# MATH 121 Formula Sheet

## Integral Facts

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Fundamental Theorem of Calculus

- Part 1: Define $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.
- Part 2: If $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

## Integration Techniques

### Substitution

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \text{ where } u = g(x)$$

### Integration by Parts

$$\int f(g)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

### Trig Integrals

$$\int \sin^m x \cos^n x\,dx$$

- $m$ is odd: use $\sin x$ times power of $\sin^2 x = 1 - \cos^2 x$, then $u = \cos x$
- $n$ is odd: use $\cos x$ times power of $\cos^2 x = 1 - \sin^2 x$, then $u = \sin x$
- Both are even: use $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$, and restart

$$\int \tan^m x \sec^n x\,dx$$

- $n$ is even: use $\sec^2 x$ times power of $\sec^2 x = \tan^2 x + 1$, then $u = \tan x$
- $m$ is odd: use $\tan x$ times power of $\tan^2 x = \sec^2 x - 1$, then $u = \sec x$

### Universal Trig Sub

If $t = \tan\dfrac{x}{2}$, then

$$dx = \frac{2}{1+t^2}\,dt \text{ and } \sin x = \frac{2t}{1+t^2} \text{ and } \cos x = \frac{1-t^2}{1+t^2}$$

### Trig Substitution

| | | |
|---|---|---|
| $\sqrt{a^2+x^2}$ | $1+\tan^2\theta = \sec^2\theta$ | $x = a\tan\theta$ |
| $\sqrt{a^2-x^2}$ | $1-\sin^2\theta = \cos^2\theta$ | $x = a\sin\theta$ |
| $\sqrt{x^2-a^2}$ | $\sec^2\theta - 1 = \tan^2\theta$ | $x = a\sec\theta$ |

- May need to complete the square first.
- Change $\theta$ back to $x$.

### Partial Fractions

$$\frac{P(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n}$$

$(x-r)^m$ corresponds to $\dfrac{A_1}{x-r} + \dfrac{A_2}{(x-r)^2} + \cdots + \dfrac{A_m}{(x-r)^m}$

$ax^2+bx+c$ corresponds to $\frac{Ax+B}{ax^2+bx+c}$

## Numerical Integration

### Midpoint Rule

$$x_i^\dagger = \frac{x_{i-1}+x_i}{2}$$

$$M_n = \sum_{i=1}^{n} f(x_i^\dagger)\Delta x = \Delta x(f(x_1^\dagger) + f(x_2^\dagger) + \cdots + f(x_n^\dagger))$$

### Trapezoid Rule

$$T_n = \frac{\Delta x}{2}(1f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + 1f(x_n))$$

### Simpson's Rule

$$S_n = \frac{\Delta x}{3}(1f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + 1f(x_n))$$

### Error

Exists $c_1 \in (a,b)$ s.t. $\displaystyle\int_a^b f(x)\,dx - M_n = \frac{f''(c_1)}{24}\frac{(b-a)^3}{n^2}$

Exists $c_2 \in (a,b)$ s.t. $\displaystyle\int_a^b f(x)\,dx - T_n = -\frac{f''(c_2)}{12}\frac{(b-a)^3}{n^2}$

If $|f''(c)| \le M$ for all $c \in (a,b)$, then

$$\left|\int_a^b f(x)\,dx - M_n\right| \le \frac{M}{24}\frac{(b-a)^3}{n^2}$$

$$\left|\int_a^b f(x)\,dx - T_n\right| \le \frac{M}{12}\frac{(b-a)^3}{n^2}$$

If $|f^{(4)}(c)| \le L$ for all $c \in (a,b)$, then for any even $n \ge 2$,

$$\left|\int_a^b f(x)\,dx - S_n\right| \le \frac{L}{180}\frac{(b-a)^5}{n^4}$$

## Improper Integrals

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

Discontinuous at $x = a$:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

$\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converges when $p > 1$ and diverges when $p \le 1$.

$\displaystyle\int_0^a \frac{1}{x^p}\,dx$ converges when $p < 1$ and diverges when $p \ge 1$.

### Limit Comparison Test

Assume $f(x) > 0$ and $g(x) > 0$. If $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ exists and is nonzero, then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

## Integration Applications

### Volumes

Disk: $V = \int_a^b \pi R(x)^2\,dx$

Washer: $V = \int_a^b \pi(R(x)^2 - r(x)^2)\,dx$

Shells: $V = \int_a^b 2\pi R(x)h(x)\,dx$

### Average Value

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

### Work

$$W = \int_a^b F(x)\,dx$$

Hooke's Law: $F = kx$
Gravity: $W = mgh = \rho Vgh$
**Center of Mass**

$$\left(\frac{N_x}{M}, \frac{N_y}{M}\right) \text{ where } M = \sum_{i=1}^{n} m_i, N_x = \sum_{i=1}^{n} m_i x_i, N_y = \sum_{i=1}^{n} m_i y_i$$

$M = \int_a^b \rho \cdot f(x)\,dx$
$N_x = \int_a^b \rho \cdot x(f(x) - g(x))\,dx$
$N_y = \int_a^b \rho \cdot \frac{1}{2}(f(x)^2 - g(x)^2)\,dx$

## Differential Equations

Separate: write $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ as $h(y)\,dy = g(x)\,dx$, then integrate
Word problems: express $A'(t)$ in terms of $A(t)$, then solve.

## Sequences

- $\{a_n\}$ has limit $L$: $\lim_{n\to\infty} a_n = L$.
- If $a_n$ is given by $f(n)$ and $\lim_{n\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(x) = L$.
- Squeeze Theorem: $l_n \le a_n \le b_n$, then if $\lim_{n\to\infty} l_n = L$ and $\lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} a_n = L$.

## Series

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} s_N$$
$$a_n = s_n - s_{n-1}$$

**Test for Divergence**
If $\{a_n\}$ does not converge to 0, then $\sum_{n=1}^{\infty} a_n$ diverges.
**Integral Test**
If $f(x)$ is positive and decreasing,
$\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.
**Geometric Series**
$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ converges if and only if $|r| < 1$.
**$p$-Series**
$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.
**Comparison Tests for Infinite Series**
Suppose $b_n \ge 0$ always.

- If $\sum_{n=1}^{\infty} b_n$ converges and $|a_n| \le b_n$, then $\sum_{n=1}^{\infty} a_n$ converges
- If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \ge b_n$, then $\sum_{n=1}^{\infty} a_n$ diverges

**Limit Comparison Test**
Suppose $b_n > 0$ always, and assume $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ exists.
If $L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
**Alternating Series Test**
Suppose $\{u_n\}$ is nonnegative, decreasing and $\lim_{n\to\infty} u_n = 0$. Then
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n u_n$$
converges.
Error: $\left|\sum_{n=1}^{\infty} a_n - s_N\right| \le |a_{N+1}|$ (absolute value of first ommited term)
**Absolute and Conditional Convergence**

- Absolutely convergence: $\sum_{n=1}^{\infty} |a_n|$ converges
- Conditional convergence: $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges
- Absolutely convergent $\implies$ convergent

**Ratio Test**
Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$.

- If $0 \le r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely
- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges
- If $r = 1$, inconclusive

**Root Test**
Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$. Same conclusion as Ratio Test.

## Power Series

- General form: $\sum_{n=0}^{\infty} A_n (x-c)^n$
- Radius of convergence: $R = \lim_{n\to\infty} \left|\frac{A_n}{A_{n+1}}\right|$
- Endpoints must be tested separately

## Taylor Series

Taylor series:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

Maclaurin series:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

**Maclaurin Series that converge for all $x \in \mathbb{R}$**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!}$$

**Maclaurin Series with $R = 1$**

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{2m+1}$$
$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

## Identities and Derivatives

**Trig**

$$\cos(2x) = \cos^2 x - \sin^2 x \qquad \sin(2x) = 2\sin x \cos x$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

**Derivatives**

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}\log|x| = \frac{1}{x}$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx} b^x = b^x \log b$$
$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\log_b x = \frac{1}{x \log b}$$