PHYS 304 Formula Sheet

Introduction

Schrödinger $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$

Probability $P(a \le x \le b) = \int_a^b |\Psi(x,t)|^2\,dx$

Expectation $\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx$

Momentum $\vec{p} = -i\hbar\frac{\partial}{\partial x}$

$\langle p\rangle = \int_{-\infty}^{\infty}\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\,dx$

Observable $\langle Q(x,p)\rangle = \int_{-\infty}^{\infty}\Psi^*(Q(x,p))\Psi\,dx$

1-D Schrödinger Equation

SOV $\Psi(x,t) = \psi(x)\varphi(t)$

$\varphi(t) = e^{-iEt/\hbar}$

TISE $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$

Hamiltonian $H(x,p) = \frac{p^2}{2m} + V(x)$

$H\psi = E\psi \qquad \langle H\rangle = E$

Wavefunction $\Psi(x,t) = \sum_n c_n\psi_n(x)e^{-iE_nt/\hbar}$

Probability $P(E_n) = |c_n|^2$

Infinite Square Well ($V = \infty$ for $x < 0, x > a$)

Wavefunction $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$

Energy $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

Harmonic Oscillator ($V = \frac{1}{2}kx^2$)

Ladder $\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$

Wavefunction $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2}$

$\psi_n(x) = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\psi_0(x)$

Energy $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

Step Potential ($V = V_0$ for $x > 0$)

Wavefunction $\psi(x) = \begin{cases} e^{ikx} + Ae^{-ikx} & x < 0 \\ Be^{ilx} & x > 0 \end{cases}$

Wavenumbers $k = \frac{\sqrt{2mE}}{\hbar} \qquad l = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

Energy $E = \frac{\hbar^2k^2}{2m}$

Coefficients $A = \frac{k-l}{k+1} \qquad B = \frac{2k}{k+1}$

$R = \frac{k-l}{k+1} \qquad T = B\sqrt{\frac{l}{k}} = \frac{2\sqrt{kl}}{k+l}$

$|R|^2 + |T|^2 = 1$

Free Particle

Wavefunction $\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}\,dk$

$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}\,dx$

Finite Well ($V = -V_0$ for $-a \le x \le a$)

Bound $\psi(x) = \begin{cases} Fe^{\kappa x} & x < -a \\ C\sin(lx) + D\cos(lx) & -a \le x \le a \\ Fe^{-\kappa x} & x > a \end{cases}$

$\kappa = \frac{\sqrt{-2mE}}{\hbar} \qquad l = \frac{\sqrt{2m(V_0+E)}}{\hbar}$

$E_n + V_0 \approx \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \quad (n = 1, 3, 5, \ldots)$

Scattering $\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ C\sin(lx) + D\cos(lx) & -a \le x \le a \\ Fe^{ikx} & x > a \end{cases}$

$k = \frac{\sqrt{2mE}}{\hbar} \qquad l = \frac{\sqrt{2m(V_0+E)}}{\hbar}$

$B = i\frac{\sin(2la)}{2kl}(l^2 - k^2)F$

$F = \frac{e^{-2ika}A}{\cos(2la) - i\frac{k^2+l^2}{2kl}\sin(2la)} \qquad T = \frac{|F|^2}{|A|^2}$

$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$

Perfect transmission: $E_n + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$

Formalism

ket $|a\rangle = [\vec{a}] \qquad \langle a|b\rangle = a_1^*b_1 + \cdots + a_n^*b_n$

$\langle f|g\rangle = \int_a^b f(x)^*g(x)\,dx \qquad \langle f|g\rangle = \langle g|f\rangle^*$

ONB $\langle i|j\rangle = \delta_{ij} \qquad c_n = \langle f_n|f\rangle$

Hermitian $\int_V \varphi_1^*(O^\dagger\varphi_2)\,dV = \int_V(O\varphi_1)^*\varphi_2\,dV$

$O^\dagger = O \qquad O_{ab}^\dagger = O_{ba}^*$

Uncertainty $\sigma_A^2\sigma_B^2 \ge \left(\frac{1}{2i}\left\langle[\hat{A},\hat{B}]\right\rangle\right)^2$

$\sigma_x\sigma_p \ge \frac{\hbar}{2}$

Dirac $\langle f| = \int f^*[\ldots]\,dx \qquad \langle b| = [b_1^*\ldots b_n^*]$

Hermitian $M^\dagger = \sum_{ij}^d [M]_{ji}^*|i\rangle\langle j|$

$\langle u|M^\dagger|v\rangle = (M|u\rangle)^\dagger|v\rangle$

3-D Schrödinger Equation

Schrödinger $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$

TISE $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$

Spherical $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

SOV $\Psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$

Angular $\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2Y}{\partial\phi^2} = -l(l+1)\sin^2\theta Y$

Radial $u = rR, -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$

Spherical Well ($V = \infty$ for $r > a$)

Bessel Solutions $u(r) = Arj_l(kr) + Brn_l(kr)$

Hydrogen ($V = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$)

$\psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3\frac{(n-l-1)!}{2n(n+l)!}}e^{-r/na}\left(\frac{2r}{na}\right)^l L_{n-l+1}^{2l+1}\left(\frac{2r}{na}\right)Y_l^m(\theta,\phi)$

Radius $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2} \approx 52.9\,\mathrm{pm}$

Energy $E_n = \frac{E_1}{n^2} \qquad E_1 = -\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 = -13.6\,\mathrm{eV}$

Photon $E_\gamma = E_i - E_f = -13.6\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) = h\nu$

Rydberg $\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad R = 1.097\times10^7\,\mathrm{m}^{-1}$

Angular Momentum

$L_x = yp_z - zp_y \qquad L_y = zp_x - xp_z \qquad L_z = xp_y - yp_x$

$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$

$L^2 = L_x^2 + L_y^2 + L_z^2 \qquad [L^2, \vec{L}] = 0$

$L_\pm = L_x \pm iL_y \qquad [L_z, L_\pm] = \pm\hbar L_\pm \quad [L^2, L_\pm] = 0$

$L^2f_l^m = \hbar^2l(l+1)f_l^m \qquad L_zf_l^m = \hbar mf_l^m$

$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$

$L_z(L_+^kf_l^m) = \hbar(m+k)(L_+^kf_l^m)$

$L_z(L_-^kf_l^m) = \hbar(m-k)(L_-^kf_l^m)$

Spin

$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$

$S^2|sm\rangle = \hbar^2s(s+1)|sm\rangle \qquad S_z|sm\rangle = \hbar m|sm\rangle$

$S_\pm = S_x \pm iS_y$

$S_\pm|sm\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}|s, m\pm1\rangle$

Spin $\frac{1}{2}$

$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \qquad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Magnetic $H = -\frac{g_s\mu_B}{\hbar}\vec{S}\cdot\vec{B} = -\frac{g_s\mu_B}{2\hbar}B_z\sigma_z$

$e^{i\frac{g_s\mu_B}{2\hbar}B_z\sigma_zT} = \cos\left(\frac{g_s\mu_B}{2\hbar}B_zT\right)\mathcal{I} + i\sin\left(\frac{g_s\mu_B}{2\hbar}B_zT\right)\sigma_z$

Eigenstates $S_z|1\rangle = \frac{\hbar}{2}|1\rangle \qquad S_z|0\rangle = -\frac{\hbar}{2}|0\rangle$

NOT $X = |1\rangle\langle0| + |0\rangle\langle1|$

Z $Z = |1\rangle\langle1| - |0\rangle\langle0|$

iY $iY = |1\rangle\langle0| - |0\rangle\langle1|$

Hadamard $h = \frac{1}{\sqrt{2}}\left(|1\rangle\langle1| - |0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle0|\right)$

Two Qubit

Tensor Product	$\|\psi\rangle_A \otimes \|\psi\rangle_B$
Spin	$\vec{S} = \vec{S}_A \otimes \mathcal{I}_B + \mathcal{I}_A \otimes \vec{S}_B$
	$\vec{S}^2 = \vec{S}_A^2 \otimes \mathcal{I}_B + \mathcal{I}_A \otimes \vec{S}_B^2 + 2\vec{S}_A \otimes \vec{S}_B$
CNOT	$\|1\rangle_{A\,A}\langle 1\| \otimes \mathcal{I}_B + \|0\rangle_{A\,A}\langle 0\| \otimes X_B$
	flips B if $\|0\rangle_A$
Density	$\rho = \|\psi\rangle\langle\psi\|$
Pure State	$\mathrm{Tr}(\rho) = 1 \quad \rho^2 = \rho$
Reduced	$\rho^{(B)} = {}_A\langle 1\|\rho\|1\rangle_A + {}_A\langle 0\|\rho\|0\rangle_A$
Entropy	$S = -\mathrm{Tr}(\rho^{(B)}) \ln(\rho^{(B)})$
	$S = -\sum_i r_i \ln(r_i)$

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