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Primal estimated sub-gradient solver for SVM

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Advanced Topics in Machine Learning

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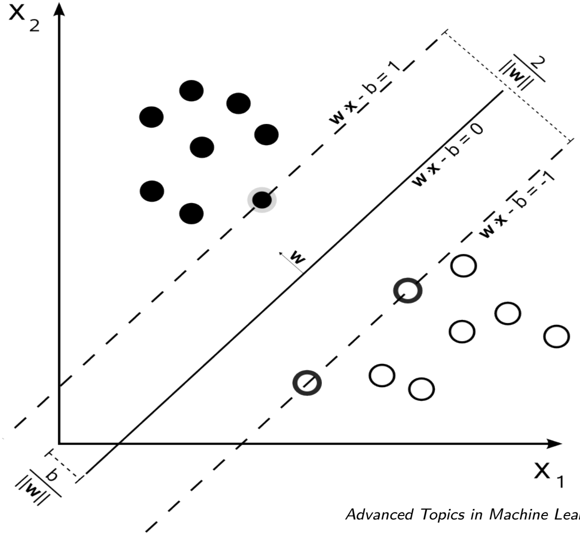
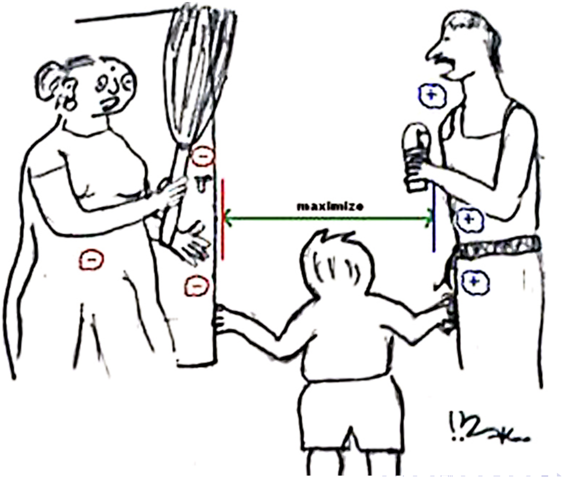
1. Introduction
2. Analysis - faster convergence rates
3. Experiments - outperforms state-of-the-art
4. Reference

Motivating example



Family Support Machine



Support Vector Machine: Primal Problem

Data:

$$\{(\boldsymbol{x}_i, y_i) \in R^d \times \{+1, -1\} : i \in S \stackrel{def}{=} \{1, 2, \ldots, n\}\}$$

- ▷ Example: $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ (assumption: $\max_i \|\boldsymbol{x}_i\|_2 \leq R$)
- ▷ Labels: $y_i \in \{+1, -1\}$

Optimization formulation of SVM:

$$\min_{\boldsymbol{w} \in \mathbb{R}^d} \{f(\boldsymbol{w}) := \hat{L}_S(\boldsymbol{w}) + \frac{\lambda}{2}\|\boldsymbol{w}\|^2\}$$

where

- ▷ $\hat{L}_A(\boldsymbol{w}) \stackrel{def}{=} \frac{1}{|A|}\sum_{i \in A} L_i$ (average loss on examples in A)

Loss Function and Subgradient

Definition

- Loss: $L_i := \ell(\langle \boldsymbol{x}_i, \boldsymbol{w} \rangle, y_i)$
- Subgradient: $l'(\langle \boldsymbol{x}_i, \boldsymbol{w} \rangle, y_i)$ with $\|l'\| \leq X$

Use the notation $z = \langle \boldsymbol{w}, \boldsymbol{x}_i \rangle$, sample loss functions:

Loss function	Subgradient
$l(z, y_i) = \max\{0, 1 - y_i z\}$	$l' = \begin{cases} -y_i x_i & \text{if } y_i z < 1 \\ 0 & \text{otherwise} \end{cases}$
$l(z, y_i) = \log(1 + e^{-y_i z})$	$l' = -\frac{y_i}{1+e^{y_i z}} \boldsymbol{x}_i$
$l(z, y_i) = \max\{0, \lvert y_i - z \rvert - \epsilon\}$	$l' = \begin{cases} \boldsymbol{x}_i & \text{if } z - y_i > \epsilon \\ -\boldsymbol{x}_i & \text{if } y_i - z > \epsilon \\ \boldsymbol{0} & \text{otherwise} \end{cases}$

Previous Work

- Dual-based methods
  - Interior Point
    - Memory: m^2, time: $m^3 log(log(1/\epsilon))$
  - Decomposition
    - Memory: m, time: super-linear in m
- Online learning & Stochastic Gradient
  - Memory: $O(1)$, time: $1/\epsilon^2$ (linear kernel)

Better rates for finite dimensional instances (Murata, Bottou)

Typically, online learning algorithms do not converge to the optimal solution of SVM

Basic Pegasos Algorithm (SGD)

1. Choose $\boldsymbol{w}_1 = 0 \in \mathbb{R}^d$
2. Iterate for $t = 1, 2, \ldots, T$
    1. Choose $A_t \subset S = \{1, 2, \ldots, n\}, |A_t| = b$, uniformly at random
    2. Set stepsize $\eta_t \leftarrow \frac{1}{\lambda t}$
    3. Update $w^{(t+1)} \leftarrow w^{(t)} - \eta_t \partial f_{A_t}(\boldsymbol{w}^{(t)})$

Theorem

For $\overline{\boldsymbol{w}} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{w}_t$, we have:

$$\mathbb{E}[f(\overline{\boldsymbol{w}})] \leq f(w^*) + c \times \frac{1 + \ln(T)}{2\lambda T}$$

where $c = 4R^2$.

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   2. Set stepsize $\eta_t \leftarrow \frac{1}{\lambda t}$
   3. Update $w^{(t+1)} \leftarrow (1 - \eta\lambda) w^{(t)} + \frac{\eta_t}{b} \sum_{i \in A_t} l' \boldsymbol{x}_i$

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Run-Time of Pegasos

- Choosing $|A_t| = 1$

  $\rightarrow$ Run-time required for Pegasos to find ϵ accurate solution

$$\tilde{O}(\frac{1}{\epsilon})$$

- Run-time does not depends on #examples, suited for learning form large datasets
- Previous, depends on "difficulty" of problem (both λ and ϵ)

Analysis from Lacoste

- Classical analysis: $\eta_t = \frac{1}{\lambda t}$
  - $\mathbb{E} f\left(\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{w}^{(t)}\right) - f(\boldsymbol{w}^*) \leq \frac{2X^2R^2}{\lambda T}(\ln T + 1)$
  - For Hinge loss $X = 1$, the result is same as before.
- New analysis: $\eta_t = \frac{2}{\lambda(t+1)}$
  - $\mathbb{E} f(\frac{2}{T(T+1)}\sum_{t=1}^{T} t w^{(t)}) - f(w^*) \leq \frac{8X^2R^2}{\lambda(T+1)}$
  - $\mathbb{E}_{i(T)}\left[\|\boldsymbol{w}^{(T+1)} - \boldsymbol{w}^*\|^2 \| \boldsymbol{w}^t\right] \leq \frac{16X^2R^2}{\lambda^2(T+1)}$
  - In this case, $\overline{w}^{(T)} \doteq \frac{2}{T(T+1)}\sum_{t=1}^{T} t\boldsymbol{w}^{(t)}$

Experiments

- 3 datasets (provided by Joachims)
  - Reuters CCAT (800K examples, 47k features)
  - Covertype (581k examples, 54 features)
  - Physics ArXiv (62k examples, 100k features)
- 4 competing algorithms
  - SVM-Perf (Joachims'06)
  - SVM-light (Joachims)
  - Norma (Kivinen, Smola, Williamson '02)
  - Zhang'04 (stochastic gradient descent)

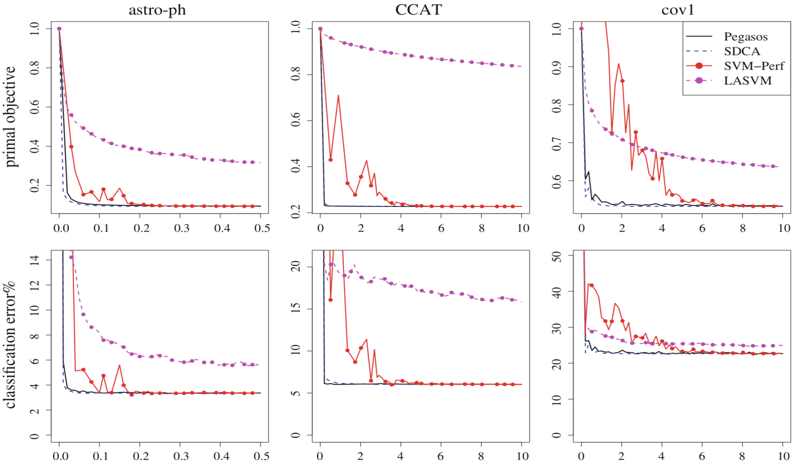
Linear kernels

Dataset	Training	Testing	Features	Sparsity(%)	λ
astro-ph	29,882	32,487	99,757	0.08	5×10^{-5}
CCAT	781,265	23,149	47,236	0.16	10^{-4}
cov1	522,911	58,101	54	22.22	10^{-6}

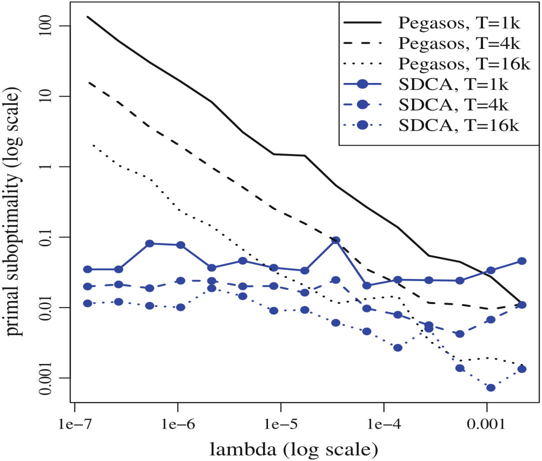
Linear kernels

Dataset	Pegasos	SDCA	SVM-Perf	LASVM
astro-ph	0.04s(3.56%)	0.03s(3.49%)	0.1s(3.39%)	54s(3.65%)
CCAT	0.16s(6.16%)	0.36s(6.57%)	3.6s(5.93%)	>18000 s
cov1	0.32s(23.2%)	0.20s(22.9%)	4.2s(23.9%)	210s(23.8%)

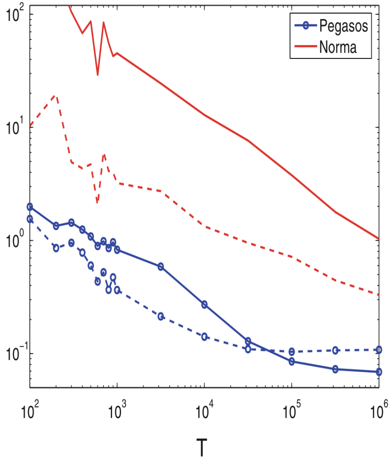
Comparison of linear SVM optimizers

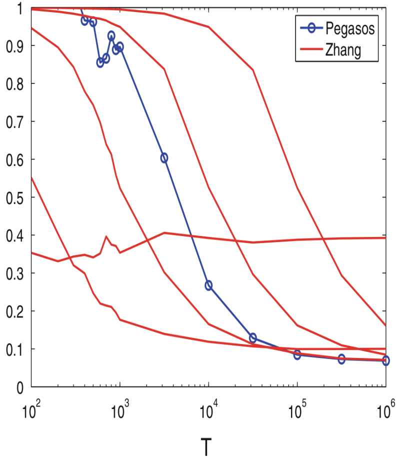


Effect of regularization parameter λ

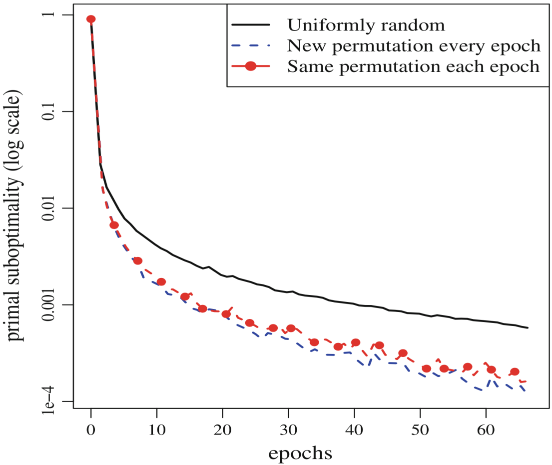


Experiments with the mini-batch variant

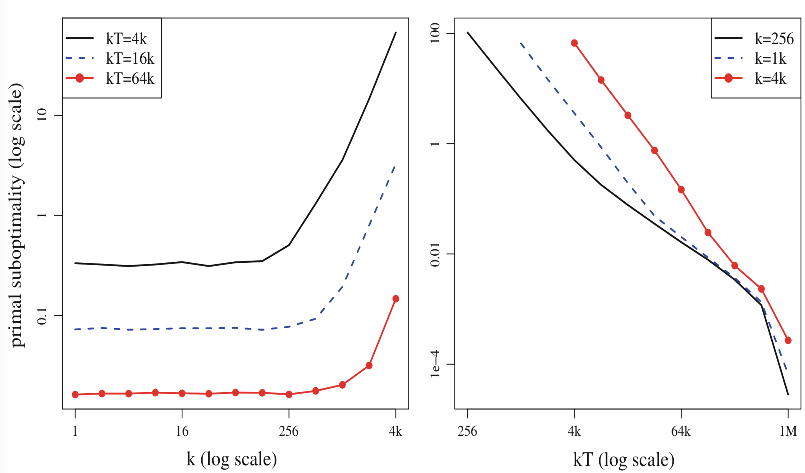




Comparison of sampling procedures



Compare to Norma and Zhang (on Physics)



Bias term

1. Popular approach: increase dimension of x
   Cons: "pay" for b in the regularization term
2. Define: $L(\boldsymbol{w}) = \min_b \sum_{(\boldsymbol{x},y)\in S} [1 - y(\langle \boldsymbol{w}, \boldsymbol{x}\rangle - b)]_+$
3. Rewrite problem: $\min_{\boldsymbol{w}} \frac{\lambda}{2}\|\boldsymbol{w}\|^2 + g(\boldsymbol{w}; S)$ where
   $g(\boldsymbol{w}; S) = \min_b \frac{1}{m} \sum_{(\boldsymbol{x},y)\in S} [1 - y(\langle \boldsymbol{w}, \boldsymbol{x}\rangle + b)]_+$
   Calculate subgradients w.r.t w and w.r.t b.
4. Search b in an outer loop
   Cons: evaluation time remain same as unbiased

Discussion

- Pegasos: Simple & Efficient solver for SVM
- Sample vs. computational complexity
  - Sample complexity: How many examples do we need as a function of VC-dim(λ), accuracy(ϵ), and confidence(δ)
  - in Pegasos, we aim at analyzing computational complexity based on λ, ϵ, δ (also in Bottou & Bousquet)
- Finding argmin vs. calculating min: It seems that Pegasos finds the argmin more easily than it requires to calculate the min value

Q&A

Thank You!

Acknowledgement:
Thanks to Martin for helpful discussions, suggestions and chips!!!

Reference

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