PHYS 350 Formula Sheet

Lagrangian

Lagrangian:

$$\mathcal{L}(q,\dot q,t) = T - U$$
$$T = \frac{1}{2}m(\dot x^2+\dot y^2+\dot z^2)$$

Euler-Lagrange Equation:

$$\left(\frac{\partial \mathcal{L}}{\partial \dot q_i}\right)^{\bullet} = \frac{\partial \mathcal{L}}{\partial q_i}$$

Beltrami Identity ($\dot{\mathcal{L}} = 0$):

$$E = \sum_i \frac{\partial \mathcal{L}}{\partial \dot q_i}\dot q_i - \mathcal{L}$$

Hamiltonian

Hamiltonian Principle:

$$S = \int_{t_1}^{t_2} \mathcal{L}(q(t),\dot q(t),t)\,\mathrm{d}t$$

Light:

$$\cos(\theta_{\max}) = \frac{n_{\min}}{n_{\max}} \qquad n_1\sin(\theta_1) = n_2\sin(\theta_2)$$

Conjugate Momenta and Hamiltonian:

$$p_i := \frac{\partial \mathcal{L}}{\partial \dot q_i} \qquad H(q,p,t) = \sum_i \dot q_i p_i - \mathcal{L}$$

Solve $p_i = \frac{\partial \mathcal{L}}{\partial \dot q_i}$ for $\dot q_i$ and substitute

Hamilton's (Cannonical) Equations:

$$\dot q_i = \frac{\partial H}{\partial p_i} = \{q_i, H\} \qquad \dot p_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}$$

Poisson Bracket:

$$\{A(q,p),B(q,p)\} = \sum_i\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right)$$
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \{A,H\} + \frac{\partial A}{\partial t}$$
$$\{A,H\} = 0 \implies A \text{ is conserved}$$

Orbital Mechanics

Kinetic Energy:

$$T = \frac{1}{2}M|\dot{\boldsymbol{x}}_{\mathrm{CM}}|^2 + \frac{1}{2}\frac{m_1m_2}{m_1+m_2}|\dot{\boldsymbol{r}}|^2$$

Reduced Mass:

$$\mu = \frac{m_1m_2}{m_1+m_2}$$

Central Potential:

$$\mathcal{L} = \frac{1}{2}\mu|\dot{\boldsymbol{r}}|^2 - U(|\boldsymbol{r}|)$$

Effective Potential Energy:

$$E = \frac{1}{2}m\dot r^2 + \underbrace{\frac{L^2}{2mr^2} + U(r)}_{U_{\mathrm{eff}}(r)}$$

Gravitational Orbit:

$$U(r) = -\frac{\alpha}{r} = -\frac{GMm}{r}$$
$$r_{\min,\max} = \frac{\alpha \pm \sqrt{\alpha^2 - 2|E|L^2/m}}{2|E|}$$
$$\frac{b^2}{ar} = 1 + e\cos\theta \qquad e = \frac{c}{a}$$
$$a = \frac{\alpha}{2|E|} \qquad b = \frac{L}{\sqrt{2m|E|}} \qquad c^2 = a^2 - b^2$$
$$T = 2\pi a^{3/2}\sqrt{m/\alpha}$$

Integrals for Time and Angle:

$$t = \int_{r(0)}^{r(t)} \frac{\mathrm{d}r}{\dot r} = \int_{r(0)}^{r(t)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - U(r)\right)}}$$
$$\varphi(t) = \int_{r(0)}^{r(t)} \frac{\dot\varphi\,\mathrm{d}r}{\dot r} = \int_{r(0)}^{r(t)} \frac{L\,\mathrm{d}r}{r^2\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - U(r)\right)}}$$

Rigid Body and Rotations

Inertial Tensor:

$$I = \begin{bmatrix} \sum m_i(y_i^2+z_i^2) & -\sum m_ix_iy_i & -\sum m_ix_iz_i \\ -\sum m_ix_iy_i & \sum m_i(x_i^2+z_i^2) & -\sum m_iy_iz_i \\ -\sum m_ix_iz_i & -\sum m_iy_iz_i & \sum m_i(x_i^2+y_i^2)\end{bmatrix}$$
$$T = \frac{1}{2}MV^2 + \frac{1}{2}\Omega^T I\Omega = \frac{1}{2}mv^2 + \frac{1}{2}I\dot\theta^2$$

Angular Momentum:

$$\boldsymbol{L} = M\boldsymbol{X}\times\boldsymbol{V} + I\boldsymbol{\Omega}$$

Time Evolution of Inertial Tensor:

$$I(t) = R(t)I(0)R(t)^T$$

Parallel Axis Theorem:

$$I = I_{\mathrm{CM}} + Md^2$$

Torque:

$$\boldsymbol{\tau} = \dot{\boldsymbol{L}} = \boldsymbol{\Omega}\times\boldsymbol{L}(t) + I(t)\dot{\boldsymbol{\Omega}}$$

Euler Equations for Diagonal Tensor $I = (\lambda_1,\lambda_2,\lambda_3)$:

$$\lambda_1\dot\Omega_1 - (\lambda_2-\lambda_3)\Omega_2\Omega_3 = \tau_1$$
$$\lambda_2\dot\Omega_2 - (\lambda_3-\lambda_1)\Omega_3\Omega_1 = \tau_2$$
$$\lambda_3\dot\Omega_3 - (\lambda_1-\lambda_2)\Omega_1\Omega_2 = \tau_3$$

Precession Ω_p:

$$\dot{\boldsymbol{\Omega}} = \boldsymbol{\Omega}_p \times \boldsymbol{\Omega}$$

Kinetic Energy of Top with $I_1 = I_2$:

$$KE = \frac{1}{2}I_1(\sin^2\theta\dot\varphi^2 + \dot\theta^2) + \frac{1}{2}I_3(\dot\psi + \dot\varphi\cos\theta)^2$$

Coupled Oscillators

Equilibrium Position from $\mathcal{L}$:

$$\frac{\partial U}{\partial q_i} = 0$$

Quadratic Expansion:

$$q_i \to (q_i)_{eq} + x_{qi}$$

Matrices:

$$\mathcal{L} = \frac{1}{2}\dot x^T M\dot x - \frac{1}{2}x^T Kx$$
$$= \frac{1}{2}\sum_{ij} M_{ij}\dot x_i\dot x_j - \frac{1}{2}\sum_{ij}K_{ij}x_ix_j$$
$$K_{ij} = \frac{\partial U(q_0)}{\partial q_i\partial q_j}$$

Normal Modes $A_\pm = \langle 1, a_\pm\rangle$

$$\det(\lambda M - K) = 0 \qquad (\lambda M - K)A = 0$$
$$\omega_I = \sqrt{\lambda_I}$$

General Solution to $M\ddot x = -Kx$:

$$x(t) = \sum_{I=1}^{n-m}(c_i^+e^{i\omega_It} + c_i^-e^{-i\omega_It})A_I + \sum_{\alpha=1}^{m}(v_0^\alpha t + x_0^\alpha)A_{0,\alpha}$$

Miscellaneous

Spherical Coordinates:

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi \qquad v^2 = \dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\dot\varphi^2$$
$$z = r\cos\theta$$

Expansions and Identities

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad \sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots$$

Sum and Difference:

$$\sin(x\pm y) = \sin x\cos y \pm \cos x\sin y$$
$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$

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