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[redacted]
Disc 2E

Math 33B - Homework 10

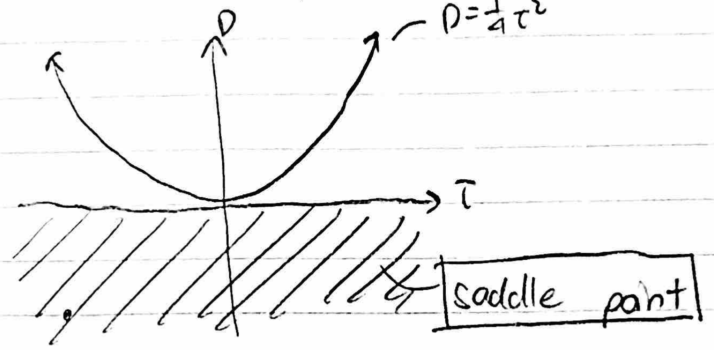
9.4.2

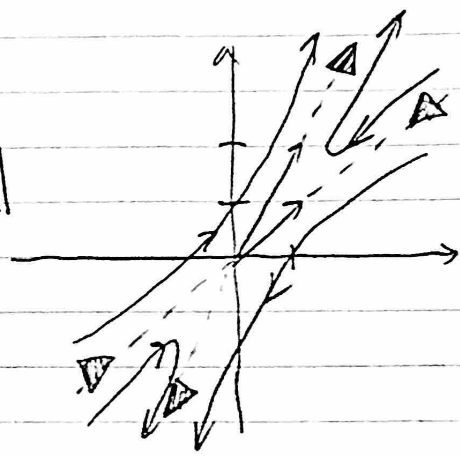
$$A = \begin{bmatrix} -16 & 9 \\ -18 & 11 \end{bmatrix} \qquad \begin{aligned} \tau &= Tr(A) = -5 \\ D &= det(A) = -14 \end{aligned}$$

$$\lambda^2 + 5\lambda - 14 = 0$$
$$(\lambda + 7)(\lambda - 2) = 0$$
$$\lambda_1 = -7, \quad \lambda_2 = 2.$$

$$D \overset{?}{=} \tfrac{1}{4}\tau^2$$
$$-14 \overset{?}{=} \tfrac{1}{4} \cdot 25$$
$$-14 < \tfrac{25}{4}$$

$$D < \tfrac{1}{4}\tau^2$$





$$A\vec{v}_1 = \lambda_1\vec{v}_1$$
$$(A - \lambda I)\vec{v}_1 = \vec{0}$$

$$\lambda = -7: \begin{bmatrix} -16+7 & 9 \\ -18 & 11+7 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-9v_{11} + 9v_{12} = 0$$
$$v_{11} = v_{12} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2: \begin{bmatrix} -16-2 & 9 \\ -18 & 11-2 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

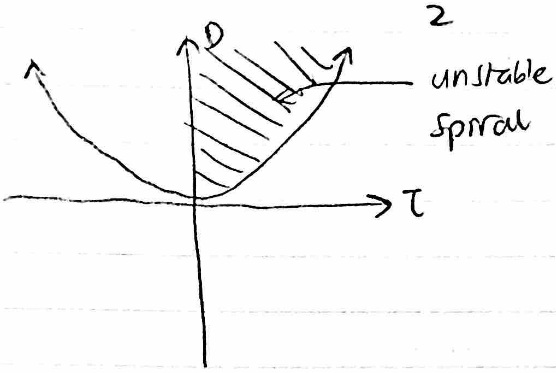
$$-18v_{21} + 9v_{22} = 0$$
$$v_{22} = 2v_{21}$$

9.4.6 $A = \begin{bmatrix} 6 & -5 \\ 10 & -4 \end{bmatrix}$ $\tau = tr(A) = 2$

$D = det(A) = 26$

$\lambda^2 - 2\lambda + 26 = 0$

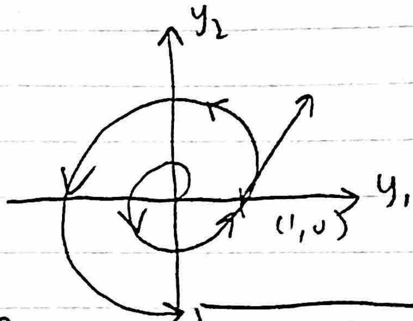
$$\lambda = \frac{2 \pm \sqrt{4 - 4(26)}}{2} = \frac{2 \pm 10i}{2} = 1 \pm 5i$$



$D \overset{?}{=} \frac{1}{4}\tau^2$

$26 > 1$

$D > \frac{1}{4}\tau^2$



$\vec{y}' = A\vec{y}$

let $\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{y}' = \begin{bmatrix} 6 & -5 \\ 10 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$

$\boxed{\text{unstable spiral / spiral source}}$

9.4.10

$A = \begin{bmatrix} -5 & 2 \\ -6 & 2 \end{bmatrix}$ $\tau = tr(A) = -3$

$D = det(A) = -10 + 12 = 2$

$\lambda^2 + 3\lambda + 2 = 0$

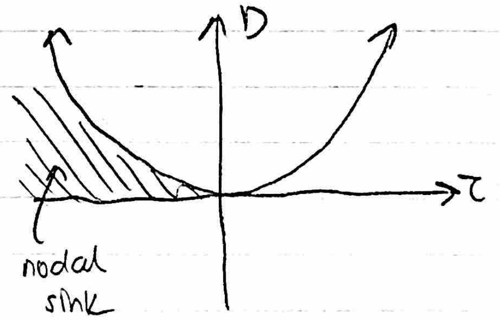
$(\lambda + 1)(\lambda + 2) = 0$

$\lambda_1 = -1, \ \lambda_2 = -2$

$D \overset{?}{=} \frac{1}{4}\tau^2$

$2 < \frac{9}{4}$

$D < \frac{1}{4}\tau^2$



For $\lambda_1 = -1$:

$(A + I)\vec{v}_1 = \begin{bmatrix} -5+1 & 2 \\ -6 & 2+1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-4v_{11} + 2v_{12} = 0$

$\frac{1}{2}v_{12} = v_{11}$

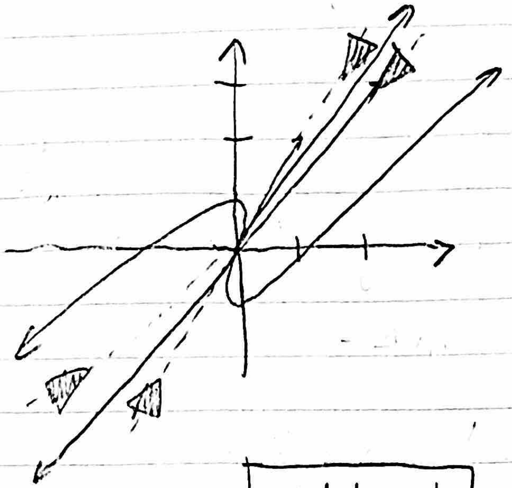
$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $r_2 = -2$:

$$(A+2I)\vec{v}_2 = \begin{bmatrix} -5+2 & 2 \\ -6 & 2+2 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3v_{21} + 2v_{22} = 0$$

$$v_{21} = \tfrac{2}{3}v_{22} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



$$\vec{y} = C_1\vec{y}_1 + C_2\vec{y}_2$$
$$= C_1 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-2t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\boxed{\text{nodal sink}}$

9.6.2

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad e^A = ?$$

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \dots$$
$$= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\boxed{e^A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}}$$

9.6.4

$$A = \begin{bmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad e^A = ?$$

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$A^2 = \begin{bmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^A = I + A + \frac{1}{2}A^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & -2 \\ -1 & \frac{5}{2} & -\frac{1}{2} \\ 1 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix}$$