University of Wisconsin—Madison

Introduction to Discrete Mathematics

Math 240

Assignment 3

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Question 1: Set identities [5 points]

Consider the sets $A = \{1, 2\}$ and $B = \{3, 4\}$. Verify that, for these sets, the following identies hold:

1. $A \setminus (B \cap A) = A$ and
2. $A \setminus (B \setminus A) = A$.

Prove that, for *arbitary* sets A and B, item 2 above remains true but item 1 may fail to hold.

Item 1:

Proof. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Then $B \cap A = \emptyset$. Therefore $A \setminus (B \cap A) = A \setminus \emptyset = A$. Now instead if A and B are arbitary sets, then $A \setminus (B \cap A)$ is not necessarily A because if $B \cap A \neq \emptyset$, then $\exists x \in A\, (x \in B \cap A)$. Therefore the cardinality of $A \setminus (B \cap A)$ is strictly less than the cardinality of A, so $A \setminus (B \cap A) \neq A$ if $A \cap B \neq \emptyset$. $\square$

Item 2:

Proof. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Then $B \setminus A = B$. Therefore $A \setminus (B \setminus A) = A \setminus B = A$. Now let us consider this identity if A and B are arbitary sets. $\forall x \in A\, (x \notin B \setminus A)$. Since $x \in A$ and $x \notin B \setminus A$, then $x \in A \setminus (B \setminus A)$. Therefore $A \subseteq A \setminus (B \setminus A)$. By definition, $A \setminus (B \setminus A) \subseteq A$. Therefore $A \setminus (B \setminus A) = A$ because they are subspaces of each other. $\square$

Question 2: Injectivity, surjectivity, and bijectivity [5 points]

For each of the functions below, determine and prove whether or not it is injective, surjective, and bijective.

1. $f : \{0,1\}^3 \to \{0,1\}^4$ is given by adding a copy of the first bit to the end of the binary string. In other words $f(xyz) = xyzx$.
2. Let $S = \{1,2,3\}$ and consider $g : \mathcal{P}(S) \to \{0,1,2,3\}$ given by $g(A) = |A|$, where recall that for any set A, $|A|$ denotes its cardinality.

Item 1:

Proof. f is injective because for any binary string $abc \in \{0,1\}^3$, $f(abc) = abcx$ where some $x = a$. Therefore for any other binary string in $\{0,1\}^3$ that is not abc, the output will not contain abc, so f is injective because $\forall (x,y) \in \{0,1\}^3$ such that $x \neq y$ $(f(x) \neq f(y))$. We will prove that f is not surjective by contradiction. For the sake of contradiction, suppose abc is a binary string in $\{0,1\}^3$ such that $f(abc) = 0011$. 0011 is in $\{0,1\}^4$. Then a must be equal to 0 and 1, which is impossible. Therefore f is not surjective. Since f is not injective and surjective, then f is not bijective. □

Item 2:

Proof. We will prove that g is not injective by contradiction. For the sake of contradiciton, suppose that g is injective. Then $\forall (A_1, A_2) \in \mathcal{P}(S)$ such that $S = \{1,2,3\}$ and $A_1 \neq A_2$ $(g(A_1) \neq g(A_2))$. However, suppose that $A_1 = \{1\}$ and $A_2 = \{2\}$. Then $g(A_1) = |A_1| = 1$ and $g(A_2) = |A_2| = 1$. Therefore $g(A_1) = g(A_2)$, which is a contradiction, so g is not injective. We will prove that g is surjective by exhaustion. The codomain of g is $0, 1, 2,$ and 3. For g to be surjective, $\forall x \in \{0,1,2,3\}\ \exists A \in \mathcal{P}(S)\ (g(A) = x)$. Let $A = \emptyset$, then $g(A) = |\emptyset| = 0$. Let $A = \{1\}$, then $g(A) = |1| = 1$. Let $A = \{1,2\}$, then $g(A) = |\{1,2\}| = 2$. Finally, let $A = \{1,2,3\}$, then $g(A) = |\{1,2,3\}| = 3$. All of these values of A are in $\mathcal{P}(S)$ and all values in $\{0,1,2,3\}$ is equal to $g(A)$ for some A. Therefore g is surjective. Since g is not injective and surjective, g is not bijective. □