Robust Treatment of Degenerate Elements in Interactive Corotational FEM Simulations

O. Civit-Flores and A. Susín

UPC-BarcelonaTech

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Outline

Interactive FEM

Interactive simulation of deformable solids using FEM

Applications:

- Virtual reality, surgery, training...
- Videogames

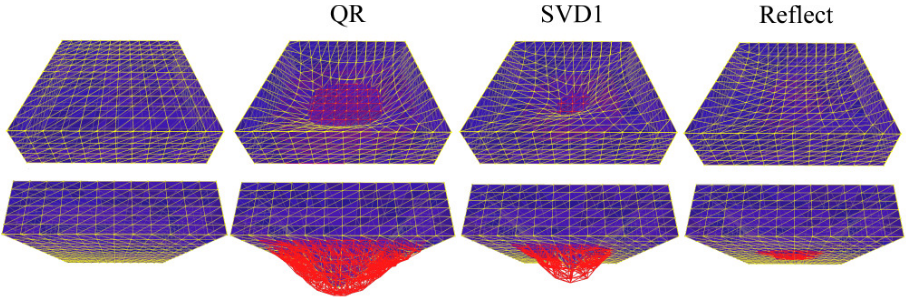
Requirements:

- User interaction
- Efficiency
- Robustness
- Realism

Contribution

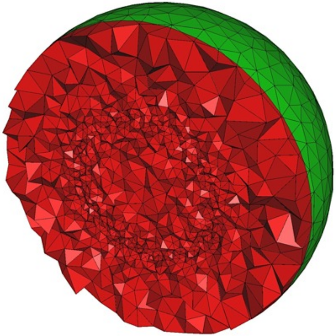
Element degeneration threatens robustness and realism:

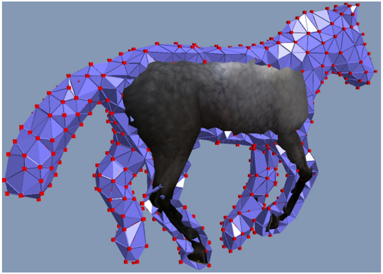
- We identify issues with existing degenerate element treatment schemes
- We propose a new method that avoids them



Outline

Partition the computational domain Ω into sub-domains Ω_i with N shared nodes





Linear FEM

Tetrahedral elements:

$$\boldsymbol{r}_e = \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_4 \end{bmatrix}, \quad \boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_4 \end{bmatrix}, \quad \boldsymbol{f}_e = \begin{bmatrix} \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_4 \end{bmatrix}$$

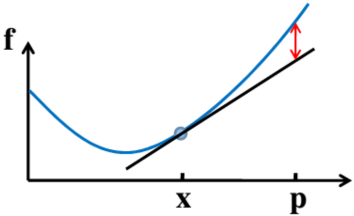
The elastic forces on the nodes are:

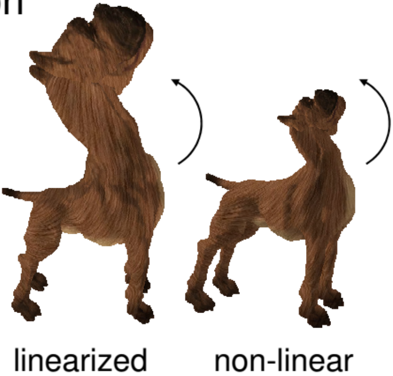
$$\boldsymbol{f}_e = -\mathcal{K}_e \boldsymbol{u}_e, \quad \boldsymbol{u}_e = \boldsymbol{x}_e - \boldsymbol{r}_e$$

Properties:

- Constant stiffness matrix $\mathcal{K}_e$
- Invariant to translation, but not to rotation

Linearization only valid close to the point of linearization

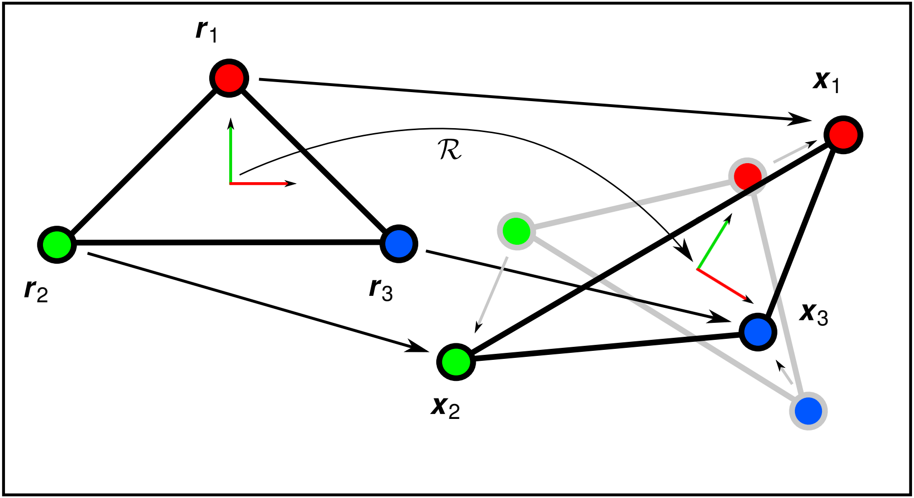




From [1]

Corotational Linear FEM (I)

Idea: Apply LFEM in a reference system local to each element



Corotational Linear FEM (II)

1. Compute deformation matrix $\mathcal{F}$:

$$\mathcal{F} = \mathcal{D}(\boldsymbol{x}_e)\mathcal{D}(\boldsymbol{r}_e)^{-1}, \quad \mathcal{D}(\boldsymbol{v}_e) = \begin{bmatrix} \boldsymbol{v}_2 - \boldsymbol{v}_1 & \boldsymbol{v}_3 - \boldsymbol{v}_1 & \boldsymbol{v}_4 - \boldsymbol{v}_1 \end{bmatrix}$$

2. Factorize into rotation and scaling:

$$\mathcal{F} = \mathcal{R}\mathcal{S}$$

3. Apply linear elasticity in local coordinates:

$$\boldsymbol{f}_e = -\mathcal{R}_e\mathcal{K}_e(\mathcal{R}_e^T\boldsymbol{x}_e - \boldsymbol{r}_e)$$

Properties:

- Geometrically non-linear
- Invariant to translation and rotation

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Dynamics and Quasistatics

Node positions $\boldsymbol{x} \in \mathbb{R}^{3N}$ are the DOF:

- Dynamics:

$$\mathcal{M}\ddot{\boldsymbol{x}} = \boldsymbol{f}_s(\boldsymbol{x}) + \boldsymbol{f}_d(\boldsymbol{x}, \dot{\boldsymbol{x}}) + \boldsymbol{f}_{\text{ext}}$$

- Quasistatics:

$$\boldsymbol{f}_s(\boldsymbol{x}) = -\boldsymbol{f}_{\text{ext}}$$

Element Degeneration

Collapse with $|\det(\mathcal{F})| < \epsilon$ or inversion with $\det(\mathcal{F}) < 0$

- Unphysical
- Unavoidable with (finite) linear forces
- Unavoidable due to discretization
- Unavoidable due to user interaction
- $\det(\mathcal{F}) < \epsilon$ affects $\mathcal{F} = \mathcal{R}\mathcal{S}$ factorization

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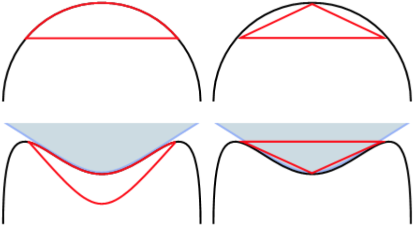

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Outline

Methods

Several methods to extract $\mathcal{R}$ from $\mathcal{F}$ are possible:

- Polar Decomposition [1]
- QR Factorization [3]
- Hybrid PD-QR [5]
- Modified Singular Value Decomposition (SVD1) [2]
- Coherent Singular Value Decomposition (SVD2) [4]
- Project/Reflect [6]
- Degeneration-Aware Polar Decomposition [**?**]

Polar Decomposition

Factorizes $\mathcal{F} = \mathcal{R}\mathcal{S}$, where $\mathcal{R}$ is orthonormal and $\mathcal{S}$ is symmetric

- Best matching, minimizes $\|\mathcal{F} - \mathcal{R}\|_F^2$
- Fails if $|\det(\mathcal{F})| \leq \epsilon$ (collapsed)
- Reflected $\mathcal{R}$ with $\det(\mathcal{R}) = -1$ if $\det(\mathcal{F}) < 0$ (inverted)

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QR Factorization

Factorizes $\mathcal{F} = \mathcal{R}\mathcal{E}$ using Gram-Schmidt orthonormalization, where $\mathcal{R}$ is orthonormal and $\mathcal{E}$ is upper-triangular

- Fast and Robust
- Handles collapsed and inverted elements seamlessly
- Induces Anisotropy
- Critical point on collapse plane

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Hybrid PD-QR

Use Polar Decomposition for undegenerate elements and QR for degenerate ones below a threshold $\det(\mathcal{F}) < \alpha$

- Inherits good properties of PD and QR...
- ...but also the drawbacks of QR...
- ...and adds discontinuity across transition

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Modified Singular Value Decomposition (SVD1)

SVD factorizes $\mathcal{F} = \mathcal{U}\hat{\mathcal{F}}\mathcal{V}^T$, where $\mathcal{U}$ and $\mathcal{V}^T$ are orthonormal and $\hat{\mathcal{F}}$ is diagonal, with singular values $\sigma_i \geq \sigma_{i+1} \geq 0$

- Handles collapsed elements
- Handles inverted elements negating the smallest σ_i
- $\bar{\mathcal{R}} = \bar{\mathcal{U}}\bar{\mathcal{V}}^T$ equivalent to “invertible” PD
- Computationally expensive
- Critical point at repeated singular values

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Coherent Singular Value Decomposition (SVD2)

Tries to improve SVD1 by enforcing temporal coherence in the inferred inversion direction (smallest σ_i)

- Better aligned forces
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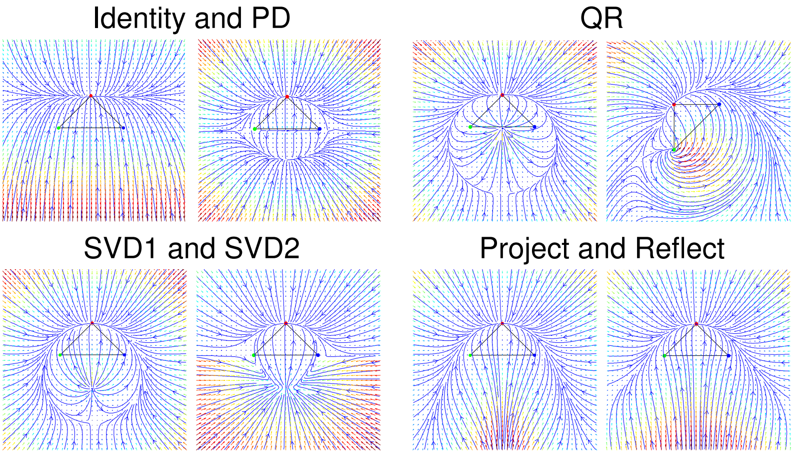
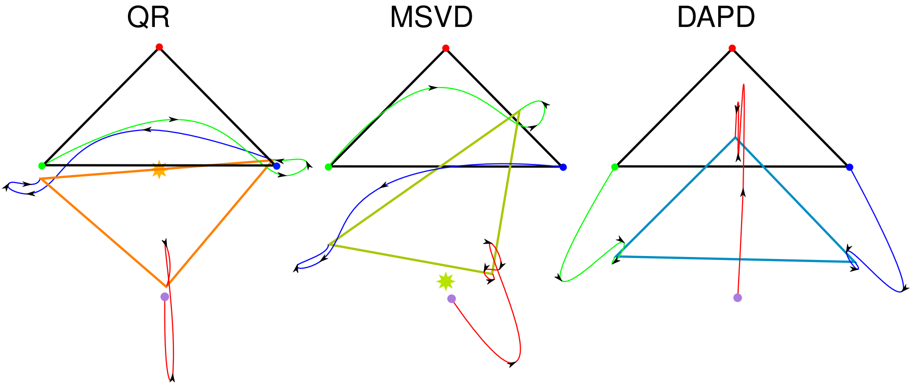
Comparison: Force Fields



Forcefields experienced by $\boldsymbol{x}_1$ (red) by in a $[-3, 3]^2$ region when $\boldsymbol{x}_2, \boldsymbol{x}_3$ (green, blue) are fixed

Comparison: Trajectories



Discussion

The main problems are:

- Discrete-time heuristic ignores trajectories: QR, SVD1, SVD2
- Critical points induce counter-intuitive rotations: QR, SVD1
- Discontinuity induces jitter: Hybrid PD-QR, SVD2

Our solution: Degeneration-Aware Polar Decomposition

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Outline

Idea

- Use continuous time to detect collapse
- Compute $\mathcal{R}$ avoiding rotation-past-collapse

Detecting collapse (I)

Given a timestep $[t_0, t_1]$:

1. New degeneration if $\det(\mathcal{F}(t_0)) > \alpha$ and $\det(\mathcal{F}(t_1)) \leq \alpha$
2. Compute $t_c \in [t_0, t_1]$ solving $\det(\mathcal{F}(t_c)) = \alpha$
3. Compute the collapse feature pair (A_c, B_c) at t_c

Details:

- In 2D only V-E case, in 3D V-F and E-E cases
- Degeneration threshold $\alpha > 0$ improves numerical robustness

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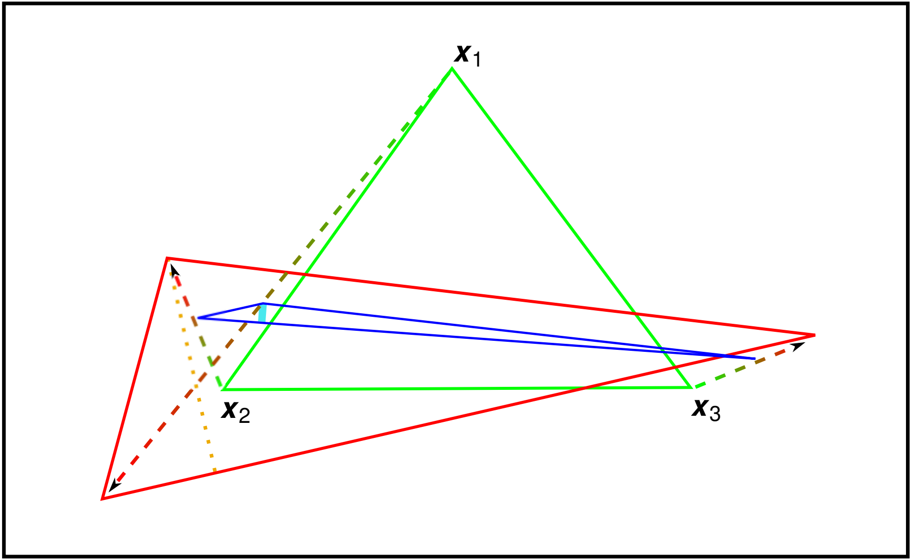
Detecting collapse (II)



Computing $\mathcal{R}$ (I)

We compute $\bar{\mathcal{R}}$ for degenerate elements as follows:

1. Compute degeneration direction $\hat{\boldsymbol{d}}_c$ from (A_c, B_c)
2. Displace nodes of feature A_c as $\bar{\boldsymbol{x}}_i = \boldsymbol{x}_i + \lambda(\alpha, \beta)\hat{\boldsymbol{d}}_c$
3. Compute $\bar{\mathcal{F}}$ from displaced nodes $\bar{\boldsymbol{x}}_i$
4. Compute $\bar{\mathcal{R}}$ using Polar Decomposition on $\bar{\mathcal{F}}$

Details:

- $\lambda(\alpha, \beta)$ computes a displacement length that reverts element degeneration and guarantees $\det(\bar{\mathcal{F}}) > \epsilon$
- Parameter β controls force alignment

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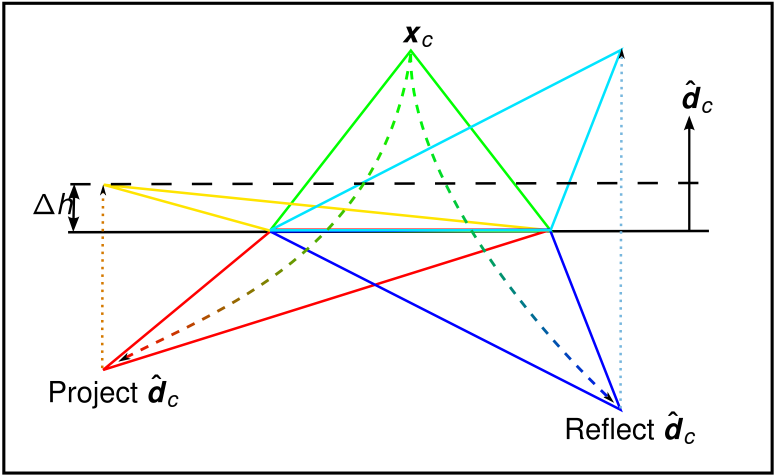
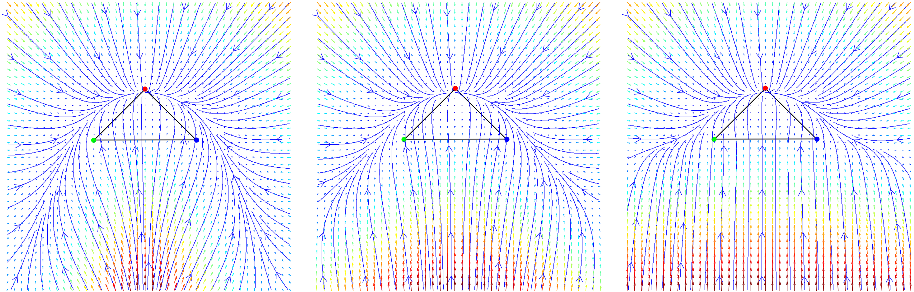


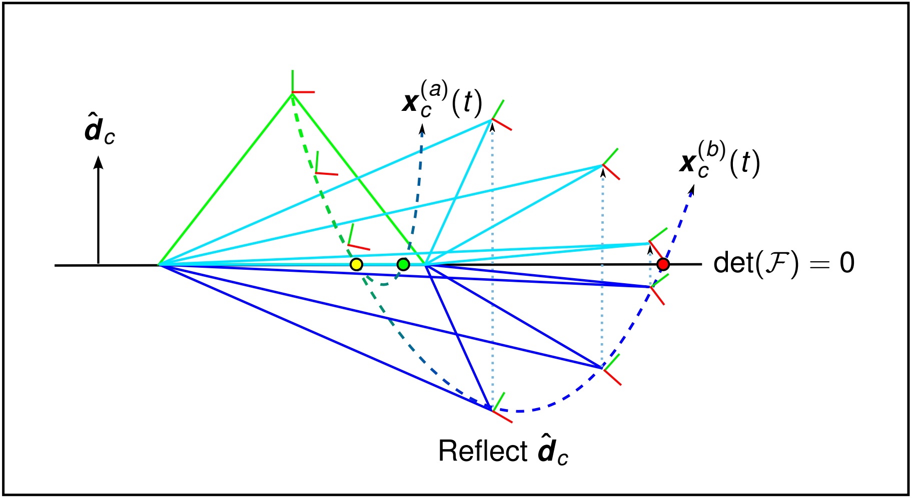
Project ($\beta = 1$) and Reflect ($\beta = 2$). Height Δh depends on α

Computing $\mathcal{R}$ (III)

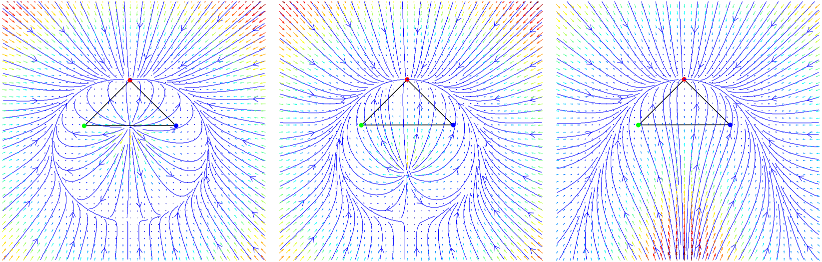


The effect of parameter β for values 1, 2 and 10

Computing $\mathcal{R}$ (VI)



Discussion

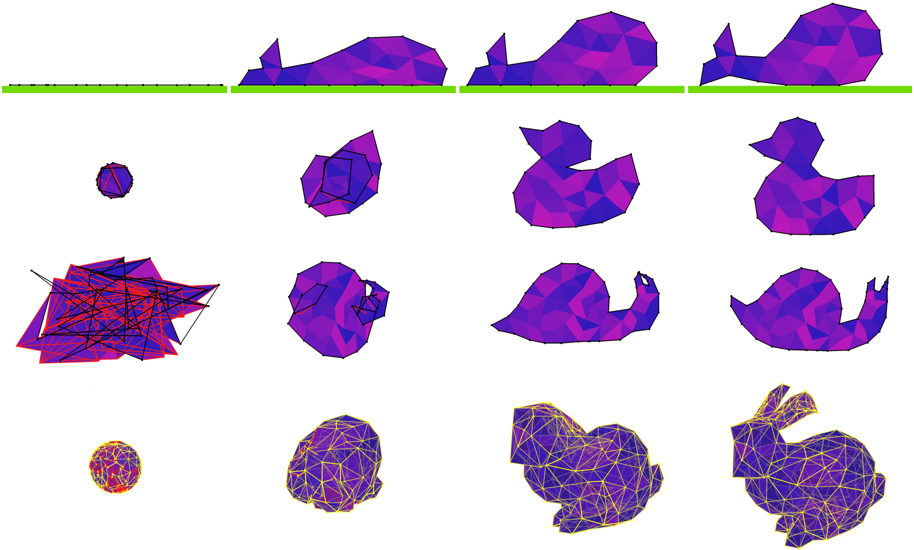


DAPD:

- Cached pair (A_c, B_c) ensures temporal coherency
- Coherently aligned recovery forces
- Continuous and differentiable everywhere...
- ...except when A_c or B_c collapse during inversion

Outline

Results



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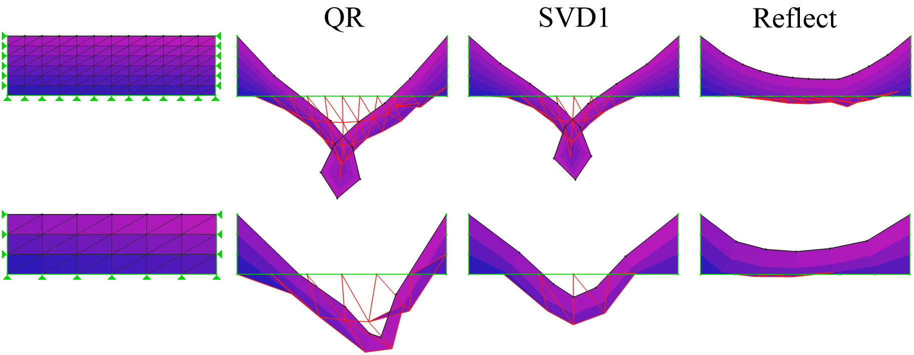

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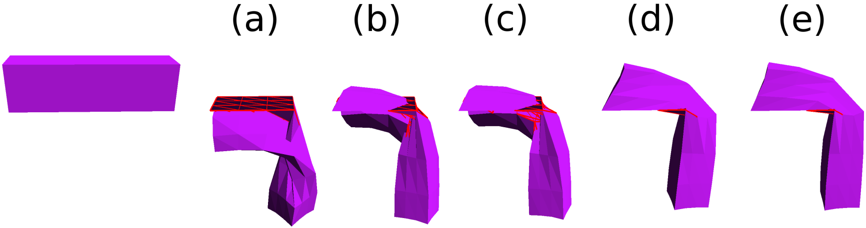

(a) QR, (b) SVD1, (c) DAPD $\beta = 1$, (d) DAPD $\beta = 5$, (e) Nonlinear

Results

`http://www.lsi.upc.edu/~ocivit/videos/DCFEM_Full.ogg`
`http://www.lsi.upc.edu/~ocivit/videos/DCNLFEM-Draft.avi`

Conclusions

Advantages of DAPD:

- Increased realism
- Shorter recovery time
- Faster than SVD1

Limitations:

- Cannot handle initially degenerate elements
- Limited to triangular/tetrahedral elements
- Slower than QR

References

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Symposium on Computer Animation 2009

O. Civit-Flores, A. Susín
Robust treatment of degenerate elements in interactive corotational FEM simulations
(to appear) Computer Graphics Forum 2014

Q&A

?