PHYS 301 Formula Sheet

Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \oint \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \qquad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

Electrostatics

Electric Field

Coulomb's law:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}$$

Electric field:

$$\mathbf{F} = Q\mathbf{E} \qquad \mathbf{E} = -\nabla V$$

E-field due to point charges:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2} \hat{\boldsymbol{\mathscr{r}}}_i$$

E-field due to continuous charge distribution:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dq = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau'$$

Gauss's law:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Electric Potential

$$V(b) - V(a) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}$$

Poisson's equation:

$$\nabla^2 V = -\rho/\epsilon_0$$

Potential due to continuous charge distribution:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} dq = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

Work and Energy in Electrostatics

Energy of point charges:

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i)$$

Energy of continuous charge distribution:

$$W = \frac{1}{2} \int \rho V \, d\tau \qquad W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

Conductors

Electric field immediately outside a conductor:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$$

Surface charge:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

Capacitors:

$$C = \epsilon_0 \frac{A}{d} \qquad Q = CV$$

$$W = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} QV$$

Potentials

Laplace's equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Separation of Variables

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ a/2 & \text{if } n = m \end{cases}$$

Legendre polynomials:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = (3x^2 - 1)/2$
- $P_3(x) = (5x^3 - 3x)/2$

Solution to Laplace in spherical (ϕ independent):

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Multipole Expansion

Axial multipole (α is between $\mathbf{r}$ and $\mathbf{r}'$):

$$\frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha) \rho(\mathbf{r}') d\tau'$$

Monopole, Dipole, Quadrupole:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \qquad Q = \int \rho(\mathbf{r}') d\tau'$$

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \qquad \mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d\tau'$$

$$V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{ij} \frac{Q_{ij} \hat{r}_i \hat{r}_j}{r^2}$$

$$Q_{ij} = \int \frac{\rho(\mathbf{r}')}{2} \left(3r_i' r_j' - (r')^2 \delta_{ij} \right) d\tau'$$

Electric Fields in Matter

Dipole torque, force, and energy:

$$\mathbf{N} = \mathbf{p} \times \mathbf{E} \qquad \mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E}) \qquad U = -\mathbf{p} \cdot \mathbf{E}$$

Bound charges:

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \rho_b = -\nabla \cdot \mathbf{P}$$

Electric displacement:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f,\text{enc}}$$

Linear Dielectrics

Polarization:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

Electric displacement:

$$\mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon \mathbf{E}$$

Energy:

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau$$

Boundary Conditions in Electrostatics

- $E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \sigma/\epsilon_0$
- $E^{\parallel}_{\text{above}} = E^{\parallel}_{\text{below}}$
- $V_{\text{above}} = V_{\text{below}}$
- $D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$
- $D^{\parallel}_{\text{above}} - D^{\parallel}_{\text{below}} = P^{\parallel}_{\text{above}} - P^{\parallel}_{\text{below}}$

Magnetostatics

Lorentz force law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Currents:

$$\mathbf{I} = \lambda \mathbf{v} \qquad \mathbf{K} = \sigma \mathbf{v} \qquad \mathbf{J} = \rho \mathbf{v}$$

Magnetic force on wire:

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B})$$

Continuity equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Biot-Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dl' = \frac{\mu_0}{4\pi} \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

Ampere's law:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

Magnetic Vector Potential

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \quad \text{where} \quad \boldsymbol{\nabla} \cdot \mathbf{A} = 0$$

Vector potential Poisson's equation:

$$\boldsymbol{\nabla}^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Vector potential when $\mathbf{J} \to \mathbf{0}$ at infinity:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$$

Multipole expansion of a current loop:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\alpha) d\mathbf{l}'$$

Magnetic dipole with vector area $\mathbf{a}$:

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{m} = I \int d\mathbf{a} = I\mathbf{a}$$

Magnetic Fields in Matter

Dipole torque, force, and energy:

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \quad \mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}) \quad U = -\mathbf{m} \cdot \mathbf{B}$$

Bound currents:

$$\mathbf{J}_B = \boldsymbol{\nabla} \times \mathbf{M} \qquad \mathbf{K}_B = \mathbf{M} \times \hat{\mathbf{n}}$$

Auxiliary field:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad \oint \mathbf{H} \cdot d\mathbf{l} = I_{f,\text{enc}}$$

Linear Media

Magnetization:

$$\mathbf{M} = \chi_m \mathbf{H}$$

Auxiliary field:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H}$$

Boundary Conditions in Magnetostatics

- $B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$
- $B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$
- $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$
- $H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$
- $H^{\parallel}_{\text{above}} - H^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}$

Miscellaneous Formulas

Electric field of dipole:

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{\mathbf{p}}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Electrodynamics

Force per unit charge:

$$\mathbf{f} = \mathbf{v} \times \mathbf{B}$$

Electromotive force:

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} \qquad \mathcal{E} = vBh$$

Flux rule:

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Vector Derivatives

Cartesian

Gradient:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\boldsymbol{\nabla} \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Spherical

Gradient:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl:

$$\boldsymbol{\nabla} \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$

$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Cylindrical

Gradient:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\boldsymbol{\nabla} \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}}$$

$$+\left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Fundamental Theorems

Gradient theorem:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\boldsymbol{\nabla} f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

Divergence theorem:

$$\int (\boldsymbol{\nabla} \cdot A) d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$$

Stoke's theorem:

$$\int (\boldsymbol{\nabla} \times A) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$$

Spherical Coordinates

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$\begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical Coordinates

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases}$$

$$\begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

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