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FINA 4140 A Computational Finance
Assignment 1
Due Date: February 6, 2014

Answer the following questions for a total of 150 points (questions are equally weighted) and show all your work carefully. You do not have to use Microsoft Excel or VBA for the assignment since all computations can also be done using programming environments such as C, C++, EViews, GAUSS, MATLAB, Octave, Ox, R, SAS, or S-PLUS. Through the course, we shall often make use of the expression "MATLAB command": in this case, MATLAB should understood as the *language* which is the common subset of both programs MATLAB and Octave. Please do not turn in the assignment in reams of unformatted computer output and without comments! Make little tables of the numbers that matter, copy and paste all results and graphs into a document prepared by typesetting system such Microsoft Word or LaTeX while you work, and add any comments and answer all questions in this document. Your assignment will be checked for correctness, completeness, and clearness of the answers. I will also check your program and run it myself. If necessary, I will ask you to give me a demo. In addition, I will probably ask you further questions regarding your work. In this case, your response will be also evaluated.

1. Suppose we have a coin that when tossed results in heads with probability p, and tails with probability $1-p$. Let X be the number of times that we need to toss the coin until r heads are obtained.

   a. What is $\Pr(X = n)$?
   b. Compute $\mathrm{E}(X)$.
   c. Compute $\mathrm{Var}(X)$.
   d. What is $\Pr(X = n|\text{1st head occurs on the 5th toss})$?

2. Suppose the continuous random vector (X, Y) has the joint probability distribution

$$f(x,y) = \begin{cases} c(4x^2y + y^2), & x \in [0,1], y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

   for some constant c.

   a. Calculate $\Pr(X + Y > 1.5)$.
   b. Calculate $\mathrm{E}(Y)$.
   c. What is $\mathrm{Cov}(X, Y)$?
   d. What is $f(x|y)$, the conditional probability density function of x given y?
   e. Compute $\mathrm{E}(X|Y = 0.5)$.

3. A manufacturer of dog food wishes to package the product in cylindrical metal cans, each of which is to contain certain volume V of food. Find the ratio of the height of the can to its radius when the amount of metal used to make the can is minimized. Assume that the thickness of the wall, top, and bottom of the can is everywhere the same, and that you can ignore the material needed for example to join the top to the wall.

4. a. Compute solution(s) of the following system of equations

$$\begin{pmatrix} 2 & 3 & 7 \\ 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

   b. Solve the following system of equations

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 3 \end{pmatrix}$$

   Will this system of equations have a solution for all values of the right hand side vector? If not, produce a vector for which this system has no solution? How is the solution of this problem related to the solution of the previous problem?

5. a. Derive the Newton iterative formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for solving $f(r) = 0$. What is the geometric interpretation of Newton's formula? Explain the rapid convergence of Newton's iteration by showing that the convergence is "quadratic."

   b. Show that Newton's method applied to $f(x) = 1/x - Q = 0$ leads to the iteration $x_{n+1} = x_n(2 - Qx_n)$ for producing reciprocals without division. Apply this iteration with $Q = e = 2.7182818$, starting with $x_0 = 0.3$ and again starting with $x_0 = 1$. How many iterations are needed for six-place accuracy?

6. (Higham, Exercise 13.2) Consider the following approach to computing a sequence of approximations $x_0, x_1, x_2, \ldots$ to $x^\star$. Given x_n, let x_{n+1} be the solution to $p_n(x) = 0$, where $p_n(x)$ is an approximation to $F(x)$ determined by the three conditions: (a) $p_n(x)$ is linear, (b) $p_n(x_n) = F(x_n)$ and (c) $p_n'(x) = F'(x_n)$. Draw a picture to illustrate this construction and then show that x_{n+1} is given by equation (13.2) of Higham (2004), i.e.,

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}.$$

   (Hence, this is an alternative derivation of Newton's method.)

7. (Higham, Exercise 1.2) Convince yourself that $\max(S(T) - E, 0) + \max(E - S(T), 0)$ is equivalent to $|S(T) - E|$ and draw the payoff diagram for this bottom straddle.

8. (Higham, Exercise 1.3) Suppose that for the same asset and expiry date, you hold a European call option with exercise price E_1 and another with exercise price E_3, where $E_3 > E_1$ and also write two calls with exercise price $E_2 := (E_1 + E_3)/2$. This is an example of a *butterfly spread*. Derive a formula for the value of this butterfly spread at expiry and draw the corresponding payoff diagram.

9. (Higham, Exercise 1.4) The holder of the bull spread with payoff diagram in Figure 1.3 of Higham (2004) would like the asset price on the expiry date to be at least as high as E_2, but, if it is, the holder does not care how much it exceeds E_2. Make similar statements about the holders of the bottom straddle in Exercise 1.2 of Higham (2004) and the butterfly spread in Exercise 1.3 of Higham (2004).

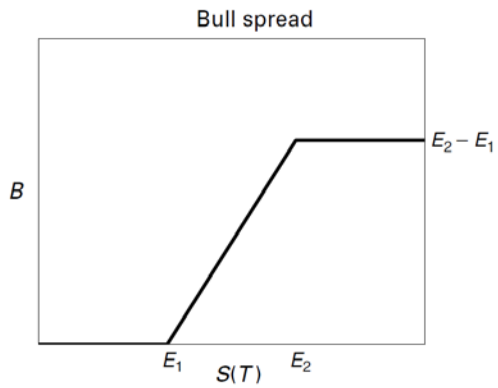

Fig. 1.3. Payoff diagram for a bull spread. Formula is $B = \max(S(T) - E_1, 0) - \max(S(T) - E_2, 0)$.

10. (Higham, Exercise 2.1) *Compound interest* works as follows. An investment D_0 at time zero when compounded m times up to time t at rate r_c becomes worth

$$D(t) = \left(1 + \frac{r_c}{m}t\right)^m D_0.$$

Show that, for a given m, the compound interest rate r_c that produces the same amount as the continuously compounded value $e^{rt}D_0$ satisfies $r_c = m(e^{rt/m} - 1)$. Use the approximation $e^x \approx 1 + x$ for smaller x to show that $r_c \approx r$ when m is large. (Note that in Higham (2004) we always work with continuously compounded interest.)

11. (Higham, Exercise 2.2) The continuously compounded interest rate formula can be derived by

    a. splitting the time interval $[0, t]$ into subintervals $[0, \delta t], [\delta t, 2\delta t], \ldots, [(L-1)\delta t, Lt]$, where $\delta t = t/L$, and

    b. assuming that the value of the investment increases by a relative amount proportional to $r\delta t$ over each subinterval.

Letting $t_i = i\delta t$, this means

$$D(t_{i+1}) = (1 + r\delta t)D(t_i)$$

and hence

$$D(t = t_L) = (1 + r\delta t)^L D_0.$$

By writing this as $D(t) = e^{L\log(1+rt/L)}D_0$ and using $\log(1+\varepsilon) = \varepsilon + O(\varepsilon^2)$ as $\varepsilon \to 0$, show that this model reproduces the formula

$$D(t) = e^{rt}D_0$$

in the limit $L \to \infty$ (i.e., $\delta t \to 0$). Show that the models

$$D(t_{i+1}) = \left(1 + r\sqrt{\delta t}\right) D(t_i)$$

and

$$D(t_{i+1}) = \left(1 + r(\delta t)^{3/2}\right) D(t_i)$$

are not consistent with continuous compounding in the limit $L \to \infty$.

12. (Higham, Exercise 2.3) Give an argument based on the no arbitrage assumption that justifies

$$C \geq S - Ee^{-rT}$$

where C is the value of a European call option, S is the underlying asset price, E is the strike price, r is the continuously compounded interest rate, and T is the expiry date.

13. (Higham, Exercise 2.4) Let P be the value of a European put option, S be the underlying asset price, E be the strike price, r be the continuously compounded interest rate, and T be the expiry date. Establish equation (2.6) of Higham (2004), i.e.,

$$P \geq \max\left(Ee^{-rT} - S, 0\right) \quad \textit{and} \quad P \leq Ee^{-rT},$$

   a. by setting up suitable portfolios and applying the arguments used to get equations (2.4)–(2.5) of Higham (2004), i.e.,

$$\begin{aligned} C &\geq \max\left(S - Ee^{-rT}, 0\right), \\ C &\leq S, \end{aligned}$$

   and,

   b. separately, by using equations (2.4)–(2.5) of Higham (2004) plus put-call parity as in equation (2.2) of Higham (2004), i.e.,

$$C + Ee^{-rT} = P + S.$$

14. (Higham, Exercise 2.5) Show that a butterfly spread with exactly the same payoff as that in Exercise 1.3 of Higham (2004) can be obtained using only a combination of European put options. Use put-call parity as in equation (2.2) of Higham (2004), i.e., $C + Ee^{-rT} = P + S$, to confirm that the two spreads have the same set-up cost.

15. (Higham, Exercise 2.6) A *forward contract*, which is similar to a *futures contract*, operates as follows. Now, at time $t = 0$, Party A agrees to purchase an asset from Party B at a specified delivery time $t = T$ for a specified price F. (Note that Party A is committed to the future purchase – by contrast, with a European call option the holder has the right, but not the obligation, to buy at the prescribed price.) Appealing to the no arbitrage assumption, show that a fair value for F is $S(0)e^{rT}$.