

UNIVERSITY OF WISCONSIN—MADISON



INTRODUCTION TO DISCRETE MATHEMATICS

MATH 240

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## Assignment 2

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*Author:*  
Raymond TIAN

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## Question 1: Proving and disproving quantified statements [5 points]

Prove or disprove the following statements.

- 1. For every real number  $x$  and  $y$  such that  $x \neq 0$  there exists a real number  $z$  such that  $xz + y = 0$ .**

*Proof.* Let  $x, y$  and  $z$  be real numbers such that  $x \neq 0$ . If we perform algebra on the equation  $xz + y = 0$ , we get

$$xz + y = 0 \tag{1}$$

$$xz = -y \tag{2}$$

$$z = -\frac{y}{x} \tag{3}$$

Since  $z$  can be written as a fraction where both the numerator ( $y$ ) and denominator ( $x$ ) are real numbers, where the denominator is nonzero, we can conclude that  $z$  is a real number. Therefore, there exists a real number  $z$  such that  $xz + y = 0$  for all real numbers  $x$  and  $y$ .  $\square$

- 2. There is an integer  $n$  such that  $2^n - 1$  is prime.**

*Proof.* Let  $n$  be an integer. We will prove that there exists an integer  $n$  such that  $2^n - 1$  is prime. We will do this by showing that  $2^n - 1$  is prime when  $n$  is equal to 3, which is an integer. By subbing in this value of  $n$  into the expression  $2^n - 1$ , we get

$$2^3 - 1 = 7 \tag{4}$$

7 is a prime number, because it is only divisible by 1 and itself. Therefore, there exists an integer  $n$  such that  $2^n - 1$  is prime.  $\square$

- 3. There is a smallest integer.**

*Proof.* Let us disprove this statement by contradiction. Assume that there is a smallest integer  $x$ . By definition, if  $x$  is an integer, then  $x - 1$  is also an integer. Since  $x - 1$  is less than  $x$ , then  $x$  is not the smallest integer, which contradicts our assumption that  $x$  is the smallest integer. Therefore, there is no smallest integer.  $\square$

## Question 2: Proving and disproving implications [5 points]

Prove or disprove the following statements.

**1. Let  $x, y, z$  be integers. If  $x|(y+z)$  then  $x|y$  and  $x|z$ .**

*Proof.* Let us disprove this statement by counterexample. Let  $x, y, z$  be integers such that  $x = 2, y = 3, z = 3$ . Subbing these values into the expression  $x|(y+z)$  gives

$$2|(3+3) \tag{5}$$

$$2|(6) \tag{6}$$

$$\tag{7}$$

Since there exists an integer value  $k$  such that  $2k = 6$ , where  $k = 3$ , we can conclude that 2 divides 6, which means that  $x|(y+z)$  when  $x = 2, y = 3, z = 3$ . However, 2 does not divide 3, because there does not exist an integer  $l$  such that  $2l = 3$ . Solving for  $l$  gives us 1.5 which is not an integer because it is between two consecutive integers. Therefore,  $x$  does not divide  $y$  or  $z$  when  $x = 2, y = 3, z = 3$ . Since the antecedent is true, but the consequent is false, we disprove the implication that if  $x|(y+z)$  then  $x|y$  and  $x|z$ .  $\square$

**2. Let  $x, y, z$  be integers. If  $x|(y+z)$  and  $x|y$  then  $x|z$ .**

*Proof.* Suppose that  $x, y, z$  are integers. If  $x|(y+z)$  and  $x|y$ , that means that there exists an integer  $k$  such that  $kx = (y+z)$ . Let us solve for  $z$  in this equation.

$$kx = (y+z) \tag{8}$$

$$kx - y = z \tag{9}$$

$$\tag{10}$$

Since  $x$  divides  $y$ , there exists an integer  $l$  such that  $lx = y$ . Subbing  $lx$  into the equation  $kx - y = z$  gives us

$$kx - lx = z \quad \text{which is equivalent to} \tag{11}$$

$$(k-l)x = z \tag{12}$$

Since  $k$  and  $l$  are integers, then  $k-l$  must be an integer. Therefore,  $x$  divides  $z$ . Therefore, if  $x|(y+z)$  and  $x|y$  then  $x|z$ .  $\square$

**3. Let  $x$  and  $y$  be positive real numbers. If  $xy > 400$  then  $x > 20$  or  $y > 20$ .**

*Proof.* Let us prove this statement by contrapositive. Suppose that  $x$  and  $y$  are real numbers such that  $x \leq 20$  and  $y \leq 20$ . Then, by using algebra,  $xy \leq 400$ . The contrapositive must be true, therefore if  $xy > 400$ , then  $x > 20$  or  $y > 20$ .  $\square$

**Question 3: Contradictions [5 points]**

Prove that  $\sqrt{3}$  is irrational by mimicking the proof from class that  $\sqrt{2}$  is irrational. Here is a handy fact we proved in class which will prove useful: For any integer  $n$ , if  $3|n^2$  then  $3|n$ .

*Proof.* Suppose for the sake of contradiction that  $\sqrt{3}$  is rational. This means that  $\sqrt{3} = \frac{x}{y}$  for some integers  $x$  and  $y$ , with  $y \neq 0$ . Moreover we may assume without loss of generality, that  $x$  and  $y$  do not have any common factors. Squaring both sides of the equation gives us  $3 = \frac{x^2}{y^2}$ . Multiplying both sides by  $y^2$  gives us  $3y^2 = x^2$ . Since  $x^2$  is a multiple of 3, then  $3|x^2$ , and using the fact that for any integer  $n$ , if  $3|n^2$  then  $3|n$ , we can conclude that  $3|x$ . This means there exists an integer  $k$  such that  $3k = x$ . Substituting  $3k$  into  $3y^2 = x^2$  gives us  $3y^2 = (3k)^2$ , which simplifies to  $3y^2 = 9k^2$ . Dividing both sides by 3 gives us  $y^2 = 3k^2$ . Since  $y^2$  is a multiple of 3, then  $3|y^2$ , therefore  $3|y$ . Since  $x$  and  $y$  are both divisible by 3, they share a common factor of 3, which contradicts the fact that  $x$  and  $y$  do not have any common factors in order for  $\sqrt{3}$  to be rational. Therefore  $\sqrt{3}$  is irrational.  $\square$

## Question 4: Proofs by cases [5 points]

Prove the following statements.

**1. Let  $x$  and  $y$  be integers. If  $xy$  and  $x + y$  are even then both  $x$  and  $y$  are even.**

*Proof.* Let us prove this statement by contrapositive and by cases. Suppose that  $x$  and  $y$  are integers such that either  $x$  or  $y$  is odd. Then,  $xy$  or  $x + y$  is odd. Since swapping  $x$  and  $y$  in either expression makes no difference, we only have to consider the cases where  $x$  is an odd integer and  $y$  is an even integer, and if  $x$  and  $y$  are both odd integers. For the first case, let us assume that  $x$  is odd and  $y$  is even. Then,  $x$  can be rewritten as  $2j + 1$  for some integer  $j$  and  $y$  can be rewritten as  $2k$  for some integer  $k$ . Then  $x + y$  can be rewritten as  $2j + 1 + 2k$  which can be simplified to  $2(j + k) + 1$ . Since  $j$  and  $k$  are integers, then  $j + k$  is an integer, therefore  $x + y$  is odd. For our second case, let us assume that  $x$  is odd and  $y$  is odd. Then,  $x$  can be rewritten as  $2j + 1$  for some integer  $j$  and  $y$  can be rewritten as  $2k + 1$  for some integer  $k$ . Subbing these values into  $xy$  gives  $(2j + 1)(2k + 1)$ . This is equivalent to  $(2j)(2k) + 2j + 2k + 1$  which can be rewritten as  $2(jk + j + k) + 1$ . Since  $j$  and  $k$  are both integers, then  $jk$  and  $j + k$  are integers, therefore  $xy$  is odd. Again, swapping  $x$  and  $y$  makes no difference, so the remaining cases are redundant. Therefore, in all cases where either  $x$  or  $y$  is odd,  $xy$  or  $x + y$  is odd. In conclusion, we prove that if either  $x$  or  $y$  are odd integers, then  $xy$  or  $x + y$  is odd. Therefore, by contrapositive, we prove that if  $xy$  and  $x + y$  are even then both  $x$  and  $y$  are even.  $\square$

**2. Let  $x$  be a real number. If  $x^2 - 3x - 10 < 0$  then  $-2 < x < 5$ .**

*Proof.* Let us prove this statement by contrapositive and by cases. Suppose that  $x$  is an integer. If  $x \leq -2$  or  $x \geq 5$  then  $x^2 - 3x - 10 \geq 0$ . We have to prove that the consequent is true given the cases where

1.  $x < -2$
  2.  $x = -2$
  3.  $x = 5$
  4.  $x > 5$
- Case 1:  $x < -2$ . Factoring  $x^2 - 3x - 10$  gives us  $(x - 5)(x + 2)$ . Since  $x < -2$ , then  $x - 5 < 0$  and  $x + 2 < 0$ . Since the product of two negative integers is a positive integer,  $x^2 - 3x - 10 \geq 0$ .
  - Case 2:  $x = -2$ . Factoring  $x^2 - 3x - 10$  gives us  $(x - 5)(x + 2)$ . Subbing in  $x = -2$  gives us  $(-2 - 5)(-2 + 2)$ , which is equal to 0, which satisfies the assumption that  $x^2 - 3x - 10 \geq 0$ .
  - Case 3:  $x = 5$ . Factoring  $x^2 - 3x - 10$  gives us  $(x - 5)(x + 2)$ . Subbing in  $x = 5$  gives us  $(5 - 5)(5 + 2)$ , which is equal to 0, which satisfies the assumption that  $x^2 - 3x - 10 \geq 0$ .
  - Case 4:  $x > 5$ . Factoring  $x^2 - 3x - 10$  gives us  $(x - 5)(x + 2)$ . Since  $x > 5$ , then  $x - 5 > 0$  and  $x + 2 > 0$ . Since the product of two positive integers is a positive integer,  $x^2 - 3x - 10 \geq 0$ .

In conclusion, we have shown that for all values of  $x$  where  $x \leq -2$  or  $x \geq 5$ ,  $x^2 - 3x - 10 \geq 0$ . Therefore, by contrapositive, we have shown that if  $x^2 - 3x - 10 < 0$  then  $-2 < x < 5$ .  $\square$

**3. Let  $x$  and  $y$  be integers. If  $x^3(y + 5)$  is odd then  $x$  is odd and  $y$  is even.**

*Proof.* Let us prove this statement by contrapositive and by cases. Suppose that  $x$  and  $y$  are integers where if  $x$  is even or  $y$  is odd, then  $x^3(y + 5)$  is even. We only need to consider two cases where

1.  $x$  is even
  2.  $y$  is odd
- Case 1:  $x$  is even. Let  $x = 2j$  for some integer  $j$ . Then  $x^3(y + 5)$  can be rewritten as  $2^3j^3(y + 5)$ . Since  $j$  is an integer, then  $j^3$  is an integer. Since  $y$  is an integer, then  $y + 5$  is an integer.  $2^3j^3(y + 5)$  can be rewritten as  $2(2^2j^3(y + 5))$  where  $2^2j^3(y + 5)$  is an integer, therefore  $x^3(y + 5)$  is even when  $x$  is even.
  - Case 2:  $y$  is odd. Let  $y = 2k + 1$  for some integer  $k$ . Then  $x^3(y + 5)$  can be rewritten as  $x^3(2k + 1 + 5)$ . Since  $x$  is an integer, then  $x^3$  is an integer. Since  $k$  is an integer, then  $2k + 1 + 5$  is an integer.  $x^3(2k + 1 + 5)$  can be rewritten as  $2((k + 3)x^3)$  where  $(k + 3)x^3$  is an integer, therefore  $x^3(y + 5)$  is even when  $y$  is odd.

In conclusion, we have shown that for if  $x$  is even or  $y$  is odd, then  $x^3(y + 5)$  is even. Therefore, by contrapositive, we have shown that if  $x^3(y + 5)$  is odd then  $x$  is odd and  $y$  is even.  $\square$