MATH 305 Notes Part 3

Term: 2024W2

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W10C1 Lecture 24 (Mar 10)

Last time, we say that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi}{6}$. In fact, $\sum_{n=1}^{\infty}\frac{1}{n^s}$ is convergent for all $s\in\mathbb{C}$ such that $\mathrm{Re}(s)>1$.

📌 **Def.** The sum, as a function of s, defines an analytic function: the zeta function.

$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$

It has an analytic extension to all $\mathbb{C}\setminus\{1\}$, where 1 is a simple pole.

Euler (1737): $\zeta(s)=\prod_{p\text{ prime}}\frac{1}{1-p^{-s}}$

Trivial zeros: $\zeta(-2n)=0$ for all integers

Riemann Hypothesis (1859): all other zeros have $\mathrm{Re}(s)=\frac{1}{2}$ (the critical line)

Remark: statistics of zeros along the critical line conjectured to be related to the statistics of energy level of quantum systems that are chaotic

💡 **Ex 34.** Compute $\oint_{|z|=1}\frac{1}{z^2\sin(z)}\,dz$

The zeros of $z^2\sin(z)$ are $z=n\pi, n\in\mathbb{Z}$.

The only zero inside of the circle is $z_0=0$.

$\oint_{|z|=1}\frac{1}{z^2\sin(z)}=2\pi i\,\mathrm{Res}\left(\frac{1}{z^2\sin(z)},0\right)$

Two methods:

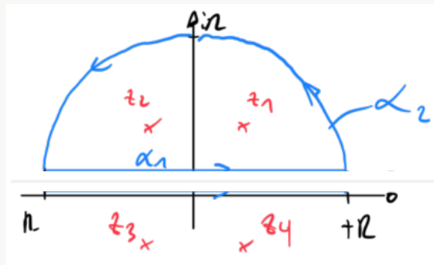
1. Pole $z_0=0$ of order $m=3$ ⇒ $\lim_{z\to z_0}\frac{1}{2}\frac{d^2}{dz^2}\frac{z^3}{z^2\sin z}=\frac{1}{6}$
2. $\frac{1}{z^2\sin(z)}=\frac{1}{z^2}\frac{1}{z-\frac{z^3}{6}+\cdots}=\frac{1}{z^3}(1-\frac{z^2}{6}+\cdots)^{-1}=\frac{1}{z^3}(1+\frac{z^2}{6}+\cdots)$

   The coefficient of $\frac{1}{z}$ is $\frac{1}{6}$.

Hence, $\oint_{|z|=1}\frac{1}{z^2\sin(z)}\,dz=2\pi i(\frac{1}{6})=\frac{i\pi}{3}$

Ex 35. Compute $I = \int_{-\infty}^{\infty} \frac{dx}{x^4+4}$.

Pick $f(z) = \frac{1}{z^4+4}$ and the contour:



$f(z)$ has simple zeros at $\pm 1 \pm i$, with $z_1 = 1+i$ and $z_2 = -1+i$ inside α.

$\oint_\alpha \frac{1}{z^4+4}\,dz = 2\pi i(\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_2))$.

$$\begin{aligned}
\mathrm{Res}(f, z_1) &= \lim_{z\to z_1} (z-z_1) f(z) \\
&= \lim_{z\to z_1} (z-z_1) \tfrac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
&= \tfrac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} \\
&= \tfrac{1}{(1+i-(-1+i))(1+i-(-1-i))(1+i-(1-i))} \\
&= \tfrac{1}{2\cdot 2(1+i)\cdot 2i} = \tfrac{1}{8(i-1)} = -\tfrac{1+i}{16}
\end{aligned}$$

Similarly, $\mathrm{Res}(f, z_2) = \frac{1-i}{16}$.

Then $\oint_\alpha f(z)\,dz = 2\pi i \cdot \frac{-1-i+1-i}{16} = \frac{\pi}{4}$.

$\frac{\pi}{4} = \oint_\alpha (f)\,dz = \int_{-R}^{R} \frac{dx}{x^4+4} + \int_{\alpha_2} f(z)\,dz$

$|\int_{\alpha_2} f(z)\,dz| \le \pi R \max_{z\in\alpha_2} \frac{1}{|z^4+4|} \le \frac{\pi R}{R^4-4} \to 0$ as $R \to \infty$.

Hence, $\int_{-\infty}^{\infty} \frac{1}{x^4+4}\,dx = \frac{\pi}{4}$.

W10C2 Midterm 2 (Mar 12)

Midterm 2 in class.

W10C3 Lecture 25 (Mar 14)

We consider integrals of the type $\int_0^{2\pi} \frac{P(\sin\varphi, \cos\varphi)}{Q(\sin\varphi,\cos\varphi)}\,d\varphi$, where P and Q are polynomials.

Idea: interpret this as an integral over the unit circle, with $\alpha(\varphi) = e^{i\varphi}$ and $\alpha'(\varphi) = ie^{i\varphi}\,d\varphi = i\alpha(\varphi)\,d\varphi$. It is helpful to note that:

- $\sin\varphi = \frac{1}{2i}(z - \frac{1}{z}), \quad z = e^{i\varphi}$
- $\cos\varphi = \frac{1}{2}(z + \frac{1}{z}), \quad z = e^{i\varphi}$
- $d\varphi = \frac{1}{iz}\,dz$

💡 **Ex 36.** Evaluate $I = \int_0^{2\pi} \frac{1}{-\frac{9}{8}+\cos^2\varphi}\,d\varphi$.

$$\begin{aligned} I &= \oint_{|z|=1} \frac{1}{-\frac{9}{8}+(\frac{1}{2}(z+\frac{1}{z}))^2}\frac{1}{iz}\,dz \\ &= \frac{4}{i}\oint_{|z|=1}\frac{1}{z(-\frac{9}{2}+z^2+2+\frac{1}{z^2})}\,dz \\ &= -4i\oint_{|z|=1}\frac{z}{z^4-\frac{5}{2}z^2+1}\,dz \\ &= -4i\oint_{|z|=1}\frac{z}{(z^2-2)(z^2-\frac{1}{2})} \end{aligned}$$

The two simple poles $\pm\sqrt{2}$ are outside the unit circle, while the two simple poles $\pm\frac{1}{\sqrt{2}}$ are inside the unit circle.

$\mathrm{Res}(f(z), \pm\frac{1}{\sqrt{2}}) = \frac{\pm\frac{1}{\sqrt{2}}}{(\frac{1}{2}-2)(\pm\frac{2}{\sqrt{2}})} = -\frac{1}{3}$

We conclude that $I = 2\pi i(\frac{4i}{3}+\frac{4i}{3}) = -\frac{16\pi}{3}$

W11C1 Lecture 26 (Mar 17)

💡 **Ex 37.** Evaluate $I = \int_0^{2\pi} \frac{\sin\theta+\cos\theta}{1+\sin^3\theta}\,d\theta$.

$I = \oint_{|z|=1} \frac{\frac{1}{2i}(z-z^{-1})+\frac{1}{2}(z+z^{-1})}{1+(\frac{1}{2i}(z-z^{-1}))^3}\frac{1}{iz}\,dz$

Find poles, evaluate residues.

The function $\sin(\frac{1}{z})$ is singular at $z_0 = 0$. We can write a series away from z_0:

$\sin(\frac{1}{z}) = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}(\frac{1}{z})^{2j+1} = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} - \cdots$

So $z_0 = 0$ is an isolated singularity, but it is not a pole of any finite order because there are arbitrary large negative powers of z.

Such singularities are called essential.

Remarks:

- The residue is well defined here, $a_{-1} = 1$, but there is no formula involving derivatives of f for it.
- $\lim_{z\to z_0}|f(z)|$ does not exist
- For any $w\in\mathbb{C}$ and $r>0$, there is a sequence of points $z_n \in B_r(z_0)$ such that $\lim_{n\to\infty} f(z_n) = w$.

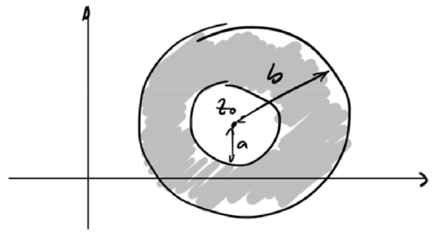
📌 **Def.** An expression of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is called a Laurent series.

- $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ is the singular part of the series
- $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is the regular part of the series

The regular part is convergent in a disc $|z-z_0|<b$.

The singular part is convergent for $\frac{1}{|z-z_0|} < \frac{1}{a} \iff |z-z_0| > a$

Overall, a Laurent series is convergent in an annulus $C(z_0, a, b) = \{z \in \mathbb{C} : a < |z-z_0| < b\}$



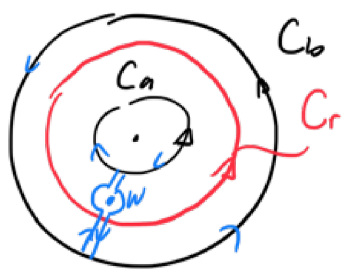
In the case of a pole of order m, the convergence is in a pointed disc.

📖 **Thm.** (Laurent's Theorem). Let $z_0 \in \mathbb{C}$ and $0 \leq a < b \leq \infty$.

If $f \in H(C(z_0, a, b))$, then $f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n$ for all $z \in C(z_0, a, b)$, and $a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$ for any $a < r < b$ and any $n \in \mathbb{Z}$.

W11C2 Lecture 27 (Mar 19)

Proof:



$\frac{f(z)}{z-w}$ is holomorphic in $C(z_0, a, b) \setminus \{w\}$.

Hence $\int_{C_b} - \int_{C_a} - \int_{|z-w|=\varepsilon} = 0$ (integral along blue path is 0).

By CIF, $\int_{|z-w|=\varepsilon} = 2\pi i f(w)$.

Hence, $f(w) = \frac{1}{2\pi i}\left(\oint_{C_b} \frac{f(z)}{z-w}\, dz - \oint_{C_a} \frac{f(z)}{z-w}\, dz \right)$

- On C_b: $(z-w)^{-1} = (z-z_0)^{-1}(1 - \frac{w-z_0}{z-z_0})^{-1}$. Note that $|\frac{w-z_0}{z-z_0}| < 1$, so we can write $(z-w)^{-1} = (z-z_0)^{-1} \sum_{n=0}^{\infty} (\frac{w-z_0}{z-z_0})^n$
- On C_a: $(z-w)^{-1} = -(w-z_0)^{-1}(1 - \frac{z-z_0}{w-z_0})^{-1}$. Note that $|\frac{z-z_0}{w-z_0}| < 1$, so we can write $(z-w)^{-1} = -(w-z_0)^{-1} \sum_{n=0}^{\infty} (\frac{z-z_0}{w-z_0})^n$

We see that C_b yields the regular part, and C_a yields the singular parts.

Remarks:

- If f is actually holomorphic in $B_b(z_0)$, then $\frac{f(z)}{(z-z_0)^{n+1}} \in H(B_b(z_0))$ for $n \leq -1$. Hence, $a_n = 0$ for all $n \leq -1$ and we recover a Taylor series.
- If there is an isolated singularity in $B_a(z_0)$, then $a_{-1} = \text{Res}(f, z_0) = \frac{1}{2\pi i} \oint f(z)\, dz$ so we recover the residue theorem.

💡 **Ex 38.** Consider $f(z) = \frac{1}{(z-1)(z+i)}$.

i. Consider $C(1, 0, \sqrt{2})$, namely $0 < |z-1| < \sqrt{2}$. We need to expand $\frac{1}{z+i}$:

$\frac{1}{z+i} = \frac{1}{(z-1)+1+i} = \frac{1}{1+i} \frac{1}{1-\frac{1-z}{1+i}} = \frac{1}{1+i} \sum_{j=0}^{\infty} (\frac{1-z}{1+i})^j$ since $|\frac{1-z}{1+i}| = \frac{|1-z|}{\sqrt{2}} < 1$.

Hence $f(z) = \frac{1}{z-1} \frac{1}{1+i} \sum_{j=0}^{\infty} (\frac{1-z}{1+i})^j = \sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{(1+i)^{n+2}} (z-1)^n$

Indeed, $z_0 = 1$ is a simple pole with $\text{Res}(f, 1) = \frac{1}{1+i}$.

ii. On $C(0, 0, 1)$, namely $0 < |z| < 1$, write $f(z) = \frac{1}{1+i}(\frac{1}{z-1} - \frac{1}{z+i})$.

$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{j=0}^{\infty} z^j \quad (|z| < 1)$

$\frac{1}{1+i} = \frac{1}{i(1+\frac{z}{i})} = \frac{1}{i} \sum_{j=0}^{\infty} (-\frac{z}{i})^j = \sum_{j=0}^{\infty} \frac{(-1)^j}{i^{j+1}} z^j$ since $|-\frac{z}{i}| < 1$.

Overall, $f(z) = -\frac{1}{1+i} \sum_{j=0}^{\infty} (1 + \frac{(-1)^j}{i^{j+1}}) z^j$

This is a Taylor series: indeed, $f \in H(B_1(0))$.

💡 **Ex 39.** Consider $g(z) = \frac{1}{z(z-1)}$ in $C(0, 1, \infty)$, namely $1 < |z| < \infty$.

$\frac{1}{z-1} = \frac{1}{z} \frac{1}{1-\frac{1}{z}} = \frac{1}{z} \sum_{j=0}^{\infty} (\frac{1}{z})^j$ since $|\frac{1}{z}| = \frac{1}{|z|} < 1$.

Hence $g(z) = \frac{1}{z^2} \sum_{j=0}^{\infty} \frac{1}{z^j} = \sum_{n=-\infty}^{-2} z^n$

Note: since both singularities lie inside the annulus, we cannot read off the residues here.

Rather, $a_{-1} = 0 = \text{Res}(g, 0) + \text{Res}(g, 1) = -1 + 1 = \frac{1}{2\pi i} \int_{|z|=r} f(z)\, dz \quad (r > 1)$

W11C3 Lecture 28 (Mar 21)

Integration with Branch Cuts

Recall that the complex integral requires by definition that the integrand is continuous along the path α ⇒ no integration along branch cuts.

Still, they can be taken advantage of.

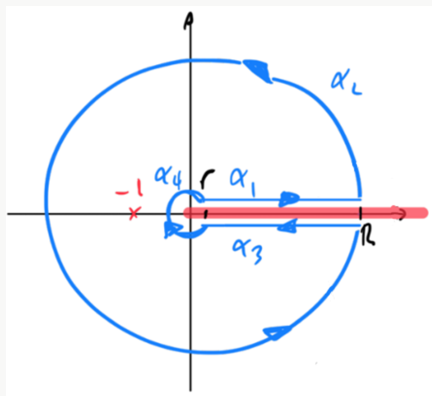
Ex 40. Consider $\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$.

We set $f(z) = \frac{1}{z^{\frac{1}{2}}(1+z)}$, but for which $z^{\frac{1}{2}}$?

Choose $z^{-\frac{1}{2}} = e^{-\frac{1}{2}(\ln|z| + i\arg(z))}$. $\arg$ takes values in $[0, 2\pi)$ and the branch cut is the positive real axis.

If x is real and positive, then $\arg(x) = 0$ and $x^{-\frac{1}{2}}$ reproduces the positive real value: $e^{-\frac{1}{2}\ln x} = \frac{1}{\sqrt{x}}$.

More precisely, we have $\lim_{\varepsilon \to 0^+}(x + i\varepsilon)^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$ and $\lim_{\varepsilon \to 0^+}(x - i\varepsilon)^{-\frac{1}{2}} = -\frac{1}{\sqrt{x}}$, so there is a discontinuity across the branch cut. So we consider the contour below:



$\alpha_1(t) = t + i\varepsilon \quad t \in [r, R]$

$\alpha_2(t) = Re^{it} \quad t \in [\theta_1, 2\pi - \theta_1] \quad \tan\theta_1 = \frac{\varepsilon}{R}$

$-\alpha_3(t) = t - i\varepsilon \quad t \in [r, R]$

$-\alpha_4(t) = re^{it} \quad t \in [\theta_2, 2\pi - \theta_2] \quad \tan\theta_2 = \frac{\varepsilon}{r}$

The branch point at $z = 0$ and the branch cut are outside of the contour.

There is one simple pole at $z = -1$, with

$\mathrm{Res}(\frac{z^{-\frac{1}{2}}}{1+z}, -1) = e^{-\frac{1}{2}(\ln|-1| + i\pi)} = e^{-i\frac{\pi}{2}} = -i$

Hence $\oint \frac{z^{-\frac{1}{2}}}{1+z}\,dz = 2\pi i(-i) = 2\pi$ by the residue theorem.

Along the individual segments, we have:

$|\int_{\alpha_2}(\cdot)\,dz| \le 2\pi R \cdot \frac{R^{-\frac{1}{2}}}{R-1} \to 0 \quad (R \to \infty)$

$|\int_{\alpha_4}(\cdot)\,dz| \le 2\pi r \cdot \frac{r^{-\frac{1}{2}}}{1-r} \to 0 \quad (r \to 0)$

$\int_{\alpha_1}(\cdot)\,dz = \int_r^R \frac{(t+i\varepsilon)^{-\frac{1}{2}}}{1+t+i\varepsilon}\,dt = \int_r^R \frac{1}{\sqrt{t}(1+t)}\,dt \quad (\varepsilon \to 0)$

$\int_{\alpha_3}(\cdot)\,dz = -\int_r^R \frac{(t-i\varepsilon)^{-\frac{1}{2}}}{1+t-i\varepsilon}\,dt = -\int_r^R \frac{-1}{\sqrt{t}(1+t)}\,dt \quad (\varepsilon \to 0)$

We get $2\pi = 2\int_0^\infty \frac{1}{\sqrt{t}(1+t)}\,dt$ in the limit $r \to 0, R \to \infty$.

We conclude that $\int_0^\infty \frac{1}{\sqrt{t}(1+t)}\,dt = \pi$.

W12C1 Lecture 29 (Mar 24)

2D Fluids

We consider 2-dimensional stationary (flow doesn't change in time) flows of incompressible and irrotational fluids.

Let the velocity field be $\vec{v} = (v_1, v_2)$. The equations for the velocity field are:

$$\begin{cases} \operatorname{div} \vec{v} = \partial_1 v_1 + \partial_2 v_2 = 0 \\ \operatorname{rot} \vec{v} = \partial_2 v_1 - \partial_1 v_2 = 0 \end{cases}$$

Define the complex function $w(z) = v_1(x, y) - i v_2(x, y)$.

Suppose w is holomorphic, then the Cauchy-Riemann equations for w are:

$$\partial_x v_1 = \partial_y(-v_2) \qquad \iff \operatorname{div} \vec{v} = 0$$
$$\partial_y v_1 = -\partial_x(-v_2) \qquad \iff \operatorname{rot} \vec{v} = 0$$

Hence w being holomorphic is equivalent to the velocity field being stationary and incompressible.

In a disc, w has an antiderivative $-\phi$. Namely, $w(z) = -\phi'(z)$, where $\phi \in H(\Omega)$.

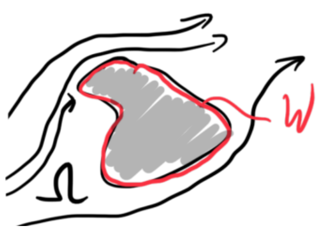
Write $\phi = \phi_1 - i\phi_2$.

$w(z) = -\phi' = -\partial_x \phi_1 + i\partial_x \phi_2$ (horizontal limit) and also $w(z) = \partial_y \phi_2 + i \partial_y \phi_1$ (vertical limit).

Hence: $v_1 = -\partial_x \phi_1 = \partial_y \phi_2$ and $v_2 = -\partial_x \phi_2 = -\partial_y \phi_1$, so we have:

$\vec{v} = \binom{v_1}{v_2} = -\nabla \phi_1$ and $\vec{v}^\perp = \binom{-v_2}{v_1} = \nabla \phi_2$.

The level curves of ϕ_2 are $\perp$ to $\nabla \phi_2 = \vec{v}^\perp$, namely the level curves of ϕ_2 are parallel to $\vec{v}$ (flow lines).



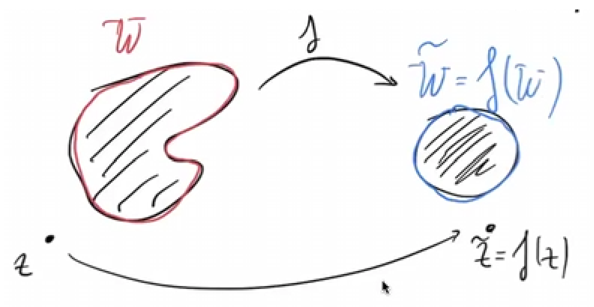
Our goal is to solve for the flow outside of a fixed body W.

The constraint is that $\vec{v}$ is tangential to the surface, namely ϕ_2 is constant on W.

Observation: if ϕ is a solution in Ω (namely $\phi \in H(\Omega)$ and ϕ_2 is constant on W), and if f is holomorphic with $f'(z) \neq 0$, then:

$\tilde{\phi}(\tilde{z}) \coloneqq \phi(f^{-1}(\tilde{z}))$ is a solution in the domain $\tilde{\Omega} = f(\Omega)$.

Key takeaway: we can deform complex domains into simple domains via holomorphic functions, which will make finding ϕ easier.



- $\tilde{\phi} \in H(\tilde{\Omega})$ because f^{-1} is holomorphic
- $\tilde{\phi}_2(\tilde{z}) = \phi_2(f^{-1}(\tilde{z}))$ is constant on $\tilde{W} = f(W)$

Transformation of the velocity field:

$\tilde{w}(\tilde{z}) = -\tilde{\phi}'(\tilde{z}) = -\phi'(f^{-1}(\tilde{z})) \cdot (f^{-1})'(\tilde{z}) = -w(z) \cdot \frac{1}{f'(z)}$

Circulation:

$Z = \oint_W w(z)\,dz = \oint_\alpha w(z)\,dz$ by Cauchy's theorem.

Remarks:

i. Z is invariant under a transformation f
ii. If $Z \neq 0$, then w has singularities inside K.
iii. The force $\vec{F} = (F_1, F_2)$ acting on K is given by $F = \frac{i\rho}{2} \oint_\alpha w^2(z)\,dz$ where $F = F_1 - iF_2$ and ρ is the density of the fluid.

The force can be read off at ∞: if $\vec{v}(\vec{x}) \to \vec{v}_\infty$ as $|\vec{x}| \to \infty$, then $\vec{F} = -\rho Z \vec{v}_\infty^\perp$.

W12C2 Lecture 30 (Mar 26)

Proof:

w is holomorphic in the annulus $C(0, r, \infty)$ and is bounded. Hence, by Liouville's theorem, its regular part reduces to a constant.

$w(z) = \sum_{n=0}^\infty \frac{w_{-n}}{z^n} = w_0 + \frac{w_{-1}}{z} + \frac{w_{-2}}{z^2} + \cdots$

with $w_0 = v_{\infty,1} - iv_{\infty,2}$ since $\lim_{|z|\to\infty} w(z) = w_0$.

We can also write $w_{-1} = \frac{1}{2\pi i} \oint_{|z|=r} w(z)\,dz = \frac{Z}{2\pi i}$.

$$
\begin{aligned}
w(z)^2 &= (w_0 + \tfrac{w_{-1}}{z} + \tfrac{w_{-2}}{z^2} + \cdots)^2 \\
&= w_0^2 + \tfrac{2w_0 w_{-1}}{z} + \cdots \\
&= w_0^2 + \tfrac{\frac{1}{i\pi}(v_{\infty,1} - iv_{\infty,2})Z}{z} + \cdots
\end{aligned}
$$

So $\text{Res}(w^2, 0) = \frac{1}{i\pi}(v_{\infty,1} - iv_{\infty,2})Z$.

$F_1 - iF_2 = \frac{i\rho}{2}\oint_{|z|=r} w^2(z)\,dz = 2\pi i \cdot \frac{i\rho}{2} \cdot \frac{1}{i\pi}(v_{\infty,1} - iv_{\infty,2})Z = \rho Z(v_{\infty,2} + iv_{\infty,1})$.

Hence $\vec{F} = \binom{F_1}{F_2} = \binom{\rho Z v_{\infty,2}}{-\rho Z v_{\infty,1}} = -\rho Z \vec{v}_\infty^{\perp}$.

💡 **Ex 41.**

Recall the Joukowsky map: $u \mapsto z(u) = u + \frac{R^2}{u}$, which is holomorphic in $\mathbb{C} \setminus \{0\}$ and $z'(u) = 1 - (\frac{R}{u})^2 \neq 0$ for $|u| > R$.

- Cylinder: $|u| = R$ is mapped to a plate.

  $u = Re^{i\theta} \mapsto z = Re^{i\theta} + Re^{-i\theta} = 2R\cos\theta$.

- Inverse of $u \mapsto z(u)$:

  $z \mapsto u(z) = \frac{1}{2}(z + \sqrt{z^2 - 4R^2})$ since $u^2 - zu + R^2 = 0$.

- Trivial flow along plate:

  $\phi(z) = -v_\infty z, v_\infty \in \mathbb{R}, w(z) = -\phi'(z) = v_\infty$.

  In turn, $Z = 0, \vec{F} = 0$.

  How does it map to the cylinder?

  $\tilde{\phi}(u) = \phi(z(u)) = -v_\infty(u + \frac{R^2}{u})$. Hence, $w(u) = v_\infty(1 - (\frac{R}{u})^2)$.

  By invariance, $\tilde{Z} = 0, \vec{f} = 0$.

- Rotate the cylinder: $u \mapsto ue^{i\alpha}$.

  $\tilde{\phi}_\alpha(u) = \phi(e^{-i\alpha}u) = -v_\infty(e^{-i\alpha}u + e^{i\alpha}\frac{R^2}{u})$ and map that back to the plate.

- Consider a $Z \neq 0$ case: $w(u) = \frac{Z}{2\pi i u}$, $F = \frac{i\rho}{2}\oint w^2\,du = 0$.

  $\phi(u) = -\frac{Z}{2\pi i}\text{Log}\,u$.

  Here $\phi_2 = -\text{Im}(\phi) = \frac{Z}{2\pi}\ln|u|$ is constant along circles.

- Superposition: $\phi(u) = -v_\infty(u + \frac{R^2}{u}) - \frac{Z}{2\pi i}\text{Log}\,u$ (Magnus effect)

  $w(u) = v_\infty(1 - (\frac{R}{u})^2) + \frac{Z}{2\pi i u}$

  Velocity at ∞: $\lim_{|u|\to\infty} w(u) = v_\infty$.

  Hence, $\vec{F} = \rho Z(0, -v_\infty)$.

W12C3 Lecture 31 (Mar 28)

We now examine the argument principle.

If $P(z)$ is a polynomial:

- if z_0 is a zero of arbitrary multiplicity m, then z_0 is a single pole of $\frac{P'(z)}{P(z)}$ and the residue is equal to m.
- $\oint_\alpha \frac{P'(z)}{P(z)}\,dz = 2\pi i(\#\text{ of zeros inside }\alpha)$ whenever α is a simple, positively oriented, closed curve

We generalize this to an arbitrary meromorphic function.

- If f has a zero of order m at z_0, then

  $f(z) = (z - z_0)^m g(z)$

  with $g \in H(B_r(z_0))$ and $g(z) \neq 0$ for all $z \in B_r(z_0)$ for some sufficiently small r.

  For any $z \in \dot{B}_r(z_0)$:

  $\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}g(z)+(z-z_0)^m g'(z)}{(z-z_0)^m g(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}$, where $\frac{g'(z)}{g(z)} \in H(B_r(z_0))$ since g has no zero there

  Hence, z_0 is a simple pole of $\frac{f'}{f}$ with $\mathrm{Res}(\frac{f'}{f}, z_0) = m$.

- If f has a pole of order m at z_0, then

  $f(z) = \frac{1}{(z-z_0)^m h(z)}$

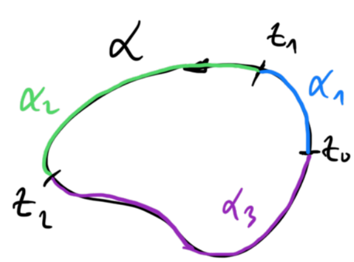
  with $h(z) \in H(B_r(z_0))$ wand $h(z) \neq 0$ for all $z \in B_r(z_0)$ for some sufficiently small r.

  For any $z \in \dot{B}_r(z_0)$:

  $\frac{f'(z)}{f(z)} = -\frac{m(z-z_0)^{m-1}h(z)+(z-z_0)^m h'(z)}{(z-z_0)^{2m}(h(z))^2} \cdot (z - z_0)^m h(z) = -\frac{m}{z-z_0} + \frac{h'(z)}{h(z)}$ where $\frac{h'(z)}{h(z)}$ is holomorphic.

  We conclude that z_0 is again a simple pole with $\mathrm{Res}(\frac{f'}{f}, z_0) = -m$.

> 📖 **Thm.** (Argument Principle). If f is meromorphic in Ω and α is a positively oriented simple closed curve in Ω such that $\mathrm{int}(\alpha) \subset \Omega$ and f has no zeros or poles on α, then
>
> $\frac{1}{2\pi i}\oint_\alpha \frac{f'(z)}{f(z)}\,dz = (\#\text{ zeros in }\mathrm{int}(\alpha)) - (\#\text{ poles in }\mathrm{int}(\alpha))$
>
> where zeros and poles are counted with their order.
>
> *Remark: why this name?*
>
> For all z such that $f(z) \notin (-\infty, 0]$:
>
> $\frac{f'(z)}{f(z)} = \frac{d}{dz}\mathrm{Log}(f(z)) = \frac{d}{dz}(\ln|f(z)| + i\mathrm{Arg}(f(z)))$



Pick a section $z_0 \to z_1$ of α such that the argument of $f(z)$ does not change by more than 2π: there is a branch of the $\log$ (or the argument) such that $z \mapsto \log^{(\alpha_1)}(f(z))$ is continuous (and hence holomorphic) along α_1. Hence:

$$\begin{aligned}\int_{\alpha_1} \frac{f'(z)}{f(z)}\,dz &= \int_{\alpha_1} \frac{d}{dz}\log^{(\alpha_1)}(z)\,dz \\ &= (\ln|f(z)| + i\arg^{(\alpha_1)}(f(z)))_{z=z_0}^{z=z_1} \\ &= (\ln|f(z_1)| - \ln|f(z_0)|) + i(\arg^{(\alpha_1)}(f(z_1)) - \arg^{(\alpha_1)}(f(z_0)))\end{aligned}$$

We repeat along further arcs $\alpha_2, \dots, \alpha_n$ until we cover all of α. Then

$$\begin{aligned}\oint_\alpha \frac{f'(z)}{f(z)}\,dz &= (\int_{\alpha_1} + \cdots + \int_{\alpha_n})(\frac{f'(z)}{f(z)})\,dz \\ &= i\Delta_\alpha \arg(f(z))\end{aligned}$$

Remarks:

- The $\ln|f(z_j)|$ cancel out in a telescopic sum
- The arguments do not cancel out, because we must pick different branches of $\arg$ along different sections

W13C1 Lecture 32 (Mar 31)

The argument principle can be written as:

$\frac{1}{2\pi}\Delta_\alpha \arg(f(z)) = (\#\text{zeros } - \#\text{poles})$

Total change of the argument of $f(z)$ along α is equal to the number of zeros minus the number of poles inside α (counted with multiplicity).

💡 **Ex 42.** Consider $f(z) = z^3$ and $g(z) = \frac{1}{z^3}$, where α is the unit circle.

Split α into 6 subarcs: $\alpha_j(t) = e^{it}, t \in [j\frac{\pi}{3}, (j+1)\frac{\pi}{3}], \quad j = 0, \dots, 5$

Then $\Delta_{\alpha_1} \arg(z^3) = \arg(e^{3i\frac{2\pi}{3}}) - \arg(e^{3i\frac{\pi}{3}}) = \pi$ (we must pick a branch of the argument that enables this)

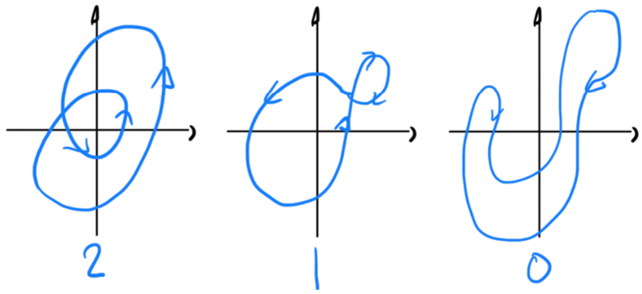
Hence $\Delta_\alpha \arg(z^3) = 6\pi$. This is consistent with the argument principle, since z^3 has a zero of order 3 at 0 and $\frac{1}{2\pi} \cdot 6\pi = 3$.

For $\frac{1}{z^3}$, we have $\Delta_{\alpha_1} \arg(z^{-3}) = \arg(e^{-3i\frac{2\pi}{3}}) - \arg(e^{-3i\frac{\pi}{3}}) = -\pi$

and overall, $\Delta_\alpha \arg(z^{-3}) = -6\pi$, which is consistent with the argument principle, since $\frac{1}{z^3}$ has a pole of order 3 at 0.

$\frac{1}{2\pi}(\text{change of argument of } f(z) \text{ along } \alpha)$ can also be interpreted as counting the number of times the closed curve $f(\alpha)$ winds around the origin.

For a general function f, a plot of the curve $f(\alpha)$ allows for a “reading” of $\frac{1}{2\pi}\Delta_\alpha \arg(f)$:

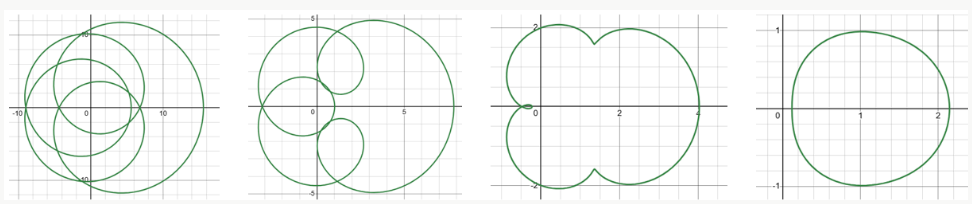


Particular case: if f is holomorphic on α and $\mathrm{int}(\alpha)$, then one easily obtains the number of zeros of f inside α by plotting $f(\alpha)$.

Ex 43. Consider $P(z) = 8z^4 + z^3 + \frac{8}{5}z^2 + 4z + 1$.

We can plot $\{P(re^{it}, t \in [0, 2\pi)\}$ for various values of r and count the number of windings.

- For $r = 1$: there are four windings, so all zeros of P have modulus < 1.
- For $r = \frac{3}{4}$: there are two windings, so two zeros lie in $\frac{3}{4} < |z| < 1$.
- For $r = \frac{1}{2}$: there is one winding, so one zero lies in $\frac{1}{2} < |z| < \frac{3}{4}$.
- For $r = \frac{1}{4}$: there are no windings, so the last zero lies in $\frac{1}{4} < |z| < \frac{1}{2}$.



Indeed, a numerical approximation yields the zeros as: $\{-0.6, -0.3, 0.38 \pm i0.75\}$.

In terms of modulus: $\{0.3, 0.6, 0.84, 0.84\}$.