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Part 2: Green's Functions of Steady State ODE

W3C1 Lecture 5 (Jan 21)

2.1 BVP for Laplace's Equation

Steady state means time independent.

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a bounded domain (open and connected) with boundary $\partial\Omega$ and unit outer normal $\hat{n}$.

We write $x = (x_1, x_2)$ or $x = (x_1, x_2, x_3)$ for a point $x \in \Omega$.

📌 **Def.** A primary example of a steady-state PDE is the following BVP of Laplace equation:

$$\begin{cases} \Delta u = f & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases} \quad \textbf{(Eq. 1)}$$

where Δ is the Laplacian $\Delta = \partial_1^2 + \cdots + \partial_n^2$.

Remark: the Laplacian is also denoted as ∇^2 in physics books, but that notation is generally avoided by mathematicians.

Problem **(1)** appears in electrostatic potential, steady state temperature, and steady state deformation of an elastic membrane.

We look for a Green's function $G(x, y) = G^x(y), \quad x, y \in \Omega$ such that the solution of **(1)** is (if $g = 0$) $u(x) = \int_\Omega G(x, y) f(y)\, dy$.

We try $\begin{cases} \Delta G^x = \delta_x(y) & y \in \Omega \\ G^x(y) = 0 & y \in \partial\Omega \end{cases}$ **(Eq. 2)**

📌 **Def.** $\delta_x(y)$ is an n-dimensional delta function centered at x.

$$\delta_x = \delta(y - x) = \begin{cases} \delta^{(1)}(y_1 - x_1)\delta^{(1)}(y_2 - x_2) & n = 2 \\ \delta^{(1)}(y_1 - x_1)\delta^{(1)}(y_2 - x_2)\delta^{(1)}(y_3 - x_3) & n = 3 \end{cases}$$

where $\delta^{(1)}$ is the one dimensional delta function.

We have $(\delta_x, \phi) = \int_{\mathbb{R}^n} \delta_x(y)\phi(y)\, dy = \phi(x)$.

📖 **Thm.** Recall the divergence theorem from vector calculus: if $u = (u_1, u_2, u_3)$ is a vector field, and $\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3$, we have

$\int_\Omega \operatorname{div} F \, dx = \int_{\partial\Omega} F \cdot \hat{n} \, dS$

It extends the fundamental theorem of calculus to higher dimensions.

W3C2 Lecture 6 (Jan 23)

By the product rule, $\partial_j(v\partial_j u) = (\partial_j v)(\partial_j u) + v\partial_j^2 u$ for $j = 1, \ldots, n$.

Summing j from 1 to n gives $\operatorname{div}(v\nabla u) = \nabla v \cdot \nabla u + v\Delta u$.

Integrating and using the divergence theorem for $F = v\nabla u$ gives

📖 **Thm.** Green's first identity: $\int_\Omega (\nabla v \cdot \nabla u + v\Delta u) \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, dS$ **(G1)**

where we have used $\nabla u \cdot \hat{n} = \frac{\partial u}{\partial n}$.

Switching v and u in **(G1)** gives $\int_\Omega (\nabla u \cdot \nabla v + u\Delta v) \, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial n} \, dS$ **(G1')**

Taking **(G1)** - **(G1')** gives

📖 **Thm.** Green's second identity: $\int_\Omega (u\Delta v - v\Delta u) \, dx = \int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}) \, dS$ **(G2)**

We now look back at the BVP **(1)**: suppose that u is a solution of **(1)** with data f, g, and G^x is the Green's function satisfying **(2)**, then taking $u = u$ and $v = G^x$ in **(G2)** gives

$\int_\Omega (u\delta_x - G^x f) \, dy = \int_{\partial\Omega} (g\frac{\partial G^x}{\partial n} - 0) \, dS$

Rearranging for $u(x)$ gives the desired solution formula **(Eq. 5)**

$u(x) = \int_\Omega G(x, y) f(y) \, dy + \int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x, y) g(y) \, dS_y$

2.2 Fundamental Solution

📌 **Def.** The Green's function for $\Omega = \mathbb{R}^n$ is called the fundamental solution.

- It gives the leading singular behavior of a Green's function $G(x, y)$ in a domain when y is close to x
- It can be used to construct $G(x, y)$ when Ω is very symmetric (method of the image)
- It solves $\Delta_y G^x = \delta_x = \delta(y - x), y \in \mathbb{R}^n$
- We expect $G(x, y) = \Phi(y - x) = h(r), r = |y - x|$ - i.e. it is translation invariant since $\mathbb{R}^n$ has no boundary, and it only depends on the distance of y and x, as Δ is rotationally invariant

📌 **Def.** $\Phi(y) = G(0,y) = h(|y|)$ is called the radial. It solves $\Delta\Phi = \delta_0$ in $\mathbb{R}^n$.

For $r = |y| > 0$, we have $0 = \Delta\Phi = \Delta_r h = (\partial_r^2 + \frac{n-1}{r}\partial_r)h$.

Using the chain rule, $\partial_j r = \frac{1}{2r} \cdot 2y_j = \frac{y_j}{r}$

$\partial_j \Phi = h' \cdot \partial_j r = h' \frac{y_j}{r}$

$\partial_j^2 \Phi = \partial_j(\frac{h'}{r})y_j + \frac{h'}{r} = (\frac{h'}{r})' \frac{y_j}{r} y_j + \frac{h'}{r}$

Summing in j gives the ODE $\Delta\Phi = r(\frac{h'}{r})' + \frac{nh'}{r} = h'' + \frac{n-1}{r}h'$

💡 **Ex.** Consider the case $n = 2$ for $y \neq x$.

$0 = \Delta_r h(r) = (\partial_r^2 + \frac{1}{r}\partial_r)h = \frac{1}{r}(rh')'$

Hence $h' = \frac{c_1}{r} \implies h = c_1 \log r + c_2$ for some constants c_1, c_2. Notice that c_2 does not change ΔG^x, so we will take $c_2 = 0$.

We find c_1 using the divergence theorem:

$1 = \int_{|y|\le\varepsilon} \delta_0(y)\,dy = \int_{|y|\le\varepsilon} \nabla\cdot\nabla\Phi(y)\,dy = \int_{|y|=\varepsilon} \frac{\partial\Phi}{\partial n}(y)\,dS_y = \int_{|y|=\varepsilon} h'(r)\,dS_y$

On the boundary, we have $h'(r) = \frac{c_1}{r} = \frac{c_1}{\varepsilon}$. Hence,

$1 = 2\pi\varepsilon \cdot \frac{c_1}{\varepsilon} \implies c_1 = \frac{1}{2\pi} \implies \Phi(y) = \frac{1}{2\pi}\log|y|$

💡 **Ex.** Consider the case $n = 3$ for $y \neq x$.

$0 = \Delta_r h(r) = (\partial_r^2 + \frac{2}{r}\partial_r)h = \frac{1}{r^2}(r^2 h')'$

Hence $h' = \frac{c_1}{r_2^2} \implies h = -\frac{c_1}{r} + c_2$ and again we take $c_2 = 0$ and find c_1 by $1 = \cdots = \int_{|y|=\varepsilon} \frac{c_1}{\varepsilon^2}\,dS_y \implies$ $1 = 4\pi\varepsilon^2 \cdot \frac{c_1}{\varepsilon^2} \implies c_1 = \frac{1}{4\pi}$

$\Phi(y) = -\frac{1}{4\pi|y|}$

Remark: the 3D fundamental solution is the electrostatic potential generated by a point charge located at x (the Coulomb potential)

In general for $n \ge 3$, $h' = \frac{c_1}{r^{n-1}}$ and $\Phi(y) = h(r) = \frac{-c_1}{(n-2)r^{n-2}}$ where $\frac{1}{c_1}$ is the area of the unit sphere in $\mathbb{R}^n$.

With the fundamental solutions, we have the solution formulas of **(1)** in $\Omega = \mathbb{R}^n$.

📖 **Thm.** Let $f \in C_C^2(\mathbb{R}^n)$, $n = 2, 3$. Let

$$u(x) = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y| f(y)\,dy & n = 2, \\ -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\,dy & n = 3 \end{cases}$$

Then $u \in C^2(\mathbb{R}^n)$ and $\Delta u = f$ in $\mathbb{R}^n$. Proof is omitted (see Evans, PDE section 2.2)

2.3 Method of Images

Recall equation **(2)** for Green's function in Ω: $\begin{cases} \partial_y^2 G^x = \delta_x(y) & y \in \Omega \\ G^x(y) = 0 & y \in \partial\Omega \end{cases}$

If we use the fundamental solution $\Phi^x(y) = \Phi(y - x)$, it satisfies the equation but not the boundary conditions. In the solution formula **(5)**, we have an extra term from the last term of **(G2)**:

$-\int_{\partial\Omega}(\frac{\partial}{\partial n}u(y))G^x(y)\, dS_y$

but we don't know $\frac{\partial y}{\partial n}$. This is why we want $G^x(y) = 0$ for $y \in \partial\Omega$.

The idea is to add a correction term: $G^x(y) = \Phi^x(y) - \phi^x(y)$ where $\phi^x(y)$ is a correction term such that

$\begin{cases} \Delta_y \phi^x = 0 & \text{in } \Omega, \\ \phi^x = \Phi^x & \text{in } \partial\Omega \end{cases}$

and $\phi^x(y) \in C^2(\Omega) \cap C^1(\overline{\Omega})$

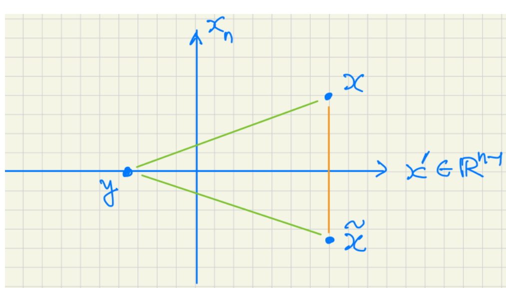
> 📖 **Thm.** If $\Omega \in \mathbb{R}^n$ is a bounded C^1 domain, then Green's function $G(x, y)$ exists. Morever, $G(x, y) = G(y, x)$.

So $G(x, y)$ exists in reasonable domains. However, to find an explicit formula, Ω needs to be very symmetric, e.g.

- half-plane, quarter-plane, half space, octant, etc
- disk, ball
- some combinations of the above, such as ½ disk.

> 💡 **Ex 1.** Half plane/space. Let $\Omega = \mathbb{R}^n_+ = \{x = (\underbrace{x_1, \dots, x_{n-1}}_{x'}, x_n) \in \mathbb{R}^n, x_n > 0\}$

Consider $x = (x', x_n)$ and the reflection point $\tilde{x} = (x', -x_n)$.



We claim that $\phi^x(y) = \Phi(y - \tilde{x}), \quad G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x})$

Check:

- $\phi^x(y)$ is smooth in $y \in \mathbb{R}^n_+$, and $\Delta_y \phi^x = 0$ since its singularity $y = \tilde{x} \notin \Omega$

- If $y \in \partial\Omega$, i.e. $y_n = 0$, then we have $|y - x| = |y - \tilde{x}|$ and so $\phi^x(y) = \Phi(y - \tilde{x}) = \Phi(y - x)$

For the solution formula **(5)**, it is useful to compute $\frac{\partial G^x}{\partial n}(y)$ when $y \in \partial\Omega$. On $\partial\Omega$ we have $y_n = 0$, so $\hat{n} = (0, \dots, 0, -1) = -e_n$, then

$$\begin{aligned}
\frac{\partial G^x}{\partial n_y}\bigg|_{y_n=0} &= -\partial_{y_n} G^x(y)\bigg|_{y_n=0} \\
&= -\partial y_n(\Phi(y-x) - \Phi(y-\tilde{x}))_{y_n=0} \\
&= -\Phi'(y-x)\frac{y_n - x_n}{|y-x|} + \Phi'(y-\tilde{x})\frac{y_n + x_n}{|y-\tilde{x}|}
\end{aligned}$$

when $y_n = 0$, we have $|y - x| = |y - \tilde{x}|$, so

$= \frac{c_1}{|y-x|^n}(-y_n + x_n + y_n + x_n)_{y_n=0} = \frac{2c_1 x_n}{|y-x|^n}$

W4C1 Lecture 7 (Jan 27)

📌 **Def.** The Poisson kernel is $P(x, y) = \frac{2x_n}{|S_1||y-x|^n}, y_n = 0$, where S_1 is the unit sphere in $\mathbb{R}^n$

Let BC mean bounded and continuous.

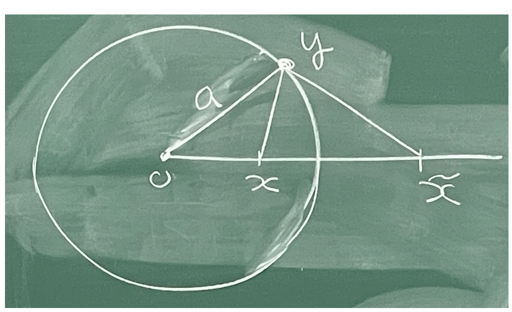
📖 **Thm.** Let $\Omega = \mathbb{R}^n_+$ with $\partial\Omega = \Gamma \approx \mathbb{R}^{n-1}$ with $n \geq 2$. If $f \in \mathrm{BC}(\Omega)$ and $g \in \mathrm{BC}(\Gamma)$, then $u(x) = \int_\Omega G(x,y)f(y)\,dy + \int_{\partial\Omega} P(x,y)g(y)\,dS_y$ is in $C^2(\Omega) \cap \mathrm{BC}(\overline{\Omega})$, and solves the BVP

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases}$$

💡 **Ex 2.** We examine Green's function in a ball $B_a \subset \mathbb{R}^n$ with radius $a > 0$, $B_a = \{x : |x| < a\}$

To find the corrector ϕ^x, let $\tilde{x} = \frac{a^2}{|x|^2}x$ be the inversion, and notice that $|\tilde{x}| \cdot |x| = a^2$.

For any $y \in \partial B_a$, there is a plane containing $0, x, \tilde{x}, y$. We have $|y| = a$.



$\triangle oxy \sim \triangle oy\tilde{x}$ because $\frac{|oy|}{|o\tilde{x}|} = \frac{|ox|}{|oy|}$ since $\frac{a}{|\tilde{x}|} = \frac{|x|}{a}$

Hence $\frac{|x-y|}{|\tilde{x}-y|} = \frac{|x|}{a}$ **(☆)**

We want a Green's function $G^x(y) = \Phi^x(y) - \phi^x(y)$, where

$$\begin{cases} \Delta\phi^x(y) = 0 & \text{in } B_a \\ \phi^x(y) = \Phi^x(y) & \text{on } \partial B_a \end{cases}$$

Candidate: $\phi^x(y) = \rho(x)\Phi(y-\tilde{x}) + \rho_1(x)$.

Since $\tilde{x}$ is outside of B_a, we have $\Delta\phi^x(y) = 0$ in B_a.

For $y \in \partial\Omega$, $\rho(x)\Phi(y-\tilde{x}) + \rho_1(x) = \Phi(y-x)$.

For $n \geq 3$, choose $\rho_1(x) = 0$, so $\rho_1(x) = 0$, then $\rho(x) = \left(\frac{|y-x|}{|y-\tilde{x}|}\right)^{2-n}$ from the boundary requirement (obtained from substituting Φ).

By **(☆)**, $\phi^x(y) = \left(\frac{|x|}{a}\right)^{2-n}\Phi(y-\tilde{x}) = \Phi\left(\frac{|x|}{a}(y-\tilde{x})\right)$

Claim: this formula also works for $n=2$: $\frac{1}{2\pi}\log\left(\frac{|x|}{a}|y-\tilde{x}|\right) = \frac{1}{2\pi}\log|y-x|$ when $|y| = a$ by **(☆)**

The Green's function for a ball B_a is $G(x,y) = \Phi(y-x) - \Phi\left(\frac{|x|}{a}|y-\tilde{x}|\right)$

In 3D: $= -\frac{1}{4\pi}\left[\frac{1}{|y-x|} - \frac{a}{|x||y-\tilde{x}|}\right]$

We now examine the Poisson kernel $\frac{\partial G^x(y)}{\partial n_y}$:

$$\begin{aligned} \frac{\partial G^x(y)}{\partial n_y} &= \frac{y}{a}\cdot\nabla G^x(y) \\ &= \frac{y}{a}\cdot\left[\Phi'(y-x)\frac{y-x}{|y-x|} - \Phi'\left(\frac{|x|}{a}|y-\tilde{x}|\right)\frac{|x|}{a}\frac{y-\tilde{x}}{|y-\tilde{x}|}\right] \\ &= \frac{a_1}{a|y-x|^n}\left[y\cdot(y-x) - \frac{|x|^2}{a^2}y\cdot(y-\tilde{x})\right] \\ &= \frac{a_1}{a|y-x|^n}\left[a^2 - \cancel{y\cdot x} - |x|^2 + \cancel{\frac{|x|^2}{a^2}(y\cdot x)\frac{a^2}{|x|^2}}\right. \\ &= \frac{c_1}{a|y-x|^n}\left[a^2 - x^2\right] = P(x,y) \end{aligned}$$

Thm. Let $n \geq 2$. If $f \in C(\overline{B_a})$, $g \in C(\partial B_a)$, then

$u(x) = \int_{B_a} G(x,y)f(y)\,dy + \int_{\partial B_a} P(x,y)g(y)\,dS_y$

is in $C^2(B_a) \cap C(\overline{B_a})$ and satisfies $\Delta u = f$ in B_a and $u = g$ on ∂B_a.

Corollary. (Mean Value Theorem). If $u \in C^2(B_a) \cap C(\overline{B_a})$ and $\Delta u = 0$ (harmonics), then $u(x)$ satisfies

$u(0) = \int_{|y|=a} \frac{c_a a^2}{a a^n} u(y)\,dS_y = \frac{1}{|\partial B_a|}\int_{|y|=a} u(y)\,dS_y$ (average of its value on the sphere $|y| = a$).

Remark: the solution in the theorem is unique by the maximal principle.

2.4 Solvability Condition & Neumann Problem

Consider the BVP

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega \end{cases}$$

Such a BVP is the Neumann problem. It is different for Ω bounded/unbounded.

Case 1: Consider the case where Ω is bounded. Recall Green's 2nd identity:

$\int_\Omega (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$

With inner product $(f, g) = \int_\Omega f(x)g(x)\,dx$, then Δ is self-adjoint, i.e. $(f\Delta g) = (\Delta f, g)$, under either $u = 0$ or $\frac{\partial u}{\partial n}$ on $\partial\Omega$ by **(G2)**.

The Neumann problem has a nonzero kernel: $u^*(x) = 1$, satisfying $\Delta u^* = 0$ and $\frac{\partial u^*}{\partial n} = 0$ on $\partial\Omega$. Hence we need a solvability condition.

Remark: when Ω is unbounded, for example $\Omega = \mathbb{R}^n_+$, then the constant function is not $L^2(\Omega)$, so it is not a kernel.

Take $u = u$ and $v = u^* = 1$ in **(G2)**:

$\int_\Omega (0 - f)\,dx = \int_{\partial\Omega} (-g)\,dS \implies \int_\Omega f\,dx = \int_{\partial\Omega} g\,dS$

is the solvability condition.

There is no Green's function, so we need a modified Green's function $G^x(y)$ which should satisfy $\Delta_y G^x(y) = \delta_x(y) - C$ in Ω and $\frac{\partial G^x}{\partial n_y} = 0$ on boundary.

for some constant C satisfying

$1 = \int_\Omega \delta_x\,dy = \int_\Omega (\Delta_y G^x(y) + C)\,dy = \int_{\partial\Omega} \hat{n}_y \cdot \nabla_y G^x\,dS_y + C|\Omega|$ where $|\Omega|$ is the volume of the domain Ω and is finite by the assumption of boundedness.

The term $\int_{\partial\Omega} \hat{n}_y \cdot \nabla_y G^x\,dS_y$ is 0 by assumption, and so $C = \frac{1}{|\Omega|}$.

Using **(G2)** with $v = G^x$ gives

$\int_\Omega (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$

$u(x) - \frac{1}{|\Omega|}\int_\Omega u\,dx - \int_\Omega G^x(y)f(y)\,dy = -\int_{\partial\Omega} G^x(y)g(y)\,dS$

The solution formula is $u(x) = \int_\Omega G(x,y)f(y)\,dy - \int_{\partial\Omega} G^x(y)g(y)\,dS_y + A$

for any constant A (note that the constant term $\frac{1}{|\Omega|}\int_\Omega u\,dx$ has been absorbed into A)

If the modified Green's function G^x is given by $G^x(y) = \Phi^x(y) - \phi^x(y)$, then

$$\begin{cases} \Delta_y \phi^x(y) = \frac{1}{|\Omega|} & \text{for } \Omega \\ \frac{\partial \phi^x(y)}{\partial n} = \frac{\partial \Phi^x}{\partial n} & \text{on } \partial\Omega \end{cases}$$

For an explicit solution $\phi^x(y)$, it is shown for $n = 2, 3$ in many textbooks, and $n \geq 4$ in a 2016 paper.

Case 2: consider an unbounded case, such as $\Omega = \mathbb{R}^n_+$ half-space. We seek $G(x,y) = \Phi^x(y) - \phi^x(y)$, satisfying

$$\begin{cases} \Delta \phi^x(y) = 0 & \text{in } \Omega \\ \frac{\partial \phi^x}{\partial n} = \frac{\partial \Phi^x}{\partial n} & \text{on } \partial\Omega \end{cases}$$

It turns out that a simple choice is $\phi^x(y) = -\Phi(y - \tilde{x})$, the negation of the fundamental solution.

The green's function for the Neumann BVP can be written as

$G_N(x,y) = \Phi(y - x) + \Phi(y - \tilde{x})$.

W4C2 Lecture 8 (Jan 30)

2.5 General Theory

Consider the BVP in $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

with Green's function $G(x,y) = \Phi^x(y) - \phi^x(y)$ satisfying

$$\begin{cases} \Delta G^x = \delta_x & \Omega \\ G^x = 0 & \partial\Omega \end{cases} \text{ and } \begin{cases} \Delta \phi^x = 0 & \Omega \\ \phi^x = \Phi^x & \partial\Omega \end{cases}$$

Properties:

1. Existence (solve for ϕ^x in MATH 516)
2. Uniqueness (by the maximal principle, MATH 400)
3. Symmetry $G(x,y) = G(y,x)$
4. Solution formula **(5)** gives the solution.

Rigorous meaning of $\Delta\Phi = \delta_0$: both are generalized functions.

For any test function $v(x)$ supported in some B_R,

$\langle \Delta\Phi, v\rangle = \int_{B_R} \Phi \Delta v \, dx = (-\int_{B_\varepsilon} + \int_{B_R \setminus B_\varepsilon}) \nabla\Phi \cdot \nabla v \, dx \quad (0 < \varepsilon < R)$

$= -\int_{B_\varepsilon} \nabla\Phi \cdot \nabla v \, dx + \int_{\partial B_\varepsilon} v \cdot \frac{\partial \Phi}{\partial n} \, dS = I_\varepsilon + J_\varepsilon$

To show that $\Delta\Phi = \delta_0$, we check that $\lim_{\varepsilon\to 0}(I_\varepsilon + J_\varepsilon) = v(0)$

Because $|\nabla\Phi(y)| \le \frac{C}{|y|^{n-1}}$, we have $|I_\varepsilon| \le C\varepsilon$

For J_ε, $\frac{\partial\Phi}{\partial n} = \frac{1}{|S_1|\varepsilon^{n-1}}$, so $v = v(0) + o(1)$ for $\varepsilon \ll 1$.

Hence $\lim_{\varepsilon\to 0}(I_\varepsilon + J_\varepsilon) = v(0)$

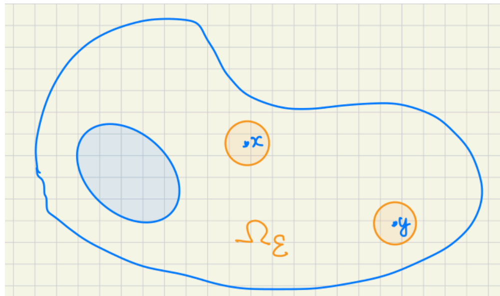
📖 **Lemma.** For f continuous at 0, $\lim_{\varepsilon\to 0} \int_{|y|=\varepsilon} f \nabla\Phi \cdot \hat{n} \, dS_y = f(0)$.

📖 **Thm.** $G(x,y) = G(y,x)$ for $x, y \in \Omega$.

Proof:

Let $\varepsilon_0 = \min(\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega)) > 0$.

Let $\Omega_\varepsilon = \Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))$



Let $u = G^x$ and $v = G^y$ in **(G2)** for Ω_ε.

$$\int_{\Omega_\varepsilon} \left[G^x \cancel{\Delta G^y} - G^y \cancel{\Delta G^x} \right] dz = \int_{\partial \Omega_\varepsilon} \left[G^x \frac{\partial G^y}{\partial n} - G^y \frac{\partial G^x}{\partial n} \right] dS_z$$

$$0 = \int_{\partial \Omega} [\, 0 - 0 \,]$$

$$+ \int_{\partial B_\varepsilon(x)} \Big[\underbrace{G^x}_{o(\varepsilon^{1-n})} \frac{\partial G^y}{\partial n} - G^y \frac{\partial G^x}{\partial n} \Big] dS_z + \int_{\partial B_\varepsilon(y)} \Big[G^x \frac{\partial G^y}{\partial n} - \overbrace{G^y}^{o(\varepsilon^{1-n})} \frac{\partial G^x}{\partial n} \Big] dS_z$$

So

$$\lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(x)} \overbrace{G^y}^{\text{nice}} \frac{\partial G^x}{\partial n}\, dS_z = \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(y)} \overbrace{G^x} \frac{\partial G^y}{\partial n}\, dS_z$$

$$\| \qquad \text{by Lemma} \qquad \|$$

$$G^y(x) \qquad\qquad G^x(y)$$

2.6 Green's Functions by Eigenfunction Expansion

> 💡 **Ex 5.** Find the Green's function in the infinite wedge of angle α, $0 < \alpha < 2\pi$, described in polar coordinates (r, θ) by $0 < \theta < \alpha, 0 < r < \infty$.

We solve $G(x,y) = G^x(y)$ which satisfies

$$\begin{cases} \Delta G^x = \delta_x(y) & \Omega \\ G^x(y) = 0 & y \in \partial\Omega \end{cases}$$

Let (r', θ') be the polar coordinates of x and let (r, θ) be the polar coordinates of y.

$\delta_x(y) = \frac{1}{r'}\delta(r-r')\delta(\theta-\theta')$

where the factor $\frac{1}{r'}$ cancels r in $dy = r\,dy\,d\theta$.

The Laplacian is $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$

For $G(r,0) = G(r,\alpha) = 0$,

$\begin{cases} (\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)G = \frac{1}{r'}\delta(r-r')\delta(\theta-\theta') \\ G(r,0) = G(r,\alpha) = 0 \end{cases}$

The eigenfunctions of ∂_θ^2 with 0-BC at $0, \alpha$ are $\sin\frac{n\pi\theta}{\alpha}, n \in \mathbb{N}$.

Try an eigenfunction expansion of the form

$G(r,\theta) = \sum_{n=1}^\infty c_n(r)\sin\frac{n\pi\theta}{\alpha}$

We have $\sum_{n=1}^\infty (c_n'' + \frac{1}{r}c_n' - \frac{n^2\pi^2}{\alpha^2}\frac{1}{r^2}c_n)\sin\frac{n\pi\theta}{\alpha} = \frac{1}{r'}\delta(r-r')\delta(\theta-\theta')$

Let's integrate the equation as such: $\int_0^\alpha [\text{eqn}]\sin\frac{k\pi\theta}{\alpha}\,d\theta$ (Fourier's trick)

$c_k'' + \frac{1}{r}ck' - \frac{k^2\pi^2}{\alpha^2}\frac{1}{r^2}c_k = \frac{2}{\alpha}\frac{1}{r'}\delta(r-r')\sin\frac{k\pi\theta'}{\alpha}$

For $r \neq r'$, the RHS is 0 and the LHS is an Euler ODE. Try $c_k = r^b \implies$

$0 = b(b-1) + b - \frac{k^2\pi^2}{\alpha^2} \implies b = \pm\frac{k\pi}{\alpha}$

$c_k(r) = \begin{cases} Ar^{\frac{k\pi}{\alpha}} + Br^{-\frac{k\pi}{\alpha}} & 0 < r < r' \\ Cr^{\frac{k\pi}{\alpha}} + Dr^{-\frac{k\pi}{\alpha}} & r' < r < \infty \end{cases}$

By finiteness as $r \to 0$ and $r \to \infty$, we need $B = C = 0$.

Continuity at $r = r'$ gives $A(r')^{\frac{k\pi}{\alpha}} = D(r')^{-\frac{k\pi}{\alpha}} = E$.

$c_k(r) = \begin{cases} E(\frac{r}{r'})^{\frac{k\pi}{\alpha}} & 0 < r < r' \\ E(\frac{r'}{r})^{\frac{k\pi}{\alpha}} & r' < r < \infty \end{cases}$

Jump condition gives

$c_k'\Big|_{r'_-}^{r'_+} = E(-\frac{2k\pi}{\alpha})\frac{1}{r'} = \frac{2}{\alpha}\frac{1}{r'}\sin\frac{k\pi\theta'}{\alpha} \implies E = -\frac{1}{k\pi}\sin\frac{k\pi\theta'}{\alpha}$

Denote $\rho = \frac{\min(r,r')}{\max(r,r')} < 1$ if $r \neq r'$, then $c_k(r) = \frac{-1}{k\pi}\sin\frac{k\pi\theta'}{\alpha}\rho^{\frac{k\pi}{\alpha}}$

$G = \sum_{n=1}^\infty \frac{-1}{n\pi} \underbrace{\sin\frac{n\pi\theta'}{\alpha}\sin\frac{n\pi\theta}{\alpha}}_{\frac{1}{2}[\cos\frac{n\pi}{\alpha}(\theta-\theta') - \cos\frac{n\pi}{\alpha}(\theta+\theta')]} \rho^{\frac{k\pi}{\alpha}}$

Let $z_1 = \rho e^{i(\theta-\theta')}$ and $z_2 = \rho e^{i(\theta+\theta')}$, so $|z_1| = |z_2| = \rho < 1$.

$G = \sum_{n=1}^\infty \frac{1}{2\pi n}\mathrm{Re}(-z_1^{\frac{n\pi}{\alpha}} + z_2^{\frac{n\pi}{\alpha}})$

Recall the Taylor series $-\log(1-w) = \sum_{n=1}^\infty \frac{1}{n}w^n, |w| < 1$, we take $w = z_j^{\frac{\pi}{\alpha}}$ for $j = 1, 2$.

Also, $\mathrm{Re}(\log\zeta) = \log|\zeta|$. Hence,

$G = \frac{1}{2\pi}\mathrm{Re}(\log(1 - z_1^{\frac{\pi}{\alpha}}) - \log(1 - z_2^{\frac{\pi}{\alpha}})) = \frac{1}{2\pi}\log\left|\frac{1-z_1^{\pi/\alpha}}{1-z_2^{\pi/\alpha}}\right|$

Exercise, simplify the formula when $\alpha = \pi, \frac{\pi}{2}$.

W5C1 (Feb 4)

Cancelled due to snow day ❄️.

W5C2 (Feb 6)

2.7 Elliptic Equations

Consider a linear 2nd order PDE of 2 variables and constant coefficients:

$Lu := au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y) \quad (*)$

In MATH 400, it is shown that, by a linear transform of independent variables, $(*)$ can be changed to

- $u_{\zeta\zeta} + u_{\eta\eta} = h$ if $D = b^2 - ac < 0$, elliptic case
- $u_{\zeta\zeta} - u_{\eta\eta} = h$ if $D > 0$, hyperbolic case
- $u_{\zeta\zeta} - u_{\eta} = h$ if $D = 0$, parabolic case

The classification is similar in higher dimensions $x \in \mathbb{R}^n, n \geq 3$.

If we consider $(*)$ in Ω, so that $Lu = f$ in Ω and $u\big|_{\partial\Omega} = 0$, and if L is elliptic, we can find Green's functions by changing variables.

More generally, we can allow variable coefficients $a(x, y), b(x, y)$, etc and still study Green's functions.

It is hard to find explicit formulae, but we can show its "estimates" and use them to show solution properties (similar to nonhomogeneous ODEs in MATH 255).