MATH 255 Formula Sheet

Definitions

Differential Equation

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$$

Ordinary Differential Equation

Derivatives are taken w.r.t. the same independent variable.

Order

The order of the highest derivative that appears.

Linear Differential Equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)$$

Only linear combination of unknown and its derivatives.

First Order ODEs

$$y'(t) = f(t, y(t))$$

Integration: $y'(t) = f(t)$

$$y(t) = \int f(t)\,dt$$

Separable Equations: $y'(t) = g(t)h(y)$

$$\int \frac{dy}{h(t)} = \int g(t)\,dt$$

Integrating Factor: $y' + p(t)y = q(t)$ **(linear)**

$$r(t) = e^{\int p(t)\,dt}$$

$$\big[r(t)y\big]' = r(t)q(t)$$

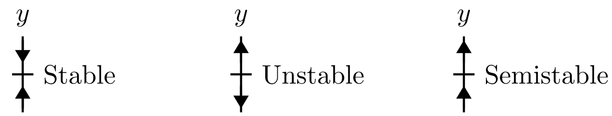
$$y = \frac{1}{r(t)}\Big(\int r(t)q(t)\,dt\Big)$$

Picard's Theorem: $y' = f(t, y)$**.**

If $f(t, y)$ is continuous near (t_0, y_0) and $\frac{\partial f}{\partial y}$ exists and is continuous near (t_0, y_0), then a solution exists locally and is unique.

Autonomous Equation: $y' = f(y)$

Critical points at $y' = f(y) = 0$.



Application: Mixing

Q: amount of salt
s: inflow concentration
r: flow rate
V: volume

$$Q' = rs - \frac{r}{V}Q$$

$$Q(t) = sV + (Q_0 - sV)e^{-\frac{r}{V}t}$$

Euler's Method: $y' = f(t, y)$

$$y(t + h) \approx y(t) + hf(t, y)$$

Second Order Linear ODEs

$$y'' + p(t)y' + q(t)y = g(t)$$

$$L[y] = g(t)$$

Homogeneous: $L[y] = 0$

If y_1 and y_2 are solutions, then $y = C_1y_1 + C_2y_2$ is also a solution.

Reduction of Order: y_1 **solves** $y'' + p(t)y' + q(t)y = 0$

Try $y_2(t) = v(t)y_1(t)$.

$$y_2(t) = y_1(t)\int \frac{e^{-\int p(t)\,dt}}{\big(y_1(t)\big)^2}\,dt$$

Constant Coefficient 2LODE: $ay'' + by' + cy = 0$

Try $y = e^{rt}$:

$$ar^2 + br + c = 0$$

$$\begin{aligned}
&y = C_1e^{r_1x} + C_2e^{r_2x} && r_1 \neq r_2\\
&y = (C_1 + C_2x)e^{r_x} && r_1 = r_2\\
&y = C_1e^{\alpha x}\cos(\beta x) + C_2e^{\alpha x}\sin(\beta x) && r_1, r_2 = \alpha \pm i\beta
\end{aligned}$$

Nonhogeneous: $L[y] = g(t) \neq 0$

$$y = y_c + y_p$$

where y_c solves the homogeneous equation and y_p is a particular solution.

Undetermined Coefficients: $L[y] = g(t)$

$s = 0, 1,$ or 2 so y_c and y_p are linearly independent.

$g(t)$	y_p
$P_n(t) = a_0t^n + \cdots + a_n$	$t^s(A_0t^n + \cdots + A_n)$
$e^{\alpha t}$	$t^sAe^{\alpha t}$
$\sin(\beta t)$ or $\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$

Variation of Parameters: $L[y] = g(t)$

Try $y_p = u_1y_1 + u_2y_2$.

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0\\ u_1'y_1' + u_2'y_2' = g(t)\end{cases}$$

Applications of 2LODE

Mechanical Oscillations

$$mx'' + cx' + kx = F(t), \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Undamped forced: $mx'' + kx = F_0\cos(\omega t)$:

$$x_c = C_1\cos(\omega t) + C_2\sin(\omega t)$$

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \quad (\omega \neq \omega_0)$$

$$x_p = \frac{F_0}{2m\omega}t\sin(\omega t) \quad (\omega = \omega_0)$$

Damped forced: $mx'' + cx' + kx = F_0\cos(\omega t)$:
Let $\rho = \frac{c}{2m}$. Call $x_p = x_{sp}$ (steady periodic).
Practical resonance: maximum amplitude of x_{sp}.

$$\omega = \sqrt{\omega^2 - 2\rho^2} \text{ or } \omega = 0$$

Laplace Transform

Definition

$$\mathcal{L}\{f(t)\} = F(s) := \int_0^\infty e^{-st}f(t)\,dt$$

$\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$ and $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$
$\mathcal{L}\{fg\} \neq \mathcal{L}\{f\}\mathcal{L}\{g\}$

Common Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
c	$\frac{c}{s}, \quad s>0$
t^n	$\frac{n!}{s^{n+1}}, \quad s>0$
e^{-at}	$\frac{1}{s+a}, \quad s>-a$
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s>0$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s>0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s>\lvert a\rvert$
$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s>\lvert a\rvert$
$e^{-at}f(t)$	$F(s+a)$

Laplace Transform of Derivatives

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$s\mathcal{L}\{g(t)\} - g(0)$
$g''(t)$	$s^2\mathcal{L}\{g(t)\} - sg(0) - g'(0)$
$g'''(t)$	$s^3\mathcal{L}\{g(t)\} - s^2g(0) - sg'(0) - g''(0)$
$g^{(n)}(t)$	$s^n\mathcal{L}\{g(t)\} - s^{n-1}g(0) - s^{n-2}g'(0) - \cdots - g^{(n-1)}(0)$

Solving ODEs

1. Laplace transform both sides
2. Plug in initial condition
3. Solve for $X(s) = \mathcal{L}\{x(t)\}$
4. $\mathcal{L}\{x(t)\} = \mathcal{L}^{-1}\{X(s)\}$

Convolution

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g = \int_0^t f(\tau)g(t-\tau)d\tau \quad (t \geq 0)$$

Linear System of ODEs

$$\vec{x}\,' = A(t)\vec{x}(t) + \vec{f}(t) \quad \Leftrightarrow \quad \vec{x}\,' = P\vec{x} + \vec{f}$$

Homogeneous

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$$

$$X(t) = \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix} \text{ (fundamental matrix)}$$

$$\vec{x} = X\vec{c}$$

Nonhomogeneous

$$\vec{x} = \vec{x}_c + \vec{x}_p = X\vec{c} + \vec{x}_p$$

Eigenvalue Method

Eigenvalue: $\det(A - \lambda I) = 0$
Eigenvector: $(A - \lambda I)\vec{v} = 0$
Solution to $\vec{x}\,' = P\vec{x}$:

$$\vec{x} = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t} + \cdots + c_n\vec{v}_ne^{\lambda_n t}$$

If complex eigenpair $\vec{x}_1 = \vec{v}_1e^{\lambda_1 t}$, $\vec{x}_2 = \overline{\vec{v}_1}$:

- Use $c_1\Re(\vec{x}_1) + c_2\Im(\vec{x}_1)$

Vector Field Behaviour

λ_1	λ_2	behaviour	stability
$+$	$+$	source	unstable
$-$	$-$	sink	stable
$+$	$-$	saddle	unstable

λ	behaviour	stability
$\pm bi$	center	ellipses
$a \pm bi, a > 0$	spiral source	unstable
$a \pm bi, a < 0$	spiral sink	stable

Multiple Eigenvalues

Generalized eigenvectors:

$$(A - \lambda I)\vec{v}_1 = \vec{v}_0 \implies \vec{x}_1 = \vec{v}_1e^{\lambda t}$$

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1 \implies \vec{x}_2 = (\vec{v}_2 + \vec{v}_1 t)e^{\lambda t}$$

$$(A - \lambda I)\vec{v}_k = \vec{v}_{k-1} \implies \vec{x}_k = \left(\sum_{i=0}^{k-1} \vec{v}_{k-i}\frac{t^i}{i!}\right)e^{\lambda t}$$

Nonlinear System of ODEs

Autonomous systems

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}}_{F(\vec{x})}$$

Critical points: $F(\vec{x}_0) = 0$.

Linearization

Suppose $\vec{p} = (x_0, y_0)$ is a critical point.
Let $u = x - x_0$ and $v = y - y_0$.
Linearized system:

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \mathbf{J}_{\vec{p}}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{\vec{p}}\begin{bmatrix} u \\ v \end{bmatrix}$$

"Almost linear" - good approximation when:

- (x_0, y_0) is isolated
- Jacobian is invertible $\Leftrightarrow$ 0 is not an eigenvalue

Classification

Refer to Vector Field Behaviour. Center behaviour is unclear because Jacobian of nonlinear system will vary.

Conservative Equation

$$x'' + f(x) = 0$$

Associated conservative system:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} y \\ -f(x) \end{bmatrix}$$

Energy function $E(x,y)$:

$$E(x,y) = \frac{1}{2}y^2 + \int_0^x f(u)\,du$$

For any solution, $E(x,y)$ is constant.

Miscellaneous Formulas

Partial Fractions

$$\frac{P(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n}$$

$(x-r)^m$ corresponds to $\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$

$ax^2 + bx + c$ corresponds to $\frac{Ax+B}{ax^2+bx+c}$

Identities and Derivatives

Trig

$$\cos(2x) = \cos^2 x - \sin^2 x \quad \sin(2x) = 2\sin x\cos x$$

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

Derivatives

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\csc x) = -\csc x\cot x \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec x) = \sec x\tan x \qquad \frac{d}{dx}\log|x| = \frac{1}{x}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}b^x = b^x\log b$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\log_b x = \frac{1}{x\log b}$$