University of Wisconsin—Madison

Introduction to Discrete Mathematics

Math 240

Assignment 2

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Question 1: Proving and disproving quantified statements [5 points]

Prove or disprove the following statements.

1. For every real number x and y such that $x \neq 0$ there exists a real number z such that $xz + y = 0$.

Proof. Let x, y and z be real numbers such that $x \neq 0$. If we perform algebra on the equation $xz + y = 0$, we get

$$xz + y = 0 \tag{1}$$
$$xz = -y \tag{2}$$
$$z = -\frac{y}{x} \tag{3}$$

Since z can be written as a fraction where both the numerator (y) and denominator (x) are real numbers, where the denominator is nonzero, we can conclude that z is a real number. Therefore, there exists a real number z such that $xz + y = 0$ for all real numbers x and y. □

2. There is an integer n such that $2^n - 1$ is prime.

Proof. Let n be an integer. We will prove that there exists an integer n such that $2^n - 1$ is prime. We will do this by showing that $2^n - 1$ is prime when n is equal to 3, which is an integer. By subbing in this value of n into the expression $2^n - 1$, we get

$$2^3 - 1 = 7 \tag{4}$$

7 is a prime number, because it is only divisible by 1 and itself. Therefore, there exists an integer n such that $2^n - 1$ is prime. □

3. There is a smallest integer.

Proof. Let us prove this statement by contradiction. Assume that there is a smallest integer x. By definition, if x is an integer, then $x - 1$ is also an integer. Since $x - 1$ is less than x, then x is not the smallest integer, which contradicts our assumption that x is the smallest integer. Therefore, there is no smallest integer. □

Question 2: Proving and disproving implications [5 points]

Prove or disprove the following statements.

1. Let x, y, z be integers. If $x|(y + z)$ then $x|y$ and $x|z$.

Proof. Let us disprove this statement by counterexample. Let x, y, z be integers such that $x = 2, y = 3, z = 3$. Subbing these values into the expression $x|(y + z)$ gives

$$2|(3 + 3) \quad (5)$$
$$2|(6) \quad (6)$$
$$(7)$$

Since there exists an integer value k such that $2k = 6$, where $k = 3$, we can conclude that 2 divides 6, which means that $x|(y + z)$ when $x = 2, y = 3, z = 3$. However, 2 does not divide 3, because there does not exist an integer l such that $2l = 3$. Solving for l gives us 1.5 which is not an integer because it is between two consecutive integers. Therefore, x does not divide y or z when $x = 2, y = 3, z = 3$. Since the antecedent is true, but the consequent is false, we disprove the implication that if $x|(y + z)$ then $x|y$ and $x|z$. □

2. Let x, y, z be integers. If $x|(y + z)$ and $x|y$ then $x|z$.

Proof. Suppose that x, y, z are integers. If $x|(y + z)$ and $x|y$, that means that there exists an integer k such that $kx = (y + z)$. Let us solve for z in this equation.

$$kx = (y + z) \quad (8)$$
$$kx - y = z \quad (9)$$
$$(10)$$

Since x divides y, there exists an integer l such that $lx = y$. Subbing lx into the equation $kx - y = z$ gives us

$$kx - lx = z \qquad \text{which is equivalent to} \quad (11)$$
$$(k - l)x = z \quad (12)$$

Since k and l are integers, then $k - l$ must be an integer. Therefore, x divides z. Therefore, if $x|(y + z)$ and $x|y$ then $x|z$. □

3. Let x and y be positive real numbers. If $xy > 400$ then $x > 20$ or $y > 20$.

Proof. Let us prove this statement by contrapositive. Suppose that x and y are real numbers such that $x <= 20$ and $y <= 20$. Then, by using algebra, $xy <= 400$. The contrapositive must be true, therefore if $xy > 400$, then $x > 20$ or $y > 20$. □

Question 3: Contradictions [5 points]

Prove that $\sqrt{3}$ is irrational by mimicking the proof from class that $\sqrt{2}$ is irrational. Here is a handy fact we proved in class which will prove useful: For any integer n, if $3|n^2$ then $3|n$.

Proof. Suppose for the sake of contradiction that $\sqrt{3}$ is rational. This means that $\sqrt{3} = \frac{x}{y}$ for some integers x and y, with $y \neq 0$. Moreover we may assume without loss of generality, that x and y do not have any common factors. Squaring both sides of the equation gives us $3 = \frac{x^2}{y^2}$. Multiplying both sides by y^2 gives us $3y^2 = x^2$. Since x^2 is a multiple of 3, then $3|x^2$, and using the fact that for any integer n, if $3|n^2$ then $3|n$, we can conclude that $3|x$. This means there exists an integer k such that $3k = x$. Substituting $3k$ into $3y^2 = x^2$ gives us $3y^2 = (3k)^2$, which simplifies to $3y^2 = 9k^2$. Dividing both sideds by 3 gives us $y^2 = 3k^2$. Since y^2 is a multiple of 3, then $3|y^2$, therefore $3|y$. Since x and y are both divisible by 3, they share a common of factor of 3, which contradicts the fact that x and y do not have any common factors in order for $\sqrt{3}$ to be rational. Therefore $\sqrt{3}$ is irrational. $\square$

Question 4: Proofs by cases [5 points]

Prove the following statements.

1. Let x and y be integers. If xy and $x+y$ are even then both x and y are even.

Proof. Let us prove this statement by contrapositive and by cases. Suppose that x and y are integers such that either x or y is odd. Then, xy or $x+y$ is odd. Since swapping x and y in either expression makes no difference, we only have to consider the cases where x is an odd integer and y is an even integer, and if x and y are both odd integers. For the first case, let us assume that x is odd and y is even. Then, x can be rewritten as $2j+1$ for some integer j and y can be rewritten as $2k$ for some integer k. Then $x+y$ can be rewritten as $2j+1+2k$ which can be simplified to $2(j+k)+1$. Since j and k are integers, then $j+k$ is an integer, therefore $x+y$ is odd. For our second case, let us assume that x is odd and y is odd. Then, x can be rewritten as $2j+1$ for some integer j and y can be rewritten as $2k+1$ for some integer k. Subbing these values into xy gives $(2j+1)(2k+1)$. This is equivalent to $(2j)(2k)+2j+2k+1$which can be rewritten as $2(jk+j+k)+1$. Since j and k are both integers, then jk and $j+k$ are integers, therefore xy is odd. Again, swapping x and y makes no difference, so the remaining cases are redundant. Therefore, in all cases where either x or y is odd, xy or $x+y$ is odd. In conclusion, we prove that if either x or y are odd integers, then xy or $x+y$ is odd. Therefore, by contrapositive, we prove that if xy and $x+y$ are even then both x and y are even. □

2. Let x be a real number. If $x^2-3x-10<0$ then $-2<x<5$.

Proof. Let us prove this statement by contrapositive and by cases. Suppose that x is an integer. If $x \leq -2$ or $x \geq 5$ then $x^2-3x-10 \geq 0$. We have to prove that the consequent is true given the cases where

1. $x < -2$
2. $x = -2$
3. $x = 5$
4. $x > 5$

- Case 1: $x<-2$. Factoring $x^2-3x-10$ gives us $(x-5)(x+2)$. Since $x<-2$, then $x-5<0$ and $x+2<0$. Since the product of two negative integers is a positive integer, $x^2-3x-10 \geq 0$.
- Case 2: $x=-2$. Factoring $x^2-3x-10$ gives us $(x-5)(x+2)$. Subbing in $x=-2$ gives us $(-2-5)(-2+2)$, which is equal to 0, which satisfies the assumption that $x^2-3x-10 \geq 0$.
- Case 3: $x=5$. Factoring $x^2-3x-10$ gives us $(x-5)(x+2)$. Subbing in $x=5$ gives us $(5-5)(5+2)$, which is equal to 0, which satisfies the assumption that $x^2-3x-10 \geq 0$.
- Case 4: $x>5$. Factoring $x^2-3x-10$ gives us $(x-5)(x+2)$. Since $x>5$, then $x-5>0$ and $x+2>0$. Since the product of two positive integers is a positive integer, $x^2-3x-10 \geq 0$.

In conclusion, we have shown that for all values of x where $x \leq -2$ or $x \geq 5$, $x^2-3x-10 \geq 0$. Therefore, by contrapositive, we have shown that if $x^2-3x-10<0$ then $-2<x<5$. □

3. Let x and y be integers. If $x^3(y+5)$ is odd then x is odd and y is even.

Proof. Let us prove this statement by contrapositive and by cases. Suppose that x and y are integers where if x is even or y is odd, then $x^3(y+5)$ is even. We only need to consider two cases where

1. x is even
2. y is odd

- Case 1: x is even. Let $x=2j$ for some integer j. Then $x^3(y+5)$ can be rewritten as $2^3j^3(y+5)$. Since j is an integer, then j^3 is an integer. Since y is an integer, then $y+5$ is an integer. $2^3j^3(y+5)$ can be rewritten as $2(2^2j^3(y+5))$ where $2^2j^3(y+5)$ is an integer, therefore $x^3(y+5)$ is even when x is even.
- Case 2: y is odd. Let $y=2k+1$ for some integer k. Then $x^3(y+5)$ can be rewritten as $x^3(2k+1+5)$. Since x is an integer, then x^3 is an integer. Since k is an integer, then $2k+1+5$ is an integer. $x^3(2k+1+5)$ can be rewritten as $2((k+3)x^3)$ where $(k+3)x^3$ is an integer, therefore $x^3(y+5)$ is even when y is odd.

In conclusion, we have shown that for if x is even or y is odd, then $x^3(y+5)$ is even. Therefore, by contrapositive, we have shown that if $x^3(y+5)$ is odd then x is odd and y is even. □