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Computational Geometry 19 (2001) 155–173

Computational Geometry
Theory and Applications

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Natural neighbor coordinates of points on a surface ☆

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Communicated by M. Bern; received 30 June 2000; received in revised form 7 November 2000

Abstract

Natural neighbor coordinates and natural neighbor interpolation have been introduced by Sibson for interpolating multivariate scattered data. In this paper, we consider the case where the data points belong to a smooth surface $\mathcal{S}$, i.e., a $(d-1)$-manifold of $\mathbb{R}^d$. We show that the natural neighbor coordinates of a point X belonging to $\mathcal{S}$ tends to behave as a local system of coordinates on the surface when the density of points increases. Our result does not assume any knowledge about the ordering, connectivity or topology of the data points or of the surface. An important ingredient in our proof is the fact that a subset of the vertices of the Voronoi diagram of the data points converges towards the medial axis of $\mathcal{S}$ when the sampling density increases. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Computational geometry; Voronoi diagrams; Medial axis; Natural neighbor interpolation; Surface reconstruction

1. Introduction

Natural neighbor coordinates and natural neighbor interpolation have been introduced by Sibson [19] for interpolating multivariate scattered data. Given a set of points $\mathcal{A} = \{A_1, \ldots, A_n\}$, the associated system of natural neighbor coordinates is a set of continuous functions $\lambda_i : \mathbb{R}^d \to \mathbb{R}$, $i = 1, \ldots, n$, defined from the Voronoi diagram of $\mathcal{A}$.

In this paper, we consider the case where the data points are scattered over a surface $\mathcal{S}$, i.e., a $(d-1)$-manifold of $\mathbb{R}^d$. We show that the set of natural neighbor coordinates of a point X belonging to $\mathcal{S}$ tends to behave as a local system of coordinates on the surface when the density of points increases. Our result does not assume any knowledge about the ordering, connectivity or topology of the data points or of $\mathcal{S}$.

This work is motivated by the many application domains where surfaces are to be reconstructed from a sample of unorganized data. Such data may be provided by various sensors or may result from a mathematical analysis. In a companion paper [6], we derive a new method for surface reconstruction based on the natural neighbor interpolation and the results of this paper.

☆ This work has been partially supported by the AFIRST program “Factory of the future”.

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0925-7721/01/$ – see front matter
PII: S0925-7721(01)00018-9

The paper is organized as follows. In Section 2, we recall the definition and the main properties of natural neighbor coordinates in $\mathbb{R}^d$. In Section 3, we consider the case where the points belong to a surface. We recall such definitions as medial axis, local feature size and Voronoi diagram of points on a surface and derive some basic results. In Section 4, we prove that some vertices of the Voronoi diagram of $\mathcal{A}$ converge towards the medial axis of $\mathcal{S}$ when the sampling density increases. Finally, in Section 5, we analyze the behaviour of the natural neighbor coordinates of a point X of $\mathcal{S}$ when the sampling density increases.

2. Natural neighbor coordinates

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a set of points. The Voronoi cell of A_i is

$$V(A_i) = \{X \in \mathbb{R}^d\colon\ \|X - A_i\| \leqslant \|X - A_j\|\ \forall j = 1, \ldots, n\},$$

where $\|X - Y\|$ denotes the Euclidean distance between points $X, Y \in \mathbb{R}^d$. The collection of Voronoi cells is called the *Voronoi diagram* of $\mathcal{A}$. Let $\mathcal{A}'$ be a subset of points of $\mathcal{A}$ whose Voronoi cells have a non-empty intersection. The convex hull $\mathrm{conv}(\mathcal{A}')$ is called a Delaunay face and the collection of all Delaunay faces is a geometric complex called the Delaunay triangulation of $\mathcal{A}$, denoted $\mathrm{Del}(\mathcal{A})$. It is well known that if there is no sphere passing through $d + 2$ points of $\mathcal{A}$, $\mathrm{Del}(\mathcal{A})$ is a simplicial complex, and that the balls circumscribing the d-simplices in $\mathrm{Del}(\mathcal{A})$ cannot contain a point of $\mathcal{A}$ in their interior.

Given a point X, we define $\mathrm{Vor}^+ = \mathrm{Vor}(\mathcal{A} \cup \{X\})$ and $\mathrm{Del}^+ = \mathrm{Del}(\mathcal{A} \cup \{X\})$. In addition, $V^+(X)$ denotes the Voronoi cell of X in Vor^+.

Definition 1. A ball is said to be *empty* if its interior does not contain any point of $\mathcal{A}$.

Definition 2. The natural neighbors of a point X with respect to $\mathcal{A}$ are the vertices other than X of the simplices of Del^+ incident to X.

Let $V(X, A_i) = V^+(X) \cap V(A_i)$ (see Fig. 1). If $V(X, A_i) \neq \emptyset$, A_i is a natural neighbor of X. Let $w_i(X)$ be the volume of $V(X, A_i)$ and let $w(X)$ be the sum, over all natural neighbors, of the $w_i(X)$.

Observe that $w_i(X)$ is bounded unless X lies outside the convex hull $\mathrm{CH}(\mathcal{A})$ of $\mathcal{A}$ and A_i is a vertex of the convex hull. In the rest of this section, we restrict our attention to points X that lie in $\mathrm{CH}(\mathcal{A})$.

Definition 3 (Sibson). The natural neighbor coordinates of X with respect to $\mathcal{A}$ are the $\lambda_i(X) = w_i(X)/w(X)$, $i = 1, \ldots, n$.

The natural neighbor coordinates have several interesting properties.

Property 1. *For any $i \leqslant n$, $\lambda_i(A_j) = \delta_{ij}$ where δ_{ij} is the Kronecker symbol.*

Property 2. *For $i = 1, \ldots, n$, $\lambda_i(X) \geqslant 0$ and $\sum_{i=1}^{n} \lambda_i(X) = 1$.*

Sibson has proved the following important property that justifies the term coordinates [19]. Besides the initial proof of Sibson, several alternative proofs are known [5,8,13].

Property 3 (Sibson). $X = \sum_i \lambda_i(X) A_i$.

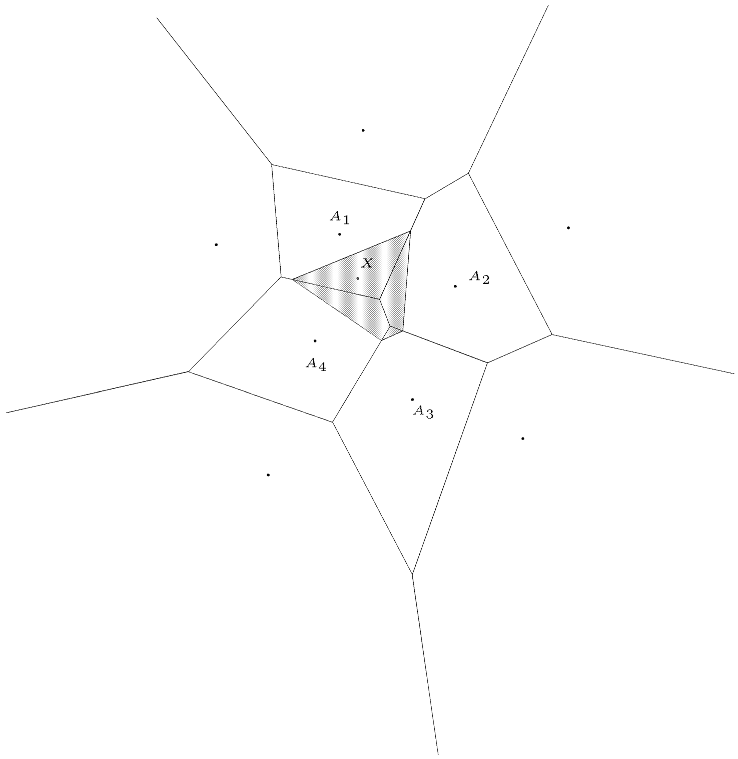

Fig. 1. X has four natural neighbors $A_1, \ldots, A_4$.

Let us consider the Delaunay balls circumscribing the d-simplices of the Delaunay triangulation of $\mathcal{A}$, called the *Delaunay balls* of $\mathcal{A}$ for short. We say that a Delaunay ball *conflicts* with a point X if X belongs to the interior of the ball. The natural neighbors of X are the vertices of the Delaunay d-simplices whose circumscribing balls conflict with X.

Let $\mathcal{D}$ denote the arrangement of the spheres bounding the Delaunay balls of $\mathcal{A}$. Any point X in the interior of a cell of $\mathcal{D}$ has the same natural neighbors. We associate to each cell Γ of $\mathcal{D}$ a set of indexes $I(\Gamma)$ as follows. If a ball circumscribing a d-simplex S of $\mathrm{Del}(\mathcal{A})$ covers Γ, we add the indexes of the vertices of S to $I(\Gamma)$. All points in the interior of Γ have the same natural neighbors which are the points of $\mathcal{A}$ with indexes in $I(\Gamma)$.

When X reaches a $(d-1)$-face of the boundary of a cell Γ of $\mathcal{D}$, X gains or looses a natural neighbor, depending whether X enters a Delaunay ball that does not cover Γ or goes out of a Delaunay ball that covers Γ. More generally, if X reaches a k-face of Γ, $0 < k < d$, $d - k$ natural neighbors of X change, some of them being gained, others being lost.

This discussion leads to the following property.

Property 4 (Combinatorial structure). *Let $\mathcal{N}$ be the equivalence relation that relates two points if they have the same natural neighbors with respect to $\mathcal{A}$. The equivalence classes of $\mathcal{N}$ are the faces of the arrangement of the spheres circumscribing the d-simplices of* $\mathrm{Del}(\mathcal{A})$.

Let Λ_i denote the *support* of λ_i, i.e., the subset of X such that $\lambda_i(X) \neq 0$.

Property 5 (Locality). *Λ_i is included in the union of the balls circumscribing the d-simplices of* $\mathrm{Del}(\mathcal{A})$ *that are incident to A_i.*

The following property is stated without proof in [19] and discussed in more detail in [11]. The formula for the gradient is due to Piper [16].

Property 6 (Differentiability). *$\lambda_i(X)$ is a function that is continuous at all X and continuously differentiable at all $X \notin \mathcal{A}$. We have*

$$\nabla w_i(X) = \frac{\mu_i(X)}{\|X - A_i\|}\overrightarrow{XC}_i,$$

where $\mu_i(X)$ and C_i are respectively the $(d-1)$-volume and the centroid of the Voronoi facet common to $V^+(X)$ and $V(X, A_i)$.

The next property is a direct consequence of the definition of the natural neighbors. Implementation details and experimental results can be found in the companion paper [6].

Property 7 (Time complexity). *The time complexity of computing the natural neighbor coordinates of a point X is the same as the time complexity of inserting a point in the Delaunay triangulation of $\mathcal{A}$.*

In view of Properties 1–3, and of the continuity of the λ_i, we say that the set of λ_i is a *coordinate system* associated to $\mathcal{A}$.

Since the pioneering work of Sibson, other Voronoi-based systems of coordinates have been proposed [8,13].

3. Sampled surfaces: definitions and preliminary results

Let $\mathcal{O}$ be a compact region of $\mathbb{R}^d$ whose boundary $\mathcal{S}$ is a smooth surface, i.e., a twice-differentiable $(d-1)$-manifold. $B(X, r)$ denotes the ball centered at X with radius r and $\Sigma(X, r)$ its bounding sphere. $\vec{n}_X$ denotes the unit normal to $\mathcal{S}$ at X directed outwards from $\mathcal{O}$ (see Fig. 2).

Let $\mathcal{A}$ denote a set of n points $A_1, \ldots, A_n$ on $\mathcal{S}$. A *local system of coordinate on $\mathcal{S}$ associated to $\mathcal{A}$* is a set of continuous functions $\sigma_i : \mathcal{S} \to \mathbb{R}$, $i = 1, \ldots, n$, such that for all $X \in \mathcal{S}$:
(1) $X = \sum_i \sigma_i(X) A_i$,
(2) for any $i \leqslant n$, $\sigma_i(A_j) = \delta_{ij}$ where δ_{ij} is the Kronecker symbol,
(3) $\sum_i \sigma_i(X) = 1$,
(4) if the surface is well sampled, for any $i \leqslant n$, the support of σ_i is a small neighborhood of A_i.
One way to determine such σ_i could be to resort to the Voronoi diagram of $\mathcal{A}$ on the surface $\mathcal{S}$ where the Euclidean distance is replaced by a Riemannian metric on $\mathcal{S}$. However such diagrams are much more complicated than Euclidean diagrams and difficult to compute [14]. Moreover, in some applications, such as surface reconstruction, the surface itself is unknown and computing Voronoi diagrams on the surface is impossible. We prefer to follow a different approach that only uses Euclidean Voronoi diagrams and natural neighbors in $\mathbb{R}^d$.

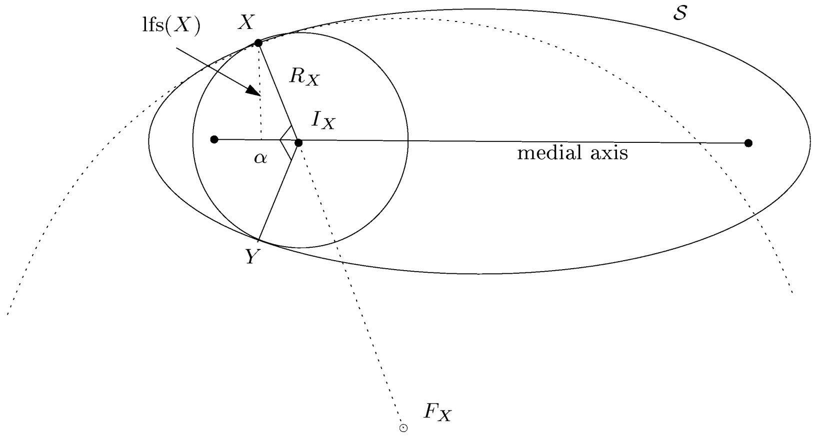

Fig. 2. For the definitions.

This section introduces some basic definitions and results, and make precise what we mean by a good sample. These results will be used in the next two sections and, in particular, in Section 5, where local systems of coordinates will be considered again.

3.1. Medial axis and local feature size

Definition 4. A ball is said to be *maximal* if (1) its interior does not intersect $\mathcal{S}$, (2) it cannot be included in a larger ball satisfying (1).

There are two maximal balls passing through a point $X \in \mathcal{S}$. We denote by B_X the one that is contained in $\mathcal{O}$, Σ_X its bounding sphere, I_X its center and R_X its radius (see Fig. 2). We use the superscript e for the other ball B^e_X, its bounding sphere Σ^e_X, its center I^e_X and its radius R^e_X.

Definition 5. The medial axis of $\mathcal{O}$ consists of the centers of the maximal balls.

Definition 6. Let $X \in \mathcal{S}$. A point on the line $\{X + t\vec{n}_X,\ t \in \mathbb{R}\}$ is called a focal point if $t = \kappa_i^{-1}(X)$ where $\kappa_i(X)$ is one of the principal curvatures of $\mathcal{S}$ at X.

For any $X \in \mathcal{S}$, let F_X be the focal point on the line $\{X + t\vec{n}_X,\ t < 0\}$ that is closer to X. I_X belongs to the line segment $[XF_X]$ and $R_X \leqslant \min_i(|\kappa_i^{-1}(X)|)$.

If I_X is distinct from F_X and therefore belongs to the relative interior of $[XF_X]$, the maximal sphere Σ_X, which is centered at point I_X and tangent to $\mathcal{S}$ at X, is tangent to $\mathcal{S}$ in at least another point $Y_X \neq X$. We denote by η_X the minimum over all such points Y_X of $\|X - Y_X\|/(2R_X)$. Observe that η_X belongs to the interval $[0, 1]$ and is 0 when $I_X = F_X$. Moreover,

$$2\eta_X^2 = 2\sin^2\frac{\alpha_X}{2} = 1 - \cos\alpha_X,$$

where α_X denotes the *bisector angle* $\angle X I_X Y_X$ of $\mathcal{S}$ at X.

Similarly, for the maximal ball B^e_X, we denote by Y^e_X one of its contact point other than X, and by η^e_X the minimum over all such points Y^e_X of $\|X - Y^e_X\|/(2R^e_X)$.

We borrow from Amenta and Bern [1] the notion of local feature size. A related notion is the r-regularity introduced by Serra [18] (see also [4,7]).

Definition 7 (Amenta and Bern). The *local feature size* $\mathrm{lfs}(X)$ at a point $X \in \mathcal{S}$ is the Euclidean distance from X to the medial axis of $\mathcal{S}$.

Lemma 8. *For any $X, Y \in \mathcal{S}$, $\mathrm{lfs}(X) \leqslant \mathrm{lfs}(Y) + \|X - Y\|$.*

Proof. $B(X, \mathrm{lfs}(Y) + \|X - Y\|)$ contains $B(Y, \mathrm{lfs}(Y))$. Since, by definition of the local feature size, the latter intersects the medial axis of $\mathcal{S}$, the same is true for the former, which proves the lemma. □

3.2. Voronoi diagram on a surface

We first define in this section the Voronoi diagram of a set of points restricted to a surface, following previous work by Chew [9] and Edelsbrunner and Shah [10].

Definition 8 (Chew). The Voronoi diagram of $\mathcal{A}$ restricted to $\mathcal{S}$ is the (curved) cell complex obtained by intersecting each face of $\mathrm{Vor}(\mathcal{A})$ with $\mathcal{S}$. We denote it by $\mathrm{Vor}_{\mathcal{S}}(\mathcal{A})$.

Similarly, we can define the Voronoi diagram of $\mathcal{A}$ restricted to $\mathcal{O}$, denoted $\mathrm{Vor}_{\mathcal{O}}(\mathcal{A})$, as the cell complex obtained by intersecting each face of $\mathrm{Vor}(\mathcal{A})$ with $\mathcal{O}$.

We denote by $V_{\mathcal{S}}(A_i)$ the cell of $\mathrm{Vor}_{\mathcal{S}}(\mathcal{A})$ consisting of the points of $\mathcal{S}$ that are closer to A_i than to any A_j, $j \neq i$. A vertex of $V_{\mathcal{S}}(\mathcal{A})$ is the intersection of an edge of the Voronoi diagram of $\mathcal{A}$ with $\mathcal{S}$. Hence, it is the center of an empty ball passing through d points of $\mathcal{A}$.

Definition 9. The Delaunay triangulation of $\mathcal{A}$ restricted to $\mathcal{S}$ is the subcomplex of $\mathrm{Del}(\mathcal{A})$ consisting of the facets of $\mathrm{Del}(\mathcal{A})$ whose dual Voronoi edges intersect $\mathcal{S}$. We denote it by $\mathrm{Del}_{\mathcal{S}}(\mathcal{A})$.

Observe that the facets of $\mathrm{Del}_{\mathcal{S}}(\mathcal{A})$ are the facets of $\mathrm{Del}(\mathcal{A})$ that can be circumscribed by an empty ball centered on $\mathcal{S}$. Such a ball is called an *$\mathcal{S}$-Delaunay ball*.

Let us now look at the natural neighbors of a point X of $\mathcal{S}$. Typically X has natural neighbors that are close to X on the surface and others that are far away, usually on both sides of the tangent plane to $\mathcal{S}$ at X.

Definition 10. The $\mathcal{S}$-natural neighbors of a point X of $\mathcal{S}$ are the vertices of the facets of $\mathrm{Del}_{\mathcal{S}}(\mathcal{A} \cup \{X\})$ that are incident to X.

3.3. ε-samples

Definition 11 (Amenta and Bern). $\mathcal{A}$ is called an *ε-sample* of $\mathcal{S}$ if $\mathcal{A} \subset \mathcal{S}$ and if, for all $X \in \mathcal{S}$, there exists a point A_i such that $\|X - A_i\| \leqslant \varepsilon\, \mathrm{lfs}(X)$. When $\varepsilon < 1/2$, the sample is said to be a *good* sample.

Lemma 9. *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a good ε-sample of $\mathcal{S}$, and X a point of $\mathcal{S}$.*

(1) *If A_i is the point of $\mathcal{A}$ closest to X, $\|X - A_i\| \leqslant (\varepsilon/(1-\varepsilon))\mathrm{lfs}(A_i)$.*

(2) *For any $\mathcal{S}$-natural neighbor of X, we have* $\|X - A_i\| \leqslant (2\varepsilon/(1-\varepsilon))\mathrm{lfs}(X)$ *and* $\|X - A_i\| \leqslant (2\varepsilon/(1-\varepsilon))\mathrm{lfs}(A_i)$.

Proof. (1) Since $\mathcal{A}$ is a good ε-sample and thanks to Lemma 8, we have $\|X - A_i\| \leqslant \varepsilon\,\mathrm{lfs}(X) \leqslant \varepsilon(\mathrm{lfs}(A_i) + \|X - A_i\|)$, which implies the first part of the lemma.

(2) Let A_i be an $\mathcal{S}$-natural neighbor of X. A_i is a vertex of some facet F of $\mathrm{Del}_{\mathcal{S}}(\mathcal{A})$ whose circumscribing $\mathcal{S}$-Delaunay ball B_F contains X. Denoting by V the center of B_F, we have $\|V - X\| \leqslant \|V - A_i\|$ and $\|V - A_i\| \leqslant \varepsilon\mathrm{lfs}(V)$ since $\mathcal{A}$ is a ε-sample.

Moreover, $\|X - A_i\| \leqslant \|X - V\| + \|V - A_i\| \leqslant 2\varepsilon\,\mathrm{lfs}(V)$. By Lemma 8,

$$\mathrm{lfs}(V) \leqslant \mathrm{lfs}(X) + \|X - V\| \leqslant \mathrm{lfs}(X) + \varepsilon\,\mathrm{lfs}(V).$$

Hence

$$\mathrm{lfs}(V) \leqslant \frac{1}{1-\varepsilon}\,\mathrm{lfs}(X).$$

Similarly,

$$\mathrm{lfs}(V) \leqslant \mathrm{lfs}(A_i) + \|V - A_i\| \leqslant \mathrm{lfs}(A_i) + \varepsilon\,\mathrm{lfs}(V).$$

Hence

$$\mathrm{lfs}(V) \leqslant \frac{1}{1-\varepsilon}\,\mathrm{lfs}(A_i). \qquad \square$$

The following lemma has been proved by Amenta and Bern [1, Lemma 5]. Although the lemma was originally stated for $d = 3$, the proof holds for any d.

Lemma 10 (Amenta and Bern). *Assume that $\mathcal{A}$ is a good ε-sample of $\mathcal{S}$. Let $X \in \mathcal{S}$ and let V be a vertex of $V^+(X)$ such that $\|V - X\| \geqslant \eta\,\mathrm{lfs}(X)$ for $\eta \geqslant \varepsilon/(1-\varepsilon)$. The angle at X between the normal to $\mathcal{S}$ at X and the vector to V (oriented so that the angle is acute) is at most* $\arcsin(\varepsilon/(\eta(1-\varepsilon))) + \arcsin(\varepsilon/(1-\varepsilon))$.

The following lemma will be useful later.

Lemma 11. *If θ is as in Lemma 10 and $\eta \geqslant 1$, then* $\cos\theta \geqslant 1 - 2\varepsilon^2/(1-\varepsilon)^2$.

Proof. $\cos\theta \geqslant \cos(2\arcsin(\varepsilon/(1-\varepsilon))) = 1 - 2\sin^2(\arcsin(\varepsilon/(1-\varepsilon))) = 1 - 2\varepsilon^2/(1-\varepsilon)^2$. $\square$

3.4. Topological balls on the surface

The following lemma extends to higher dimensions an analogous result proved by Amenta et al. [2] for $d = 2$.

Proposition 12. *Let $B = B(X, R)$ be a ball that intersects $\mathcal{S}$. If $B \cap \mathcal{S}$ is not a topological ball, then B contains a point of the medial axis of $\mathcal{S}$.*

Proof. The result is clearly true if X lies on the medial axis. Consider the other case and assume that $B \cap \mathcal{S}$ is not empty nor a topological ball. Let X be the center of B and R be its radius. We denote by

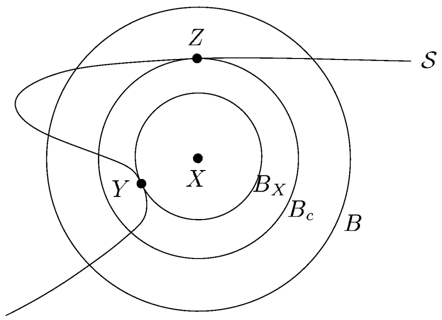

Fig. 3. For the proof of Proposition 13.

B_X the maximal ball centered at X and by R_X its radius. B_X is tangent to $\mathcal{S}$ in a unique (since X does not belong to the medial axis) point Y. $B_X \cap \mathcal{S} = \{Y\}$ and is therefore a topological ball. It follows from Lemma A.1 and Theorem A.2 that the distance function $\delta_X(Y) = \|X - Y\|^2$ has a critical point Z, with $R_X < \|X - Z\| \leqslant R$ (see Fig. 3). By Lemma A.1 again, $B_c = B(X, \|X - Z\|)$ is tangent to $\mathcal{S}$ at Z. One of the two maximal balls $B_X(Z)$ or $B_X^e(Z)$ tangent to $\mathcal{S}$ at Z is contained in B_c (since it cannot contain Y) and therefore in B, and its center belongs to the medial axis of $\mathcal{S}$. This proves the lemma. $\square$

Proposition 13. *For any $X \in \mathcal{S}$ and any $r < \operatorname{lfs}(X)$, $\mathcal{S} \cap B(X, r)$ is a topological $(d-1)$-ball.*

Proof. By definition of $\operatorname{lfs}(X)$, $B(X, r)$ cannot intersect the medial axis of $\mathcal{S}$. Proposition 12 then implies that $\mathcal{S} \cap B(X, r)$ is a topological $(d-1)$-ball. $\square$

Corollary 14. *$\mathcal{S}$ cuts $B(X, \operatorname{lfs}(X))$ into two regions, one entirely inside $\mathcal{O}$ and one entirely outside $\mathcal{O}$.*

4. Approximation of the medial axis

In this section, we show that the centers of a subset of the Delaunay balls converge towards the medial axis when the sampling density increases. This is an extension to higher dimensions of a result proved in the plane by Schmitt [17] (see also [7]). More precisely, Schmitt proved that, when ε tends to 0, the centers of all the Delaunay circles converge towards the medial axis of $\mathcal{S}$. This result does not extend in higher dimensions. Indeed, Amenta, Bern and Eppstein have shown that, even in three dimensions, the centers of some Delaunay balls may be far away from the medial axis [2]. Propositions 16–18 below provide convergence results that hold in any dimension. We first prove the following technical lemma.

Given are two spheres Σ and Σ_X of the same radius R_X and passing through a point X, and a point A (see Fig. 4). Let I and I_X denote the centers of Σ and Σ_X, θ the angle [1] $\angle I X I_X$, Σ_A the sphere tangent to Σ at X and passing through A, C its center and α be the angle $\angle X I_X A$.

[1] In the paper, angles are taken in the interval $[0, 2\pi)$.

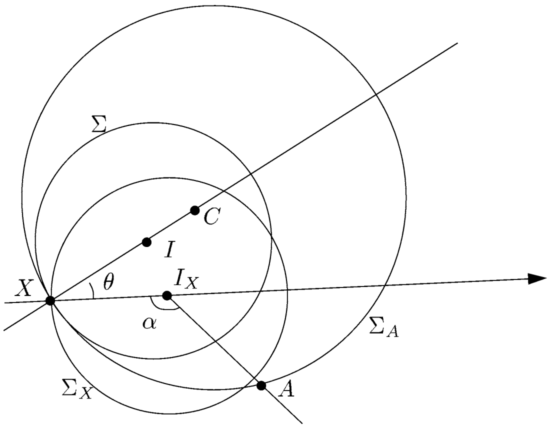

Fig. 4. For Lemma 15.

Lemma 15. *Assume that A lies at distance $R_X(1+\rho)$ from I_X, for some small positive ρ, $\theta = c\rho(1+\mathrm{O}(\rho))$ for some* (*positive*) *constant c, and $\cos\alpha \leqslant \cos\theta/(1+\rho)$. Then $\overrightarrow{XI}\cdot\overrightarrow{XA} > 0$ and we have for any point P on the line segment $[IC]$,*

$$\|I_X - P\| \leqslant \sqrt{c^2 + \frac{(1+c)^2}{(1-\cos\alpha)^2}}\,R_X\rho\big(1+\mathrm{O}(\rho)\big).$$

Proof. For the proof that $\overrightarrow{XI}\cdot\overrightarrow{XA} > 0$, refer to Fig. 5. H denotes the hyperplane passing through X and tangent to Σ at X, and J the projection of I_X onto H. The portion of the ball centered at I_X of radius $R_X(1+\rho)$ that is in the halfspace H^- limited by H not containing I is contained in the cone of revolution with apex at I_X and a 2ϕ apex angle. We have $R_X\cos\theta = R_X(1+\rho)\cos\phi$. When $\cos\alpha \leqslant \cos\phi = \cos\theta/(1+\rho)$, A cannot belong to H^-, which implies the first part of the lemma.

Let $R = R_X(1+\rho)$. For simplicity, we take X as the origin of the reference frame and XI_X as the first axis (see Fig. 4). Moreover, we choose the second axis so that I lies in the plane defined by the first two axis. If C and r denote respectively the center and the radius of Σ_A, we have

$$I_X = \begin{pmatrix} R_X \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad I = \begin{pmatrix} R_X\cos\theta \\ R_X\sin\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad A = \begin{pmatrix} R_X - R\cos\alpha \\ \phi_2 R\sin\alpha \\ \phi_3 R\sin\alpha \\ \vdots \\ \phi_d R\sin\alpha \end{pmatrix},$$

with $r > 0$ and $\sum_{i=2}^d \phi_i^2 = 1$. From $r^2 = \|C - A\|^2$, simple computations lead to

$$r = R_X\frac{(1-\cos\alpha) + (1-\cos\alpha)\rho + \rho^2/2}{(1-\cos\alpha)\cos\theta - \cos\alpha\cos\theta\rho + \phi_2\sin\alpha\sin\theta(1+\rho)}.$$

With $\theta = c\rho(1+\mathrm{O}(\rho))$, we get

$$r \leqslant R_X\left(1 + \frac{\rho - \phi_2\sin\alpha\sin\theta + \mathrm{O}(\rho^2)}{1-\cos\alpha}\right) \leqslant R_X\left(1 + \frac{1+c}{1-\cos\alpha}\rho(1+\mathrm{O}(\rho))\right).$$

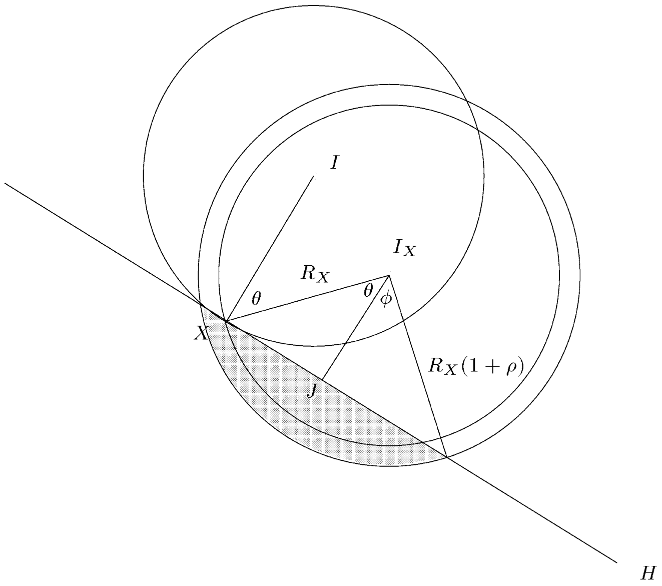

Fig. 5. For the first part of Lemma 15.

Finally,

$$\begin{aligned}\|I_X - C\| &= \left((r\cos\theta - R_X)^2 + r^2\sin^2\theta\right)^{1/2}\\ &\leqslant \left((R_X - r)^2 + rR_X\theta^2\right)^{1/2}\\ &\leqslant \sqrt{c^2 + \frac{(1+c)^2}{(1-\cos\alpha)^2}}R_X\rho\bigl(1+\mathrm{O}(\rho)\bigr)\\ &\stackrel{\text{def}}{=} M.\end{aligned}$$

The claim is therefore proved for $P = C$. Since we also have

$$\|I_X - I\| = 2R_X\sin\frac{\theta}{2} \leqslant R_X\theta = cR_X\rho(1+\mathrm{O}(\rho)) \leqslant M,$$

the claim is also proved for any point P on the line segment $[IC]$. □

The following lemma shows that, if the radius of a Delaunay d-simplex S incident to a given point $X \in \mathcal{S}$ is greater than the radius R_X of the maximal sphere Σ_X, then the circumcenter of S is close to the medial axis of $\mathcal{S}$.

Proposition 16. *Let $\mathcal{A}$ be a good ε-sample of $\mathcal{S}$. Let X be a point of $\mathcal{S}$ such that $\eta_X \geqslant (3/2)\sqrt{\varepsilon}$ and let S be a d-simplex of* Del$^+$ *incident to X. If the circumcenter V of S satisfies $\overrightarrow{XV}\cdot\overrightarrow{XI}_X > 0$ and is at distance ηR_X from X, for $\eta \geqslant 1$, we have*

$$\|V - I_X\| \leqslant \omega_X(\eta)R_X\varepsilon\bigl(1+\mathrm{O}(\sqrt{\varepsilon})\bigr), \quad \textit{where } \omega_X(\eta) = \sqrt{\left(1+\frac{1}{\eta}\right)^2 + \frac{1}{\eta_X^4}\left(1+\frac{1}{2\eta}\right)^2}.$$

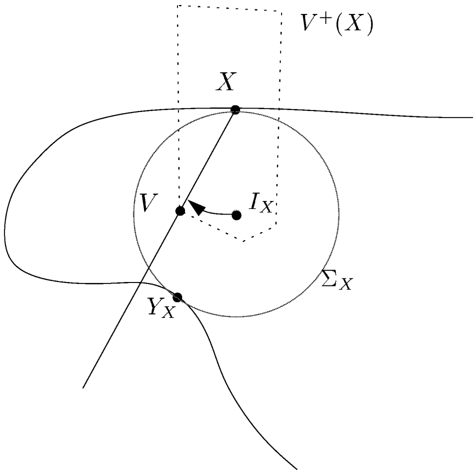

Fig. 6. For the proof of Proposition 16.

Remark 1. Observe that, since $V_{\mathcal{O}}(X)$ contains I_X, a d-simplex as in the lemma exists for any X. A similar result has been independently obtained by Amenta and Kolluri [3] in the special case where S is the d-simplex incident to X whose circumcenter (called a pole) is the farthest from X.

Proof. Since $\eta_X \neq 0$, I_X is not a focal point of $\mathcal{S}$ and therefore Σ_X is tangent to $\mathcal{S}$ at two distinct points, X and at some other point $Y_X \neq X$ (see Fig. 6). Let Σ be a moving and deformable sphere that initially coincides with Σ_X, I its center, and $Y \in \Sigma$ the point that initially coincides with Y_X. We first rotate Σ around X until its center lies on the ray going from X towards V. Let I' denote the new position of I and Σ' the corresponding sphere.

By Lemma 11, the angle θ between the vectors $\overrightarrow{XV}$ and $\overrightarrow{XI_X}$ is at most

$$\arcsin\left(\frac{\varepsilon}{\eta(1-\varepsilon)}\right) + \arcsin\left(\frac{\varepsilon}{1-\varepsilon}\right) = \left(1+\frac{1}{\eta}\right)\varepsilon\bigl(1+\mathrm{O}(\varepsilon)\bigr).$$

Since V is farther from X than I_X, I' lies between X and V and Σ' does not contain any point of $\mathcal{A}$ in its interior. We then grow Σ until it passes through $A_1, \ldots, A_d$ and X. More precisely, we move the center I of Σ along the line XV towards V while keeping Σ passing through X. We stop when $I = V$, i.e., when Σ coincides with the sphere circumscribing S. During this second motion, Σ cannot grow much. Indeed, since $\mathcal{A}$ is an ε-sample, there exists a sample point A_i at distance at most $\varepsilon \operatorname{lfs}(Y_X) \leqslant \varepsilon R_{Y_X} = \varepsilon R_X$ from Y_X. The sample point A_i is therefore at distance at most $R_X(1+\varepsilon)$ from I_X. Since, at the end of the motion, Σ is a Delaunay sphere, the interior of the ball bounded by Σ cannot contain A_i. We now intend to apply Lemma 15 to Σ_X, Σ' and A_i, with $\rho = \varepsilon$ and $c = 1 + 1/\eta$.

Before applying the lemma, we need to show that $\cos\alpha_i \leqslant \cos\theta/(1+\varepsilon)$ where $\alpha_i = \angle X I_X A_i$. If we note $\alpha = \angle X I_X Y_X$ and $\phi = \angle Y_X I_X A_i$, we have $\sin\phi \leqslant \varepsilon \operatorname{lfs}(Y_X)/R_X \leqslant \varepsilon$, $\alpha_i \geqslant \alpha - \phi$ and, since $1 - \cos\alpha = 2\eta_X^2$, we get $1 - \cos\alpha_i \geqslant 2\eta_X^2 - \varepsilon \sin\alpha \geqslant 2\eta_X^2 - \varepsilon$. Hence $\cos\alpha_i \leqslant 1 - 2\eta_X^2 + \varepsilon$. Moreover, using Lemma 11 and $\varepsilon \leqslant 1/2$,

$$\frac{\cos\theta}{1+\varepsilon} \geqslant \left(1 - 2\frac{\varepsilon^2}{(1-\varepsilon)^2}\right)\frac{1}{1+\varepsilon} \geqslant 1 - 3.5\varepsilon.$$

Therefore, if $\eta_X^2 \geqslant (9/4)\varepsilon$, $\cos\alpha_i \leqslant \cos\theta/(1+\varepsilon)$ and we can apply Lemma 16. Observing that

$$\begin{aligned}\frac{1}{1-\cos\alpha_i} &\leqslant \frac{1}{1-\cos\alpha-\varepsilon\sin\alpha} = \frac{1}{2\sin^2\alpha/2(1-\varepsilon\cot\alpha/2)} \leqslant \frac{1}{2\eta_X^2(1-\varepsilon/\eta_X)}\\ &\leqslant \frac{1}{2\eta_X^2(1-\sqrt{\varepsilon/2})},\end{aligned}$$

we finally obtain

$$\begin{aligned}\|I-I_X\| &\leqslant \sqrt{c^2+\frac{(1+c)^2}{(1+\cos\alpha_i)^2}}R_X\varepsilon\bigl(1+\mathrm{O}(\varepsilon)\bigr)\\ &\leqslant \sqrt{\left(1+\frac{1}{\eta}\right)^2+\frac{1}{\eta_X^4}\left(1+\frac{1}{2\eta}\right)^2}R_X\varepsilon\bigl(1+\mathrm{O}\bigl(\sqrt{\varepsilon}\,\bigr)\bigr). \qquad \square\end{aligned}$$

The following lemma shows that the circumcenters of the Delaunay d-simplices that have a long edge are close to the medial axis.

Proposition 17. *Let $\mathcal{A}$ be a good ε-sample of $\mathcal{S}$. Let X be a point of $\mathcal{S}$ such that $\eta_X \geqslant (3/2)\sqrt{\varepsilon}$, and let A_r be a natural neighbor of X lying at distance $2\eta R_X$ from X for $\eta \geqslant \sqrt[3]{(\pi/2)\varepsilon}$. The circumcenter V of any d-simplex incident to $[XA_r]$ and such that $\overrightarrow{XV}\cdot\overrightarrow{XI_X} > 0$ satisfies*

$$\|V-I_X\| \leqslant \widetilde{\omega}_X(\eta)R_X\varepsilon\bigl(1+\mathrm{O}\bigl(\sqrt[3]{\varepsilon}\,\bigr)\bigr),$$

where

$$\widetilde{\omega}_X(\eta) = \max\left(\omega_X(\eta'),\ \left(1+\frac{1}{\eta}\right)\sqrt{1+\frac{1}{\eta^4}}\right),$$

$\eta' = \|X-V\|/R_X$ and ω_X is as in Proposition 16.

Proof. For convenience, let $r=1$ and $S=[A_1\dots A_d X]$ be a d-simplex of Del^+ incident to $[XA_1]$. Let V be the circumcenter of S.

The proof is similar to the proof of Lemma 16. Let Σ be a moving and deformable sphere that initially coincides with Σ_X, I its center, and $Y\in\Sigma$ the point that initially coincides with Y_X. We first rotate Σ around X until its center lies on the ray going from X towards V. Let I' denote the new position of I and Σ' the corresponding sphere.

By Lemma 10, the angle θ between the vector $\overrightarrow{VX}$ and the normal to $\mathcal{S}$ at X is at most

$$\arcsin\left(\frac{\varepsilon}{\eta(1-\varepsilon)}\right)+\arcsin\left(\frac{\varepsilon}{1-\varepsilon}\right) = \left(1+\frac{1}{\eta}\right)\varepsilon\bigl(1+\mathrm{O}(\varepsilon)\bigr).$$

If I' lies between X and V, the previous lemma shows that

$$\|V-I_X\| \leqslant \omega_X(\eta')R_X\varepsilon\bigl(1+\mathrm{O}\bigl(\sqrt{\varepsilon}\,\bigr)\bigr).$$

Otherwise, Σ contains $A_1,\dots,A_d$. We shrink Σ until it passes through $A_1,\dots,A_d$ and X. More precisely, we move the center I of Σ as above along the line XV towards V while keeping Σ passing through X. We stop when $I = V$. During this second motion, Σ cannot be shrunk much. Indeed, since A_1 lies in the interior of the ball bounded by Σ after the first motion, we have

$$\|I_X - A_i\| \leqslant \|I_X - I'\| + \|I' - A_1\| \leqslant \|I_X - I'\| + R_X \leqslant R_X\left(1 + 2\sin\frac{\theta}{2}\right) \leqslant R_X(1+\theta).$$

We wish to apply Lemma 16 to Σ_X, Σ' and A_1 (with $c = 1$ and $\rho = \theta$).

Let $\alpha_1 = \angle X I_X A_1$. We bound α_i using the fact that edge $[XA_1]$ is long. Since A_1 does not lie inside Σ_X, we have

$$\sin\left(\frac{\alpha_1}{2}\right) = \frac{\|X - A_1'\|}{2R_X} \geqslant \frac{\|X - A_1\| - \|A_1 - A_1'\|}{2R_X},$$

where A_1' is the intersection of Σ_X and the line segment $[A_1 I_X]$. Since $\|A_1 - A_1'\| \leqslant R_X\theta$, we have $\sin(\alpha_1/2) \geqslant \eta - \theta/2$. In order to apply Lemma 15, we need to show that $\cos\alpha_1 \leqslant \cos\theta/(1+\theta)$. Since $\cos\theta/(1+\theta) > 1 - \theta - \theta^2/2$ and $\cos\alpha_1 \leqslant 1 - 2(\eta - \theta/2)^2$, the inequality holds if $2\eta^2 - 2\theta\eta - \theta \geqslant 0$. The following claim will be proved below.

Claim. $2\eta^2 - 2\theta\eta - \theta \geqslant 0$ *holds when* $\eta \geqslant \sqrt[3]{\pi/2}$.

We can now apply Lemma 16 and, observing that $\theta/\eta = \mathrm{O}(\sqrt[3]{\varepsilon})$, we obtain

$$\begin{aligned}\|I - I_X\| &\leqslant \sqrt{1 + \frac{4}{(1-\cos\alpha_1)^2}}\, R_X\theta\big(1 + \mathrm{O}(\theta)\big)\\ &\leqslant \left(1 + \frac{1}{\eta}\right)\sqrt{1 + \frac{1}{\eta^4}}\, R_X\varepsilon\big(1 + \mathrm{O}\big(\sqrt[3]{\varepsilon}\big)\big).\end{aligned}$$

This achieves the proof of the lemma. □

Proof of the claim. We prove that $\theta \leqslant 2\eta^2/(1+2\eta)$. By Lemma 10, θ is at most

$$\arcsin\left(\frac{\varepsilon}{\eta(1-\varepsilon)}\right) + \arcsin\left(\frac{\varepsilon}{1-\varepsilon}\right),$$

which is no more than $\pi\varepsilon/(2(1-\varepsilon))(1+1/\eta)$ since $\arcsin x \leqslant (\pi/2)x$ for $x \in (0, 1)$. It therefore suffices to show that

$$\frac{\pi\varepsilon}{2(1-\varepsilon)}\left(1 + \frac{1}{\eta}\right) \leqslant \frac{2\eta^2}{1+2\eta}.$$

With $\gamma = 1 + 1/\eta$ and $\zeta = (4/\pi)(1-\varepsilon)/\varepsilon$, this reduces to $\gamma^3 - \gamma - \zeta \leqslant 0$, which is true for

$$\gamma \leqslant \frac{1}{\sqrt[3]{2}}\left(\zeta + \sqrt{\zeta^2 - \frac{4}{27}}\right)^{1/3} + \frac{\sqrt[3]{2}}{3}\left(\zeta + \sqrt{\zeta^2 - \frac{4}{27}}\right)^{-1/3} \leqslant \sqrt[3]{\zeta} + 1.$$

It follows that the claim holds for $\eta \geqslant \sqrt[3]{(\pi/4)\varepsilon/(1-\varepsilon)}$, and, since $\varepsilon < 1/2$, also for $\eta \geqslant \sqrt[3]{(\pi/2)\varepsilon}$. □

Propositions 16 and 17 state that certain vertices of $V^+(\mathcal{A})$ converge towards the medial axis of $\mathcal{S}$. The next lemma states a similar result for the vertices of $V(X, A_r)$ provided that A_r is sufficiently far from X.

Proposition 18. *Let $\mathcal{A}$, X and A_r be as in Proposition 18. Any vertex of $V(X, A_r)$ such that $\overrightarrow{XV} \cdot \overrightarrow{XI_X} > 0$ satisfies $\|V - I_X\| \leqslant \overline{\omega}(\eta) R_X \varepsilon (1 + \mathrm{O}(\sqrt[3]{\varepsilon}))$ where $\overline{\omega}(\eta) = 1/\eta^2 + \widetilde{\omega}_X(\eta)$, and $\widetilde{\omega}_X$ is defined as in Proposition 17.*

Proof. Consider first the face F of Vor^+ that is common to $V^+(X)$ and $V(X, A_r)$. The vertices of F are the circumcenters of the d-simplices of Del^+ incident to $[XA_r]$. By Proposition 17, the vertices of F all lie in a ball B centered at I_X whose radius is $R \leqslant \omega(\eta) R_X \varepsilon (1 + \mathrm{O}(\sqrt[3]{\varepsilon}))$. This proves the claim for the vertices of $V(X, A_r)$ that are also vertices of $V^+(X)$. Moreover, by convexity of F and B, the claim holds for any point of F.

Let us consider now a vertex V of $V(X, A_r)$ that is not a vertex of F. Therefore, V lies strictly inside $V^+(X)$ and we have $\|V - X\| < \|V - A_r\|$. Let Σ_V be the sphere centered at V and passing through A_r. We have $\|V - A_r\| \leqslant \|V - X\| + \varepsilon\, \mathrm{lfs}(X)$ since the interior of the ball bounded by Σ_V contains X but not the sample point closest to X. Hence,

$$\|V - A_r\| - \varepsilon\, \mathrm{lfs}(X) \leqslant \|V - X\|. \tag{1}$$

Refer to Fig. 7. Let Q be the affine hull of F, i.e., the bisector hyperplane of X and A_r, let P be the orthogonal projection of V onto Q, and let W be the point of intersection of Q and the line segment $[A_r V]$. We first observe that $W \in F$. Indeed, $[A_r V]$ is contained in $V(A_r)$ and $F = V(A_r) \cap Q$. Moreover, we have

$$\|V - A_r\|^2 - \|V - X\|^2 = 2(A_r - X) \cdot \left(\frac{A_r + X}{2} - V\right) = 2\|X - A_r\|\|V - P\|.$$

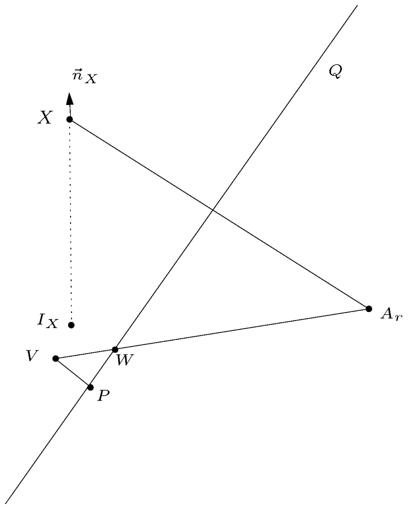

Fig. 7. For the proof of Proposition 18.

Moreover,

$$\frac{\|V-P\|}{\|V-W\|}=\frac{\frac{1}{2}\|X-A_r\|}{\|W-A_r\|}.$$

We therefore get

$$\|V-X\|^2=\|V-A_r\|^2-\|X-A_r\|^2\frac{\|V-W\|}{\|W-A_r\|}.$$

By squaring the two sides of Eq. (1), we obtain

$$-2\|V-A_r\|\varepsilon\,\mathrm{lfs}(X)+\varepsilon^2\,\mathrm{lfs}^2(X)\leqslant-\|X-A_r\|^2\frac{\|V-W\|}{\|W-A_r\|},$$

and, since $\|V-A_r\|=\|V-W\|+\|W-A_r\|$,

$$\|V-W\|\leqslant 2\frac{\|W-A_r\|^2}{\|X-A_r\|^2}\,\mathrm{lfs}(X)\varepsilon\bigl(1+\mathrm{O}(\varepsilon)\bigr). \tag{2}$$

Since $W\in F$, as observed above, W is at distance $R\leqslant\omega(\eta)R_X\varepsilon(1+\mathrm{O}(\sqrt[3]{\varepsilon}))$ from I_X. Hence, we have

$$\|W-A_r\|=\|X-W\|\leqslant\|X-I_X\|+\|I_X-W\|\leqslant R_X+R.$$

This last inequality together with Eq. (2) and $\|X-A_r\|\geqslant 2\eta R_X$, leads to

$$\begin{aligned}\|V-W\| &\leqslant \frac{1}{2\eta^2}\bigl(1+\omega(\eta)\varepsilon\bigl(1+\mathrm{O}\bigl(\sqrt[3]{\varepsilon}\,\bigr)\bigr)\bigr)\,\mathrm{lfs}(X)\varepsilon\bigl(1+\mathrm{O}(\varepsilon)\bigr)\\ &\leqslant \frac{1}{\eta^2}\,\mathrm{lfs}(X)\varepsilon\bigl(1+\mathrm{O}\bigl(\sqrt[3]{\varepsilon}\,\bigr)\bigr)\\ &\leqslant \frac{1}{\eta^2}R_X\varepsilon\bigl(1+\mathrm{O}\bigl(\sqrt[3]{\varepsilon}\,\bigr)\bigr).\end{aligned}$$

Finally,

$$\|I_X-V\|\leqslant\|I_X-W\|+\|V-W\|\leqslant\left(\omega(\eta)+\frac{1}{\eta^2}\right)R_X\varepsilon\bigl(1+\mathrm{O}\bigl(\sqrt[3]{\varepsilon}\,\bigr)\bigr),$$

which proves the lemma. □

Remark 2. Propositions 16–18 are still valid if one considers $\mathbb{R}^d\setminus\mathcal{O}$ instead of $\mathcal{O}$. More precisely, if one replaces $\overrightarrow{XV}\cdot\overrightarrow{XI_X}>0$ by $\overrightarrow{XV}\cdot\overrightarrow{XI_X}<0$ in the propositions, the same bounds hold provided that R_X, I_X and η_X are replaced by R_X^e, I_X^e and η_X^e. However, since R_X^e can be arbitrarily large, the results are only meaningful when only a bounded region of the plane (containing $\mathcal{S}$) is considered. For the lemmas to hold, the boundary $\mathcal{B}$ of that region must be smooth and $\mathcal{A}$ must be a ε-sample of $\mathcal{S}\cup\mathcal{B}$.

5. Natural neighbor coordinates of points on $\mathcal{S}$

The set of natural neighbor coordinates of a point X of $\mathcal{S}$ (computed in $\mathbb{R}^d$) do not constitute a local system of coordinates on $\mathcal{S}$. Indeed, the support of the λ_i is *not* a small neighborhood of A_i even if the sampling is very dense. This is illustrated in Fig. 8.

In this section, we show however that the set of natural neighbor coordinates of $X \in \mathcal{S}$ tends to behave as a local coordinate system on the surface $\mathcal{S}$ when the density of points increases, i.e., when ε tends to 0.

Theorem 19. *Let $\mathcal{A}$ be a ε-sample of $\mathcal{S}$. Let X be a point of $\mathcal{S}$ such that $\eta_X \geqslant \frac{3}{2}\sqrt{\varepsilon}$ and let $N_\eta(X)$ denote the set of indexes of the natural neighbors of X lying at distance $< 2\eta R_X$ from X, for $\eta \geqslant \sqrt[3]{(\pi/2)\varepsilon}$. We have*

$$\sum_{i \notin N_\eta(X)} \lambda_i(X) \leqslant d\overline{\omega}(\eta)\varepsilon\big(1 + \mathrm{O}\big(\sqrt[3]{\varepsilon}\,\big)\big), \tag{3}$$

where $\overline{\omega}(\eta)$ is defined as in Proposition 18.

Proof. We first consider the case where the $\mathcal{S}$-natural neighbors of X all lie in the hyperplane $H(X)$ passing through X and normal to $\vec{n}_X$. Let H^i denote the halfspace limited by $H(X)$ that contains I_X (i.e., opposite to $\vec{n}_X$) and H^e the other halfspace.

Under this assumption, $V^+_{\mathcal{S}}(X)$ is a convex $(d-1)$-polytope contained in $H(X)$. Let $v(X)$ denote its area, i.e., $(d-1)$-dimensional volume. Since $I_X \in V^+_{\mathcal{O}}(X)$, the volume of $V^+_{\mathcal{O}}(X)$ is at least $(1/d)R_X v(X)$.

When points are appropriately added on a bounding box (see Remark 2), a similar inequality holds for the portion of $V^+(X)$ that is outside $\mathcal{O}$: we simply need to replace R_X by the radius R^e_X of the other maximal sphere tangent to $\mathcal{S}$ at X. We therefore have

$$w(X) \geqslant \frac{1}{d}\big(R_X + R^e_X\big)v(X). \tag{4}$$

Let A_r be a natural neighbor of X lying at distance $\geqslant 2\eta \max(R_X, R^e_X)$ from X.

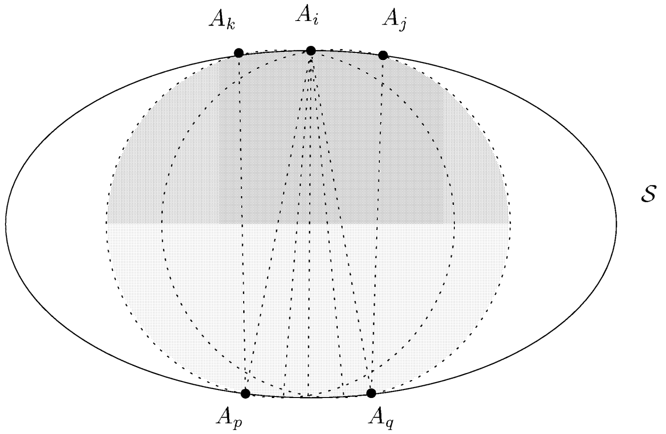

Fig. 8. The grey region is the union of the bounded Delaunay balls passing through A_i. Its intersection with $\mathcal{S}$ is the support of A_i. It consists of two arcs, A_jA_k and A_pA_q.

We denote by C the cylinder intersecting $\mathcal{S}$ along $V_{\mathcal{S}}^{+}(X)$ and whose axis is parallel to $\vec{n}(X)$. It follows from Propositions 17 and 18 that all the vertices of $V(X, A_r)$ that lie in H^i belong to a portion $\mathcal{Z}$ of C. $\mathcal{Z}$ contains I_X and has height $\leqslant \overline{\omega}(\eta) R_X \varepsilon(1+\mathrm{O}(\sqrt[3]{\varepsilon}))$. Hence, $V_{\mathcal{O}}(X, A_r) = V(X, A_r)$ and we have

$$\sum_{r \notin N_\eta(X)} \mathrm{vol}\big(V_{\mathcal{O}}(X, A_r)\big) \leqslant \mathrm{vol}(\mathcal{Z}) \leqslant v(X)\overline{\omega}(\eta) R_X \varepsilon\big(1+\mathrm{O}\big(\sqrt[3]{\varepsilon}\,\big)\big).$$

The same inequality holds for the vertices of $V(X, A_r)$ that lie in H^e provided that R_X is replaced by R_X^e. Therefore, we have

$$\sum_{r \notin N_\eta(X)} \mathrm{vol}\big(V(X, A_r)\big) \leqslant v(X)\overline{\omega}(\eta)\big(R_X + R_X^e\big)\varepsilon\big(1+\mathrm{O}\big(\sqrt[3]{\varepsilon}\,\big)\big). \tag{5}$$

From Eqs. (4) and (5), we get

$$\sum_{r \notin N_\eta(X)} \lambda_r(X) = \frac{\sum_{r \notin N_\eta(X)} \mathrm{vol}(V(X, A_r))}{w(X)} \leqslant d\overline{\omega}(\eta)\varepsilon\big(1+\mathrm{O}\big(\sqrt[3]{\varepsilon}\,\big)\big). \tag{6}$$

We now show that the same result holds when the $\mathcal{S}$-natural neighbors of X do not belong to the hyperplane $H(X)$. Let $A_1, \ldots, A_m$ be the vertices of these facets other than X and $A_1', \ldots, A_m'$ their projection onto $H(X)$.

It is easy to see that $\|A_i - A_i'\| \leqslant (2\varepsilon^2/(1-\varepsilon)^2)\,\mathrm{lfs}(X)$. Indeed, $\|A_i - A_i'\| \leqslant \|A_i - X\| \sin\theta$, where $\theta = \angle A_i X A_i'$. Since A_i does not belong to the two balls of radius $\mathrm{lfs}(X)$ that are tangent to $\mathcal{S}$ at X,

$$\sin\theta \leqslant \frac{\|A_i - X\|}{2\,\mathrm{lfs}(X)}.$$

With $\|A_i - X\| \leqslant (2\varepsilon/(1-\varepsilon))\,\mathrm{lfs}(X)$ (Lemma 9), we finally get $\|A_i - A_i'\| \leqslant (2\varepsilon^2/(1-\varepsilon)^2)\mathrm{lfs}(X)$.

Since the element of area dv on a surface is the element of area in the tangent plane, Eq. (4) holds up to second order terms in ε. The rest of the proof is unchanged. We conclude that Eq. (6) still holds when the $\mathcal{S}$-natural neighbors of X do not belong to the hyperplane $H(X)$. □

Corollary 20. *Let X and $N_\eta(X)$ be as in Theorem 19. We have*

$$X = \sum_{i \in N_\eta(X)} \lambda_i(X) A_i + Y \quad \textit{with } \|Y\| = d\overline{\omega}(\eta)\varepsilon\big(1+\mathrm{O}\big(\sqrt[3]{\varepsilon}\,\big)\big).$$

Proof. We have

$$X = \sum_i \lambda_i(X) A_i = \sum_{i \in N_\eta(X)} \lambda_i(X) A_i + \sum_{i \notin N_\eta(X)} \lambda_i(X) A_i.$$

By the previous result,

$$\lambda = \sum_{i \notin N_\eta(X)} \lambda_i(X) = d\overline{\omega}(\eta)\varepsilon\big(1+\mathrm{O}\big(\sqrt[3]{\varepsilon}\,\big)\big).$$

Hence,

$$X = \sum_{i \in N_\eta(X)} \lambda_i(X) A_i + Y$$

where Y belongs to the convex hull,

$$\mathrm{CH}\big(\{\lambda A_i,\ i \notin N_\eta(X)\}\big) = d\overline{\omega}(\eta)\varepsilon\big(1 + \mathrm{O}\big(\sqrt[3]{\varepsilon}\,\big)\big)\mathrm{CH}\big(\{A_i,\ i \notin N_\eta(X)\}\big).$$

Since, the diameter of $\mathrm{CH}(\{A_i,\ i \notin N_\eta(X)\})$ remains bounded, the result follows. □

6. Concluding remarks

Our main result, Theorem 19, has been stated and proved for natural neighbor coordinates. However, its proof can be adapted to take care of other coordinate systems based on Voronoi diagrams, e.g., the Laplace coordinates introduced by Hiyoshi and Sugihara [13].

The major restriction of this work is to assume that the surface is smooth. It would be interesting to extend our results to the case of non-smooth surfaces.

Acknowledgements

We are indebted to Mariette Yvinec for a thorough reading of the manuscript. Dominique Attali is also acknowledged for discussions on this and related topics.

Appendix A. Distance to a manifold and Morse theory

Let f be a smooth function on a smooth manifold $\mathcal{S}$. A point $Y \in \mathcal{S}$ is called *critical* if all the partial derivatives of f are zero at Y. A critical point is called *degenerate* if the determinant of the Hessian matrix is zero at Y. A smooth function on a smooth manifold is called a *Morse function* if all its critical points are nondegenerate.

Let X be a fixed point of $\mathbb{R}^d$ and consider the distance function δ_X that associates to any point $Y \in \mathcal{S}$ its squared distance to X. The following lemma is well known (see, for instance, [12, Section 9.4].

Lemma A.1. *A point $Y \in \mathcal{S}$ is a critical point of δ_X iff the vector $\overrightarrow{XY}$ is normal to $\mathcal{S}$ at Y. Moreover, δ_X has a degenerate critical point exactly when X is a focal point of $\mathcal{S}$. Hence, δ_X is a Morse function when X is not a focal point of $\mathcal{S}$. In particular, δ_X is a Morse function for any $X \in \mathcal{S}$.*

The following theorem is a basic theorem in Morse theory (see [15] for a proof).

Theorem A.2. *Let ϕ be a smooth function on a compact smooth surface $\mathcal{S}$ and assume that, for two reals $a < b$, $\phi^{-1}([a, b]) = \{x\colon\ a \leqslant \phi(x) \leqslant b\}$ contains no critical point. Then $\phi_a = \{x\colon\ \phi(x) \leqslant a\}$ and $\phi_b = \{x\colon\ \phi(x) \leqslant b\}$ are diffeomorphic.*

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