MATH 257 Formula Sheet

ODE Review

Separable ODEs: $y' = P(x)Q(y)$

$$\int \frac{\mathrm{d}y}{Q(y)} = \int P(x)\,\mathrm{d}x + C$$

Integrating Factor: $y' + P(x)y = Q(x)$

$$\mu(x) = e^{\int P(x)\,\mathrm{d}x}$$

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,\mathrm{d}x + C\right]$$

Constant Coefficients: $ay'' + by' + cy = 0$

$$y = e^{rx} \implies ar^2 + br + c = 0$$

$$y(x) = \begin{cases} c_1e^{r_1x} + c_2e^{r_2x} & \Delta > 0 \\ (c_1 + c_2x)e^{r_0x} & \Delta = 0 \\ e^{\lambda x}[c_1\cos(\mu x) + c_2\sin(\mu x)] & \Delta < 0 \end{cases}$$

Cauchy-Euler: $x^2y'' + axy' + by = 0$

$$y = x^r \implies r(r-1) + ar + b = 0$$

$$y(x) = \begin{cases} c_1x^{r_1} + c_2x^{r_2} & \Delta > 0 \\ (c_1 + c_2\ln|x|)x^{r_0} & \Delta = 0 \\ x^{\lambda}[c_1\cos(\mu\ln|x|) + c_2\sin(\mu\ln|x|)] & \Delta < 0 \end{cases}$$

Series Solutions

Power Series:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = L$$

$L < 1$ implies convergence

Taylor Series:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad \sinh(x) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \qquad \cosh(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

Power Series Solution:

$$y = \sum_{n=0}^{\infty}a_nx^n \qquad y' = \sum_{n=1}^{\infty}a_nnx^{n-1}$$

$$y'' = \sum_{n=2}^{\infty}a_nn(n-1)x^{n-2}$$

Singular Points: $P(x)y'' + Q(x)y' + R(x)y = 0$

x_0 is a singular point if $P(x_0) = 0$

Polynomial Test for Regular Singular Points:

Finite $\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)}, \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} \implies$ RSP

Frobenius Series Solution:

$$y = \sum_{n=0}^{\infty}a_nx^{n+r} \qquad y' = \sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2}$$

PDE Intro

Type	PDE	Surface	Name
Parabolic	$u_t = u_{xx}$	$T = X^2$	Heat
Elliptic	$u_{xx} + u_{yy} = k$	$X^2 + Y^2 = k$	Laplace
Hyperbolic	$u_{tt} = c^2u_{xx}$	$T^2 = C^2X^2$	Wave

Finite Difference Approximations:

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x)$$

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + O(\Delta x^2)$$

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} + O(\Delta x^2)$$

u_N^k: position N, time k

Fourier Series

Fourier Series: $f(x)$ defined on $[-L, L]$

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_n = \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{n\pi x}{L}\right)\mathrm{d}x$$

$$b_n = \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{n\pi x}{L}\right)\mathrm{d}x$$

Fourier Sine and Cosine Series: $f(x)$ defined on $[0, L]$

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)$$

$$a_n = \tfrac{2}{L}\int_0^L f(x)\cos\left(\tfrac{n\pi x}{L}\right)\mathrm{d}x$$

$$Sf(x) = \sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \tfrac{2}{L}\int_0^L f(x)\sin\left(\tfrac{n\pi x}{L}\right)\mathrm{d}x$$

Boundary Value Problems $X'' + \lambda^2X = 0$

Dirichlet: $X(0) = 0 = X(L)$

$$\begin{cases}\lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots \\ X_n(x) = \sin\left(\frac{n\pi x}{L}\right)\end{cases}$$

Neumann: $X'(0) = 0 = X'(L)$

$$\begin{cases}\lambda_n = \frac{n\pi}{L}, \quad n = 0, 1, 2, \ldots \\ X_n(x) = \cos\left(\frac{n\pi x}{L}\right)\end{cases}$$

Periodic: $X(-L) = X(L), X'(-L) = X'(L)$

$$\begin{cases}\lambda_n = \frac{n\pi}{L}, \quad n = 0, 1, 2\ldots \\ X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\end{cases}$$

Mixed A: $X(0) = 0 = X'(L)$

$$\begin{cases}\lambda_n = \frac{(2n+1)\pi}{2L}, \quad n = 0, 1, 2, \ldots \\ X_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right)\end{cases}$$

Mixed B: $X'(0) = 0 = X(L)$

$$\begin{cases}\lambda_n = \frac{(2n+1)\pi}{2L}, \quad n = 0, 1, 2, \ldots \\ X_n(x) = \cos\left(\frac{(2n+1)\pi x}{2L}\right)\end{cases}$$

Heat Equation $u_t = \alpha^2u_{xx}$

Separation of Variables:

$$\frac{\dot{T}}{\alpha^2T} = \frac{X''}{X} = -\lambda^2 \implies T = e^{-\lambda^2t}, X'' + \lambda^2X = 0$$

Inhomogeneous Heat Equation:

Use BC to find $w(x,t)$, then $u(x,t) = w(x,t) + v(x,t)$

Eigenfunction Expansions:

$$v(x,t) = \sum_{n=1}^{\infty}v_n(t)X_n(x), \text{ solve ODE for } v_n(t)$$

Wave Equation $u_{tt} = c^2u_{xx}$

D'Alembert's Solution: $u(x,0) = f(x), u_t(x,0) = g(x)$

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,\mathrm{d}s$$

Separation of Variables:

$$\frac{\ddot{T}}{c^2T} = \frac{X''}{X} = -\lambda^2 \implies \ddot{T} + \lambda^2c^2T = 0, X'' + \lambda^2X = 0$$

Physical Phenomena:

String: $c = \sqrt{T_0/\rho_0}$ Bar: $c = \sqrt{E/\rho}$

Laplace's Equation $\Delta u = 0$

2D Cartesian:

$$u_{xx} + u_{yy} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{const}$$

2D Polar:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

$$u(r,\theta) = [A_0 + \alpha_0 \ln r] + \sum_{n=1}^{\infty}\left[A_n r^{\lambda_n} + \alpha_n r^{-\lambda_n}\right]\cos(\lambda_n\theta)$$
$$+ \sum_{n=1}^{\infty}\left[B_n r^{\lambda_n} + \beta_n r^{-\lambda_n}\right]\sin(\lambda_n\theta)$$

Sturm-Liouville

Sturm-Liouville Eigenvalue Problem:

$$\begin{cases} [p(x)y']' - q(x)y + \lambda r(x)y = 0, & a < x < b \\ a_1 y(a) + a_2 y'(a) = 0, & b_1 y(b) + b_2 y'(b) = 0 \end{cases}$$

Converting to SL Form:

$$-P(x)y'' - Q(x)y + R(x)y = \lambda y$$

$$\mu(x)P(x) = p(x) \text{ and } \mu(x)Q(x) = p'(x)$$

$$\mu(x) = \frac{e^{\int \frac{Q}{P}\,\mathrm{d}x}}{P}$$

Eigenfunction Expansion:

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad a < x < b$$

$$c_n = \frac{\int_a^b f(x)\phi_n(x)r(x)\,\mathrm{d}x}{\int_a^b \phi_n^2(x)r(x)\,\mathrm{d}x}$$

Identities

Double Angle:

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

Squared Angle:

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

Sum and Difference:

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

Product to Sum:

$$\cos x\cos y = \tfrac{1}{2}(\cos(x+y) + \cos(x-y))$$

$$\sin x\sin y = \tfrac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\sin x\cos y = \tfrac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\cos x\sin y = \tfrac{1}{2}(\sin(x+y) - \sin(x-y))$$

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