Syzygies of the Veronese

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1 Introduction

Let R be a ring, and let M be a finitely generated module over R. Let $a_1, a_2, ..., a_n$ be a set of generators for M, and let $f_1 : R^{\oplus n} \to M$ be the map that takes the free generators to the a_i's and extends R-linearly. We can think of the kernel of f_1 as the set of relations between the a_i's. If it remains finitely generated, we may take another set of generators $b_1, b_2, ..., b_k$ of $\ker f_1$ and define a new map $f_2 : R^{\oplus k} \to \ker f_1$. This yields the relations between the generators of $\ker f_1$, which can be thought of as the "relations between the relations" on M. Continuing this process, we might imagine that we can capture much of the complexity of M by analyzing these recursively defined relations. One might also imagine that this process can be standardized by choosing a mimimal set of generators for each module, and defining that set to be the image of the generators of the next free module.

In working through this process, we form one of the most essential tools in homological algebra, the projective resolution. Given a ring R, a module M, and an exact sequence of the form

$$... \to P_n \to P_{n-1} \to ... \to M$$

where each P_i is projective (here we choose free modules), one can compute essential invariants for a module, such as the Tor groups.

Interestingly, we can say much more about these free resolutions when we set R to be the polynomial ring over a field, M to be a graded module, and require that each map in the free resolution respects the graded structure. Hilbert's basis theorem allows us to avoid pathological cases where any of these modules is not finitely generated, and Hilbert's Syzygy Theorem guarantees that the free resolution is not required to be infinitely long. The nice conclusions that these assumptions bring form the motivation and fundamental results in the study of Syzygies.

In section 2, we give basic background surrounding graded rings and modules, and discuss the nice results afforded by the constraints of graded structure. In particular, we introduce minimal free resolutions, discuss a set of powerful invariants named the *Betti Numbers*, and their connection to the Hilbert function for a graded module. We also introduce the Veronese embedding, an important space in Algebraic Geometry, which serves as a case study for this theory that we treat in the next section.

Section 3 is dedicated to computing the minimal free resolution of $\text{Veronese}(n, d)$ in the simplest case, when $n = 1$. We do so by building up the theory of the Eagon-Northcott complex, a chain complex that can automatically serve as the minimal graded free resolution for a wide class of modules, provided that it is exact.

2 Foundational Motivation, Definitions, Results, and Notation

For the sake of this exposition, all rings are commutative with unity.

2.1 Graded Rings and Modules

To begin, we must build some of the basic theory of graded rings and modules.

Definition 2.1.1. A *graded ring* is a ring S whose abelian group structure decomposes as a countable direct sum $S = \oplus_{i=-\infty}^{\infty} S_i$ and whose multiplicative structure satisfies $S_i S_j \subset S_{i+j}$ for all $i, j \geq 0$. The nonzero elements of S_i are referred to as *homogenous of degree* i. We call S_i the ith *graded component of* S.

Definition 2.1.2. A *graded module* is a module M over a graded ring S whose abelian group structure decomposes as a countable direct sum $M = \oplus_{i=-\infty}^{\infty} M_i$ and whose module structure satisfies $S_i M_j \subset M_{i+j}$ for all $i, j \geq 0$. We similarly call M_i the *ith graded component of* M, and also refer to the nonzero elements of M_i as *homogenous of degree* i.

Such a structure is compatible with finite generation in the following sense:

Lemma 2.1.3. *Let M be a finitely generated graded module over a graded ring S. Then M has a finite generating set consisting of homogenous elements.*

Proof. Let $x_1, x_2, ..., x_n$ be a finite generating set. Then each x_i is a sum of homogenous elements $c_{i,1}, c_{i,2}, ..., c_{i,m_i}$ by the direct sum decomposition. Taking all the elements $c_{i,j}$ yields a finite generating set of homogenous elements. □

Example 2.1.4. The motivating example for a graded ring is the polynomial ring $S = k[x_1, x_2, ..., x_n]$, where the subgroup S_i is the set of homogenous forms of degree i (polynomials that only have degree i monomials). With this defined graded structure, S is easily seen to satisfy the definition of a graded ring.

Example 2.1.5. Let S be a graded ring. Taking S to be a module over itself and giving it the same graded structure (letting the ith homogenous elements in the module structure be the ith homogenous elements in the ring structure) clearly makes it into a graded module over itself.

Example 2.1.6. Let M be a graded module over a graded ring S. For $n \in \mathbb{Z}$, let $M(n)$ be the graded module whose ungraded module structure is the same as that of M, and whose graded structure is defined by $(M(n))_k = M_{n+k}$. Shifting each of the homogenous degrees of M in this way is called *twisting* M, and the module $M(n)$ is read M *twisted by* n.

Lemma 2.1.7. *Let M, M' be graded modules. Then $M \oplus M'$ has a graded structure defined by $(M \oplus M')_i = M_i + M'_i$.*

Proof. It's easily checked from the definitions that $M \oplus M' = \bigoplus_{i=-\infty}^{\infty} M_i + M'_i$ and that $R_i(M_j + M'_j) \subset M_{i+j} + M'_{i+j}$, from which the proposition follows immediately. □

The prior example and proposition give rise to a basic class of graded modules that is important for our purposes:

Example 2.1.8. Let S be a graded ring. Then $\oplus_{j=1}^{n} S(n_j)$ is a graded module, with ith homogenous component $\sum_{j=1}^{n} S_{i-n_j} e_j$, where e_j is the basis element in the summand $S(n_j)$.

Definition 2.1.9. Let M be a graded module and let $N \subset M$ be a submodule. If $N = \bigoplus_{i=\infty}^{\infty} M_i \cap N$ (that is, if the modules in the direct sum have trivial pairwise intersection and sum to all of N), then N is called a *graded submodule* of M. If this is the case, N is a graded module with the obvious graded structure $N_i = M_i \cap N$. (The only thing to be checked is that $R_i N_j \subset M_{i+j} \cap N = N_{i+j}$, which is clear from the definitions). Note that to check if N is a graded submodule, it suffices to check that the sum of the modules N_i form N, as the N_i's will always have trivial pairwise intersection because the M_i's do.

Lemma 2.1.10. *Let M be a graded module and let $x \in M$ be homogenous. Then $(x) \subset M$ is a graded submodule.*

Proof. Let $N = (x)$, and assume that $x \in M_i$. Then $xM_{j-i} \subset M_j$, so $xM_{j-i} \subset N \cap M_j = N_j$. Then we have $N = xM = \sum_{j=-\infty}^{\infty} xM_{j-i} \subset \sum_{j=-\infty}^{\infty} N_j$, so N is the sum of the modules $N_j = N \cap M_j$, showing that N is a graded submodule. $\square$

Lemma 2.1.11. *Let N and N' be graded submodules of M. Then $N + N'$ is a graded submodule of M.*

Proof. Note that $N_i + N_i' \subset (N + N') \cap M_i = (N + N')_i$. Then $N + N' = \sum_{i=-\infty}^{\infty} N_i + N_i' \subset \sum_{i=-\infty}^{\infty} (N + N')_i$ because N and N' are graded submodules, so we have $\sum_{i=-\infty}^{\infty} (N + N')_i = N + N'$, and thus the sum is a graded submodule. $\square$

Again, these past two lemmas culminate an important class of graded submodules for our purposes.

Example 2.1.12. Let $x_1, x_2, ..., x_k$ each lie in some graded component of a graded module M. Then $(x_1, x_2, ..., x_k)$ is a graded submodule of M.

For illustrative purposes, we investigate a case where a submodule doesn't admit this graded structure.

Example 2.1.13. Let M be a graded module over S, and let $m \in M$ not lie in any graded component. Then (m) is not a graded submodule of M.

Proof. Since M is the direct sum of its graded components, we can write $m = \sum_{i=1}^{n} m_{\ell_i}$, where $m_{\ell_i} \in M_{\ell_i}$ and the ℓ_i's are in increasing order. Since m doesn't lie in any M_ℓ, the sum has at least two terms. Let $s \in S$ be nonzero, and write $s = \sum_{j=1}^{k} s_{t_j}$ with $s_{t_j} \in S_{t_j}$, where the t_j's are in increasing order and the sum isn't empty because s is nonzero. We see that $sm = \sum_{i=1}^{n} \sum_{\ell=1}^{k} s_{t_j} m_{\ell_i}$. Since the t's and ℓ's are in increasing order, we have that $s_{t_k} m_{\ell_n}$ is the unique summand in $M_{t_k+\ell_n}$ (of highest graded degree) and $s_{t_1} m_{\ell_1}$ is the unique summand in $M_{t_1+\ell_1}$ (of lowest degree). Since there are at least two ℓ_i's, these terms are distinct and lie in different graded components. Thus xm doesn't lie in any graded component.

This shows that for $x \neq 0$, xm doesn't lie in a graded component, so $(m) \cap M_i = 0$ for all i. Thus (m), which is not the sum of these zero modules, is not a graded submodule. $\square$

Now that we've discussed graded submodules of graded modules, we wish to provide quotients by such objects with a canonical graded structure, which we will now do.

Lemma 2.1.14. *Let M be a graded module and let $N \subset M$ be a graded submodule. Then M/N has a graded structure where, letting $q : M \to M/N$ be the quotient map, $(M/N)_i = q(M_i)$.*

Proof. The modules $q(M_i)$ generate M/N as an abelian group because the modules M_i generate M, and q is surjective.

We must now show that they have pairwise trivial intersection. Let $i \neq j$ and let $m' \in q(M_i) \cap q(M_j)$. This is to say that for some $m_i \in M_i$ and $m_j \in M_j$, $q(m_i) = q(m_j) = m'$. This means $q(m_i - m_j) = 0$, so $m_i - m_j \in N$. Since N is a graded submodule, we can then write $m_i - m_j = \sum_{k=0}^{t} n_{c_k}$ where each n_{c_k} lies in $N_{c_k} = M_{c_k} \cap N$ by the graded structure on N. The equality $m_i - m_j = \sum_{k=a}^{b} n_{c_k}$ then equates two sums of elements lying in distinct M_i's, so by the direct sum decomposition $M = \oplus_{i=-\infty}^{\infty} M_i$, we must have $n_i = m_i$, $n_j = -m_j$, and $n_{c_k} = 0$ for all c_k other than i or j.

This gives that $m_i = n_i \in N$, and $m_j = -n_j \in N$, so $m' = q(m_i) = q(m_j) = 0$. This shows that $M/N = \oplus_{i=-\infty}^{\infty} q(M_i)$ as a direct sum of abelian groups.

We also then have $R_i q(M_j) = q(R_i M_j) = q(M_{i+j})$, so the subgroups $q(M_i)$ satisfy all necessary properties to induce a graded structure on M. □

Remark 2.1.15. The graded structures of Lemmas 2.1.7 and 2.1.14 are the standard graded structures placed on a direct sum of graded modules and a quotient of a graded module by a graded submodule respectively. Unless otherwise indicated, we will always implicitly endow these constructions with these graded structures.

Now that our required objects are defined, we define morphisms between them:

Definition 2.1.16. Let M and N be graded modules over a graded ring S, and let $f : M \to N$ be a module homomorphism. If in addition, $f(M_i) \subset N_{i+d}$ for all $i \in \mathbb{Z}$ and a fixed integer d, then f is *a homogeneous homomorphism of degree d* or simply a *degree-d map*.

As one might imagine, this definition fits well with our definition of graded submodules:

Lemma 2.1.17. *Let $f : M \to N$ be a degree-d map of graded modules. Then $\ker f$ and $\operatorname{im} f$ are graded submodules of M and N respectively.*

Proof. Let $I = \operatorname{im} f$. We need to check that $\sum_{i=-\infty}^{\infty} I \cap N_i = I$. This holds because $f(M_i) \subset I \cap N_{i+d}$ for all i, so $I = f(M) = f(\sum_{i=-\infty}^{\infty} M_i) = \sum_{i=-\infty}^{\infty} f(M_i) \subset \sum_{i=-\infty}^{\infty} I \cap N_{i+d}$.

Similarly let $K = \ker f$. We need to check that $\sum_{i=-\infty}^{\infty} K \cap M_i = K$. Let $k \in K$ be any element and write $k = \sum_{i=1}^{n} k_{j_i}$, where the j_i's are distinct and $k_{j_i} \in M_{j_i}$ for all i. Then $0 = q(k) = q(\sum_{i=1}^{n} k_{j_i}) = \sum_{i=1}^{n} q(k_{j_i})$. Since q is a degree-d map, for all i, $q(k_{j_i}) \in N_{j_i+d}$, so the sum $\sum_{i=1}^{n} q(k_{j_i})$ is a sum of terms in distinct homogenous components of N. By the direct sum decomposition of N, each term must thus be 0. This gives that $k_{j_i} \in K$ and thus $k_{j_i} \in M_{j_i} \cap K$ for all i, so $k = \sum_{i=1}^{n} k_{j_i} \in \sum_{i=-\infty}^{\infty} M_i \cap K$. This was shown for an arbitrary element of K, so we have $K \subset \sum_{i=-\infty}^{\infty} M_i \cap K$ and we're done. □

It also fits with our definition of the grading on quotients.

Lemma 2.1.18. *Let M be a graded module and let N be a graded submodule. Let $q : M \to M/N$ be the quotient map. Then q is a degree-0 map.*

Proof. This follows directly from the definition of the graded structure on M/N. □

Interestingly, a version of Nakayama's lemma holds here that is much easier to prove. This will be important for our later discussion of graded free resolutions. Note that when R is a positively graded ring ($R_i = 0$ for $i < 0$), $\bigoplus_{i=1}^{\infty} R_i$ is an ideal, as products of elements in this sum with other elements in R remain in this sum.

Proposition 2.1.19. *(Nakayama's Lemma, Graded Case) Let R be a positively graded ring and let $I = \bigoplus_{i=1}^{\infty} R_i$ be the aforementioned ideal. Let M be a graded module over R and let $m_1, m_2, ..., m_k \in M$ each lie in graded components. Then $\overline{m_1}, \overline{m_2}, ..., \overline{m_k}$ generate M/IM if and only if $m_1, m_2, ..., m_k$ generate M. ([4], Lemma 1.4)*

Proof. The direction in which we assume the elements generate M is trivial. Assume instead that $\overline{m}_1, \overline{m}_2, ..., \overline{m}_k$ generate M/IM. Let $N \subset M$ be the submodule generated by the m_i's, and note that our assumption gives $I + N = M$, so $I(M/N) = M/N$.

By Lemma 2.1.14, M/N has a graded structure. Assume for contradiction that M/N is nonzero, and let $m \in M/N$ be a nonzero homogenous element of minimal degree. Since multiplication by a nonzero element of I increases the degree of the lowest degree term in the direct sum decomposition by at least 1, we see that $m \notin I(M/N)$, so $m \notin M/N$, which is a contradiction.

We conclude that $M/N = 0$, so $M = N$, as desired.

□

We finish this chapter with a technical lemma that will quickly find use in section 2.3.

Lemma 2.1.20. *Let $f : M \to N$ be a degree-d map of graded modules. Then $f(M_c) = f(M) \cap N_{c+d}$.*

Proof. We see $f(M_c) \subset N_{c+d}$ and thus $f(M_c) \subset f(M) \cap N_{c+d}$ by the definition of a degree-d map.

We show the other inclusion. Let $n \in f(M) \cap N_{c+d}$ be arbitrary, and let m lie in $f^{-1}(n)$. As usual, decompose m by writing $m = \sum_{i=1}^{k} m_{j_i}$ where each m_{j_i} lies in M_{j_i}, and see $n = q(m) = \sum_{i=1}^{k} q(m_{j_i})$. Since each m_{j_i} lies in M_{j_i} and f is a degree-d map, $f(m_{j_i}) \in N_{j_i+d}$ for all i. Thus $n = \sum_{i=1}^{k} q(m_{j_i})$ presents n as a sum of terms in distinct homogenous components of N. Thus we must have $q(m_c) = n$ and $q(m_{j_i}) = 0$ for $j_i \neq c$ by the direct sum decomposition. Then $m_c \in M_c$ and $q(m_c) = n$, so $n \in f(M_c)$.

This shows both inclusions, completing the proof.

□

2.2 Graded Free Modules and the Polynomial Ring

Before discussing the fundamental theory of syzygies, we briefly seek to specialize our theory of graded modules to the case we wish to focus on: graded free modules over the polynomial ring. In this exposition, the polynomial ring will always be assumed to have standard grading (the grading of Example 2.1.3).

Remark 2.2.1. Let S be a ring and let $f : S^n \to S^m$ be a module homomorphism. Let $e_1, e_2, ..., e_n$ be the standard basis for S^n and let $E_1, E_2, ..., E_m$ be the standard basis for S^m.

For $1 \leq j \leq n$, f maps e_j to some element of S^m–say $f(e_j) = \sum_{i=1}^{m} a_{i,j} E_i$ where each $a_{i,j}$ is an element of S. Then for an arbitrary element $b = \sum_{j=1}^{n} b_j e_j \in S^n$, we have $f(b) = \sum_{j=1}^{n} \sum_{i=1}^{m} b_j a_{i,j} E_j$. In analogy with the formula for matrix-vector multiplication in linear algebra, we associate such a map with an $m \times n$ matrix $A = (a_{i,j})$ of entries in S.

We then give a lemma that we immediately specialize to maps of free modules:

Proposition 2.2.2. *Let S be a graded ring and let $f : F \to M$ be an (ungraded) module homomorphism, where M is a graded module and $F = S^n$ is free and finitely generated. Let $e_1, e_2, ..., e_n$ be a basis for F.*

Then F can be given a graded module structure of the form $\bigoplus_{i=1}^n S(-a_i)$ (where in this expression, S has the standard graded module structure over itself as seen in Example 2.1.4) such that f is a degree-0 map if and only if for all $1 \leq i \leq n$, $f(e_i) \in M_{a_i}$.

Proof. We first assume that for all $1 \leq i \leq n$, $f(e_i) \in M_{a_i}$. Then give F the grading $\oplus_{i=1}^n S(-a_i)$. Elements in the dth homogenous component of $S(-a_i)$ take the form c_ie_i with $c_i \in S_{d-a_i}$, so elements in the dth homogenous component of F take the form $c = \sum_{i=1}^n c_ie_i$ with $c_i \in S_{d-a_i}$ for each i. Then we compute $f(c) = \sum_{i=1}^n c_if(e_i)$. Since $f(e_i) \in M_{a_i}$ for each i, $c_if(e_i) \in M_d$ for each i, so $f(c) \in M_d$ as a sum of terms in M_d. This shows that f sends F_d to M_d, so it is a degree-0 map.

Assume conversely that F can be given the graded structure $F = \bigoplus_{i=1}^n S(-a_i)$ for which f is a degree-0 map. Then e_i lies in the a_i'th graded degree, so we must have $f(e_i) \in M_{a_i}$ for all i. □

Corollary 2.2.3. *Let $f : F \to \bigoplus_{i=1}^n S(-a_i)$ be an (ungraded) module homomorphism where $F = S^m$ is free and finitely generated. Let $M = (m_{i,j})$ be the matrix associated with this map. Then F can be given a graded module structure of the form $F = \bigoplus_{i=1}^m S(-b_j)$ with respect to which f is a degree-0 map if and only if for all $1 \leq j \leq m$ and $1 \leq i \leq n$, $m_{i,j} \in S_{b_j - a_i}$.*

Proof. This follows immediately from Prop. 2.2.2, where $M = \bigoplus_{i=1}^n S(-a_i)$. □

We can specialize even further for polynomial rings:

Corollary 2.2.4. *Let $S = k[x_1, x_2, ..., x_n]$ be a polynomial ring and let $f : F \to \bigoplus_{i=1}^n S(-a_i)$ be an (ungraded) module homomorphism where $F = S^m$ is free and finitely generated. Let $M = (m_{i,j})$ be the matrix associated with this map. Then F can be given a graded module structure of the form $F = \bigoplus_{i=1}^m S(-b_i)$ with respect to which f is a degree-0 map if and only if for all $1 \leq j \leq m$ and $1 \leq i \leq n$, $m_{i,j}$ is a homogenous $b_j - a_i$ form.*

Proof. The c-forms are the degree c homogenous components of S, and the corollary follows immediately from 2.2.3. □

Finally we make a note that will have important consequences in the next subchapter:

Lemma 2.2.5. *Let $S = k[x_1, x_2, .., x_n]$ be a polynomial ring and let M be a graded module over S. Then for each $i \in \mathbb{Z}$, M_i is a k-vector subspace of M. If $f : M \to N$ is a degree-d map, each restriction $f : M_i \to N_{i+d}$ is a linear map.*

Proof. Since the elements of k are have degree 0 in S, $kM_i \subset M_i$ for all i, showing that M_i is a k-vector space. Linearity of the restriction of f then follows directly from the fact that f is an S-module homomorphism and $k \subset S$. □

2.3 Hilbert's Function and Betti Numbers

Throughout all of chapter 2.3, let S be the polynomial ring.

The Hilbert function and the Betti numbers are a set of extremely natural invariants for finitely generated graded modules over the polynomial ring, which we investigate by use of a couple of somewhat surprising results that we discuss in this chapter. First, we define the Hilbert function:

Definition 2.3.1. Let M be a finitely generated graded module over S. We define the *Hilbert function of* M to be $H_M(d) = \dim_k M_d$, noting that M_d is a subspace by Lemma 2.2.5.

When M is finitely generated, we can see that this dimension is always finite:

Lemma 2.3.2. *Let M be a finitely generated graded module over S. Then $H_M(d)$ is finite for all $d \in \mathbb{Z}$.*

Proof. Pick a finite homogenous generating set $m_1, m_2, ..., m_k$ for M (which can be done by Lemma 2.1.3). For each i, let m_i lie in M_{d_i}. Define a surjective map of modules $f : F = S^k \to M$ that takes the basis $e_1, e_2, ..., e_k$ of S^n to $m_1, m_2, ..., m_k$. By Proposition 2.2.2, we can give F a graded structure $F = \bigoplus_{i=1}^k S(-d_i)$ with respect to which f is a degree-0 map. By Lemma 2.19, given this structure, $f(F_d) = f(F) \cap M_d = M_d$ for all integers d, so such a restriction of f is a linear surjection onto M_d.

As a vector space, S_c has finite dimension for all c (having a basis consisting of the finitely many degree-i homogenous forms). Then $(S(-a_i))_c$, and thus F_c has finite dimension for all integers c. So M_d is the image of a linear map from a finite dimensional vector space for each d, and thus finite dimensional.

□

We then ask how we could attempt to compute this function. Given a finitely generated graded module M, we can pick a finite generating set $m_1, m_2, ..., m_k$ for M consisting of homogenous elements. Through use of Prop. 2.2.2, we may form a surjection $F_0 \to M$ where $F = \bigoplus_{i=1}^k S(-a_i)$ is a free module and the map has degree 0. Since S is Noetherian by the Hilbert Basis theorem, the kernel of this map is also finitely generated, and we may create another map surjecting a free module onto its kernel. Continuing in this fashion, we create a free resolution

$$\to F_n \to ... \to F_1 \to F_0 \to M \to 0$$

where all the maps are degree 0. This gives us our next definition:

Definition 2.3.3. Define a *graded free resolution* to be a free resolution

$$\to F_n \to ... \to F_1 \to F_0$$

of graded finitely generated free modules where each free module is given a graded structure of the form $F_i = \bigoplus_{j=1}^{k_i} S(-a_j)$, F_0 only has generators in degree 0, and all the maps are degree 0.

Such a construction restricts to a free resolution of vector spaces in each graded component.

Lemma 2.3.4. *Let $... \to F_n \to F_{n-1} \to ... \to F_1 \to F_0$ be a graded free resolution. Then for any integer d, the chain $... \to (F_n)_d \to (F_{n-1})_d \to ... \to (F_1)_d \to (F_0)_d$, where the maps are restrictions of the initial maps, is an exact sequence of vector spaces.*

Proof. Let $\phi_i : F_i \to F_{i-1}$ be the maps in this graded complex and let $\phi_{i,d} : (F_i)_d \to (F_{i-1})_d$ be the restricted maps. Then we have

$$\ker(\phi_{i,d}) = \ker(\phi_i) \cap (F_i)_d = \phi_{i+1}(F_{i+1}) \cap (F_i)_d = \phi_{i+1}((F_{i+1})_d) = \operatorname{im}(\phi_{i+1,d})$$

where the third equality holds by Lemma 2.1.20. This shows that the restricted maps form an exact sequence. □

It is then tempting to claim that for a graded free resolution $F_\bullet$ of M, $H_M(d) = \sum_i (-1)^i H_{F_i}(d)$. The issue is that this sum might be infinite and divergent. This problem is solved by a foundational theorem in the study of Syzygies, the aptly named Hilbert Syzygy Theorem. Before discussing this theorem, we need to make a quick aside to discuss minimal free resolutions. These will be our standard choice of free resolution for computing Hilbert's function because of a surprising uniqueness result discussed below, but also serve as essential tool in proving the Syzygy Theorem.

Definition 2.3.5. Let $\mathfrak{m} \subset S = k[x_1, x_2, ..., x_n]$ be the maximal ideal $(x_1, ..., x_n)$. A graded free resolution

$$\to F_n \xrightarrow{\varphi_n} ... \to F_1 \xrightarrow{\varphi_1} F_0$$

is *minimal* if for all i, $\varphi_i(F_i) \subset \mathfrak{m}F_{i-1}$. This is equivalent to the matrix representation of each $\varphi_i(F_i)$ having no entries in k since each entry must be homogenous.

This may initially feel like a strange definition of minimality, but it can quickly be seen to be equivalent to a more natural definition.

Proposition 2.3.6. *A graded free resolution*

$$\to F_n \xrightarrow{\varphi_n} ... \to F_1 \xrightarrow{\varphi_1} F_0$$

is minimal if and only if each φ_i takes the standard basis of F_i to a minimal generating set of its image. ([4], Corollary 1.5)

Proof. Let $\mathfrak{m} \subset S = k[x_1, x_2, ..., x_n]$ be the maximal ideal $(x_1, ..., x_n)$. Form the right exact sequence $F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \operatorname{im}(\varphi_{n-1}) \to 0$. By tensoring with $S/\mathfrak{m}$, we get another right exact sequence $F_n/\mathfrak{m}F_n \xrightarrow{\varphi'_n} F_{n-1}/\mathfrak{m}F_{n-1} \xrightarrow{\varphi'_{n-1}} \operatorname{im}(\varphi_{n-1})/\mathfrak{m}\operatorname{im}(\varphi_{n-1}) \to 0$.

By definition, $F_\bullet$ is minimal if and only if for each φ_n, the induced map φ'_n is 0. This holds if and only if each induced map φ'_{n-1} is an isomorphism onto its image, by the right exact sequence above. By Nakayama's lemma, this holds if and only if each φ_i maps the standard basis of F_i to a minimal generating set of its image.

□

From Prop. 2.3.6, it's then clear that any finitely generated free module over S has a minimal graded free resolution–just undergo the same process used to construct any free resolution, but insist on choosing a minimal generating set for each module when defining the map out of each free module. With these results in hand, we may finally approach the proof of the Syzygy theorem.

Theorem 2.3.7. *(Hilbert's Syzygy Theorem) Let M be a finitely generated free module over S, a polynomial ring on n indeterminates. Then M has a finite graded free resolution of finitely generated modules of the form*

$$0 \to F_n \to ... \to F_1 \xrightarrow{\varphi} F_0 \to F_0/\operatorname{im}(\varphi) \cong M$$

([4], Theorem 1.1)

Proof. This proof makes use of Corollary 3.5.6, a much later result that nevertheless doesn't rely on Hilbert's Syzygy theorem. While there are certainly other proofs that the Koszul complex provides a free resolution of $(x_1, x_2, ..., x_n) \subset S$, we find it redundant to provide a separate one here.

Let $\mathfrak{m} = (x_1, x_2, ..., x_n) \subset S$. The proof hinges around computing $\mathrm{Tor}^S_{n+1}(M, k)$, where $k = S/\mathfrak{m}$ in two different ways. First, Corollary 3.5.8 yields the existence of a free resolution

$$0 \to G_n \to ... \to G_1 \xrightarrow{\psi} G_0 \to G_0/\operatorname{im}(\psi) \cong k$$

of k. Tensoring this resolution with M and noting that the the module in the $n+1$st homological degree is 0 clearly yields that $\mathrm{Tor}^S_{n+1}(M, k) = 0$. Alternatively, let

$$... \to F_{n+1} \to F_n \to ... \to F_1 \xrightarrow{\varphi} F_0 \to F_0/\operatorname{im}(\varphi) \cong M$$

be a minimal free resolution of M. Tensoring this resolution with k yields a new complex

$$... \to F_{n+1} \otimes k \to F_n \otimes k \to ... \to F_1 \otimes k \xrightarrow{\varphi} F_0 \otimes k$$

Since the image of each map $F_{i+1} \to F_i$ is contained in $\mathfrak{m}F_i$ by definition of the minimal free resolution, the image of each map $F_{i+1} \otimes k \to F_i \otimes k$ is contained in $\mathfrak{m}F_i \otimes k = F_i \otimes \mathfrak{m}k = F_i \otimes 0 = 0$. Thus all the maps in the tensor complex are zero maps, so $H_i(F_\bullet \otimes k) = F_i \otimes k$ for all i. In particular, $0 = \mathrm{Tor}^S_{n+1}(M, k) = H_{n+1}(F_\bullet \otimes k) = F_{n+1} \otimes k$, so the minimal free resolution vanishes in the $n+1$st homological degree. Truncating the resolution there then yields a finite graded free resolution of the desired form.

□

This immediately gives rise to our desired result:

Proposition 2.3.8. *Let M be a finitely generated module over S. Let it have finite graded free resolution*

$$0 \to F_n \to ... \to F_1 \to F_0$$

Then for any integer d, $H_M(d) = \sum_{i=0}^{n} (-1)^i H_{F_n}(d)$.

Proof. Lemma 2.3.4 gives that

$$0 \to (F_n)_d \to (F_{n-1})_d \to ... \to (F_1)_d \to (F_0)_d$$

is an exact sequence of k-vector spaces. The result follows by easy induction and the rank-nullity theorem. □

In fact, when the graded degrees of each F_n are known, a more precise formulation can be given:

Proposition 2.3.9. *Let M be a finitely generated module over S, a polynomial ring on $n+1$ indeterminates. Let it have finite graded free resolution*

$$0 \to F_n \to ... \to F_1 \to F_0 \to M \to 0$$

where for each i, $F_i = \bigoplus_j S(-a_{i,j})$. Then for any integer d, $H_M(d) = \sum_{i=0}^{n} (-1)^i \sum_j \binom{n+d-a_{i,j}}{n}$. ([4], Corollary 1.2)

Proof. Lemma 2.3.8 yields that $H_M(d) = \sum_{i=0}^{n}(-1)^i H_{F_n}(d)$, so it suffices to show that $H_{F_n}(d) = \sum_j \binom{n+d-a_{i,j}}{n}$. Noting the direct sum decomposition of F_n, it suffices to show that $H_{S(-a_{i,j})}(d) = \binom{n+d-a_{i,j}}{n}$, which is equivalent to showing that $H_S(d) = \binom{n+d}{n}$. We note that $H_S(d)$ is simply the number of monomials of degree d, which can be computed combinatorially as the number of ways to place d balls in $n+1$ labeled boxes, which is known to be $\binom{n+d}{n}$. ☐

This gives rise to an interesting corollary, explaining why the Hilbert function is sometimes referred to as the *Hilbert Polynomial*.

Corollary 2.3.10. *Let M be a finitely generated module over S a polynomial ring on $n+1$ indeterminates. Let it have finite graded free resolution*

$$0 \to F_n \to \ldots \to F_1 \to F_0$$

where for each i, $F_i = \bigoplus_j S(-a_{i,j})$. Then there exists a polynomial P_M such that for $d \geq \max_{i,j}\{a_{i,j} - n\}$, $P_M(d) = H_M(d)$ ([4], Corollary 1.3).

Proof. By definition of the binomial coefficient, for $d \geq a_{i,j} - n$, $\binom{n+d-a_{i,j}}{n}$ agrees with a polynomial of degree n over d. By Prop. 2.3.9 then, for $d \geq \max_{i,j}\{a_{i,j} - n\}$, $H_M(d)$ agrees with a sum of polynomials. ☐

We then have an effective method of computing the Hilbert function–find a minimal free resolution and plug the generators in to Prop. 2.3.9. This gives us access to the computation of a concrete invariant for finitely generated modules over S. It turns out, however, that minimal free resolutions give us access to even finer invariants. This is as a result of the following deeply surprising uniqueness result, that will not be proven here:

Theorem 2.3.11. *Let M be a finitely generated graded S-module. Let $F_\bullet$ and $G_\bullet$ be minimal graded free resolutions of M. Then there is an isomorphism $F_\bullet \to G_\bullet$ of chain complexes consisting of degree-0 maps inducing the identity map on M. ([4], Theorem 1.6)*

This result is incredibly strong. It shows that not only can minimal graded free resolutions be used to compute the Hilbert polynomial, but the minimal graded free resolution is *itself an invariant* for M. We give a few examples of conclusions we can make from here. First, the minimal free resolution is embedded in any other free resolution:

Proposition 2.3.12. *Let $G_\bullet$ be any graded free resolution that resolves to M. Let $F_\bullet$ be its minimal free resolution. Then $F_\bullet$ is a direct summand of $G_\bullet$. ([4], Theorem 1.6)*

Proof. Let $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$. If $G_\bullet$ is not the minimal free resolution of M, for some map $G_{i+1} \to G_i$, some basis element e of G_{i+1} is mapped outside $\mathfrak{m}G_i$. Let it get mapped to $c_1 \in G_{i+1} \backslash \mathfrak{m}G_i$. By use of Nakayama's lemma (examining the preimages of $G_i \to G_i/\mathfrak{m}G_i$), we can form a new homogenous basis $c_1, c_2, \ldots, c_k \in G_i$ that includes c_1. Reassigning G_i this new free basis, we see that the map from G_{i+1} to G_i restricts to an isomorphism of the form

$$H_\bullet = 0 \to S(-a) \to S(-a) \to 0$$

where the first $S(-a)$ is generated by e, and the next is generated by c_1. This allows us to form a short exact sequence of the form

$$0 \to H_\bullet \to G_\bullet \to G_\bullet/H_\bullet \to 0$$

where the quotient is exact because $H_\bullet$ and $G_\bullet$ both are, and which splits because all modules in all sequences are free. Then $G_\bullet/H_\bullet$ is a different graded free resolution for M that is a summand of $G_\bullet$ with strictly less generators in two homological degrees. Recursively continuing this process, we can give a direct sum decomposition of $G_\bullet$ as $G'_\bullet \oplus K_\bullet$ for which the maps

$$G'_{n+1} \to G'_n \to ... \to G'_1 \to G'_0$$

all map G'_i into $\mathfrak{m}G'_{i-1}$. Arguing as in Hilbert's Syzygy theorem, we may tensor this complex with $k = S/(x_1, x_2, .., x_n)$ and perform the same Tor computation to see that in fact, G'_{n+1} must be 0. Then $G'_\bullet$ is easily seen to be the direct sum of the complexes

$$G''_\bullet = ... \to G_m \to ... \to G'_{n+1} = ... \to G_m \to ... \to 0$$

and

$$G'''_\bullet = 0 \to G'_n \to ... \to G'_0$$

This expresses $G_\bullet$ as the direct sum $K_\bullet \oplus G''_\bullet \oplus G'''_\bullet$, where $G'''_\bullet$ must be minimal because each G'_i maps into $\mathfrak{m}G_{i-1}$ by construction.

□

This gives rise to an even more natural formulation for the minimal graded free resolution:

Proposition 2.3.13. *Let $F_\bullet$ be the minimal graded free resolution for M. Then if $G_\bullet$ is any free resolution for M, for all i, G_i has at least as many free generators as F_i.*

Proof. By Prop. 2.3.12, there's a surjective map of chain complexes $G_\bullet \to F_\bullet$. This restricts to a surjective map of free modules $G_i = S^{m_i} \to F_i = S^{n_i}$ in each homological degree. Letting $\mathfrak{m} \subset S$ be a maximal ideal, this descends to a surjective map $(S/\mathfrak{m})^{m_i} \to (S/\mathfrak{m})^{n_i}$, which is a surjection of $S/\mathfrak{m}$-vector spaces and thus we must have $m_i \geq n_i$. □

While the minimal graded free resolution is indeed an invariant for M, specifying each module and map in the resolution can often result in giving an unwieldy amount of information. As a result, we define the following invariant, which is still finer than the Hilbert function:

Definition 2.3.14. Let $F_\bullet$ be the minimal free resolution of a finitely generated module M over S. Let $F_i = \bigoplus_{j=-\infty}^{\infty} S(-j)^{\beta_{i,j}}$. We define the (i, j)th Betti number to be $\beta_{i,j}$, the number of free generators for F_i that lie in the jth graded degree.

We can then perform the computation of Prop. 2.3.9 to produce the Hilbert function from the Betti numbers.

Proposition 2.3.15. *Let M be a finitely generated module over S with Betti numbers $\beta_{i,j}$. Let $B_j = \sum_{i\geq 0}(-1)^i \beta_{i,j}$. Then $H_M(d) = \sum_j B_j \binom{r+d-j}{r}$. ([4], Corollary 1.10)*

Proof. This is simply a restatement of Prop. 2.3.9 with our new language. □

2.4 The Veronese Embeddings

Rather than associating these invariants to abstract finitely generated graded modules over the polynomial ring, we may associate them with important geometric spaces through the machinery of algebraic geometry. In algebraic geometry, one of the most fundamental spaces is a *projective variety*. The ideal associated to any such variety is a homogenous ideal, which is an ideal of $S = k[x_0, x_1, ..., x_n]$ that is also a graded submodule. Then the projective coordinate ring S/I is a finitely generated graded module, so we can apply our theory of Hilbert's function, Betti numbers, and the minimal free resolution here to associate invariants with geometric spaces rather than modules.

The entire remainder of our exposition will be focused on work attempting to compute minimal free resolutions and the Betti numbers of the projective coordinate rings of a single collection of spaces. Specifically, we study the Veronese embeddings, a collection of varieties defined below.

Definition 2.4.1. Define the variety $\mathrm{Veronese}(n, d)$ to be the image of the embedding $\mathbb{P}^n \to \mathbb{P}^{\binom{d+n}{n}-1}$ defined by

$$[x_0, x_1, x_2, ..., x_n] \mapsto [x_0^d, x_0^{d-1}x_1, x_0^{d-1}x_2, ..., x_0^{d-1}x_n, x_0^{d-2}x_1^2, .., x_n^d]$$

whose component functions in homomogenous coordinates are all of the monomials of degree d in $n + 1$ variables.

3 The $n = 1$ Case

3.1 Notation on Matrices

In analyzing the $n = 1$ case of the Veronese embedding, we begin by defining some basic notions on matrices that will be used in our discussion of the Eagon-Northcott complex below.

Definition 3.1.1. For $a < b$, let $M(a, b)$ denote the set of increasing injective maps from the set $\{1, 2, ..., a\}$ to the set $\{1, 2, ..., b\}$. In this exposition, when A is a r by t matrix (with entries in any set), a *submatrix* of A is an r' by t' matrix B with $r' \leq r$, $t' \leq t$, and $B_{i,j} = A_{\sigma(i),\tau(j)}$ for functions $\sigma \in M(r', r)$, $\tau \in M(t', t)$. A *maximal square submatrix* is a an r' by t' submatrix with $r' = t' = \min(r, t)$.

Definition 3.1.2. Let R be a ring and let A be an r by t matrix with entries in R. We define I_A to be the ideal generated by the determinants of the maximal square submatrices of A. If $\alpha : R^t \to R^r$ is an R-linear map, we define $I_\alpha = I_A$, where A is the matrix associated to α as described in Remark 2.2.1. In this exposition, when these definitions are invoked, we will always have $r \leq t$, so $I_\alpha = I_A$ will be generated by the determinants of the r by r submatrices.

3.2 The Ideal in the $n = 1$ Case

In the $n = 1$ case, the map defining the Veronese embedding becomes

$$[x_0, x_1] \mapsto [x_0^d, x_0^{d-1}x_1, ..., x_0x_1^{d-1}, x_1^d]$$

Computing the minimal free resolution of the projective coordinate ring of this variety is much more tractable than in the general case because of the following result:

Theorem 3.2.1. *Let $V_n \subset \mathbb{P}^{n+1}$ be* $\mathrm{Veronese}(1,d)$*. The ideal $I(V_n)$ of $S = k[x_0, x_1, ..., x_d]$ associated with V_n is I_A, where*

$$A = \begin{bmatrix} x_0 & x_1 & ... & x_{d-1} \\ x_1 & x_2 & ... & x_d \end{bmatrix}$$

([4], Proposition 6.1)

Proof. The generators of I_A take the form $x_i x_j - x_{i+1}x_{j-1}$ for $i < j-1$. Direct computation shows that each of these generators vanishes on the Veronese surface, so I_A vanishes on the Veronese surface, and thus $I_A \subset I(V_n)$. This means that the quotient map $S \to S/I(V_n)$ descends to a map $S/I_A \to S/I(V_n)$. Further, note that $I(V_n)$ is a graded submodule of S because it's the ideal of a projective module, and I_A is a graded submodule of S because it's generated by homogenous 2-forms. Thus $S \to S/I(V_n)$ is a degree-0 map of graded modules by Lemma 2.1.18, and thus $S/I_A \to S/I(V_n)$ is also a degree-0 map by the definition of the graded structure for a quotient by a graded submodule.

Let $\alpha : S \to k[x_0, x_1]$ be the algebraic extension of the map $\alpha(x_i) = x_0^{d-i}x_1^i$. This is the map of rings induced by the map of varieties defining V_n, so the kernel of α is $I(V_n)$. This shows that α maps S_t, the subgroup of homogenous t-forms, onto $(k[x_0, x_1])_{dt}$. We note that the descent of α to the quotient yields an embedding of $S/I(V_n)$ into $k[x_0, x_1]$ by the first isomorphism theorem. Since $(S/I(V_n))_t$ is the image of S_t under the quotient, we then have that α maps $(S/I(V_n))_t$ onto $(k[x_0, x_1])_{dt}$ in a k-linear bijection. Since the dimension of $(k[x_0, x_1])_{dt}$ is $dt + 1$ as a k-vector space, the same can be said about $(S/I(V_n))_t$.

We then investigate the dimension of $(S/I_A)_t$. Certainly $\{\overline{x^\alpha} : \sum_{i=0}^{d} \alpha_i = t\}$, using multi-index exponent notation, is a k-linear spanning set for $(S/I_A)_t$. However, some of these multi-indices α yield the same generators. Specifically, if α_i and α_j are both nonzero and $0 < i \leq j < d$, we may decrease α_i and α_j by 1 and increase α_{i-1} and α_{j+1} by 1 to yield a new multi-index α' where $\overline{x^\alpha} = \overline{x^{\alpha'}}$ (where the generator $x_{i-1}x_{j+1} - x_i x_j$ is responsible for the identification of these equivalence classes in the quotient). By continuing to "push indices outward" in this fashion, we can conclude that for any multi-index α, there's a multi-index β such that $\overline{x^\alpha} = \overline{x^\beta}$ and $\sum_{i=1}^{d-1} \beta_i \leq 1$ (since if this sum were larger, we would be able to push indices outward as above). Thus we have that $\{\overline{x^\beta} : \sum_{i=0}^{d} \beta_i = t, \sum_{i=1}^{d-1} \beta_i \leq 1\}$ is a k-linear spanning set for $(S/I_A)_t$.

If $\sum_{i=1}^{d-1} \beta_i = 0$, then x^β takes the form $x_0^a x_d^{t-a}$. There are $t+1$ such monomials. If $\sum_{i=1}^{d-1} \beta_i = 1$, then x^β takes the form $x_0^a x_i x_d^{t-a-1}$. There are $t(d-1) = td - t$ such monomials, making for a total of $(td - t) + (t + 1) = td + 1$ monomials. This means $\{\overline{x^\beta} : \sum_{i=0}^{d} \beta_i = t, \sum_{i=1}^{d-1} \beta_i = 1\}$ contains at most $td + 1$ generators and thus $(S/I_A)_t$ has a k-linear spanning set of size at most $td + 1$. This means the dimension of $(S/I_A)_t$ is at most $td + 1$.

By Lemma 2.2.5, the S-module homomorphism $S/I_A \to S/I(V_n)$ restricts to a k-linear map $(S/I_A)_t \to (S/I(V_n))_t$ for each t. The map is surjective because the original homomorphism is surjective, and the image has dimension $td + 1$ while the preimage has dimension at most $td + 1$. Thus we may conclude that the preimage has dimension $td + 1$ and the k-linear map is a vector space isomorphism.

Then the canonical quotient map $S/I_A \to S/I(V_n)$ restricts to vector space isomorphisms between summands in $\bigoplus_{t=0}^{\infty} (S/I_A)_t \to \bigoplus_{t=0}^{\infty} (S/I(V_n))_t$. Thus it must be a vector space isomorphism between both direct sums as a whole, so the quotient $S/I_A \to S/I(V_n)$ is a bijection, and thus $I_A = I(V_n)$, as desired.

□

As a bonus, this proof gives us Hilbert's function for $I(V_n)$.

Corollary 3.2.2. *When* $V_n = \text{Veronese}(1, d)$, $H_{I(V_n)}(t) = td + 1$.

Proof. Evident from the previous proof. □

This result allows us to make use of invaluable theory surrounding ideals defined by the determinants of matrices. The main tool we'll use here is the Eagon-Northcott Complex.

3.3 The Eagon-Northcott Complex

The Eagon-Northcott Complex serves as a generalization of the Koszul complex. We first define this special case below, to motivate the definition of the Eagon-Northcott complex and help define its differential.

Definition 3.3.1. Let S be a ring.

Let $\alpha : S^k \to S$ be the map defined by the matrix

$$\begin{bmatrix} a_1 & a_2 & \dots & a_k \end{bmatrix}$$

The *Koszul Complex associated to* α is the graded complex $K_\bullet$ for which $K_i = \bigwedge^i S^k$ and for which the differential $\Delta : \bigwedge^d S^k \to \bigwedge^{d-1} S^k$ is defined on decomposable elements by

$$\Delta(X_1 \wedge X_2 \wedge \dots \wedge X_d) = \sum_{j=1}^{d} (-1)^{j+1} a_j X_1 \wedge \dots \wedge \widehat{X}_j \wedge \dots \wedge X_d$$

and extended S-linearly to the entirety of $\bigwedge^d S^k$. The analogous map on tensor products $\Delta' : \otimes^d S \to \otimes^{d-1} S$ is clearly a R-module map, and a direct computation shows that Δ' sends simple tensors with repeated elements to sums of simple tensors with repeated elements (as the terms in which the repeated elements are removed cancel with eachother), so Δ is a well-defined map on the exterior products.

An easy computation on alternating sums shows that $\Delta \circ \Delta = 0$, so this is a valid chain complex.

We note that the last map $\Delta : \wedge^1 S^k = S^k \to S$ maps the ith generator of S^k to $(-1)^{i+1} a_i$, so the image of the last differential is $\text{im}(\alpha) = (a_1, a_2, \dots, a_k)$. When S is a polynomial ring and the a_i's are its indeterminates, we will later prove that $K_\bullet$ is exact, and thus a free resolution. In this case, the exact sequence resolves S/I, where $I = (a_1, a_2, \dots, a_k)$ is the ideal of maximal minors of the $1 \times k$ matrix defining α. This will be the behavior we generalize in the Eagon-Northcott complex.

For the sake of later computations, we prove the following useful lemma on the differentials for Koszul complexes associated with arbitrary rows of a matrix.

Lemma 3.3.2. *Let* S *be a ring and let*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,t} \\ a_{2,1} & a_{2,2} & \dots & a_{2,t} \\ \dots & \dots & \dots & \dots \\ a_{r,1} & a_{r,2} & \dots & a_{r,t} \end{bmatrix}$$

be a $t \times r$ *matrix with entries in* S*. Let* Δ_i *be the differential of the Koszul complex associated to the* i*th row of* A*. For* $1 \le i, j \le t$, $\Delta_i \circ \Delta_j + \Delta_j \circ \Delta_i : \wedge^d S^k \to \wedge^{d-2} S^k$ *is the zero map. ([2], Equation 3.3)*

Proof. Let $X_1, X_2, ..., X_k$ be an S-linear basis for S^k, and let $X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_d} \in \wedge^d S^k$ be an arbitrary basis vector. Then

$$\Delta_i \circ \Delta_j(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_d}) = \sum_{n=1}^{d} (-1)^{n+1} a_{j,n} \Delta_i(X_{i_1} \wedge X_{i_2} \wedge ... \widehat{X}_{i_n} ... \wedge X_{i_d})$$

Expanding Δ_i as well, this becomes a sum of all terms of the form

$$(-1)^{w+n+1} a_{j,n} a_{i,m} X_{i_1} \wedge X_{i_2} \wedge ... \widehat{X}_{i_n} ... \widehat{X}_{i_m} ... \wedge X_{i_d}$$

where we may well have $m < n$ but the term was written in this fashion for definiteness, and $w = m + 1$ if $m < n$ but $w = m$ if $m > n$. Performing the same computation in the other order, $\Delta_j \circ \Delta_i(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_d})$ becomes a sum of all terms of the form

$$(-1)^{w+m+1} a_{j,n} a_{i,m} X_{i_1} \wedge X_{i_2} \wedge ... \widehat{X}_{i_n} ... \widehat{X}_{i_m} ... \wedge X_{i_d}$$

where $w = n$ if $m < n$ but $w = n + 1$ if $m > n$. Examining these terms shows that they are entirely identical except for a swapped sign (since the power of -1 is always off by one), so the terms cancel and thus the sums cancel. This means that $\Delta_i \circ \Delta_j + \Delta_j \circ \Delta_i$ sends all basis elements of $\wedge^d S^k$ to 0, so it is the zero map.

□

With these definitions, we may now define the Eagon-Northcott complex.

Definition 3.3.3. Let S be a ring and let $t \geq r > 0$ be integers. Let $\alpha : S^t \to S^r$ be an S-module homomorphism associated to the matrix (a_{ij}).

We now define $E^\alpha_\bullet$, the *Eagon-Northcott complex associated with* α ([2], Page 190). The modules in the complex are

$$E^\alpha_c = \begin{cases} \wedge^r S^r & \text{if } c = 0 \\ \wedge^{r+c-1} S^t \otimes \operatorname{Sym}^{c-1} S^r & \text{otherwise} \end{cases}$$

The last differential will be the map $\wedge^r S^t \xrightarrow{\wedge^r \alpha} \wedge^r S^r$. Before defining the other differentials, we define some auxiliary maps:

Let $\alpha_i : S^t \to S$ be the map associated to the matrix

$$\begin{bmatrix} a_{i,1} & a_{i,2} & ... & a_{i,t} \end{bmatrix}$$

Let Δ_i be the differential for the Koszul complex associated with α_i. Let $X_1, X_2, .., X_t$ form a basis for S^t and let $Y_1, Y_2, .., Y_r$ form a basis for S^r. Then for $k > 0$, $d : E_{k+1} \to E_k$, the differential for the remainder of $E_\bullet$, is defined on simple tensors of basis elements by

$$d(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+k}} \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}) = \sum_j \Delta_j(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+k}}) \otimes Y_1^{\nu_1} ... Y_j^{\nu_j - 1} ... Y_r^{\nu_r}$$

where $\sum_{i=1}^n \nu_i = c$ and above, we only sum over the j for which $\nu_j \neq 0$.

We seek to build some theory before showing that the Eagon-northcott complex satisfies $d \circ d = 0$, and thus is truly a complex. First however, we give some extremal examples.

Example 3.3.4. Note that when $r = 1$ and $\alpha = [a_1, a_2, ..., a_n]$ is a row matrix, $\mathrm{Sym}^{c-1} S^r$ is a copy of k for all c, so the differential becomes Δ_1 and the Eagon-Northcott complex reduces to the Koszul complex.

Example 3.3.5. In the other extremal case when $r = t$, the complex reduces to $0 \to \wedge^r S^t \cong S \xrightarrow{\wedge^r \alpha} \wedge^r S^r \cong S \to 0$, where the map takes $1 \in S$ to the determinant of the matrix of α, as we will see in Prop. 3.3.6.

The next proposition will be instrumental in showing that $d \circ d = 0$ at the end of the Eagon-Northcott complex.

Proposition 3.3.6. *Let S be a ring, $t \geq r > 0$ be integers. Let $\alpha : S^r \to S^t$ be an S-module homomorphism with corresponding matrix $A = (a_{i,j})$. Let $c_1, c_2, ..., c_t$ be the standard basis for S^t. Let $c_{i_1} \wedge c_{i_2} \wedge .... \wedge c_{i_r} \in E_1^\alpha$. Then*

1) $E_0^\alpha \cong S$ and

2) under this identification, $d(c_1 \wedge c_2 \wedge ... \wedge c_{i_r})$ is the determinant of the maximal submatrix of A formed by taking the columns $i_1, i_2, ..., i_r$.

Proof. Let $b_1, b_2, ..., b_r$ be the standard basis for S^r. Recall that as modules, $E_1^\alpha = \wedge^r S^t$ (ignoring the tensor with $\mathrm{Sym}^0(S^r) \cong S$). and $E_0^\alpha = \wedge^r S^r$.

By a standard result on exterior products, $\wedge^r S^t$ has an S-linear basis consisting of $e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_p}$ with $1 \leq i_1 < i_2 < ... < i_p \leq q$, where the e_j's are basis elements for S^t. From this it's clear that $E_0^\alpha = \wedge^r S^r$ has a basis consisting of the singleton $\{b_1 \wedge b_2 \wedge ... \wedge b_r\}$. Then $S \cong E_0^\alpha$ under the S-linear extension of the map $1 \mapsto b_1 \wedge b_2 \wedge ... \wedge b_r$.

Note then that $d : E_1^\alpha \to E_0^\alpha$ is the map $\wedge\alpha : \wedge^r S^t \to \wedge^r S^r$. We compute by definition

$$d(c_{i_1} \wedge c_{i_2} \wedge ... \wedge c_{i_r}) = \alpha(c_{i_1}) \wedge \alpha(c_{i_2}) \wedge ... \wedge \alpha(c_{i_r}) = \sum_{j=1}^{r} a_{i_1 j} b_j \wedge \sum_{j=1}^{r} a_{i_2 j} b_j \wedge ... \wedge \sum_{j=1}^{r} a_{i_r j} b_j$$

Let Γ be the set of of functions from $\{i_1, i_2, ..., i_r\}$ to $\{1, 2, ..., r\}$. Expanding the above sum, we have

$$d(c_{i_1} \wedge c_{i_2} \wedge ... \wedge c_{i_r}) = \sum_{\gamma \in \Gamma} a_{i_1 \gamma(i_1)} b_{\gamma(i_1)} \wedge a_{i_2 \gamma(i_2)} b_{\gamma(i_2)} \wedge ... \wedge a_{i_n \gamma(i_n)} b_{\gamma(i_n)}$$

$$= \sum_{\gamma \in \Gamma} \left(\prod_{j=1}^{n} a_{i_j \gamma(i_j)} \right) b_{\gamma(i_1)} \wedge b_{\gamma(i_2)} \wedge ... \wedge b_{\gamma(i_n)}$$

If $\gamma \in \Gamma$ isn't a bijection, then it fails to be injective because it's a map between finite sets of the same size. Then the associated element $\left(\prod_{j=1}^{n} a_{i_j \gamma(i_j)} \right) b_{\gamma(i_1)} \wedge b_{\gamma(i_2)} \wedge ... \wedge b_{\gamma(i_n)}$ of the sum is an exterior product containing two identical basis elements b_k, so it vanishes. Thus we only sum over the bijections $\gamma \in \Gamma$, which are maps $i_k \mapsto \sigma(k)$ for permutations $\sigma \in S_r$. Thus we rewrite the expression

$$d(c_{i_1} \wedge c_{i_2} \wedge ... \wedge c_{i_r}) = \sum_{\sigma \in S_r} \left(\prod_{j=1}^{r} a_{i_j \sigma(j)} \right) b_{\sigma(1)} \wedge b_{\sigma(2)} \wedge ... \wedge b_{\sigma(r)}$$

By performing the necessary transpositions, we note that $b_{\sigma(1)} \wedge b_{\sigma(2)} \wedge ... \wedge b_{\sigma(r)} = \operatorname{sgn}(\sigma) b_1 \wedge b_2 \wedge ... \wedge b_r$ for any $\sigma \in S_r$. Thus we rewrite

$$d(c_{i_1} \wedge c_{i_2} \wedge ... \wedge c_{i_r}) = \sum_{\sigma \in S_r} \left(\operatorname{sgn}(\sigma) \prod_{j=1}^{r} a_{i_j \sigma(j)} \right) b_1 \wedge b_2 \wedge ... \wedge b_n$$

and under the above discussed identification of E_0^α with S,

$$d(c_{i_1} \wedge c_{i_2} \wedge ... \wedge c_{i_r}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{r} a_{i_j \sigma(j)}$$

The right hand side of this expression is the determinant of the maximal square submatrix of (a_{ij}) with columns $i_1, i_2, ..., i_r$.

□

This gives the following important corollary:

Corollary 3.3.7. *Let S be a ring and let $\alpha : S^r \to S^t$ be an S-module homomorphism with corresponding matrix A. Then $d(E_1^\alpha) \subset E_0^\alpha \cong S$ is the ideal $I_\alpha = I_A$.*

Proof. The previous proposition shows that $d : E_1^\alpha \to E_0^\alpha$ takes the generators of E_1^α to the generators of I_A. □

In particular, if $E_\bullet^\alpha$ happens to be a free resolution, it resolves S/I_α. A large portion of this section will be dedicated to showing that when S and α are respectively the ring and map defined by the matrix in Theorem 3.2.1, $E_\bullet^\alpha$ is a free resolution. It will easily been seen to be a minimal free resolution, and allow us to read off the Betti numbers in the $n = 1$ case. For now though, we must first show that the Eagon-Northcott complex satisfies the essential condition necessary to be a complex.

Proposition 3.3.8. *As defined above, the differential d is an S-module homomorphism and satisfies $d \circ d = 0$, so $E_\bullet^\alpha$ is a valid chain complex.*

Proof. We check that $d \circ d$ sends basis elements to 0.

Let $X_1, X_2, ..., X_t$ be the standard basis for S^t and let $Y_1, Y_2, ..., Y_r$ be the standard basis for S^r. Justifying that $d \circ d = 0$ when neither differential is the last in the complex follows from the fact that $\Delta_i \circ \Delta_j + \Delta_j \circ \Delta_i = 0$ by Lemma 3.3.2. and that $\Delta_i \circ \Delta_i = 0$ because the Koszul complex is a complex. Noting this, we compute

$$d \circ d(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+k}} \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r})$$

$$= \sum_{i \neq j} \Delta_i \circ \Delta_j (X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+k}}) \otimes Y_1^{\nu_1} ... Y_i^{\nu_i - 1} ... Y_j^{\nu_j - 1} ... Y_r^{\nu_r}$$

$$+ \sum_{i} \Delta_i \circ \Delta_i (X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+k}}) \otimes Y_1^{\nu_1} ... Y_i^{\nu_i - 2} ... Y_r^{\nu_r}$$

$$= \sum_{i<j} (\Delta_i \circ \Delta_j + \Delta_j \circ \Delta_i)(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+k}}) \otimes Y_1^{\nu_1} ... Y_i^{\nu_i - 1} ... Y_j^{\nu_j - 1} ... Y_r^{\nu_r}$$

$$+\sum_i \Delta_i \circ \Delta_i(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+k}}) \otimes Y_1^{\nu_1}...Y_i^{\nu_i-2}...Y_r^{\nu_r} = 0$$

Performing this justification at the end of this complex is a little more involved. Let $A = (a_{i,j})$ be the matrix of α with respect to the X_i's and Y_j's. Let $X_{i_1} \wedge ... \wedge X_{i_{r+1}} \otimes Y_j$ be an arbitrary basis element of E_2^α. We show that this basis element is sent to 0. By definition, we compute

$$d \circ d(X_{i_1} \wedge ... \wedge X_{i_{r+1}} \otimes Y_j) = d(\Delta_j(X_{i_1} \wedge ... \wedge X_{i_{r+1}}))$$

$$= \sum_{k=1}^{r+1} (-1)^{k+1} a_{j,i_k} d\left(X_{i_1} \wedge ... \wedge \hat{X}_k \wedge ... \wedge X_{i_{r+1}}\right)$$

Letting M_k be the maximal submatrix of A given by columns $i_1, i_2, ..., i_{k-1}, i_{k+1}, ..., i_{r+1}$, Prop. 3.3.6 yields

$$= \sum_{k=1}^{r+1} (-1)^{k+1} a_{j,i_k} \det(M_k)$$

Up to a negative sign, this is a computation of the determinant by minors across the bottom row of an $(r+1) \times (r+1)$ matrix whose (p,q)th entry is (a_{p,i_q}) for $p \leq r$, and whose $(r+1,q)$th entry is (a_{j,i_q}). Thus this computes the determinant of a matrix whose jth row and last row are the same, which is 0. This shows that $d \circ d$ sends all basis elements in E_2^α to 0, so it is the 0 map at the end of the complex as well.

We have thus covered all cases, and shown that the Eagon-Northcott complex is actually a chain complex.

□

3.4 Useful Auxiliary Chain Complexes

For the rest of this section, let S be a ring and let A be a matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & ... & a_{1,t} \\ a_{2,1} & a_{2,2} & ... & a_{2,t} \\ ... & ... & ... & ... \\ a_{r,1} & a_{r,2} & ... & a_{r,t} \end{bmatrix}$$

for $t \geq r > 0$ with entries in S. This subsection is focused around exhibiting $E_\bullet^A$ as the middle of a short exact sequence of chain complexes, a presentation that proves instrumental for later inductive arguments:

Theorem 3.4.1. *If $r < t$, there is an a exact sequence of chain complexes $0 \to E_\bullet^{L(A)} \to E_\bullet^A \to C_\bullet^{A,L,M} \to 0$ where the map $E_\bullet^{L(A)} \to E_\bullet^A$ is the inclusion of Remark 3.4.3. Additionally at each index, $0 \to E_q^{L(A)} \to E_q^A \to C_q^{A,L,M} \to 0$ splits. Both of these chain complexes will be defined below. ([2], Lemma 1)*

Definition 3.4.2. Define $L(A)$ to be the submatrix

$$L(A) = \begin{bmatrix} a_{1,1} & a_{1,2} & ... & a_{1,t-1} \\ a_{2,1} & a_{2,2} & ... & a_{2,t-1} \\ ... & ... & ... & ... \\ a_{r,1} & a_{r,2} & ... & a_{r,t-1} \end{bmatrix}$$

with the last column removed and define $M(A)$ to be the submatrix

$$M(A) = \begin{bmatrix} a_{2,1} & a_{2,2} & \dots & a_{2,t-1} \\ a_{3,1} & a_{3,2} & \dots & a_{3,t-1} \\ \dots & \dots & \dots & \dots \\ a_{r,1} & a_{r,2} & \dots & a_{r,t-1} \end{bmatrix}$$

with both the first row and last column removed.

The following subcomplex will then form the first complex in our short exact sequence containing $E_\bullet^A$.

Remark 3.4.3. When $r < t$, there's an obvious inclusion $E_\bullet^{L(A)} \to E_\bullet^A$ which, on individual modules, is the S-linear extension of the map $E_p^{L(A)} \to E_p^A$ that takes basis elements of $E_p^{L(A)}$ to symbolically identical basis elements of E_p^A. The fact that this is a map of chain complexes follows easily from the definitions.

For use in the rest of this chapter we wish to define an auxiliary collection of maps $\mu_q : E_q^{L(A)} \to E_q^{M(A)}$ for $1 \leq q \leq t - r + 1$. Note that we do not define μ_0 even though the Eagon-Northcott complex has a nonzero module in the 0th homological degree. Note on the other hand that for $q \geq t - r + 1$, $E_q^{L(A)} = \wedge^{r+q-1} S^{t-1} \otimes \operatorname{Sym}^{q-1} S^r = 0$ because $r + q - 1 \geq r + (t - r + 1) - 1 = t$. In particular, μ_{t-r+1} is the zero map out of the zero module.

It remains to define μ_q for $1 \leq q \leq t - r$.

Definition 3.4.4. Let $Y_1, Y_2, ..., Y_r$ serve as the basis for S^r. To stay consistent with entries of the matrix, since $M(A)$ removes the first row of $L(A)$, we take $Y_2, Y_3, ..., Y_r$ to be the basis for S^{r-1} in the image of μ_{q+1}.

Choosing X to be an arbitrary member of the standard basis for $\wedge^{r+q-1} S^{t-1}$, define $\mu_q : \wedge^{r+q-1} S^{t-1} \otimes \operatorname{Sym}^{q-1} S^r \to \wedge^{r+q-2} S^{t-1} \otimes \operatorname{Sym}^{q-1} S^{r-1}$ for $1 \leq q \leq t-r$ to be the R-linear extension of the map

$$\mu_{q+1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}) = \begin{cases} 0 & \nu_1 \neq 0 \\ \Delta_1 \otimes \operatorname{id}(X \otimes Y_2^{\nu_2} ... Y_r^{\nu_r}) & \nu_1 = 0 \end{cases}$$

where Δ_1 is the differential of the Koszul complex associated with the first row of $L(A)$ (as in Lemma 3.3.2) in the appropriate homological degree. ([2], Equation 4.5)

The most important property of these maps μ is the following:

Lemma 3.4.5. *Let $1 \leq q \leq t-r$. The map $d_{q+1} \circ \mu_{q+1} + \mu_q \circ d_{q+1} : E_{q+1}^{L(A)} \to E_q^{M(A)}$, where d_{q+1} is to be interpreted as the differential in the appropriate Eagon-Northcott complex, is the zero map. ([2], Equation 4.7).*

Proof. As always, we perform a computation on basis elements. Let X be an arbitrary member of the standard basis for $\wedge^{r+q} S^{t-1}$, and let $Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}$ be a basis vector of $\operatorname{Sym}^q S^r$. We compute

$$d_{q+1} \circ \mu_{q+1} + \mu_q \circ d_{q+1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r})$$

When $\nu_1 > 1$, $\mu_{q+1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}) = 0$ and $d_{q+1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r})$ is a sum of tensors between terms of the form $\Delta_i(X)$ and monomials with a positive power of Y_1. All of these are killed by μ_q, so we have the desired conclusion $d_{q+1} \circ \mu_{q+1} + \mu_q \circ d_{q+1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}) = 0$.

When $\nu_1 = 1$, again $\mu_{q+1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}) = 0$, and

$$d_{q+1}(X \otimes Y_1^1 Y_2^{\nu_2} ... Y_r^{\nu_r}) = \sum_{j=1}^{n} \Delta_j(X) \otimes Y_1^1 Y_2^{\nu_2} ... Y_j^{\nu_j - 1} ... Y_r^{\nu_r}$$

All terms in this sum have a positive power of Y_1 except when $j = 1$, when the term of the sum is $\Delta_1(X) \otimes Y_2^{\nu_2} ... Y_r^{\nu_r}$. We thus have that μ_{q+1} kills all other terms and sends $\Delta_1(X) \otimes Y_2^{\nu_2} ... Y_r^{\nu_r}$ to $\Delta_1 \circ \Delta_1(X) \otimes Y_2^{\nu_2} ... Y_r^{\nu_r} = 0$, so again $d_{q+1} \circ \mu_{q+1} + \mu_q \circ d_{q+1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}) = 0$ because both summands send the element to 0.

Lastly consider the case $\nu_1 = 0$. In this case,

$$d_{q+1} \circ \mu_{q_1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}) = \sum_{j=2}^{n} \Delta_1 \circ \Delta_j(X) \otimes Y_2^{\nu_2} ... Y_j^{\nu_j - 1} ... Y_r^{\nu_r}$$

and

$$d_{q+1} \circ \mu_{q_1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}) = \sum_{j=2}^{n} \Delta_j \circ \Delta_1(X) \otimes Y_2^{\nu_2} ... Y_j^{\nu_j - 1} ... Y_r^{\nu_r}$$

so

$$d_{q+1} \circ \mu_{q+1} + \mu_q \circ d_{q+1}(X \otimes Y_1^{\nu_1} Y_2^{\nu_2} ... Y_r^{\nu_r}) = \sum_{j=2}^{n} (\Delta_j \circ \Delta_1 + \Delta_j \circ \Delta_1)(X) \otimes Y_2^{\nu_2} ... Y_j^{\nu_j - 1} ... Y_r^{\nu_r}$$

which is 0 by Lemma 3.3.2. This proves the desired claim because the map takes all basis elements to 0.

□

We use these maps to build a new complex that will serve as the third complex in our short exact sequence.

Definition 3.4.6. Define $C_\bullet^{A,L,M}$ to be a complex whose modules are

$$C_q^{A,L,M} = \begin{cases} 0 & q = 0 \\ E_1^{M(A)} & q = 1 \\ E_q^{M(A)} \oplus E_{q-1}^{L(A)} & q > 1 \end{cases}$$

and whose nonzero differentials $\delta_i : C_q^{A,L,M} \to C_{q-1}^{A,L,M}$ are defined by

$$\delta_q(a, b) = \begin{cases} d_2(a) + \mu_1(b) & q = 2 \\ (d_q(a) + \mu_{q-1}(b), d_{q-1}(b)) & q > 2 \end{cases}$$

where d is taken to be the differential in the appropriate Eagon-Northcott complex. ([2], Equation 4.8)

Proposition 3.4.7. *As defined above, $C_\bullet^{A,L,M}$ is a complex. That is, $\delta_q \circ \delta_{q-1} = 0$. ([2], Page 194)*

Proof. If $q = 3$,
$$\delta_3 \circ \delta_2(a, b) = d_3 \circ d_2(a) + d_2 \circ \mu_2(b) + \mu_1 \circ d_2(b) = 0$$
because d is the differential for the Eagon-Northcott complex and $d_2 \circ \mu_2 + \mu_1 \circ d_2 = 0$ by Lemma 3.4.4. If $q > 3$, then

$$\delta_q \circ \delta_{q-1}(a, b) = (d_q \circ d_{q-1}(a) + d_{q-1} \circ \mu_{q-1}(b) + \mu_{q-2} \circ d_{q-1}(b), d_q \circ d_{q-1}(b)) = 0$$

for identical reasons. $\square$

With these constructions in tow, we finally come to this subchapter's desired result:

Proof. (Of Theorem 3.4.1) We must first define the second map of chain complexes, which we name $\phi : E^A_\bullet \to C^{A,L,M}_\bullet$. Note that $C^{A,L,M}_0 = 0$, so $\phi_0 = 0$. We then define the maps $\phi_{q+1} : E^A_{q+1} \to C^{A,L,M}_{q+1}$ for $0 \leq q \leq t - r$, noting that for $q > t - r$, $E^A_{q+1} = 0$ because the exterior product vanishes. We first set up notation.

We note by definition that $E^A_{q+1} = \wedge^{r+q} S^t \otimes \mathrm{Sym}^q S^r$, and

$$C^{A,L,M}_{q+1} = E^{M(A)}_{q+1} \oplus E^{L(A)}_q = (\wedge^{r+q-1} S^{t-1} \otimes \mathrm{Sym}^q S^{r-1}) \oplus (\wedge^{r+q-1} S^{t-1} \otimes \mathrm{Sym}^{q-1} S^r)$$

To disambiguate our notation, we use different letters for the generators of all six of the different relevant free S-modules. For $E^A_{q+1} = \wedge^{r+q} S^t \otimes \mathrm{Sym}^q S^r$, let $B_1, B_2, ..., B_t$ be a basis for S^t and $C_1, C_2, ..., C_r$ be a basis for S^r. This makes

$$B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}} \otimes C_1^{\nu_1} ... C_r^{\nu_r}$$

a basis for E^A_{q+1}, where the i_j's are assumed to be in strictly increasing order.

For $E^{M(A)}_{q+1} = \wedge^{r+q-1} S^{t-1} \otimes \mathrm{Sym}^q S^{r-1}$, let $U_1, U_2, ..., U_{t-1}$ be a basis for S^{t-1} and let $V_2, V_3, ..., V_r$ be a basis for S^{r-1} so elements of the form

$$U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r+q-1}} \otimes V_2^{\nu_2} ... V_r^{\nu_r}$$

with the i_j's increasing form a basis for $E^{M(A)}_q$.

Finally, for $E^{L(A)}_q = \wedge^{r+q-1} S^{t-1} \otimes \mathrm{Sym}^{q-1} S^r$, let $X_1, X_2, ..., X_{t-1}$ be a basis for S^{t-1} and let $Y_1, Y_2, ..., Y_r$ be a basis for S^r, so that elements of the form

$$X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_1^{\nu_1} ... Y_r^{\nu_r}$$

with the i_j's increasing form a basis for $E^{L(A)}_{q-1}$. Here, we have $\nu_1 + \nu_2 + ... + \nu_r = q - 1$, so this basis is empty when $q = 0$.

Identifying the elements of direct summands of $C^{A,L,M}_{q+1}$ with elements in the module itself, we find that tensors of the form $U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r+q-1}} \otimes V_2^{\nu_2} ... V_r^{\nu_r}$ and $X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_1^{\nu_1} ... Y_r^{\nu_r}$ form a basis of $C^{A,L,M}_{q+1}$, where there are no generators of the second form when $q = 0$. Now define ϕ_{q+1} by

$$\phi_{q+1}(B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}} \otimes C_1^{\nu_1} ... C_r^{\nu_r}) = \begin{cases} 0 & i_{r+q} \neq t \\ U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r+q-1}} \otimes V_2^{\nu_2} ... V_r^{\nu_r} & i_{r+q} = t, \nu_1 = 0 \\ X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_1^{\nu_1 - 1} ... Y_r^{\nu_r} & i_{r+q} = t, \nu_1 > 0 \end{cases}$$

We must first show that ϕ is a map of complexes, which amounts to showing that $\delta_{q+1} \circ \phi_{q+1} = \phi_q \circ d_{q+1}$. As usual, we compute this on basis elements of the form

$$B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}} \otimes C_1^{\nu_1} ... C_r^{\nu_r}$$

We see that if $i_{r+q} \neq t$, then $d_{q+1}(B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}} \otimes C_1^{\nu_1} ... C_r^{\nu_r})$ becomes an alternating sum of tensors with a wedge that also doesn't include B_t. Thus both sides of the equality are killed by ϕ, so we have $\delta_{q+1} \circ \phi_{q+1} = \phi_q \circ d_{q+1}$ in this case. Assume then that $i_{r+q} = t$.

If $\nu_1 = 0$, then $\delta_{q+1} \circ \phi_{q+1}(B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}} \otimes C_1^{\nu_1} ... C_r^{\nu_r})$ yields $\delta_{q+1}(U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r+q-1}} \otimes V_2^{\nu_2} ... V_r^{\nu_r}) = d_{q-1}(U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r+q-1}} \otimes V_2^{\nu_2} ... V_r^{\nu_r})$, and it's easy to see that applying $\phi_q \circ d_{q+1}$ will yield the same result.

If $\nu_1 \geq 2$, then

$$\begin{aligned}
&\delta_{q+1} \circ \phi_{q+1}(B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}} \otimes C_1^{\nu_1} ... C_r^{\nu_r}) \\
&= \delta_{q+1}(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_1^{\nu_1 - 1} ... Y_r^{\nu_r}) \\
&= \mu_q(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_1^{\nu_1 - 1} ... Y_r^{\nu_r}) + d_q(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_1^{\nu_1 - 1} ... Y_r^{\nu_r}) \\
&= d_q(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_1^{\nu_1 - 1} ... Y_r^{\nu_r})
\end{aligned}$$

noting that μ kills its input because $\nu_1 - 1 > 0$. It's again easy to see that applying $\phi_q \circ d_{q+1}$ will yield the same result.

If $\nu_1 = 1$, we have

$$\begin{aligned}
&\delta_{q+1} \circ \phi_{q+1}(B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}} \otimes C_1^{1} ... C_r^{\nu_r}) \\
&= \delta_{q+1}(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_2^{\nu_2} ... Y_r^{\nu_r}) \\
&= \mu_q(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_2^{\nu_2} ... Y_r^{\nu_r}) + d_q(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_2^{\nu_2} ... Y_r^{\nu_r}) \\
&= \Delta_1(U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r+q-1}}) \otimes V_2^{\nu_2} ... V_r^{\nu_r} + d_q(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_2^{\nu_2} ... Y_r^{\nu_r})
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&d_{q+1}(B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}} \otimes C_1^{1} ... C_r^{\nu_r}) \\
&= \Delta_1(B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}}) \otimes C_2^{\nu_2} ... C_r^{\nu_r} + \sum_{j=2}^{r} \Delta_r(B_{i_1} \wedge B_{i_2} \wedge ... \wedge B_{i_{r+q}}) \otimes C_1^{1} ... C_j^{\nu_j - 1} ... C_r^{\nu_r}
\end{aligned}$$

where ϕ_q takes the first term of the above sum to $\Delta_1(U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r+q-1}}) \otimes V_2^{\nu_2} ... V_r^{\nu_r}$, and takes the rest of the sum to $d_q(X_{i_1} \wedge X_{i_2} \wedge ... \wedge X_{i_{r+q-1}} \otimes Y_2^{\nu_2} ... Y_r^{\nu_r})$. Thus again we have that the results of $\delta_{q+1} \circ \phi_{q+1}$ and $\phi_q \circ d_{q+1}$ align. We've checked all cases, so we have that these maps align on all basis elements, and thus are equal.

It's then fairly immediate that these two maps form an exact sequence. Letting $i : E_\bullet^{L(A)}$ be the inclusion, it's clear that i is injective. The new map ϕ_q is surjective on modules because by inspection, it has every basis vector of $C_q^{A,L,M}$ in its image. Lastly, this sequence is exact at $E_\bullet^A$ because the basis elements killed by ϕ_q are precisely those in the image of i_q, and ϕ_q restricts to a bijection from the basis elements it doesn't kill to the basis elements of $C_q^{A,L,M}$. The associated short exact sequences of modules split because the modules are all free.

□

We finish off this subsection with a result on $C_\bullet^{A,L,M}$, which uses the following computation from linear algebra.

Lemma 3.4.8. *Let*

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,n} \\ m_{2,1} & m_{2,2} & \dots & m_{2,n} \\ \dots & \dots & \dots & \dots \\ m_{n,1} & m_{n,2} & \dots & m_{n,n} \end{bmatrix}$$

be a matrix with entries in S. Let $M_{i,j}$ be the cofactor of $m_{i,j}$ and let $a \neq b$. Then $\sum_{j=1}^{n} m_{i,a} M_{i,b}$ = 0

Proof. Up to a negative sign, this sum can be computed as the determinant of a matrix that is identical to M, except with row b replaced with row a. This determinant is 0 because the matrix has two identical rows. □

Proposition 3.4.9. *Let D be the determinant of*

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,r} \\ a_{2,1} & a_{2,2} & \dots & a_{2,r} \\ \dots & \dots & \dots & \dots \\ a_{r,1} & a_{r,2} & \dots & a_{r,r} \end{bmatrix}$$

Then $D^r C_1^{A,L,M} \subset \delta(C_2^{A,L,M})$ ([2], Lemma 2).

Proof. By construction, $C_1^{A,L,M} = E_1^{M(A)} = \wedge^{r-1} R^{t-1} \otimes \operatorname{Sym}^0 R^{r-1}$. We also see

$$\delta(C_2^{A,L,M}) = d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$$

Now let $U_1, U_2, ..., U_{t-1}$ form a basis for R^{t-1} and let $V_1, V_2, ..., V_{r-1}$ form a basis for R^{r-1}. We see that elements of the form

$$U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r-1}} \otimes 1$$

where the i_j are increasing form a basis for $C_1^{A,L,M}$. Our goal is then to show that

$$D^s(U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r-1}} \otimes 1) \in d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$$

for all choices of the i_j. Let T_n be the set of basis elements $U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r-1}} \otimes 1$ where n of the indices $i_1, i_2, ..., i_{r-1}$ do not lie in $\{1, 2, ..., r\}$. We prove that for basis elements in T_n,

$$D^{n+1}(U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r-1}} \otimes 1) \in d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$$

by induction on n.

We see that T_0 consists of elements of the form $U_1 \wedge U_2 \wedge ... \wedge \widehat{U}_m \wedge ... \wedge U_r \otimes 1$. To show the desired result here, we make use of the equations

$$\mu_1(U_1 \wedge U_2 \wedge ... \wedge U_r \otimes 1) = \sum_{j=1}^{r} (-1)^{j+1} a_{1,j} (U_1 \wedge U_2 \wedge ... \wedge \widehat{U}_j \wedge ... \wedge U_r \otimes 1)$$

$$d_2(U_1 \wedge U_2 \wedge ... \wedge U_r \otimes V_2) = \sum_{j=1}^{r} (-1)^{j+1} a_{2,j} (U_1 \wedge U_2 \wedge ... \wedge \widehat{U}_j \wedge ... \wedge U_r \otimes 1)$$

$$\dots$$

$$d_r(U_1 \wedge U_2 \wedge ... \wedge U_r \otimes V_2) = \sum_{j=1}^{r}(-1)^{j+1} a_{r,j}(U_1 \wedge U_2 \wedge ... \wedge \widehat{U}_j \wedge ... \wedge U_r \otimes 1)$$

Let $A_{i,j}$ be the cofactor of $a_{i,j}$ in the square matrix whose determinant yields D. We multiply the ith equation by $A_{i,m}$ and add all the results. The left hand side clearly evaluates to an element in $d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$. The right hand side becomes

$$\sum_{j=1}^{r}(-1)^{j+1} \sum_{i=1}^{r} A_{i,m} a_{i,j}(U_1 \wedge U_2 \wedge ... \wedge \widehat{U}_j \wedge ... \wedge U_k \otimes 1)$$

and by Lemma 3.4.8, the inner sum vanishes when $m \neq j$. Thus this reduces to

$$= \pm \sum_{i=1}^{r} A_{i,m} a_{i,m}(U_1 \wedge U_2 \wedge ... \wedge \widehat{U}_j \wedge ... \wedge U_k \otimes 1) = \pm D(U_1 \wedge U_2 \wedge ... \wedge \widehat{U}_j \wedge ... \wedge U_k \otimes 1)$$

This shows that

$$D(U_1 \wedge U_2 \wedge ... \wedge \widehat{U}_j \wedge ... \wedge U_k \otimes 1) \in d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$$

which shows our desired inductive result for all elements of T_0. This completes our base case.

In the inductive step, we assume the desired statement for $n = k - 1 \geq 0$, and we show it for $n = k$. Pick $r < j_1 < j_2 < ... < j_k \leq t - 1$. We show that

$$D^k(U_{k+2} \wedge ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k}) \in d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$$

and claim without loss of generality that this shows the statement for all basis elements in T_k. We now make use of the equations

$$\mu_1(U_{k+1} \wedge ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k} \otimes 1) = L_1 + \sum_{j=k+1}^{r} (-1)^{j+1} a_{1,j}(U_{k+1} \wedge ... \widehat{U}_j ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k} \otimes 1)$$

$$d_2(U_{k+1} \wedge ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k} \otimes V_2) = L_2 + \sum_{j=k+1}^{r} (-1)^{j+1} a_{2,j}(U_{k+1} \wedge ... \widehat{U}_j ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k} \otimes 1)$$

$$...$$

$$d_r(U_{k+1} \wedge ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k} \otimes V_2) = L_r + \sum_{j=k+1}^{r} (-1)^{j+1} a_{r,j}(U_{k+1} \wedge ... \widehat{U}_j ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k} \otimes 1)$$

where each L_i is a linear combination of terms in T_{k-1} (consisting of the terms where the entry removed is one of the j_i's instead of one of the initial ones). We multiply the ith equation by $A_{i,k+1}$ and add all the results. Again, the left hand side yields an element in $d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$. By an identical argument to the base case, the right hand side evaluates to the expression

$$\pm D(U_{k+2} \wedge ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k}) + \sum_{i=1}^{r} A_{i,k+1} L_i$$

which thus lies in $d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$. Multiplying both sides by D^{k-1},

$$D^k(U_{k+2} \wedge ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k}) + \sum_{i=1}^{r} D^{k-1} A_{i,k+1} L_i \in d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$$

which gives

$$D^k(U_{k+2} \wedge ... \wedge U_r \wedge U_{j_1} \wedge ... \wedge U_{j_k}) \in d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$$

since each L_i lies in T_{k+1}, so each $D^{k-1}L_i$ lies in $d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$ for each i by the inductive hypothesis.

Thus our induction is complete, and we have shown that for basis elements in T_n,

$$D^{n+1}(U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r-1}} \otimes 1) \in d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$$

All basis elements are contained in $\bigcup_{n=0}^{r-1} T_n$ because there are only $r-1$ indices, so this shows

$$D^r(U_{i_1} \wedge U_{i_2} \wedge ... \wedge U_{i_{r-1}} \otimes 1) \in d_2(E_2^{M(A)}) + \mu_1(E_1^{L(A)})$$

for all basis elements in $C_1^{A,L,M}$ and thus $D^r C_1^{A,L,M} \subset \delta_2(C_2^{A,L,M})$ as desired.

□

3.5 Freeness of the Eagon-Northcott Complex

We are now ready to give a sufficient condition for exactness of the Eagon-Northcott Complex. First we work through two important building blocks, the second of which makes essential use of our main result in Section 3.4.

Lemma 3.5.1. *Let*

$$\left[a_1, a_2, ..., a_t\right]$$

be a vector of entries in S and let $K_\bullet$ be the associated Koszul complex. Let M be any S-module. Then for any n, $(a_1, a_2, ..., a_t)H_n(K_\bullet \otimes M) = 0$. ([2], Lemma 3).

Proof. Let $X_1, X_2, ..., X_t$ be a basis for S^t.

Let $\sum_{i=1}^t B_i \otimes m_i$ be an arbitrary cycle in the nth homological degree. The condition that the element is a cycle is to say that $\sum_{i=1}^t \Delta(B_i) \otimes m_i = 0$. We compute for fixed $1 \leq j \leq t$,

$$\sum_{i=1}^{t} \Delta(X_j \wedge B_i) \otimes m_i = \sum_{i=1}^{t} a_j B_i \otimes m_i + \sum_{i=1}^{t} X_j \wedge \Delta(B_i) \otimes m_i = a_j \sum_{i=1}^{t} B_i \otimes m_i + 0$$

because $\sum_{i=1}^t \Delta(B_i) \otimes m_i = 0$. This exhibits $a_j \sum_{i=1}^t B_i \otimes m_i$ as a boundary, so $a_j H_n(K_\bullet \otimes M) = 0$. Since this holds for all j, we have the desired conclusion $(a_1, a_2, ..., a_t)H_n(K_\bullet \otimes M) = 0$.

□

Proposition 3.5.2. *As in section 3.4, let $\alpha : S^t \to S^r$ for $t \geq r$ be a map represented by a matrix*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,t} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,t} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r,1} & a_{r,2} & \cdots & a_{r,t} \end{bmatrix}$$

Let N be any S-module. There exists an integer h dependent only on t and r (and notably not on A, E, or even the ring S) for which $I_A^h H_n(E_\bullet^A \otimes N) = 0$ for all n ([2], Proposition 2).

Proof. In the case $r = 1$, $E_\bullet^A$ is the Koszul complex and I_A is the ideal generated by the elements of A. Lemma 3.5.1 then directly shows that $I_A H_n(E_\bullet^A \otimes N) = 0$.

In the case $t = r$, $E_\bullet^A$ takes the form $0 \to S \xrightarrow{\times I_A = \det A} S \to 0$ by Example 3.3.5. Then $E_\bullet^A \otimes N$ is the complex $0 \to N \xrightarrow{\times I_A} N \to 0$. In this case, if $m \in E_1^A \otimes N = N$ is a cycle, then $I_A m = 0$, so $I_A(H_1(E_\bullet^A \otimes N)) = 0$. On the other hand, if $m \in E_0^A \otimes N = N$ is any element, then $I_A m$ is the image of $m \in E_1^A \otimes N$ under the differential, so $I_A N$ consists entirely of boundaries and thus $I_A H_0(E_\bullet^A \otimes N) = 0$.

We have thus proven the proposition in the cases $r = 1$ and $t = r$, where $h = 1$ in both cases. We prove this proposition in the general case by induction, where our base case consists of the cases $r = 1$ and $t = r$, and in our inductive step we assume the proposition has been shown for all $r' \times t'$ matrices with either $r' < r$ or $r' = r$ and $t' < t$. The base case is shown above, and we now tackle the inductive step.

First, let D be the determinant of the maximal submatrix

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,r} \\ a_{2,1} & a_{2,2} & \dots & a_{2,r} \\ \dots & \dots & \dots & \dots \\ a_{r,1} & a_{r,2} & \dots & a_{r,r} \end{bmatrix}$$

By the inductive assumption, for some h' dependent on only t and r, $I_{L(A)}^{h'} H_n(E_\bullet^{L(A)} \otimes N) = 0$ and $I_{M(A)}^{h'} H_n(E_\bullet^{M(A)} \otimes N) = 0$. Since $D \in I_{L(A)} \subset I_{M(A)}$, we have that $D^{h'} H_n(E_\bullet^{L(A)} \otimes N) = D^{h'} H_n(E_\bullet^{M(A)} \otimes N) = 0$.

We then wish to inspect the homology of $C_\bullet^{A,L,M} \otimes N$. Note that $C_n^{A,L,M} \otimes N = (E_n^{M(A)} \oplus E_{n-1}^{L(A)}) \otimes N = E_n^{M(A)} \otimes N \oplus E_{n-1}^{L(A)} \otimes N$. Let δ' be the differential on $C_\bullet^{A,L,M} \otimes N$ and let $d' : E_n^{M(A)} \otimes N \to E_{n-1}^{M(A)} \otimes N$ and $d' : E_n^{L(A)} \otimes N \to E_{n-1}^{L(A)} \otimes N$ be the differentials, where we don't provide indices or distinguish between the differentials on the two complexes in order to avoid notational clutter.

Let $\mu' : E_n^{L(A)} \otimes N \to E_n^{M(A)} \otimes N$ be $\mu \otimes \mathrm{id}$. Recall by Lemma 3.4.5 that $d \circ \mu + \mu \circ d : E_n^{L(A)} \to E_{n-1}^{M(A)}$ is the zero map (with appropriate indices for homological degree). We can then conclude immediately that $d' \circ \mu' + \mu' \circ d' = 0$ (with appropriate indices for homological degree). We can also immediately see that for $e_n^M \in E_n^{M(A)} \otimes N$ and $e_{n-1}^L \in E_{n-1}^{L(A)} \otimes N$, $\delta'(e_n^M, e_{n-1}^L) = (d' e_n^M + \mu' e_{n-1}^L, d' e_{n-1}^L)$.

With this notation set up, let us first investigate the homology $H_n(C_\bullet^{A,L,M} \otimes N)$ with $n > 2$. We wish to show that in such a case, $D^{2h'} H_n(C_\bullet^{A,L,M} \otimes N) = 0$. Let $(e_n^M, e_{n-1}^L) \in C_n^{A,L,M} \otimes N = (E_n^{M(A)} \otimes N) \oplus (E_{n-1}^{L(A)} \otimes N)$ be an arbitrary cycle. This means that $\delta(e_n^M, e_{n-1}^L) = (d' e_n^M + \mu' e_{n-1}^L, d' e_{n-1}^L) = 0$.

Since $D^k H_{n-1}(E_\bullet \otimes N) = 0$ and e_{n-1}^L is a cycle (because $d' e_{n-1}^L = 0$), we have that $D^{h'} e_{n-1}^L$ is a boundary by the inductive hypothesis. That is, for some $e_n^L \in E_n^{L(A)}$, $d' e_n^L = D^{h'} e_{n-1}^L$. Since $d' e_n^M + \mu' e_{n-1}^L = 0$ from above, this gives the chain of equalities

$$0 = D^{h'}(d' e_n^M + \mu' e_{n-1}^L) = d'(D^{h'} e_n^M) + \mu'(D^{h'} e_{n-1}^L) = d'(D^{h'} e_n^M) + \mu'(d' e_n^L) = d'(D^{h'} e_n^M) - d'(\mu' e_n^L)$$

$$= d'(D^{h'}e_n^M - \mu' e_n^L)$$

where the second-to-last equality holds because $d'\mu' + \mu' d' = 0$, as discussed above. Note that the equality chain shows that $D^{h'}e_n^M - \mu' e_n^L \in E_n^{M(A)} \otimes N$ is a cycle. Then because $D^{h'}H_n(E_\bullet^{M(A)} \otimes N) = 0$, we can write $D^{h'}(D^{h'}e_n^M - \mu' e_n^L) = d' e_{n+1}^M$ for some element $e_{n+1}^M \in E_{n+1}^{M(A)} \otimes N$. Rearranging this equality, we have

$$D^{2h'}e_n^M = d' e_{n+1}^M + \mu'(D^{h'}e_n^L)$$

We may then compute

$$\delta'(e_{n+1}^M, D^{h'}e_n^L) = (d' e_{n+1}^M + \mu'(D^{h'}e_n^L), d'(D^{h'}e_n^L)) = (D^{2h'}e_n^M, D^{2h'}e_{n-1}^L) = D^{2h'}(e_n^M, e_{n-1}^L)$$

where we recall that $d' e_n^L = D^{h'}e_{n-1}^L$ for the sake of the second equality. This exhibits $D^{2h'}(e_n^M, e_{n-1}^L) \in C_n^{A,L,M} \otimes N$ as a boundary where (e_n^M, e_{n-1}^L) was an arbitrary cycle, so we may conclude $D^{2h'}H_n(C_\bullet^{A,L,M} \otimes N) = 0$ for $n > 2$.

In the $n = 2$ case, the only added wrinkle is that $\delta'(e_2^M, e_1^L) = d' e_2^M + \mu' e_1^L$, so it's not known a priori that $d' e_1^L = 0$. However, we do have that since (e_2^M, e_1^L) is a cycle, $d' e_2^M + \mu' e_1^L = 0$. Applying d' again and noting that $d' \circ d' = 0$, we have $d'\mu' e_1^L = 0$. We then assert that the following diagram commutes:

$$\begin{array}{ccc}
E_1^{L(A)} & \xrightarrow{d} & E_0^{L(A)} \cong S \\
\Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \cong} \\
E_1^{M(A)} & \xrightarrow[d]{} & E_0^{M(A)} \cong S
\end{array}$$

This is a computation on basis elements where essentially, the top row computes the determinant of a matrix while the composition of the left column and bottom row compute the same determinant by minors. This shows that under the identification of $E_0^{L(A)}$ with $E_0^{M(A)}$, $d\mu = d$. The same applies after taking tensors: under the identification of $E_0^{L(A)} \otimes N$ with $E_0^{M(A)} \otimes N$, $d'\mu' = d'$. Thus since $d'\mu' e_1^L = 0$, $d' e_1^L = 0$. This is the only missing piece necessary to assert that the argument from the $n > 2$ case works, so we can now apply the argument verbatim.

This shows that for $n \geq 2$, $D^{2h'}H_n(C_\bullet^{A,L,M} \otimes N) = 0$. We also have that $D^r H_1(C_\bullet^{A,L,M} \otimes N) = 0$ by Prop. 3.4.9 and $H_0(C_\bullet^{A,L,M} \otimes N) = 0$ because $C_0^{A,L,M} = 0$. Thus for $v = \max\{2h', r\}$ and all n, $D^v H_n(C_\bullet^{A,L,M} \otimes N) = 0$. Now since the exact sequence

$$0 \to E_\bullet^{L(A)} \to E_\bullet^A \to C_\bullet^{A,L,M} \to 0$$

is a split exact sequence of free modules at each row, the induced maps

$$0 \to E_\bullet^{L(A)} \otimes N \to E_\bullet^A \otimes N \to C_\bullet^{A,L,M} \otimes N \to 0$$

also form a short exact sequence. A chunk of the resulting long exact sequence on homology reads

$$H_n(E_\bullet^{L(A)} \otimes N) \to H_n(E_\bullet^A \otimes N) \to H_n(C_\bullet^{A,L,M} \otimes N)$$

and since $D^{h'}H_n(E_\bullet^{L(A)} \otimes N) = D^v H_n(C_\bullet^{A,L,M} \otimes N) = 0$, $D^{v+h'}H_n(E_\bullet^A \otimes N) = 0$.

We may generalize this conclusion to all maximal subdeterminants, as swapping columns of A gives an Eagon-Northcott complex that is easily seen to be isomorphic. Since I_A is generated by $\binom{t}{r}$ such subdeterminants, $I_A^{(v+h')\binom{t}{r}}$ is generated by terms with at least $v+h'$ powers of some maximal subdeterminant. Thus we may conclude $I_A^{(v+h')\binom{t}{r}} H_n(E_\bullet^A \otimes N) = 0$, where v and h' only depend on t and r. This concludes our inductive step, and proves the statement as a whole.

□

We give a brief aside to discuss a definition in commutative algebra that we use to give a sufficient condition for exactness of the Eagon-Northcott complex.

Definition 3.5.3. Let R be a ring, M be an R-module, and I be an ideal of R. An *M-regular sequence in I* is a sequence of elements $x_1, x_2, ..., x_n \in I$ such that for each i, x_i is not a zero divisor on $M/(x_1, x_2, ..., x_{i-1})M$.

Definition 3.5.4. Let R be a ring, M be an R-module, and I be an ideal of R. The *grade of I on M*, denoted $\mathrm{grade}(I, M)$, is the maximum length of all M-regular sequences in I.

These are the final pieces that allow us to discuss the homology of the Eagon-Northcott complex. Everything culminates in the following central theorem:

Theorem 3.5.5. *Assume that S is Noetherian and let M be a finitely generated S module. Assume further that $I_A E \neq E$. Let q be the largest n such that $H_n(E_\bullet^A \otimes M) \neq 0$. Then*

$$\mathrm{grade}(I_A, M) + q \leq t - r + 1$$

([2], Theorem 1)

Proof. We prove this by induction. Assume first that $\mathrm{grade}(I_A, M) = 0$. Then we must show $q \leq t - r + 1$, which holds easily because for $q > t - r + 1$, $E_q^A = 0$, completing the base case.

We now approach the inductive step. Let $\mathrm{grade}(I_A, M) = k$ and assume the statement for modules of grade $k-1$. Let $x_1, x_2, ..., x_k$ be a maximal regular sequence for M, and form the short exact sequence $0 \to M \xrightarrow{x_1} M \to M/x_1 \to 0$. This induces a short exact sequence of complexes

$$0 \to E_\bullet^A \otimes M \xrightarrow{x_1} E_\bullet^A \otimes M \to E_\bullet^A \otimes M/x_1 \to 0$$

which yields a long exact sequence on homology of the form

$$... \to H_n(E_\bullet^A \otimes M) \xrightarrow{x_1} H_n(E_\bullet^A \otimes M) \to H_n(E_\bullet^A \otimes M/x_1) \to ...$$

By definition of grade, we have $\mathrm{grade}(I_A, M/x_1) = k-1$. Letting q' be the largest n such that $H_n(E_\bullet^A \otimes M/x_1) \neq 0$, the inductive hypothesis yields $k - 1 + q' \leq t - r + 1$. From the long exact sequence, we have an exact sequence

$$H_{n+1}(E_\bullet^A \otimes M/x_1) \to H_n(E_\bullet^A \otimes M) \xrightarrow{x_1} H_n(E_\bullet^A \otimes M)$$

where when $n \geq q'$, $n + 1 > q'$ so the sequence becomes

$$0 \to H_n(E_\bullet^A \otimes M) \xrightarrow{x_1} H_n(E_\bullet^A \otimes M)$$

This means that multiplication by x_1 is an injective map on $H_n(E_\bullet^A \otimes M)$, and thus multiplication by x_1^c is an injective map on $H_n(E_\bullet^A \otimes M)$ for any $c \geq 0$.

On the other hand, since $x_1 \in I_A$, for some integer h we have $x_1^h H_n(E_\bullet^A \otimes M) = 0$ by Prop. 3.5.2. Thus we have an injective map from $H_n(E_\bullet^A \otimes M)$ which is the zero map, so $H_n(E_\bullet^A \otimes M) = 0$ for $n \geq q'$. This means that $q < q'$, so $q + 1 \leq q'$. Plugging into our above inequality, we have

$$k + q = k - 1 + q + 1 \leq k - 1 + q' \leq t - r + 1$$

Since $k = \mathrm{grade}(I_A, M)$, this completes our inductive step and thus our induction as a whole, proving the statement. □

In the original paper by Eagon and Northcott, this is in fact shown to be an equality. We didn't pursue the strongest version of this result because this bound suffices for our purposes, namely due to the following corollary:

Corollary 3.5.6. *Assume that S is Noetherian and M is a finitely generated S-module for which $\mathrm{grade}(I_A, M) \geq t - r + 1$. Then $E_\bullet^A \otimes M$ is a free resolution of M/I_AM. ([2], Corollary to Theorem 1)*

Proof. Theorem 3.5.5 gives that for $n > t - r + 1 - \mathrm{grade}(I_A, M) \geq 0$, $H_n(E_\bullet^A \otimes M) = 0$, so this is a free resolution of

$$E_0^A \otimes M/d(E_1^A \otimes M) = M/d(E_1^A \otimes M)$$

Prop. 3.3.5 shows that $d(E_1^A) = I_A$, so $d(E_1^A \otimes M) = I_AM$, and thus the complex is a free resolution of M/I_AM. □

We can specialize further to the following case:

Corollary 3.5.7. *Assume that S is Noetherian and $\mathrm{grade}(I_A, S) \geq t - r + 1$. Then $E_\bullet^A$ is a free resolution of S/I_A ([2], Theorem 2).*

Proof. Choose $E = S$ in Corollary 3.5.6. □

This shows that if we can prove that $\mathrm{grade}(I_A, S) \geq t - r + 1$ when $S = k[x_1, x_2, ..., x_{n+1}]$ and A is the matrix in Theorem 3.2.1, then $E_\bullet^A$ presents a free resolution of our desired projective coordinate ring. We will prove this in the next subchapter, but first we give an easy application of this theorem to the Koszul complex.

Corollary 3.5.8. *Let $S = k[x_1, x_2, ..., x_n]$ and let A be the $1 \times n$ matrix $[x_1, x_2, ..., x_n]$. Then $E_\bullet^A$, the Koszul complex of this map, is a free resolution of $S/I_A = S/(x_1, x_2, ..., x_n) = k$.*

Proof. By Cor. 3.5.7, we only require $\mathrm{grade}(I_A, S) \geq n - 1 + 1 = n$ to make this conclusion. This is clear, since $I_A = (x_1, x_2, ..., x_n)$ and $x_1, x_2, ..., x_n$ is a maximal regular sequence in S. □

3.6 Results in the $n = 1$ case

We are finally ready to finish our computations in the $n = 1$ case. Our work culminates in the following theorem, which will be proven near the end of this section

Theorem 3.6.1. *Let $S = k[x_0, x_1, ..., x_d]$ and let A be the matrix*

$$\begin{bmatrix} x_0 & x_1 & ... & x_{d-1} \\ x_1 & x_2 & ... & x_d \end{bmatrix}$$

Then $E^A_\bullet$ is the minimal graded free resolution of S/I_A, the ideal of $\mathrm{Veronese}(1, d)$ with respect to a graded structure in which all free generators of E^A_k are in graded degree $k + 1$ for $k \geq 1$, and the free generators of E^A_0 are in graded degree 0.

First, we discuss results from commutative algebra to help us show the grade condition from the previous subchapter.

Definition 3.6.2. Let R be a ring and let $\mathfrak{p} \subset R$ be a prime ideal.

The *height* of $\mathfrak{p}$, denoted $\mathrm{ht}(\mathfrak{p})$, is the maximum length n of a chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset ... \subset \mathfrak{p}_n = \mathfrak{p}$$

More generally, if $I \subset R$ is any proper ideal, the *height* of I is $\min\{\mathrm{ht}(\mathfrak{p}) : I \subset \mathfrak{p}, \mathfrak{p} \text{ prime}\}$.

This definition helps us analyze the grade of an ideal through the following connection which we do not prove.

Proposition 3.6.3. *Let R be a Cohen-Macaulay ring and $I \neq R$ an ideal. Then $\mathrm{grade}(I, R) = \mathrm{ht}(I)$. ([1], Cor. 2.1.4).*

We then give another definition discuss and cite a theorem that will help us compute height.

Definition 3.6.4. A *maximal sequence of prime ideals* in R is a sequence of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset ... \subset \mathfrak{p}_k \subset R$ such that $\mathfrak{p}_k$ is maximal, $\mathfrak{p}_0$ is minimal, and for all $0 \leq i < k$, there are no primes $\mathfrak{p}'$ satisfying $\mathfrak{p}_i \subsetneq \mathfrak{p}' \subsetneq \mathfrak{p}_{i+1}$.

Proposition 3.6.5. *Let R be a domain that is a finitely generated algebra over a field. Then the length of any maximal chain of primes is the same. In particular, the dimension of R can be computed as the length of any maximal chain. ([3], Chapter 13, Theorem A)*

Corollary 3.6.6. *Let R be a domain that is a finitely generated algebra over a field and let $\mathfrak{p} \subset R$ be a prime ideal. Then $\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$.*

Proof. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset ... \subset \mathfrak{p}_k = \mathfrak{p}$ be a chain of ideals witnessing the height of $\mathfrak{p}$, and let $0 \subset \mathfrak{p}'_{k+1} \subset \mathfrak{p}'_{k+2} \subset ... \subset \mathfrak{p}'_{k+n}$ be a chain of ideals witnessing the dimension of $R/\mathfrak{p}$. Lift this second chain to R and concatenate the chains to form a new chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset ... \subset \mathfrak{p}_{k+n}$$

The conditions for this chain to be maximal are satisfied because the first chain was a maximum length chain of primes from 0 to $\mathfrak{p}$, and the next chain lifts to a maximum length chain of primes containing $\mathfrak{p}$. Thus $k + n$ is the dimension of R by Prop. 3.6.5, where k is the height of $\mathfrak{p}$ and n is the dimension of $R/\mathfrak{p}$.

□

By use of this result, we may now compute the grade of the desired ideal by computing its codimension.

Lemma 3.6.7. *Let $S = k[x_0, x_1, ..., x_d]$ and let A be the matrix*

$$\begin{bmatrix} x_0 & x_1 & \dots & x_{d-1} \\ x_1 & x_2 & \dots & x_d \end{bmatrix}$$

where by Theorem 3.2.1, I_A is the ideal of $\mathrm{Veronese}(1, d)$*. Then* $\dim(R/I_A) = 2$*.*

Proof. Rather than projective space, we may view $V_d = \mathrm{Veronese}(1, d)$ as the image of a map $f : \mathbb{A}^2 \to \mathbb{A}^{d+1}$ defined in the same way by $(x, y) \mapsto (x^d, x^{d-1}y, ..., y^d)$. Then R/I_A is the affine coordinate ring of V_d, so its dimension is the topological dimension of V_d.

Any chain of closed sets $C_0 \subsetneq C_1 \subsetneq ... \subsetneq C_k = V_d$ then gives a chain of closed sets $f^{-1}(C_0) \subsetneq f^{-1}(C_1) \subsetneq ... \subsetneq f^{-1}(C_k)$ where adjacent sets in the chain remain distinct because f is surjective onto V_d. This shows $\dim(S/I_A) = \dim(V_d) \leq \dim(\mathbb{A}^2) = 2$. On the other hand, we recall that the Veronese embedding induces a map of rings $k[x_0, x_1, ..., x_d] \to k[x, y]$ defined by $x_i \mapsto x^{d-i}y^i$. Letting the image of this map be R, we have that $S/I_A \cong R$ and we note that R is the k-algebra generated by $\{x^d, x^{d-1}y, ..., y^d\}$.

It's then clear that

$$0 \subset (x^d) \subset (x^d, x^{d-1}y, ..., y^d) \subset R$$

forms an increasing chain of prime ideals in R, so $\dim(R) = \dim(S/I_A) \geq 2$, and thus $\dim(R) = 2$. □

Corollary 3.6.8. *Let S, I_A be as in Lemma 3.6.7. Then* $\mathrm{grade}(I_A, S) = n - 1$*.*

Proof. Since S is a domain and a finitely generated algebra over a field and I_A is a prime ideal as the ideal of a variety, $\mathrm{ht}(I_A) + \dim(S/I_A) = \dim(S)$. We see $\dim(S) = n + 1$ and by Lemma 3.6.7, $\dim(S/I_A) = 2$, so $\mathrm{ht}(I_A) = n - 1$. Then by Prop. 3.6.3, $\mathrm{grade}(I_A, S) = \mathrm{ht}(I_A) = n - 1$ because the polynomial ring is Cohen-Macaulay. □

With all the theory we have built up, we can finally conclude

Proof. (Of Theorem 3.6.1) By Corollary 3.5.7 and 3.6.8, $E^A_\bullet$ is a free resolution of S/I_A. As usual, take basis elements for each E^A_k of the form $X \otimes Y$, where X is a wedge of $k + 1$ basis elements for S^{n-1} with strictly increasing indices, and Y is a monomial of degree $k - 1$.

Examining the definition of the differential in the Eagon-Northcott complex shows that the matrix representing each map $d : E^A_k \to E^A_{k-1}$ with respect to this basis only has entries in A (which are individual indeterminates) when $k \geq 2$, but has determinants of maximal submatrices as entries for $k = 1$ (which are homogenous 2-forms).

By Cor. 2.2.4, since each entry in the matrix of the map $d : E^A_1 \to E^A_0$ is a homogenous 2-form, we can give E^A_0 and E^A_1 graded structures for which the former only has generators in degree 0, the latter only has generators in degree 2, and d has degree 0. By further applications of Cor. 2.2.4 to the maps $d : E^A_k \to E^A_{k-1}$ with $k \geq 2$, we can give each E^A_k a graded structure where all of its generators are in degree $k + 1$.

Finally, the Eagon-Northcott complex is minimal because all of the entries in the matrix representing each map lie in the maximal ideal.

□

This gives us direct access to the Betti numbers:

Corollary 3.6.9. *Let I_A be the ideal of* $\mathrm{Veronese}(1, d)$. *Let $\beta_{i,j}$ be its Betti numbers. Then*

$$\beta_{i,j} = \begin{cases} 1 & i = 0, j = 0 \\ i\binom{d}{i+1} & j = i+1, i \leq d-1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. This is simply a count of the generators in the Eagon-Northcott complex. By Prop. 3.3.6, E_0^A has one free generator which is in degree 0 by our construction in the previous theorem, explaining the first case computed above. When $k \geq 1$, $E_k^A = \wedge^{k+1} S^d \otimes \mathrm{Sym}^{k-1} S^2$.

We note that $\wedge^{k+1} S^d$ has $\binom{d}{k+1}$ generators because choosing increasing sets of indices is the same as choosing unordered sets, and $\mathrm{Sym}^{k-1} S^2$ has k generators, being the k monomials of degree $k-1$ on two variables. Thus E_k^A has $k\binom{d}{k+1}$ generators which are all in degree $k+1$ by Theorem 3.6.9, which explains the second case computed above.

All other Betti numbers are 0 because each module has no generators in any other graded degree, and for $i > d-1$, $\wedge^{i+2-1} S^d = 0$.

□

References

[1] Winfrid Burns and Jurgen Herzog. *Cohen-Macaulay Rings*. Cambridge University Press, 1998.

[2] J. A. Eagon and D. G. Northcott. "Ideals Defined by Matrices and a Certain Complex Associated with Them". In: *Proceedings of the Royal Society of London* 269.1337 (1962), pp. 179–204.

[3] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*. Springer, 1995.

[4] David Eisenbud. *Geometry of Syzygies*. Springer, 2005.