IMPERIAL-CAMBRIDGE
MATHEMATICS
COMPETITION

ICMC 9 — Round One

23 November 2025

Name: ______________________________

Contestant ID: ______________________________

University: ______________________________

Instructions:

- Do not turn over until told to do so.
- You will have 3 hours to solve 6 problems, each of which carries 10 marks.
- Use a black or blue pen or a dark pencil. Rulers, compasses, protractors, and erasers may be used but will not be required. All electronic devices, including calculators, are prohibited.
- Drinks are allowed, but food is prohibited.
- Write your solution to each problem on a different page. At the top of each page, write down the question number, your initials, and your contestant number. Use both sides whenever possible. Write clearly and not too faintly – your work will be scanned for marking.
- Problems are listed roughly in order of difficulty. Proofs are expected for all problems even if they only ask for an answer.
- One complete solution will be awarded more marks than several unfinished attempts.
- You may not leave the contest venue early unless exceptional circumstances arise.
- Do not take away the problems sheet or any rough work when leaving the venue.

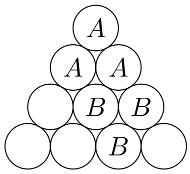
Declaration:

I hereby declare that the work I am submitting is entirely my own, and that I will not discuss or disclose the contents of this paper until 19:00, 23 November 2025, UTC Time.

Signature: ______________________________

Problem 1. Tony draws $1 + 2 + 3 + \cdots + 2026$ unit circles arranged in a triangular lattice, forming an equilateral triangular array with 2026 rows. A *triple* consists of 3 mutually tangent circles, and a pair of triples is considered *disjoint* if they do not share a circle. Given a collection of disjoint triples, we call a circle *unused* if it is not in any of the triples. Over all collections of disjoint triples, what is the smallest possible number of unused circles?

(As an example, the diagram below shows a pair of disjoint triples in an equilateral triangular array with 4 rows, leaving 4 circles unused.)



Problem 2. ICMC is turning 9 years old! To celebrate, Dylan buys 9 candles to put on a birthday cake. He would like to place 8 of the candles in distinct positions so as to form two squares $ABCD$ and $EFGH$. Is it possible to do this so that, regardless of where he places the ninth candle P, it is true that

$$PA^2 + PB^2 + PC^2 + PD^2 = PE^2 + PF^2 + PG^2 + PH^2?$$

(Assume the birthday cake is the Euclidean plane, and each candle is a distinct point.)

Problem 3. Let a, b, and c be positive integers with $\gcd(a, b, c) = 1$ such that

$$2a^2 - b^2 - c^2 + 2bc - ab - ac = 0.$$

Show that a is either an odd square number or two times an even square number.

Problem 4. Daniel and Andrija play a game in the Euclidean plane. Daniel chooses a set of 2025 distinct points such that no three are collinear. Andrija then draws m lines in the plane, none of which pass through any of Daniel's points. Andrija's lines split the plane into regions, and he wins if each region contains at most 1 point. Find the smallest m such that Andrija has a winning strategy regardless of Daniel's choice of points.

Problem 5. Does there exist a twice-differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f''(x) > f(x) > f'(x) > 0 \qquad \text{for all } x \in \mathbb{R}?$$

Problem 6. Ishan has a fair coin with an equal chance of landing on heads (H) or tails (T), and would like to simulate a fair 5-sided die using a *correspondence*. A correspondence assigns which (possibly infinite) set of finite sequences of coin tosses correspond to each face of the die. Using a correspondence, Ishan will toss the coin until his sequence of coin tosses matches one assigned to a face f of the die exactly (i.e. not just as a subsequence), at which point Ishan stops tossing the coin and declares the result of the simulated die-roll to be f. In order for the correspondence to be well-defined, no sequence can be assigned to more than one face, and no assigned sequence may be the start of any other assigned sequence. For example, if the sequences H, TT, THT, and THH are assigned to faces of the die, then no other sequences may be assigned.

Over all possible correspondences, what is the smallest expected number of times Ishan will need to toss the coin in order to simulate a fair 5-sided die?