Probability vector, Markov chains, stochastic matrix

Section 4.9 Applications to Markov Chains

- A **Probability vector** is a vector with nonnegative entries that add up to 1.
- A **stochastic matrix** is a square matrix whose columns are probability vectors.
- A **Markov chain** is a sequence of probability vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \cdots$, together with a stochastic matrix P, such that

$$\mathbf{x}_1 = P\mathbf{x}_0, \mathbf{x}_2 = P\mathbf{x}_1, \mathbf{x}_3 = P\mathbf{x}_2, \cdots$$

A Markov chain of vectors in $\mathbb{R}^n$ describes a system or a sequence of experiments. $\mathbf{x}_k$ is called **state vector**. An example is the crunch and munch breakfast problem.

Example: model for population movement

Consider a population movement between city and suburbs in a metropolitan region governed by the *migration matrix* M:

	From		
	City	Suburbs	To:
M=	.95	.03	City
	.05	.97	Suburbs

- Each column vector of M is a probability vector.
- M is a stochastic matrix
- Suppose that the population in 2006 is: 600,000 in the city and 400,000 in the suburbs.

The initial state (probability vector) is: $\mathbf{x}_0 = \begin{bmatrix} .60 \\ .40 \end{bmatrix}$

Find the population distribution in the year 2007 and 2008.
Solution. The population distribution in 2007 is given by $\mathbf{x}_1 = M\mathbf{x}_0$, i.e.,

$$\begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} .60 \\ .40 \end{bmatrix} = \begin{bmatrix} .582 \\ .418 \end{bmatrix}.$$

- 58.2% of the region will live in the city in 2007.
- 41.8% of the region will live in the suburbs in 2007.

$\mathbf{x}_2 = M\mathbf{x}_1$,

To find the population distribution in the year 2008, compute $\mathbf{x}_2 = M\mathbf{x}_1$, i.e.,

$$\begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} .582 \\ .418 \end{bmatrix} = \begin{bmatrix} .565 \\ .435 \end{bmatrix}.$$

- 56.5% of the region will live in the city in 2008.
- 43.5% of the region will live in the suburbs in 2008.

Steady state vectors

- If P is a stochastic matrix, then a **steady state vector (or equilibrium vector)** for P is a probability vector $\mathbf{v}$ such that
$$\boxed{P\mathbf{v} = \mathbf{v}}$$
.

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Rent-a-Lemon has three locations from which to rent a car for one day: Airport, downtown and the valley

Daily Migration:

Rented From				
Airport	Downtown	Valley		
.95	.02	.05	Airport	**Returned To**
.03	.90	.05	Downtown	
.02	.08	.90	Valley	

Airport

Downtown ***Valley***

$$M = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix}$$

(migration matrix)

$$\mathbf{x}_0 = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} \quad \begin{matrix} \text{(initial fraction of cars at } \textit{airport}\text{)} \\ \text{(initial fraction of cars } \textit{downtown}\text{)} \\ \text{(initial fraction of cars at } \textit{valley} \text{ location)} \end{matrix}$$

(initial distribution vector which is a *probability vector*)

Interpretation of $M\mathbf{x}_0$

$$M\mathbf{x}_0 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix} \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} =$$

$$.5 \begin{bmatrix} .95 \\ .03 \\ .02 \end{bmatrix} + .3 \begin{bmatrix} .02 \\ .90 \\ .08 \end{bmatrix} + .2 \begin{bmatrix} .05 \\ .05 \\ .90 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \qquad\qquad & \uparrow & \qquad\qquad & \uparrow \\ \text{Redistribution} & & \text{Redistribution} & & \text{Redistribution} \\ \text{of airport} & & \text{of downtown} & & \text{of valley} \\ \text{cars} & & \text{cars} & & \text{cars} \end{matrix}$$

Distribution after one day=

$$\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix}\begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} = \begin{bmatrix} 0.491 \\ 0.295 \\ 0.214 \end{bmatrix}$$

$$\mathbf{x}_{k+1} = M\mathbf{x}_k \text{ for } k = 0, 1, 2, \ldots$$

(Markov Chain)

Distribution after two days=

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix}\begin{bmatrix} 0.491 \\ 0.295 \\ 0.214 \end{bmatrix} = \begin{bmatrix} 0.483 \\ 0.290 \\ 0.226 \end{bmatrix}$$

$$\mathbf{x}_3 = M\mathbf{x}_2 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix}\begin{bmatrix} 0.483 \\ 0.290 \\ 0.226 \end{bmatrix} = \begin{bmatrix} 0.475 \\ 0.287 \\ 0.236 \end{bmatrix}$$

$$\mathbf{x}_4 = M\mathbf{x}_3 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix}\begin{bmatrix} 0.475 \\ 0.287 \\ 0.236 \end{bmatrix} = \begin{bmatrix} 0.468 \\ 0.284 \\ 0.244 \end{bmatrix}$$

$$\vdots$$

$$\mathbf{x}_{49} = M\mathbf{x}_{48} = \begin{bmatrix} 0.417 \\ 0.278 \\ 0.305 \end{bmatrix}$$

$$\mathbf{x}_{50} = M\mathbf{x}_{49} = \begin{bmatrix} 0.417 \\ 0.278 \\ 0.305 \end{bmatrix} \text{ (long term distribution)}$$

$$\vdots$$

$\mathbf{x} = \begin{bmatrix} 0.417 \\ 0.278 \\ 0.305 \end{bmatrix}$ is called a **steady state vector** since $\mathbf{x} = M\mathbf{x}$

Finding the Steady State Vector

$$M\mathbf{x} = \mathbf{x}$$

$$M\mathbf{x} = I\mathbf{x}$$

$$M\mathbf{x} - I\mathbf{x} = \mathbf{0}$$

$$(M - I)\mathbf{x} = \mathbf{0}$$

Solve $(M - I)\mathbf{x} = \mathbf{0}$ to find the steady state vector. Note that the solution $\mathbf{x}$ must be a *probability vector*.

EXAMPLE: Suppose that 3% of the population of the U.S. lives in the State of Washington. Suppose the migration of the population into and out of Washington State will be constant for many years according to the following migration probabilities. What percentage of the total U.S. population will eventually live in Washington?

From :			
WA	Rest of U.S.	**To**:	
$.9$	$.01$	WA	
$.1$	$.99$	Rest of U.S.	

Solution

$$M = \begin{bmatrix} .9 & .01 \\ .1 & .99 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \text{\% of people in WA} \\ \text{\% in rest of U.S.} \end{bmatrix}$$

$$M - I = \begin{bmatrix} .9 & .01 \\ .1 & .99 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.01 \\ 0.1 & -0.01 \end{bmatrix}$$

Solve $(M-I)\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} -0.1 & 0.01 & 0 \\ 0.1 & -0.01 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.1x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$$

One solution: $\mathbf{x} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$

Solution we want has entries which add up to one:

$$\mathbf{x} = \begin{bmatrix} 1/11 \\ 10/11 \end{bmatrix} \approx \begin{bmatrix} 0.091 \\ 0.909 \end{bmatrix}$$

The convergence of the state vectors to the steady state (fixed point of the the stochastic matrix) occurs very frequently. The feature required for this convergence is called regularity of P.

- A stochastic matrix P is **regular** if, for some integer k, $P^k > 0$ (all entries are strictly positive).

Example. Let $P = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$ Then P^2 is

$$\begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} = \begin{bmatrix} .37 & .26 & .33 \\ .45 & .70 & .45 \\ .18 & .04 & .22 \end{bmatrix}.$$

Hence P is regular.

Theorem

If P is an $n \times n$ regular stochastic matrix, then P has a unique steady state vector $\mathbf{v}$. Further, if $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, 2, \cdots$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{v}$.

Remark. The initial state does not affect the long time behavior of the Markv chain.

Application to Markov Chains

EXAMPLE Consider the migration matrix $M = \begin{bmatrix} .95 & .90 \\ .05 & .10 \end{bmatrix}$

and define $\mathbf{x}_{k+1} = M\mathbf{x}_k$. It can be shown that

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$$

converges to a steady state vector $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. Why?

The answer lies in examining the corresponding eigenvectors.

First we find the eigenvalues:

$$\det(M - \lambda I) = \det\left(\begin{bmatrix} .95 - \lambda & .90 \\ .05 & .10 - \lambda \end{bmatrix}\right) = \lambda^2 - 1.05\lambda + 0.05$$

So solve

$$\lambda^2 - 1.05\lambda + 0.05 = 0$$

By factoring

$$\lambda = 0.05, \ \lambda = 1$$

It can be shown that the eigenspace corresponding to $\lambda = 1$ is span$\{\mathbf{v}_1\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the eigenspace corresponding to $\lambda = 0.05$ is span$\{\mathbf{v}_2\}$ where $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Note that

$$M\mathbf{v}_1 = \mathbf{v}_1,$$

and so $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ is our steady state vector.

Then for a given vector $\mathbf{x}_0$,

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

$$\mathbf{x}_1 = M\mathbf{x}_0 = M(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1M\mathbf{v}_1 + c_2M\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2(0.05)\mathbf{v}_2$$

$$\mathbf{x}_2 = M\mathbf{x}_1 = M(c_1\mathbf{v}_1 + c_2(0.05)\mathbf{v}_2) = c_1M\mathbf{v}_1 + c_2(0.05)M\mathbf{v}_2 = c_1\mathbf{v}_1 +$$

and in general

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(0.05)^k\mathbf{v}_2$$

and so $\lim_{k\to\infty}\mathbf{x}_k = \lim_{k\to\infty}\left(c_1\mathbf{v}_1 + c_2(0.05)^k\mathbf{v}_2\right) = c_1\mathbf{v}_1$

and this is the steady state when $c_1 = \frac{1}{2}$.