ETH zürich

Adaptive Probabilities in Stochastic Optimization Algorithms

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Problem

Empirical Risk Minimization

$$\min_{\boldsymbol{w} \in \mathbb{R}^n} f(\boldsymbol{w})$$

$$f(\boldsymbol{w}) := \ell(\boldsymbol{w}) + \lambda r(\boldsymbol{w}) \tag{1}$$

where

$$\ell(\boldsymbol{w}) := \frac{1}{n} \sum_{i=1}^{n} \ell(\langle \boldsymbol{x}_i, \boldsymbol{w} \rangle, y_i).$$

and

$$r(\boldsymbol{w}) := \frac{1}{2} \|\boldsymbol{w}\|_2^2$$

Here, $\ell(., y_i) : \mathbb{R} \to \mathbb{R}$ is a loss function and $r(.)$ takes the role of a regularizer.

Problem cont.

Partial Objective Function

$$f(\boldsymbol{w}, i) = \ell(\langle \boldsymbol{x}_i, \boldsymbol{w} \rangle, y_i) + \lambda r(\boldsymbol{w}).$$

- $\boldsymbol{x}_i$: feature vector of sample i
- y_i: label of sample i
- $\boldsymbol{w}$: solution of objective function
- η: stepsize for updating $\boldsymbol{w}$
- χ_i: subgradient of $f(\boldsymbol{w}, i)$

Part A

Non-Uniform Sampling Algorithms

Non-Uniform

Define p_i as the probability that sample i will be selected with $\sum_{j=1}^{n} p_j = 1$. We use $\boldsymbol{g}_i$ (the weighted subgradient with $\boldsymbol{g}_i = \frac{\chi_i}{np_i}$).

$$\mathbb{E}[\boldsymbol{g}(\boldsymbol{w})] = \sum_{i=1}^{n} \frac{\chi_i}{n} = \nabla f(\boldsymbol{w})$$

NonUnifSGD

Algorithm 1: Non-Uniform Stochastic Gradient Discent

Input: $\lambda > 0$, $p_i = \frac{\|\mathbf{x}_i\|}{\sum_{j=1}^{n} \|\mathbf{x}_j\|}, \forall i \in \{1, \ldots, n\}$.

Data: $\{(\mathbf{x}_i, y_i)\}_{i=1}^{n}$.

Initialize: $\mathbf{w}^1 = \mathbf{0}$.

for $t = 1, 2, \ldots, T$

  Sample i_t from $\{1, \ldots, n\}$ based on $\mathbf{p}$;

  Set stepsize $\eta_t \leftarrow \frac{1}{\lambda t}$;

  Set $\chi_{i_t}^t(\mathbf{w}^t) \leftarrow \ell'(\langle \mathbf{w}^t, \mathbf{x}_{i_t} \rangle, y_{i_t}) \mathbf{x}_{i_t} + \lambda \nabla r(\mathbf{w}^t)$;

  Set $\mathbf{g}_{i_t}^t \leftarrow \frac{\chi_{i_t}^t(\mathbf{w}^t)}{n p_{i_t}}$;

  Set $\mathbf{w}^{t+1} \leftarrow \mathbf{w}^t - \eta_t \mathbf{g}_{i_t}^t$;

end

Output: $\mathbf{w}^{T+1}$

Convergence Theorem

Theorem

Suppose f is a λ-strongly convex function. If we choose the stepsize $\eta_t = \frac{1}{\lambda t}$, then after T iterations of NonUnifSGD (Algorithm 1) starting with $\mathbf{w}^1 = \mathbf{0}$, it holds that

$$\mathbb{E}\left[f\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{w}^t\right)\right] - f(\mathbf{w}^*) \leq \frac{1}{2\lambda T}\sum_{t=1}^{T}\frac{\mathbb{E}[\|\mathbf{g}_{i_t}^t\|^2]}{t}$$

where $\mathbf{g}_{i_t}^t = \frac{\chi_{i_t}^t(\mathbf{w}^t)}{np_{i_t}}$ and the expectation is taken with respect to the distribution $\mathbf{p}$.

Two corollaries

Definition

Define $G := \max_{i,t}\{\|\chi_i^t(\mathbf{w}^t)\|^2\}$ $(i = 1 \ldots n,\ t = 1 \ldots T)$.
Define $W := \max_{i,t}\{\mathbb{E}[\|\chi_i^t(\mathbf{w}^t)\|^2]\}$ $(i = 1 \ldots n,\ t = 1 \ldots T)$.

Corollary

Assume that $\max_t\{\|\chi_{i_t}^t(\mathbf{w}^t)\|^2\} \leq G$ *or* $\mathbb{E}[\|\chi_{i_t}^t(\mathbf{w}^t)\|^2] \leq W$ *for all* t *and* $p_i > \epsilon$ *for all* $i = \{1 \ldots, n\}$,

$$\mathbb{E}\left[f\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{w}^t\right)\right] - f(\mathbf{w}^*) \leq \frac{1}{2\lambda T}\sum_{t=1}^{T}\frac{G}{\epsilon n t} \leq \frac{G(\ln T + 1)}{2\lambda\epsilon n T} \text{ or}$$

$$\mathbb{E}\left[f\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{w}^t\right)\right] - f(\mathbf{w}^*) \leq \frac{1}{2\lambda T}\sum_{t=1}^{T}\frac{W}{n^2\epsilon^2 t} \leq \frac{W(\ln T + 1)}{2\lambda T n^2\epsilon^2}$$

Another Theorem for SGD

Theorem

Suppose f is a λ-strongly convex function. If we choose the stepsize $\eta_t = \frac{2}{\lambda(t+1)}$, then after T iterations of NonUnifSGD (Algorithm 1) with starting point $\boldsymbol{w}^1 = \mathbf{0}$, it holds that the weighted average of the iterates satisfies

$$\mathbb{E}\Big[f\Big(\frac{2}{T(T+1)}\sum_{t=1}^{T} t\boldsymbol{w}^t\Big)\Big] - f(\boldsymbol{w}^*) \leq \frac{2}{\lambda(T+1)} \max_t \mathbb{E}[\|\boldsymbol{g}_{i_t}^t\|^2]$$

where $\boldsymbol{g}_{i_t}^t = \frac{\chi_{i_t}^t(\boldsymbol{w}^t)}{np_{i_t}}$, and the expectation is taken with respect to the distribution $\boldsymbol{p}$.

Dual Problem

Dual Objective Function

$$\max_{\boldsymbol{\alpha}\in\mathbb{R}^n} D(\boldsymbol{\alpha}) := \frac{1}{n}\sum_{i=1}^{n} -\ell_i^*(-\alpha_i) - \lambda r^*(\boldsymbol{v}(\boldsymbol{\alpha})).$$

The relationship between primal variable $\boldsymbol{w}$ and dual variable $\boldsymbol{\alpha}$ is

$$\boldsymbol{w}(\boldsymbol{\alpha}) := \nabla r^*(\boldsymbol{v}(\boldsymbol{\alpha})), \boldsymbol{v}(\boldsymbol{\alpha}) := \frac{1}{\lambda n}\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$.

NonUnifSDCA

Algorithm 2: Non-Uniform Stochastic Dual Coordinate Ascent

Input: $\lambda > 0$, $p_i = \frac{\|\boldsymbol{x}_i\|}{\sum_{j=1}^{n} \|\boldsymbol{x}_j\|}$, $\forall i \in \{1, \dots, n\}$.

Data: $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^{n}$

Initialize: $\alpha^1 = \mathbf{0}$, $\boldsymbol{w}^1 = \mathbf{0}$.

for $t = 1, 2, \dots, T$

- Sample i_t from $\{1, \dots, n\}$ based on $\boldsymbol{p}$;
- Calculate $\Delta\alpha_{i_t}^t =$ $\arg\max_{\Delta\alpha_{i_t}^t} [-\frac{\lambda n}{2} \|\boldsymbol{w}^t + \frac{1}{\lambda n} \Delta\alpha_{i_t}^t \boldsymbol{x}_{i_t}\|^2 - \ell_{i_t}^*(-(\alpha_{i_t}^t + \Delta\alpha_{i_t}^t))]$;
- Set $\alpha_{i_t}^{t+1} \leftarrow \alpha_{i_t}^t + \Delta\alpha_{i_t}^t$;
- Set $\boldsymbol{w}^{t+1} \leftarrow \boldsymbol{w}^t + \frac{1}{\lambda n} \Delta\alpha_{i_t}^t \boldsymbol{x}_{i_t}$;

end

Output: $\boldsymbol{w}^{T+1}$

Part B

Adaptive Sampling Algorithms

Idea behind SGD

According to the SGD theorem, we can reduce the convergence rate by solving the following optimization problem:

$$\min \mathbb{E}[\|\boldsymbol{g}_{i_t}^t\|^2].$$

By the Cauchy-Schwarz inequality and the fact that $\sum_{i=1}^n p_i = 1$,

$$\mathbb{E}[\|\boldsymbol{g}_{i_t}^t\|^2] = \sum_{i=1}^n \frac{\|\chi_i\|^2}{n^2 p_i} = (\sum_{i=1}^n \frac{\|\chi_i\|^2}{n^2 p_i})(\sum_{i=1}^n p_i) \geq (\sum_{i=1}^n \frac{\|\chi_i\|}{n})^2.$$

The above inequality holds when

$$p_i = \frac{\|\chi_i\|}{\sum_{j=1}^n \|\chi_i\|}.$$

AdaSGD

Algorithm 5: AdaSGD (Adaptive Non-Uniform Stochastic Gradient Discent)

Input: $\lambda > 0$
Data: $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$
Initialize: $\boldsymbol{w}^0 = \boldsymbol{0}$, probabilities

$$p_i = \frac{\|\boldsymbol{x}_i\|^2 + \sqrt{\lambda}}{\sum_{j=1}^n \|\boldsymbol{x}_j\|^2 + \sqrt{\lambda}}, c_i = 0, \forall i \in \{1, \dots, n\}.$$

for $t = 1, 2, \dots, T$
  Sample i_t from $\{1, \dots, n\}$ based on $\boldsymbol{p}$;
  Set $\eta_t \leftarrow$ …
  Calculate …
$c_i \leftarrow \max\{c_i, \|\chi_i\|\};$
  Set $\chi_{i_t}^t(\boldsymbol{w}^t) \leftarrow \ell' \boldsymbol{x}_{i_t} + \lambda \nabla r(\boldsymbol{w}^t)$;
  if *(t-1) mod n* $\geq$ *n-k* **then**
    for $i = 1, 2, \dots, n$
      Calculate $\ell'(\langle \boldsymbol{x}_i, \boldsymbol{w}^t \rangle, y_i)$;
      Set $\chi_i \leftarrow \ell'(\langle \boldsymbol{x}_i, \boldsymbol{w}^t \rangle, y_i)\boldsymbol{x}_i + \lambda \nabla r(\boldsymbol{w}^t)$;
      Set $c_i \leftarrow \max\{c_i, \|\chi_i\|\}$;
    end
  end
  if *t mod n* = 0 **then**
    Option I: Run Algorithm 3 (Aggressive Update);
    Option II: Run Algorithm 4 (Conservative Update);
  end
  Set $\boldsymbol{g}_{i_t}^t \leftarrow \frac{\chi_{i_t}^t(\boldsymbol{w}^t)}{n p_{i_t}}$;
  Set $\boldsymbol{w}^{t+1} \leftarrow \boldsymbol{w}^t - \eta_t \boldsymbol{g}_{i_t}^t$;
end
Output: $\boldsymbol{w}^{T+1}$

Two Updates

Algorithm 3: Aggressive Probability Update

for $j = 1, \ldots, n$
  Set $p_j \leftarrow \frac{c_j}{\sum_{k=1}^{n} c_k}$;
end

Algorithm 4: Conservative Probability Update

for $j = 1, \ldots, n$
  Set $p_j \leftarrow \frac{\tilde{c}_j}{\sum_{k=1}^{n} \tilde{c}_k}$;
end

For all $i \in \{1, \ldots, n\}$, $\tilde{c}_i = \max\{1, c_i\}$.

AdaSVRG

We add a $\tilde{\boldsymbol{w}}$ (which denotes the $\boldsymbol{w}$ of last epoch) for a new update equation. Therefore, we get

$$\boldsymbol{w}^{t+1} := \boldsymbol{w}^{t} - \eta_t[\boldsymbol{g}_{i_t}^{t}(\boldsymbol{w}^{t}) - \boldsymbol{g}_{i_t}^{t}(\tilde{\boldsymbol{w}}) - \nabla f(\tilde{\boldsymbol{w}})]$$

The expectation of the update function is still the same as before, because

$$\mathbb{E}[\boldsymbol{g}(\boldsymbol{w}) - \boldsymbol{g}(\tilde{\boldsymbol{w}}) + \nabla f(\tilde{\boldsymbol{w}})] = \mathbb{E}[\boldsymbol{g}(\boldsymbol{w})] - \mathbb{E}[\boldsymbol{g}(\tilde{\boldsymbol{w}})] + \nabla f(\tilde{\boldsymbol{w}}) = \nabla f(\boldsymbol{w}).$$

Idea behind SDCA

Definition

Define the gap of point i as

$$\sigma_i^t = \ell(\boldsymbol{x}_i^\top \boldsymbol{w}^t) + \ell^*(-\alpha_i^t) + \alpha_i^t \boldsymbol{x}_i^\top \boldsymbol{w}^t$$

where $\boldsymbol{w}^t$ here is assumed to be the corresponding primal vector for the current $\boldsymbol{\alpha}^t$, that is $\boldsymbol{w}^t(\boldsymbol{\alpha}^t) := \frac{1}{\lambda n} \sum_{i=1}^n \alpha_i \boldsymbol{x}_i^t$.

The **duality gap** between the primal objective and dual objective at the t-th iteration is defined as

$$f(\boldsymbol{w}^t) - D(\boldsymbol{\alpha}^t) = \frac{1}{n} \sum_{i=1}^n \sigma_i^t.$$

AdaSDCA (Duality Gap)

Algorithm 7: AdaSDCA (Adaptive Non-uniform Stochastic Dual Coordinate Ascent)

Input: $\lambda > 0$
Data: $\{(x_i, y_i)\}_{i=1}^n$
Initialize: $\alpha^1 = \mathbf{0}$, $w^1 = \mathbf{0}$, probabilities $p_i = \frac{1+\frac{1}{\lambda n \gamma_i}}{n+\sum_{j=1}^n \frac{1}{\lambda n \gamma_j}}$ or $p_i = \frac{\|x_i\|}{\sum_{j=1}^n \|x_j\|}$, $c_i = 0$, $\forall i \in \{1, \dots, n\}$.
for $t = 1, 2, \dots, T$
  Sample i_t from $\{1, \dots, n\}$ based on p;
  Calculate $\Delta\alpha^t_{i_t}$ using following formulas:
  $\Delta\alpha^t_{i_t} = \arg m$ … $)]$;
  Set $\alpha^{t+1}_{i_t} \leftarrow \alpha^t_{i_t} + \Delta\alpha^t_{i_t}$;
  Set $w^{t+1} \leftarrow w^t + \frac{1}{\lambda n}\Delta\alpha^t_{i_t} x_{i_t}$;
  if *(t-1) mod n* $\geq$ *n-k* **then**
    for $i = 1, 2, \dots, n$
      Calculate $\sigma^t_i \leftarrow \ell(x_i^\top w^t) + l^*(-\alpha^t_i) + \alpha^t_i \langle x_i, w^t \rangle$;
      Set $c_i \leftarrow \max\{c_i, \sigma^t_i\}$;
    end
  end
  if *t mod n* $= 0$ **then**
    Option I: Run Algorithm 3 (Aggressive Update);
    Option II: Run Algorithm 4 (Conservative Update);
  end
end
Output: w^{T+1}

Set $c_i \leftarrow \max\{c_i, \sigma^t_i\}$;

AdaSDCAS (Subgradient)

Algorithm 8: AdaSDCAS (Adaptive Non-uniform Stochastic Dual Coordinate Ascent by Subgradient)

Input: $\lambda > 0$
Data: $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$
Initialize: $\boldsymbol{\alpha}^1 = \mathbf{0}$, $\boldsymbol{w}^1 = \mathbf{0}$, probabilities $p_i = \frac{1+\frac{1}{\lambda n \gamma_i}}{n+\sum_{j=1}^n \frac{1}{\lambda n \gamma_j}}$ or

$p_i = \frac{\|\boldsymbol{x}_i\|}{\sum_{j=1}^n \|\boldsymbol{x}_j\|}$, $c_i = 0, \forall i \in \{1, \ldots, n\}$.

for $t = 1, 2, \ldots, T$
  Sample i_t from $\{1, \ldots, n\}$ based on $\boldsymbol{p}$;
  Calculate $\Delta\alpha_{i_t}^t$ using following formulas:
  $\Delta\alpha_{i_t}^t = \arg\mathrm{m}$ [illegible] $))]$;
  Set $\alpha_{i_t}^{t+1} \leftarrow \alpha$ [illegible]
  Set $\boldsymbol{w}^{t+1} \leftarrow \boldsymbol{w}^t + \frac{1}{\lambda n}\Delta\alpha_{i_t}^t \boldsymbol{x}_{i_t}$;
  if *(t-1) mod n* $\geq$ *n-k* **then**
    for $i = 1, 2, \ldots, n$
      Calculate $\ell'(\langle \boldsymbol{x}_i, \boldsymbol{w}^t \rangle, y_i)$;
      Set $\chi_i^t \leftarrow \ell'(\langle \boldsymbol{x}_i, \boldsymbol{w}^t \rangle, y_i)\boldsymbol{x}_i + \lambda \nabla r(\boldsymbol{w}^t)$;
      Record $c_i \leftarrow \max\{c_i, \|\chi_i^t\|\}$;
    end
  end
  if *t mod n* $= 0$ **then**
    Option I: Run Algorithm 3 (Aggressive Update);
    Option II: Run Algorithm 4 (Conservative Update);
  end
end
Output: $\boldsymbol{w}^{T+1}$

$c_i \leftarrow \max\{c_i, \|\chi_i^t\|\}$;

Part C

Discussions and Experiments

Datasets for empirical study

Dataset	Training(n)	Test	Features (d)	Sparsity($\frac{\text{nnz}}{nd}$)
rcv1	20,242	677,399	47,236	0.16%
astro-ph	29,882	32,487	99,757	0.08%

- **rcv1** is a corpus from Reuters news stories.
- **astro-ph** is astronomy data.

Cost per epoch and properties of algorithms

ALGORITHM	cost of an epoch	non-uniform	adaptive
NonUnifSGD	**nnz**	✓	✗
NonUnifSDCA	**nnz**	✓	✗
AdaSGD	$(k+1)$ nnz	✓	✓
AdaSVRG	$nd + k$ nnz	✓	✓
AdaSDCA	$(k+1)$ nnz	✓	✓
AdaSDCAS	$(k+1)$ nnz	✓	✓
AdaGrad (by Duchi)	$2nd$	✗	✗
AdaSDCA (by Csiba)	n nnz	✓	✓
AdaSDCA+ (by Csiba)	2 nnz	✓	✓

nnz: is the number of nonzero elements of the matrix consisting of all the samples in the dataset.

Test Error with Different Values of λ

`rcv1`	1e-2	5e-3	**1e-3**	5e-4	1e-4
Test Error	0.05160	0.04833	**0.04713**	0.04913	0.05693

`astro-ph`	1e-2	5e-3	**1e-3**	5e-4	1e-4
Test Error	0.04103	0.03715	**0.03441**	0.03586	0.04371

Verifying the Convergence of Duality Gap

Table: Average duality gap at different epochs for $\lambda = 0.001$

#epoch	duality gap on `rcv1`	duality gap on `astro-ph`
1	0.0863765	0.0883917
3	0.0105347	6.13163e-03
10	1.7485e-04	3.93673e-05
20	2.21547e-05	6.24779e-07
50	3.12797e-06	6.7474e-10
100	**5.47897e-07**	**1.43083e-12**

Performance Metrics

Definition

The **primal sub-optimality** of algorithm is defined as $f(\boldsymbol{w}) - f(\boldsymbol{w}^*)$.

Definition

Test error is the error rate on test dataset.

We calculate the value by

$$\ln(f(\boldsymbol{w}) - f(\boldsymbol{w}^*) + \epsilon).$$

Performance of Two Updating Algorithms

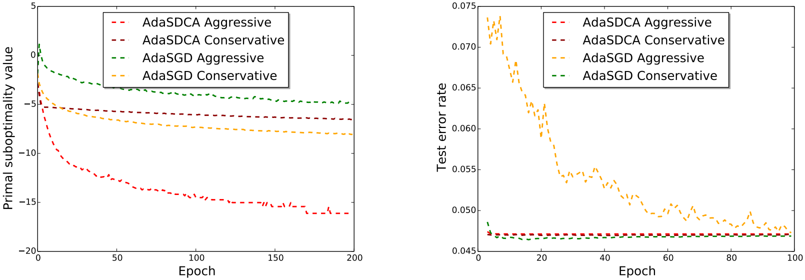
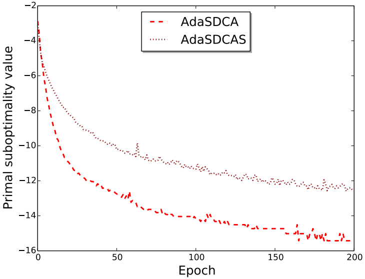


Figure: Comparison of two updating algorithms for AdaSGD and AdaSDCA on `rcv1`

Different Adaptive Strategies for AdaSDCA



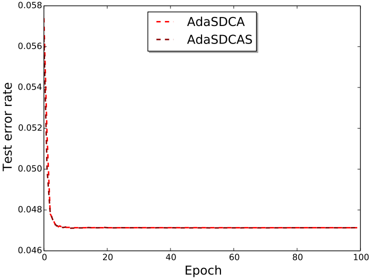


Figure: Comparison of AdaSDCA and AdaSDCAS on `rcv1`

Comparison of Adaptive Algorithms

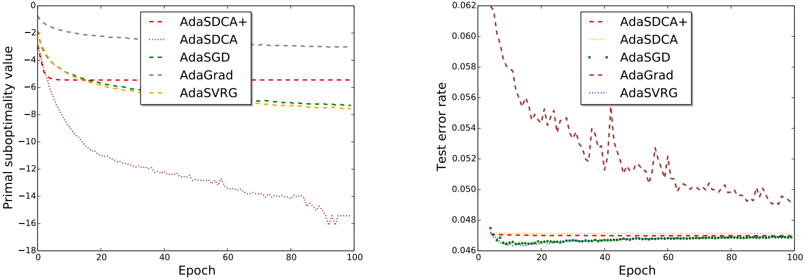


Figure: Comparison of five adaptive algorithms on `rcv1`

Comparison of Adaptive Algorithms cont.

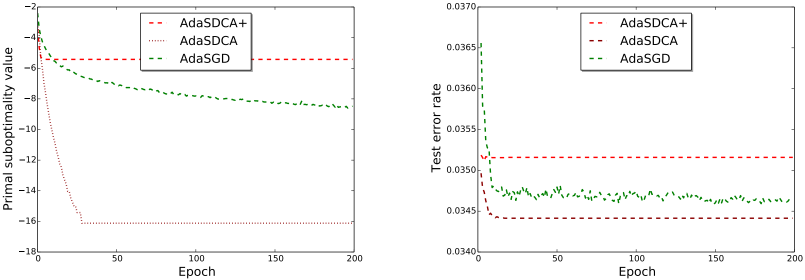


Figure: Comparison of three adaptive algorithms on `astro-ph`

Comparison of Average Time

Table: Detailed training time and total running time per epoch

rcv1	Training time(s)	Total running time(s)
AdaSGD	0.04765	0.2059
AdaSDCA	0.05042	0.2064
NonUnifSGD	**0.04244**	**0.1988**
NonUnifSDCA	0.04716	0.2037

astro-ph	Training time(s)	Total running time(s)
AdaSGD	0.07236	0.1363
AdaSDCA	0.07050	0.1343
NonUnifSGD	**0.06284**	**0.1259**
NonUnifSDCA	0.07054	0.1339

Comparison of Time

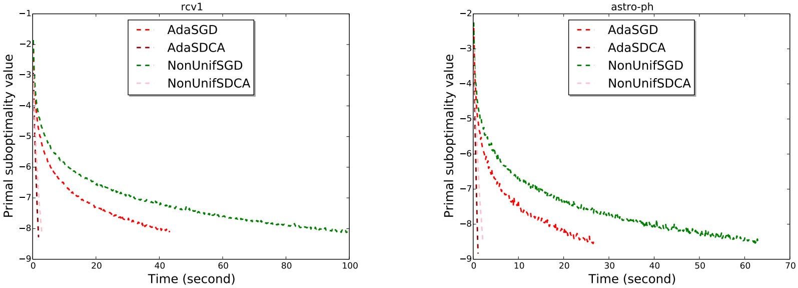


Figure: Comparison of the total running time to reach the same optimality

Same Level of Optimality

Table: The number of epochs taken to reach the same level of optimality

rcv1	AdaSDCA	NonUnifSDCA	AdaSGD	NonUnifSGD
#epochs	**9**	35	210	500

astro-ph	AdaSDCA	NonUnifSDCA	AdaSGD	NonUnifSGD
#epochs	**8**	28	195	500

Comparison of Vector Operation

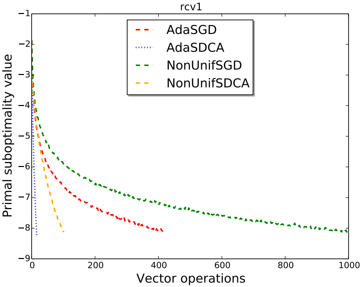
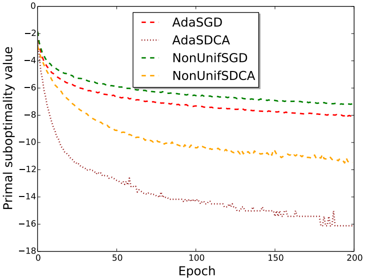


Figure: Comparison of the vector operations taken to reach the same optimality

Adaptive vs. Non-Uniform Algorithms



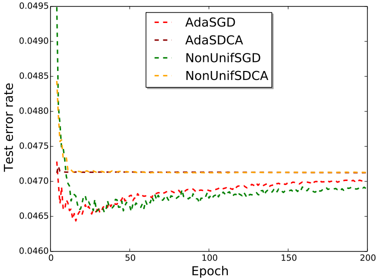


Figure: Comparison of adaptive algorithms with non-adaptive algorithms on `rcv1`

Adaptive vs. Non-Uniform Algorithms cont.

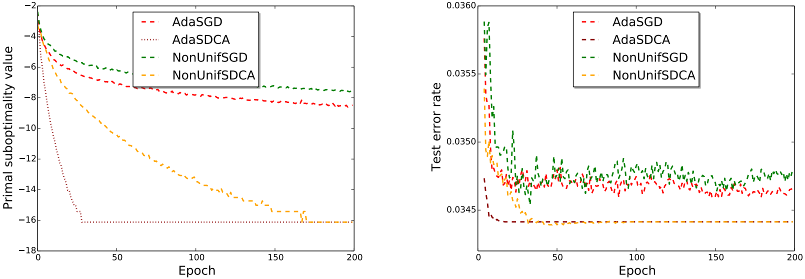


Figure: Comparison of adaptive algorithms with non-adaptive algorithms on `astro-ph`

Summary

- Conservative Update works better on AdaSGD while Aggressive Update works better on AdaSDCA.
- AdaSDCA (adaptive algorithm with duality gap) performs better than AdaSDCAS (adaptive algorithm with subgradient).
- AdaSDCA has the best performance among all the adaptive algorithms (AdaSDCA, AdaSGD, AdaSVRG, AdaGrad and AdaSDCA+) and AdaSGD is the second best.
- AdaSVRG achieves a slightly better performance per epoch than AdaSGD but sacrifices the running time on sparse datasets.

Reference

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Q&A

Thank You!

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