MATH 307 Formula Sheet

LU Decomposition

$A = LU$

Properties

- $\text{rank}(A) = \text{rank}(U)$
- If A is square, then $\det(A) = \det(U)$

Norms

Vector Norms

Frobenius Norm

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{i,j}|^2 \right)^{1/2}$$

Matrix Norm

$$\|A\| = \max_{\mathbf{x} \neq \mathbf{0}} \left(\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \right) = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

$$\left\|A^{-1}\right\| = \frac{1}{\min\limits_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|}$$

$$\text{cond}(A) = \|A\| \left\|A^{-1}\right\| = \frac{\text{max stretch}}{\text{min stretch}}$$

$$\frac{\|\Delta \mathbf{x}\|}{\|\mathbf{x}\|} \leq \text{cond}(A) \frac{\|\Delta \mathbf{b}\|}{\|\mathbf{b}\|}$$

Interpolation

Polynomial Interpolation

For $d+1$ points $(t_0, y_0), (t_1, y_1), \ldots, (t_d, t_d)$,

$$p(t) = c_0 + c_1 t + \cdots + c_d t^d$$

Cubic Spline Interpolation

$$p_k(t) = a_k(t - t_{k-1})^3 + b_k(t - t_{k-1})^2 + c_k(t - t_{k-1}) + d$$

$$a_k + b_k + c_k + d_k = d_{k+1}$$

$$3a_k + 2b_k + c_k = c_{k+1}$$

$$6a_k + 2b_k = 2b_{k+1}$$

$$\left[\begin{array}{c|c|c|c|c} A(L_1) & B & & & \\ \hline & A(L_2) & B & & \\ \hline & & \ddots & \ddots & \\ \hline & & & A(L_{N-1}) & B \\ \hline T & & & & V \end{array}\right] \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ \vdots \\ a_N \\ b_N \\ c_N \end{bmatrix} = \begin{bmatrix} y_1 - y_0 \\ 0 \\ 0 \\ \vdots \\ y_N - y_{N-1} \\ 0 \\ 0 \end{bmatrix}$$

$$A(L) = \begin{bmatrix} L^3 & L^2 & L \\ 3L^2 & 2L & 1 \\ 6L & 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$A(L) = \begin{bmatrix} L^3 & L^2 & L \\ 3L^2 & 2L & 1 \\ 6L & 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Differential Equations

$$y'' + p(t)y' + q(t)y = r(t)$$

$$y(t_0) = \alpha \qquad y(t_f) = \beta$$

$$h = \frac{t_f - t_0}{N+1}$$

Approximation to $y'' = r(t)$: solve $A\mathbf{y} = \mathbf{b}$.

$$A = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} h^2 r_1 - \alpha \\ h^2 r_2 \\ \vdots \\ h^2 r_{N-1} \\ h^2 r_N - \beta \end{bmatrix}$$

General approximation: solve $A\mathbf{y} = \mathbf{b}$.

$$A = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & & \ddots & & \\ & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & a_N & b_N \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} h^2 r_1 - a_1 \alpha \\ h^2 r_2 \\ \vdots \\ h^2 r_{N-1} \\ h^2 r_N - c_N \beta \end{bmatrix}$$

$$a_k = 1 - \frac{hp_k}{2} \qquad b_k = h^2 q_k - 2 \qquad c_k = 1 + \frac{hp_k}{2}$$

Vector Spaces

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

$$R(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} = \text{span}\{\text{columns of } A\}$$

$$\dim(R(A)) = \text{rank}(A)$$

$$R(A) = \text{span}\{l_1, \ldots, l_r\}$$

$$\dim(R(A)) + \dim(N(A)) = n$$