IMPERIAL-CAMBRIDGE MATHEMATICS COMPETITION

9th Edition (2025–2026)

ROUND ONE

Official Solutions*

Last updated: 2025-11-24 01:21, UTC Time

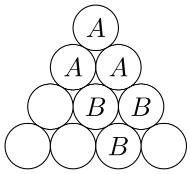
*A solution may receive full or partial marks even if it does not appear in this booklet.

Problem 1.

(proposed by Andrija Živadinović)

Tony draws $1 + 2 + 3 + \cdots + 2026$ unit circles arranged in a triangular lattice, forming an equilateral triangular array with 2026 rows. A *triple* consists of 3 mutually tangent circles, and a pair of triples is considered *disjoint* if they do not share a circle. Given a collection of disjoint triples, we call a circle *unused* if it is not in any of the triples. Over all collections of disjoint triples, what is the smallest possible number of unused circles?

(As an example, the diagram below shows a pair of disjoint triples in an equilateral triangular array with 4 rows, leaving 4 circles unused.)



Notes on Marking. None yet.

Solution 1.

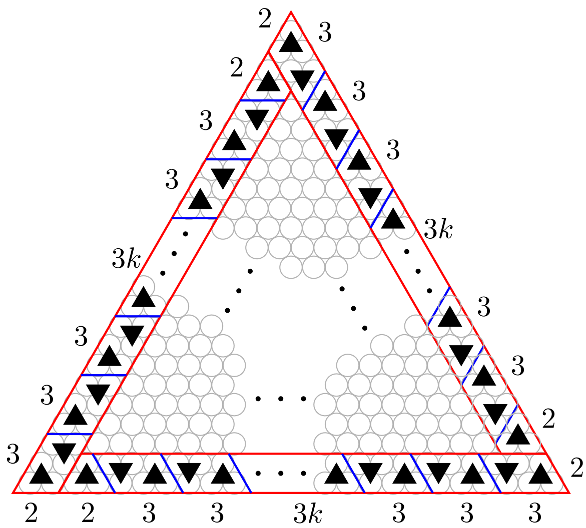
(solution by Tony Wang)

We prove that the answer is 1. We start by proving that the answer must be at least 1. This is true because the total number of circles in the equilateral triangular array is

$$1 + 2 + \cdots + 2026 = \frac{2026 \times 2027}{2} = 1013 \times 2027,$$

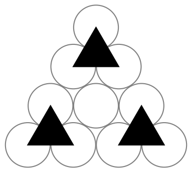
which, modulo 3, works out to be $2 \times 2 \equiv 1 \pmod 3$.

We now need to construct a collection of disjoint triples that use all but one circle. To do this, we note that, given that the side length of the equilateral triangular array is a number congruent to 1 modulo 3, we can use the following construction along the edge:



Note that this construction works whenever the side length of the equilateral triangle is $1 \pmod 3$ since the triplets along the border have a total length of $3n + 2 + 2 \equiv 1 \pmod 3$.

Using this construction essentially reduces the side length of the equilateral triangle by 6, since it reduces the triangle on each side by 2. By repeating this construction 337 times, we will end up with an equilateral triangular array of side length 4, and we can tile that triangular array as follows:



This leaves 1 circle unused. $\square$

Problem 2. *(proposed by Daniel Naylor)*

ICMC is turning 9 years old! To celebrate, Dylan buys 9 candles to put on a birthday cake. He would like to place 8 of the candles in distinct positions so as to form two squares $ABCD$ and $EFGH$. Is it possible to do this so that, regardless of where he places the ninth candle P, it is true that

$$PA^2 + PB^2 + PC^2 + PD^2 = PE^2 + PF^2 + PG^2 + PH^2?$$

(Assume the birthday cake is the Euclidean plane, and each candle is a distinct point.)

Notes on Marking. None yet.

Solution 1. *(solution by Daniel Naylor)*

For a square S in the plane, consisting of points A, B, C, D, let $f_S : \mathbb{R}^2 \to \mathbb{R}$ denote the function $f(P) = PA^2 + PB^2 + PC^2 + PD^2$. Then the question is asking us whether we can find squares S_1, S_2 with no overlapping points, such that the functions f_{S_1}, f_{S_2} are the same.

For the square S with coordinates $(1,1), (1,-1), (-1,-1), (-1,1)$, we have

$$\begin{aligned} f_S(x,y) &= ((x-1)^2 + (y-1)^2) \\ &\quad + ((x-1)^2 + (y+1)^2) \\ &\quad + ((x+1)^2 + (y+1)^2) \\ &\quad + ((x+1)^2 + (y-1)^2) \\ &= 4x^2 + 4y^2 + 8 \end{aligned}$$

By rotating and translating, we see that for any unit square S with centre O, we have $f_S(P) = 4OP^2 + 8$. Thus if Dylan places $ABCD$ and $EFGH$ to form two unit squares with a common centre, then the desired equation is satisfied for all points P. One possible way of doing this is by using a regular octagon. $\square$

Solution 2. *(solution by Tony Wang)*

The equation of a general quadratic in the plane can be written as $f(P) = a \cdot PB^2 + c$, where a and c are real numbers, and B is a point in the plane. If we write $P = (x,y)$ and $B = (x_0, y_0)$, then this can also be written as $f\Big((x,y)\Big) = a\Big((x-x_0)^2 + (y-y_0)^2\Big) + c$.

Lemma: The sum of two quadratics in the plane with leading coefficients a_0 and a_1 is another quadratic in the plane with leading coefficient $a_0 + a_1$.

Proof. We can compute the sum of two quadratics $a_0\Big((x-x_0)^2 + (y-y_0)^2\Big) + c_0$ and $a_1\Big((x-x_1)^2 + (y-y_1)^2\Big) + c_1$. This turns out to be

$$(a_0 + a_1)\left[\left(x - \frac{a_0x_0 + a_1x_1}{a_0 + a_1}\right)^2 + \left(y - \frac{a_0y_0 + a_1y_1}{a_0 + a_1}\right)^2\right] + c,$$

for some real constant c. We note that this is in the form of a general quadratic in the plane, and its leading coefficient is $a_0 + a_1$, as desired. $\square$

By the lemma, the expression $f(P) = PA^2 + PB^2 + PC^2 + PD^2$ will have the form $f(P) = 4 \cdot PX^2 + c$ for some point X. However, note that squares in the plane have four degrees of freedom (since squares can be uniquely determined by choosing two points to be its diameter, and each point has two degrees of freedom), and yet the equation has three degrees of freedom (two for X, and one for c). This implies that there must be two squares $ABCD$ and $EFGH$ which result in the same quadratic in the plane. Hence, for those two squares, the expressions $PA^2 + PB^2 + PC^2 + PD^2$ and $PE^2 + PF^2 + PG^2 + PH^2$ will be identical. $\square$

Comment. It turns out that, given a square $ABCD$, we can write $PA^2 + PB^2 + PC^2 + PD^2$ as $4 \cdot PX^2 + c$, where X is the centre of the square, and c is twice the area of the square. This implies that two squares result in the same quadratic in the plane if and only if they are the same size and share the same centre – that is, if and only if one is a rotation of the other about its centre.

Problem 3. *(proposed by Andrija Živadinović)*

Let a, b, and c be positive integers with $\gcd(a,b,c) = 1$ such that

$$2a^2 - b^2 - c^2 + 2bc - ab - ac = 0.$$

Show that a is either an odd square number or two times an even square number.

Notes on Marking. None yet.

Solution 1. *(solution by Tony Wang and Daniel Naylor)*

The equation can be rewritten in the form $a(2a-x) = y^2$, where $x = b+c$ and $y = b-c$. Write $a = kn^2$, where k is square-free (that is, each prime factor divides k at most once). We now have

$$k(2kn^2 - x) = \left(\frac{y}{n}\right)^2.$$

Since k is square-free and yet $k(2kn^2) - x)$ is a square, each prime factor of k must also appear in $2kn^2 - x$. From here, we deduce that $k \mid 2kn^2 - x$, or $k \mid x$. Each prime factor dividing k hence also appears in y, and so we also have $k \mid y$. Now, we have $k \mid x + y = 2b$ and $k \mid x - y = 2c$. Since $k \mid a \mid 2a$ by definition, we must have $k \mid \gcd(2a, 2b, 2c) = 2$, as $\gcd(a,b,c) = 1$. So, $k = 1$ or 2. We now have two cases:

- Case 1: $k = 1$. If $2 \nmid y$, we must have $n \equiv 1 \pmod 2$. If $2 \mid y$, so $2 \mid x$, we write $x = 2x_1$, $y = 2y_1$, so the last equation becomes $n^2 - x_1 = 2\left(\frac{y_1}{n}\right)^2$. If we had $2 \mid n$, then we must have $2 \mid x_1$, but as $n \mid y_1$, we also have $2 \mid y_1$, so $4 \mid x$, and $4 \mid y$. Then $2 \mid b = \frac{x+y}{2}$ and $2 \mid c = \frac{x-y}{2}$, but since we also have $2 \mid n \mid a$, we get $\gcd(a,b,c) \geq 2$, a contradiction. Thus, it must be $n \equiv 1 \pmod 2$.
- Case 2: $k = 2$. We then have $2(4n^2 - x) = (\frac{y}{n})^2$, so $2 \mid y$, and hence $2 \mid x$. Writing $x = 2x_1$ and $y = 2y_2$, we get $2n^2 - x_1 = (\frac{y_1}{n})^2$. Since $2 \mid a$, $\gcd(a,b,c) = 1$, and $b \equiv c \pmod 2$ (as $2 \mid x$, $2 \mid y$), we must have $b \equiv c \equiv 1 \pmod 2$. If $2 \nmid n$, then, by the above, $2 \mid x_1$ if and only if $2 \mid y_1$. Since $b \equiv c \equiv 1 \pmod 2$, at least one of x_1 and y_1 must be even (if $b \equiv c \pmod 4$, that is y_1, otherwise that is x_1), so by the above, both x_1 and y_1 are even. We then have $2 \mid b = x_1 + y_1$ and $2 \mid c = x_1 - y_1$, giving us a contradiction, as $2 \mid a$ and $\gcd(a,b,c) = 1$. Thus, it must be the case that $2 \mid n$.

Hence, a must be either an odd square number or two times an even square number. $\square$

Comment. Continuing from here, we can parametrise all solutions: the general solutions for (a,b,c) are, for all $k \in \mathbb{N}$:

$$\left((2k-1)^2,\ (2k-1)^2 - \frac{d^2 - (2k-1)d}{2},\ (2k-1)^2 + \frac{d^2 - (2k-1)d}{2}\right)$$

for $|d| \leq 2k - 1$ and $\gcd(d, 2k-1) = 1$;

$$\left(2 \cdot (2k)^2,\ 2 \cdot (2k)^2 - d^2 + 2kd,\ 2 \cdot (2k)^2 - d^2 - 2kd\right),$$

for $|d| \leq 2k$ and $\gcd(d, 2k) = 1$.

Problem 4.

(proposed by Dylan Toh and Tony Wang)

Daniel and Andrija play a game in the Euclidean plane. Daniel chooses a set of 2025 distinct points such that no three are collinear. Andrija then draws m lines in the plane, none of which pass through any of Daniel's points. Andrija's lines split the plane into regions, and he wins if each region contains at most 1 point. Find the smallest m such that Andrija has a winning strategy regardless of Daniel's choice of points.

Notes on Marking. None yet.

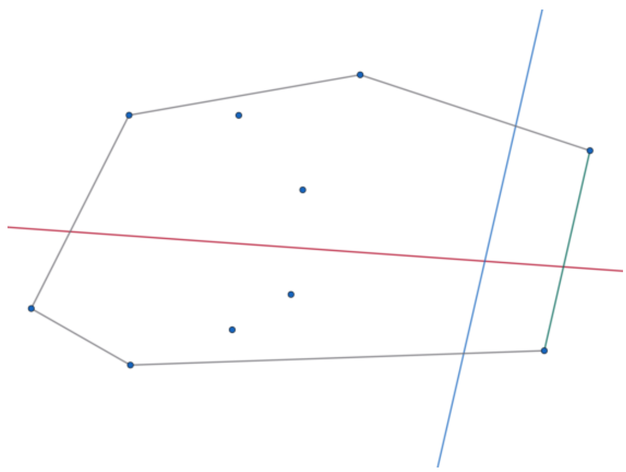
Solution 1.

(solution by Andrija Živadinović)

We will prove that, in general, for $n \in \mathbb{N}$, given $2n-1$ or $2n$ points in a plane such that no three are collinear we need n lines to guarantee we can split the plane in regions using them, such that there is at most 1 point in each region.

To see that $n-1$ line is not enough, consider $2n-1$ vertices of a regular $(2n-1)$-gon in the plane (and if we have $2n$ points, put the last one anywhere in the plane, satisfying the problem conditions). To split two adjacent vertices of this polygon into separate regions, we need to have at least one of the lines intersecting the side of this polygon connecting them. As any line not containing any of the vertices of this polygon can intersect at most 2 different sides of this polygon, using $n-1$ line we can intersect at most $2n-2 < 2n-1$ sides of this polygon, so at least one pair of adjacent vertices will not be split into two separate regions. Thus, $n-1$ line is not enough.

Now we prove that n lines is enough. Since the problem is strictly harder for $2n$ points than for $2n-1$ points, assume we are given $2n$ points in the plane, no three of which are collinear. First, we can choose an arbitrary line not parallel to any line joining two of the points, and then translate the this line until it is positioned such that there are exactly n points on each side of it (red line in the picture below). This is possible by the discrete intermediate value theorem. We then repeat the following process: we draw a line that splits exactly one point on each side of the red line from the rest of the currently unsplit points. After $n-1$ iterations of this, all $2n$ points will be split into separate regions using exactly n lines. Hence, it remains to prove that such a line can be found.



To prove this, consider the convex hull of all the currently unsplit points. As there is at least one such point on each side of the red line, there must be at least one side of this convex hull intersecting the red line. The line determined by that side contains exactly one of the currently unsplit points on each side of the red line, and all the other currently unsplit points are on the same side of it. Thus, translating this line a little towards the rest of the currently unsplit points without changing its direction will give us a line with desired property. Hence, we are done.

Since $n = 2025$ given in the problem, the smallest m for which Andrija has a winning strategy regardless of Daniel's choice of points is $m = 1013$. $\square$

Problem 5. *(proposed by Tony Wang)*

Does there exist a twice-differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f''(x) > f(x) > f'(x) > 0 \qquad \text{for all } x \in \mathbb{R}?$$

Notes on Marking. None yet.

Solution 1. *(solution by Tony Wang)*

We show that the answer is no. First, note that since f is increasing and strictly positive, it must to converge to some value $c \geq 0$ as $x \to -\infty$. We now split into two cases depending on the value of c:

- If $c = 0$, then there exists a $d < 0$ sufficiently small such that $f(d) < f(0) - f'(0)$. We also have $\frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x) - f'(x)\Big) = f'(x) - f''(x) < 0$, so $f(x) - f'(x)$ is decreasing. Putting these together yields $f(d) - f'(d) > f(0) - f'(0) \implies 0 > f'(d)$, a contradiction.
- If $c > 0$, then $f''(x) > c$ for all real numbers x. Hence, $f'(x)$ increases with gradient at least c everywhere, so $f'(x) \to -\infty$ as $x \to -\infty$. $\square$

Solution 2. *(solution by Daniel Naylor)*

Since $f(x) > 0$, we can perform the substitution $f(x) = e^{g(x)}$. The given inequality then transforms to

$$(g'(x)^2 + g''(x))e^{g(x)} > e^{g(x)} > g'(x)e^{g(x)} > 0.$$

Dividing by $e^{g(x)}$ (which is necessarily positive), we get

$$g'(x)^2 + g''(x) > 1 > g'(x) > 0.$$

Now let $h(x) = g'(x)$, so

$$h(x)^2 + h'(x) > 1 > h(x) > 0.$$

A consequence of this is that $h'(x) > 1 - h(x)^2 > 1 - 1^2 > 0$. Hence for $x < 0$, we have $h(x) < h(0)$, and so for $x < 0$ we have

$$h'(x) > 1 - h(x)^2 > 1 - h(0)^2 > 0.$$

Using this,

$$h(x) < h(0) + x(1 - h(0)^2),$$

so if $x \ll 0$, we have $h(x) < 0$, a contradiction. $\square$

Solution 3. *(solution by Contestants)*

Since $f(x) > f'(x) > 0$ for all x, we have $0 < \frac{f'(x)}{f(x)} < 1$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f'(x)}{f(x)}\right) = \frac{f''(x)}{f(x)} - \left(\frac{f'(x)}{f(x)}\right)^2 > 0,$$

so $\frac{f'(x)}{f(x)}$ must be strictly increasing. As $x \to -\infty$ we get $\frac{f'(x)}{f(x)} \to C < 1$, where $C \neq 1$ as it is strictly increasing. Hence,

$$\lim_{x \to -\infty} \frac{\mathrm{d}}{\mathrm{d}x} \big(f'(x)/f(x) \big) \geq 1 - C^2 > 0$$

contradicting the convergence of f'/f as $x \to -\infty$

Problem 6. *(proposed by Tony Wang)*

Ishan has a fair coin with an equal chance of landing on heads (H) or tails (T), and would like to simulate a fair 5-sided die using a *correspondence*. A correspondence assigns which (possibly infinite) set of finite sequences of coin tosses correspond to each face of the die. Using a correspondence, Ishan will toss the coin until his sequence of coin tosses matches one assigned to a face f of the die exactly (i.e. not just as a subsequence), at which point Ishan stops tossing the coin and declares the result of the simulated die-roll to be f. In order for the correspondence to be well-defined, no sequence can be assigned to more than one face, and no assigned sequence may be the start of any other assigned sequence. For example, if the sequences H, TT, THT, and THH are assigned to faces of the die, then no other sequences may be assigned.

Over all possible correspondences, what is the smallest expected number of times Ishan will need to toss the coin in order to simulate a fair 5-sided die?

Notes on Marking. None yet.

Solution 1. *(solution by Tony Wang)*

We will show that the answer is 3.6. We will model each correspondence as a rooted binary tree, with leaf nodes corresponding to assigned sequences. For the purposes of this proof, the root of the tree will be at the top, and the tree will grow downwards. Call a correspondence that achieves the smallest expected coin tosses *optimal*.

Lemma 1: No two leaves at the same level of an optimal correspondence can be assigned to the same face.

> *Proof.* Suppose for the sake of contradiction that an optimal correspondence has two leaves V_1 and V_2 at the same level assigned to the same face. If those two leaves are not children of the same parent vertex, then swap the sibling of V_1 with V_2 in the tree. This doesn't change the probabilities or the expected value of the correspondence.
>
> Next, combine V_1 and V_2 into their shared parent vertex, so that their parent vertex is now assigned to the same face V_1 and V_2 were assigned to. This doesn't change the probabilities of the correspondence, and it strictly decreases the expected value. That is, the result is still a valid correspondence, but it has a lower expected value of coin tosses and hence it is not optimal, a contradiction. $\square$

Lemma 2: The optimal correspondence assigns a single sequence of each length in the set $S = \{3, 4, 7, 8, 11, 12, \ldots\}$ to each face of the die.

> *Proof.* As the sum of reciprocals of powers of two must be equal to 0.2, and because each power of two can only be used once as per Lemma 1, we need to find a set of exponents $S \in \mathbb{Z}^+$ such that
>
> $$0.2 = \sum_{n \in S} 2^{-n}.$$
>
> However, there is only one way to do this, which is clear once we consider that the binary representation of 0.2 is $0.001100110011\ldots_2$. Note that this also corresponds to the set $\{3, 4, 7, 8, 11, 12, \ldots\}$, as desired. $\square$

Hence, we can calculate the optimal correspondence as follows:

- At the root level, there is one unassigned vertex.
- At level 1, there are two unassigned vertices.
- At level 2, there are four unassigned vertices.
- At level 3, five of the eight leaves are assigned, one to each face, leaving three unassigned vertices.
- At level 4, five of the six leaves are assigned, one to each face, leaving one unassigned vertex.
- At level 5, we repeat from level 1, since we now have two assigned vertices, and the set S also repeats every four levels.

Since this is a recursive correspondence, the expected value E of coin tosses satisfies the equation

$$E = 3 \cdot \frac{5}{8} + 4 \cdot \frac{5}{16} + (E+4) \cdot \frac{1}{16},$$

and so we calculate that $E = 3.6$. □

Solution 2. *(solution by Daniel Naylor)*

Claim: A correspondence naturally induces a family $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$ with each S_i being a multi-set of positive integers and having $\sum_{x \in S_i} 2^{-x} = \frac{1}{5}$, and moreover every such family induces a correspondence.

The important feature of this claim is the 'moreover' part at the end of it: in other words, what this claim is saying is that the subtle conditions about "no assigned sequence may be the start of any other assigned sequence" are in fact not hard to satisfy once you have found suitable elements for your sets S_i.

Proof. The first part is clear, so we only prove the 'moreover' part. Given such a family $\mathcal{S}$, we will iteratively construct a rooted binary tree, with each leaf being an element of a set S_i. To do this, we start with all the 1s in $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$, then all the 2s, then all the 3s, and so on. When we move on to a new number, we extend all unused leaves of the tree down. This process can continue forever (or as long as needed) precisely because of the assumed condition on the sums of 2^{-x} terms. □

We now want to minimise the expected number of coin flips over all families $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$ as described in the claim. To do this, note that

$$\text{expected number of coin flips} = \sum_{i=1}^{5} \sum_{x \in S_i} \frac{x}{2^x},$$

so it suffices to find an optimal set S_1, as we can then take $S_i = S_1$ for the other i. By 'optimal', we mean a set S_1 that minimises $\sum_{x \in S_1} \frac{x}{2^x}$.

Note that $\sum_{x \in S_1} \frac{x}{2^x}$ equals

$$\sum_{x \in S_1} \frac{x}{2^x} = \sum_{x \in S_1} \sum_{\substack{n \in \mathbb{N} \\ x \geq n}} 2^{-x} = \sum_{n \in \mathbb{N}} \sum_{\substack{x \in S_1 \\ x \geq n}} 2^{-x}. \qquad (*)$$

Now note that

$$\sum_{\substack{x \in S_1 \\ x \geq n}} 2^{-x} = \frac{1}{5} - \sum_{\substack{x \in S_1 \\ x < n}} 2^{-x},$$

and note that for each fixed n, the left hand side achieves its minimum when S_1 is the set induced by the binary expansion of $\frac{1}{5}$. Thus, if we take S_1 to be the set corresponding to the binary expansion of $\frac{1}{5}$, then S_1 pointwise minimises the sum on the right hand side of ($*$), and hence this choice of S_1 minimises the desired quantity.

Calculating the expected number of flips in this case is left as an exercise for the interested reader.

Comment. The same solution works with 5 replaced by any number, i.e. considering binary expansions always gives an optimal strategy. The only part that changes is calculating the expected number of flips using the binary expansion. □