde{x} = (x', -x_n)$ if $x = (x', x_n)$ and let $\phi(x,y,t) = \Gamma(y - \tilde{x}, t)$.

The singularity $(\tilde{x}, 0)$ is outside if $x_n > 0$.

Check:

$(\partial_t - \Delta_y)\phi = 0 \quad y \in \Omega$

$\phi(x,y,t) = \Gamma(x-y,t) \quad y \in \partial\Omega$ (because $|x-y| = |\tilde{x} - y|$ when $y_n = 0$)

$\phi(x,y,t) = 0 \quad t < 0$

Hence, $G(x,y,t) = \Gamma(x-y,t) - \Gamma(\tilde{x}-y,t)$

The solution formula is

$u(x,t) = \int_{\mathbb{R}^n_+} G(x,y,t)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^n_+} G(x,y,t-s)f(y,s)\,dy\,ds$

Remark: we don't know an explicit formula for Green's function of heat equation in a disk.

Ex 2. (finite rod) Consider $0 < x < L, t > 0$:

$$\begin{cases} u_t - u_{xx} = 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

The method of images does not work here, so we try eigen function expansion.

The eigenfunctions of ∂_x^2 with 0-BC (Dirichlet) are:

$\phi_n(x) = \sin\frac{n\pi x}{L}, \lambda_n = -(\frac{n\pi}{L})^2, n \in \mathbb{N}$

Try $G(x,y,t) = \sum_{n=1}^{\infty} g_n(x,t)\phi_n(y)$ with $x, y \in (0,L)$

We need $(\partial_t - \Delta_y)G = \delta_x(y)\delta_0(t)$

$\implies \sum_{n=1}^{\infty}(\partial_t + (\frac{n\pi}{L})^2)g_n(x,t)\phi_n(y) = \delta_x(y)\delta_0(t)$

Using Fourier trick $\int_0^L \phi_n\phi_k\,dy = \begin{cases} \frac{L}{2} & n=k \\ 0 & n \neq k \end{cases}$, we get

$(\partial_t + (\frac{n\pi}{L})^2)g_n(x,t) = \frac{2}{L}(\delta_x(y)\delta_0(t), \phi_n(y)) = \frac{2}{L}\phi_n(x)\delta_0(t)$

For $t < 0$, we require $g_n(x,t) = 0$.

For $t > 0$, we have the ODE $(\partial_t + (\frac{n\pi}{L})^2)g_n = 0 \implies g_n(x,t) = c(x)e^{-(\frac{n\pi}{L})^2 t}$

We determine $c(x)$ be a jump condition:

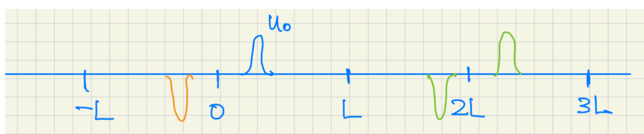
$\frac{2}{L}\phi_n(x) = \int_{0^-}^{0^+} \frac{2}{L}\phi_n(x)\delta_0(t)\,dt = \int_{0^-}^{0^+}(\partial_t g_n + (\frac{n\pi}{L})^2 g_n)\,dt = [g_n]_{0^-}^{0^+} = c(x)$

Hence,

$g_n(x,t) = \frac{2}{L}\phi_n(x)e^{-(\frac{n\pi}{L})^2 t}$

$G(x,y,t) = \sum_{n=1}^{\infty} \frac{2}{L}\phi_n(x)e^{-(\frac{n\pi}{L})^2 t}\phi_n(y)$

Remarks (Lecture 11 Feb 13)



For a given $u_0(x)$ defined on $[0,L]$, first do an odd extension to $[-L,L]$, then do a periodic extension to $\mathbb{R}$.

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty}\Gamma(x-y,t)\tilde{u_0}(y)\,dy \\
&= \sum_{n\in\mathbb{Z}} \begin{cases} \int_0^L \Gamma(x-y-2nL,t)u_0(y)\,dy \\ -\int_{-L}^0 \Gamma(x-y-2nL,t)u_0(-y)\,dy \end{cases} \\
&= \sum_{n\in\mathbb{Z}} \int_0^L (\Gamma(x-y-2nL,t) - \Gamma(x+y-2nL,t))u_0(y)\,dy \\
&= \int_0^L G(x-y,t)u_0(y)\,dy
\end{aligned}
$$

where $G(x,t) = \sum_{n\in\mathbb{Z}}[\Gamma(x-y-2nL,t) - \Gamma(x+y-2nL,t)]$

Source: Haberman (11.3.36) p. 533

3.4 BVPs of Heat Equation

Consider **(1)** with nonzero BC, $g \neq 0$.

By considering $\tilde{u} = u - v$ where v solves **(1)** for data f, u_0 and $g = 0$, we may assume $f = 0$, $u_0 = 0$, and focus on (with $u = \tilde{u}$)

$$
\begin{cases} u_t - \Delta u = 0 \quad x \in \Omega, \quad t > 0 \\ u|_{t=0} = 0 \\ u|_{x_n=0} = g(x', t) \end{cases} \quad \textbf{(Eq. 10)}
$$

Recall **(2)**, the heat equation version of **(G2)**:

$(Lu, v) = (u, L^*v) + \int_a^b \int_{\partial\Omega}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS_y\,ds + \int_\Omega [uv]_a^b\,dy$ **(2)**

If u is a solution of **(10)** and $v(y,s) = G(x,y,t-s)$,

$$\underset{0}{\underset{\|}{(Lu, v)}} = u(x,t) + \int_0^t \int_{\partial\Omega} \Big(g\frac{\partial v}{\partial n} - \overset{0}{\overset{\|}{v}}\,\overbrace{\frac{\partial u}{\partial n}}^{\text{no info}} \Big)\, dS_y\, ds + 0$$

We get $u(x,t) = -\int_0^t \int_{\partial\Omega} g(y,s) \frac{\partial G}{\partial n_y}(x,y,t-s)\, dy\, ds = \int_0^t \int_{\partial\Omega} P(x,y,t-s)\, dy\, ds$

where $P(x,y,t) = -\frac{\partial}{\partial n_y} G(x,y,t)|_{y\in\partial\Omega}$ is the Poisson kernel for the heat equation.

Ex 3. In $\Omega = R^n_+$, **(10)** becomes

$$\begin{cases} u_t - \Delta u = 0 \quad x \in \mathbb{R}^n_+, \quad t > 0 \\ u|_{t=0} = 0 \\ u|_{x_n=0} = g(x', t) \end{cases}$$

By **Ex 1**, the Green's function is $G(x,y,t) = \Gamma(x-y,t) - \Gamma(\tilde{x}-y,t)$ where $x = (x', x_n)$ and $\tilde{x} = (x', -x_n)$.

We have $n_y = -(0,\dots,1)$ and $-\frac{\partial}{\partial n_y} = \partial_{y_n}$

The Poisson kernel is

$$\begin{aligned} P(x,y,t) &= -\frac{\partial}{\partial n_y} G(x,y,t)|_{y_n=0} \\ &= \partial_{y_n}(\Gamma(x-y,t) - \Gamma(\tilde{x}-y,t))|_{y_n=0} \\ &= -\partial_{x_n}\Gamma(x-y,t) - \partial_{x_n}\Gamma(\tilde{x}-y,t)|_{y_n=0} \\ &= -2\partial_{x_n}\Gamma(x-y,t) \end{aligned}$$

For example, when $n=2$, we have $\Gamma(x,y) = \frac{1}{4\pi t} e^{-\frac{x^2}{4t}}$ and $-2\partial_{x_2}\Gamma(x,t) = \frac{x_2}{t}\frac{1}{4\pi t} e^{-\frac{x^2}{4t}}$

The solution is $u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{x_2}{4\pi(t-s)^2} e^{-\frac{(x_1-y_1)^2 + x_2^2}{4(t-s)}} g(y_1, s)\, dy_1\, ds$

Remark: when $\Omega = \mathbb{R}^n_+$, the Poisson kernel is

- Laplace equation: $2\partial_n \Phi(x)$
- Heat equation: $-2\partial_n \Gamma(x,t)$

They would have the same sign if we chose $-\Delta\Phi = \delta_0(x)$

Exercise: find Green's function and Poisson kernel in $\mathbb{R}^n_+ \times \mathbb{R}$ with Neumann BC.

W6C2 Lecture 11 (Feb 13)

3.5 Green's Functions for Wave Equations

The wave equation in $\Omega \subset \mathbb{R}^n$ is

$$\begin{cases} \partial_t^2 - c^2\Delta u = f & x \in \Omega, t > 0 \\ u = g & x \in \partial\Omega \\ u = u_0, \partial_t u = u_1 & t = 0 \end{cases} \quad \textbf{(Eq. 11)}$$

where $c > 0$ is the wave speed.

The Green's function $G(x,t,y,s) = G^{x,t}(y,s)$ solves for a fixed x, t:

$$\begin{cases} (\partial_s^2 - c^2\Delta y)G = \delta_x(y)\delta_t(s) & y \in \Omega, s \in \mathbb{R} \\ G = 0 & y \in \partial\Omega \\ G = 0 & s > t \end{cases} \quad \textbf{(Eq. 12)}$$

where the last line is due to causality.

Translation invariance in t gives $G(x,t,y,s) = G(x,t-s,y,0)$.

The linear differential operator for the wave equation $L = \partial_t^2 - c^2\Delta$ is self-adjoint ($L^* = L$).

Recall that $(u,v) = \int_a^b \int_\Omega u(x,t)v(x,t)\,dx\,dt$.

The wave equation version of Green's identity is

$$\begin{aligned} (u, Lv) = {} & (Lu, v) \\ & + c^2 \int_a^b \int_{\partial\Omega} -u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n}\,ds\,dt \quad \textbf{(Eq. 13)} \\ & + \int_\Omega [uv_t - u_t v]_{t=a}^b\,dx \end{aligned}$$

If u is a solution of **(11)**, and $v(y,s) = G^{x,t}(y,s)$ solves **(12)**, then **(13)** with $a = 0$ becomes (using causality)

$$\begin{aligned} u(x,t) = {} & \int_0^t \int_\Omega G^{x,t}(y,s)f(y,s)\,dy\,ds \\ & - c^2 \int_0^t \int_{\partial\Omega} g\frac{\partial G^{x,t}}{\partial n}\,dS_y\,ds \quad \textbf{(Eq. 14)} \\ & + \int_\Omega (u_1 G^{x,t} - u_0\partial_s G^{x,t})(y,0)\,dy \end{aligned}$$

Note that $\partial_s G^{x,t}(y,0) = -\partial_t G(x,t,y,0)$ because $G^{x,t}(y,s) = G(x,t-s,y,0)$.

3.6 Fundamental Solution for Wave Equation in $\mathbb{R}^n$

When $\Omega = \mathbb{R}^n$, we have translation invariance in both x and t.

$G(x,t,y,s) = G(x-y,t-s,0,0) = K(x-y,t-s)$ where $K(x,t) = G(x,t,0,0)$ is the fundamental solution.

By **(12)** for $\Omega = \mathbb{R}^n$:

$$\begin{cases} (\partial_t^2 - c^2\Delta)K = \delta_0(x)\delta_0(t) & x \in \mathbb{R}^n, t \in \mathbb{R} \\ K(x,t) = 0 & t < 0 \end{cases} \quad \textbf{(Eq. 15)}$$

From **(14)** with $t = 0$:

$$\begin{cases} K(x,0) = 0 \\ \partial_t K(x,0) = \delta_0(x) \end{cases} \quad \textbf{(Eq. 16)}$$

The solution u of **(11)** is given by **(14)**:

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} G^{x,t}(y,s)f(y,s)\,dy\,ds + \int_{\mathbb{R}^n} (u_1 G^{x,t} - u_0\partial_s G^{x,t})(y,0)\,dy.$$

In terms of the fundamental solution, this becomes

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)\,dy\,ds + \int_{\mathbb{R}^n} u_1 K(x-y,t) + u_0\partial_t K(x-y,t)\,dy \quad \textbf{(Eq. 17)}$$

Notice the sign change where $\partial_s = -\partial_t$.

We can solve $K(x,t)$ using Laplace transforms for $n = 1, 2, 3$. Fourier transform is also useful.

Case 1: $\mathbb{R}^1$

We first consider $\Omega = \mathbb{R}$, $n = 1$.

We have $K_{tt} - c^2 K_{xx} = 0$ for $x \in \mathbb{R}, t > 0$.

Write $K(x,t) = f(x - ct) + g(x + ct)$ as the sum of two waves with velocities $\pm c$.

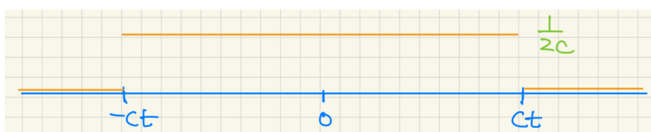
The condition $K(x,0) = 0$ imposes $g(x) = -f(x)$, so $K(x,t) = f(x - ct) - f(x + ct)$.

Hence,

$$\begin{aligned}
\partial_t K(x,0) &= -c(f'(x-ct) + f'(x+ct))\big|_{t=0} \\
&= \delta_0(x) \\
f'(x) &= -\frac{1}{2c}\delta_0(x) \\
f(x) &= -\frac{1}{2c}H(x) + c_1
\end{aligned}$$

where $H(x)$ is the Heaviside step function.

$K(x,t) = \frac{1}{2c}[H(x+ct) - H(x-ct)] = \frac{1}{2c}\mathbf{1}_{[-ct,ct]}$



The support of $K(x,t)$ is $[-ct, ct]$ which is expanding at speed c. Also, $K(x,t) = K(-x,t)$.

In the solution formula **(17)**, we need $\partial_t K$:

$\partial_t K(x,t) = \frac{1}{2}[H'(x+ct) + H'(x-ct)] = \frac{1}{2}[\delta(x+ct) + \delta(x-ct)]$

The solution $u(x,t)$ assuming $f = 0$ for simplicity is

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}^n} u_1(y)K(x-y,t) + u_0(y)\partial_t K(x-y,t)\,dy \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\,dy + \frac{1}{2}[u_0(x+ct) + u_0(x-ct)]
\end{aligned}$$

We recover d'Alembert's formula.

Exercise: find the Green's function for $\Omega = (0,\infty)$ by the method of images.

Case 2: $\mathbb{R}^3$

Since the equation $(\partial_t^2 - c^2\Delta)K(x,t) = \delta_0(x)\delta_0(t)$ is rotationally symmetric, we expect $K(x,t) = \hat{K}(\rho,t)$ for $\rho = |x|$. For clarity let's drop the hat and write $K(\rho,t)$.

The equation for $t > 0$ becomes

$\partial_t^2 - c^2(K_{\rho\rho} + \frac{2}{\rho}K_\rho) = 0 \quad (\rho, t > 0)$

Try a solution of the form $K = h\rho^a$.

$K_\rho = h'\rho^a + ah\rho^{a-1}$

$K_{\rho\rho} = h''\rho^a + 2ah'\rho^{a-1} + a(a-1)h\rho^{a-1}$

In $\mathbb{R}^n$: $\Delta_{\text{rad}} = K_{\rho\rho} + \frac{n-1}{\rho} K_\rho = \rho^a [h'' + \frac{2a+n-1}{\rho} h' + \frac{a(a-1+n-1)}{\rho^2} h]$

The dream is that $2a + n - 1 = 0 = a - 1 + n - 1$. For $n = 3$ this means that $a = -1$.

Hence $K = \frac{h}{\rho}$, and we need $\frac{1}{\rho}(\partial_t^2 h - c^2 h'') = 0$.

Try a sum of moving waves: $K(x,t) = \frac{f(\rho - ct) + g(\rho + ct)}{\rho}$.

IC **(16)** gives $g(x) = -f(x)$ like before, so

$\partial_t K(x,0) = -\frac{c}{\rho}(f'(\rho - ct) + f'(\rho + ct))_{t=0} = -\frac{2c}{\rho} f'(\rho) = \delta_0(x)$

$1 = \int_{x \in \mathbb{R}^3, |x|<1} \delta_0(x)\, dx$

$\quad = \int_0^1 -\frac{2c}{\rho} f'(\rho) 4\pi\rho^2 d\rho$

$\quad = -8\pi c \int_0^1 f'(\rho) \rho \, d\rho$

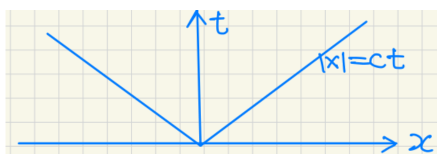
$\quad = 8\pi c \int_0^1 f(\rho)\, d\rho$

We can take $f(\rho) = \frac{1}{4\pi c}\delta(\rho)$, noting that $\int_0^1 \delta(\rho)\, d\rho = \frac{1}{2}$.

$K(x,t) = \frac{1}{4\pi c \rho}[\delta(\rho - ct) - \cancel{\delta(\rho + ct)}]$ where the second δ is cancelled because $\rho + ct > 0$ always.

Our fundamental solution in $\mathbb{R}^3$ is $K(x,t) = \frac{1}{4\pi c|x|}\delta(|x| - ct)$.

This is a distribution, not a function, supported in the cone $|x| = ct$.



Solution formula (Kirchoff's formula):

Assume $f = 0$ for simplicity. By **(14)**,

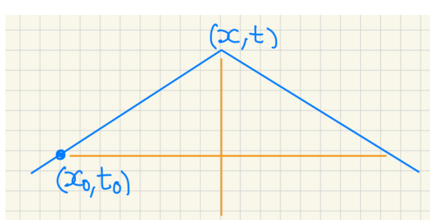
$u(x,t) = \int_{\mathbb{R}^3} u_1(y) K(x-y,t) + u_0(y)\partial_t K(x-y,t)\, dy$

$\qquad = \int_{|y-x|=ct} \frac{u_1(y)}{4\pi c^2 t}\, dS_y + \partial_t (\int_{|y-x|=ct} \frac{u_0(y)}{4\pi c^2 t}\, dS_y)$

Remark: The value of u at (x,t) only depends on the values of the data u_0, u_1 on the sphere $\{y \in \mathbb{R}^3 : |y - x| = ct\}$.

Huygens' principle ($\mathbb{R}^3$ only)

A concentrated source at location x_0 and time t_0 only influences the position x at a later time t if $|x - x_0| = c(t - t_0)$.



Remark: (alternative approach in $\mathbb{R}^3$) we can solve the solution formula directly, and then extract the fundamental solution. See Fritz John PDE 5.1a p. 126-129.

Method of Descent (from 3D to 2D)

In 2D, the wave equation is $(\partial_t^2 - \Delta_2)u = 0, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1$.

For $t > 0$, think of $u = u(x_1, x_2, x_3, t)$ and $u_j = u_j(x_1, x_2, x_3), \quad j = 0, 1$ with no dependence on x_3.

In 3D, the wave equation is $(\partial_t^2 - \Delta_3)u = 0$ for $t > 0$.

By the 3D Kirchoff formula,

$u(x,t) = \int_{|y-x|=ct} \frac{u_1(y)}{4\pi c^2 t}\, dS_y + \partial_t(\int_{|y-x|=ct} \frac{u_0(y)}{4\pi c^2 t}\, dS_y) = I_1 + \partial_t I_0$, where both u_1 and u_0 have no dependence in y_3.

Let's write $x_3 = 0, x = (x', 0), y = (y', y_3)$.

$c^2t^2 = |y-x|^2 = |y' - x'|^2 + y_3^2$

Hence $y_3 = \pm h(y')$ where $h(y') = \sqrt{c^2t^2 - |y' - x'|^2}$.

The sphere $|y - x| = ct$ in $\mathbb{R}^3$ is the graph $y_3 = \pm h(y'), \quad y' \in \mathbb{R}^2$.

For such a graph $y_3 = h(y')$, the area element (MATH 317) is

$dS = \sqrt{1 + |\nabla h(y')|^2} dy'$

For our h,

$$\begin{aligned}
\nabla h &= \frac{1}{2h}(-2(y' - x')) \\
|\nabla h| &= \frac{|y'-x'|}{h} \\
1 + |\nabla h|^2 &= \frac{c^2t^2}{h^2} \\
dS &= \frac{ct}{h} dy'
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{|y-x|=ct} \frac{u_1(y)}{4\pi c^2 t}\, dS \\
&= 2\int_{|y'-x'|\le ct} \frac{u_1(y')}{4\pi c^2 t} \frac{ct}{h(y')}\, dy' \\
&= \int_{|y'-x'|\le ct} \frac{u_1(y')}{2\pi c\sqrt{c^2t^2 - |y'-x'|^2}}\, dy' \\
&= \int_{\mathbb{R}^2} K(x - y', t) u_1(y')\, dy'
\end{aligned}$$

We conclude that:

$K(x,t) = \frac{\mathbf{1}_{|x|<ct}}{2\pi c\sqrt{c^2t^2 - |x|^2}}$ in $\mathbb{R}^2$.

The solution formula (Poisson's Formula) for the wave equation in $\mathbb{R}^2$ is

$u(x,t) = I_1 + \partial_t I_0$ where

$$\begin{aligned}
I_j(x,t) &= \int_{|y-x|\le ct} \frac{u_j(y)}{2\pi c\sqrt{c^2t^2 - |y-x|^2}}\, dy \\
&= \int_{\mathbb{R}^2} K(x-y,t) u_j(y)\, dy, \quad j = 0, 1
\end{aligned}$$

W7 Reading Break

No classes due to reading break.