MATH 317 Formula Sheet

C^n: continuous 0^{th} to n^{th} order partial derivatives.

Curves

velocity	$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}(t) = \frac{ds}{dt}(t)\hat{\mathbf{T}}(t)$
unit tangent	$\hat{\mathbf{T}}(t) = \frac{\mathbf{v}(t)}{\lvert\mathbf{v}(t)\rvert}$ (general parameterization)
	$\hat{\mathbf{T}}(s) = \frac{d\mathbf{r}}{ds}(s)$ (arc length parameterization)
acceleration	$\mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2}(t) = \frac{d^2 s}{dt^2}(t)\hat{\mathbf{T}}(t) + \kappa(t)\,\lvert\mathbf{v}(t)\rvert^2\,\hat{\mathbf{N}}(t)$
speed	$\frac{ds}{dt}(t) = \lvert\mathbf{v}(t)\rvert = \left\lvert\frac{d\mathbf{r}}{dt}(t)\right\rvert$
arc length	$s(T) = \int_0^T \lvert\mathbf{v}(t)\rvert\, dt = \int_0^T \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$
curvature	$\kappa(t) = \left\lvert\frac{d\hat{\mathbf{T}}}{dt}(t)\right\rvert / \frac{ds}{dt}(t) = \frac{\lvert\mathbf{v}(t)\times\mathbf{a}(t)\rvert}{(\frac{ds}{dt}(t))^3} = \frac{\lvert\mathbf{v}(t)\times\mathbf{a}(t)\rvert}{\lvert\mathbf{v}(t)\rvert^3}$
	$\kappa(s) = \left\lvert\frac{d\varphi}{ds}(s)\right\rvert = \left\lvert\frac{d\hat{\mathbf{T}}}{ds}(s)\right\rvert$
unit normal	$\hat{\mathbf{N}}(t) = \frac{d\hat{\mathbf{T}}}{dt}(t) / \left\lvert\frac{d\hat{\mathbf{T}}}{dt}(t)\right\rvert$ $\quad \hat{\mathbf{N}}(s) = \frac{d\hat{\mathbf{T}}}{ds}(s)/\kappa(s)$
radius of curvature	$\rho(t) = \frac{1}{\kappa(t)}$
center of curvature	$\mathbf{r}(t) + \rho(t)\hat{\mathbf{N}}(t)$
binormal	$\hat{\mathbf{B}}(t) = \hat{\mathbf{T}}(t)\times\hat{\mathbf{N}}(t) = \frac{\mathbf{v}(t)\times\mathbf{a}(t)}{\lvert\mathbf{v}(t)\times\mathbf{a}(t)\rvert}$
torsion	$\tau(t) = \frac{(\mathbf{v}(t)\times\mathbf{a}(t))\cdot\frac{d\mathbf{a}}{dt}(t)}{\lvert\mathbf{v}(t)\times\mathbf{a}(t)\rvert^2}$

Curvature Formulas in Two Dimensions

$$\kappa(t) = \frac{\left\lvert\frac{dx}{dt}(t)\frac{d^2y}{dt^2}(t) - \frac{dy}{dt}(t)\frac{d^2x}{dt^2}(t)\right\rvert}{\left[(\frac{dx}{dt}(t))^2 + (\frac{dy}{dt}(t))^2\right]^{3/2}} \qquad \kappa(x) = \frac{\left\lvert\frac{d^2y}{dx^2}(x)\right\rvert}{\left[1 + (\frac{dy}{dx}(x))^2\right]^{3/2}}$$

Vector Fields

Field Lines

$$\text{Solve: } \frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}(t))$$

$$\text{Integrate: } \frac{dx}{v_1(x,y)} = \frac{dy}{v_2(x,y)} \quad \text{or} \quad \frac{dx}{v_1(x,y,z)} = \frac{dy}{v_2(x,y,z)} = \frac{dz}{v_3(x,y,z)}$$

Conservative Vector Fields

Definition	$\mathbf{F}$ is conservative iff there exists φ such that $\mathbf{F} = \nabla\varphi$.
Screening Test	$\nabla\times\mathbf{F} = \mathbf{0}$

If $\mathbf{F}$ is C^0 on a connected open set $U \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$, then the following are equivalent:

- There exists φ such that $\mathbf{F} = \nabla\varphi$.
- $\oint_C \mathbf{F}\cdot dr = 0$ for any closed curve C.
- $\int \mathbf{F}\cdot dr$ is path independent.

If $\mathbf{F}$ is C^1 on a simply-connected open set $U \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$, then:

- $\mathbf{F}$ is conservative iff $\nabla\times\mathbf{F} = \mathbf{0}$.

Line Integrals

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz) = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(P_1) - \varphi(P_0) \qquad \text{(conservative)}$$

Parameterizations

Surface	$\mathbf{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$ where $(u,v)\in D\subseteq\mathbb{R}^2$
Spherical	$(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$
	$dV = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$
Cylindrical	$(\rho\cos\theta, \rho\sin\theta, z)$
	$dV = r\,dr\,d\theta\,dz$

Surface Integrals

$$\iint_S \rho\,dS \quad \text{(area)} \qquad \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS \quad \text{(flux)}$$

$\mathbf{r}(u,v)$	$\hat{\mathbf{n}}\,dS = \pm\frac{\partial\mathbf{r}}{\partial u}(u,v)\times\frac{\partial\mathbf{r}}{\partial v}(u,v)\,du\,dv$ $dS = \left\lvert\frac{\partial\mathbf{r}}{\partial u}(u,v)\times\frac{\partial\mathbf{r}}{\partial v}(u,v)\right\rvert\,du\,dv$
$z = f(x,y)$	$\hat{\mathbf{n}}\,dS = \pm\left[-f_x(x,y)\hat{\mathbf{\imath}} - f_y(x,y)\hat{\mathbf{\jmath}} + \hat{\mathbf{k}}\right]\,dx\,dy$ $dS = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,dx\,dy$
$G(x,y,z) = K$ $G_z(x,y,z) \neq 0$	$\hat{\mathbf{n}}\,dS = \pm\frac{\nabla G(x,y,z)}{\nabla G(x,y,z)\cdot\hat{\mathbf{k}}}\,dx\,dy$ $dS = \left\lvert\frac{\nabla G(x,y,z)}{\nabla G(x,y,z)\cdot\hat{\mathbf{k}}}\right\rvert\,dx\,dy$

Gradient, Divergence, and Curl

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

$$\nabla \times (\nabla f) = 0$$

Vector Potentials

Definition	$\mathbf{G}$ is a vector potential for $\mathbf{F}$ if $\mathbf{F} = \nabla \times \mathbf{G}$
Screening Test	$\nabla \cdot \mathbf{F} = 0$

$$\mathbf{G} = \left\langle \int_0^z F_2(x,y,\tilde{z})\,\mathrm{d}\tilde{z} + M(x,y), -\int_0^z F_1(x,y,\tilde{z})\,\mathrm{d}\tilde{z} + N(x,y), 0 \right\rangle, \text{ where } \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = F_3(x,y,0)$$

Take $M(x,y) = 0$ and $N(x,y) = \int_0^x F_3(\tilde{x},y,0)\,\mathrm{d}\tilde{x}$.

$$\mathbf{G} = \left\langle \int_0^z F_2(x,y,\tilde{z})\,\mathrm{d}\tilde{z}, -\int_0^z F_1(x,y,\tilde{z})\,\mathrm{d}\tilde{z} + \int_0^x F_3(\tilde{x},y,0)\,\mathrm{d}\tilde{x}, 0 \right\rangle$$

Divergence Theorem

- V is a bounded solid with a piecewise smooth surface ∂V,
- $\mathbf{F}$ is C^1 in V:

$$\iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \iiint_V \nabla \cdot \mathbf{F}\,\mathrm{d}V$$

where $\hat{\mathbf{n}}$ is the outward unit normal of ∂V.

$$\iint_{\partial V} \hat{\mathbf{n}} * \tilde{\mathbf{F}}\,\mathrm{d}S = \iiint_V \nabla * \tilde{\mathbf{F}}\,\mathrm{d}V$$

where $* = \cdot$ or $\times$ or nothing and $\tilde{\mathbf{F}} = \mathbf{F}$ or f.

Green's Theorem

- $R \subset \mathbb{R}^2$ is connected and open,
- Bounded by ∂R: finite # of simple, closed, piecewise-smooth curves oriented consistently with R,
- $\mathbf{F}$ is C^1 on R:

$$\oint_{\partial R} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_{\partial R} \Big(F_1(x,y)\,\mathrm{d}x + F_2(x,y)\,\mathrm{d}y\Big) = \iint_{\mathcal{R}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

Stoke's Theorem

- S is a piecewise-smooth, oriented surface,
- ∂S consists of finite number of piecewise smooth, simple curves that are oriented consistently with $\hat{\mathbf{n}}$.
- $\mathbf{F}$ is C^1 on S:

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

Identities and Derivatives

Trig

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\sin(2x) = 2\sin x \cos x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Derivatives

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}\log|x| = \frac{1}{x}$$

$$\frac{d}{dx}b^x = b^x \log b$$

$$\frac{d}{dx}\log_b x = \frac{1}{x \log b}$$