Part 1: Green's Functions of Linear ODE

W1C1 Lecture 1 (Jan 7)

1.1 Green's Functions

The idea of Green's functions works for all linear equations:

- Linear systems
- Linear ODE
- Linear PDE

💡 **Ex 1.** Consider the linear system $Au = f$ **(Eq. 1)**, where A is an $n \times n$ matrix, $u \in \mathbb{R}^n$ (solution) and $f \in \mathbb{R}^n$ (data/source).

If A is invertible, let $A^{-1} = G$, then $u = Gf$.

Component wise, write
$u = (u_i)_{i=1}^n$ and $f = (f_j)_{j=1}^n$ and $G = (G_{ij})_{i,j=1}^n$. Then $u_i = \sum_{j=1}^n G_{ij} f_j$ **(Eq. 2)**.

We can call
G the Green's function of $Au = f$.

Recall the standard basis
$e_j = (\cdots 0 \cdots, \underbrace{1}_{j^{\text{th}}}, \cdots 0 \cdots)^\intercal$.

If
$f = e_j$ for a fixed j, then $u = u^{(j)} = (G_{ij})_{i=1}^n =$ jth column of G. This is the solution if data is concentrated at j.

$$G = (u^{(1)} \mid u^{(2)} \mid \cdots \mid u^{(n)})$$

We can write
$u = \sum_{j=1}^n f_j u^{(j)}$ as a linear combination of $u^{(j)}$.

For ODEs, we replace sums with integration, and G_{ij} with $G(x, y)$.

💡 **Ex 2.** Consider the ODE BVP

$\begin{cases} u'' = f & 0 < x < 1 \\ u(0) = 0, & u(1) = 0 \end{cases}$ **(Eq. 3)**.

It cannot be solved as an IVP because we do not know $u'(0)$. Show that the system has a unique solution.

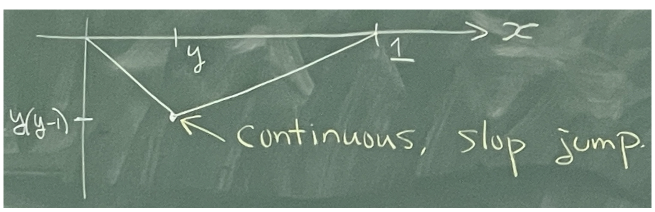
<u>**Uniqueness:**</u> consider two solutions u_1, u_2 for the same f, so $u_1'' = u_2'' = f$ and $u_1 = u_2 = 0$ at $x = 0, 1$.

Let
$w = u_1 - u_2$, so $w'' = u_1'' - u_2'' = f - f = 0$. Hence $w = ax + b$.

$w(0) = 0$ implies that $b = 0$, and $w(1) = 0$ implies that $a = 0$. Therefore $w = 0$ as required.

<u>**Existence:**</u> we claim that the solution is $u(x) = \int_0^1 G(x,y)f(y)\,\mathrm{d}y$ **(Eq. 4)** where $G(x,y) =$
$\begin{cases} y(x-1) & 0 \le y \le x \le 1 \text{ (I)} \\ x(y-1) & 0 \le x \le y \le 1 \text{ (II)} \end{cases}$

For a fixed
$y \in (0,1)$, the graph of G is as shown:



We now check that u is a solution to the BVP.

$u(0) = 0$ because $G(0,y) = 0$.

$u(1) = 0$ because $G(1,y) = 0$.

We have
$u(x) = \int_0^x y(x-1)f(y)\,\mathrm{d}y + \int_x^1 x(y-1)f(y)\,\mathrm{d}y$.

Recall:
<u>Fundamental Theorem of Calculus:</u> $\frac{d}{dx}\int_0^x g(y)\,dy = g(x)$ and $\frac{d}{dx}\int_x^1 g(y)dy = -g(x)$.

$u'(x) = \cancel{\left(y(x-1)f(y)\right)_{y=x}} - \cancel{\left(x(y-1)f(y)\right)_{y=x}} + \int_0^x yf(y)\,dy + \int_x^1 (y-1)f(y)\,dy$

$u''(x) = \big(yf(y) - (y-1)f(y)\big)_{y=x} = f(x)$ as desired.

Remarks:

i. **Eq. 4** is similar to **Eq. 4**, but sum → integral and G_{ij} → $G(x, y)$.

ii. $G(x, y) = G(y, x)$, so G is symmetric.

iii. $G(x, y)$ would be the solution u if f is concentrated at location y. The general solution is the linear combination of those solutions with source concentrated at one point (delta function).

Benefits of Green's functions:

1. It is clear how force at one point y affects the solutions at all points x.
2. It is easier to implement numerically than solving the DE
3. Analytically, it can be used to derive qualitative properties of the solution

HW 1 Problem: Show that if $f(x)$ is bounded, that is, $|f(x)| \le M$ for $0 \le x \le 1$, then the solution $u(x)$ of Exercise 2 satisfies $|u(x)| \le Cx$ for $0 < x < \frac{1}{2}$.

W1C2 Lecture 2 (Jan 9)

1.2 Generalized Functions

📌 **Def.**

a. Let Ω be an open set in $\mathbb{R}^n$. A function ϕ on Ω, $\phi \in C^k(\Omega)$ if all derivatives of ϕ up to order k exist and are continuous in Ω.

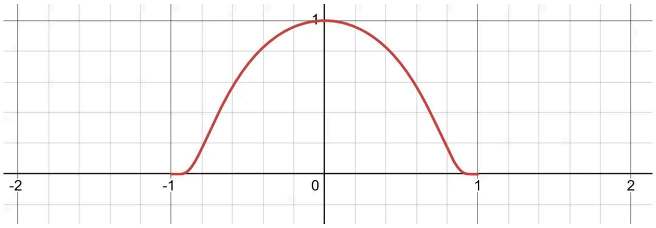
b. $\phi \in C^k_C(\Omega)$ if it has compact support in Ω, as in $\operatorname{supp}(\phi) = \{x : \phi(x) \neq 0\} \ll \Omega$. We have $\operatorname{dist}(\operatorname{supp}(\phi), \partial\Omega) > 0$.

c. $\phi \in C^\infty(\Omega)$ if $\phi \in C^k(\Omega)\, \forall\, k$.

$\phi \in C^\infty_C(\Omega)$ if $\phi \in C^k_C(\Omega)\, \forall\, k$. We call these functions test functions.

💡 **Ex 3.** Consider

$$\phi(x) = \begin{cases} e^{-\frac{x^2}{1-x^2}} = e^{1-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$



- monotone

Claim: $\phi^{(n)}(1_-) = 0 \, \forall \, n$

Proof: $\phi^{(n)}(x) = \dfrac{\text{polynomial}}{(1-x^2)^m} e^{-\frac{1}{1-x^2}} \sim y^m e^{-y}$ where $y = \frac{1}{1-x^2}$

Then as $x \to 1_-$, $y \to +\infty$ so $y^m e^{-y} \to 0$ as claimed.

Hence $\phi \in C_C^\infty(\mathbb{R})$, $\phi \in C_C^\infty((-2,2))$, $\phi \notin C_C^\infty((-2,2))$, $\text{supp}(\phi) = [-1,1]$

Let $A = \int_{-1}^{1} \phi(x) \, dx$ and define $\psi(x) = \frac{1}{A}\phi(x)$. For $\varepsilon > 0$, define $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi(\frac{x}{\varepsilon})$.

📌 **Def.** $\psi_\varepsilon \in C_C^\infty$, $\int_{-\infty}^{\infty} \psi_\varepsilon(x) \, dx = 1$ for all $\varepsilon > 0$. They are examples of mollifiers.

📌 **Def.** given $f(x), g(x)$, the convolution is defined as $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy$.

Remarks:

i. If $f(x) = 0 = g(x)$ for $x < 0$, then $f * g(x) = \int_0^x f(x-y)g(y) \, dy$ for $x > 0$.

ii. $(f * g)' = f * g'$. In general, $(f * g)^{(k)} = f * (g^{(k)})$

📖 **Thm.** Convolution with mollifiers gives smooth approximations. For a given $f(x)$, let $f_\varepsilon(x) = f * \psi_\varepsilon(x)$.

- $f_\varepsilon \in C^\infty(\mathbb{R})$, since $f_\varepsilon^{(k)} = f * (\psi_\varepsilon)^{(k)}$
- $f_\varepsilon \to f$ as $\varepsilon \to 0$ in many senses (see below).

For example, if $f \in L^2(\mathbb{R})$ (square integrable), we have $||f||_{L^2(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |f(x)|^2 \, dx\right)^{\frac{1}{2}}_{<\infty}$ then $f_\varepsilon \in L^2$ and $||f_\varepsilon - f||_{L^2(\mathbb{R})} \to 0$ as $\varepsilon \to 0_+$.

Consider L^q if $q < \infty$, we have $||f||_{L^q} = \left(\int_{-\infty}^{\infty} |f(x)|^q \, dx\right)^{\frac{1}{q}}$.

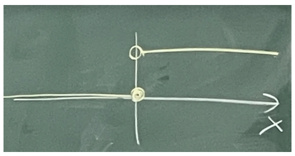
We have $||f||_{L^\infty} = \text{esssup}_{x \in \mathbb{R}} |f(x)|$.

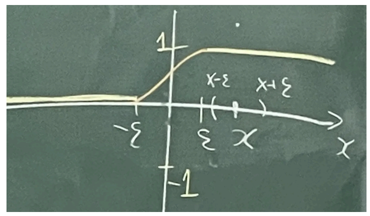
- Essential: forget measure 0 points
- Supremum ~ max

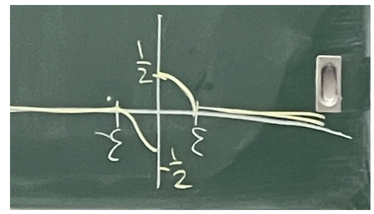
📌 **Def.** Recall the Heaviside function:

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Its mollification is $H_\varepsilon(x) = H * \psi_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} H(x-y)\psi_\varepsilon(y)\,dy$







We have $||H - H_\varepsilon||_{L^q} \begin{cases} \to 0 & \text{if } q < \infty \\ = \frac{1}{2} & \text{if } q = \infty \end{cases}$

📌 **Def.** A generalized function (a distribution) is a linear functional (maps functions to functions) on $C_C^\infty(\Omega)$ where $\Omega = \mathbb{R}^n$ or its open subsets.

$f : \phi \in C_C^\infty(\Omega) \mapsto f(\phi) \in \mathbb{R}$

Linear: $f(a\phi + b\psi) = af(\phi) + bf(\psi)$

Other notation: $f(\phi) = f[\phi] = (f, \phi) = \langle f, \phi \rangle$

💡 **Ex 4.**

a. Any integrable function $f(x)$ defines a distribution by

   $f(\phi) = (f, \phi) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx \,\forall\, \phi \in C_C^\infty$

   Remark: (f, ϕ) is the L^2 inner product if $f, \phi \in L^2(\mathbb{R})$

b. $f(x) = \begin{cases} \frac{1}{|x|} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$ is not a distribution because $\int_{-\infty}^{\infty} \frac{1}{|x|}\phi(x)\,dx$ is not defined if $\phi \in C_C^\infty, \phi(0) \neq 0$.

💡 **Ex 5.** The Dirac δ function is a distribution

$\langle \delta, \phi \rangle = \phi(0) \,\forall\, \phi \in C_C^\infty$.

For a fixed y, $\delta y(x) = \delta(x-y)$ and $\langle \delta y, \phi \rangle = \phi(y)$.

Operations on distributions:

i. (multiplication by smooth functions) If f is a distribution, $a(x) \in C^\infty(\mathbb{R})$, then $\langle af, \phi \rangle = \langle f, a(x)\phi \rangle$. Hence $\forall \phi \in C_C^\infty \implies a\phi \in C_C^\infty$.

ii. (derivatives) By IBP, $\int_{-\infty}^{\infty} f' \phi \, dx = \cancel{[f\phi]_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} f \phi' \, dx$.

For a distribution f, $\langle f', \phi \rangle := -\langle f, \phi' \rangle \,\forall\, \phi \in C_C^\infty$

📌 **Def.** A sequence of distributions $f_k, k \in \mathbb{N}$ is said to converge weakly to a distribution f if $\lim_{k\to\infty} \langle f_k, \phi \rangle = \langle f, \phi \rangle \,\forall\, \phi \in C_C^\infty$.

i. (limits)

💡 **Ex 6.**

a. $\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0) \,\forall\, \phi \in C_C^\infty$

b. If $F(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$ then we claim that $F' = H$ (weak derivative).

c. What is H'?

b): $\langle F', \phi \rangle = -\langle F, \phi' \rangle = -\int_{-\infty}^{\infty} F(x)\phi'(x)\, dx = -\int_0^\infty x\phi'(x)\, dx = \cancel{-[x\phi(x)]_0^\infty} + \int_0^\infty \phi(x)\, dx = \int_{-\infty}^{\infty} H(x)\phi(x)\, dx = \langle H, \phi \rangle$.

c): $\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty 1\phi'(x)\, dx = \cancel{-\phi(\infty)} + \phi(0) = \langle \delta, \phi \rangle$. Hence, $H' = \delta$

💡 **Ex 7.** The mollifiers ψ_ε converges weakly to the dirac δ function.

$\langle \psi_\varepsilon, \phi \rangle = \int_{-\infty}^{\infty} \psi_\varepsilon(x)\phi(x)\, dx \to \phi(0) = \langle \delta, \phi \rangle$.

Remark: the rectangular pulse seen in ODE courses also converges to the dirac δ function.

💡 **Ex 8.** For $f_k(x) = \cos kx$, its oscillation increases as $k \to \infty$. Examine $\langle f_k, \phi \rangle$.

$\langle f_k, \phi \rangle = \int_{-\infty}^{\infty} \cos(kx)\phi(x)\, dx \to 0$ as $k \to \infty$, due to cancellation (Riemann Lebesque theorem). f_k converges weakly to 0.

Note that f_k has no pointwise limit, but it has a weak limit.

W2C1 Lecture 3 (Jan 14)

1.3 BVP of ODE

- Consider $Lu = a(x)u'' + b(x)u' + c(x)u = f$ **(Eq. 6)** for $0 < x < T$ with boundary conditions

  $\alpha_0 u(0) + \beta_0 u'(0) = 0, \quad \alpha_1 u(T) + \beta_1 u'(T) = 0$ **(Eq. 7)**

  where $a(x) \neq 0, b(x), c(x), \alpha_i, \beta_i$ are given, and α_i, β_i are not both 0.

  We also write $L = a\partial_x^2 + b\partial_x + c$.

💡 **Ex 9.** (homework) steady state temperature along a rod $[0, T]$ with thermal conductivity $\rho(x) > 0$ and heat source $f(x)$ with end point temperatures fixed.

$(\rho(x)u')' = f, \quad u(0) = u(T) = 0$

📌 **Def.** Consider a dual BVP motivated by IVP:

For $u, v \in C^2([0, T])$ - recall the L^2 inner product $(u, v) = \int_0^T u(x)v(x)\,dx$.

$$\begin{aligned}
(Lu, v) &= \int_0^T (au'' + bu' + cu)v\,dx \\
&= \int_0^T -u'(av)' - u(bv)' + cuv\,dx + [u'av + buv]_0^T \\
&= \int_0^T u\underbrace{[(av)'' - (bv)' + cv]}_{L^*v}\,dx + \underbrace{[u'av + buv - u(av)']_0^T}_{\text{B.T (boundary term)}} \\
&= (u, L^*v) + \text{B.T.}
\end{aligned}$$

We call L^* the adjoint operator of L. We have $(Lu, v) = (u, L^*v)$ if boundary terms $\text{B.T.} = 0$ which is achieved if v satisfies suitable boundary conditions.

For simplicity, at $x = 0$, let $(\alpha, \beta) = (\alpha_0, \beta_0)$ so we have $\alpha u + \beta u' = 0$.

Hence $\exists k$ such that $u = -\beta k, u' = \alpha k$

$$\begin{aligned}
\text{B.T.} &= u'av + buv - u(av)' \\
&= k[\alpha av - b\beta v + \beta(a'v + av')] \\
&= k[\underbrace{(\alpha a - b\beta + \beta a')}_{\tilde{\alpha}}v + [\underbrace{\beta a}_{\tilde{\beta}}\, v']
\end{aligned}$$

This equals 0 if $\tilde{\alpha}v + \tilde{\beta}v' = 0$ **(Eq. 8)**. This is the dual boundary condition.

The dual BVP of **(Eq. 6&7)** is $L^*v = g$ with dual BC **(Eq. 8)**

Remark (matrix analog): let $x, y \in \mathbb{R}^n$, $A = [n \times n]$, then

- $(x, y) = x \cdot y = x^T y$

- $(x, Ay) = x^T Ay = (A^T x)^T y = (A^T x, y)$ and the transpose A^T is the adjoint operator of A

> 📌 **Def.** An operator L is self-adjoint if $L = L^*$ (including same B.C.)

For our $L = a\partial_x^2 + b\partial_x + c$,

$L^* = (av)'' - (bv)' + cv = av'' + 2a'v' + a''v - bv' - b'v + cv$

To have $L = L^* \iff \begin{cases} b = 2a' - b \\ c = a'' - b' + c \end{cases} \iff a' = b$

This is called a Sturm-Liouville operator.

In this case, $Lu = (au')' + cu$. We claim that the dual BC is equal to the BC.

$\begin{cases} \tilde{\alpha} = a\alpha - \cancel{b\beta} + \cancel{a'\beta} \\ \tilde{\beta} = a\beta \end{cases}$

Remark: $(Lu, v) = \underbrace{\int_0^T -au'v' + cuv\,dx}_{\text{symmetric}} + \underbrace{[au'v]_0^T}_{\text{symmetric, need to check}=[av'u]_0^T} = (u, Lv)$

1.4 Green's Functions for ODE BVP

We want to solve **(Eq. 6&7)** with a formula

$u(x) = \int_0^T G(x, y) f(y)\,dy$.

Treat x as a parameter (during IBP in y) and denote $G^x(y) = G(x, y)$.

Then $u(x) = \int_0^T G^x(y) f(y)\,dy = (G^x, f) = (G^x, Lu) = (L^* G^x, u) + \text{B.T.}$

We need G^x to satisfy:

- Dual BC **(Eq. 8)** such that B.T. = 0
- $L^* G^x = \delta_x = \delta(y - x)$

i.e. G^x solves the dual BVP with force δ_x.

> 💡 **Ex 10.** Consider the BVP **(Eq. 3&4)** in Ex2: $u'' = f \quad (0 < x < 1), \quad u(0) = 0 = u(1)$

Then $Lu = u'', \quad L = \partial_x^2$ and $a(x) = 1, b(x) = 0, a' = b$ (self adjoint).

The Green's function $G(x, y) = G^x(y)$ satisfies (for a fixed $0 < x < 1$)

- $\partial_y^2 G^x(y) = \delta_x(y) \quad 0 < y < 1$
- $G^x(0) = G^x(1) = 0$

Remark: now L is understood as acting on distributions as $G^x(y)$ can not be in C_y^2

For $0 < y < x$ or $x < y < 1$, $\partial_y^2 G^x = 0$, so then

$$G^x(y) = \begin{cases} Ay + B & 0 < y < x \\ Cy + D & x < y < 1 \end{cases}$$ where A, B, C, D depend on x

By $G^x(0) = 0 \implies B = 0$

By $G^x(1) = 0 \implies C + D = 0$

Hence $G^x(y) = \begin{cases} Ay & 0 < y < x \\ C(y-1) & x < y < 1 \end{cases}$

To decide A and C, we use two conditions:

1. (continuity) $G^x(x_-) = G^x(x_+)$ otherwise $\partial_y^2 G^x$ would be more singular than δ_1. Recall **Ex 6.** where the Heaviside function $H(x)$ is not continuous at 0, and $H''(x) = \delta'$ is more singular than δ.

   Hence $Ax = C(x-1)$

2. (jump) $\partial G^x(x+\varepsilon) - \partial G^x(x-\varepsilon) = \int_{x-\varepsilon}^{x+\varepsilon} \partial^2 G^x \, dy = \int_{x-\varepsilon}^{x+\varepsilon} \delta_x \, dy = 1$

   Hence $C - A = 1$

Solving gives $(C-1)x = C(x-1) \implies C = x, A = x - 1$

$$G^x(y) = \begin{cases} (x-1)y & 0 < y < x \\ x(y-1) & x < y < 1 \end{cases}$$

This is how the Green's function in **Ex 2.** is derived.

> 💡 **Ex 11.** Use the Green's function method to solve
>
> $x^2 u''(x) + 2xu'(x) - 2u(x) = f(x), \quad 0 < x < 1$
>
> with boundary conditions
>
> $u(0) = u(1) = 0.$
>
> *Remark:* if $u(x)$ is too wild, then f has to satisfy the compatibility condition $f(0) = 0$

Write $L = x^2\partial_x^2 + 2x\partial_x - 2$ which is self-adjoint since $ax = x^2, a' = 2x = b(x)$. Hence $G^x(y)$ satisfies

$$\begin{cases} LG^x = \partial_x \\ G^x(0) = G^x(1) = 0 \end{cases}$$

For $y \neq x$, $L_y G^x = 0$. This is an Euler ODE, so try y^r, then

$$\begin{aligned} y^2 r(r-1)y^{r-2} + 2yry^{r-1} - 2y^r &= 0 \\ r^2 - r + 2r - 2 &= 0 \\ (r+2)(r-1) &= 0 \end{aligned}$$

which gives y and $\frac{1}{y^2}$. Hence

$$G^x(y) = \begin{cases} Ay + \frac{B}{y^2} & 0 < y < x \\ Cy + \frac{D}{y^2} & x < y < 1 \end{cases}$$

Boundary conditions give

$G^x(0) = 0 \implies B = 0$

$G^x(1) = 0 \implies C + D = 0$

Hence

$$G^x(y) = \begin{cases} Ay & 0 < y < x \\ C(y - \frac{1}{y^2}) & x < y < 1 \end{cases}$$

To solve for A and C, we use the same two conditions:

1. (continuity) $Ax = C(x - \frac{1}{x^2})$
2. (jump)

$$\begin{aligned} y^2\partial_y^2 G^x + 2y\partial_y G^x - 2G^x &= \delta_x \\ x^2(\partial_y G^x(x_+) - \partial_y G^x(x_-)) &= 1 \\ x^2(C(1 + \tfrac{2}{x^3}) - A) &= 1 \end{aligned}$$

Solving gives $C(x^2 + \frac{2}{x}) - C(x^2 - \frac{1}{x}) = 1 \implies C = \frac{x}{3}, A = \frac{1}{3}(x - \frac{1}{x^2})$

$$G^x(y) = \begin{cases} \frac{1}{3}(x - \frac{1}{x^2})y & 0 < y < x \\ \frac{1}{3}x(y - \frac{1}{y^2}) & x < y < 1 \end{cases}$$

and the solution is given by

$u(x) = \int_0^1 G^x(y)f(y)\,dy = \int_0^x \frac{1}{3}(x - \frac{1}{x^2})yf(y)\,dy + \int_x^1 \frac{1}{3}x(y - \frac{1}{y^2})f(y)\,dy$

Remark: we can easily check that $Lu = f$ and $u(1) = 0$. It is harder to check $u(0) = 0$ due to the singularities in the integrands.

Let's check that $\lim_{x \to 0_+} u(x) = 0$, assuming that $f(y)$ is continuous.

Take a small $\varepsilon > 0$ so that for $0 < x < \varepsilon$,

$3u(x) \approx -\frac{1}{x^2}\int_0^x yf(y)\,dy - x\int_x^1 \frac{1}{y^2}f(y)\,dy + O(1)$ where $O(1) \to 0$ as $\varepsilon \to 0$.

Since f is continuous, $\int_x^1 \frac{1}{y^2}f(y)\,dy = \int_x^\varepsilon \frac{1}{y^2}\underbrace{f(y)}_{f(0)+O(1)}\,dy + \underbrace{\int_\varepsilon^1 \frac{f(y)}{y^2}\,dy}_{C_\varepsilon,\text{ large}}$

$$\begin{aligned} 3u(x) &= -\tfrac{1}{2}f(0) + O(1) - x\{(f(0) + O(1))(-\tfrac{1}{\varepsilon} + \tfrac{1}{x}) + C_\varepsilon\} \\ &= -\tfrac{3}{2}f(0) + (f(0) + O(1))\tfrac{x}{\varepsilon} + xC_\varepsilon \end{aligned}$$

For small ε, we have $\lim_{x \to 0} 3u(x) = -\frac{3}{2}f(0)$ so we need $f(0) = 0$.

W2C2 Lecture 4 (Jan 16)

💡 **Ex 12.** (Nonhomogeneous BC). Consider

$$\begin{cases} u'' = f(x) & 0 < x < 1 \\ u(0) = 0, & u(1) = \gamma \end{cases}$$

We can use the Green's function for the homogeneous BVP from **Ex 10.**

$$G(x,y) = G^x(y) = \begin{cases} y(x-1) & 0 < y < x \\ x(y-1) & x < y < 1 \end{cases}.$$

$$\begin{aligned} u(x) &= (u, \delta_x) = (u, \partial_y^2 G^x) \\ &= -(u', \partial_y G^x) + [u \partial_y G^x]_0^1 \\ &= (u'', G^x) + [u\partial_y G^x - u' G^x]_0^1 \\ &= (f, G^x) + \gamma \partial_y G^x(1) \\ &= \int_0^1 G(x,y) f(y)\, dy + \gamma x \end{aligned}$$

> 💡 **Ex 13.** Consider a Sturm-Liouville BVP
>
> $$\begin{cases} Lu = (a(x)u')' + c(x)u = f & 0 < x < 1 \\ \alpha_0 u(0) + \beta_0 u'(0) = 0, & \alpha_1 u(1) + \beta_1 u'(1) = 0 \end{cases}$$
>
> with $a(x) > 0$.

It is self-adjoint. The Green's function is

$(a\partial G^x)' + cG^x = \delta_x + BC$ at $y = 0, 1$.

For $j = 0, 1$, fix any $w_j(y) \neq 0$ (independet of x) and consider

$\begin{cases} Lw_j = 0 \\ \alpha_j w_j(j) + \beta_j w_j'(j) = 0 \end{cases}$, so w_j satisfies the boundary conditions at $y = j$.

This is a 2nd order linear ODE IVP with one BC, so there is a 1 dimensional family of solutions.

For $y \neq x$, $LG^x = 0$, so

$$G^x(y) = \begin{cases} c_0 w_0(y) & 0 \leq y < x \\ c_1 w_1(y) & x < y \leq 1 \end{cases}$$

for some $c_j \neq 0$ to be determined by matching conditions.

1. continuity at $y = x$: $c_0 w_0(x) = c_1 w_1(x)$
2. jump condition at $y = x$:

   $a(x)[\partial G^x(x_+) - \partial G^x(x_-)] = 1$

   $c_1 w_1'(x) - c_0 w_0'(x) = \frac{1}{a(x)}$

Solving gives $c_1 \underbrace{(w_0 w_1' - w_0' w_1)}_{W} = \frac{w_0}{a}$ at x.

> 📌 **Def.** The <u>Wronskian</u> is $W = W[w_0, w_1](x) = \begin{vmatrix} w_0(x) & w_0'(x) \\ w_1(x) & w_1'(x) \end{vmatrix}$.

📖 **Lemma.** For solutions w_0, w_1 of $Lw = 0$, $a(y)W(y)$ is constant in y.

> *Proof:*
>
> $$\begin{aligned}\tfrac{d}{dy}a(y)W(y) &= (w_0 a w_1\text{'} - w_1 a w_0\text{'})\text{'} \\ &= \cancel{w_0\text{'} a w_1\text{'}} + w_0(a w_1\text{'})\text{'} - \cancel{w_1\text{'} a w_0\text{'}} - w_1(a w_0\text{'})\text{'} \\ &= w_0(-cw_1) - w_1(-cw_0) = 0\end{aligned}$$

There are two possibilities:

1. $W(x) = 0$, which would imply that $W(y) = 0 \,\forall\, y$ by the lemma. In this case, $w_0(y)$ and $w_1(y)$ are proportional, and they both satisfy both BC. We cannot solve for c_0, c_1, so there is no Green's function (nontrivial kernel: nonzero solution of homogeneous BVP).
2. $W(x) \neq 0$, so $c_0 = \frac{w_1(x)}{aW}, c_1 = \frac{w_0(x)}{aW}$. The Green's function is

   $G^x(y) = \frac{1}{aW}\begin{cases} w_1(x)w_0(y) & 0 < y < x \\ w_0(x)w_1(y) & x < y < 1 \end{cases}$.

   $u(x) = \frac{1}{aW}[w_1(x)\int_0^x w_0(y)f(y)\,dy + w_0(x)\int_x^1 w_1(y)f(y)\,dy$.

Remarks:

i. Using the variation of parameters method in ODEs, we would obtain the same formula
ii. $G(x, y) = G(y, x)$ is symmetrical, which is a general property for self-adjoint problems

1.5 Modified Green's Function

In **Ex 13.**, we have a case with no Green's function, $W[w_0, w_1] = 0$, i.e. a nontrivial solution u^* such that $Lu^* = 0$ and u^* satisfies both BCs.

If u solves $Lu = f$ and u satisfies both BCs, then $0 = (u, Lu^*) = (Lu, u^*) = (f, u^*)$ so $(f, u^*) = 0$ is a necessary solvability condition for existence of u. In fact, it is also sufficient.

📖 **Thm.** (Fredholm alternative) For the Sturm-Liouville BVP, either

a. there is exactly one solution for every f, or
b. there is a nonzero solution u^* of $Lu^* = 0$. In this case, there is a solution for the given f if $(f, u^*) = 0$.

Remark: in case b) with $(f, u^*) = 0$, the solution u of $Lu = f$ is nonunique as $Lv = f$ where $v = u + ku^*$ for any $k \in \mathbb{R}$ is also a solution. Hence a general solution formula will involve a free parameter k, unless we specify an extra condition on u, such as $(u, u^*) = 0$. However, this is not always imposed.

If $G(x,y)$ is a modified Green's function, a solution is given by $u(x) = \int_0^1 G(x,y)f(y)\,dy$ when $(u^*, f) = 0$. Thus, $u(x) = (G^x, f) = (G^x, Lu) = (LG^x, u)$. We require that

$$\begin{cases} L_y G^x = \delta_x(y) + c(x)u^*(y) \\ \text{same BC on } G^x \text{ at } y = 0, 1 \end{cases}$$

The new term $c(x)u^*(y)$ allows the RHS to satisfy the solvability condition:

$$0 = (\delta_x(y) + c(x)u^*(y), u^*(y)) = u^*(x) + c(x)(u^*, u^*) \implies c(x) = \tfrac{-u^*(x)}{(u^*,u^*)}$$

With the new term,

$$(LG^x, y) = (\delta_x + c(x)u^*, u) = u(x) - \underbrace{\frac{u^*(x)}{(u^*, u^*)}}_{k}(u^*, u)$$

If we impose that $(u^*, u) = 0$, then $k = 0$, but in general it is a free parameter.

💡 **Ex 14.** Solve $u''(x) = f(x), 0 < x < 1$ with boundary conditions $u'(0) = u'(1) = 0$.

Clearly $u^*(x) = 1$ solves the homogeneous problem (as typical for Neumann BC).

The solvability condition is $0 = (f, u^*) = \int_0^1 f(x)\,dx$.

We have $c(x)u^*(y) = \frac{-1}{(u^*,u^*)}1 = -1$

The modified Green's function satisfies

$$\begin{cases} \partial_y^2 G^x(y) = \delta_x(y) - 1 \\ \partial_y G^x(0) = 0 = \partial_y G^x(1) \end{cases}$$

For $y \neq x, \partial_y^2 G^x = -1$ so

$$G^x(y) = \begin{cases} -\frac{1}{2}y^2 + Ay + B & 0 < y < x \\ -\frac{1}{2}y^2 + Cy + D & x < y < 1 \end{cases}$$

The boundary conditions imply that $A = 0$ and $C = 1$.

Continuity gives $B = x + D$, so

$$G^x(y) = -\frac{1}{2}y^2 + \begin{cases} x + D & 0 < y < x \\ y + D & x < y < 1 \end{cases}$$

Jump condition already holds, because $1 = [\partial_y G^x]_{x_-}^{x_+} = (-y + 1) - (-y + 0) = 1$

We have one free, useless parameter D.

Recall that $(LG^x, u) = u(x) - ku^*(x)$ where $k = \frac{(u^*,u)}{(u^*,u^*)}$.

Hence if $Lu = f$ and $(f, u^*) = 0$, then

$$
\begin{aligned}
u(x) &= (G^x, f) + ku^*(x) \\
&= \int_0^1 G(x,y) f(y)\, dy + k \\
&= \int_0^x (-\tfrac{1}{2}y^2 + x + D) f(y)\, dy + \int_x^1 (-\tfrac{1}{2}y^2 + y + D) f(y)\, dy + k \\
&= \int_0^x x f(y)\, dy + \int_x^1 y f(y)\, dy - \frac{1}{2}\int_0^1 y^2 f(y)\, dy + \cancel{D\int_0^1 f(y)\, dy} + k \\
&= \int_0^x x f(y)\, dy + \int_x^1 y f(y)\, dy + k_1
\end{aligned}
$$

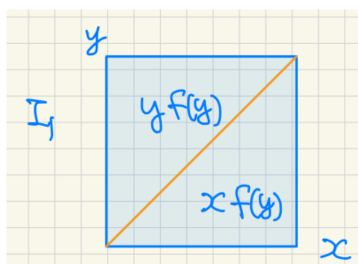
where $k_1 = k - \frac{1}{2}\int_0^1 y^2 f(y)\, dy$ is a free parameter in the solution formula, and D is useless and can be chosen $= 0$.

Check: $u'(x) = xf(x) + \int_0^x f(y)\, dy - xf(x)$ so $u'(0) = 0$ and $u'(1) = \int_0^1 f(y)\, dy = 0$. Also, $u''(x) = f(x)$.

If $k = 0$, do we have $(u, u^*) = 0$?

$$
(u, u^*) = (u, 1) = \int_0^1 \left(\int_0^x x f(y)\, dy + \int_x^1 y f(y)\, dy \right) dx - \frac{1}{2}\int_0^1 y^2 f(y)\, dy = I_1 + I_2
$$

where I_1 is a 2D integral.



Use a change of integration order,

$I_1 = \int_0^1 (\int_0^y y f(y)\, dx + \int_y^1 x f(y)\, dx)\, dy = \int_0^1 (y^2 f(y) + \frac{1-y^2}{2} f(y))\, dy = \frac{1}{2}\int_0^1 y^2 f(y)\, dy$

Hence $(u, u^*) = I_1 + I_2 = 0$ when $k = 0$, as expected.

Remark: the same idea works when L is not self adjoint. Case (b) of the Fredholm alternative $\Longrightarrow$ nonzero solution of both homogeneous BVP and homogeneous dual BVP:

$$
\begin{cases} Lu^* = 0, & \text{BC for } u^* \\ L^* v^* = 0, & \text{dual BC for } v^* \end{cases}
$$

which is more complicated and beyond the scope of this course.

W3C1 Lecture 5 (Jan 21)

1.6 Green's Function and Eigenfunction Expansion

Consider the Sturm-Liouville eigenvalue problem:

$$\begin{cases} L\phi = \lambda\phi & 0 < x < T \\ \alpha_0\phi(0) + \beta_0\phi'(0) = 0, & \alpha_1\phi(T) + \beta_1\phi'(T) = 0 \end{cases}$$

Where $L\phi = -(a(x)\phi')' + c(x)\phi$ with $a(x) > 0$.

It has real eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ each with multiplicity 1, and corresponding eigenfunctions ϕ_j that each solve the eigenvalue problem.

They can be taken as orthonormal, so $(\phi_j, \phi_k) = \delta_{jk}$ where δ is the Kronecker δ.

If same L with other BC (for example, periodic) is self-adjoint in the sense that $(Lu, v) = (u, Lv)$ for all u, v satisfying BCs, then the eigenvalues may have finite multiplicity $1 < m < \infty$. Such eigenvalue problems are not called Sturm-Liouville.

💡 **Ex 15.** $L = -\partial_x^2$ on $(0,1)$, $Lu = u''$.

a. Dirichlet BC $u(0) = u(1) = 0$:

   $\phi_k(x) = \sqrt{2}\sin k\pi x, \lambda_k = k^2\pi^2, k \in \mathbb{N}$

b. Neumann BC $u'(0) = u'(1) = 0$:

   $\lambda_k = k^2\pi^2, k \in \mathbb{N}_0$ with eigenfunctions

   $$\begin{cases} \psi_0(x) = 1, \\ \psi_k(x) = \sqrt{2}\cos k\pi x, k \geq 1 \end{cases}$$

c. Periodic BC $u(0) = u(1), u'(0) = u'(1)$

   $\lambda_k = k^2\pi^2, k \in N_0$, eigenfunctions include ϕ_k, ψ_k and λ_k has multiplicity 2 for $k \geq 1$.

Eigenfunction expansion: any square integrable function $f(x)$ on $[0, T]$ can be expanded as

$f(x) = \sum_{j=0}^{\infty} c_j\phi_j(x)$ where $c_j = (\phi_j, f)$ are the fourier coefficients.

The convergence is in the L^2 sense, not pointwise, i.e.

$\int_0^T |f(x) - f_k(x)|^2\, dx \to 0$ as $k \to \infty$, where $f_k(x)$ is the kth partial sum.

Parseval's equality holds: $||f||^2 = \int_0^T |f(x)|^2\, dx = \sum_{j=0}^{\infty} |c_j|^2$.

The series in **Ex 15.** are the a) fourier sine series, b) fourier cosine series, c) fourier series

Let's express the Green's function $G(x, y)$ by an eigenfunction expansion:

$G(x, y) = G^x(y) = \sum_{j=0}^{\infty} c_j(x)\phi_j(y)$.

Boundary conditions are automatically satisfied. We must also have

$\delta_x = L_y G^x(y) = \sum_j c_j(x) L\phi_j(y) = \sum_j c_j(x)\lambda_j\phi_j(y)$.

Taking the inner product with ϕ_j gives $\phi_j(x) = (\delta_x, \phi_j) = c_j(x)\lambda_j$

Hence $c_j(x) = \frac{1}{\lambda_j}\phi_j(x)$ and $G(x, y) = \sum_{j=0}^{\infty} \frac{1}{\lambda_j}\phi_j(x)\phi_j(y)$

Remarks:

i. $G(x, y)$ is symmetric, as expected

ii. The solution is $u(x) = \int_0^T G(x,y) f(y)\, dy = \sum_{j=0}^{\infty} \frac{1}{\lambda_j} (f, \phi_j) \phi_j(x)$

iii. If some $\lambda_j = 0$, then $G(x,y)$ will not be defined. We use the modified Green's function, and require the solvability condition $(f, \phi_j) = 0$ in order for u to be defined.