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FINA 4140 A Computational Finance
Assignment 2
Due Date: February 27, 2014

Answer the following questions for a total of 150 points (10 points for first question and 20 points for remaining questions) and show all your work carefully. You do not have to use Microsoft Excel or VBA for the assignment since all computations can also be done using programming environments such as C, C++, EViews, GAUSS, MATLAB, Octave, Ox, R, SAS, or S-PLUS. Through the course, we shall often make use of the expression "MATLAB command": in this case, MATLAB should understood as the *language* which is the common subset of both programs MATLAB and Octave. Please do not turn in the assignment in reams of unformatted computer output and without comments! Make little tables of the numbers that matter, copy and paste all results and graphs into a document prepared by typesetting system such Microsoft Word or LaTeX while you work, and add any comments and answer all questions in this document. Your assignment will be checked for correctness, completeness, and clearness of the answers. I will also check your program and run it myself. If necessary, I will ask you to give me a demo. In addition, I will probably ask you further questions regarding your work. In this case, your response will be also evaluated.

1. (Higham, Exercise 16.1) Consider the discrete asset price model used in the binomial method. Show that it may be written in the form of equation (6.2) of Higham (2004), i.e.,
$$S(t_{i+1}) = S(t_i) + \mu\Delta t S(t_i) + \sigma\sqrt{\Delta t} Y_i S(t_i)$$
if we let Y_i be defined as in equation (6.10) of Higham (2004), i.e.,
$$Y_i = \begin{cases} \frac{u-1-\mu\Delta t}{\sigma\sqrt{\Delta t}}, & \text{with probability } p, \\ \frac{d-1-\mu\Delta t}{\sigma\sqrt{\Delta t}}, & \text{with probability } 1-p. \end{cases}$$

2. (Higham, Exercise 16.2) Starting from equation (16.4) of Higham (2004), i.e.,
$$\log\left(\frac{S(n\Delta t)}{S_0}\right) = n\log(d) + \log\left(\frac{u}{d}\right)\sum_{i=1}^{n} R_i$$
where R_i is a Bernoulli random variable with parameter p, Show that
$$\mathrm{E}\left(\log\left(\frac{S(n\Delta t)}{S_0}\right)\right) = n\log(d) + \log\left(\frac{u}{d}\right)np$$
and
$$\mathrm{Var}\left(\log\left(\frac{S(n\Delta t)}{S_0}\right)\right) = \left(\log\left(\frac{u}{d}\right)\right)^2 np(1-p).$$

Hence, obtain equations (16.5)–(16.6) of Higham (2004), i.e.,

$$\begin{aligned} p\log(u) + (1-p)\log(d) &= \left(r - \frac{1}{2}\sigma^2\right)\Delta t, \\ \log\left(\frac{u}{d}\right) &= \sigma\sqrt{\frac{\Delta t}{p(1-p)}}, \end{aligned}$$

where we have assumed (or required) the mean of $\log\left(S(n\Delta t)/S_0\right)$ to be $\left(r - \frac{1}{2}\sigma^2\right)\Delta t$ and the variance to be $\sigma^2 n\Delta t$.

3. (Higham, Exercise 16.3) Show that setting $p = \frac{1}{2}$ in equations (16.5)–(16.6) of Higham (2004) produces equation (16.7) of Higham (2004), i.e.,

$$u = \exp\left(\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right) \quad \text{and} \quad d = \exp\left(-\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right).$$

4. (Higham, Exercise 16.4) For the parameter u and d in equation (16.7) of Higham (2004), show that

$$u = 1 + \sigma\sqrt{\Delta t} + r\Delta t + O(\Delta t^{3/2}) \quad \text{and} \quad d = 1 - \sigma\sqrt{\Delta t} + r\Delta t + O(\Delta t^{3/2})$$

as $\Delta t \to 0$. Show also that the corresponding u and d parameters in equation (16.9) of Higham (2004), i.e.,

$$u = \exp\left(r\Delta t\right)\left(1 + \sqrt{\exp\left(\sigma^2\Delta t\right) - 1}\right) \quad \text{and} \quad d = \exp\left(r\Delta t\right)\left(1 - \sqrt{\exp\left(\sigma^2\Delta t\right) - 1}\right)$$

have the same expansions up to $O(\Delta t^{3/2})$. Hint: recall that $\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$ and $e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$ as $x \to 0$.

5. (Higham, Exercise 16.5) We know from Exercise 6.2 of Higham (2004) that if Y_i in equation (16.10) of Higham (2004) has zero mean and unit variance, we recover the continuous asset price model in the limit $\Delta t \to 0$. Set $\mu = r$ and $p = \frac{1}{2}$ and show that requiring $\mathrm{E}(Y_i) = 0$ and $\mathrm{Var}(Y_i) = 1$ in equation (16.10) of Higham (2004) leads to

$$u = 1 + \sigma\sqrt{\Delta t} + r\Delta t \quad \text{and} \quad d = 1 - \sigma\sqrt{\Delta t} + r\Delta t.$$

Note that these values agree with those in Exercise 16.4 of Higham (2004) up to $O(\Delta t^{3/2})$.

6. (Higham, Exercise 16.6) Returning to the recurrence (16.3) of Higham (2004), i.e.,

$$V_n^i = e^{-r\Delta t}\left(pV_{n+1}^{i+1} + (1-p)V_n^{i+1}\right), \quad 0 \le n \le i, \quad 0 \le i \le M-1,$$

we see that for $M = 1$

$$V_0^0 = e^{-r\Delta t}\left(pV_1^1 + (1-p)V_0^1\right),$$

and for $M = 2$

$$\begin{aligned} V_0^0 &= e^{-r\Delta t}\left(pV_1^1 + (1-p)V_0^1\right) \\ &= e^{-r\Delta t}\left(pe^{-r\Delta t}\left(pV_2^2 + (1-p)V_1^2\right) + (1-p)e^{-r\Delta t}\left(pV_1^2 + (1-p)V_0^2\right)\right) \\ &= e^{-2r\Delta t}\left(p^2V_2^2 + 2p(1-p)V_1^2 + (1-p)^2V_0^2\right). \end{aligned}$$

Similarly for $M = 3$ we find that

$$V_0^0 = e^{-3r\Delta t}\left(p^3 V_3^3 + 3p^2(1-p)V_2^3 + 3p(1-p)^2 V_1^3 + (1-p)^3 V_0^3\right).$$

The coefficients $\{1,1\}$, $\{1,2,1\}$, $\{1,3,3,1\}$ are familiar from Pascal's triangle. Having spotted this connection, prove by induction that

$$V_0^0 = e^{-rT}\sum_{k=0}^{M}\binom{M}{k}p^k(1-p)^{M-k}V_k^M,$$

where $\binom{M}{k}$ denotes the binomial coefficient

$$\binom{M}{k} = \frac{M!}{k!(M-k)!}.$$

7. Design and implement an algorithm for calculating the value $V^{(M)}$ of a European or American option.

   INPUT: r (interest rate), σ (volatility), T (time to expiration in years), K (strike price), S (price of asset), and the choices *put* or *call*, and *European* or *American*

   The program should compute a price for the option chosen. In a first step the program should compute the time increment $\Delta t = T/M$ as well as the parameters β, u, d and q, which belong to the underlying binomial model. The program should create an output consisting of the values of these parameters and the price of the option chosen. Control the mesh size $\Delta t = T/M$ adaptively. For example, calculate V for $M = 8$ and $M = 16$ and in case of a significant change in V use $M = 32$ and possibly $M = 64$. Test examples:

   a. put, European, $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = 10$, $S = 5$
   b. put, American, $S = 9$, otherwise as in (a)
   c. call, otherwise as in (a)
   d. The mesh size control must be done carefully and has little relevance to error control. To make this evident, calculate for the test numbers in (a) a sequence of $V^{(M)}$ values, say for $M = 100, 101, 102, \ldots, 150$, and plot the error $\left|V^{(M)} - 4.430465\right|$.
   e. (Optional) Extend your program to evaluate Binary (Digital) Call and Binary (Digital) Put using parameters as in (a). Note: A binary option is a type of option where the payoff is either some fixed amount of some asset or nothing at all. The two main types of binary options are the *cash-or-nothing binary option* and the *asset-or-nothing binary option*. The cash-or-nothing binary option pays some fixed amount of cash Q if the option expires in-the-money while the asset-or-nothing pays the value of the underlying security. Thus, the options are binary in nature because there are only two possible outcomes. They are also called all-or-nothing options, digital options (more common in forex/interest rate markets), and Fixed Return Options (FROs) (on the American Stock Exchange). Binary options is

usually European-style with the following payoff:

$$
\begin{aligned}
\text{Binary (digital) cash-or-nothing call option} &: \quad C_T = \begin{cases} Q & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K \end{cases} \\
\text{Binary (digital) cash-or-nothing put option} &: \quad P_T = \begin{cases} Q & \text{if } S_T < K \\ 0 & \text{if } S_T \geq K \end{cases} \\
\text{Binary (digital) asset-or-nothing call option} &: \quad C_T = \begin{cases} S_T & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K \end{cases} \\
\text{Binary (digital) asset-or-nothing put option} &: \quad P_T = \begin{cases} S_T & \text{if } S_T < K \\ 0 & \text{if } S_T \geq K \end{cases}
\end{aligned}
$$

American binary options exist also, but these automatically exercise whenever the price "touches" the strike price, yielding very different behavior. For this question, you can work with just the European-style binary cash-or-nothing option with $Q = 1$, and the valuation of European-style binary asset-or-nothing option is optional.

8. The equation $ud = 1$ has established a kind of symmetry for the grid in binomial method. As an alternative, one may anchor the grid by requiring (for even M)

$$S_0 u^{M/2} d^{M/2} = K.$$

   a. Give a geometrical interpretation.

   b. Derive from the "moment matching conditions"

$$
\begin{aligned}
e^{r\Delta t} &= pu + (1-p)d \\
e^{2r\Delta t + \sigma^2 \Delta t} &= pu^2 + (1-p)d^2
\end{aligned}
$$

   and $ud = \gamma$ for some constant γ (not necessarily $\gamma = 1$) the relation

$$u = \beta + \sqrt{\beta^2 - \gamma} \quad \text{for} \quad \beta := \frac{1}{2}\left(\gamma e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\right).$$

   c. Show that the solution is given by

$$ud = \gamma := \exp\left(\frac{2}{M} \ln \frac{K}{S_0}\right).$$