Primal estimated sub-gradient solver for SVM

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Advanced Topics in Machine Learning

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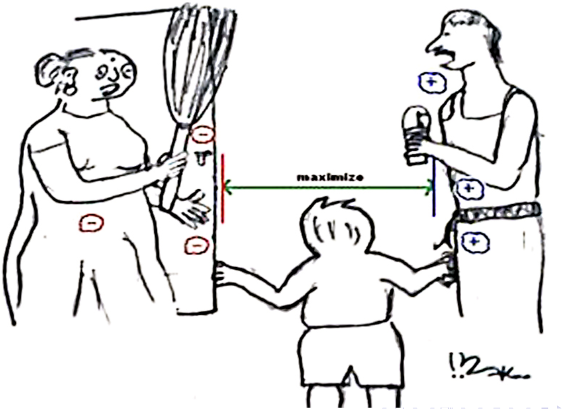
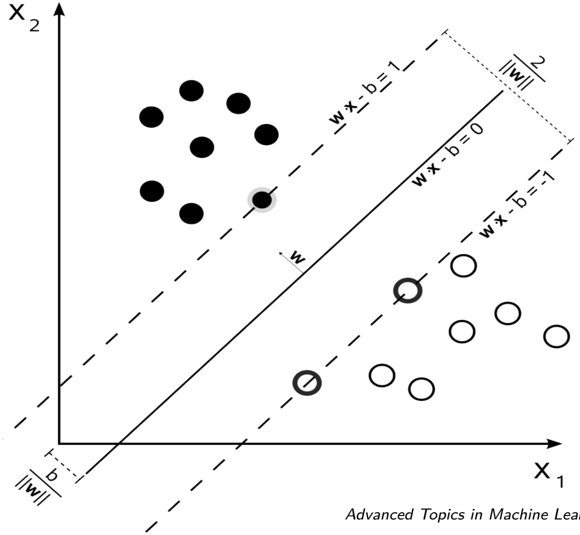
Family Support Machine



From Peter Richtárik's slides

Motivating example



Support Vector Machine: Primal Problem

Data:

$$\{(\boldsymbol{x}_i, y_i) \in \mathbb{R}^d \times \{+1, -1\} : i \in S \stackrel{def}{=} \{1, 2, \ldots, n\}\}$$

- ▷ Example: $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ (assumption: $\max_i \|\boldsymbol{x}_i\|_2 \leq R$)
- ▷ Labels: $y_i \in \{+1, -1\}$

Optimization formulation of SVM:

$$\min_{\boldsymbol{w} \in \mathbb{R}^d} \{f(\boldsymbol{w}) := \hat{L}_S(\boldsymbol{w}) + \frac{\lambda}{2}\|\boldsymbol{w}\|^2\}$$

where

- ▷ $\hat{L}_A(\boldsymbol{w}) \stackrel{def}{=} \frac{1}{|A|}\sum_{i \in A} L_i$ (average loss on examples in A)

Loss Function and Subgradient

Definition

- Loss: $L_i := \ell(\langle \boldsymbol{x}_i, \boldsymbol{w} \rangle, y_i)$
- Subgradient: $l'(\langle \boldsymbol{x}_i, \boldsymbol{w} \rangle, y_i)$ with the assumption of $\|l'\| \leq \mathbb{L}$

Use the notation $z = \langle \boldsymbol{w}, \boldsymbol{x}_i \rangle$, sample loss functions:

Loss function	Subgradient
$l(z, y_i) = \max\{0, 1 - y_i z\}$	$l' = \begin{Bmatrix} -y_i \boldsymbol{x}_i & \text{if } y_i z < 1 \\ 0 & \text{otherwise} \end{Bmatrix}$
$l(z, y_i) = \log(1 + e^{-y_i z})$	$l' = -\frac{y_i}{1+e^{y_i z}} \boldsymbol{x}_i$
$l(z, y_i) = \max\{0, \lvert y_i - z \rvert - \epsilon\}$	$l' = \begin{Bmatrix} \boldsymbol{x}_i & \text{if } z - y_i > \epsilon \\ -\boldsymbol{x}_i & \text{if } y_i - z > \epsilon \\ \boldsymbol{0} & \text{otherwise} \end{Bmatrix}$

Previous Work

- Dual-based methods
  - Interior Point
    - Memory: n^2, time: $n^3 \log(\log(1/\epsilon))$, run time per iteration n^3
  - Decomposition
    - Memory: n, time complexity: super-linear in n, slow convergence
- Online learning & Stochastic Gradient
  - Memory: $O(1)$, time: $1/\epsilon^2$ (linear kernel), run-time per iteration: O(d)

Better rates for finite dimensional instances (Murata, Bottou)

Typically, online learning algorithms do not converge to the optimal solution of SVM

Basic Pegasos Algorithm (SGD)

1. Choose $\boldsymbol{w}_1 = 0 \in \mathbb{R}^d$
2. Iterate for $t = 1, 2, \ldots, T$
   1. Choose $A_t \subset S = \{1, 2, \ldots, n\}, |A_t| = k$, uniformly at random
   2. Set stepsize $\eta_t \leftarrow \frac{1}{\lambda t}$
   3. Update $w^{(t+1)} \leftarrow w^{(t)} - \eta_t \partial f_{A_t}(\boldsymbol{w}^{(t)})$

Theorem

For $\overline{\boldsymbol{w}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{w}_t$, we have:

$$\mathbb{E}[f(\overline{\boldsymbol{w}})] \leq f(w^*) + c \cdot \frac{1 + \ln(T)}{2\lambda T}$$

where $c = 4R^2$.

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   3. Update $w^{(t+1)} \leftarrow (1 - \eta_t \lambda) w^{(t)} + \frac{\eta_t}{k} \sum_{i \in A_t} l' \boldsymbol{x}_i$

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Run-Time of Pegasos

- Choosing $|A_t| = 1$
  $\rightarrow$ Run-time required for Pegasos to find ϵ accurate solution

$$\tilde{O}(\frac{d}{\lambda\epsilon})$$

- Run-time does not depends on #examples, suited for learning form large datasets
- Depends on "difficulty" of problem (both λ and ϵ)

How to achieve this?

$$\|\boldsymbol{w}^{(t+1)} - \boldsymbol{w}^*\|^2 = \|\boldsymbol{w}^{(t)} - \eta_t \boldsymbol{\chi}_i^{(t)} - \boldsymbol{w}^*\|^2$$
$$= \|\boldsymbol{w}^{(t)} - \boldsymbol{w}^*\|^2 + \eta_t^2 \|\boldsymbol{\chi}_i^{(t)}\|^2 - 2\eta_t \boldsymbol{\chi}_i^{(t)}(\boldsymbol{w}^{(t)} - \boldsymbol{w}^*)$$

Taking the expectation on both sides over the recent step

$$\mathbb{E}_{i^{(t)}}[\|\boldsymbol{w}^{(t+1)} - \boldsymbol{w}^*\|^2 \mid \boldsymbol{w}^t]$$
$$= \|\boldsymbol{w}^{(t)} - \boldsymbol{w}^*\|^2 + \eta_t^2 \mathbb{E}_{i^{(t)}} \|\boldsymbol{\chi}_i^{(t)}\|^2 - 2\eta_t \boldsymbol{\chi}_i^{(t)}(\boldsymbol{w}^{(t)} - \boldsymbol{w}^*)$$
$$\leq \|\boldsymbol{w}^{(t)} - \boldsymbol{w}^*\|^2 + \eta_t^2 \mathbb{E}_{i^{(t)}} \|\boldsymbol{\chi}_i^{(t)}\|^2 - 2\eta_t [f(\boldsymbol{w}^{(t)}) - f(\boldsymbol{w}^*) + \frac{\lambda}{2}\|\boldsymbol{w}^{(t)} - \boldsymbol{w}^*\|^2].$$

Re-arranging and taking expectation over the whole process again

$$\mathbb{E} f(\boldsymbol{w}^{(t)}) - f(\boldsymbol{w}^*) \leq \frac{\eta_t}{2} \mathbb{E}_{i^{(t)}} \|\boldsymbol{\chi}_i^{(t)}\|^2 + \frac{1 - \lambda\eta_t}{2\eta_t} \mathbb{E}\|\boldsymbol{w}^{(t)} - \boldsymbol{w}^*\|^2$$
$$- \frac{1}{2\eta_t} \mathbb{E}\left[\|\boldsymbol{w}^{(t+1)} - \boldsymbol{w}^*\|^2 \mid \boldsymbol{w}^t\right]$$

Lemma

With the Lipschitz assumption on $l(y, u)$ that $\|l'(y, u)\| \leq \mathbb{L}$, and assuming that $\|\boldsymbol{x}_i\| \leq R \;\forall i$ where i is picked according to p_i, it holds that

$$\mathbb{E}_{i^{(t)}} \|\boldsymbol{\chi}_i^{(t)}\| \leq 4\mathbb{L}^2 R^2$$

where $l'(y, u)$ denotes any subgradient with respect to the second variable.

How to prove it?

Minkowski inequality

$$\sqrt{\mathbb{E}(X+Y)^2} \leq \sqrt{\mathbb{E}X^2} + \sqrt{\mathbb{E}Y^2}$$

$$\chi_i^{(t)}(\boldsymbol{w}^{(t)}) = l'\boldsymbol{x}_i + \lambda \boldsymbol{w}^{(t)}, \boldsymbol{w}^{(t+1)} = \boldsymbol{w}^{(t)} - \eta_t \chi_i^{(t)}$$

$$\sqrt{\mathbb{E}_{i^{(t)}} \|\chi_i^{(t)}\|^2} \leq \sqrt{\mathbb{E}_{i^{(t)}} \|l'\boldsymbol{x}_i\|^2} + \lambda \sqrt{\mathbb{E}_{i^{(1:t-1)}} \|\boldsymbol{w}^{(t)}\|^2}$$

$$\sqrt{\mathbb{E}_{i^{(1:t)}} \|\boldsymbol{w}^{(t+1)}\|^2} \leq (1-\lambda\eta_t)\sqrt{\mathbb{E}_{i^{(1:t-1)}} \|\boldsymbol{w}^{(t)}\|^2} + \eta_t \sqrt{\mathbb{E}_{i(t)} \|l'\boldsymbol{x}_i\|^2}$$

Why we don't need projection?

Analysis from Lacoste-Julien et.al.[2]

- Classical analysis: $\eta_t = \frac{1}{\lambda t}$
  - $\mathbb{E} f\left(\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{w}^{(t)}\right) - f(\boldsymbol{w}^*) \leq \frac{2\mathbb{L}^2 R^2}{\lambda T}(\ln T + 1)$
  - For Hinge loss $\mathbb{L} = 1$, the result is same as before.
- New analysis: $\eta_t = \frac{2}{\lambda(t+1)}$
  - $\mathbb{E} f(\frac{2}{T(T+1)}\sum_{t=1}^{T} t w^{(t)}) - f(w^*) \leq \frac{8\mathbb{L}^2 R^2}{\lambda(T+1)}$
  - $\mathbb{E}_{i(T)}\left[\|\boldsymbol{w}^{(T+1)} - \boldsymbol{w}^*\|^2 \|\boldsymbol{w}^t\right] \leq \frac{16\mathbb{L}^2 R^2}{\lambda^2(T+1)}$
  - In this case, $\overline{w}^{(T)} \doteq \frac{2}{T(T+1)}\sum_{t=1}^{T} t\boldsymbol{w}^{(t)}$

Stochastic Dual Coordinate Ascent

- Dual problem

$$\max_{\boldsymbol{\alpha} \in \mathbb{R}^n, 0 \leq \boldsymbol{\alpha}_i \leq 1} D(\boldsymbol{\alpha}) := \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\alpha}_i - \frac{1}{2\lambda n^2} \sum_{i=1}^{n} \boldsymbol{\alpha}^T \boldsymbol{Q} \boldsymbol{\alpha}.$$

  where

$$\boldsymbol{Q} \in \mathbb{R}^{n \times n}, \boldsymbol{Q}_{i,j} = y_i y_j \langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle,$$

- Relationship with primal variable:

$$\boldsymbol{w}(\boldsymbol{\alpha}) := \frac{1}{\lambda n} \sum_{i=1}^{n} \boldsymbol{\alpha}_i y_i \boldsymbol{x}_i.$$

- Traditional SDCA

$$\boldsymbol{\alpha}_i^{(t+1)} = \boldsymbol{\alpha}_i^{(t)} + \boldsymbol{\delta}_i^{(t)} e_i,$$

Experiments

- 3 datasets (provided by Joachims)
  - Reuters CCAT (800K examples, 47k features)
  - Covertype (581k examples, 54 features)
  - Physics ArXiv (62k examples, 100k features)
- 4 competing algorithms
  - SVM-Perf (Joachims'06)
  - SVM-light (Joachims)
  - Norma (Kivinen, Smola, Williamson '02)
  - Zhang'04 (stochastic gradient descent)

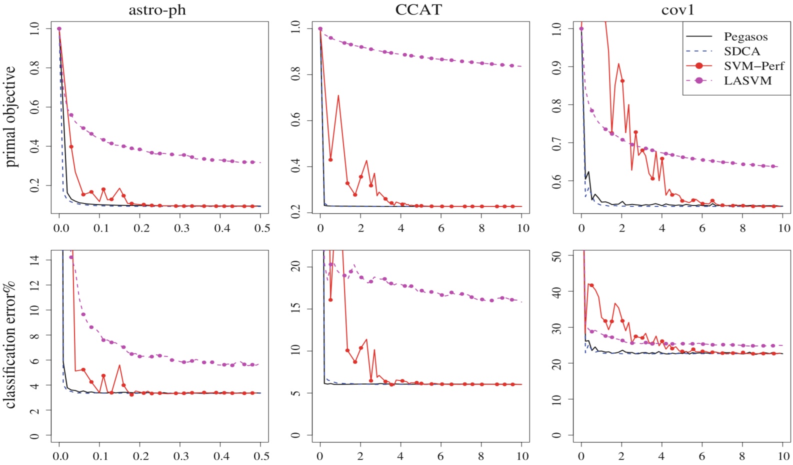
Dataset characteristics

Dataset	Training	Testing	Features	Sparsity(%)	λ
astro-ph	29,882	32,487	99,757	0.08	5×10^{-5}
CCAT	781,265	23,149	47,236	0.16	10^{-4}
cov1	522,911	58,101	54	22.22	10^{-6}

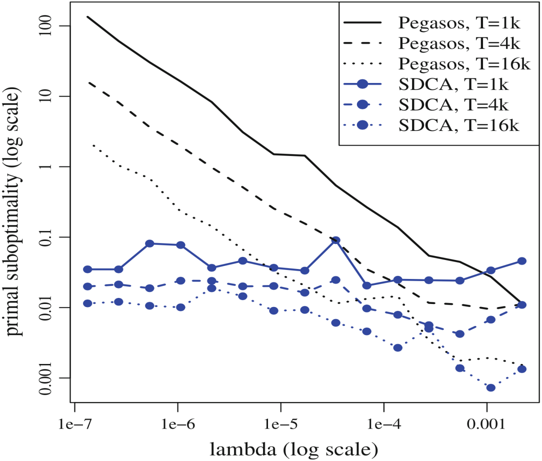
Training runtime and test error

Dataset	Pegasos	SDCA	SVM-Perf	LASVM
astro-ph	0.04s(3.56%)	0.03s(3.49%)	0.1s(3.39%)	54s(3.65%)
CCAT	0.16s(6.16%)	0.36s(6.57%)	3.6s(5.93%)	$>$18000 s
cov1	0.32s(23.2%)	0.20s(22.9%)	4.2s(23.9%)	210s(23.8%)

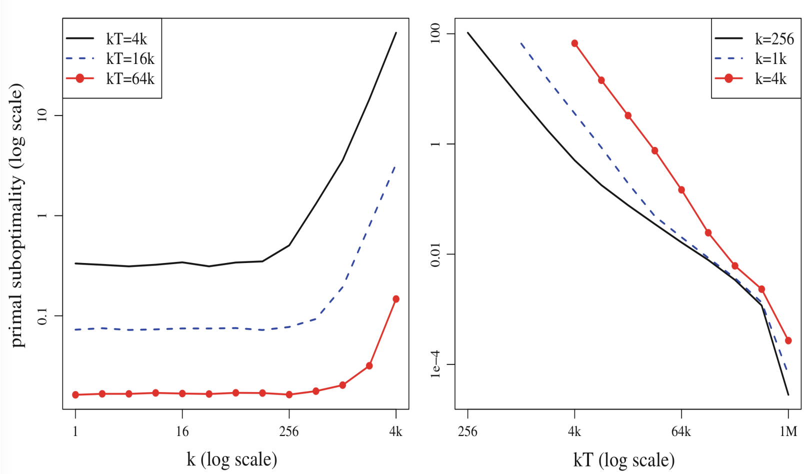
Comparison of linear SVM optimizers



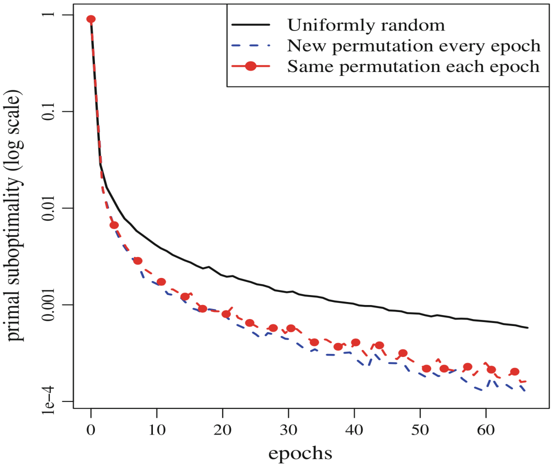
Effect of regularization parameter λ



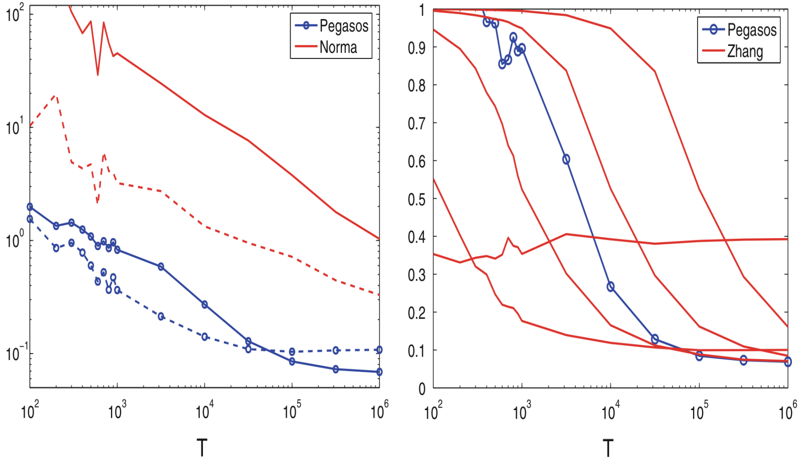
Experiments with the mini-batch variant



Comparison of sampling procedures



Compare to Norma and Zhang (on Physics)



Kernels

The basic Pegasos algorithm can easily be implemented using only kernel evaluations.

- For each t let $\alpha_{t+1} \in R^n$ be the vector such that $\alpha_{t+1}[j]$ counts how many times example j has been selected so far and we had a non-zero loss on it, namely, $\alpha_{t+1}[j] = \ |t' \leq t : i_{t'} = j \ \wedge \ y_j \langle \boldsymbol{w}_{t'}, \phi(\boldsymbol{x}_j) \rangle < 1|.$
- Represent $\boldsymbol{w}_{t+1} = \frac{1}{\lambda t} \sum_{j=1}^{m} \alpha_{t+1}[j] y_j \phi(\boldsymbol{x}_j)$
- Cons: overall runtime $\tilde{O}(nd/(\lambda\epsilon))$

Summary

- Pegasos: Simple & Efficient solver for SVM
- Faster convergence rate
  - Choose of different step size
- Extension
  - Bound on ∇f is enough
  - Non-uniform sampling

Reference

[1] Shalev-Shwartz, Shai, et al. "Pegasos: Primal estimated sub-gradient solver for svm." Mathematical programming 127.1 (2011): 3-30.

[2] Lacoste-Julien, Simon, Mark Schmidt, and Francis Bach. "A simpler approach to obtaining an o $(1/t)$ convergence rate for the projected stochastic subgradient method." arXiv preprint arXiv:1212.2002 (2012).

[3] Takáč Martin, et al. "Mini-batch primal and dual methods for SVMs." arXiv preprint arXiv:1303.2314 (2013).

[4] Zhao, Peilin, and Tong Zhang. "Stochastic optimization with importance sampling." arXiv preprint arXiv:1401.2753 (2014).

Q&A

Thank You!

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