1. Problem 1

   Put Y_i as given in the equation (6.10) to (6.2).

   Then with probability p,

   $S(t_{i+1})$

   $= S(t_i) + \mu\Delta t S(t_i) + \sigma\sqrt{\Delta t} Y_i S(t_i)$

   $= S(t_i) + \mu\Delta t S(t_i) + \sigma\sqrt{\Delta t}\left(\frac{u-1-\mu\Delta t}{\sigma\sqrt{\Delta t}}\right) S(t_i)$

   $= S(t_i)(1 + \mu\Delta t + u - 1 - \mu\Delta t)$

   $= uS(t_i)$

   With probability $1 - p$,

   $S(t_{i+1})$

   $= S(t_i) + \mu\Delta t S(t_i) + \sigma\sqrt{\Delta t} Y_i S(t_i)$

   $= S(t_i) + \mu\Delta t S(t_i) + \sigma\sqrt{\Delta t}\left(\frac{d-1-\mu\Delta t}{\sigma\sqrt{\Delta t}}\right) S(t_i)$

   $= S(t_i)(1 + \mu\Delta t + d - 1 - \mu\Delta t)$

   $= dS(t_i)$

2. Problem 2

$$\begin{aligned}
\log\left(\frac{S(n\Delta t)}{S_0}\right) &= n\log(d) + \log\left(\frac{u}{d}\right)\sum_{i=1}^{n} R_i \\
E\left[\log\left(\frac{S(n\Delta t)}{S_0}\right)\right] &= E\left[n\log(d)\right] + E\left[\log\left(\frac{u}{d}\right)\sum_{i=1}^{n} R_i\right] = n\log(d) + \log\left(\frac{u}{d}\right) np
\end{aligned}$$

Note:

$$E\left[\left(\sum_{i=1}^{n} R_i\right)^2\right] = Var\left(\sum_{i=1}^{n} R_i\right) + \left(E\left[\sum_{i=1}^{n} R_i\right]\right)^2 = \sum_{i=1}^{n} Var(R_i) + n^2p^2 = np(1-p) + n^2p^2$$

$$\begin{aligned}
E\left[\left(\log\left(\frac{S(n\Delta t)}{S_0}\right)\right)^2\right] &= n^2(\log(d))^2 + 2n\log(d)log\left(\frac{u}{d}\right)E\left[\sum_{i=1}^{n} R_i\right] + \left(\log\left(\frac{u}{d}\right)\right)^2 E\left[\left(\sum_{i=1}^{n} R_i\right)^2\right] \\
&= n^2(\log(d))^2 + 2n\log(d)\log\left(\frac{u}{d}\right)(np) + \left(\log\left(\frac{u}{d}\right)\right)^2\left(np(1-p) + n^2p^2\right) \\
&= \left(n\log(d) + \log\left(\frac{u}{d}\right)np\right)^2 + \left(\log\left(\frac{u}{d}\right)\right)^2(np(1-p)) \\
Var\left(\log\left(\frac{S(n\Delta t)}{S_0}\right)\right) &= E\left[\left(\log\left(\frac{S(n\Delta t)}{S_0}\right)\right)^2\right] - \left(E\left[\log\left(\frac{S(n\Delta t)}{S_0}\right)\right]\right)^2 = \left(\log\left(\frac{u}{d}\right)\right)^2(np(1-p))
\end{aligned}$$

Given that the mean of $\log\left(\frac{S(n\Delta t)}{S_0}\right)$ is $\left(r - \frac{1}{2}\sigma^2\right)n\Delta t$ and the variance is $\sigma^2 n\Delta t$
$\left(r - \frac{1}{2}\sigma^2\right)n\Delta t = E\left[\log\left(\frac{S(n\Delta t)}{S_0}\right)\right] = n\log(d) + \log\left(\frac{u}{d}\right)np = n\log(d) + np(\log(u) - \log(d))$
$= n((1-p)\log(d) + p\log(u))$
Therefore, $\left(r - \frac{1}{2}\sigma^2\right)\Delta t = (1-p)\log(d) + p\log(u)$

$$\begin{aligned}
\sigma^2 n\Delta t &= Var\left(\log\left(\frac{S(n\Delta t)}{S_0}\right)\right) = \left(\log\left(\frac{u}{d}\right)\right)^2(np(1-p)) \\
\left(\log\left(\frac{u}{d}\right)\right)^2 &= \frac{\sigma^2 n\Delta t}{(np(1-p))} \\
\log\left(\frac{u}{d}\right) &= \sigma\sqrt{\frac{\Delta t}{p(1-p)}}
\end{aligned}$$

3. Problem 3

Put $p = \frac{1}{2}$ in the equations (16.5)-(16.6). Then,

$\frac{1}{2}\log(u) + \left(1 - \frac{1}{2}\right)\log(d) = \left(r - \frac{1}{2}\sigma^2\right)\Delta t$ and $\log\left(\frac{u}{d}\right) = \sigma\sqrt{\frac{\Delta t}{\frac{1}{2}\left(1-\frac{1}{2}\right)}}$

$\frac{1}{2}\left(\log(u) + \log(d)\right) = \left(r - \frac{1}{2}\sigma^2\right)\Delta t$ and $\frac{1}{2}\left(\log(u) - \log(d)\right) = \sigma\sqrt{\Delta t}$

Hence,

$\log(u)$

$= \frac{1}{2}\left(\log(u) - \log(d)\right) + \frac{1}{2}\left(\log(u) + \log(d)\right)$

$= \sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t$

$\Rightarrow u = \exp\left(\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right)$

$\log(d)$

$= -\frac{1}{2}\left(\log(u) - \log(d)\right) + \frac{1}{2}\left(\log(u) + \log(d)\right)$

$= -\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t$

$\Rightarrow d = \exp\left(-\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right)$

4. Problem 4

From equation (16.7),
$u = \exp\left(\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right)$ and $d = \exp\left(-\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right)$
Since $e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$ as $x \to 0$,
as $\Delta t \to 0$,

$$\begin{aligned}
u &= 1 + \sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \frac{1}{2}\left[\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right]^2 + O\left(\left(\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right)^3\right) \\
&= 1 + \sigma\sqrt{\Delta t} + r\Delta t - \frac{1}{2}\sigma^2\Delta t + \frac{1}{2}\sigma^2\Delta t + O(\Delta t^{3/2}) \\
&= 1 + \sigma\sqrt{\Delta t} + r\Delta t + O(\Delta t^{3/2}) \\
d &= 1 - \sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \frac{1}{2}\left[-\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right]^2 + O\left(\left(-\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right)^3\right) \\
&= 1 - \sigma\sqrt{\Delta t} + r\Delta t - \frac{1}{2}\sigma^2\Delta t + \frac{1}{2}\sigma^2\Delta t + O(\Delta t^{3/2}) \\
&= 1 - \sigma\sqrt{\Delta t} + r\Delta t + O(\Delta t^{3/2})
\end{aligned}$$

From equation (16.9),
$u = \exp(r\Delta t)(1 + \sqrt{\exp(\sigma^2\Delta t) - 1})$ and $d = \exp(r\Delta t)(1 - \sqrt{\exp(\sigma^2\Delta t) - 1})$
Since $e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$ as $x \to 0$,
as $\Delta t \to 0$,

$$\begin{aligned}
u &= \left(1 + r\Delta t + O(\Delta t^2)\right)\left(1 + \sqrt{1 + \sigma^2\Delta t + \frac{1}{2}\sigma^4\Delta t^2 + O(\Delta t^3) - 1}\right) \\
&= (1 + r\Delta t)\left(1 + \sqrt{\sigma^2\Delta t + \frac{1}{2}\sigma^4\Delta t^2}\right) + O(\Delta t^{3/2}) \\
&= (1 + r\Delta t)\left(1 + \sigma\sqrt{\Delta t}\sqrt{1 + \frac{1}{2}\sigma^2\Delta t}\right) + O(\Delta t^{3/2})
\end{aligned}$$

Since $\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$ as $x \to 0$,

$$\begin{aligned}
u &= (1 + r\Delta t)\left(1 + \sigma\sqrt{\Delta t}\left(1 + \frac{1}{4}\sigma^2\Delta t + O(\Delta t^2)\right)\right) + O(\Delta t^{3/2}) \\
&= (1 + r\Delta t)(1 + \sigma\sqrt{\Delta t}) + O(\Delta t^{3/2}) \\
&= 1 + \sigma\sqrt{\Delta t} + r\Delta t + O(\Delta t^{3/2})
\end{aligned}$$

$$\begin{aligned}
d &= \left(1 + r\Delta t + O(\Delta t^2)\right)\left(1 - \sqrt{1 + \sigma^2\Delta t + \frac{1}{2}\sigma^4\Delta t^2 + O(\Delta t^3) - 1}\right) \\
&= (1 + r\Delta t)\left(1 - \sqrt{\sigma^2\Delta t + \frac{1}{2}\sigma^4\Delta t^2}\right) + O(\Delta t^{3/2}) \\
&= (1 + r\Delta t)\left(1 - \sigma\sqrt{\Delta t}\sqrt{1 + \frac{1}{2}\sigma^2\Delta t}\right) + O(\Delta t^{3/2}) \\
&= (1 + r\Delta t)\left(1 - \sigma\sqrt{\Delta t}\left(1 + \frac{1}{4}\sigma^2\Delta t + O(\Delta t^2)\right)\right) + O(\Delta t^{3/2}) \\
&= (1 + r\Delta t)(1 - \sigma\sqrt{\Delta t}) + O(\Delta t^{3/2}) \\
&= 1 - \sigma\sqrt{\Delta t} + r\Delta t + O(\Delta t^{3/2})
\end{aligned}$$

5. Problem 5

Y_i

$= \begin{cases} \frac{u-1-\mu\Delta t}{\sigma\sqrt{\Delta t}}, & \text{for probability } p \\ \frac{d-1-\mu\Delta t}{\sigma\sqrt{\Delta t}}, & \text{for probability } 1-p \end{cases}$

$= \begin{cases} \frac{u-1-r\Delta t}{\sigma\sqrt{\Delta t}}, & \text{for probability } \frac{1}{2} \\ \frac{d-1-r\Delta t}{\sigma\sqrt{\Delta t}}, & \text{for probability } \frac{1}{2} \end{cases}$

$E[Y_i] = 0$

$\frac{1}{2}\left(\frac{u-1-r\Delta t}{\sigma\sqrt{\Delta t}}\right) + \frac{1}{2}\left(\frac{d-1-r\Delta t}{\sigma\sqrt{\Delta t}}\right) = 0$

$u + d = 2(1 + r\Delta t)$

$\frac{1}{2}(u + d) = 1 + r\Delta t$

$Var(Y_i) = 1$

$\frac{1}{2}\left(\frac{u-1-r\Delta t}{\sigma\sqrt{\Delta t}} - 0\right)^2 + \frac{1}{2}\left(\frac{d-1-r\Delta t}{\sigma\sqrt{\Delta t}} - 0\right)^2 = 1$

$u^2 + d^2 + 2(1 + r\Delta t)^2 - 2(u + d)(1 + r\Delta t) = 2\sigma^2\Delta t$

$u^2 + d^2 = 2(1 + r\Delta t)^2 + 2\sigma^2\Delta t$

$(u - d)^2$

$= 2(u^2 + d^2) - (u + d)^2$

$= 4(1 + r\Delta t)^2 + 4\sigma^2\Delta t - 4(1 + r\Delta t)^2$

$= 4\sigma^2\Delta t$

With the assumption $u \geq d$, we have

$u - d = 2\sigma\sqrt{\Delta t}$

$\frac{1}{2}(u - d) = \sigma\sqrt{\Delta t}$

$u = \frac{1}{2}(u + d) + \frac{1}{2}(u - d) = 1 + \sigma\sqrt{\Delta t} + r\Delta t$

$d = \frac{1}{2}(u + d) - \frac{1}{2}(u - d) = 1 - \sigma\sqrt{\Delta t} + r\Delta t$

6. Problem 6

Given $M = 1, 2, 3$ are true, we can assume the statement is true for $M = n$, for some constant n, i.e. $V_0^0 = e^{-nrT} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} V_k^n$

Note that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

$$
\begin{aligned}
V_0^0 &= e^{-nrT} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} V_k^n \\
&= e^{-nrT} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (e^{-rT})(pV_{k+1}^{n+1} + (1-p)V_k^{n+1}) \\
&= e^{-(n+1)rT} \left(\sum_{k=0}^{n} \binom{n}{k} p^{k+1} (1-p)^{n-k} V_{k+1}^{n+1} + \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k+1} V_k^{n+1} \right) \\
&= e^{-(n+1)rT} \left(\sum_{k=1}^{n+1} \binom{n}{k-1} p^k (1-p)^{n-k+1} V_k^{n+1} + \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k+1} V_k^{n+1} \right) \\
&= e^{-(n+1)rT} \left(p^{n+1} V_{n+1}^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) p^k (1-p)^{n-k+1} V_k^{n+1} + (1-p)^{n+1} V_0^{n+1} \right) \\
&= e^{-(n+1)rT} \left(p^{n+1} V_{n+1}^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} p^k (1-p)^{n-k+1} V_k^{n+1} + (1-p)^{n+1} V_0^{n+1} \right) \\
&= e^{-(n+1)rT} \left(\sum_{k=0}^{n+1} \binom{n+1}{k} p^k (1-p)^{n-k+1} V_k^{n+1} \right)
\end{aligned}
$$

$M = n + 1$ is true. Therefore, by induction, $V_0^0 = e^{-MrT} \sum_{k=0}^{M} \binom{M}{k} p^k (1-p)^{M-k} V_k^M$

7. Problem 7

The algorithm for calculating the value $V^{(M)}$ of a European or American option (including binary option) is implemented in q7algorithm.cpp

The values obtained using the algorithm are as follows:

(a) Price=4.430403

(b) Price=1.434657

(c) Price=0.012758

(d) $V^{(M)}$ values:

M	$V^{(M)}$	M	$V^{(M)}$	M	$V^{(M)}$
100	4.430215	117	4.429885	134	4.430247
101	4.429936	118	4.430253	135	4.430058
102	4.430189	119	4.430009	136	4.430277
103	4.430071	120	4.430234	137	4.429988
104	4.430114	121	4.430117	138	4.430282
105	4.430165	122	4.430180	139	4.430092
106	4.429991	123	4.430197	140	4.430262
107	4.430220	124	4.430092	141	4.430175
108	4.429838	125	4.430247	142	4.430216
109	4.430235	126	4.429969	143	4.430236
110	4.429987	127	4.430268	144	4.430144
111	4.430209	128	4.430008	145	4.430276
112	4.430104	129	4.430260	146	4.430047
113	4.430142	130	4.430113	147	4.430295
114	4.430188	131	4.430223	148	4.430056
115	4.430034	132	4.430192	149	4.430291
116	4.430238	133	4.430155	150	4.430143

Plot of errors $|V^{(M)} - 4.430465|$:

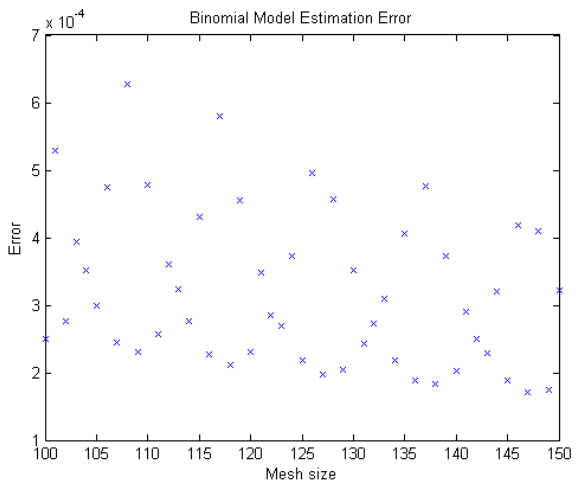

Figure 1: Plot of errors

(e) Prices of the following European options:
Binary Cash-or-nothing Call: 0.012060
Binary Cash-or-nothing Put: 0.929704
Binary Asset-or-nothing Call: 0.133363
Binary Asset-or-nothing Put: 4.866637

8. Problem 8

(a) It gives a symmetric recombining tree.

(b) From $e^{r\Delta t} = pu + (1-p)d$,

$$e^{-r\Delta t} = \frac{1}{pu + (1-p)d}$$

From $e^{2r\Delta t + \sigma^2 \Delta t} = pu^2 + (1-p)d^2$,

$$e^{(r+\sigma^2)\Delta t} = e^{-r\Delta t}(pu^2 + (1-p)d^2) = \frac{pu^2 + (1-p)d^2}{pu + (1-p)d}$$

Given $ud = \gamma$, let $\beta = \frac{1}{2}\left(\gamma e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\right) = \frac{1}{2}\left(\frac{ud}{pu+(1-p)d} + \frac{pu^2+(1-p)d^2}{pu+(1-p)d}\right) = \frac{1}{2}\left(\frac{pu^2+ud+(1-p)d^2}{pu+(1-p)d}\right)$

$$\begin{aligned}
u - \beta &= u - \frac{1}{2}\left(\frac{pu^2 + ud + (1-p)d^2}{pu + (1-p)d}\right) \\
&= \frac{2pu^2 + 2(1-p)ud - pu^2 - ud - (1-p)d^2}{2(pu + (1-p)d)} \\
&= \frac{pu^2 + (1-2p)ud - (1-p)d^2}{2(pu + (1-p)d)} \\
&= \sqrt{\frac{(pu^2 + (1-2p)ud - (1-p)d^2)^2}{4(pu + (1-p)d)^2}} \\
&= \sqrt{\frac{p^2u^4 + (2p - 4p^2)u^3d + (6p^2 - 6p + 1)u^2d^2 + (-4p^2 + 6p - 2)ud^3 + (1-p)^2d^4}{4(pu + (1-p)d)^2}} \\
&= \sqrt{\frac{(pu^2 + ud + (1-p)d^2)^2 - 4ud(pu + (1-p)d)^2}{4(pu + (1-p)d)^2}} \\
&= \sqrt{\frac{(pu^2 + ud + (1-p)d^2)^2}{4(pu + (1-p)d)^2} - ud} \\
&= \sqrt{\beta^2 - \gamma}
\end{aligned}$$

Therefore, $u - \beta = \sqrt{\beta^2 - \gamma}$, which gives $u = \beta + \sqrt{\beta^2 - \gamma}$

(c) Since $ud = \gamma$,
$S_0 u^{\frac{M}{2}} d^{\frac{M}{2}} = K$
$(ud)^{\frac{M}{2}} = \frac{K}{S_0}$
$\gamma^{\frac{M}{2}} = \frac{K}{S_0}$
$\frac{M}{2} \ln \gamma = \ln \frac{K}{S_0}$
$\ln \gamma = \frac{2}{M} \ln \frac{K}{S_0}$
$\gamma = \exp\left(\frac{2}{M} \ln \frac{K}{S_0}\right)$