MATH 305 Formula Sheet

Complex Numbers

Definition	$z = x + iy$
Conjugate	$\bar{z} = x - iy$
Modulus	$\lvert z\rvert = \sqrt{x^2 + y^2}$
Polar Form	$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$
Arguments	$\arg(z) \in [0, 2\pi)$
	$\mathrm{Arg}(z) \in (-\pi, \pi]$
Exponential	$e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$
Unity	$z = e^{\frac{2\pi i k}{n}} \iff z^n = 1$
Triangle	$\lvert z + w\rvert \leq \lvert z\rvert + \lvert w\rvert$
	$\lvert z - w\rvert \geq \lvert\lvert z\rvert - \lvert w\rvert\rvert$

Complex Functions

Translation by $\vec{w}$	$f(z) = z + w$
Rotation CCW by φ	$f(z) = e^{i\varphi} z$
Scaling by λ	$f(z) = \lambda z$
Reciprocal $\frac{1}{z}$	$\dot{B}_1(0) \mapsto \lvert z\rvert > 1$
	$B_1(1) \mapsto \mathrm{Re}(z) > \frac{1}{2}$

Limits, Continuity, Differentiability

Continuity	$\lim_{z \to z_0} f(z) = f(z_0)$
Differentiability	$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$ exists
Cauchy-Riemann	$\partial_x u(x, y) = \partial_y v(x, y)$
	$\partial_y u(x, y) = -\partial_x v(x, y)$
Harmonics	$f \in H(\Omega) \implies \Delta u = \Delta v = 0$
Branch Cuts	$\mathrm{Log}(z)$ along $(-\infty, 0]$
	$\log(z)$ along $[0, \infty)$

Elementary Functions

Trig	$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$
	$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$
Hyperbolic	$\cosh z = \frac{1}{2}(e^z + e^{-z})$
	$\sinh z = \frac{1}{2}(e^z - e^{-z})$
Logarithm	$\mathrm{Log}(z) = \ln\lvert z\rvert + i\mathrm{Arg}_{(-\pi,\pi]}(z)$
	$\log(z) = \ln\lvert z\rvert + i\arg_{[0,2\pi)}(z)$
Roots	$z^\alpha = e^{\alpha \mathrm{Log}(z)}$

Integration

Curve	$\alpha : [a, b] \to \mathbb{C}$
Contour Integral	$\int_\alpha f(z)\, dz = \int_a^b f(\alpha(t))\alpha'(t)\, dt$
Length	$\ell(\alpha) = \int_a^b \lvert\alpha'(t)\rvert\, dt$
Bound	$\left\lvert\int_\alpha f(z)\, dz\right\rvert \leq \ell(\alpha) \max_{z\in\alpha}\lvert f(z)\rvert$
Antiderivative	$F'(z) = f(z)$
Closed Integral	$\oint_\alpha f(z)\, dz = 0$
Loop Reciprocal	$\oint_\alpha \frac{a}{z - z_0}\, dz = 2\pi i a \,\forall\, \alpha$ around z_0
Cauchy Integral	$\oint_\alpha \frac{f(z)}{z - w}\, dz = 2\pi i f(w) \,\forall\, \alpha$ around w
Cauchy Derivative	$f^{(k)}(w) = \frac{k!}{2\pi i} \oint_\alpha \frac{f(z)}{(z - w)^{k+1}}\, dz$
MVP	$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{it})\, dt$

Analytic Functions

Geometric	$\sum_{n=0}^\infty z^n = \frac{1}{1-z}, \lvert z\rvert < 1$
Taylor	$f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$
Residue	$\mathrm{Res}(f, z_0) = a_{-1}$
Theorem	$\oint_\alpha f(z)\, dz = 2\pi i \sum_{j=1}^N \mathrm{Res}(f, z_j)$
Simple	$\mathrm{Res}(f, z_0) = \lim_{z\to z_0}(z - z_0)f(z)$
General	$\mathrm{Res} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]$
Derivative	$\mathrm{Res}(\frac{f(z)}{g(z)}, z_0[\,]) = \frac{f(z_0)}{g'(z_0)}$
Basel	$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$
Zeta	$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$
Trig	$\int_0^{2\pi} \frac{P(\sin\varphi, \cos\varphi)}{Q(\sin\varphi, \cos\varphi)}\, d\varphi$
	$\sin\varphi = \frac{1}{2i}(z - \frac{1}{z}), \quad z = e^{i\varphi}$
	$\cos\varphi = \frac{1}{2}(z + \frac{1}{z}), \quad z = e^{i\varphi}$
	$d\varphi = \frac{1}{iz}\, dz$
	$\int_0^{2\pi} \implies \oint_{\lvert z\rvert=1}$
Laurent	$f(z) = \sum_{n=-\infty}^\infty a_n(z - z_0)^n$
	$a_n = \frac{1}{2\pi i} \oint_{\lvert z - z_0\rvert = r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$
Regular	$\sum_{n=0}^\infty a_n(z - z_0)^n$
Singular	$\sum_{n=-\infty}^{-1} a_n(z - z_0)^n$
Argument	$\frac{1}{2\pi i} \oint_\alpha \frac{f'(z)}{f(z)}\, dz = \#\text{zeros} - \#\text{poles}$
	$\frac{f'(z)}{f(z)} = \frac{d}{dz}(\ln\lvert f(z)\rvert + i\mathrm{Arg}(f(z)))$
	$\oint_\alpha \frac{f'(z)}{f(z)}\, dz = i\Delta_\alpha \arg(f(z))$
	$\frac{1}{2\pi}\Delta_\alpha \arg(f(z)) = \#\text{zeros} - \#\text{poles}$

Series

Sine	$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} z^{2n+1}$
Cosine	$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} z^{2n}$
Exp	$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^\infty \frac{z^n}{n!}$
Log	$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} z^n$

Identities

$\sin^2 z + \cos^2 z = 1$

$\cosh^2 z - \sinh^2 z = 1$

$\sec^2 z = 1 + \tan^2 z$

$\sin 2z = 2 \sin z \cos z$

$\cos 2z = \cos^2 z - \sin^2 z$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

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Theorems and Definitions

1. (Limit) Consider $f : \Omega \to \mathbb{C}$. For $z_0 \in \Omega$, we write $\lim_{z \to z_0} f(z) = L$ if for all $\varepsilon > 0$, there exists a radius $\delta > 0$ such that $|f(z) - L| < \varepsilon$ for all z such that $|z - z_0| < \delta$.
2. (Continuity) f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.
3. (Differentiability) f is differentiable at z_0 if $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.
4. (Cauchy-Riemann) Let $f = u + iv$. Then f is differentiable at z_0 if and only if $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$ at z_0.
5. (Holomorphic) If f is differentiable for all $z \in \Omega$, then f is holomorphic on Ω and we write $f \in H(\Omega)$. If $f \in H(\mathbb{C})$ then f is entire.
6. (Harmonics) If $f \in H(\Omega)$, then $\Delta u(x, y) = \Delta v(x, y) = 0$.
7. (Curves) A smooth parameterized curve is a function $\alpha : [a, b] \to \mathbb{C}$ such that α' exists and is nonzero for $t \in [a, b]$.
8. (Closed and Simple) α is closed if $\alpha(a) = \alpha(b)$ and simple if $\alpha(t_1) \neq \alpha(t_2)$ for $a < t_1 < t_2 < b$.
9. (Fundamental Theorem of Calculus) Let $\alpha \in \Omega$. If F is an antiderivative of f in a neighbourhood of α, then $\int_\alpha f(z)\, dz = F(\alpha(b)) - F(\alpha(a))$.
10. (Closed Integral) If f has an antiderivative and α is a closed curve, then $\oint_\alpha f(z)\, dz = 0$.
11. (Cauchy's Theorem) If f is holomorphic in a simply connected domain Ω, then $\oint_\alpha f(z)\, dz = 0$ for all simple closed curves $\alpha \in \Omega$.
12. (Cauchy's Integral Formula) Let α be a simple closed curve and f be holomorphic on α and its interior. Then $\oint_\alpha \frac{f(z)}{z-w}\, dz = 2\pi i f(w)$ for all w inside α.
13. (Mean Value Property) For a holomorphic function $f(w)$, we have $f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{it})\, dt$.
14. (Maximum Modulus Principle) If f is a non-constant holomorphic function on a domain Ω, then $|f|$ cannot reach a local maximum in Ω.
15. (Liouville's Theorem) If f is entire and bounded, then it is constant.
16. (Rouche's Theorem) Let f be holomorphic on α and its interior. If $|f(z) - 1| < 1$ for all $f \in \alpha$, then $f(z) \neq 0$ for all z in the interior of α.
17. (Fundamental Theorem of Algebra) A polynomial of degree n has exactly n roots in $\mathbb{C}$.
18. (Analyticity) If $f \in H(\Omega)$, then f is equal to its Taylor series at $z_0 \in \Omega$.
19. (Identity Theorem) Let Ω be a domain and α be a curve in Ω. If $f, g \in H(\Omega)$ are so that $f(z) = g(z)$ for all $z \in \alpha$, then $f(z) = g(z)$ for all $z \in \Omega$.
20. (Residue Theorem) $\oint_\alpha f(z)\, dz = 2\pi i \sum_{j=1}^{N} \mathrm{Res}(f, z_j)$
21. (Argument Principle) If f is meromorphic in Ω and α is a positively oriented simple closed curve in Ω such that $\mathrm{int}(\alpha) \subset \Omega$ and f has no zeros or poles on α, then $\frac{1}{2\pi i} \oint_\alpha \frac{f'(z)}{f(z)}\, dz = (\#\text{ zeros in } \mathrm{int}(\alpha)) - (\#\text{ poles in } \mathrm{int}(\alpha))$ where zeros and poles are counted with their order.

Quick Definitions

1. (Bounded) $\Omega \subset \mathbb{C}$ is bounded if there exists $r > 0$ such that $|z| \leq r$ for all $z \in \Omega$.
2. (Open) $\Omega \subset \mathbb{C}$ is open if for all $z \in \Omega$, there exists $r > 0$ such that $B_r(z) \subset \Omega$.
3. (Connected) $\Omega \subset \mathbb{C}$ is connected if there is a continuous path between any two points in Ω.
4. (Simply-Connected) $\Omega \subset \mathbb{C}$ is simply-connected if it is connected and every closed curve in Ω can be shrunk to a point.
5. (Branch Cuts) A branch cut is a curve on which a function is discontinuous.
6. (Zero) A point $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$ is a zero of f. z_0 is a zero of order m if $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$.
7. (Pole) f has a pole of order m at z_0 if $\frac{1}{f}$ is holomorphic in $B_r(z_0)$ and z_0 is a zero of order m of $\frac{1}{f}$. If $m = 1$, then z_0 is a simple pole.
8. (Meromorphic) If $f \in H(\Omega \setminus \{z_1, \ldots, z_n\})$, then f is meromorphic, as in it has finite poles.
9. (Residue) Let $f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z - z_0) + \cdots$. Then the coefficient a_{-1} is called the residue of f at z_0.
10. (Essential) If f has an essential singularity at z_0, then z_0 is a pole of infinite order.
11. (Laurent Series) $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ is a Laurent series. $\sum_{n=-\infty}^{-1} a_n(z - z_0)^n$ is the singular part of the series, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is the regular part of the series