1. Problem 1

   (a) i. Since $W^{(1)}$ and $W^{(2)}$ are white noise processes, we have:
   $W^{(1)}(s) - W^{(1)}(t) \sim N(0, s-t)$ and $W^{(2)}(s) - W^{(2)}(t) \sim N(0, s-t)$ for all s, t with $s > t$
   Hence, $\rho\left(W^{(1)}(s) - W^{(1)}(t)\right) \sim N(0, \rho^2(s-t))$
   $\sqrt{1-\rho^2}\left(W^{(2)}(s) - W^{(2)}(t)\right) \sim N(0, (1-\rho^2)(s-t))$
   For all s, t with $s > t$, $W(s) - W(t) = \rho\left(W^{(1)}(s) - W^{(1)}(t)\right) + \sqrt{1-\rho^2}\left(W^{(2)}(s) - W^{(2)}(t)\right)$
   Since $W^{(1)}$ and $W^{(2)}$ are independent, summing the two independent normal distributions gives:
   $W(s) - W(t) \sim N(0, \rho^2(s-t) + (1-\rho^2)(s-t)) = N(0, s-t)$
   Therefore, W is a white noise process.

   ii. For all s, t with $s > t$,
   define $X = W(s) - W(t), X_1 = W^{(1)}(s) - W^{(1)}(t), X_2 = W^{(2)}(s) - W^{(2)}(t)$
   Since W, $W^{(1)}$ and $W^{(2)}$ are white noise processes,
   $\mathrm{E}[X] = \mathrm{E}[X_1] = \mathrm{E}[X_2] = 0, \mathrm{Var}(X) = \mathrm{Var}(X_1) = \mathrm{Var}(X_2) = s - t$
   Also, since $W^{(1)}$ and $W^{(2)}$ are independent, $\mathrm{Cov}(X_1, X_2) = 0$
   $\mathrm{E}[X_1 X_2] = \mathrm{Cov}(X_1, X_2) + E[X_1]E[X_2] = 0 + 0 = 0$
   $X = W(s) - W(t) = \left(\rho W^{(1)}(s) + \sqrt{1-\rho^2}W^{(2)}(s)\right) - \left(\rho W^{(1)}(t) + \sqrt{1-\rho^2}W^{(2)}(t)\right)$
   $= \rho\left(W^{(1)}(s) - W^{(1)}(t)\right) + \sqrt{1-\rho^2}\left(W^{(2)}(s) - W^{(2)}(t)\right) = \rho X_1 + \sqrt{1-\rho^2}X_2$
   $\mathrm{Cov}(X, X_1) = \mathrm{E}[XX_1] - \mathrm{E}[X]\mathrm{E}[X_1] = \mathrm{E}[XX_1] = \mathrm{E}\left[\rho X_1^2 + \sqrt{1-\rho^2}X_2 X_1\right]$
   $= \rho\mathrm{E}\left[X_1^2\right] + \sqrt{1-\rho^2}\mathrm{E}[X_2 X_1] = \rho\mathrm{E}\left[X_1^2\right]$
   $= \rho\mathrm{Var}(X_1) + \rho\mathrm{E}[X_1]^2 = \rho\mathrm{Var}(X_1) = \rho(s-t)$
   Therefore, the correlation between W and $W^{(1)}$
   $= \frac{\mathrm{Cov}(X, X_1)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(X_1)}} = \frac{\rho(s-t)}{s-t} = \rho$

   iii. Correlation between W and $W^{(2)}$ is $\sqrt{1-\rho^2}$

   (b) Since $t_{i+1} > t > t_i$, $t_{i+1} - t_i > t - t_i > 0$, $1 = \frac{t_{i+1}-t_i}{t_{i+1}-t_i} > \frac{t-t_i}{t_{i+1}-t_i} = \lambda > 0$
   As $\lambda(1-\lambda) > 0$, we can set $\alpha = \sqrt{\lambda(1-\lambda)(t_{i+1} - t_i)}$ and

   $$W(t) = (1-\lambda)W(t_i) + \lambda W(t_{i+1}) + \alpha\omega$$

   where $\omega \sim N(0,1)$ independent from $W(t_i)$ and $W(t_{i+1})$
   i.e. $\mathrm{E}[\alpha\omega] = 0$, $\mathrm{Var}(\alpha\omega) = \alpha^2$, $\mathrm{Cov}(W(t_{i+1}) - W(t_i), \omega) = 0$
   Now verify the statistical properties of $W(t) - W(t_i)$ and $W(t_{i+1}) - W(t)$:

   i. $\mathrm{E}[W(t) - W(t_i)] = \mathrm{E}[\lambda(W(t_{i+1}) - W(t_i)) + \alpha\omega] = \lambda\mathrm{E}[W(t_{i+1}) - W(t_i)] + \mathrm{E}[\alpha\omega] = 0$
   $\mathrm{E}[W(t_{i+1}) - W(t)] = \mathrm{E}[(1-\lambda)(W(t_{i+1}) - W(t_i)) - \alpha\omega] = (1-\lambda)\mathrm{E}[W(t_{i+1}) - W(t_i)] - \mathrm{E}[\alpha\omega] = 0$

   ii. $\mathrm{Var}(W(t) - W(t_i)) = \mathrm{Var}(\lambda(W(t_{i+1}) - W(t_i)) + \alpha\omega)$
   $= \mathrm{Var}(\lambda(W(t_{i+1}) - W(t_i))) + \mathrm{Var}(\alpha\omega) + 2\mathrm{Cov}(\lambda(W(t_{i+1}) - W(t_i)), \alpha\omega)$
   $= \lambda^2\mathrm{Var}(W(t_{i+1}) - W(t_i)) + \alpha^2 + 2\lambda\alpha\mathrm{Cov}(W(t_{i+1}) - W(t_i), \omega)$
   $= \lambda^2(t_{i+1} - t_i) + \lambda(1-\lambda)(t_{i+1} - t_i) + 0 = \lambda(t_{i+1} - t_i) = t - t_i$
   $\mathrm{Var}(W(t_{i+1}) - W(t)) = \mathrm{Var}((1-\lambda)(W(t_{i+1}) - W(t_i)) - \alpha\omega)$
   $= \mathrm{Var}((1-\lambda)(W(t_{i+1}) - W(t_i))) + \mathrm{Var}(\alpha\omega) - 2\mathrm{Cov}((1-\lambda)(W(t_{i+1}) - W(t_i)), \alpha\omega)$
   $= (1-\lambda)^2\mathrm{Var}(W(t_{i+1}) - W(t_i)) + \alpha^2 - 2(1-\lambda)\alpha\mathrm{Cov}(W(t_{i+1}) - W(t_i), \omega)$
   $= (1-\lambda)^2(t_{i+1} - t_i) + \lambda(1-\lambda)(t_{i+1} - t_i) - 0 = (1-\lambda)(t_{i+1} - t_i) = t_{i+1} - t$

iii. Since $W(t_{i+1}) - W(t_i) \sim N(0, t_{i+1} - t_i)$,
$E\left[(W(t_{i+1}) - W(t_i))\right] = 0, \mathrm{E}\left[(W(t_{i+1}) - W(t_i))^2\right] = t_{i+1} - t_i$
Also, as $\omega \sim N(0,1)$ is independent from $W(t_i)$ and $W(t_{i+1})$, we have
$\mathrm{E}\left[(W(t_{i+1}) - W(t_i))\omega\right] = \mathrm{Cov}(W(t_{i+1}) - W(t_i), \omega) = 0$
Since $\mathrm{E}\left[W(t) - W(t_i)\right] = \mathrm{E}\left[W(t_{i+1}) - W(t)\right] = 0$,
$\mathrm{Cov}(W(t) - W(t_i), W(t_{i+1}) - W(t)) = \mathrm{E}\left[(W(t) - W(t_i))(W(t_{i+1}) - W(t))\right]$
$= \mathrm{E}\left[(\lambda(W(t_{i+1}) - W(t_i)) + \alpha\omega)((1-\lambda)(W(t_{i+1}) - W(t_i)) - \alpha\omega)\right]$
$= \mathrm{E}\left[\lambda(1-\lambda)(W(t_{i+1}) - W(t_i))^2 + (1-2\lambda)(W(t_{i+1}) - W(t_i))\alpha\omega - \alpha^2\omega^2\right]$
$= \lambda(1-\lambda)\mathrm{E}\left[(W(t_{i+1}) - W(t_i))^2\right] + (1-2\lambda)\alpha\mathrm{E}\left[(W(t_{i+1}) - W(t_i))\omega\right] - \alpha^2\mathrm{E}\left[\omega^2\right]$
$= \lambda(1-\lambda)(t_{i+1} - t_i) + 0 - \lambda(1-\lambda)(t_{i+1} - t_i) = 0$
Hence we have shown that $W(t) - W(t_i)$ and $W(t_{i+1}) - W(t)$ are uncorrelated.
$W(t) - W(t_i)$ is normally distributed, since $W(t) - W(t_i) = \lambda(W(t_{i+1}) - W(t_i)) + \alpha\omega$, where $\lambda(W(t_{i+1}) - W(t_i))$ and $\alpha\omega$ are normally distributed and independent. Sum of independent normally distributed random variables is also normally distributed. The same holds for $W(t_{i+1}) - W(t)$.
Since $W(t) - W(t_i)$ and $W(t_{i+1}) - W(t)$ are normally distributed, zero correlation implies that they are independent.

Thus, we have shown that $W(t) - W(t_i)$ and $W(t_{i+1}) - W(t)$ have the correct statistical properties.

2. Problem 2

(a) $\displaystyle\int_0^t dW(\tau) = \lim_{N\to\infty}\sum_{i=0}^{N-1}(W(t_{i+1}) - W(t_i)) = \lim_{N\to\infty}(W(t_N) - W(t_0)) = W(t) - W(0)$, where $t_i = \frac{i}{N}t$

$$\int_0^t \tau\, dW(\tau) = \lim_{N\to\infty}\sum_{i=0}^{N-1} t_i\,(W(t_{i+1}) - W(t_i))$$
$$= \lim_{N\to\infty}\sum_{i=0}^{N-1}(t_{i+1}W(t_{i+1}) - t_iW(t_i)) - \lim_{N\to\infty}\sum_{i=0}^{N-1}(t_{i+1}W(t_{i+1}) - t_iW(t_{i+1}))$$
$$= t_NW(t_N) - t_0W(0) - \lim_{N\to\infty}\sum_{i=0}^{N-1}(t_{i+1} - t_i)W(t_{i+1}) = tW(t) - \int_0^t W(\tau)\, d\tau, \text{ where } t_i = \tfrac{i}{N}t$$

(b) Referring to Excel,
$N = 1000, t = 1$, hence we define $t_i = \frac{i}{1000}$ and W_i calculated by:
$W(0) = 0$, $W(t_{i+1}) = W(t_i) + \frac{\omega_i}{\sqrt{1000}}$ for $0 \le i < 1000$
where $\omega_i \sim N(0,1)$ is independent of $\{t_i\}$
Then, $W(t_i)$ approximates a white noise path.
$\displaystyle\int_0^1 W\, dW$ is approximated by $\displaystyle\sum_{i=0}^{999} W(t_i)(W(t_{i+1}) - W(t_i))$
According to our simulation, $\displaystyle\sum_{i=0}^{999} W(t_i)(W(t_{i+1}) - W(t_i)) = -0.27790$
$W(1) = -0.67581$, $\frac{1}{2}W(1)^2 = 0.22836$, $\frac{1}{2}W(1)^2 - \frac{1}{2} = -0.27164$
Therefore, $\frac{1}{2}W(1)^2 - \frac{1}{2}$ is closer.

(c) $N = 1000, t = 1$, hence we define $t_i = \frac{i}{1000}$
Using Euler-Maruyama method, $X(t)$ can be approximated by:
$X(0) = 0$, $X(t_{i+1}) = X(t_i) - 2(X(t_i) - 1)\frac{1}{1000} + 4X(t_i)(W(t_{i+1}) - W(t_i))$ for $0 \le i < 1000$
According to our simulation, $X(0) = 0$
$X(\frac{1}{4}) = X(t_{250}) = 0.052331$
$X(\frac{1}{2}) = X(t_{500}) = 0.234539$
$X(\frac{3}{4}) = X(t_{750}) = 0.418817$
$X(1) = X(t_{1000}) = 0.356394$

3. Problem 3

(a) From the definition of Ito's integral,

$$\int_0^t \sigma\, dW(s) = \lim_{N\to\infty} \sum_{i=0}^{N-1} \sigma\left(W(t_{i+1}) - W(t_i)\right)$$

$$= \lim_{N\to\infty} \sigma \sum_{i=0}^{N-1} \left(W(t_{i+1}) - W(t_i)\right)$$

$$= \lim_{N\to\infty} \sigma\left(W(t) - W(0)\right)$$

$$= \sigma W(t)$$

where $t_i = \frac{i}{N} t$

Hence, $X(t) = X_0 + \mu t + \sigma W(t) = X_0 + \int_0^t \mu\, ds + \sigma W(t)$

$$= X_0 + \int_0^t \mu\, ds + \int_0^t \sigma\, dW(s)$$

Therefore, $X(t) = X_0 + \mu t + \sigma W(t)$ is a solution to the SDE $dX = \mu\, dt + \sigma\, dW$

$\Pr(X(1) - X_0 > 0.35) = \Pr(\mu(1-0) + \sigma(W(1) - W(0)) > 0.35)$

$= \Pr(0.1 + 0.25(W(1) - W(0)) > 0.35)$

$= \Pr(W(1) - W(0) > 1)$

Since $W(1) - W(0) \sim N(0,1)$, we have:

$\Pr(X(1) - X_0 > 0.35) = \Pr(W(1) - W(0) > 1) = 1 - \Phi(1) = 0.1587$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$ denotes the CDF of $N(0,1)$

(b) i. From Ito's Lemma, the SDE for $s(t) = \log(S(t))$ is:

$$ds = \left(\frac{\partial \log(S)}{\partial S}\mu S + 0 + \frac{1}{2}\frac{\partial^2 \log(S)}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial \log(S)}{\partial S}\sigma S\, dW$$

$$= \left(\frac{1}{S}\mu S + \frac{1}{2}\frac{-1}{S^2}\sigma^2 S^2\right) dt + \frac{1}{S}\sigma S\, dW$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW$$

$$= (0.1 - 0.25^2/2)\, dt + 0.25\, dW = 0.06875\, dt + 0.25\, dW$$

ii. From Part(a), the solution to the SDE is:

$s(t) = s(0) + 0.06875t + 0.25W(t)$

$S(t) = e^{s(t)} = e^{s(0)} e^{0.06875t + 0.25W(t)}$

$= e^{0.06875t + 0.25W(t)}$

iii. $\Pr(S(1) > 1.3) = \Pr(s(1) > \log(1.3)) = \Pr(s(1) - s(0) > \log(1.3) - s(0))$

$= \Pr(0.06875 + 0.25(W(1) - W(0)) > \log(1.3))$

$= \Pr(W(1) - W(0) > (\log(1.3) - 0.06875)/0.25)$

$= 1 - \Phi((\log(1.3) - 0.06875)/0.25) = 1 - \Phi(0.7745)$

$= 0.2193$

$\Pr(S(1) < 0.7) = \Pr(s(1) < \log(0.7)) = \Pr(s(1) - s(0) < \log(0.7) - s(0))$

$= \Pr(0.06875 + 0.25(W(1) - W(0)) < \log(0.7))$

$= \Pr(W(1) - W(0) < (\log(0.7) - 0.06875)/0.25)$

$= \Phi((\log(0.7) - 0.06875)/0.25) = \Phi(-1.7017)$

$= 0.0444$

4. Problem 4

   Let n be the number of observations per year Fama assumes to make:

   $\frac{1}{7000n} = 5.733 \times 10^{-7}$

   $n = 249.184 \approx 250$

   This means Fama assumes one observation per trading day.

5. Problem 5

(a) $d_1 = \dfrac{\ln(S/E) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \dfrac{\ln(S/E)}{\sigma\sqrt{T-t}} + \left(\dfrac{2r+\sigma^2}{2\sigma}\right)\sqrt{T-t}$

$d_2 = \dfrac{\ln(S/E)}{\sigma\sqrt{T-t}} + \left(\dfrac{2r-\sigma^2}{2\sigma}\right)\sqrt{T-t}$

As $t \to T^-$, $T - t \to 0$, $\lim\limits_{t\to T^-} d_1 = \lim\limits_{t\to T^-} d_2 = \lim\limits_{t\to T^-} \dfrac{\ln(S/E)}{\sigma\sqrt{T-t}}$

Case 1: $S(T) > E$, $\ln(S(T)/E) > 0$, hence $\lim_{t\to T^-} d_1 = \lim_{t\to T^-} d_2 = +\infty$

$C(S,T) = S(T)N(+\infty) - Ee^{-r(T-T)}N(+\infty) = S(T) - E$

Case 2: $S(T) = E$, $\ln(S(T)/E) = 0$, hence $\lim_{t\to T^-} d_1 = \lim_{t\to T^-} d_2 = 0$

$C(S,T) = S(T)N(0) - Ee^{-r(T-T)}N(0) = S(T)/2 - E/2 = 0$

Case 3: $S(T) < E$, $\ln(S(T)/E) < 0$, hence $\lim_{t\to T^-} d_1 = \lim_{t\to T^-} d_2 = -\infty$

$C(S,T) = S(T)N(-\infty) - Ee^{-r(T-T)}N(-\infty) = 0 - 0 = 0$

Hence, $C(S,T) = \max(S(T) - E, 0)$

(b) As $S \to 0^+$, $\ln(S/E) \to -\infty$

$\lim\limits_{S\to 0^+} d_1 = \lim\limits_{S\to 0^+} \dfrac{\ln(S/E) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = -\infty$

$\lim\limits_{S\to 0^+} d_2 = \lim\limits_{S\to 0^+} \dfrac{\ln(S/E) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = -\infty$

$C(0,t) = 0 - Ee^{-r(T-t)}N(-\infty) = 0 - 0 = 0 \ \forall 0 \le t \le T$

(c) As $S \to \infty$, $\ln(S/E) \to \infty$

$\lim\limits_{S\to\infty} d_1 = \lim\limits_{S\to\infty} \dfrac{\ln(S/E) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \infty$

$\lim\limits_{S\to\infty} d_2 = \lim\limits_{S\to\infty} \dfrac{\ln(S/E) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \infty$

$C(S,t) \approx SN(\infty) - Ee^{-r(T-t)}N(\infty) = S - Ee^{-r(T-t)}$ for large S

Since $S >> Ee^{-r(T-t)}$, $C(S,t) \approx S$ for large S.

6. Problem 6

(a) $d_1 = \dfrac{\ln(S/E) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \dfrac{\ln(S/E)}{\sigma\sqrt{T-t}} + \dfrac{(r+\sigma^2/2)\sqrt{T-t}}{\sigma}$

$\lim\limits_{t\to T^-} d_1 = \lim\limits_{t\to T^-} \left(\dfrac{\ln(S/E)}{\sigma\sqrt{T-t}} + \dfrac{(r+\sigma^2/2)\sqrt{T-t}}{\sigma}\right) = \lim\limits_{t\to T^-} \dfrac{\ln(S/E)}{\sigma\sqrt{T-t}}$

For $S(T) > E$, $\lim_{t\to T^-} d_1 = +\infty$ and $\lim\limits_{t\to T^-} \dfrac{\partial C}{\partial S} = \lim\limits_{t\to T^-} N(d_1) = N(+\infty) = 1$

For $S(T) = E$, $\lim_{t\to T^-} d_1 = 0$ and $\lim\limits_{t\to T^-} \dfrac{\partial C}{\partial S} = \lim\limits_{t\to T^-} N(d_1) = N(0) = \dfrac{1}{2}$

For $S(T) < E$, $\lim_{t\to T^-} d_1 = -\infty$ and $\lim\limits_{t\to T^-} \dfrac{\partial C}{\partial S} = \lim\limits_{t\to T^-} N(d_1) = N(-\infty) = 0$

(b) For $S(T) > E$, $\lim_{t\to T^-} d_1 = +\infty$ and $\lim\limits_{t\to T^-} \dfrac{\partial P}{\partial S} = \lim\limits_{t\to T^-} (N(d_1) - 1) = N(+\infty) - 1 = 0$

For $S(T) = E$, $\lim_{t\to T^-} d_1 = 0$ and $\lim\limits_{t\to T^-} \dfrac{\partial P}{\partial S} = \lim\limits_{t\to T^-} (N(d_1) - 1) = N(0) - 1 = -\dfrac{1}{2}$

For $S(T) < E$, $\lim_{t\to T^-} d_1 = -\infty$ and $\lim\limits_{t\to T^-} \dfrac{\partial P}{\partial S} = \lim\limits_{t\to T^-} (N(d_1) - 1) = N(-\infty) - 1 = -1$

7. Problem 7

(a) $d_1 = \dfrac{\log(S/E) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, d_2 = d_1 - \sigma\sqrt{T-t}$

$\dfrac{\partial d_1}{\partial S} = \dfrac{\partial d_2}{\partial S} = \dfrac{1}{S\sigma\sqrt{T-t}}$

$N'(d_1) = \dfrac{\partial N(d_1)}{\partial S} = \dfrac{\partial N(d_1)}{\partial d_1}\dfrac{\partial d_1}{\partial S} = \dfrac{n(d_1)}{S\sigma\sqrt{T-t}}$

$N'(d_2) = \dfrac{\partial N(d_2)}{\partial S} = \dfrac{\partial N(d_2)}{\partial d_2}\dfrac{\partial d_2}{\partial S} = \dfrac{n(d_2)}{S\sigma\sqrt{T-t}}$

$\dfrac{N'(d_1)}{N'(d_2)} = \dfrac{n(d_1)}{n(d_2)} = \dfrac{\exp(-d_1^2/2)/\sqrt{2\pi}}{\exp(-d_2^2/2)/\sqrt{2\pi}}$

$= \exp\left(\dfrac{d_2^2 - d_1^2}{2}\right)$

$= \exp\left((d_2 - d_1)\dfrac{d_2 + d_1}{2}\right)$

$= \exp\left((-\sigma\sqrt{T-t})\dfrac{\log(S/E) + r(T-t)}{\sigma\sqrt{T-t}}\right)$

$= \exp\left(-\log(S/E) - r(T-t)\right)$

$\log\left(\dfrac{SN'(d_1)}{e^{-r(T-t)}EN'(d_2)}\right)$

$= \log\left(\dfrac{S}{e^{-r(T-t)}E}\right) + \log\left(\dfrac{N'(d_1)}{N'(d_2)}\right)$

$= \log(S/E) + r(T-t) - \log(S/E) - r(T-t)$

$= 0$

Hence,

$\dfrac{SN'(d_1)}{e^{-r(T-t)}EN'(d_2)} = 1$

$SN'(d_1) = e^{-r(T-t)}EN'(d_2)$

$SN'(d_1) - e^{-r(T-t)}EN'(d_2) = 0$

(b) $\dfrac{\partial N(d_1)}{\partial E} = -\dfrac{n(d_1)}{E\sigma\sqrt{T-t}} = -\dfrac{S}{E}\dfrac{\partial N(d_1)}{\partial S} = -\dfrac{S}{E}N'(d_1)$

$\dfrac{\partial N(d_2)}{\partial E} = -\dfrac{n(d_2)}{E\sigma\sqrt{T-t}} = -\dfrac{S}{E}\dfrac{\partial N(d_2)}{\partial S} = -\dfrac{S}{E}N'(d_2)$

$\dfrac{\partial C}{\partial E} = S\dfrac{\partial N(d_1)}{\partial E} - Ee^{-r(T-t)}\dfrac{\partial N(d_2)}{\partial E} - e^{-r(T-t)}N(d_2)$

$= -\dfrac{S}{E}\left(SN'(d_1) - e^{-r(T-t)}EN'(d_2)\right) - e^{-r(T-t)}N(d_2)$

$= -e^{-r(T-t)}N(d_2)$

Now $e^{-r(T-t)} > 0$ and $N(d_2) > 0$ ($d_2 > -\infty$), hence we have:

$$\frac{\partial C}{\partial E} = -e^{-r(T-t)}N(d_2) < 0$$

It shows that the value of an European call option is strictly decrerasing as the strike price increases, other factors (time to maturity, risk free rate, ...) remaining constant

8. Problem 8

(a) The argument is not valid. It said the reason of lying above the hockey stick was the expected increase in asset price.

Indeed, the option price $C(S,t)$ has a lower limit as $\max(S(t)-E,0)$. The true reason that the price lies above is that an European option provides a potential positive payoff to the holder when the stock price is above the strike at maturity, which requires an additional cost and it is known as the time value of the option. The value $\max(S(t)-E,0)$ is only the intrinsic value of the option.

(b) Replacing σ by $-\sigma$,

$d_1^{\text{new}} = \dfrac{\ln(S/E)+(r+\sigma^2/2)(T-t)}{-\sigma\sqrt{T-t}} = -d_1, d_2^{\text{new}} = \dfrac{\ln(S/E)+(r-\sigma^2/2)(T-t)}{-\sigma\sqrt{T-t}} = -d_2$

$C(-\sigma) = SN(-d_1) - Ee^{-r(T-t)}N(-d_2)$

$= -\left(Ee^{-r(T-t)}N(-d_2) - SN(-d_1)\right)$

$= -P(\sigma)$

9. Problem 9

(a) $\mathrm{E}[X]$

$= \int_0^\infty x f(x)\, dx$

$= \lambda \int_0^\infty x e^{-\lambda x}\, dx$

$= \lambda \left(\left[-\frac{x e^{-\lambda x}}{\lambda} \right]_0^\infty - \int_0^\infty -\frac{e^{-\lambda x}}{\lambda}\, dx \right)$

$= \left[-x e^{-\lambda x} \right]_0^\infty + \int_0^\infty e^{-\lambda x}\, dx$

$= \lim_{x\to\infty} (-x e^{-\lambda x}) - 0 + \left[-\frac{e^{-\lambda x}}{\lambda} \right]_0^\infty$

$= -\lim_{x\to\infty} (\frac{x}{e^{\lambda x}}) + 0 - \frac{-1}{\lambda}$

$= -\lim_{x\to\infty} (\frac{1}{\lambda e^{\lambda x}}) + \frac{1}{\lambda}$

$= \frac{1}{\lambda}$

$\mathrm{E}\left[X^2\right]$

$= \int_0^\infty x^2 f(x)\, dx$

$= \lambda \int_0^\infty x^2 e^{-\lambda x}\, dx$

$= \lambda \Gamma(3) \left(\frac{1}{\lambda}\right)^3 \int_0^\infty \frac{1}{\Gamma(3)\left(\frac{1}{\lambda}\right)^3} x^{3-1} e^{-\lambda x}\, dx$

Since $\frac{1}{\Gamma(3)\left(\frac{1}{\lambda}\right)^3} x^{3-1} e^{-\lambda x}$ represents the pdf of a Gamma distribution, we have:

$\int_0^\infty \frac{1}{\Gamma(3)\left(\frac{1}{\lambda}\right)^3} x^{3-1} e^{-\lambda x}\, dx = 1$

Therefore, $\mathrm{E}\left[X^2\right] = \lambda \Gamma(3) \left(\frac{1}{\lambda}\right)^3 = \frac{2}{\lambda^2}$

$\mathrm{Var}(X) = \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$

$= \frac{1}{\lambda^2}$

(b) Refer to q9b.m

The program is similar to ch03 in Chapter 3 of Higham (2004)

The required analogue of ch03 is shown as follows

```
clf

dsig = 0.25;
dx = 0.5;
[X,LAMBDA] = meshgrid(0:dx:20,1:dsig:5);
Y = LAMBDA.*exp(-LAMBDA.*X);
waterfall(X,LAMBDA,Y)
xlabel('x')
ylabel('\lambda')
zlabel('f(x)')
title('exp(\lambda) density for various \lambda')
```

10. Problem 10

(a) $N(x) = \frac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-s^2/2}\, ds$

$= \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{0} e^{-s^2/2}\, ds + \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{x} e^{-s^2/2}\, ds$

$= \dfrac{1}{2} + \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{x} e^{-s^2/2}\, ds$

$= \dfrac{1}{2} + \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{0}^{x/\sqrt{2}} e^{-(\sqrt{2}t)^2/2}\, d\sqrt{2}t$

$= \dfrac{1}{2} + \dfrac{1}{\sqrt{\pi}} \displaystyle\int_{0}^{x/\sqrt{2}} e^{-t^2}\, dt$

$= \dfrac{1 + \text{erf}(x/\sqrt{2})}{2}$

(b) Given that $\displaystyle\int_{-\infty}^{z(p)} f(x)\, dx = p$, we can write it as $F(z(p)) = p$, which means $z(p) = F^{-1}(p)$

For normal distribution,

$N(x) = y = \frac{1+\text{erf}(x/\sqrt{2})}{2}$

$\text{erf}(x/\sqrt{2}) = 2y - 1$

$x/\sqrt{2} = \text{erfinv}(2y - 1)$

$N^{-1}(y) = x = \sqrt{2}\text{erfinv}(2y - 1)$

Therefore, $z(p) = N^{-1}(p) = \sqrt{2}\text{erfinv}(2p - 1)$

11. Problem 11

(a) $f(x) = \lambda e^{-\lambda x}, x > 0$
$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} \lambda e^{-\lambda t}\,dt$
$= \left[-e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x}$
Hence, using the Inverse Transform Method, we have

$$U = F(x) = 1 - e^{-\lambda x}$$

where U is a generated $U(0,1)$ sample.
$e^{-\lambda x} = 1 - U$
$-\lambda x = \log(1-U)$
$x = -\frac{\log(1-U)}{\lambda}$
Let $V = 1 - U$. Then $U = 1 - V$, $\frac{dU}{dV} = -1$, $f_V(v) = f_U(1-u)|-1| = 1$, i.e $V \sim U(0,1)$
Hence, $x = -\frac{\log V}{\lambda}, V \sim U(0,1)$
Therefore, samples from exponential distribution with parameter λ can be generated by $-\frac{\log(\xi_i)}{\lambda}$, where $\{\xi_i\}$ are $U(0,1)$ samples.

(b) The definition is in effect $F(z(p)) = p$
From 11(a), $F(x) = 1-e^{-\lambda x}$ for exponential distribution. With parameter $\lambda = 1$, we have $F(x) = 1-e^{-x}$.
Hence
$1 - e^{-z(p)} = p$
$e^{-z(p)} = 1 - p$
$-z(p) = \log(1-p)$
$z(p) = -\log(1-p)$

12. Problem 12

    Refer to q12.cpp

    The algorithm implemented in q12.cpp is as follows:

```
Input: M = modulus, a = multiplier, b = increment, N[0] = Initial "seed"
  for i = 1, 2, ..., M+5 do //M+5 ensures at least 1 period is generated
    N[i] = (a * N[i-1] + b) mod M
    U[i] = N[i] / M
  end for

  start = 1
  end = 2
  while N[start] <> N[end] do
    if end > M do
      start = start + 1
      end = start
    end if
    end = end + 1
  end while
  P = end - start //Period of the sequence

  sum = 0
  for i = start, start + 1, ..., end - 1 do
    sum = sum + U[i]
  end for
  mean = sum / P //Average of one period of the real random numbers

  sum = 0
  for i = start, start + 1, ..., end - 1 do
    sum = sum + (U[i] - mean) * (U[i] - mean)
  end for
  var = sum / P //Variance of the sequence of one period of the random numbers
Output: P, mean, var
```

13. Problem 13

    Refer to q13.cpp

    The algorithm implemented in q13.cpp is as follows:

    Assume given a random number generator Ugen().

```
Main program:
  for n = 2,3,4,5,6 do
    start = time_now() //time_now() gets the system time
    BoxMuller1(10^n) //Box-Muller method
    end = time_now()
    time_elapsed[n,method1] = end - start

    start = time_now() //time_now() gets the system time
    BoxMuller2(10^n) //Marsagalia polar method
    end = time_now()
    time_elapsed[n,method2] = end - start
  end for
  plot(n,time_elapsed[n,method1], n,time_elapsed[n,method2])

BoxMuller1:
Input: n = Number of random numbers to be generated
  if n mod 2 != 0 do // Box-Muller method generates 2 numbers each time
    n = n + 1
  end if

  for i = 1,2,..., n/2 do
    U1 = Ugen()
    U2 = Ugen()
    theta = 2 * pi * U2 // pi is constant
    rho = sqrt(-2*ln(U1))
    Z1 = rho * cos(theta) // Z1,Z2 are generated random variables
    Z2 = rho * sin(theta)
  end for
Output: None, since we only need to know the time needed

BoxMuller2: // Marsaglia Polar Method
Input: n = Number of random numbers to be generated
  if n mod 2 != 0 do // Box-Muller method generates 2 numbers each time
    n = n + 1
  end if

  for i = 1,2,..., n/2 do
    do
      U1 = Ugen()
      U2 = Ugen()
      V1 = U1 * 2 - 1
      V2 = U2 * 2 - 1
      W = V1 * V1 + V2 * V2
    while W >= 1 OR W <= 0

    Z1 = V1 * sqrt(-2 * ln(W) / W) // Z1,Z2 are generated random variables
```

```
    Z2 = V2 * sqrt(-2 * ln(W) / W)
  end for
Output: None, since we only need to know the time needed
```

Plot of elapsed time of the two versions of Box-Muller algorithm:

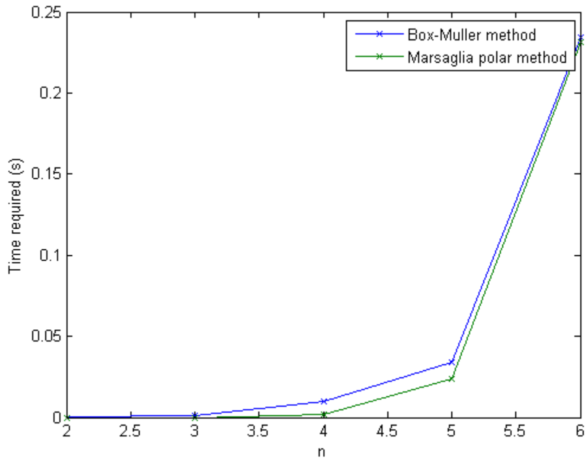

Figure 1: Plot of elapsed time