5️⃣

Part 5: Eigenvalue Problems

W11C2 Lecture 18 (Mar 20)

5.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Consider the heat and wave equations with variable coefficients.

$r(x)u_t = \nabla \cdot (p(x)\nabla u) - q(x)u$ **(Eq. 1)**

$r(x)u_{tt} = \nabla \cdot (p(x)\nabla u) - q(x)u$ **(Eq. 2)**

where $r(x) > 0, p(x) > 0, q(x) \in \mathbb{R}$ are smooth functions.

The boundary conditions are

$a\frac{\partial u}{\partial n} + bu = 0, \quad x \in \partial\Omega$ **(Eq. 3)**

where a, b are real constants with $a^2 + b^2 > 0$.

Suppose we look for special solutions of the form, for some $\lambda \geq 0$:

- $u(x,t) = \phi(x)e^{-\lambda t}$ for **(1)** - decay component
- $u(x,t) = \phi e^{\pm i\sqrt{\lambda}t}$ for **(2)** - plane waves

We get $L\phi(x) = -\nabla \cdot (p(x)\nabla u) + q(x)u = \lambda r(x)\phi \quad x \in \Omega$ **(Eq. 4)**

The boundary conditions become $a\frac{\partial \phi}{\partial n} + b\phi = 0$.

> 📌 **Def. (4)** is an eigenvalue problem for the differential operator L. If $\phi(x)$ is a nonzero solution for some $\lambda \in \mathbb{C}$, we call ϕ an eigenfunction, and λ the corresponding eigenvalue.

💡 **Ex 1.** (Bessel's equation of order $n \geq 0$):

$x^2y'' + xy' + (x^2 - n^2)y = 0, \quad x > 0$

The solutions are Bessel's function of the 1st kind:

$J_n(x) \approx c_n x^n$ for $x \sim 0$ and $J_n(x) \to 0$ as $x \to \infty$. Oscillatory with infinitely many roots.

Let $L_n y = -(xy')' + \frac{n^2}{x}y$, so $L_n J_n = xJ_n$.

Let α_{nk} be the kth positive zero of $J_n(x)$.

$\phi_k(x) = J_n(\frac{\alpha_{nk}}{a}x)$ satisfies:

$\begin{cases} L_n\phi_k = \lambda_k x\phi, & \lambda_k = (\frac{\alpha_{nk}}{a})^2 \\ \phi_k(0) = 0, \phi_k(a) = 0 \end{cases}$

For this eigenvalue problem, the coefficient functions are:

$p(x) = x, q(x) = \frac{n^2}{x}, r(x) = x.$

Basic properties of eigenvalue problem (4):

a. L is self-adjoint with respect to the inner product:

$(f, g) = \int_\Omega f(x)\overline{g(x)}\,dx, \|f\|_{L^2(\Omega)} = (f, f)^{\frac{1}{2}}$

Claim: $(Lf, g) = (f, Lg)$ if both f and g satisfy the BC. As a consequence, all eigenvalues are real.

Proof: If $L\phi = \lambda\phi$, then:

$\lambda(\phi, \phi) = (\lambda\phi, \phi) = (L\phi, \phi) = (\phi, L\phi) = (\phi, \lambda\phi) = \bar{\lambda}(\phi, \phi) \implies \lambda \in \mathbb{R}$

Remarks:

i. Consider a matrix A that is symmetric ($a_{ij} = a_{ji}$) and self-adjoint ($a_{ij} = \overline{a_{ji}}$). Then all eigenvalues are real, and defining inner product as $(x, y) = x \cdot \bar{y}$, we have $(Ax, y) = (x, Ay)$.

ii. Since $L\mathrm{Re}(\phi) = \mathrm{Re}(L\phi) = \lambda r \mathrm{Re}(\phi)$, we may assume that ϕ is real valued.

b. All eigenvalues form an infinite sequence:

$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots, \quad \lambda_j \to \infty \text{ as } j \to \infty$

The eigenspace of an eigenvalue λ is $E_\lambda = \{\phi : L\phi = \lambda r\phi\}$, the set of all eigenfunctions of λ. We call $\dim E_\lambda$ the multiplicity of λ. It is always finite. We repeat each eigenvalue according to its multiplicity.

c. Eigenfunctions of different eigenvalues are orthogonal in another inner product:

$\lambda_j \neq \lambda_k \implies ((\phi_j, \phi_k))_r := \int_\Omega \phi_j \overline{\phi_k} r(x)\,dx = 0$

Note $c_1(f, f) \leq ((f, f))_r \leq c_2(f, f)\,\forall f$ for some $0 < c_1 < c_2$ if $\min r(x) > 0$.

We may and will normalize ϕ_j such that:

$$((\phi_j, \phi_k))_r = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

If $\dim E_\lambda > 1$, we may use the Gram-Schmidt method to choose an orthonormal set of eigenfunctions from E_λ.

d. The eigenfunctions are complete, meaning any $u \in L^2(\Omega)$ can be written as
   $u(x) = \sum_{j=1}^{\infty} c_j \phi_j(x), \quad c_j = (\!(u, \phi_j)\!)_r$
   in the sense that $\lim_{N\to\infty} \|u(x) - \sum_{j=1}^{N} c_j \phi_j\|_{L^2(\Omega)} = 0$
e. If $q(x) \geq 0, ab \geq 0$, then all $\lambda_j \geq 0$. $\lambda_1 = 0 \iff q(x) \equiv 0, ab = 0$ (ϕ_1 is constant in this case).

The proof of properties (b) and (d) are beyond this course. We show (a), (c), and (e).

Proof of (a):

$$\begin{aligned}(Lu, v) - (u, Lv) &= \int_\Omega [(-\nabla \cdot (p\nabla u) + \cancel{qu})\bar{v} - u(-\nabla \cdot (p\nabla \bar{v}) + \cancel{q\bar{v}})]\, dx \\ &= \int_\Omega \nabla \cdot [-p\bar{v}\nabla u + up\nabla \bar{v}]\, dx \\ &= \int_{\partial\Omega} p(\bar{v}\tfrac{\partial u}{\partial n} - u\tfrac{\partial \bar{v}}{\partial n}) dS_x\end{aligned}$$

If $a = 0$, then $u = v = 0$, and if $a \neq 0$, then we have $\bar{v}(-\frac{b}{a}u) - u(-\frac{b}{a}\bar{v}) = 0$. Hence, $(Lu, v) - (u, Lv) = 0$ so $(Lu, v) = (u, Lv)$.

Proof of (c):

If $L\phi_j = \lambda_j r\phi_j, L\phi_k = \lambda_k r\phi_k, \lambda_j \neq \lambda_k$, and they satisfy the BC, we have

$$\begin{aligned}0 &= (L\phi_j, \phi_k) - (\phi_j, L\phi_k) \\ &= (\lambda_j r\phi_j, \lambda_k) - (\phi_j, \lambda_k r\phi_k) \\ &= (\lambda_j - \lambda_k)(r\phi_j, \phi_k)\end{aligned}$$

and so $(\!(\phi_j, \phi_k)\!)_r = 0$.

Proof of (e):

$$\begin{aligned}(u, Lu) &= \int_\Omega u[-\nabla \cdot (p\nabla \bar{u}) + q\bar{u}]\, dx \\ &= \int_\Omega -\nabla \cdot [up\nabla \bar{u}] + p|\nabla u|^2 + q|u|^2\, dx \\ &= \int_\Omega (p|\nabla u|^2 + q|u|^2)\, dx - \int_{\partial\Omega} pu\tfrac{\partial \bar{u}}{\partial n}\, dS\end{aligned}$$

Using the BC, on $\partial\Omega$, we have $u\frac{\partial \bar{u}}{\partial n} = -b_1|u|^2$ where $b_1 = \begin{cases} 0 & \text{if } a = 0 \\ \frac{b}{a} & \text{if } a \neq 0 \end{cases}$. Hence,

$(u, Lu) = \int_\Omega (p|\nabla u|^2 + q|u|^2)\, dx + b_1 \int_{\partial\Omega} p|u|^2\, dS$ **(Eq. 5)**

Note that $\operatorname{sgn} b_1 = \operatorname{sgn} ab = \begin{cases} 0 & ab = 0 \\ 1 & ab > 0 \\ -1 & ab < 0 \end{cases}$.

If λ_k is an eigenvalue with eigenfunction ϕ_k, then:

$\lambda_k = \lambda_k(\!(\phi_k, \phi_k)\!)_r = (\phi_k, \lambda_k r\phi_k) = (\phi_k, L\phi_k)$ **(Eq. 6)**

For the first eigenvalue λ_1, by **(5)**,

$\lambda_1 = \int_\Omega (p|\nabla\phi_1|^2 + q\phi_1^2)\, dx + b_1 \int_\Omega p\phi_1^2\, dS \geq 0$ if $q(x) \geq 0$ and $b_1 \geq 0$.

$\lambda_1 = 0$ only if $q(x) \equiv 0, b_1 = 0, \phi_1 = \text{const}$, which proves (e).

W12C1 Lecture 19 (Mar 25)

5.2 Variational Principle for Eigenvalues and Rayleigh Quotient

In this section we only consider real-valued functions and the special cases:

$$(a,b) = \begin{cases} (0,1) & u|_{\partial\Omega} = 0 \quad \text{Dirichlet BC} \\ (1,0) & \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 \;\; \text{Neumann BC} \end{cases}$$

Then $b_1 = 0$ in **(5)**.

📌 **Def.** For $Lu = -\nabla \cdot (p\nabla u) + qu$, let the energy be

$E(u) = \int_\Omega (p|\nabla u|^2 + qu^2)\, dx$

Then **(5)** under Dirichlet/Neumann BC ($ab = 0$) becomes $(u, Lu) = E(u)$.

Suppose u is any (nice) function satisfying the BC (D/N). By completeness of eigen-functions,

$u(x) = \sum_{j=1}^\infty c_j \phi_j(x)$ **(Eq. 7)**

$(\!(u,u)\!)_r = (\!(\sum_j c_j\phi_j, \sum_k c_k\phi_k)\!)_r = \sum_{j,k} c_j c_k \underbrace{(\!(\phi_j, \phi_k)\!)_r}_{\delta_{jk}} = \sum_j c_j^2$ **(Eq. 8)**

$E(u) = (u, Lu) = \sum_{j,k} c_j c_k (\phi_j, L\phi_k) = \sum_{j,k} c_j c_k (\phi_j, \lambda_k r \phi_k) = \sum_{j,k} c_j c_k \lambda_k \underbrace{(\!(\phi_j, \phi_k)\!)_r}_{\delta_{jk}} = \sum_j c_j^2 \lambda_j$ **(Eq. 9)**

Since $\lambda_j \geq \lambda_1$, for all j, $E(u) \geq \lambda_1 \sum_j c_j^2 = \lambda_1 (\!(u,u)\!)_r \implies \lambda_1 \leq \frac{E(u)}{(\!(u,u)\!)_r}$.

Equality is achieved when all $c_j = 0$ for all $\lambda_j > \lambda_1$, then $u \in E_{\lambda_1}$.

📖 **Thm 1.** (First eigenvalue). The lowest eigenvalue of L with D/N BC is

$\lambda_1 = \min_{u \neq 0 \text{ satisfies BC}} \frac{E(u)}{(\!(u,u)\!)_r}$ **(Eq. 10)**

Remarks:

i. The quantity $\frac{E(u)}{(\!(u,u)\!)_r}$ is called the Rayleigh quotient.

ii. This formula can be used to find lower and upper bounds of λ_1 (see **Ex. 2**).

iii. Alternatively, $\lambda_1 = \lambda_1^* := \min_{(\!(v,v)\!)_r = 1} E(v)$

Proof:

Clearly $\lambda_1 \leq \lambda_1^*$. To show $\lambda_1 \geq \lambda_1^*$ for all $u \neq 0$, when $(\!(u,u)\!)_r \neq 1$, let $v = \frac{u}{(\!(u,u)\!)_r^{1/2}}$ then $(\!(v,v)\!)_r = 1$, $E(v) = \frac{E(u)}{(\!(u,u)\!)_r}$.

💡 **Ex 2.** Let $0 < \varepsilon < 1$. Find an upper bound for the lowest eigenvalue (with $r \equiv 1$) of $L = -\frac{d^2}{dx^2} + \varepsilon x$ on $[0, 1]$ with 0-BC at $x = 0, 1$.

We have $\Omega = (0, 1)$, $p(x) = 1$, $q(x) = \varepsilon x$, $r(x) \equiv 1$, hence $(\!(u, u)\!)_r = (u, u)$.

By **Thm 1** and $r \equiv 1$, we have

$\lambda_1 \leq \frac{E(u)}{(u,u)} = \frac{\int_0^1 u_x^2 + \varepsilon x u^2 \, dx}{\int_0^1 u^2 \, dx}$ for any $u \not\equiv 0$ satisfying the BC.

First try: $u(x) = x(1-x)$, so $(u, u) = \int_0^1 x^2(1-x)^2 \, dx = \int_0^1 x^2 - 2x^3 + x^4 \, dx = [\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5]_0^1 = \frac{1}{30}$.

$$
\begin{aligned}
E(u) &= \int_0^1 (1-2x)^2 + \varepsilon x(x - x^2)^2 \, dx \\
&= \int_0^1 1 - 4x + 4x^2 + \varepsilon(x^3 - 2x^4 + x^5) \, dx \\
&= [x - 2x^2 + \tfrac{4}{3}x^4 + \varepsilon(\tfrac{x^4}{4} - \tfrac{2x^5}{5} + \tfrac{x^6}{6})]_0^1 \\
&= \tfrac{1}{3} + \tfrac{\varepsilon}{60}
\end{aligned}
$$

Hence $\lambda_1 \leq \frac{\frac{1}{3} + \frac{\varepsilon}{60}}{\frac{1}{30}} = 10 + \frac{\varepsilon}{2}$.

Second try, when $\varepsilon \ll 1$:

For $\varepsilon = 0$, eigenfunctions of $L_0 = -\frac{d^2}{dx^2}$ are $\phi_k(x) = \sin(k\pi x), k \in \mathbb{N}$ with eigenvalues $\lambda_k = k^2\pi^2$. Let's try $u(x) = \phi_1(x) = \sin(\pi x)$, so $(u, u) = \int_0^1 \sin^2(\pi x) \, dx = \frac{1}{2}$.

$$
\begin{aligned}
E(u) &= \int_0^1 (\pi \cos(\pi x))^2 + \varepsilon x \sin^2(\pi x) \, dx \\
&= \tfrac{1}{2}\pi^2 + \varepsilon \int_0^1 x \sin^2(\pi x) \, dx \\
&= \tfrac{1}{2}\pi^2 + \tfrac{\varepsilon}{4}
\end{aligned}
$$

Hence $\lambda_1 \leq \frac{\frac{1}{2}\pi^2 + \frac{\varepsilon}{4}}{\frac{1}{2}} = \pi^2 + \frac{\varepsilon}{2}$ which is better since $\pi^2 < 10$.

We also have a lower bound for λ_1:

$\lambda_1 = \min \frac{E(u)}{(u,u)} \geq \min \frac{\int_0^1 u_x^2 \, dx}{(u,u)} = \lambda_1 \text{ of } L_0 = \pi^2$.

Higher eigenvalues

Suppose u is a function in Ω satisfying the BC, and

$(\!(u, \phi_j)\!)_r = 0, \quad j = 1, 2, \ldots, n-1$

Then $c_1 = c_2 = \ldots = c_{n-1} = 0$. By **(7)** - **(9)**,

$u = \sum_{j=n}^{\infty} c_j \phi_j(x)$ and $(\!(u, u)\!)_r = \sum_{j=n}^{\infty} c_j^2$.

$E(u) = \sum_{j=n}^{\infty} \lambda_j c_j^2 \geq \lambda_n \sum_{j=n}^{\infty} c_j^2 = \lambda_n (\!(u, u)\!)_r$, so $\lambda_n \leq \frac{E(u)}{(\!(u,u)\!)_r}$

with equality only if $c_j = 0$ for all j with $\lambda_j \geq \lambda_n \implies u \in E_{\lambda_n}$.

📖 **Thm 2.** (Higher eigenvalues) The nth eigenvalue of the operator L with D/N BC is

$\lambda_n = \min_{u \text{ satisfies BC}, (\!(u,\phi_j)\!)_r = 0 \text{ for } j \leq n-1} \frac{E(u)}{(\!(u,u)\!)_r}$ **(Eq. 11)**

Remarks:

i. For general BC $a\frac{\partial u}{\partial n} + bu = 0$, if $ab \geq 0$, we can add $b_1 \int_{\partial\Omega} |u|^2\, dS$ to $E(u)$ and define $E(u) = (u, Lu) = \text{RHS of } (5)$.

ii. In **Thm 2**, λ_n is given inductively, which is not convenient since we need to first know $\phi_1, \ldots, \phi_{n-1}$.

📖 **Thm 3.** (Courant max-min principle)

$\lambda_n = \max_{f_1, \ldots, f_{n-1}} [\min_{u \in \mathcal{A}, (\!(u,f_j)\!)_r = 0 \text{ for } j \leq n-1} \frac{E(u)}{(\!(u,u)\!)_r}]$ **(Eq. 12)**

where $u \in \mathcal{A}$ means u satisfies the boundary conditions.

💡 **Ex 3.** Consider the matrix $A = \frac{1}{2}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

$\lambda_1 = 1, \phi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2, \phi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda_3 = 3, \phi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Define $(x, y) = x \cdot y$, then

$\lambda_1 = \min_{x \neq 0} \frac{Ax \cdot x}{|x|^2}$

$\lambda_2 = \min_{0 \neq x \perp \phi_1} \frac{Ax \cdot x}{|x|^2} = \max_y \min_{0 \neq x \perp y} \frac{Ax \cdot x}{|x|^2}$

$\lambda_3 = \min_{0 \neq x \perp \phi_1, \phi_2} \frac{Ax \cdot x}{|x|^2} = \max_{y_1, y_2} \min_{0 \neq x \perp y_1, y_2} \frac{Ax \cdot x}{|x|^2}$

Proof of Thm 3:

Let λ_n^* denote the max-min value in **Thm 3**. Clearly $\lambda_n \leq \lambda_n^*$ by taking $f_j = \phi_j$.

To show $\lambda_n \geq \lambda_n^*$, let $f_1, \ldots, f_{n-1}$ be any functions on Ω satisfying the BC.

We now look for nonzero $(a_1, \ldots, a_n)$ such that

$u(x) = \sum_{j=1}^n a_j \phi_j(x) \perp f_1, \ldots, f_{n-1}$

$0 = (\!(u, f_k)\!)_r = \sum_{j=1}^n a_j (\!(\phi_j, f_k)\!)_r, \quad k = 1, \ldots, n-1.$

a system of $n-1$ linear equations for n variables.

By matrix algebra from MATH 221, such a system as at least one set of nonzero solutions.

For this u,

$E(u) = (u, Lu) = \sum_{j=1}^n \lambda_j a_j^2 \leq \lambda_n \sum_{j=1}^n a_j^2 = \lambda_n (\!(u,u)\!)_r$.

Hence, $\min_{(u,f_j)=0, j\leq n-1} \frac{E(u)}{(\!(u,u)\!)_r} \leq \lambda_n$.

Hence, $\lambda_n^* = \max[\text{above expression}] \leq \lambda_n$.

Remark:

The min-max principle is also valid. $\lambda_n = \min_{f_1,\dots,f_n}[\max_{u\in\text{span}\{f_1,\dots,f_n\}} \frac{E(u)}{(\!(u,u)\!)_r}]$

W12C2 Lecture 20 (Mar 27)

We revisit the Euler-Lagrange equations.

Denote $M(u) = (\!(u,u)\!)_r = \int_\Omega u^2 r(x)\,dx$.

From **Thm 1** and its remark, $\lambda_1 = \min_{0\neq u\in\mathcal{A}} \frac{E(u)}{M(u)} = \min_{u\in\mathcal{A}, M(u)=1} E(u)$

The 2nd form is constrained minimization. Hence, u satisfies the E-L equation, for some μ:

$E\text{'}(u) = \mu M\text{'}(u)$ where $E\text{'} = \frac{\delta E}{\delta u}$.

$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} M(u+\varepsilon h) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega (u+\varepsilon h)^2 r(x)\,dx = \int_\Omega 2uhr\,dx \implies M'(u) = 2ur$.

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u+\varepsilon h) &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega p(\nabla|u+\varepsilon h|^2) + q|u+\varepsilon h|^2\,dx \\ &= \int_\Omega 2p\nabla u\cdot\nabla h + 2quh\,dx \\ &= \int_\Omega (2Lu)h\,dx \implies E'(u) = 2Lu\end{aligned}$$

The E-L equation becomes $2Lu = \mu(2ur) \implies Lu = \mu ru, \mu = \lambda_1$.

We can also compute via the 1st form:

$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{E(u+\varepsilon h)}{M(u+\varepsilon h)} = \frac{E'(u)M(u) - E(u)M'(u)}{M^2(u)} = \frac{2LuM(u) - E(u)2ru}{M^2(u)}$, so $Lu = \lambda ru$ with $\lambda = \frac{E(u)}{M(u)} = \lambda_1$.

For higher eigenvalues λ_n, by **Thm 2**,

$\lambda_n = \min_{0\neq u\in\mathcal{A}, (\!(u,\phi_j)\!)_r=0, j\leq n-1} \frac{E(u)}{M(u)} = \min_{u\in\mathcal{A}, (\!(u,\phi_j)\!)_r=0, j\leq n-1, M(u)=1} E(u)$

Recall extrema under multiple constraints:

If extrema of $f(x), x\in\mathbb{R}^n$, subject to constraints $g_1(x) = 0, \cdots, g_m(x) = 0$, $1\leq m < n$, happens at x_0 and $\{\nabla g_1(x_0), \dots, \nabla g_m(x_0)\}$ are linearly independent, then

$\nabla f(x_0) = \mu_1 \nabla g_1(x_0) + \cdots + \mu_m \nabla g_m(x_0)$ for some Lagrange multipliers $\mu_1, \dots, \mu_m$,

💡 **Ex 4.** Let $f = x + 2y + 3z, g_1 = x^2 + z^2 - 1, g_2 = y^2 + z^2 - 1$. Minimize f subject to the constraints $g_1 = g_2 = 0$.

By Lagrange multipliers:

$\nabla f = (1,2,3) = \lambda\nabla g_1 + \mu\nabla g_2 = \lambda(2x, 0, 2z) + \mu(0, 2y, 2z)$

We have five equations: $1 = 2\lambda x, 2 = 2\mu y, 3 = 2(\lambda+\mu)z, g_1 = 0, g_2 = 0$.

Five equations, five unknowns. The solution is $\vec{x}_\pm = \pm\frac{1}{\sqrt{13}}(2,2,3)$.

We have $f(\vec{x}_\pm) = \pm\frac{1}{\sqrt{13}}(2+4+9) = \pm\frac{15}{\sqrt{13}} \implies \min_{g_1=g_2=0} f = -\frac{15}{\sqrt{13}}$.

The same is true for functionals. By the 2nd form of λ_n,

$E'(u) = \mu_1 M_1'(u) + \cdots + \mu_{n-1} M_{n-1}'(u) + \mu_n M'(u)$

where $M_j(u) = (\!(u, \phi_j)\!)_r = \int_\Omega u\phi_j r(x)\, dx, \quad 1 \le j < n.$

$\frac{d}{d\varepsilon}\big|_{\varepsilon=0} M_j(u + \varepsilon h) = \int_\Omega h\phi_j r\, dx$ so $M_j'(u) = r\phi_j$.

E-L equation becomes $2Lu = \mu_1 r\phi_1 + \cdots + \mu_{n-1} r\phi_{n-1} + \mu_n 2ru$.

For $j = 1, \ldots, n-1$, compute $(2Lu, \phi_j)$:

$(2Lu, \phi_j) = (2u, L\phi_j) = (2u, \lambda_j r\phi_j) = 0$ because $(r\phi_k, \phi_j) = \delta_{kj} \implies (ru, \phi_j) = 0$.

Hence $\mu_j = 0$ for $j = 1, \ldots, n-1$ and $Lu = \mu_n ru$ so $\mu_n = \lambda_n$.

5.3 Eigenvalue Bounds by Comparison

In this section we consider Dirichlet BC only. We will obtain bounds of eigenvalues λ_n of $Lu = -\nabla \cdot (p(x)\nabla u) + q(x)u$ with weight $r(x)$ in Ω, by comparing it with another simpler eigenvalue problem.

Comparison in coefficients

Suppose the coefficients satisfy: $0 < p_{\min} \le p(x) \le p_{\max}$, $q_{\min} \le q(x) \le q_{\max}$, $0 < r_{\min} \le r(x) \le r_{\max}$ for $x \in \Omega$.

Denote by $\lambda_{n,\min}$ the n-th eigenvalue of

$$\begin{cases} -\nabla \cdot p_{\min}\nabla\phi + q_{\min}\phi = \lambda_{n,\min} r_{\max}\phi & x \in \Omega \\ \phi = 0 & x \in \partial\Omega \end{cases}$$

and by $\lambda_{n,\max}$ the n-th eigenvalue of

$$\begin{cases} -\nabla \cdot p_{\max}\nabla\phi + q_{\max}\phi = \lambda_{n,\max} r_{\min}\phi & x \in \Omega \\ \phi = 0 & x \in \partial\Omega \end{cases}$$

📖 **Thm 4.** $\lambda_{n,\min} \le \lambda_n \le \lambda_{n,\max}$

Proof:

For any admissible u,

$E(u) = \int_\Omega p|\nabla u|^2 + q|u|^2\, dx \le \int_\Omega p_{\max}|\nabla u|^2 + q_{\max}|u|^2\, dx =: E_{\max}(u)$

$(\!(u,u)\!)_r = \int_\Omega u^2 r\, dx \ge \int_\Omega u^2 r_{\min} dx =: (\!(u,u)\!)_{r_{\min}}$

so that $\frac{E(u)}{(\!(u,u)\!)_r} \le \frac{E_{\max}(u)}{(\!(u,u)\!)_{r_{\min}}}$.

Also note that for any admissible $f_1, \ldots, f_{n-1}$,

$(\!(u, f_j)\!)_r = 0 \iff (\!(u, \tilde{f}_j)\!)_{r_{\min}} = 0$ where $\tilde{f}_j(x) = \frac{r(x)}{r_{\min}} f_j(x)$ is also admissible.

By the min-max principle (**Thm 3**, not **Thm 2**),

$$\begin{aligned} \lambda_n &= \max_{f_1,\ldots,f_{n-1}\in\mathcal{A}} \min_{u\in\mathcal{A}, (\!(u,f_j)\!)_r = 0} \frac{E(u)}{(\!(u,u)\!)_r} \\ &= \max_{\tilde{f}_1,\ldots,\tilde{f}_{n-1}\in\mathcal{A}} \min_{u\in\mathcal{A}, (\!(u,\tilde{f}_j)\!)_{r_{\min}} = 0} \frac{E(u)}{(\!(u,u)\!)_r} \\ &\le \max_{\tilde{f}_1,\ldots,\tilde{f}_{n-1}\in\mathcal{A}} \min_{u\in\mathcal{A}, (\!(u,\tilde{f}_j)\!)_{r_{min}} = 0} \frac{E_{\max}(u)}{(\!(u,u)\!)_{r_{\min}}} = \lambda_{n,\max} \end{aligned}$$

By the same argument, $\lambda_n \geq \lambda_{n,\min}$.

Remark: it is hard to prove **Thm 4** using **Thm 2**.

💡 **Ex 5.** Find the upper and lower bounds for the n-th eigenvalue of

$L = -\Delta + \varepsilon|x|^2$ in $\Omega = (0,1)^2 \subset \mathbb{R}^2$ with zero BC. Here $\varepsilon \geq 0$ and $x = (x_1, x_2)$.

We have $p(x) = 1, q(x) = \varepsilon|x|^2, r(x) = 1$.

$0 \leq q(x) \leq 2\varepsilon = q(1,1)$.

So λ_n is sandwiched between the n-th Dirichlet eigenvalue of $L_0 = -\Delta$ and $L_{2\varepsilon} = -\Delta + 2\varepsilon$, i.e. $\mu_n \leq \lambda_n \leq \mu_n + 2\varepsilon, n \in \mathbb{N}$, where $\mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$ is a reordering of $\{\lambda_{k,j} = (k^2 + j^2)\pi^2, \quad k, j \in \mathbb{N}\} = \{2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, \dots\}$ counting multiplicity and $\lambda_{k,j}$ are eigenvalues of L_0 with eigenfunction

$\phi_{k,j}(x) = \sin(k\pi x_1)\sin(j\pi x_2)$

W13C1 Lecture

Comparison in domains

Let coefficients p, q, r be defined in Ω. We now specify the domain dependence and denote by $\lambda_n(\Omega)$ the n-th eigenvalue of $L = -\nabla \cdot (p\nabla u) + qu$ in Ω with weight $r(x)$ and Dirichlet BC.

📖 **Thm 5.** If $\tilde{\Omega} \subset \Omega$ is a subdomain, then $\lambda_n(\tilde{\Omega}) \geq \lambda_n(\Omega) \quad \forall n \in \mathbb{N}$, the larger set has smaller eigenvalues.

Proof:

We use the max-min principle (**Thm 3**). Denote by $\mathcal{A}(\Omega)$ the set of admissible functions in Ω. $u \in \mathcal{A}(\Omega) \implies u|_{\partial\Omega} = 0$.

Fix any $f_1, \dots, f_{n-1} \in \mathcal{A}(\Omega)$.

For any $u \in \mathcal{A}(\tilde{\Omega})$ in $\tilde{\Omega} \subset \Omega$ with $(\!(u, f_j)\!)_{r,\tilde{\Omega}} = 0, \quad j = 1, \dots, n-1$, consider its extension

$$\hat{u}(x) = \begin{cases} u(x) & x \in \tilde{\Omega} \\ 0 & x \notin \tilde{\Omega} \end{cases}$$

Then $\hat{u} \in \mathcal{A}(\Omega)$ and it satisfies:

- $(\!(\hat{u}, f_j)\!)_{r,\Omega} = 0, \quad j = 1, \dots, n-1$
- $E_\Omega(\hat{u}) = E_{\tilde{\Omega}}(u)$
- $(\!(\hat{u}, \hat{u})\!)_{r,\Omega} = (\!(u, u)\!)_{r,\tilde{\Omega}}$

So

$$\min_{\substack{\hat{u}\in \mathcal{A}(\Omega)\\ ((\hat{u}, f_j))_{r,\Omega}=0\\ j=1,\dots,n-1}} \frac{E_\Omega(\hat{u})}{M_\Omega(\hat{u})} \le \min_{\substack{\hat{u}\in \mathcal{A}(\Omega)\\ ((\hat{u}, f_j))_{r,\Omega}=0\\ j=1,\dots,n-1\\ \&\ \hat{u}(x)=0 \text{ in } \Omega\setminus\tilde{\Omega}}} \frac{E_\Omega(\hat{u})}{M_\Omega(\hat{u})}$$

(over a smaller set)

$$= \min_{\substack{u\in \mathcal{A}(\tilde{\Omega})\\ ((u, f_j))_{r,\tilde{\Omega}}=0\\ j=1,\dots,n-1}} \frac{E_{\tilde{\Omega}}(u)}{M_{\tilde{\Omega}}(u)}$$

Consider both restriction / extension

$$\le \max_{g_1,\dots,g_{n-1}\in \mathcal{A}(\tilde{\Omega})} \left[\min_{\substack{u\in \mathcal{A}(\tilde{\Omega})\\ ((u, g_j))_{r,\tilde{\Omega}}=0\\ j=1,\dots,n-1}} \frac{E_{\tilde{\Omega}}(u)}{M_{\tilde{\Omega}}(u)} \right] = \lambda_n(\tilde{\Omega}).$$

Since $f_j \in \mathcal{A}(\Omega)$ arbitrary,

$$\lambda_n = \max_{f_1,\dots,f_{n-1}\in \mathcal{A}(\Omega)} \left(\text{LHS}\right) \le \text{RHS} = \lambda_n(\tilde{\Omega}). \qquad \square$$

Remark:

For the admissible set $\mathcal{A}(\Omega)$, it is better to choose the Sobolev space:

$\mathcal{A}(\Omega) = H_0^1(\Omega)$ (1: # of derivatives, 0: boundary conditions)

$\mathcal{A}(\Omega) = \{f \in L^2(\Omega) : \nabla f \in L^2(\Omega), f|_{\partial\Omega} = 0\}$

where ∇ is a weak (distributional) derivative. This is the set of square integrable functions with square integrable weak derivatives and 0-BC.

It has better properties than $\mathcal{A}_1(\Omega) = \{u \in C^2(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$.

- One property: if $\tilde{\Omega} \subset \Omega$ and $u \in H_0^1(\tilde{\Omega})$, then the extension

  $$\hat{u}(x) = \begin{cases} x & x \in \tilde{\Omega} \\ 0 & x \in \Omega \notin \tilde{\Omega} \end{cases}$$

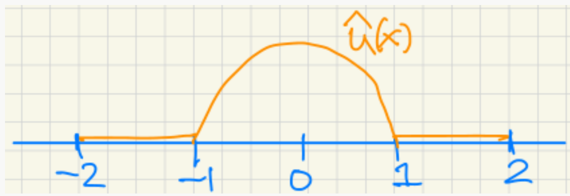
  is in $H_0^1(\Omega)$. This is not true for $\mathcal{A}_1(\Omega)$.

- Another property: any bounded sequence in $H_0^1(\Omega)$ has a subsequence that converges in $H_0^1(\Omega)$.

💡 **Ex 6.** Let $\tilde{\Omega} = (-1, 1) \subset \Omega = (-2, 2)$.

$u(x) = 1 - x^2 \in H_0^1(-1, 1)$.

$$\hat{u}(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & 1 < |x| < 2 \end{cases}$$



Note that $\hat{u}(x) \in H_0^1(-2, 2)$ but $\hat{u}(x) \notin C^1(-2, 2)$.

Exercise:

Find the weak derivative $u'(x)$ in $(-2, 2)$ and verify it is a distributional derivative by definition.