University of Wisconsin—Madison

Introduction to Discrete Mathematics

Math 240

Assignment 4

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Question 1: Examples of big O notation [5 points]

Prove the folowing statements.

1. $n^3 = O(e^n)$

   Proof. Suppose that $n_0 = 1$ and $c = 24$. We will show that for any $n \geq 1$, $n^3 \leq 24 \cdot e^n$. For every integer value of n such that $n \geq 1$, $n^3 \leq n^4$. The Taylor expansion of the exponential function is given by

   $$e^n = \sum_{i=0}^{\infty} \frac{n^k}{k!} = 1 + n + \frac{n^2}{2} + \frac{n^3}{6} + \frac{n^4}{24} + \dots$$

   Since $n \geq 1$, then $n^4 < 24 \cdot e^n$. Combining these two inequalities gives us

   $$n^3 \leq n^4 < 24 \cdot e^n$$

   Therefore for $n \geq 1$, $n^3 \leq 24 \cdot e^n$, so $n^3 = O(e^n)$. $\square$

2. $n^3 + 3n^2 - 4 = \Theta(n^3)$

   Proof. Suppose that $n_0 = 2$ and $c = 1$. We will show that for any $n \geq 2$, $n^3 + 3n^2 - 4 \geq n^3$. Since $n \geq 2$, then $n^3 + 3n^2 - 4 \geq n^3 + 8 > n^3$. Therefore for $n \geq 2$, $n^3 + 3n^2 - 4 \geq n^3$, so $n^3 + 3n^2 - 4 = \Omega(n^3)$. Suppose that $n_0 = 1$ and $c = 100$. We will show that for any $n \geq 1$, $n^3 + 3n^2 - 4 \leq 100n^3$. Since $n \geq 1$, then $n^3 + 3n^2 - 4 \leq 100n^3$. Therefore for $n \geq 1$, $n^3 + 3n^2 - 4 \leq c \cdot n^3$, so $n^3 + 3n^2 - 4 = O(n^3)$. Since $n^3 + 3n^2 - 4 = O(n^3)$ and $n^3 + 3n^2 - 4 = \Omega(n^3)$, then by definition, $n^3 + 3n^2 - 4 = \Theta(n^3)$. $\square$

3. $n^{3/2} \neq \Omega(n^2)$

   Proof. Suppose that $n_0 = 1$ and $c = 1$. For the sake of contradiction, we assume that $n^{3/2} = \Omega(n^2)$. Then for every $n \geq 1$, $n^{3/2} \geq n^2$. However if we set $n = 2$, then the previous inequality becomes $2\sqrt{2} \geq 4$, which is false. Therefore by contradiction, $n^{3/2} \neq \Omega(n^2)$. $\square$

Question 2: Some rules of big O notation [5 points]

1. $f = O(g)$ if and only if $g = \Omega(f)$.

   Proof. We will prove this statement by proving both sides of the biconditional. If $f = O(g)$, then there exists real numbers $n_0 > 0$ and $c > 0$, such that for every integer $n \geq n_0$, $f(n) \leq c \cdot g(n)$, which is equivalent to $c \cdot g(n) \geq f(n)$. Since c is positive, dividing both sides of the previous inequality by c gives us $g(n) \geq \frac{1}{c} \cdot f(n)$. Since $\frac{1}{c}$ is a real number, then $g = \Omega(f)$. Therefore $f = O(g) \implies g = \Omega(f)$. Now, if $g = \Omega(f)$, then there exists real numbers $n_0 > 0$ and $c > 0$, such that for every integer $n \geq n_0$, then $g(n) \geq c \cdot f(n)$, which is equivalent to $c \cdot f(n) \leq g(n)$. Since c is positive, dividing both sides of the previous inequality by c gives us $f(n) \leq \frac{1}{c} \cdot g(n)$. Since $\frac{1}{c}$ is a real number, then $f = O(g)$. Therefore $g = \Omega(f) \implies f = O(g)$. Since $f = O(g) \implies g = \Omega(f)$ and $g = \Omega(f) \implies f = O(g)$, then $f = O(g) \iff g = \Omega(f)$. □

2. If $f_1 = \Omega(g_1)$ and $f_2 = \Omega(g_2)$ then $f_1 f_2 = \Omega(g_1 g_2)$.

   Proof. If $f_1 = \Omega(g_1)$, there exists real numbers $n_{1_0} > 0$ and $c_1 > 0$, such that for every integer $n_1 \geq n_{1_0}$, $f_1(n_1) \geq c_1 \cdot g_1(n_1)$. Similarly, if $f_2 = \Omega(g_2)$, there exists real numbers $n_{2_0} > 0$ and $c_2 > 0$, such that for every integer $n_2 \geq n_{2_0}$, $f_2(n_2) \geq c_2 \cdot g_2(n_2)$. Multiplying these two inequalities together gives us

   $$f_1(n_1)f_2(n_2) \geq (c_1 \cdot g_1(n_1)) \cdot (c_2 \cdot g_2(n_2)) \tag{1}$$
   $$f_1(n_1)f_2(n_2) \geq c_1 c_2 \cdot g_1(n_1) \cdot g_2(n_2) \tag{2}$$

   Since $c1$ and $c2$ are both real numbers, then $c1c2$ is a real number. Additionally, we can assume without loss of generality that if $n_{1_0} \leq n_{2_0}$, then

   $$f_1(n)f_2(n) \geq c \cdot g_1(n)g_2(n)$$

   where $c = c1c2$ and for any integer $n \geq n_{2_0}$. Therefore $f_1 f_2 = \Omega(g_1 g_2)$. □

3. If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

   Proof. If $f = O(g)$ there exists positive real numbers n_0, c such that for every integer $n \geq n_0, f(n) \leq c \cdot g(n)$. If $g = O(h)$ there exists positive real numbers m_0, d such that for every integer $m \geq m_0, g(m) \leq d \cdot h(m)$. Dividing both sides of the first inequality by c gives us $\frac{f(n)}{c} \leq g(n)$. Subbing this value of $g(n)$ into the second inequality gives us $\frac{f(m)}{c} \leq d \cdot h(m)$, which is equivallent to $f(m) \leq c \cdot d \cdot h(m)$. Since c, d are integers, $c \cdot d$ is an integer, therefore $f = O(h)$. □

Question 3: Examples of finite state machines [5 points]

Note that I will only be writing the answers for Question 3.

1. (a) 000 $\rightarrow$ 000
   (b) 010 $\rightarrow$ 000
   (c) 101 $\rightarrow$ 100
   The output string has leading 1s that are in the same position as the input string's leading 1s. The remainder of the output string are 0s.

2. (a) 100011 $\rightarrow$ No
   (b) 1111 $\rightarrow$ Yes
   (c) 0010 $\rightarrow$ No
   (d) 1100 $\rightarrow$ No
   The FSM only accepts input strings where every single character of the input string is 1.

3. (a) 100011 $\rightarrow$ No
   (b) 0000 $\rightarrow$ No
   (c) 0010 $\rightarrow$ No
   (d) 1100 $\rightarrow$ Yes
   The FSM only accepts input strings where the first character is 1 and the last character is 0.

Question 4: Designing finite state machines [5 points]

I will only be writing the answers. Since the problem did not specify to draw the FSMs, **I will write the proper notation for the FSMs in the form of tuples.**

1. This FSM will be defined by the tuple (Q, q_0, I, A, δ) where Q represents the finite set of states, $q_0 \in Q$ represents the starting state, I represents the finset set of inputs, $A \subseteq Q$ represents the accepting states, and $\delta\colon Q \times I \to Q$ represents the transition function. Let $Q = \{\alpha, \beta, \gamma\}$, $q_0 = \alpha$, $I = \{0, 1\}$ and $A = \alpha$. Let δ be defined by:

$$\forall q \in Q\ (\delta(q, 0) = q) \tag{3}$$
$$\delta(\alpha, 1) = \beta \tag{4}$$
$$\delta(\beta, 1) = \gamma \tag{5}$$
$$\delta(\gamma, 1) = \alpha \tag{6}$$

   This is a valid FSM.

2. This FSM will be defined by the tuple (Q, q_0, I, O, δ) where Q represents the finite set of states, $q_0 \in Q$ represents the starting state, I represents the finite set of inputs, O represents the finite set of outputs, and $\delta\colon Q \times I \to Q \times O$ represents the transition function. Let $Q = \{alpha\}$, $q_0 = \alpha$, $I = \{0, 1\}$, and $O = \{0, 1\}$. Let δ be defined by:

$$\forall i \in I\ (\delta(\alpha, i) = (\alpha, i))$$

   This is a valid FSM.