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MATH 305 Notes Part 2

Term: 2024W2

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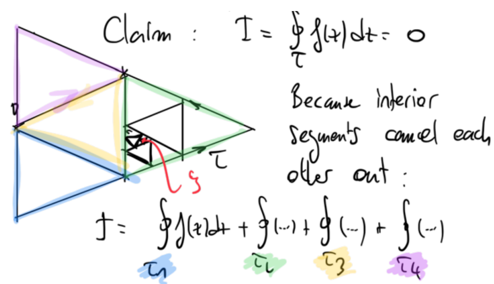
W5C1 Lecture 13 (Feb 3)

Proof of Cauchy's Theorem:

Part 1: Pick a triangle τ of perimeter L as α.



Then $I = \oint_{\tau_1} f(z)\,dz + \oint_{\tau_2} f(z)\,dz + \oint_{\tau_3} f(z)\,dz + \oint_{\tau_4} f(z)\,dz$

Let $I^{(1)}$ be the integral of greatest modulus, so $|I| \le 4|I^{(1)}|$.

Likewise, $|I^{(1)}| \le 4|I^{(2)}|$.

Overall, $|I| \le 4^n|I^{(n)}|$ after the n-th step.

In the limit, there is just one point ζ left inside all triangles.

We have $f(z) = f(\zeta) + f'(\zeta)(z-\zeta) + (z-\zeta)r(z,\zeta)$ where:

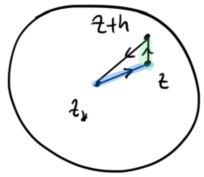
$\lim_{z\to\zeta} r(z,\zeta) = \lim_{z\to\zeta}\left(\frac{f(z)-f(\zeta)}{z-\zeta} - f'(\zeta)\right) = 0$ by definition of differentiability.

We have:

- $\oint_{\tau^n} f(\zeta)\,dz = f(\zeta)\oint_{\tau^n} dz = 0$
- $\oint_{\tau^n} f'(\zeta)(z-\zeta)\,dz = f'(\zeta)\oint_{\tau^n}(z-\zeta)\,dz = 0$
- $|\oint_{\tau^n} r(z,\zeta)(z-\zeta)\,dz| \le M_n(\frac{1}{2^n}L)(\frac{1}{2^n}L)$, where
  - M_n is the maximum of $|r(z,z_0)|$ along τ^n and $\to 0$ as $n\to\infty$
  - $\frac{1}{2^n}L$ is the max of $|z-\zeta|$ along τ^n
  - $\frac{1}{2^n}L$ is $l(\tau^n)$

We conclude that $|I| = 4^n M_n \frac{L^2}{4^n} = M_n L^2 \to 0$ as $n\to\infty$, hence $I = \oint_\tau f(z)\,dz = 0$.

Part 2: to prove Cauchy's theorem, it suffices to construct an antiderivative.



We claim that $F(z) = \int_{[z_*, z]} f(z)\, dz$ is an antiderivative of f.

By Part 1, $F(z) + \int_{[z, z+h)} f(w)\, dw - F(z+h) = 0$.

We parameterize $[z, z+h)$ as $\alpha(t) = z + th, t \in [0,1]$.

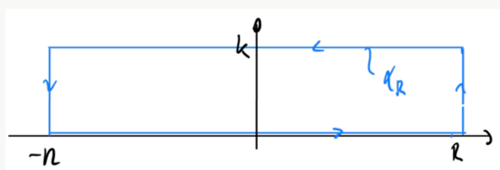
Hence $F(z+h) - F(z) = \int_0^1 f(z+th) \cdot h\, dt$, and so

$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \to 0} \int_0^1 f(z+th)\, dt = f(z)$, as desired.

W5C2 Lecture 14 (Feb 5)

💡 **Ex 22.** (Computing real integrals). Compute $\int_{-\infty}^{\infty} e^{-x^2} \cos(2kx)\, dx$ for $k \in \mathbb{R}$.

Let $f(z) = e^{-z^2}$ which is entire, and consider the contour



By CT, $\oint_{\alpha_R} f(z)\, dz = 0$ for all $R > 0$. Let's parameterize each segment:

$$0 = \underbrace{\int_{-R}^{R} f(x)\, dx}_{(1)} + \underbrace{\int_0^k f(R+it) i\, dt}_{(2)} - \underbrace{\int_{-R}^{R} f(x+ik)\, dx}_{(3)} - \underbrace{\int_0^k f(-R+it) i\, dt}_{(4)}$$

We have

- $\lim_{R \to \infty} (1) = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$
- $\lim_{R \to \infty} |(2)| = \lim_{R \to \infty} |\int_0^k e^{-(R+it)^2}| dt = 0$, same for (4)
- $\lim_{R \to \infty} (3) = \int_{-\infty}^{\infty} e^{-(x+ik)^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2+k^2} (\cos(2kx) - i \sin(2kx))\, dx$

Hence, $e^{k^2} \int_{-\infty}^{\infty} e^{-x^2} (\cos(2kx) - i\sin(2kx))\, dx = \sqrt{\pi}$

Matching real and imaginary components gives

- $\int_{-\infty}^{\infty} e^{-x^2} \cos(2kx) = e^{-k^2} \sqrt{\pi}$
- $\int_{-\infty}^{\infty} e^{-x^2} \sin(2kx)\, dx = 0$

Remark: replacing x with $\frac{x}{\sqrt{2}}$ so dx becomes $\frac{1}{\sqrt{2}}\, dx$, and k with $\frac{k}{\sqrt{2}}$ gives

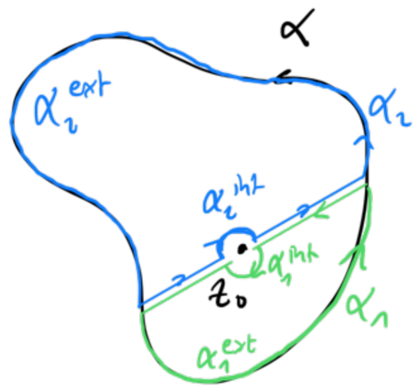
$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{ikx}\, dx = e^{-\frac{k^2}{2}}$

The Fourier transform of a Gaussian is a Gaussian.

W5C3 Lecture 15 (Feb 7)

Cauchy's Integral Formula and Consequences

We already know that $\oint_{|z-z_0|=r} \frac{a}{z-z_0}\, dz = 2\pi i a$. We now consider deformations of the circle.



Since $\frac{a}{z-z_0}$ is holomorphic on $\mathbb{C} \setminus \{z_0\}$, Cauchy's Theorem implies that $0 = \oint_{\alpha_1} \frac{a}{z-z_0}\, dz = \oint_{\alpha_2} \frac{a}{z-z_0}\, dz$.

Adding the two contours, we get that

$\oint_\alpha \frac{a}{z-z_0}\, dz = \oint_{\alpha_1} \frac{a}{z-z_0}\, dz + \oint_{\alpha_2} \frac{a}{z-z_0}\, dz + \oint_{|z-z_0|=r} \frac{a}{z-z_0}\, dz = 2\pi i a$

📖 **Result:** for any simple smooth curve α with z_0 in the interior of α,

$\oint_\alpha \frac{a}{z-z_0}\, dz = 2\pi i a$

📖 **Thm.** (Cauchy's Integral Formula). Let α be a simple closed curve and f be holomorphic on α and its interior.

Let w be inside of α, and we consider $F(z) = \frac{f(z)}{z-w}$, which is holomorphic on $\Omega \setminus \{w\}$.

Working as the previous example, we conclude that $\oint_\alpha F(z)\, dz = \oint_{|z-w|=\varepsilon} F(z)\, dz$ for any sufficiently small ε.

Parameterize the circle as $w + \varepsilon e^{it}$ where $t \in [0, 2\pi)$, we get

$\oint_\alpha F(z)\, dz = \int_0^{2\pi} \frac{f(w+\varepsilon e^{it})}{\varepsilon e^{it}} i\varepsilon e^{it}\, dt = i \int_0^{2\pi} f(w + \varepsilon e^{it})\, dt.$

Letting $\varepsilon \to 0^+$, we conclude that $\oint_\alpha \frac{f(z)}{z-w}\, dz = 2\pi i f(w)$

Remark: the value of the holomorphic function f on the whole domain inside the closed curve α are completely determined by the values of f on α (the boundary of the domain).

The assumption that α is simple ensures that it "winds around" the point w only once.

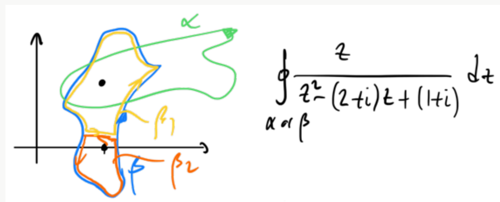
💡 **Ex 23.**

a. Evaluate $\oint_{|z-2|=3} \frac{e^z+\sin(z)}{z}\, dz$

   $f(z) = e^z + \sin(z)$ is holomorphic in the disc $|z-2| \leq 3$. Since $z = z - 0$ and 0 belongs in the disc, we conclude that

   $\oint_{|z-2|=3} \frac{e^z+\sin(z)}{z}\, dz = 2\pi i(e^0 + \sin 0) = 2\pi i.$

b. Evaluate the following contour integrals:



   The denominator has two zeros:

   $z^2 - (2+i)z + (1+i) = (z-(1+i))(z-1)$

   - Case α: we apply Cauchy's integral formula with $f_1(z) = \frac{z}{z-1}$ to get

     $\oint_\alpha (\dots)\, dz = 2\pi i f_1(1+i) = 2\pi(1+i)$

   - Case β: now there are two zeros, so we compute the integrals along β_1 and β_2, and add the results.

     For β_1, use $f_1(z) = \frac{z}{z-1}$ to get $\oint_{\beta_1} (\dots)\, dz = 2\pi i f_1(1+i) = 2\pi(1+i)$

     For β_2, use $f_2(z) = \frac{z}{z-(1+i)}$ to get $\oint_{\beta_2} (\dots)\, dz = 2\pi i f_2(1) = -2\pi$

   Overall, $\oint_\beta (\dots)\, dz = \oint_{\beta_1} (\dots)\, dz + \oint_{\beta_2} (\dots)\, dz = 2\pi(1+i) - 2\pi = 2\pi i$

W6C1 Lecture 16 (Feb 10)

Differentiation

From Cauchy's Integral Formula (CIF) $f(w) = \frac{1}{2\pi i} \oint_\alpha \frac{f(z)}{z-w}\, dz$, it follows that

$f'(w) = \frac{1}{2\pi i} \oint_\alpha \frac{f(z)}{(z-w)^2}\, dz$

$f''(w) = \frac{2}{2\pi i} \oint_\alpha \frac{f(z)}{(z-w)^3}\, dz$

$f^{(k)}(w) = \frac{k!}{2\pi i} \oint_\alpha \frac{f(z)}{(z-w)^{k+1}}\, dz$ (Cauchy's Differentiation Formula)

In particular, if f is holomorphic, then so is f', and so is f'', etc, so f is infinitely differentiable, and we have a formula for all $f^{(k)}$.

Disc/Circle

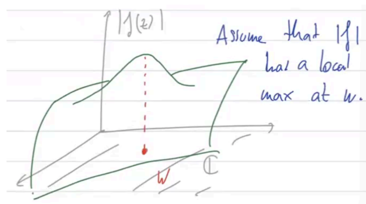
Let $f \in H(B_R(w))$, and let $r < R$. Then

$f(w) = \frac{1}{2\pi i} \int_{|z-w|=r} \frac{f(z)}{z-w}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(w+re^{it})}{re^{it}} ire^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(w+re^{it})\, dt$ (MVP)

Notice that this is equal to the mean of f along the circle.

📖 **Thm.** (Mean Value Property). For a holomorphic function $f(w)$, we have $f(w) = \frac{1}{2\pi}\int_0^{2\pi} f(w + re^{it})\, dt$.

The value of the function is equal to the mean of that holomorphic function on a small circle around it.



Assume that $|f(w)|$ is maximal, then $|f(w + re^{it})| < |f(w)|$ for all $t \in [0, 2\pi)$ and sufficiently small r.

Taking the modulus of (MVP) gives

$|f(w)| < \frac{1}{2\pi}\int_0^{2\pi} |f(w + re^{it})|\, dt = \frac{1}{2\pi}\int_0^{2\pi} |f(w)|\, dt = |f(w)|$

which is a contradiction: hence, $|f|$ has to be constant.

📖 **Thm.** (Maximum Modulus Principle). If f is a non-constant holomorphic function on a domain Ω, then $|f|$ cannot reach a local maximum inside Ω.

If Ω is a bounded domain and f is continuous all the way to the boundary, then the max $|f|$ must be attained on the boundary.

📖 **Thm.** (Liouville's Theorem). If f is entire and bounded, then it is constant.

Proof:

f bounded means $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

We have seen: $f^{(k)}(z) = \frac{k!}{2\pi i}\oint_{|z|=R} \frac{f(z)}{z^{k+1}}\, dz$ and so

$|f^{(k)}(z)| \leq \frac{k!}{2\pi} \cdot 2\pi R \cdot \frac{M}{R^{k+1}}$ for all $R > 0$ (Cauchy's estimate)

$|f^{(k)}(z)| \leq \frac{MK!}{R^k}$

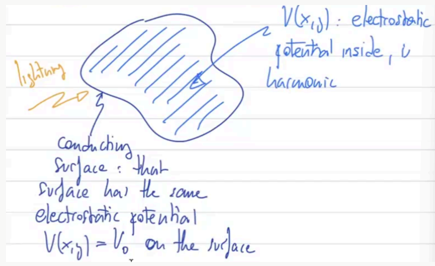
For $k = 1$: $|f'(z)| \leq \frac{Mk!}{R}$. We can take R to be arbitrarily large, so $f'(z) = 0$ for all $z \in \mathbb{C}$, namely f is constant.

Revisiting MVP

Taking the real and imaginary parts of MVP: yields that MVP applies to harmonic functions.

For instance, the steady state temperature distributions, electrostatic/gravitational potentials, pressure in slow viscous fluids:

1. Satisfy the mean value property
2. Have no local extrema (max or min) inside any finite domain

> **Ex 24.** (Faraday's cage effect)
>
> 
>
>
> Surface potential is constant $V(x,y) = V_0$.
>
> Electrostatic potential is harmonic, so $V(x,y)$ cannot have a maximum or a minimum, hence $V(x,y) = V_0$ everywhere.
>
> Since potential is constant, there is no electric field, so the person inside is safe.

W6C2 Midterm 1 (Feb 12)

Midterm 1 in class.

W6C3 Lecture 17 (Feb 14)

> **Ex 25.** Find the maximum modulus of the function $f(z) = z^n e^{z^2}$ in the disc $B_1(0)$, for $n = 0, 1, 2, \ldots$
>
> MMP: $\max\{|f(z)| : z \in B_1(0)\} = \max\{|f(z)|, |z| = 1\}$ (on the boundary)
>
> $= \max\{e^{\mathrm{Re}(z^2)} : |z| = 1\} = \max\{e^{\cos 2\varphi} : \varphi \in [0, 2\pi)\}$
>
> Max attained when $\cos 2\varphi = 1 \iff \varphi = 0, \pi$
>
> Hence $\max\{|z^n e^{z^2}| : z \in B_1(0)\} = e = f(\pm 1)$
>
> *Remark:* the minimum modulus would be $f(0) = 0$ (for $n = 1, 2, \ldots$)
>
> In general, if $|f(z)|$ reaches a minimum inside a domain Ω, say at z_0, then z_0 is a zero of f.
>
> - If $f(z) \neq 0$ for all $z \in \Omega$, then $\frac{1}{f(z)}$ is holomorphic in $\Omega \implies$ minimum modulus principle holds
> - For the example when $n = 0$, minimum modulus would be $f(\pm i) = e^{-1}$ attained by $\varphi = \frac{\pi}{2}, \frac{3\pi}{2}$

📖 **Thm.** (Rouche's Theorem).

Let f be holomorphic on α and its interior.

If $|f(z)-1|<1$ for all $z \in \alpha$, then f does not have any zeros inside α.

$f(z) \neq 0$ for all z in the interior of α.

Proof:

Apply the MMP to $f(z)-1$:

$\max(|f(z)-1| : z \in \text{int } \alpha) = \max(|f(z)-1| : z \in \alpha) < 1$ by assumption.

If the distance between $f(z)$ and 1 is less than 1, then $f(z)$ does not have zeros in the interior.

💡 **Ex 26.** The characteristic polynomial of a matrix M is

$p(t) = -t^5 + \frac{1}{12}t^4 - \frac{1}{4}t^2 + \frac{1}{6}t + 2$

Claim: all eigenvalues lie outside the unit disc.

This is equivalent to $p(t)$ has no zero inside $B_1(0)$.

Let $f(z) = \frac{1}{2}p(z)$. Then

$|f(z)-1| \leq \frac{1}{2}|z|^5 + \frac{1}{24}|z|^4 + \frac{1}{8}|z|^2 + \frac{1}{12}|z|$

On the unit circle $|z|=1$: $|f(z)-1| \leq \frac{3}{4} < 1$

By Rouche's theorem, $f(z)$ has no zero inside $B_1(0)$.

📖 **Thm.** (Fundamental Theorem of Algebra).

A polynomial of degree n has exactly n roots in $\mathbb{C}$.

Proof:

$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_n \neq 0$.

We will show that $P(z)$ has one root. This suffices since then $P(z) = (z-z_0)Q(z)$, and we can repeat the argument on $Q(z)$ which is a polynomial with degree $n-1$.

Assume by contradiction that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $\frac{1}{P(z)}$ is entire. Moreover:

$$\frac{P(z)}{z^n} = a_n + \underbrace{\left(\frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)}_{\to 0 \text{ as } |z| \to \infty}$$

So $|P(z)| > \frac{1}{2}|a_n||z|^n$ for sufficiently large $|z|$, say $|z| > R$.

Hence, $\frac{1}{|P(z)|} < \frac{2}{|a_n|}\frac{1}{R^n}$ for $|z| > R$.

For $|z| \leq R$, in that finite disc, $\frac{1}{P(z)}$ is continuous and $\frac{1}{|P(z)|}$ attains its maximum on the boundary. Furthermore, we showed earlier that $\frac{1}{|P(z)|}$ is bounded outside the disc too.

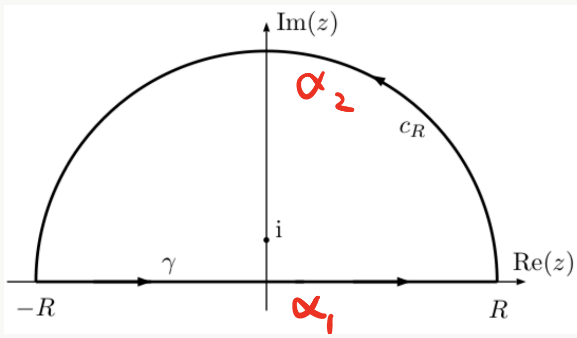
Overall, $\frac{1}{P(z)}$ is entire and bounded. By Liouville's Theorem, it must be constant, which is a contradiction.

W7 Reading Break

Reading break!

W8C1 Lecture 18 (Feb 24)

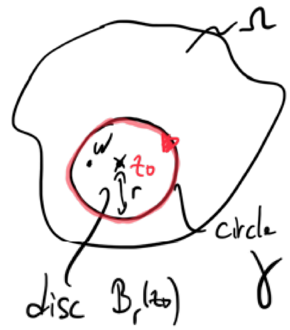
Went through review slides on Cauchy's theorem and consequences

> 💡 **Ex 27.** (HW5 Q5)
>
> Evaluate $\int_{-\infty}^{\infty} \frac{\cos\theta}{1+x^2}\,dx$ where $\theta > 0$.
>
> 
>
>
> Consider the contour above, and let $f(z) = \frac{e^{i\theta z}}{1+z^2}$.
>
> Factor $f(z)$ as $f(z) = \frac{e^{i\theta z}}{(z+i)(z-i)} = \frac{g(z)}{z-i}$ where $g(z) = \frac{e^{i\theta z}}{z+i}$.
>
> By CIF, $\oint_\alpha f(z)\,dz = \oint_\alpha \frac{g(z)}{z-i}\,dz = 2\pi i g(i) = 2\pi i \cdot \frac{e^{-\theta}}{2i} = \pi e^{-\theta}$.
>
> Along α_1, $\int_{\alpha_1} f(z)\,dz = \int_{-R}^{R} f(z) 1\,dt = \int_{-R}^{R} \frac{e^{i\theta t}}{1+t^2}\,dt$.
>
> In the limit $R \to \infty$, we get $\int_{-\infty}^{\infty} \frac{e^{i\theta t}}{1+t^2}\,dt$.
>
> Along α_2, we have $|\int_{\alpha_2} f(z)\,dz| = |\int_0^\pi \frac{e^{i\theta Re^{it}}}{1+R^2e^{2it}} iRe^{it}\,dt| \le R\pi \max\{|\frac{e^{i\theta Re^{it}}}{1+R^2e^{2it}}|, t \in [0,\pi]\}$
>
> For the numerator, $|e^{i\theta Re^{it}}| = |e^{i\theta R(\cos t + i \sin t)}| = |e^{-\theta R \sin t}| \le 1$ since $\sin t > 0$.
>
> For the denominator, $|1 + R^2 e^{2it}| \ge R^2 - 1$.
>
> Altogether, $|\int_{\alpha_2} f(z)\,dz| \le R\pi \cdot \frac{1}{R^2-1}$ tends to 0 as $R \to \infty$.
>
> Matching components, we get $\int_{-\infty}^{\infty} \frac{\cos\theta}{1+x^2}\,dx = \pi e^{-\theta}$

W8C2 Lecture 19 (Feb 26)

Analytic Functions

- Geometric series: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ whenever $|r| < 1$
- For any $z \in B_1(0)$, $\sum_{n=0}^{\infty} |z|^n$ convergent ⇒ $\sum_{n=0}^{\infty} z^n = (1-z)^{-1}$ convergent.



Now we apply Cauchy's Integral Formula with $f \in H(\Omega)$, $z_0 \in \Omega$, and $B_r(z_0) \subset \Omega$.

For any $w \in B_r(z_0)$, $f(w) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-w}\, dz$

Note that $|w - z_0| < r = |z - z_0|$ for any $z \in \gamma$.

Hence $\frac{1}{z-w} = \frac{1}{(z-z_0)-(w-z_0)} = \frac{1}{z-z_0}\left(\frac{1}{1-\frac{w-z_0}{z-z_0}}\right)$ with $|\frac{w-z_0}{z-z_0}| < 1$.

We may write $\frac{1}{z-w} = \frac{1}{z-z_0} \sum_{n=0}^{\infty} (\frac{w-z_0}{z-z_0})^n$.

Plugging into CIF gives $f(w) = \sum_{n=0}^{\infty} \underbrace{(\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz)}_{a_n}(w-z_0)^n = \sum_{n=0}^{\infty} a_n (w-z_0)^n$

a_n does not depend on w. We have written $f(w)$ as a power series, and by definition it is analytic.

Observe that $a_n = \frac{f^{(n)}}{n!}$ by Cauchy's Differentiation Formula.

We recover a Taylor series: $f(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(w-z_0)^n$, analogous to Calc 1.

📖 **Thm.** (Analyticity) If $f \in H(\Omega)$, then f is equal to its Taylor series at $z_0 \in \Omega$.

The Taylor series is convergent in the largest disc around z_0, which is completely inside Ω.

Hence, holomorphic ⇒ analytic.

Note: if f is entire, then its Taylor series around any z_0 converges for all $z \in \mathbb{C}$.

💡 **Ex 28.** $f(z) = e^z = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!}$ (where we have picked $z_0 = 0$) is convergent for all $z \in \mathbb{C}$.

Note that $f' \in H(\Omega)$, so $f'(w) = \sum_{n=0}^{\infty} \frac{(f')^{(n)}(z_0)}{n!}(w-z_0)^n = f'(z_0) + f''(z_0)(w-z_0) + \frac{1}{2}f^{(3)}(z_0)(w-z_0)^2 + \dots$

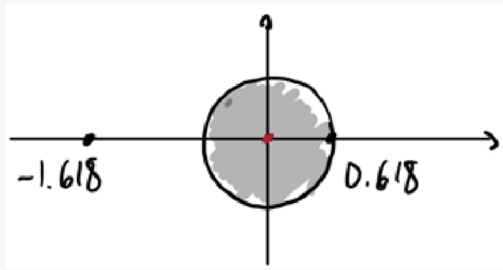
This is the term-by-term differentiation of $f(z_0) + f'(z_0)(w-z_0) + \frac{f'(z_0)}{2}(w-z_0)^2 + \dots$

In other words, a Taylor series defines a differentiable function. Hence, analytic ⇒ holomorphic.

To conclude, analytic ⇔ holomorphic.

💡 **Ex 29.** The Fibonacci numbers are defined as the Taylor coefficients of $(1-z-z^2)^{-1}$ around $z_0 = 0$.

The roots of $1-z-z^2$ are $-\frac{1\pm\sqrt{5}}{2}$ (golden ratio).



We have $(1-z-z^2)^{-1} = \sum_{n=0}^{\infty} F_n z^n$

$1 = (1-z-z^2)\sum_{n=0}^{\infty} F_n z^n = \sum_{n=0}^{\infty} F_n z^n - \sum_{n=0}^{\infty} F_n z^{n+1} - \sum_{n=0}^{\infty} F_n z^{n+2}$

$1 = F_0 + (F_1 - F_0)z + \sum_{n=2}^{\infty}(F_n - F_{n-1} - F_{n-2})$

Matching terms gives $F_0 = 1, F_1 = F_0 = 1, F_n = F_{n-1} + F_{n-2} \quad (n \geq 2)$

$\{F_i\} = 1, 1, 2, 3, 5, 8, 13, \ldots$

W8C3 Lecture 20 (Feb 28)

Recall that the derivatives of $\text{Log}(z)$ are $z^{-1}, -z^{-2}, 2z^{-3}, \ldots$

In general, $\frac{d^j}{dz^j}\text{Log}(z) = (-1)^{j-1}(j-1)!z^{-j}$ for all $j \in \mathbb{N}$.

The Taylor series of $\text{Log}(z)$ at $z_0 = 1$ is:

$\text{Log}(z) = 0 + (z-1) - \frac{(z-1)^2}{2!} + 2!\frac{(z-1)^2}{3!} - \ldots = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}(z-1)^j}{j}$

Since $\text{Log}(z) \in H(\mathbb{C} \setminus (-\infty, 0])$, the series converges in $B_1(1)$.

Differentiating the series, we get:

$1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots = \sum_{n=0}^{\infty}[-(z-1)]^n = \frac{1}{1-[-(z-1)]} = \frac{1}{z}$, as expected.

Zeros and Poles

📌 **Def.** A point $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$ is called a zero of f.

If f is analytic in a neighbourhood of a zero z_0: then $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for some $R > 0$ and all $z \in B_R(z_0)$. Hence, $f(z_0) = a_0 = 0$. There are two possibilities:

1. all $a_n = 0$, so $f(z) = 0 \, \forall \, z$.
2. there is a $m \geq 1$ such that $a_0 = a_1 = \ldots = a_{m-1} = 0$ but $a_m \neq 0$.

   Then $f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \ldots$ and we say z_0 is a zero of order m.

Factoring, we get $f(z) = (z - z_0)^m [a_m + a_{m+1}(z - z_0) + \ldots] = (z - z_0)^m g(z)$, where $g(z)$ is analytic and $g(z_0) \neq 0$.

Remark: since $a_n = \frac{1}{n!} f^{(n)}(z_0)$, we conclude that z is a zero of order m if and only if:

$$\begin{cases} f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0 \\ f^{(m)}(z_0) \neq 0 \end{cases}$$

💡 **Ex 30.** $P(z) = -(2 - 2i) + (5 - 3i)z - 3z^2 - (1 - i)z^3 + z^4$

Factor as $P(z) = (z - 1)^2 (z + 2)(z - (1 - i))$.

So 1 is a zero of multiplicity 2, while -2 and $(1 - i)$ are zeros of multiplicity 1, also called simple zeros.

A zero of finite order of an analytic function is always isolated (there exists a neighbourhood with no other zeros).

Since $g(z_0) = a_m \neq 0$ and g is continuous, then $g(z) \neq 0$ in a small enough disc around z_0. This implies that $f(z)$ is non-zero in that disc.

📖 **Thm.** (Identity Theorem). Let Ω be a domain and α be a curve in Ω. If $f, g \in H(\Omega)$ are so that $f(z) = g(z)$ for all $z \in \alpha$, then $f(z) = g(z)$ for all $z \in \Omega$.

Proof: α is a set of zeros of $f - g$, and they are not isolated, so $f - g$ must be identically 0.

Remark: this explains why trigonometric identities hold for all $z \in \mathbb{C}$ if they hold on the real line.

More generally: the curve α above can be replaced by a set of points that accumulate somewhere.

Summarizing: in a neighbourhood of a zero, an analytic function behaves as $f(z) \simeq a_m (z - z_0)^m$ for some $m \geq 1$.

W9C1 Lecture 21 (Mar 3)

📌 **Def.** f has an isolated singularity at z_0 if $f \in H(\dot{B}_r(z_0))$ for some $r > 0$

📌 **Def.** A function f has a pole of order m at z_0 if $\frac{1}{f}$ is holomorphic in $B_r(z_0)$ and z_0 is a zero of order m of $\frac{1}{f}$.

In particular, z_0 is an isolated singularity.

If $m = 1$, then z_0 is a simple pole.

💡 **Ex 31.** Let $f(z) = \frac{\sin z}{(2z-1)(z+(1-2i))^2}$

- Zeros at $z = n\pi, n \in \mathbb{Z}$

  Focus on $n = 0$. We have $f(0) = 0$ and $f'(0) \neq 0$. Hence $z_0 = 0$ is a zero of order 1.

- Taylor series:

  $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{(iz)^n}{n!} - \frac{(-iz)^n}{n!}\right) = \frac{1}{i}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(iz)^{2n+1}$
  $\quad = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(-1)^n z^{2n+1} = z - \frac{z^3}{6} + \cdots$

  $(2z-1)(z+(1-2i))^2 = (4i+3) + (-8-4i)z + (3-8i)z^2 + 2z^3$

  $\frac{1}{(2z-1)(z+(1-2i))^2} = \frac{1}{4i+3}\frac{1}{1+\frac{(-8-4i)}{4i+3}z+\cdots} = \frac{1}{4i+3}(1 - \frac{(-8-4i)}{4i+3}z + \cdots)$

  where we have used $\frac{1}{1-u} \sim 1 + u$. Hence,

  $f(z) = (z - \frac{z^2}{6} + \cdots)(\frac{1}{4i+3}(1 + \frac{8+4i}{4i+3}z + \cdots)) = \frac{1}{4i+3}z + \cdots$

  We recover that $z_0 = 0$ is a zero of order 1.

- Poles:

  $\frac{1}{f(z)} = \frac{(2z-1)(z+(1-2i))^2}{\sin z}$ has two zeros $z_1 = \frac{1}{2}$ (order 1) and $z_2 = -1+2i$ (order 2).

  Hence $z_1 = \frac{1}{2}$ is a pole of order 1 (simple pole), and $z_2 = -1+2i$ is a pole of order 2.

In general, if f has a pole of order m, then

$f(z) = \frac{1}{(z-z_0)^m g(z)}$

Since $g(z)$ does not have a zero in a neighbourhood of z_0, $\frac{1}{g(z)}$ is holomorphic and so:

$f(z) = \frac{1}{(z-z_0)^m}(A_0 + A_n(z-z_0) + A_2(z-z_0)^2 + \cdots) = \frac{1}{(z-z_0)^m}h(z)$

In the presence of a pole of order m at z_0, $f(z)$ has a power series representation in $\dot{B}_r(z_0)$, but it includes negative powers, down to $(z-z_0)^{-m}$.

Remarks:

- If z_0 is a pole, then $\lim_{z\to z_0}|f(z)| = \infty$
- If z_0 is a pole of order m, then $(z-z_0)^k f(z)$ has a pole z_0 of order $m-k$, provided $k < m$.

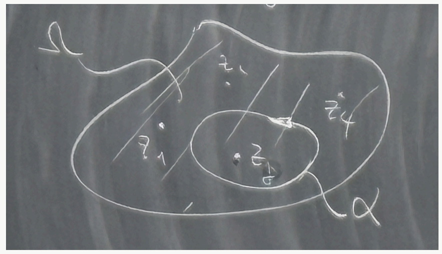
W9C2 Lecture 22 (Mar 5)

📌 **Def.** Let $f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$

Then the coefficient a_{-1} is called the residue of f at z_0.

We write $a_{-1} = \mathrm{Res}(f, z_0)$.

📌 **Def.** If $f \in H(\Omega \setminus \{z_1, \dots, z_n\})$, then f is called meromorphic (finite singularities).



Consider α that encloses z_j. In that case, z_j is the only singularity inside α, so:

$\oint_\alpha f(z)\,dz = \sum_{n=1}^m \oint_\alpha \frac{a_{-n}}{(z-z_j)^n} + \underbrace{\oint_\alpha \sum_{n=0}^\infty a_n(z-z_j)^n\,dz}_{(*)}$

By Cauchy's Theorem, $(*) = 0$.

Recall from W4C2 that $\displaystyle\oint_\alpha \frac{a_{-n}}{(z-z_j)^n}\,dz = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i a_{-1} & \text{if } n = 1 \end{cases}$

Hence, $\oint_\alpha f(z)d(z) = 2\pi i a_{-1} = 2\pi i \operatorname{Res}(f, z_j)$

📖 **Thm.** (Residue Theorem)

$\oint_\alpha f(z)\,dz = 2\pi i \sum_{j=1}^N \operatorname{Res}(f, z_j)$

Remarks:

i. No singularity ⇒ $\oint_\alpha f(z)\,dz = 0$ (Cauchy's Theorem)

ii. No singularity ⇒ $\frac{f(z)}{z-w}$ has a single pole at w, and

$\frac{f(z)}{z-w} = \frac{1}{z-w}(f(w) + f'(w)(z-w) + \cdots) = \frac{f(w)}{z-w} + f'(w) + \cdots$

Hence, $\operatorname{Res}(\frac{f(z)}{z-w}, w) = f(w)$, so $\oint_\alpha \frac{f(z)}{z-w}\,dz = 2\pi i f(w)$ (Cauchy's Integral Formula)

Computing residue

i. (Simple pole)

$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + \cdots \implies (z-z_0)f(z) = a_{-1} + a_0(z-z_0) + \cdots$

$\lim_{z\to z_0}(z-z_0)f(z) = a_{-1} = \operatorname{Res}(f, z_0)$

ii. (General case, pole of order m)

$(z-z_0)^m f(z) = a_{-m} + \cdots + a_{-1}(z-z_0)^{m-1} + \cdots$

Taking $m-1$ derivatives:

$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)$

Ex 32.

Consider $f(z) = \frac{\sin(z)}{(2z-1)(z+(1-2i))^2}$

$\operatorname{Res}(f, \frac{1}{2}) = \lim_{z \to \frac{1}{2}} (z - \frac{1}{2}) f(z) = \lim_{z \to \frac{1}{2}} \frac{\sin z}{2(z+(1-2i))^2} = \frac{\sin(\frac{1}{2})}{2(\frac{3}{2}-2i)^2}$

W9C3 Lecture 23 (Mar 7)

iii. (Special case: $\operatorname{Res}(\frac{f(z)}{g(z)}, z_0)$ where z_0 is a simple zero of g)

$$\operatorname{Res}(\frac{f(z)}{g(z)}, z_0) = \lim_{z \to z_0} (z - z_0) \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{z - z_0}{g(z) - g(z_0)} f(z_0) = \frac{1}{g'(z_0)} f(z_0) = \frac{f(z_0)}{g'(z_0)}$$

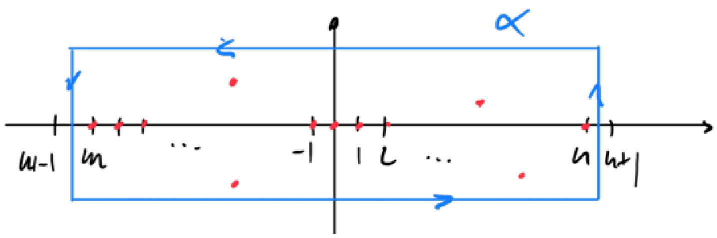
Application: calculating series $\sum_{n=0}^{\infty} f(n)$ where $f(n) \in \mathbb{R}$

Let $\varphi(z) = \pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)}$, then this function has simple poles at $n \in \mathbb{Z}$

$\operatorname{Res}(\varphi, n) = \pi \frac{\cos(\pi n)}{\pi \sin(\pi n)} = 1$

Hence, if f does not have a pole of n, then $\operatorname{Res}(f(z)\varphi(z), n) = f(n)$

Pick the curve below:

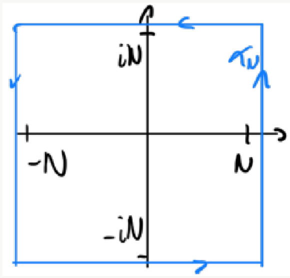


$\frac{1}{2\pi i} \oint_{\alpha} \varphi(z) f(z)\, dz = \sum_{j=-m}^{n} f(j) + \sum(\text{other residues coming from poles of } f)$

💡 **Ex 33.** Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Basel Problem, Euler 1734)

We start with $f(z) = \frac{1}{z^2+a^2}$ where $a \notin i\mathbb{Z}$

f has simples poles at $\pm ia$. Consider the contour α below (where α goes right between N and $N+1$)



We have $\lim_{N\to\infty} \oint_{\alpha_N} \pi \cot(\pi z) \frac{1}{z^2+a^2}\, dz = 0.$

This is because $|\cot(\pi z)|$ is bounded away from the singularities.

$|\frac{1}{z^2+a^2}| \leq \frac{1}{|z|^2-|a|^2} \sim \frac{1}{N^2}$

The length is $l(\alpha_N) = 4(2N+1)$

So $|\oint_{\alpha_N} \pi \cot(\pi z) \frac{1}{z^2+a^2}\, dz| \lesssim \frac{8N}{N^2} \to 0$ as $N \to \infty$

We conclude that:

$0 = \sum_{j\in\mathbb{Z}} \frac{1}{j^2+a^2} + \mathrm{Res}(\varphi f, ia) + \mathrm{Res}(\varphi f, -ia)$

Note that $\mathrm{Res}(\varphi f, \pm ia) = \frac{\pi}{\pm 2ia} \cot(\pm \pi i a)$

So $\sum_{j\in\mathbb{Z}} \frac{1}{j^2+a^2} = -\frac{\pi}{ia} \cot(i\pi a) = -\frac{\pi}{a} \frac{e^{-\pi a}+e^{\pi a}}{e^{-\pi a}-e^{\pi a}} = \frac{\pi}{a} \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}$

Move $j = 0$ to the RHS and note that j and $-j$ are symmetric:

$\sum_{j=1}^{\infty} \frac{1}{j^2+a^2} = \frac{1}{2}(\frac{\pi}{a} \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} - \frac{1}{a^2})$

Taking the limit $a \to 0$ gives:

$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$ (to take limit, use a series expansion)