ELEC 221 Formula Sheet

Signals Basics

DT signals	$x[n], n \in \mathbb{Z}$
CT signals	$x(t), t \in \mathbb{R}$
IV transformation	$x(t) \to x(\alpha t + \beta)$
	1. Shift by β
	2. Compress by $\|\alpha\|$
	3. Reverse if $\alpha < 0$
Periodic, period T	$x(t+T) = x(t)$
Odd	$x(-t) = -x(t)$
Even	$x(-t) = x(t)$

Systems Basics

DT systems	$x[n] \to y[n]$
CT systems	$x(t) \to y(t)$
1. Memoryless	$y(t_0)$ depends only on $x(t_0)$
2. Invertible	distinct $x(t)$ map to distinct $y(t)$
3. Causal	$y(t_0)$ depends only on $x(t)$ for $t \leq t_0$
4. Stable	bounded input $\implies$ bounded output
5. Linear	$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$
6. Time-invariant	$x(t - t_0) \to y(t - t_0)$

DT Impulse and Convolution Sum

DT unit impulse	$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$
DT unit step	$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
Relations	$\delta[n] = u[n] - u[n-1]$
	$u[n] = \sum_{m=0}^{\infty} \delta[n-m] = \sum_{k=-\infty}^{n} \delta[k]$
Sampling	$x[k] = x[n]\delta[n-k]$
Weighted sum	$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
Impulse response	$\delta[n-k] \to h_k[n]$
	$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$
Time-invariant	$\delta[n] \to h[n]$
	$\delta[n-k] \to h[n-k]$
Convolution	$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
	$y[n] = x[n] * h[n]$

Convolution Properties

Associative	$x[n] * (h_1[n] * h_2[n])$
	$= (x[n] * h_1[n]) * h_2[n]$
Commutative	$x[n] * h[n] = h[n] * x[n]$
Distributive	$x[n] * (h_1[n] + h_2[n])$
	$= x[n] * h_1[n] + x[n] * h_2[n]$

CT Impulse and Convolution Integral

CT unit impulse	$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$
	$\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$
CT unit step	$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$
Relations	$\delta(t) = \frac{du(t)}{dt}$
	$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$
Weighted integral	$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$
Impulse response	$\delta(t) \to h(t)$
Convolution	$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$
	$y(t) = x(t) * h(t)$

Systems Properties

1. Memoryless	$h(t) = K\delta(t)$
2. Invertible	$h_i(t) * h(t) = \delta(t)$
3. Stable	$\|x(t)\| \leq B \implies$
	$\|y(t)\| \leq B \int_{-\infty}^{\infty} \|h(\tau)\|\, d\tau < \infty$
4. Causality	$h(t) = 0$ for $t < 0$

Fourier Series

Euler Identities	$e^{j\theta} = \cos\theta + j\sin\theta$
	$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$
	$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$
$x(t) = e^{st}$	$y(t) = e^{st}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)\, d\tau$
	$y(t) = x(t) \cdot H(s)$
System function	$H(s) = \int_{-\infty}^{\infty} e^{-s\tau}h(\tau)\, d\tau$
Frequency response	$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}h(\tau)\, d\tau$
Synthesis equation	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$
	$y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t}$
Real $x(t)$	$c_{-k}^* = c_k$

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k\cos(k\omega t) + b_k\sin(k\omega t))$$

Analysis equation	$c_k = \frac{1}{T}\int_T e^{-jk\omega t}x(t)\, dt$

Dirichlet Conditions

For a periodic $x(t)$ over one period, if

1. $x(t)$ is single-valued
2. $x(t)$ is absolutely integrable
3. $x(t)$ has a finite maxima, minima, discontinuities

$\implies$ Fourier series converges to $x(t)$ where continuous, and half the jump where discontinuous.

Fourier Series Properties

Linear	$z(t) = Ax(t) + By(t) \implies c_k = Aa_k + Bb_k$
Time shift	$x(t - t_0) \implies c_k' = e^{-jk\omega t_0}c_k$
Time scale	$x(\alpha t) \implies T' = \frac{T}{\alpha} \quad \omega' = \omega\alpha$
Product	$z(t) = x(t)y(t) \implies c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Parseval	$\frac{1}{T}\int_T \|x(t)\|^2\, dt = \sum_{k=-\infty}^{\infty}\|c_k\|^2$

DT Fourier Series

Complex	$x[n] = Ce^{jwn} = C\cos(\omega n) + jC\sin(\omega n)$
Periodic	$e^{j\omega n} = e^{j(\omega + 2\pi)n} \iff 0 \leq \omega < 2\pi$
	$\omega = \frac{2\pi}{N}$, N is the fundamental period
Harmonics	$x_k[n] = e^{jk\frac{2\pi}{N}n}, \quad k = 0, 1, \ldots, N-1$
$x[n] = e^{j\omega n}$	$y[n] = e^{j\omega n}\sum_{k=-\infty}^{\infty} e^{-j\omega k}h[k]$
	$y[n] = x[n]H(e^{j\omega})$
Frequency response	$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} e^{-j\omega k}h[k]$
Synthesis equation	$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi}{N}n}$
Analysis equation	$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$

Filters

1. Low-pass: $H(j\omega) = 0$ for $\omega > \omega_c$
2. High-pass: $H(j\omega) = 0$ for $\omega < \omega_c$
3. Band-pass: $H(j\omega) = 0$ for $\omega < \omega_1$ or $\omega > \omega_2$

Fourier Transform

Synthesis	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega$
Analysis	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$
Impulse	$H(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}h(t)\, dt$
	$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}H(j\omega)\, d\omega$
Impulse train	$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(w - k\omega_0)$

Fourier Transform Properties

Linear	$ax(t) + by(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bY(j\omega)$
Time shift	$x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}X(j\omega)$
Time scale	$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Time reversal	$x(-t) = X(-j\omega)$
Conjugation	$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$
Convolution	$y(t) = h(t) * x(t) \to Y(j\omega) = H(j\omega)X(j\omega)$

Differentiation and Integration

Magnitude $|Y(j\omega)| = |X(j\omega)||H(j\omega)|$

Phase $\sphericalangle Y(j\omega) = \sphericalangle X(j\omega) + \sphericalangle H(j\omega)$

Differentiation $\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$

Integration $\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$

Unit impulse $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$

Unit step $u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$

Generic ODE

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^k}{\sum_{k=0}^{N} \alpha_k (j\omega)^k} = \frac{\text{X coeffs}}{\text{Y coeffs}}$$

Filter Behavior

Step response $s(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$

1st order $T\frac{dy(t)}{dt} + y(t) = x(t)$

$H(j\omega) = \frac{1}{1+j\omega T}$

$s(t) = (1 - e^{-t/T})u(t)$

2nd order $\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$

$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$

$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_+)(j\omega - c_-)}$

$c_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

$s(t) = \left[1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{c_+ t}}{c_+} - \frac{e^{c_- t}}{c_-}\right)\right]u(t)$

DT Fourier Transform

Synthesis $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\, d\omega$

Analysis $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Convergence $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$ or $\sum_{n=-\infty}^{\infty}|x[n]|^2 < \infty$

Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} = \frac{\text{X coeffs}}{\text{Y coeffs}}$$

Sampling

Sampling $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

$x_p(t) = x(t)p(t)$

Impulse train $P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$

Freq domain $X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$

Nyquist rate $\omega_s \geq 2\omega_M$

CT-DT Conversion

Freq resp $X_d(e^{j\Omega}) = X_p(j\frac{\Omega}{T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\frac{\Omega - 2\pi k}{T})$

DT Sampling

Sampling $p[n] = \sum_{k=-\infty}^{\infty}\delta[n - kN]$

$$x_p[n] = \begin{cases} x[n], & n \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

Impulse train $P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$

Freq domain $X_p(j\omega) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})$

Decimation $x_b[n] = x[nN]$

$X_b(e^{j\omega}) = X_p(e^{j\frac{\omega}{N}})$

Interpolation add $N - 1$ points between

Modulation

Modulation $Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega - \theta))d\theta$

Amplitude (AM) $c(t) = e^{j(\omega_c t + \theta_c)}$

Sinusoidal AM $c(t) = \cos(\omega_c t + \theta_c)$

Laplace Transform

Laplace $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\, dt$

$X(s) = \mathcal{F}[e^{-\sigma t}x(t)] \qquad s = \sigma + j\omega$

1. ROC no $j\omega$ axis $\iff$ FT does not converge
2. ROC of rational $\mathcal{L}$ contains no poles
3. $x(t)$ finite duration and absolutely integrable
   $\implies$ ROC = s-plane
4. $x(t)$ left sided and $\mathrm{Re}(s) = \sigma_0$ in ROC
   $\implies$ s s.t. $\mathrm{Re}(s) < \sigma_0$ in ROC
5. $x(t)$ right sided and $\mathrm{Re}(s) = \sigma_0$ in ROC
   $\implies$ s s.t. $\mathrm{Re}(s) > \sigma_0$ in ROC
6. $x(t)$ two sided, then ROC is a strip or does not exist

Laplace Transform Properties

Linear $ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\longleftrightarrow} aX_1(s) + bX_2(s)$

ROC $\supseteq R_1 \cap R_2$

Time shift $x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0}X(s)$ ROC $= R$

Time scale $x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right)$ ROC $= aR$

Time reversal $x(-t) = X(-s)$ ROC $= -R$

Differentiation $\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$ ROC $\supseteq R$

$-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds}$ ROC $= R$

Conjugation $x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(-s^*)$ ROC $= R$

Convolution $x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s)$

ROC $\supseteq R_1 \cap R_2$

Systems

Causal ROC is a right-half plane

Stable for rational $H(s)$, stable iff ROC contains $j\omega$ and there aren't more zeros than poles

ODE

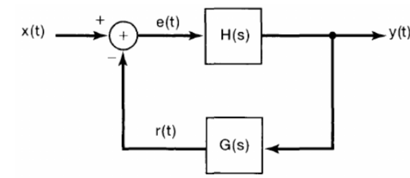
$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} = \frac{\text{X coeffs}}{\text{Y coeffs}}$$

\# of zeros at infinity = $\deg(D)$-$\deg(N)$

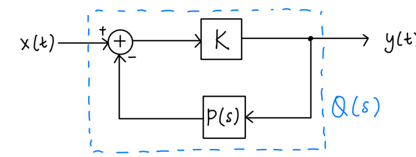
Feedback Systems

Feedback



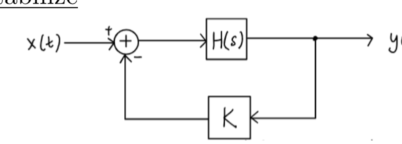
$Q(s) = \frac{H(s)}{1 + G(s)H(s)}$

Inverse



$Q(s) = \frac{K}{1 + KP(s)} \approx \frac{1}{P(s)}$ for large K

Stabilize

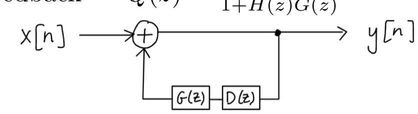


$H(s) = \frac{b}{s-a} \implies Q(s) = \frac{b}{s - a + Kb}$. Stable if $K > \frac{a}{b}$

Z Transform

Transform $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

Feedback $Q(z) = \frac{H(z)}{1 + H(z)G(z)}$



- $D(z)$ is a system that causes a delay of K steps
- $G(z)$ is a system with gain g

$Q(z) = \frac{z^k}{z^k - g}$

Miscellaneous

Geometric $\sum_{k=0}^{N} a^k = \frac{1 - a^{N+1}}{1 - a}$

Compiled December 22, 2024

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	$\begin{matrix} a_k \\ b_k \end{matrix}$
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t - \tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\begin{matrix} \mathcal{Re}\{a_k\} \\ j\mathcal{Im}\{a_k\} \end{matrix}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n - r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n - 1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\begin{matrix} \mathcal{Re}\{a_k\} \\ j\mathcal{Im}\{a_k\} \end{matrix}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{Ev}\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{Od}\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega$		

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\displaystyle\sum_{k=-\infty}^{n} x[k]$	$\dfrac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\displaystyle\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\displaystyle\sum_{n=-\infty}^{+\infty}\lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$		

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\displaystyle\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi\displaystyle\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi\displaystyle\sum_{l=-\infty}^{+\infty}\delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos\omega_0 n$	$\pi\displaystyle\sum_{l=-\infty}^{+\infty}\{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin\omega_0 n$	$\dfrac{\pi}{j}\displaystyle\sum_{l=-\infty}^{+\infty}\{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi\displaystyle\sum_{l=-\infty}^{+\infty}\delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & N_1 < \lvert n\rvert \leq N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi\displaystyle\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \dfrac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N\sin[2\pi k/2N]}$, $k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \dfrac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N, \ldots$
$\displaystyle\sum_{k=-\infty}^{+\infty}\delta[n - kN]$	$\dfrac{2\pi}{N}\displaystyle\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \dfrac{1}{N}$ for all k
$a^n u[n]$, $\lvert a\rvert < 1$	$\dfrac{1}{1 - ae^{-j\omega}}$	—
$x[n]\begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\dfrac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \displaystyle\sum_{k=-\infty}^{+\infty}\pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n]$, $\lvert a\rvert < 1$	$\dfrac{1}{(1 - ae^{-j\omega})^2}$	—
$\dfrac{(n+r-1)!}{n!(r-1)!}a^n u[n]$, $\lvert a\rvert < 1$	$\dfrac{1}{(1 - ae^{-j\omega})^r}$	—

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to\infty} sX(s)$$

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{R}e\{s\} < -\alpha$
10	$\delta(t - T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{R}e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{R}e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n\text{ times}}$	$\frac{1}{s^n}$	$\mathcal{R}e\{s\} > 0$