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FINA 4140 A Computational Finance
Assignment 3
Due Date: April 3, 2014

Answer the following questions for a total of 150 points (10 points for first eleven question and 20 points for last two questions) and show all your work carefully. You do not have to use Microsoft Excel or VBA for the assignment since all computations can also be done using programming environments such as C, C++, EViews, GAUSS, MATLAB, Octave, Ox, R, SAS, or S-PLUS. Through the course, we shall often make use of the expression "MATLAB command": in this case, MATLAB should understood as the *language* which is the common subset of both programs MATLAB and Octave. Please do not turn in the assignment in reams of unformatted computer output and without comments! Make little tables of the numbers that matter, copy and paste all results and graphs into a document prepared by typesetting system such Microsoft Word or LaTeX while you work, and add any comments and answer all questions in this document. Your assignment will be checked for correctness, completeness, and clearness of the answers. I will also check your program and run it myself. If necessary, I will ask you to give me a demo. In addition, I will probably ask you further questions regarding your work. In this case, your response will be also evaluated.

1. a. Suppose $W^{(1)}$ and $W^{(2)}$ are independent white noise processes, i.e., $W^{(1)}(t)-W^{(1)}(s)$ and $W^{(2)}(t') - W^{(2)}(s')$ are independent for all s, t, s', t'. For some ρ, $-1 \leq \rho \leq 1$, let

$$W = \rho W^{(1)} + \sqrt{1-\rho^2}W^{(2)}.$$

   i. Show W is a white noise process.
   ii. Show the correlation between $W^{(1)}$ and W is ρ, i.e., $W^{(1)}(t) - W^{(1)}(s)$ and $W(t) - W(s)$ have correlation coefficient ρ for all s, t.
   iii. What is the correlation between W and $W^{(2)}$?

   b. If the values $W(t_i)$ and $W(t_{i+1})$ of a white noise process have been computed and $t_i < t < t_{i+1}$, how would you compute $W(t)$? Hint: Let

$$\lambda = \frac{t-t_i}{t_{i+1}-t_i} \qquad and \qquad W(t) = (1-\lambda)W(t_i) + \lambda W(t_{i+1}) + \alpha w$$

   where α will depend on λ and w is an independent $\mathcal{N}(0,1)$ variable. Verify $W(t) - W(t_i)$ and $W(t_{i+1}) - W(t)$ have the correct statistical properties.

2. a. Use the white noise paths to approximate $\int_0^1 W dW$, i.e., for $N = 1000$, and $W_i = W(i/N)$ compute $\sum_{i=0}^{N-1} W_i(W_{i+1} - W_i)$. Which is closer, $\frac{1}{2}W(1)^2$ or $\frac{1}{2}W(1)^2 - \frac{1}{2}$? (If your experiment is inconclusive use a larger value of N.)

   b. Use the definition of an Itô integral to show

$$\int_0^t dW(\tau) = W(t) - W(0)$$

and

$$\int_0^t \tau dW(\tau) = tW(t) - \int_0^t W(\tau)d\tau$$

c. Suppose

$$\begin{aligned} dX &= -2(X(t)-1)dt + 4X(t)dW \\ X(0) &= 0 \end{aligned}$$

Use (stochastic) Euler's method to compute the values of $X(t)$ at $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ when W is the particular white noise path computed in the previous exercise.

3. a. Use the definition of the solution of the SDE to show the solution of

$$dX = \mu dt + \sigma dW, \quad X(0) = X_0$$

is

$$X(t) = X_0 + \mu t + \sigma W(t)$$

If $\mu = 0.1$ and $\sigma = 0.25$, what is the probability that $X(1) - X_0 > 0.35$ (respectively < 0.35)? Remember $W(t) - W(0) \sim \mathcal{N}(0, t)$.

b. A share price $S(t)$ is modeled by

$$dS = \mu S(t)dt + \sigma S(t)dW$$

with $\mu = 0.1$ and $\sigma = 0.25$. The present price at $t = 0$ is $S(0) = 1$.

i. What is the SDE for $s(t) = \log(S(t))$?

ii. Write down a formula for $S(t)$. (Hint: $S(t) = e^{s(t)}$ and you have a formula for $s(t)$. Be careful, as the μ's are different!)

iii. What is the probability that $S(1)$ is more than 30% higher that $S(0)$? (Hint: $S(1) = e^{s(1)}$ and $S(1) > 1.3$ iff $s(1) >$?) What is the probability that $S(1)$ is less that 30% lower that $S(0)$?

4. (Higham, Exercise 5.1) Consider the following quote from Eugene Fama, who was Myron Scholes' thesis adviser, which can be found in (Lowenstein, 2001, page 71).

"If the population of price changes is strictly normal, on the average for any stock ... an observation more than five standard deviations from the mean should be observed about once every 7000 years. In fact such observations seem to occur about once every three to four years."

Given that for $X \sim \mathcal{N}(\mu, \sigma^2)$, $\Pr(|X - \mu| > 5\sigma) = 5.733 \times 10^{-7}$, deduce how many observations per year Fama is implicitly assuming to be made.

5. (Higham, Exercise 8.3) Confirm that $C(S, t)$ in equation (8.19)

$$C(S, t) = S\mathcal{N}(d_1) - Ee^{-r(T-t)}\mathcal{N}(d_2)$$

where $\mathcal{N}(\cdot)$ is the $\mathcal{N}(0, 1)$ distribution function,

$$d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \textit{and} \quad d_2 = \frac{\ln(S/E) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},$$

satisfies equations (8.16), (8.17) and (8.18), i.e.,

$$\begin{aligned} C(S,T) &= \max(S(T)-E,0), \\ C(0,t) &= 0, \qquad \forall 0 \le t \le T, \\ C(S,t) &\approx S, \qquad \text{for large } S. \end{aligned}$$

Hint: to deal with (8.16), take the limit $t \to T^-$, to deal with (8.17) take the limit $S \to 0^+$, and to deal with (8.18) take the limit $S \to \infty$.

6. a. (Higham, Exercise 9.3) Using the expression (9.1), i.e., delta of a European call under BSM model:

$$\frac{\partial C}{\partial S} = \mathcal{N}(d_1)$$

confirm the limiting behavior for $\partial C(S,t)/\partial S$ displayed in expression (9.7):

$$\lim_{t \to T^-} \frac{\partial C(S,t)}{\partial S} = \begin{cases} 1, & \text{if } S(T) > E, \\ \frac{1}{2}, & \text{if } S(T) = E, \\ 0, & \text{if } S(T) < E. \end{cases}$$

b. (Higham, Exercise 9.4) Using the expression (9.2), i.e., delta of a European put under BSM model:

$$\frac{\partial P}{\partial S} = \mathcal{N}(d_1) - 1$$

confirm the limiting behavior for $\partial P(S,t)/\partial S$ displayed in expression (9.8):

$$\lim_{t \to T^-} \frac{\partial P(S,t)}{\partial S} = \begin{cases} 0, & \text{if } S(T) > E, \\ -\frac{1}{2}, & \text{if } S(T) = E, \\ -1, & \text{if } S(T) < E. \end{cases}$$

7. a. (Higham, Exercise 10.2) Verify the identity

$$\log\left(\frac{S\mathcal{N}'(d_1)}{e^{-r(T-t)}E\mathcal{N}'(d_2)}\right) = 0,$$

and hence derive equation (10.1):

$$S\mathcal{N}'(d_1) - e^{-r(T-t)}E\mathcal{N}'(d_2) = 0.$$

b. (Higham, Exercise 10.6) Using equation (10.1), show that the partial derivative $\partial C/\partial E$ (which, sadly, does not have a Greek name) satisfies

$$\frac{\partial C}{\partial E} = -e^{-r(T-t)}\mathcal{N}(d_2).$$

Deduce that $\partial C/\partial E < 0$ and interpret this result.

8. a. (Higham, Exercise 11.1) Consider the following 'explanation' of why the Black–Scholes European call option value curve $C(S,t)$ lies above the payoff hockey stick $\max(S(t)-E,0)$, for $t < T$.

"Since $\mathrm{E}(S(t)) = S_0 e^{\mu t}$, the asset price generically drifts upwards. Hence, on average, the asset price will increase between time t and expiry, so the time t value is greater than $\max(S(t)-E,0)$."

Is this argument valid?

b. (Higham, Exercise 11.5) Show that 'Call$(-\sigma) = -$Put(σ)', that is, replacing σ in equation (8.19) by $-\sigma$ is equivalent to evaluating $-P(S,t)$ in equation (8.24). This relation is sometimes called *put–call supersymmetry*.

9. a. (Higham, Exercise 3.4) A continuous random variable X with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0, \end{cases}$$

where $\lambda > 0$, is said to have the *exponential distribution with parameter* λ. Show that in this case $\mathrm{E}(X) = 1/\lambda$. Show also that $\mathrm{E}(X^2) = 2/\lambda^2$ and hence find an expression for $\mathrm{Var}(X)$.

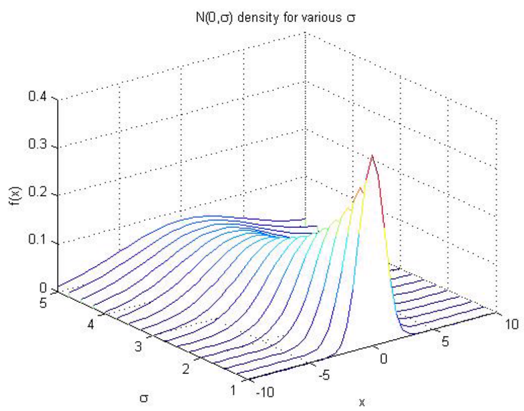
b. (Higham, Programming Exercise 3.2) Consider the MATLAB Program `ch03` for Chapter 3 of Higham (2004):

```
%CH03    Program for Chapter 3
%
% Illustrates Normal distribution

clf

dsig = 0.25;
dx = 0.5;
mu = 0;
[X,SIGMA] = meshgrid(-10:dx:10,1:dsig:5);
Z = exp(-(X-mu).^2./(2*SIGMA.^2))./sqrt(2*pi*SIGMA.^2);
waterfall(X,SIGMA,Z)
xlabel('x')
ylabel('\sigma')
zlabel('f(x)')
title('N(0,\sigma) density for various \sigma')
```

which produces the following figure:



Write an analogue of `ch03` for the exponential density function defined in Exercise 3.4. That is, write a program to produce three-dimensional plot of the $f(x)$ as λ varies.

10. a. (Higham, Exercise 4.1) Some scientific computing packages offer a black-box routine

to evaluate the *error function*, erf, defined by equation (4.8) of Higham (2004), i.e.,

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Show that the $\mathcal{N}(0,1)$ distribution function $N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{s^2}{2}} ds$ can be evaluated as

$$N(x) = \frac{1 + \mathrm{erf}\left(x/\sqrt{2}\right)}{2}$$

b. (Higham, Exercise 4.3) For a given density function $f(x)$ and a given $0 < p < 1$ define the *pth quantile* of f as $z(p)$ in equation (4.6) of Higham (2004), i.e.,

$$\int_{-\infty}^{z(p)} f(x)dx = p.$$

Show that the quantile $z(p)$ for the $\mathcal{N}(0,1)$ distribution function can be written as

$$z(p) = \sqrt{2}\,\mathrm{erfinv}(2p-1)$$

Here, erfinv is the inverse error function; so $\mathrm{erfinv}(x) = y$ means $\mathrm{erf}(y) = x$.

11. a. (Higham, Exercise 4.2) Show that samples from the exponential distribution with parameter λ, as described in Higham (2004) Exercise 3.4, may be generated as $-(\log(\xi_i))/\lambda$, where the $\{\xi_i\}$ are $\mathcal{U}(0,1)$ samples.

b. (Higham, Exercise 4.4) For a given density function $f(x)$ and a given $0 < p < 1$ define the *pth quantile* of f as $z(p)$ in equation (4.6) of Higham (2004), i.e.,

$$\int_{-\infty}^{z(p)} f(x)dx = p.$$

In the case where $f(x)$ is the density for the exponential distribution with parameter $\lambda = 1$, show that the quantile $z(p)$ satisfies $z(p) = -\log(1-p)$.

12. Write a function that implements a linear congruent generator $LCG(a,b,M,N_0)$ with the following inputs and outputs:

inputs: M is the modulus, a the multiplier, b the increment, N_0 the initial "seed"

outputs: P = period of the sequence, $mean$ = average of one period of the real random numbers, var = variance of the sequence of one period of the random numbers, one period and three $(P+3)$ integer numbers.

13. Having a generator that produces a uniform distributed random numbers, write a generator which samples from a standard normal distribution. Implement both versions of the Box–Muller algorithm. Then compare the speed of these versions. Namely, produce a loop for generating 10^n, $n = 2, \ldots, 6$ random numbers and then plot at the same graph the elapsed time as a function of n obtained for both versions.