Trig Sum/Difference

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Derivatives

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\sec^2 x$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\log|x| = \frac{1}{x}$$

$$\frac{d}{dx}b^x = b^x \log b$$

$$\frac{d}{dx}\log_b x = \frac{1}{x \log b}$$

Limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

Intermediate Value Theorem
Suppose that $f(x)$ is continuous on the closed interval $[a,b]$. Then for any number Y between $f(a)$ and $f(b)$, there exists some number $c \in [a,b]$ such that $f(c) = Y$.

Definition of the Derivative

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

Squeeze Theorem
Let $l(x), f(x), u(x)$ be functions that satisfy $l(x) \le f(x) \le u(x)$ near $x = a$. If $\lim_{x\to a} l(x) = L$ and $\lim_{x\to a} u(x) = L$, then $\lim_{x\to a} f(x) = L$.

Logarithmic Differentiation

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(\log|f(x)|)$$

Extreme Value Theorem
If $f(x)$ is continuous on the closed interval $[a,b]$, then $f(x)$ is bounded on $[a,b]$.

Summation Notation

$$P(x) = \sum_{k=0}^{d} a_k x^k$$

Related Rates
Write equivalence and differentiate w.r.t. t.

Percentage Rate of Change

$$K(t) = \frac{f'(t)}{f(t)}$$

Exponential Decay

$$Q'(t) = -kQ(t)$$

$$Q(t) = Q(0)e^{-kt}$$

$$\text{Half life } = \frac{\log 2}{k}$$

Newton's Law of Cooling

$$T'(t) = K(T(t) - A)$$

$$T(t) = (T(0) - A)e^{kt} + A$$

Population Growth

$$P'(t) = bP(t)$$

$$P(t) = P_0 e^{bt}$$

Logistic Growth

$$P'(t) = b\Big(1 - \frac{P(t)}{K}\Big)P(t)$$

$$P(t) = \frac{KP_0 e^{bt}}{K - P_0 + P_0 e^{bt}}$$

Mean Value Theorem
If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists some number $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Triangle Inequality

$$|x+y| \le |x| + |y|$$

Equal Derivatives Fact
If $f'(x) = g'(x)$ on (a,b), then $f(x)$ and $g(x)$ differ by a constant on (a,b).

Taylor Polynomials

$$T_n(x) = \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(x-a)^k$$

$$T_1(x) = f(a) + f'(a)(x-a)$$

Lagrange Remainder Formula
Let $n \in \mathbb{N}$. If $f(x)$ is $(n+1)$-times differentiable, then there exists c between a and x such that

$$\begin{aligned} R_n(x) &= f(x) - T_n(x) \\ &= \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1} \end{aligned}$$

If $|f^{(n+1)}(c)| \le M$ for all c between a and x, then

$$\begin{aligned} |R_n(x)| &= |f(x) - T_n(x)| \\ &\le \frac{M}{(n+1)!}|x-a|^{n+1} \end{aligned}$$

Generalized Mean Value Theorem
Let $F(x)$ and $G(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that $\frac{F'(c)}{G'(c)} = \frac{F(b)-F(a)}{G(b)-G(a)}$.

First Derivative Test
Let $x = c$ be a critical or singular point.

- If $f'(x) > 0$ to the left and $f'(x) < 0$ to the right, then $f(x)$ has a local maximum at $x = c$.
- If $f'(x) > 0$ to the right and $f'(x) < 0$ to the left, then $f(x)$ has a local minimum at $x = c$.
- If $f'(x)$ has the same sign to the left and to the right, then $f(x)$ does not have a local extremum at $x = c$.

Inference

- If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(x)$ has its global minimum at $x = c$.

Second Derivative Test

- If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$.

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Convexity
$f(x)$ is convex on $[A,B]$ if, for any $a, b \in \mathbb{R}$ such that $A \le a \le b \le B$, and for every $0 < t < 1$, we have

$$(1-t)f(a) + tf(b) \ge f((1-t)a + tb)$$

Equivalent statements for twice-differentiable functions:

- $f''(x) \ge 0$
- $f'(x)$ is increasing
- If $c \in (A,B)$ and $l(x)$ is the tangent line approximation to $f(x)$ at $x = c$, then $f(x) \ge l(x)$ for all $x \in [A,B]$.

Curve Sketching

- $f(x)$: domain, asymptotes, symmetries, intercepts, positive/negative
- $f'(x)$: critical and singular points, local extrema, vertical tangent lines, increasing/decreasing
- $f''(x)$: convex/concave, inflection points

Indeterminate Forms

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 1^\infty, 0^0, \infty^0$$

L'Hopital's Rule
For $\frac{0}{0}$ and $\frac{\infty}{\infty}$ indeterminate forms,

$$\lim_{x\to *}\frac{f(x)}{g(x)} = \lim_{x\to *}\frac{f'(x)}{g'(x)}$$

Polynomial Indeterminate Case

$$\lim_{x\to\infty}(\sqrt[d]{P(x)} - x) = \frac{c_1}{d}$$

Exponentiation Limit Law

If $\lim_{x\to *} f(x) = F$ with $F > 0$ and $\lim_{x\to *} g(x) = G$, then $\lim_{x\to *}(f(x)^{g(x)}) = F^G$.

Composition with Exponential Limit Law

- If $\lim_{x\to *} g(x) = G$, then $\lim_{x\to *}(e^{g(x)}) = e^G$
- If $\lim_{x\to *} g(x) = \infty$, then $\lim_{x\to *}(e^{g(x)}) = \infty$
- If $\lim_{x\to *} g(x) = -\infty$, then $\lim_{x\to *}(e^{g(x)}) = 0$

Composition with Logarithm Limit Law

- If $\lim_{x\to *} f(x) = F$, then $\lim_{x\to *}(\log f(x)) = \log F$
- If $\lim_{x\to *} f(x) = \infty$, then $\lim_{x\to *}(\log f(x)) = \infty$
- If $\lim_{x\to *} f(x) = 0$ and $f(x) > 0$ near $x = *$, then $\lim_{x\to *}(\log f(x)) = -\infty$

∞ ^ Bounded Away from 0 Limit Law	If $\lim\limits_{x\to *} f(x) = \infty$ and $g(x)$ is bounded below away from 0 near $x = *$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = \infty$. If $\lim\limits_{x\to *} f(x) = \infty$ and $g(x)$ is bounded above away from 0 near $x = *$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = 0$.
∞ ^ Convergent Limit Law	If $\lim\limits_{x\to *} f(x) = \infty$ and $\lim\limits_{x\to *} g(x) = G$ with $G > 0$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = \infty$. If $\lim\limits_{x\to *} f(x) = \infty$ and $\lim\limits_{x\to *} g(x) = G$ with $G < 0$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = 0$.
∞ ^ ±∞ Limit Law	If $\lim\limits_{x\to *} f(x) = \infty$ and $\lim\limits_{x\to *} g(x) = \infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = \infty$. If $\lim\limits_{x\to *} f(x) = \infty$ and $\lim\limits_{x\to *} g(x) = -\infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = 0$.
0 ^ Bounded Away from 0 Limit Law	If $\lim\limits_{x\to *} f(x) = 0$ and $f(x) > 0$ near $x = *$, and $g(x)$ is bounded below away from 0 near $x = *$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = 0$. If $\lim\limits_{x\to *} f(x) = 0$ and $f(x) > 0$ near $x = *$, and $g(x)$ is bounded above away from 0 near $x = *$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = \infty$.
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0 ^ ±∞ Limit Law	If $\lim\limits_{x\to *} f(x) = 0$ and $f(x) > 0$ near $x = *$, and $\lim\limits_{x\to *} g(x) = \infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = 0$. If $\lim\limits_{x\to *} f(x) = 0$ and $f(x) > 0$ near $x = *$, and $\lim\limits_{x\to *} g(x) = -\infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = \infty$.
Bounded Below Away from 1 ^ ±∞ Limit Law	If $f(x)$ is bounded below away from 1 near $x = *$, and $\lim\limits_{x\to *} g(x) = \infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = \infty$. If $f(x)$ is bounded below away from 1 near $x = *$, and $\lim\limits_{x\to *} g(x) = -\infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = 0$.
Convergent ^ ±∞ Limit Law	If $\lim\limits_{x\to *} f(x) = F$ with $F > 1$, and $\lim\limits_{x\to *} g(x) = \infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = \infty$. If $\lim\limits_{x\to *} f(x) = F$ with $F > 1$, and $\lim\limits_{x\to *} g(x) = -\infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = 0$.
Bounded Above Away from 1 ^ ±∞ Limit Law	If $f(x) > 0$ near $x = *$ and $f(x)$ is bounded above away from 1 near $x = *$, and $\lim\limits_{x\to *} g(x) = \infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = 0$. If $f(x) > 0$ near $x = *$ and $f(x)$ is bounded above away from 1 near $x = *$, and $\lim\limits_{x\to *} g(x) = -\infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = \infty$.
Convergent ^ ±∞ Limit Law	If $\lim\limits_{x\to *} f(x) = F$ with $0 < F < 1$, and $\lim\limits_{x\to *} g(x) = \infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = 0$. If $\lim\limits_{x\to *} f(x) = F$ with $0 < F < 1$, and $\lim\limits_{x\to *} g(x) = -\infty$, then $\lim\limits_{x\to *}\left(f(x)^{g(x)}\right) = \infty$.