A Fixed Time-Step Approach for Multibody Dynamics with Contact and Friction

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***Abstract*— We present a fixed time-step algorithm for the simulation of multi-rigid-body dynamics with joints, contact, collision, and friction. The method solves a linear complementarity problem (LCP) at each step. We show that the algorithm can be obtained as the stiff limit of fixed time-step schemes applied to regularized contact models. We do not perform collision detection. Instead, a noninterpenetration constraint is replaced by its linearization, which, together with a judicious choice of the active constraints, guarantees geometrical constraint stabilization without the need to perform a reduction of the time step to detect new collision or stick-slip transition events. Partially elastic collisions are accommodated by a suitable modification of the free term of the LCP.**

I. Introduction

Simulating the dynamics of a system with several rigid bodies and with joint, contact (noninterpenetration), and friction constraints is an important part of virtual reality and robotics simulations.

If the simulation has only joint constraints, then the problem is a differential algebraic equation (DAE) [1], [2], which is a widely studied and used computational paradigm. However, the nonsmooth nature of the noninterpenetration and friction constraints requires the use of specialized techniques. By and large there are two ways to approach this nonsmoothness: regularization approaches and hard constraint (complementarity) approaches.

The regularization approach [3], [4], [5] consists of smoothing the nonsmoothness in the description of noninterpenetration and frictional constraints and creating a DAE, for which substantial analytical and software tools exist, an important advantage. Sometimes the smoothing is based on some physical interpretation, as is the case when using a nonlinear spring and damper model as a replacement for the noninterpetration constraint [5]. The immediate disadvantage of the regularization approach is that the resulting DAE can be quite stiff.

The hard constraint approach has been used in either an acceleration-force setup [6], [7], [8] or a velocity-impulse time-stepping approach [9], [10], [11], [12]. The latter has the advantage that it always produces a solution that satisfies the constraint simulations and avoids the Coulomb friction model inconsistencies that are apparent in the acceleration-force approach. We note that, when the value of the time step is set to 0, the linear complementarity problem (LCP) of the velocity-impulse approach is the same as the one used in the compression phase of multiple collision resolution [13]. The advantage of the hard constraint approach is that there are no additional parameters to tune and there are no model stability issues. This gain comes, however, at the cost of a more difficult subproblem to solve, that is, a potentially nonconvex LCP.

All of the hard constraint approaches mentioned above are based on collision detection. A decision, based on geometrical computations, is made about which pairs of bodies are in contact and which features are active at the current time. Then the LCP is set up to compute the new acceleration or the new velocity, and this information is used to compute the future value of the position for the intended time step. If a new collision occurs within the time step, the simulation is backtracked to the first collision and is restarted after applying a collision resolution technique [13], [11].

Although the hard constraint approach leads to a stable simulation, the amount of computation needed per unit of target time step is impossible to predict because there is no conceptual upper bound on the number of collisions that can occur per unit of time. An extreme example is a rigid ball bouncing on a flat, rigid surface with a restitution coefficient strictly between 0 and 1, which sustains an infinite number of collisions in a finite amount of time. Of course, this situation can be accommodated by turning the restitution coefficient to 0 if the normal velocity is below a certain threshold; but our example shows that there is no upper bound on the number of collisions and, therefore, on the number of backtracking steps. Moreover, the LCP subproblems are expensive compared with one step of an explicit method applied to a regularization formulation. The fact that bouncing substantially worsens the performance of hard constraint approaches has been noted before in comparison with impulse-based simulation [14].

We note that regularization approaches are not immune to increased computation in the case of locally high density of events, even though the effect is not explicit. In this case, proximity of an event such as collision is manifested by a sudden increase in the penalty term that leads to terminal instability unless the timestep is reduced dramatically or the system is treated implicitly. In the latter case, the time step may also be reduced because the existence of a solution

to the nonlinear equation defining the integration method is guaranteed only for sufficiently small time steps [15].

The fact that the amount of computation per unit of time step may be locally unpredictible creates an important obstacle for applications that are intended to run, eventually, in an interactive fashion. It is therefore useful to investigate whether one can define an approach where, once the time step is fixed, the amount of computation necessary to advance the simulation for that one time-step is upper bounded, while maintaining the stability of the system.

A stable, fixed time-step approach can be realized in two ways, based on the approaches presented above. One can integrate the equations that result from the penalty method implicitly, or one can use a time-stepping approach. We show that, in this context, the two approaches lead to essentially the same subproblems to be solved at every step, and we therefore concentrate on the hard constraints approach. We show how partially elastic collisions as well as contact, friction, and joint constraints can be accommodated by this approach. We have found that, for this method, constraint stabilization can be achieved at no additional cost, as in [16].

In the following we will restrict ourselves to first-order integration methods. This restriction is justifiable since, if we do not plan to perform collision detection, the method cannot exceed order 1 anyway [15].

II. Extremal Analysis of a Penalty Model

Consider a multi-rigid-body system whose state is quantified by the position vector q and the velocity vector v. The external and inertial forces are denoted by $k(t, q, v)$. We assume that the system has a constant, positive-definite mass matrix M. This assumption is not essential, but it simplifies our notation. Such a mass matrix can be obtained in the Newton Euler body coordinates.

A noninterpenetration constraint is represented by the signed distance between two bodies functions, $\Phi(q)$ [17]. The noninterpenetration constraint becomes $\Phi(q) \geq 0$. The mapping $\Phi(q)$ is generally not differentiable everywhere even for simple shapes [16]. For smooth and strictly convex bodies, the mapping $\Phi(q)$ is differentiable in a neighborhood of the feasible set $\{q|\Phi(q) \geq 0\}$. To simplify the discussion, we assume that $\Phi(q)$ is differentiable at every point where it is evaluated, and we defer to future work the case when $\Phi(q)$ is nonsmooth, which appears for both nonconvex smooth and nonsmooth bodies.

We consider that the system is subject to m noninterpenetration constraints. The feasible set for the entire system is represented by

$$\Phi^{(j)}(q) \geq 0, \ j = 1, 2, \ldots, m. \tag{2.1}$$

To enforce these constraints, we use a penalty method [5] that allows the noninterpenetration constraints to be violated but creates a reaction force that prevents more severe interpenetration. For one noninterpenetration constraint j, the modulus of the reaction force is

$$\theta^{(j)}(q) = \gamma^{(j)} \left(\Phi_-^{(j)}(q)\right)^b, \tag{2.2}$$

where $b > 1$ is some appropriate exponent and $\gamma^{(j)} > 0$ is the penalty parameter. Here the quantity $\Phi_-^{(j)}(q)$ represents the negative part, that is, the constraint violation, of $\Phi^{(j)}(q)$. Its algebraic expression is

$$\Phi_-^{(j)}(q) = \frac{-\Phi^{(j)}(q) + |\Phi^{(j)}(q)|}{2}.$$

We can add to (2.2) a damping term whose effect is to produce dissipation when a collision occurs. For the present development we restrict our model to the elastic-type force from (2.2). By using Hertzian contact theory, it is determined that, in three dimensions, the appropriate exponent is $b = \frac{3}{2}$ [5]. Note that, when $b > 1$, the function $\theta^{(j)}(q)$ is continuously differentiable.

For the total reaction force from the noninterpenetration constraints to be of the potential type, its direction must be $\nabla_q(\Phi^{(j)})(q)$. With these choices and the use of Newton's law, the dynamics of the system becomes

$$\begin{array}{rcl} \frac{dq}{dt} & = & v. \\ M\frac{dv}{dt} & = & k(t, q, v) + \sum_{j=1}^m \theta^{(j)}(q)\nabla_q\Phi^{(j)}(q). \end{array} \tag{2.3}$$

We now look for numerical schemes for the system (2.3). The concern is that the stiffness that appears through the force modulus $\theta^{(j)}(q)$ could lead to numerical instability. To alleviate this concern, we consider two stiffness accommodating approaches: (1) $\theta^{(j)}(q)$ is treated implicitly and (2) $\theta^{(j)}(q)$ is treated linearly implicitly. In the following, we consider h_l, $t^{(l)}$, $q^{(l)}$, and $v^{(l)}$, to be the current time step, time, position, and velocity, respectively. We have that $t^{(l+1)} - t^{(l)} = h_l$. For generality of the setup we allow h_l to vary, but our method works just as well for a fixed time step.

A. Implicit Approach

We obtain the following time-stepping scheme:

$$\begin{array}{rcl} q^{(l+1)} & = & q^{(l)} + h_l v^{(l+1)}. \\ M\frac{v^{(l+1)} - v^{(l)}}{h_l} & = & k(t^{(l)}, q^{(l)}, v^{(l)}) \\ & + & \sum_{j=1}^m \theta^{(j)}(q^{(l+1)})\nabla_q\Phi^{(j)}(q^{(l+1)}). \end{array} \tag{2.4}$$

Consider the optimization problem

$$\begin{array}{rcl} \min_v \psi(v, \Gamma) & = & \frac{1}{2}v^T M v - v^T \left(M v^l + h_l k(t^l, q^l, v^l)\right) \\ & + & \sum_{j=1}^m \frac{1}{b+1}\gamma^{(j)} \left(\Phi_-^{(j)}(q^{(l)} + h_l v)\right)^{b+1}, \end{array} \tag{2.5}$$

where $\Gamma = \left(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(m)}\right)$.

One can immediately be see that any local solution v^* of the optimization problem (2.5) is a solution $v^{(l+1)}$ of (2.4). In effect, the discretized version of Newton's law in (2.4) is precisely the optimality conditions for the optimization problem (2.5). This property is related to the one of variational integrators [18].

We are interested in the situation where $\gamma^{(j)}$ is so large that it results in stiffness that is much more severe than the intended

time step could acommodate. So we wish to determine what happens if we let $\gamma^{(j)} \to \infty$, $j = 1, 2, \ldots, m$. We denote by v_n the solution (and, in case of multiplicity, the global solution) of (2.5) when $\Gamma_n = \Gamma = \left(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(m)}\right) = (n, n, \ldots, n)$. One solution must exist because the objective function is bounded below as a result of the existence of the quadratic term and to the nonnegativity of the penalty term.

We assume that the set defined by the constraints (2.1) is feasible, that is, that there exists some q^* such that $\Phi^{(j)}(q^*) \geq 0$, for $j = 1, 2, \ldots, m$. This implies that $\Phi_-^{(j)}(q^*) = 0$, for $j = 1, 2, \ldots, m$. In this case it is immediate that, since v_n is the optimal solution of (2.5),

$$\begin{aligned} & \frac{1}{2} v_n^T M v_n - v_n^T \left(M v^l + h_l k(t^l, q^l, v^l) \right) \\ \leq \quad & \psi(v_n, \Gamma_n) \leq \psi\left(\frac{q^* - q^l}{h_l}, \Gamma_n \right) \end{aligned}$$

Therefore, the sequence v_n is upper bounded uniformly with n. Since q^* is feasible, the last term in the preceding sequence of inequalities does not depend on Γ_n. Therefore v_n admits a limit point v_*.

Another consequence is that the term $\sum_{j=1}^m \frac{1}{b+1} n \left(\Phi_-^{(j)}(q^{(l)} + h_l v_n) \right)^{b+1}$ is upper bounded uniformly with respect to n. Taking the limit, we obtain that $\Phi_-^{(j)}(q^{(l)} + h_l v^*) = 0$, that is, that the point $q^{(l)} + h v^*$ is feasible.

Finally, if the set of vectors $\{\nabla_q \Phi(j)\}$ is linearly independent, then from (2.3) one can see that $\theta^{(j)}(q^l + h v_n)$ is also uniformly bounded, and we can assume, after eventually restricting to a subsequence, that $\theta^{(j)}(q^l + h_l v_n) \to c^{(j)} \geq 0$ as $n \to \infty$.

Since $(a_-)^b a = -(a_-)^{(b+1)}$ for any real number a, we obtain that

$$\begin{aligned} & \theta^{(j)}\left(q^l + h_l v_n)\right) \Phi^{(j)}\left(q^{(l)} + h_l v_n\right) \\ = \quad & -\theta^{(j)}\left(q^l + h_l v_n)\right) \Phi_-^{(j)}\left(q^{(l)} + h_l v_n\right). \end{aligned}$$

From our preceding results, this implies that $c^{(j)} \Phi^{(j)}\left(q^{(l)} + h_l v_*\right) = 0$.

After we replace all these limit relations in (2.3) and we associate $v^{(l+1)}$ to v_*, and $c^{(j),(l+1)}$ to $c^{(j)}$, we obtain the following nonlinear complementarity based time-stepping scheme:

$$\begin{array}{rcl} q^{(l+1)} & = & q^{(l)} + h_l v^{(l+1)}. \\ M \frac{v^{(l+1)} - v^{(l)}}{h_l} & = & k(t^{(l)}, q^{(l)}, v^{(l)}) \\ & + & \sum_{j=1}^m c^{(j),(l+1)} \nabla_q \Phi^{(j)}(q^{(l+1)}) \\ 0 & \leq & c^{(j),(l+1)} \\ 0 & \leq & \Phi^{(j)}(q^{(l+1)}) \\ 0 & = & c^{(j),(l+1)} \Phi^{(j)}(q^{(l+1)}), \end{array} \tag{2.6}$$

which is precisely the time-stepping scheme from [9], for the frictionless case.

B. Linearly Implicit Approach

In the linearly implicit case, the stiffness in $\theta^{(j)}(q^{(l+1)})$ is accommodated by linearization. To obtain meaningful results, we write $\theta^{(j)}(q)$ in the form $\theta^{(j)}(q) = \gamma^{(j)} \epsilon(\Phi^{(j)}(q)) \left|\Phi^{(j)}(q)\right|^b$, where

$$\epsilon(x) = \left\{ \begin{array}{ll} 1 & x \leq 0 \\ 0 & x > 0. \end{array} \right.$$

We approximate $\theta^{(j)}(q^{(l+1)})$ by linearizing $\Phi^{(j)}(q^{(l+1)})$ at the point $q^{(l)}$, and we use that $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$, as well as the approximation

$$\begin{aligned} \Phi^{(j)}(q^{(l+1)}) & \approx \widehat{\Phi}^{(j),(l)}(v^{(l+1)}) \\ & = \Phi^{(j)}(q^{(l)}) + h_l \nabla_q \Phi^{(j)}(q^{(l)})^T v^{(l+1)}, \end{aligned}$$

to obtain that

$$\begin{aligned} \theta^{(j)}(q^{(l+1)}) & \approx \hat{\theta}^{(j)}(v^{(l+1)}) \\ & = \gamma^{(j)} \epsilon\left(\widehat{\Phi}^{(j),(l)}(v^{(l+1)})\right) \left|\widehat{\Phi}^{(j),(l)}(v^{(l+1)})\right|^b. \end{aligned}$$

The numerical scheme becomes

$$\begin{array}{rcl} q^{(l+1)} & = & q^{(l)} + h_l v^{(l+1)}. \\ M \frac{v^{(l+1)} - v^{(l)}}{h_l} & = & k(t^{(l)}, q^{(l)}, v^{(l)}) \\ & & + \sum_{j=1}^m \hat{\theta}^{(j)}(v^{(l+1)}) \nabla_q \Phi^{(j)}(q^{(l)}). \end{array} \tag{2.7}$$

Note that the gradient of Φ is evaluated at $q^{(l)}$, since its linearization would result in an $O(h_l)$ term that disappears in the limit.

If $b > 1$, we use that $\frac{d}{dt}|t|^a = sgn(t)|t|^{a-1}$, which is true whenever $a > 1$ to obtain that

$$\begin{aligned} & \nabla_{v^{l+1}} \tfrac{1}{b+1} \epsilon\left(\widehat{\Phi}^{(j),(l)}(v^{(l+1)})\right) \left|\widehat{\Phi}^{(j),(l)}(v^{(l+1)})\right|^{b+1} \\ = \quad & -\epsilon\left(\widehat{\Phi}^{(j),(l)}(v^{(l+1)})\right) \left|\widehat{\Phi}^{(j),(l)}(v^{(l+1)})\right|^b. \end{aligned}$$

We therefore get that $v^{(l+1)}$ is a solution of the following optimization problem:

$$\begin{aligned} & \min_v \psi(v, \Gamma) = \tfrac{1}{2} v^T M v - v^T \left((M v^l + h_l k(t^l, q^l, v^l) \right) \\ + \quad & \sum_{j=1}^m \tfrac{1}{b+1} \gamma^{(j)} \epsilon\left(\widehat{\Phi}^{(j),(l)}(v))\right) \left|\widehat{\Phi}^{(j),(l)}(v))\right|^{b+1}. \end{aligned} \tag{2.8}$$

By using the same techniques as in the fully implicit case, we obtain the following LCP-based time-stepping scheme in the stiff limit of $\gamma^{(j)} \to \infty$:

$$\begin{array}{rcl} q^{(l+1)} & = & q^{(l)} + h_l v^{(l+1)}. \\ M \frac{v^{(l+1)} - v^{(l)}}{h_l} & = & k(t^{(l)}, q^{(l)}, v^{(l)}) \\ & + & \sum_{j=1}^m c^{(j),(l+1)} \nabla_q \Phi^{(j)}(q^{(l+1)}) \\ 0 & \leq & c^{(j),(l+1)} \\ 0 & \leq & \Phi^{(j)}(q^{(l)}) + h_l \nabla \Phi(q^{(l)})^T v^{(l+1)} \\ 0 & = & c^{(j),(l+1)} \left(\Phi^{(j)}(q^{(l)}) + h_l \nabla \Phi(q^{(l)})^T v^{(l+1)} \right). \end{array} \tag{2.9}$$

This is precisely the time-stepping scheme from [16] when joint and frictional constraints are not present. It has been shown not only that this scheme is stable if the ratio between consecutive time-steps is bounded below (the velocity stays bounded uniformly as the time step goes to 0), but also that

it achieves constraint stabilization without the need to solve a problem where feasibility is enforced exactly at $q^{(l+1)}$ in a nonlinear fashion. A similar constraint stabilization result is achieved if we replace the constraint $0 \leq \Phi^{(j)}(q^{(l)}) + h_l \nabla\Phi(q^{(l)})^T v^{(l+1)}$ by $0 \leq \gamma\Phi^{(j)}(q^{(l)}) + h_l \nabla\Phi(q^{(l)})^T v^{(l+1)}$, where γ is a parameter in $(0, 1]$ [19].

C. Discussion

We have obtained that, in the very stiff limit of the penalty method (when the parameters $\gamma^{(j)}$ approach ∞, in relation to the size of the time step), we recover complementarity-based time-stepping schemes. A similar result (that was submitted for publication after this work) can be obtained for the case when there is Coulomb friction acting at a contact [20].

We point out that, for many simulation schemes, if the penalty parameter is appropriately chosen and the velocities are not exceedingly large, then one can use an explicit integration method that produces quite accurate results [5], especially when there is only one contact. But if one desires to create a general-purpose simulation environment that is computationally efficient, then a stable scheme is required for a large variety of examples. It is difficult to find the appropriate penalty parameter, especially in a multicontact regime, where it is conceivable that the penalty parameters should be chosen differently for each contact.

If, in order to accommodate a wide range of applications for a fixed target time-step, one goes to an implicit approach, then choosing a large penalty parameter in order to prevent interpenetration for a large class of examples results essentially in an complementarity-based time-stepping scheme, as shown in this section. For this reason, in the rest of the paper we work with an LCP-based approach.

III. Accommodating the caveats of a fixed time-step model

As is to be expected, although a fixed time-step approach has the obvious advantage of solving a predictable number of subproblems per step, it also presents some unwanted side-effects. In this section we describe them and discuss possible ways to avoid them.

When describing some of the issues we will discuss collisions. We say that a collision occurs if both $\Phi(q^{(l)}) > 0$ and $\Phi(q^{(l)}) + h_l \nabla\phi^{(j)^T} v^{(l+1)} > 0$ but $c^{(j),(l+1)} > 0$ (the multiplier for the normal force at the next step).

A. Larger Number of Constraints

Either the nonlinear (2.6) or the linearized formulation (2.9) has the problem that all constraints need to be considered. If we do not intend to backtrack when a constraint is violated, then all constraints that could become active need to be included on the list of constraints to be enforced. A simple and provably correct strategy, at least in the limit of sufficiently small time step, is to define the active set as

$$\mathcal{A} = \left\{ j | \Phi^{(j)}(q) \leq \epsilon \right\},$$

where $\epsilon > 0$ is a fixed parameter. In our numerical experiments, we used a parameter ϵ that was dependent on the product between the norm of the velocity and the size of the time step, and we did not encounter any difficulties.

B. Collisions that Occur during One Time Step are Simultaneous

Simultaneous collisions are best observed if the mappings $\Phi^{(j)}(q)$ are linear. If $\Phi^{(j)}(q^{(l+1)}) > 0$ but $c_n^{(j),(l+1)} > 0$ then, from (2.9) $\Phi^{(j)}(q^{(l+1)}) = 0$, but $\Phi^{(j)}(q^{(l)} + t(q^{(l+1)} - q^{(l)})) > 0$ for any $t < 1$. So the distance can switch from positive to 0 only at the end of one interval. In some sense, this is one feature that makes the method work: By forcing all collisions occuring during one time-step to be simultaneous, we avoid having to treat them sequentially, which could require an uncontrollable amount of computational effort.

If collisions are isolated in time (which is not the case with the bouncing ball example at the beginning of this paper), then, as the time step goes to 0, they will eventually be resolved individually, so this is not so much of an issue. But for many bodies, it is unlikely that the user is willing to take a sufficiently small time step that will isolate the collisions, since this may lead to a large amount of computation, especially if each step involves solving an LCP.

In light of the extremal analysis of the penalty model, this situation is unavoidable for any method that attempts to simulate with a fixed time step. In our (subjective) experience, in animation applications this effect is invisible for time steps of 0.05 and below.

C. Dissipation of Energy

If we use the scheme (2.9) to simulate a ball falling on a table, we can show (as we later show in an example) that the ball will stick to the table, even though we started with a non-dissipative penalty model to justify (2.9). Therefore (2.9) can accommodate only plastic collisions. This situation is to be expected, because backward Euler type schemes are dissipative. The use a symplectic method like implicit midpoint on the penalty approach followed by taking the parameters $\gamma^{(j)}$ to ∞ does not completely remedy this problem. We will address this issue by using an explicit energy restitution model.

D. Dependency of the Impact Velocity on the Time Step

We will prove by an example that the dependence of the impact velocity on the time step is, by far, the most subtle effect of a fixed time-step approach. We assume that a ball moves horizontally without friction, starting at $x = 0$ with velocity 1, and it encounters a wall at $x = 1$. We assume that there is no gravity so, in effect, this is one-dimensional motion. We assume that the collision response mechanism is of the Newton type [13]: A portion e of the normal velocity is returned to the system. The exact velocity solution to this problem is clearly

$$\dot{x}(t) = \begin{cases} 1 & t \in [0, 1] \\ -e & t \in [1, \infty) \end{cases} .$$

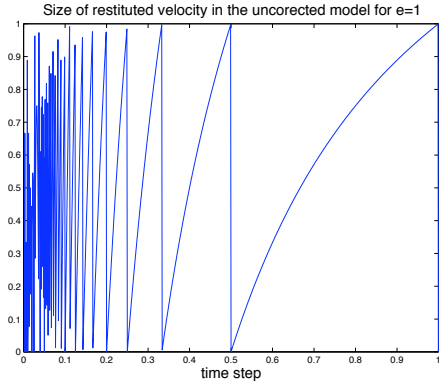

Fig. 1. Velocity after collision for the fixed time-step uncorrected method.

We now assume that we apply the linearized time-stepping scheme (2.9) with constant time step h. We have only one noninterpenetration constraint: $\Phi(x) = 1 - x \geq 0$. We define $L = \lceil \frac{1}{h} \rceil - 1$, where $\lceil \cdot \rceil$ denotes the ceiling function: the smallest integer larger than or equal to the argument of the function. Clearly, the constraint is inactive and $\dot{x}^l = 1$ and $x^l = lh$ for $l \leq L$. Since $x^L + h > 1$, we must have a collision at time L. From (2.9) we get that (since a collision must occur)

$$\dot{x}^{L+1} = 1 - c^{(1),L}, \quad 1 - x^L - h\dot{x}^{L+1} = 0.$$

The solution to this problem is $\dot{x}^{L+1} = \frac{1}{h} - \lceil \frac{1}{h} \rceil + 1 \in (0, 1]$ and $c^{(1),(L)} = -\frac{1}{h} + \lceil \frac{1}{h} \rceil \in [0, 1)$. The velocity solution after impact will be $\dot{x}^{(l)} = -e - e\frac{1}{h} + e\lceil \frac{1}{h} \rceil$ for $l \geq L + 2$ instead of its exact solution $\dot{x}^{(l)} = -e$. As $h \to 0$, the error is anywhere in the range $[0, e)$. For example the error can be $\frac{e}{2}$ for arbitrarily small values of h! This effect is seen in Figure 1 for $e = 1$.

If the collision is inelastic (e=0), then the error after collision is 0; but if the collision is partially elastic, then we may introduce an $O(1)$ error for arbitrarily small values of the time step. The situation does not improve if we use a collision model based on the Poisson hypothesis [13], [11]: That part of the compression impulse is restituted (even if, at time $L + 1$, we consider the collision instantaneous instead of embedding it in a time-stepping scheme). Indeed, the Lagrange multiplier $c^{(1),(L)}$ that enters an impulse restitution model suffers from exactly the same effect.

IV. A VELOCITY RESTITUTION MODEL FOR FIXED TIME-STEP SCHEMES

The simultaneous contact issue is unavoidable for fixed time-step schemes. We will address the last two issues that appear in a fixed time-step scheme by defining an appropriate velocity restitution (Newton) model. Since we cannot use the velocity at the time of the collision to compute the restituted velocity, because of the lack of convergence effect, we will use the normal velocity computed with the velocity vector at the previous time (for our example, time L), before the collision occurred. In doing so, we may incur an $O(h)$ error which is unavoidable anyway if we do not detect events such as collisions with superior accuracy [15]. If the number of collisions is finite, then these errors disappear in the limit. If a collision occurs at contact (j), at time $l-1$, then we replace the linearization of the contact constraint from (2.9) by

$$\Phi^{(j)}(q^{(l)}) + h_l \nabla\Phi^{(j)}(q^{(l)})^T v^{(l+1)} + h_l \Lambda^{(j),(l)} \geq 0$$

where, after computing the modified normal velocity $v_n^{(j),(l)} = \nabla\Phi^{(j)^T}(q^{(l)})v^{(l-1)}$, we define

$$\Lambda^{(j),(l)} = e^{(j)} \begin{cases} v_n^{(j),(l)} & v_n^{(j),(l)} < -TOL \\ 0 & v_n^{(j),(l)} \geq -TOL. \end{cases} \tag{4.10}$$

Here $e^{(j)}$ is the restitution coefficient at noninterpenetration constraint (j). For ease of notation, if a collision does not occur, we still use the parameter $\Lambda^{(j),(l)}$, though we will assign it a 0 value. The parameter TOL is used as a truncation parameter to remove exceedingly small bounces. If we apply this approach to our one-body example, we get that the velocity following the collision is $-e$, the exact solution. In the general case (where the velocity before and after the collision is not constant), we get an error of $O(h)$.

The Newton approach has the advantage that, if it is used as a collision resolution technique (where $\Phi^{(j)}(q^{(l)})$ and h_l are removed from the linearization), it is guaranteed not to increase the kinetic energy when the scheme is slightly modified to include the dissipation terms from $v^{(l)}$ [21]. We cannot guarantee a good energy behavior for our scheme (i.e., that the energy will decrease in an isolated system), though we have never seen energy increases with this approach in our examples.

A. The Time-Stepping LCP

Including frictional and joint constraints, and using the same notations as in [11], [22], [16] and, by and large, the same notation as in [9], [10], we obtain that $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$ and that $v^{(l+1)}$ is the solution of the following linear complementarity problem, where we use the notation: $Q^{(l)} = Mv^{(l)} + h_l k\left(t^{(l)}, q^{(l)}, v^{(l)}\right)$

$$\begin{bmatrix} M & -\widetilde{\nu} & -\widetilde{n} & -\widetilde{D} & 0 \\ \widetilde{\nu}^T & 0 & 0 & 0 & 0 \\ \widetilde{n}^T & 0 & 0 & 0 & 0 \\ \widetilde{D}^T & 0 & 0 & 0 & \widetilde{E} \\ 0 & 0 & \widetilde{\mu} & -\widetilde{E}^T & 0 \end{bmatrix} \begin{bmatrix} v^{(l+1)} \\ c_\nu \\ c_n \\ \widetilde{\beta} \\ \lambda \end{bmatrix} + \begin{bmatrix} -Q^{(l)} \\ \Upsilon \\ \Delta + \Lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho \\ \widetilde{\sigma} \\ \zeta \end{bmatrix} \tag{4.11}$$

$$\begin{bmatrix} c_n \\ \widetilde{\beta} \\ \lambda \end{bmatrix}^T \begin{bmatrix} \rho \\ \widetilde{\sigma} \\ \zeta \end{bmatrix} = 0, \qquad \begin{bmatrix} c_n \\ \widetilde{\beta} \\ \lambda \end{bmatrix} \geq 0, \qquad \begin{bmatrix} \rho \\ \widetilde{\sigma} \\ \zeta \end{bmatrix} \geq 0\,. \tag{4.12}$$

Here $\widetilde{\nu} = [\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(m)}]$ are the gradients of the joint constraints; $c_\nu = [c_\nu^{(1)}, c_\nu^{(2)}, \ldots, c_\nu^{(m)}]^T$ are the multipliers of the joint constraints; $\widetilde{n} = [n^{(j_1)}, n^{(j_1)}, \ldots, n^{(j_s)}]$ are

the gradients of the active noninterpenetration constraints; $c_n = [c_n^{(j_1)}, c_n^{(j_2)}, \ldots, c_n^{(j_s)}]^T$ are the multipliers (normal impulses) corresponding to the interpenetration constraints; $\widetilde{\beta} = [\beta^{(j_1)T}, \beta^{(j_2)T}, \ldots, \beta^{(j_s)T}]^T$ is the aggregate of vectors of tangential impulses, $\widetilde{D} = [D^{(j_1)}, D^{(j_2)}, \ldots, D^{(j_s)}]$, the tangent vectors corresponding to a discretization of the friction cone; $\lambda = [\lambda^{(j_1)}, \lambda^{(j_2)}, \ldots, \lambda^{(j_s)}]^T$ are the multipliers of the conical constraint; $\widetilde{\mu} = diag(\mu^{(j_1)}, \mu^{(j_2)}, \ldots, \mu^{(j_s)})^T$ is a diagonal matrix whose diagonal is made of the Coulomb friction coefficients, $\Upsilon = \frac{1}{h_l}\left(\Theta^{(1)}, \Theta^{(2)}, \ldots, \Theta^{(m)}\right)^T$ where $\Theta^{(i)}$ is the value of the joint constraint i at $q^{(l)}$, $\Delta = \frac{1}{h_l}\left(\Phi^{(j_1)}, \Phi^{(j_2)}, \ldots, \Phi^{(j_s)}\right)^T$; $\Lambda = \left(\Lambda^{(j_1)}, \Lambda^{(j_2)}, \ldots, \Lambda^{(j_s)}\right)^T$ is the vector of restitution factors from (4.10),

$$\widetilde{E} = \begin{bmatrix} E^{(j_1)} & 0 & 0 & \cdots & 0 \\ 0 & E^{(j_2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E^{(j_s)} \end{bmatrix},$$

where $E^{(j)}$ is a vector of ones of the same dimension as the number of columns in $D^{(j)}$ and number of elements in $\beta^{(j)}$. Note that the joint constraints are also enforced by linearization. Here $\mathcal{A} = \{j_1, j_2, \ldots, j_s\}$ are the active contact constraints. The vector inequalities in (4.12) are to be understood componentwise. We use the $\widetilde{\ }$ notation to indicate that the quantity is obtained by properly adjoining blocks that are relevant to the aggregate joint or contact constraints. The problem is called mixed LCP because it contains both equality and complementarity constraints.

We call our model fixed time-step, although h_l is allowed to vary, because the time step does not need to be reduced to 0 in the event of a collision and the collision resolution mechanism is integrated in the time-stepping scheme. In effect, the time step should not be reduced to 0 because it may lead to large velocities due to the fact that Δ and Υ contain an $\frac{1}{h_l}$ factor. If all restitution coefficients are always 0 ($\Lambda = 0$), then the scheme has been proven to be stable and to stabilize constraints.

V. Numerical Results

We have applied this approach to a two-dimensional system whose initial configuration is a cannonball arrangement of 66 disks of radius 3 on a horizontal plank bounded by two slanted walls. The friction coefficient is 0.15, the restitution coefficient is 0.4, and the time step is constant 0.05. The simulation was run for 20 seconds.

From Figure 2 we see that the time needed to solve the LCP (4.11–4.12) is correlated with the number of active contacts. This result is to be expected, because the size of the LCP is proportional to the number of active contacts. We also see that most of the time, less than 0.5 seconds were needed to solve the LCP on a 1.7 GHz Pentium IV running Windows 2000. The largest increase in the number of contacts in one time step was 7. This means that an event-driven method may have taken seven times as many LCPs to solve for the same time step, which would have substantially hurt the performance.

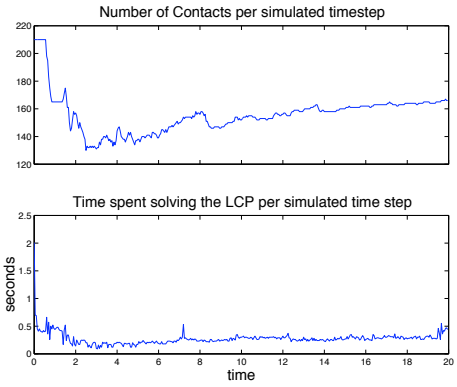

Fig. 2. Number of active contacts and LCP computing time per timestep.

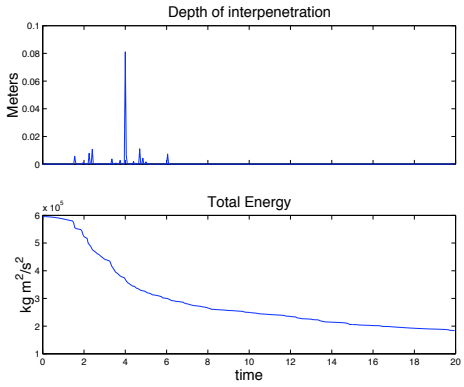

Fig. 3. Depth of interpenetration (constraint violation) and total energy.

From Figure 3 we see that neither the constraint violation nor the total energy increases uncontrollably. In effect, the total energy decreases steadily, as should happen in the continuous time limit. In addition we see that the schemes achieves constraint stabilization: constraint violations are rapidly corrected. This effect is proved for 0 restitution coefficient in [16]. The maximum constraint violation is about 8 centimeters, but it should be kept in mind that the body radius is 3 meters. Four frames of the simulation are presented in Figure 4.

For successful application of the fixed time-step method we plan to address several issues in the near future. The method should be extended to nonsmooth shapes, which are ubiquitous in applications. Also, a better strategy is needed to predict the future active set, which would result in even smaller constraint violation.

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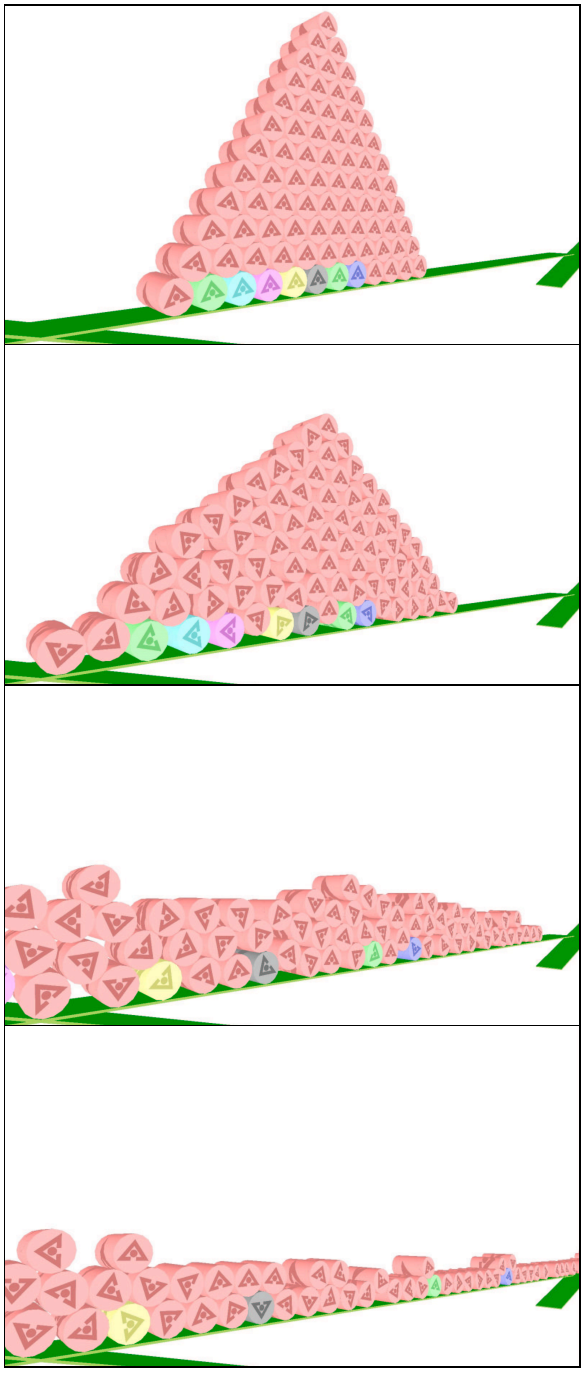

Fig. 4. Four frames of a two-dimensional cannonball arrangement simulation involving 66 bodies

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